New Higher Order Iterative Method for Multiple Roots of Nonlinear Equations

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Abstract In this paper, we propose a new higher order iterative method to find multiple roots of nonlinear equations. The combination of Taylor's series, Newton's method and the composition approach are used to derive the new method. It requires three evaluations of the function and two evaluations of the derivative of the function per iteration. The theoretical convergence of the proposed method is proved in the main theorem which establishes sixth order of convergence. We compare the developed method with well-known equivalent existing methods by taking various numerical examples. The numerical results demonstrate the better efficiency of the developed method as compared to some standard iterative methods.

Keywords Multiple roots · Nonlinear equation · Iterative methods · Error

1 Introduction

Solving nonlinear equations is one of the most important problems in applied mathematics, engineering and science. Sometimes, analytical methods are not applicable to solve nonlinear equations. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a nonlinear differentiable function defined on an open interval $\mathbb{D}$ such that

$$\psi(x) = 0 \tag{1}$$

We use iterative method for solving such nonlinear equations (1), which is defined as

$$x_{n+1} = P(\psi)(x_n) \;\; for \;\; n \; = \; 1, \; 2, \; 3, \ldots \tag{2}$$

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where $P(\psi)$ is called the iterative function. Newton's method (NM) [1, 2] is perhaps the most popular root-finding method for solving nonlinear equations, and it is given by

$$x_{n+1} = x_n - \frac{\psi(x_n)}{\psi'(x_n)} \tag{3}$$

This is quadratically convergent in some neighbourhood of simple roots. Let α be the root of equation (1) with multiplicity $\nu > 1$, i.e. $\psi^{(i)}(\alpha) = 0$ for $i = 1, 2, ..., \nu - 1$ and $\psi^{(\nu)}(\alpha) \neq 0$. When used for finding multiple roots of such nonlinear equations, Newton's method (3) is linearly convergent. The modified Newton's method [3] is written as

$$x_{n+1} = x_n - \nu\frac{\psi(x_n)}{\psi'(x_n)} \tag{4}$$

This is quadratically convergent for the equation having multiple roots with multiplicity $\nu > 1$. Many researchers have developed iterative methods using the modified Newton method for solving nonlinear equations having multiple roots.

In 2013, Thukral [4] developed the following new six-order method (TM for short) for finding multiple roots of a nonlinear equation:

$$\begin{aligned} y_n &= x_n - \nu\frac{\psi(x_n)}{\psi'(x_n)} \\ z_n &= x_n - \nu\left(\sum_{i=1}^{3} i\left(\frac{\psi(y_n)}{\psi(x_n)}\right)^{i/\nu}\right)\left(\frac{\psi(x_n)}{\psi'(x_n)}\right) \\ x_{n+1} &= z_n - \nu\left(\sum_{i=1}^{3} i\left(\frac{\psi(y_n)}{\psi(x_n)}\right)^{i/\nu}\right)^2\left(\frac{\psi(z_n)}{\psi(x_n)}\right)^{\nu^{-1}}\left(\frac{\psi(x_n)}{\psi'(x_n)}\right) \end{aligned} \tag{5}$$

where $n \in \mathbb{N}$.

Geum et al. [5] also developed a new sixth-order method (GM for short) in 2016 which is written as follows:

$$\begin{aligned} y_n &= x_n - \nu\frac{\psi(x_n)}{\psi'(x_n)} \\ z_n &= x_n - \nu Q_\psi(s_n)\frac{\psi(x_n)}{\psi'(x_n)} \\ x_{n+1} &= x_n - \nu K_\psi(s_n, t_n)\frac{\psi(x_n)}{\psi'(x_n)} \end{aligned} \tag{6}$$

where Q_ψ and K_ψ are weight functions, $s_n = \left(\frac{\psi(y_n)}{\psi(x_n)}\right)^{\frac{1}{\nu}}$ and $t_n = \left(\frac{\psi(z_n)}{\psi(x_n)}\right)^{\frac{1}{\nu}}$.

In 2017, Qudsi et al. [6] developed the following method (QM for short) of sixth order:

$$
\begin{aligned}
y_n &= x_n - t \\
z_n &= x_n - t\left(1 + \frac{\psi(y_n)}{\psi(x_n)}\left(1 + 2\frac{\psi(y_n)}{\psi(x_n)}\right)\right) \\
x_{n+1} &= x_n - t\left(1 + \frac{\psi(y_n)}{\psi(x_n)}\left(1 + 2\frac{\psi(y_n)}{\psi(x_n)}\right) + \frac{\psi(z_n)}{\psi(x_n)}\left(1 + 2\frac{\psi(y_n)}{\psi(x_n)}\right)\right)
\end{aligned} \tag{7}
$$

where $t = \frac{2\psi^2(x_n)}{\psi(x_n+\psi(x_n))-\psi(x_n-f(x_n))}$.

