Positivity Preserving Rational Quartic Spline Zipper Fractal Interpolation Functions

Vijay and A. K. B. Chand

Abstract In this paper, we introduce a class of novel C^1-rational quartic spline zipper fractal interpolation functions (RQS ZFIFs) with variable scalings, where rational spline has a quartic polynomial in the numerator and a cubic polynomial in the denominator with two shape control parameters. We derive an upper bound for the uniform error of the proposed interpolant with a C^3 data generating function, and it is shown that our fractal interpolant has $O(h^2)$ convergence and can be increased to $O(h^3)$ under certain conditions. We restrict the scaling functions and shape control parameters so that the proposed RQS ZFIF is positive, when the given data set is positive. Using this sufficient condition, some numerical examples of positive RQS ZFIFs are presented to support our theory.

Keywords Fractals · Positivity · Rational quartic spline · Zipper · Zipper smooth fractal function

AMS subject classifications 28A80 · 41A05 · 41A20 · 41A25 · 65D10

1 Introduction

To find a nice interpolation curve with various attributes is an active area of research in numerical analysis, approximation theory, wavelets, classical and discrete geometry, engineering design, civil engineering and computer science. From the last many decades, researchers have come up with various types of interpolants that have advantages over one another. Polynomial interpolations are preferred when the original function is sufficiently smooth. For some fixed order of smoothness, different types of spline (polynomial/trigonometric/exponential/rational) interpolants are

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used. Rational spline interpolants with shape parameters are more flexible over other type of spline interpolants, and hence popular in geometric modelling problems for discrete data visualization. These have been utilized from animated films to simulated surgery. For the classical positivity preserving rational splines one can see [1, 15, 16, 23, 25, 33]. Schmidt and Heß in [25] discussed positive interpolation with quadratic and rational quadratic spline and observed that rational quadratic splines have an advantage over quadratic splines. Sakai and Schmidt in [23] presented a class of C^2 positivity-preserving rational spline using two local control parameters with the cubic numerator and linear denominator. Using cubic numerator and quadratic denominator, Abbas et al. in [1] constructed a C^2 rational cubic spline with three shape parameters. They derived the shape feature of data using a single shape parameter and the other two shape parameters were left free for the designer to adjust the shape of positive curves as per industrial requirements. Hussain and Sarfraz in [16] constructed a C^1 piecewise rational cubic spline with four parameters to visualize positive data set. Two parameters are constrained for the presentation of positive curves through positive data while the other two provide extra freedom to vary the curve shape as needed. Han in [15] presented a piecewise rational spline with the quartic numerator and quadratic denominator. He derived the shape-preservation properties like positivity, monotonicity and convexity of the interpolant. But these non-recursive classical interpolants are either smooth or piecewise smooth and consequently, they are not differentiable at the finite number of points. But if the data is taken from an irregular and non-smooth function, these classical interpolants are not good approximants for it.

Non-smooth and irregular curves such as profiles of mountain ranges, tops of clouds, lightning, ECG curves, turbulence, etc. cannot be interpolated by classical interpolants. The term fractal was given by Mandelbrot [19] to unify the irregular and complex structures. After that many researchers worked on it and expand its theory. To construct fractals, Hutchinson [17] introduced the concept of iterated function system (IFS). The fractal-based theory is a new tool to analyse various non-linear complex phenomena in nature, sciences and engineering. With the help of some parameters, we can easily model most of these complex phenomena by using self-referential rules. Using the theory of IFS, Barnsley [5] created fractal interpolation functions (FIFs) to generate non-smooth and irregular curves from their data points [6] and proved the existence and uniqueness of fractal interpolation function for a hyperbolic IFS with fixed parameters. Barnsley and Harrington [7] constructed r-times differentiable polynomial spline with fixed type of boundary conditions to interpolate functions that have fractality in their higher-order derivatives. For all kinds of boundary conditions, Chand and Kapoor [8] constructed cubic spline FIFs using moments. For application of FIF in data visualization, Chand and collaborators have proposed shape-preserving fractal interpolants, see for instance [9, 10, 12, 18, 29, 30]. Akhtar et al. in [20] introduced a group of fractal functions on the unit sphere through a linear bounded fractal operator and presented some approximation properties. Balasubramani et al. [4] constructed rational cubic spline α-fractal functions with three shape parameters that can preserve positivity and monotonicity. They have also found the conditions on the IFS parameters so that the proposed interpolant is constrained between two piecewise linear functions. But most of the development in shape-preserving FIF theory, the authors have used constant scaling factors, whereas fractal functions with variable scalings provide more flexibility. Using variable scaling, Wang and Shan [32] generated FIFs to approximate functions with less self-similarity and studied their analytical properties such as smoothness, stability and sensitivity. Gowrisankar and Guru Prem Prasad [14] investigated Riemann-Liouville fractional calculus of quadratic FIF with constant as well as variable scaling factors.