Moreover, Singh et al. [7] developed a new fourth-order method in 2015. In the year 2019, Bhel et al. [8] developed multiple roots version of Ostrowski's method having fourth order of convergence. W. H. Chanu et al. also proposed an iterative method of fifth order in [9], Qudsi et al. [10] developed a new iterative method of order six, Kattri [11] proposed a new sixth-order iterative method, etc. In this work, we have introduced a higher order iterative method for solving nonlinear equations having multiple roots. In the following sections, we present the development of our new method, numerical results and conclusion.

2 Development of the Method

In this section, we propose a new sixth-order method for determining the multiple roots of nonlinear equation (1) with multiplicity $\nu > 1$ as follows:

$$
\begin{aligned}
y_n &= x_n - \frac{2\nu}{\nu+1}\frac{\psi(x_n)}{\psi'(x_n)} \\
z_n &= x_n - \frac{\psi(y_n)(\nu^2-1) - \left(\frac{\nu-1}{\nu+1}\right)^{\nu}(\nu(\nu-4)-1)\psi(x_n)}{4\left(\frac{\nu-1}{\nu+1}\right)^{\nu}\psi'(x_n)} \\
x_{n+1} &= z_n - \nu\frac{\psi(z_n)}{\psi'(z_n)}
\end{aligned} \tag{8}
$$

Theorem 1 *Let $\alpha \in \mathbb{R}$ be a multiple root of multiplicity ν of a sufficiently differentiable function $\psi : \mathbb{D} \to \mathbb{R}$ in an open interval $\mathbb{D}$ which is a subset of $\mathbb{R}$. Let x_0 be an initial guess of the root α. Then, the method defined by (8) has six orders of convergence.*

Proof Let α be a root of multiplicity ν of $\psi(x) = 0$ and let $e_n = x_n - \alpha$ be the error at nth iteration. Then, using Taylor expansion, we get

$$
\psi(x_n) = \left(\frac{\psi^{(\nu)}(\alpha)}{\nu!}\right)e_n^{\nu}[1 + C_1e_n + C_2e_n^2 + C_3e_n^3 + C_4e_n^4 + C_5e_n^5 + C_6e_n^6 + O[e_n]^7] \tag{9}
$$

$$\psi'(x_n) = \left(\frac{\psi^{(\nu)}(\alpha)}{(\nu-1)!}\right)e_n^{\nu-1}\left[1 + \left(\frac{\nu+1}{\nu}\right)C_1 e_n + \left(\frac{(\nu+2)}{\nu}\right)C_2 e_n^2 + O[e_n]^7\right] \tag{10}$$

$$\tilde{e_n} = y_n - \alpha = B_1 e_n + B_2 e_n^2 + B_3 e_n^3 + B_4 e_n^4 + B_5 e_n^5 + B_6 e_n^6 + O[e_n]^7 \tag{11}$$

where

$$B_1 = \frac{\nu-1}{\nu+1},$$
$$B_2 = \frac{2C_1}{\nu+\nu^2},$$
$$B_3 = \frac{2\left(\frac{2\nu C_2}{1+\nu} - C_1^2\right)}{\nu^2}$$

$$B_4 = \frac{2((1+\nu)^2 C_1^3 - \nu(4+3\nu)C_1C_2 + 3\nu^2 C_3)}{\nu^3(1+\nu)}$$
$$B_5 = \frac{1}{\nu^4(1+\nu)}\Big(-2((1+\nu)^3 C_1^4 - 2\nu(1+\nu)(3+2\nu)C_1^2C_2$$
$$+ 2\nu^2(3+2\nu)C_1C_3 + 2\nu^2((2+\nu)c_2^2 - 2\nu C_4))\Big)$$
$$B_6 = \frac{1}{\nu^5(1+\nu)}\Big(2((1+\nu)^4 C_1^5 - \nu(1+\nu)^2 \times (8+5\nu)C_1^3C_2$$
$$+ \nu^2(1+\nu)(9+5\nu)C_1^2C_3 + \nu^2 C_1((2+\nu)(6+5\nu)C_2^2 - \nu(8+5\nu)C_4)$$
$$+ \nu^3(-(12+5\nu)C_2C_3 + 5\nu C_5))\Big)$$