Aseev [2] conceptualized the notion of the zipper, which is the generalization of the IFS. Several interesting topological and structural properties of zipper are studied related to dendrites and self-similar continua by Tetenov and his group [3, 24, 26–28]. Similar to fractal interpolants, zipper fractal interpolant as an attractor of a suitable zipper can give details on arbitrarily small scales. Chand et al. [11] introduced affine zipper fractal interpolants. They constructed affine zipper interpolants inscribed in a rectangle and found a basis for the affine zippers fractal interpolation function for a prescribed data set. Zipper fractal interpolants can be non-differentiable in a dense set of an interval. The construction of smooth zipper FIFs is proposed recently in Reddy [22], where certain derivative of smooth zipper FIF is a typical fractal function. Thus, zipper fractal interpolants can be smooth or non-smooth, and smooth zipper fractal interpolants may be used to generalize traditional non-recursive spline interpolants. In this work, we have come up with a novel C^1-rational quartic spline zipper fractal interpolation function with variable scaling functions and studied its positivity preserving property.

The main points of our work are as follows: First, we formulate a class of novel C^1-rational quartic spline (RQS) with two families of shape control parameters with the help of a binary vector, and then using that RQS and the theory of zipper, we derive a new type of fractal interpolant with variable scaling functions named rational quartic spline zipper fractal interpolation function (RQS ZFIF) in Sect. 2. In Sect. 3, we glean that our RQS and RQS ZFIF converge to a C^3 data generating function with the order $O(h^2)$ as $h \to 0$, and under additional assumptions on IFS parameters, we can increase the order of convergence up to $O(h^3)$. To get a strictly positive RQS ZFIF or RQS for a strictly positive data set, we derive sufficient conditions on the shape control parameters and the variable scaling functions in Sect. 4 and give some numerical examples to reinforce our theory. In Sect. 5, we summarize our work.

2 Construction of RQS ZFIFs

In this section, we will construct a new type of C^1-rational quartic spline using a binary vector called a signature, and then we will construct a class of novel C^1-RQS ZFIF with the help of our new rational quartic spline and the theory of the zipper.

Follows are some notation for this paper: Let $I := [a, b] \subset \mathbb{R}$. For $j \in \mathbb{N}$, let $\mathbb{N}_j := \{1, 2, 3, \ldots, j\}$, and $\mathbb{N}_j^0 := \{0, 1, 2, 3, \ldots, j\}$. For $j \in \mathbb{N} \cup \{0\}$, $C^j(I)$ is the Banach space of real valued functions having j continuous derivatives defined on I, and for $g \in C^j(I)$, $\|g\|_j := \max\{\|g^{(r)}\|_\infty : r = 0, 1, 2, \ldots, j\}$. For $g \in C(I)$, $\|g\|_\infty := \max\{|g(x)| : x \in I\}$.

Let a set of interpolation points $\{(x_i, y_i) \in I \times \mathbb{R} : i \in \mathbb{N}_N\}$ with increasing abscissae be given with $a = x_1$ and $b = x_N$. Let $[k_1, k_2]$ be a large compact interval in $\mathbb{R}$ such that $y_i \in [k_1, k_2] \forall i \in \mathbb{N}_N$. For a binary vector $\epsilon := (\epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}) \in \{0, 1\}^{N-1}$, let $L_i : I \to I_i := [x_i, x_{i+1}]$, $i = 1, 2, \ldots, N-1$, be contractive homeomorphisms such that

$$\begin{aligned} &L_i(x_1) = x_{i+\epsilon_i}, \quad L_i(x_N) = x_{i+1-\epsilon_i}, \\ &|L_i(x) - L_i(x^*)| \le r|x - x^*|, \quad \forall x, x^* \in I, \end{aligned} \tag{1}$$

for some $0 \le r < 1$.

For $0 \le \theta := \frac{x-x_1}{x_N-x_1} \le 1$ and $Q_i(\theta) = w_i(1-\theta)^3 + (w_i + u_i)(1-\theta)^2\theta + (w_{i+1} + u_{i+1})(1-\theta)\theta^2 + w_{i+1}\theta^3$, where w_i and u_i are the shape control parameters, let

$$\begin{aligned} P_{i1}(\theta) &= \frac{w_i(1-\theta)^3 + (w_i + u_i)(1-\theta)^3\theta + (w_{i+1} + u_{i+1})(1-\theta)^2\theta^2}{Q_i(\theta)}, \\ P_{i2}(\theta) &= \frac{(w_i + u_i)(1-\theta)^2\theta^2 + (w_{i+1} + u_{i+1})(1-\theta)\theta^3 + w_{i+1}\theta^3}{Q_i(\theta)}, \\ P_{i3}(\theta) &= \frac{w_i(1-\theta)^3\theta}{Q_i(\theta)}, \quad P_{i4}(\theta) = \frac{-w_{i+1}(1-\theta)\theta^3}{Q_i(\theta)}, \quad i \in \mathbb{N}_{N-1}, \end{aligned} \tag{2}$$

Then, for each $j \in \mathbb{N}_4$, $P_{ij} \in C^1(I)$ and satisfies

$$\begin{aligned} &P_{i1}(0) = 1,\ P_{i1}(1) = 0,\ P'_{i1}(0) = 0,\ P'_{i1}(1) = 0, \\ &P_{i2}(0) = 0,\ P_{i2}(1) = 1,\ P'_{i2}(0) = 0,\ P'_{i2}(1) = 0, \\ &P_{i3}(0) = 0,\ P_{i3}(1) = 0,\ P'_{i3}(0) = 1,\ P'_{i3}(1) = 0, \\ &P_{i4}(0) = 0,\ P_{i4}(1) = 0,\ P'_{i4}(0) = 0,\ P'_{i4}(1) = 1. \end{aligned} \tag{3}$$

Let $h_i := x_{i+1} - x_i$, $|I| := x_N - x_1$, and $h_i^* := x_{i+1-\epsilon_i} - x_{i+\epsilon_i}$. Now consider the function

$$P_\epsilon(L_i(x)) = P_{i1}(\theta)y_{i+\epsilon_i} + P_{i2}(\theta)y_{i+1-\epsilon_i} + h_i^* P_{i3}(\theta)d_{i+\epsilon_i} + h_i^* P_{i4}(\theta)d_{i+1-\epsilon_i} = \frac{P_i(\theta)}{Q_i(\theta)}, \tag{4}$$

where

$$\begin{aligned} &P_i(\theta) = \sum_{k=0}^{4} A_{ik}(1-\theta)^{4-k}\theta^k, \\ &A_{i0} = w_i y_{i+\epsilon_i}, \quad A_{i1} = u_i y_{i+\epsilon_i} + w_i(2y_{i+\epsilon_i} + h_i^* d_{i+\epsilon_i}), \\ &A_{i2} = (u_i + w_i)y_{i+1-\epsilon_i} + (u_{i+1} + w_{i+1})y_{i+\epsilon_i}, \\ &A_{i3} = u_{i+1}y_{i+1-\epsilon_i} + w_{i+1}(2y_{i+1-\epsilon_i} - h_i^* d_{i+1-\epsilon_i}), \quad A_{i4} = w_{i+1}y_{i+1-\epsilon_i}. \end{aligned} \tag{5}$$