$$\psi(y_n) = e_n^{\nu}\frac{\psi^{(\nu)}(\alpha)}{\nu!}[((\frac{\nu-1}{\nu+1})^{\nu} + D_1 e_n + D_2 e_n^2 + D_3 e_n^3) + \frac{1}{3\nu^2}D_4 e_n^4$$
$$+ \frac{1}{15\nu^4}D_5 e_n^5 + O[e_n]^6] \tag{12}$$

where

$$D_1 = \frac{(\frac{\nu-1}{\nu+1})^{\nu}(\nu^2+3)C_1}{\nu^2-1}$$
$$D_2 = \frac{(\nu-1)^{\nu-1}(\nu+1)^{-\nu-2}(-2(\nu+1)^2C_1^2 + \nu(3+\nu(11+\nu+\nu^2))C_2)}{\nu})$$
$$D_3 = \frac{1}{3\nu^2}\Big((\nu^2-1)^{\nu-2}(\nu+1)^{-\nu-3}(2(\nu+1)^4(3\nu-4)C_1^3 - 24(\nu-1)\nu^2$$

$$\times (\nu+1)(\nu+2)C_1C_2 + 3(\nu-1)\nu^2(7+\nu(14+\nu(24+\nu(2+\nu))))C_3)\Big)$$

$$D_4 = (\nu-1)^{\nu-3}(\nu+1)^{-\nu-4}(-2(\nu+1)^4(3+\nu(3+\nu(-9+(-2+3\nu))))C_1^4$$
$$+ 6(\nu-1)\nu(\nu+1)^3(-2+\nu(-15+\nu(4+5\nu)))C_1^2C_2 - 12(-1+\nu)^2\nu^2$$
$$\times (\nu+1)C_1C_3 + 3(-1+\nu)^2\nu^2(-8(\nu+1)^2(-1+\nu(2+\nu))C_2^2$$
$$+ \nu(7+\nu(37+\nu(42+\nu)))C_4))$$

$$D_5 = (\nu-1)^{\nu-4}(\nu+1)^{-\nu-5}(2(\nu+1)^5(\nu+2)(-9+\nu(-2+\nu(22+15(1$$
$$+ (-3+\nu)\nu))))C_1^5 - 20(\nu-1)\nu^2(\nu+1)^4(30+\nu(-11+\nu(-49+\nu$$
$$\times (9\nu+5))))C_1^3C_2 + 30\nu^2(\nu^2-1)^2(-18+\nu(-7+\nu(-34+\nu(-16+$$
$$\times \nu(20+7\nu))))) + 120(\nu-1)^2\nu^2(\nu+1)^2(2+\nu(\nu+1)(-9+2\nu(\nu+1)))C_2^2$$
$$- 2(\nu-1)\nu^2(\nu+4)(\nu^2+1)C_4 + 15(\nu-1)^3\nu^4(-8(\nu+1)^2(-2$$
$$+ 3\nu(\nu+3))C_2C_3 + (11+\nu(44+\nu(115+\nu(80+\nu(65+\nu(\nu+4))))))C_5))$$

Using the expression of Eqs. (9), (10), (11) and (12) in the second step of the proposed method defined in Eq. (8), we get

$$\hat{e}_n = z_n - \alpha = E_1e_n^3 + E_2e_n^4 + E_3e_n^5 + E_4e_n^6 + O[e_n]^7 \tag{13}$$

where

$$E_1 = \frac{1}{2\nu^2(1+\nu)}\big((1+\nu)^2C_1^2 - 2(-1+\nu)\nu C_2\big)$$

$$E_2 = \frac{1}{6((-1+\nu)\nu^3(1+\nu)^2)}\big((1+\nu)^4(-7+6\nu)C_1^3 - 6(-1+\nu)\nu(1+\nu)$$
$$\times (-1+\nu(4+3\nu))C_1C_2 + 6(-1+\nu)^2\nu^2(1+3\nu)C_3\big)$$

$$E_3 = \frac{1}{6(-1+\nu)^2\nu^4(1+\nu)^3}\big((1+\nu)^4(10+\nu(4+\nu(-22-3\nu+9\nu^2)))C_1^4$$
$$- 6(-1+\nu)\nu(1+\nu)^3(-1+\nu(-13+4\nu+6\nu^2))C_1C_2 + 12(-1+\nu)^2\nu^2$$
$$\times (1+\nu^2(2+\nu)(2+3\nu))C_1C_3 + 6\nu^2((-1+\nu^2)^2(-4+$$
$$\times \nu(5+3\nu))C_2^2 - 2(-1+\nu)^3\nu(1+\nu(2+3\nu))C_4)\big)$$

$$E_4 = \frac{1}{30(-1+\nu)^3\nu^5(1+\nu)^4}\big(-(1+\nu)^5(-68+\nu(-33+\nu(202$$
$$+ \nu(87+10\nu(-23-3\nu+6\nu^2)))))C_1^3 + 5(-1+\nu)\nu(1+\nu)^4(20+$$
$$\times \nu(147+\nu(-77+\nu(-227+15\nu(3+4\nu)))))C_1^3C_2 - 15\nu^2(-1+\nu^2)^2(-25+$$
$$\times \nu(-10+\nu(-52+\nu(-26+5\nu(9+4\nu)))))C_1^2C_3 + 60(-1+\nu)^2\nu^2(1+\nu)C_1$$
$$\times ((-1+\nu)^2(5+\nu(-16+\nu(-12+5\nu(2+\nu))))C_2^2 + (-1$$
$$+ \nu)\nu(-1+\nu(6+5\nu^2(2+\nu)))C_4) + 30(-1+\nu)^3\nu^3(5(1+$$
$$\times \nu)^2(-1+\nu(-2+\nu(5+2\nu)))C_2C_3 + 2\nu(1+4\nu-5\nu^4C_5))\big)$$

Using the expression of z_n from Eq. (13) in the third step of the proposed method defined by Eq. (8), we get

Table 1 Test function with initial guess x_0 and multiplicity ν

Test function $\psi(x)$	Initial guesses	Multiplicity
$\psi_1(x) = (cos(x) + x)^{15}$	–0.9	15
$\psi_2(x) = ((x-1)^{10} - 1)^6$	–0.1	6
$\psi_3(x) = (x^3 + x + 1)^6$	–0.8	10
$\psi_4(x) = (sin(x^2) - x^2 + 1)^{66}$	1.6	66
$\psi_5(x) = (2 - x + e^{3+x-x^2})^9$	2.49	9

Table 2 Comparison of various iterative methods

$\|\psi(x_n)\|$	TM	GM	QM	NPM
$\|\psi_1(x_n)\|$	4.4266×10^{-9}	$2.6244 \times 10^{-22646}$	2.2574×10^{-1923}	$1.2291 \times 10^{-24937}$
$\|\psi_2(x_n)\|$	1.9439	4.4745×10^{-2050}	8.8343×10^{-4052}	3.0102×10^{-3150}
$\|\psi_3(x_n)\|$	6.1462×10^{-6}	$1.4433 \times 10^{-10001}$	2.5167×10^{-917}	$2.6274 \times 10^{-12555}$
$\|\psi_4(x_n)\|$	5.5942×10^{-19}	$1.0763 \times 10^{-53877}$	2.0935×10^{-349}	$3.7107 \times 10^{-66773}$
$\|\psi_5(x_n)\|$	6.6823×10^{-26}	$1.8141 \times 10^{-22160}$	9.9962×10^{-3692}	$2.2123 \times 10^{-31653}$

$$e_{n+1} = \frac{C_1((\nu+1)^2 C_1^2 - 2(\nu-1)\nu C_2)^2}{4\nu^5(1+\nu)^2} e_n^6 + O[e_n]^7 \tag{14}$$

Equation (14) shows that the newly developed method defined by (8) has sixth order of convergence.

3 Numerical Results

In this section, we analyse the computational efficiency of the introduced iterative method (8) using several test functions and compare it with other existing methods. In Table 2, we have displayed the comparison of the convergence of the methods. Table 2 shows the absolute residual error ($|\ \psi(x_n)\ |$) of the functions after four full iterations of the methods have been completed. We have compared the newly proposed method (NPM for short) defined in Eq. (8) with the methods given in Eqs. (5), (6) and (7) denoted by TM [4], GM [5] and QM [6], respectively. Mathematica 11.3 software has been used to generate the numerical results in Table 2.

4 Conclusion

We have introduced a new sixth-order iterative method based on Newton's method for finding multiple roots of nonlinear equations. We compare the newly introduced method with existing methods having the same convergence order using some examples of nonlinear equations. The results given in Table 2, have demonstrated the superiority of the introduced method as compared to the existing methods even though the same examples with the same initial guess are used. It affirms that the introduced iterative method has smaller $|\psi(x)|$ and simple asymptotic error terms. Therefore, the introduced method is efficient than the other equivalent methods in comparison to finding multiple roots of nonlinear equations.

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