Then, the RQS $P_\epsilon \in C^1(I)$ and satisfies $P_\epsilon(L_i(x_1)) = y_{i+\epsilon_i}$, $P_\epsilon(L_i(x_N)) = y_{i+1-\epsilon_i}$, $P'_\epsilon(L_i(x_1)) = d_{i+\epsilon_i}$, and $P'_\epsilon(L_i(x_N)) = d_{i+1-\epsilon_i}$. From (1), we can easily obtain that it also interpolates the given data, i.e. $\forall i \in \mathbb{N}$, $P_\epsilon(x_i) = y_i$, and $P'_\epsilon(x_i) = d_i$ for arbitrary signature ϵ, where d_i's are called derivative parameters. If the given data set $\{(x_i, y_i) : i \in \mathbb{N}_N\}$ is without the derivative parameters, then they must be calculated either from the data or by some appropriate methods. The arithmetic mean method (amm) and the geometric method (gmm) are popular choices for calculating derivatives from data. For details of these methods, see [10].

Remark 1 (i) If our shape control parameters w_i and u_i for $i \in \mathbb{N}$, are fixed and $w_i \neq w_{i+1}$ for $i \in \mathbb{N}_{N-1}$, then we can generate 2^{N-1} different rational quartic spline interpolation functions using different signatures for N numbers of data points.
(ii) If $\epsilon_i = 0$, for all $i \in \mathbb{N}_{N-1}$, then our rational quartic spline $P_\epsilon(x)$ reduces to the rational quartic spline $R(x)$ defined in [33].

Definition 1 A zipper with vertices $(v_1, v_2, \ldots, v_N)$ and signature $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}) \in \{0, 1\}^{N-1}$ is a collection of some non-surjective maps with a complete metric space is denoted by $\Lambda := \{X; W_i : i \in \mathbb{N}_{N-1}\}$, where for each $i \in \mathbb{N}_{N-1}$, W_i satisfies $W_i(v_1) = v_{i+\epsilon_i}$ and $W_i(v_N) = v_{i+1-\epsilon_i}$.

If there exists a compact set $\Gamma \subset X$ such that

$$\Gamma = \bigcup_{j=1}^{N-1} W_j(\Gamma),$$

then Γ is called the attractor or fractal corresponding to the zipper Λ.

Let $H := I \times [k_1, k_2]$. Construct $N - 1$ continuous functions $F_i : H \to \mathbb{R}$ such that

$$F_i(x, y) = \alpha_i(x)y + (P_\epsilon(L_i(x)) - \alpha_i(x)B_i(x)),$$

where $\alpha_i \in C^1(I)$ such that $\|\alpha_i\|_1 < 1$, and $B_i \in C^1(I)$ such that

$$\begin{aligned}
&B_i(x) = P_{i1}(\theta)y_1 + P_{i2}(\theta)y_N + |I|P_{i3}(\theta)d_1 + |I|P_{i4}(\theta)d_N = \frac{P_i^*(\theta)}{Q_i(\theta)},\\
&P_i^*(\theta) = \sum_{k=0}^{4} A_{ik}^*(1-\theta)^{4-k}\theta^k,\\
&A_{i0}^* = w_i y_1, \quad A_{i1}^* = u_i y_1 + w_i(2y_1 + |I|d_1),\\
&A_{i2}^* = (u_i + w_i)y_N + (u_{i+1} + w_{i+1})y_1,\\
&A_{i3}^* = u_{i+1}y_N + w_{i+1}(2y_N - |I|d_N), \quad A_{i4}^* = w_{i+1}y_N.
\end{aligned} \tag{6}$$

Now, for each $i \in \mathbb{N}_{N-1}$, B_i satisfies $B_i(x_1) = y_1$, $B_i(x_N) = y_N$, $B'_i(x_1) = d_1$, and $B'_i(x_N) = d_N$. Therefore, we have

$$\begin{aligned}
&F_i(x_1, y_1) = y_{i+\epsilon_i}, \quad F_i(x_N, y_N) = y_{i+1-\epsilon_i},\\
&|F_i(x, y) - F_i(x, y^*)| \le \|\alpha_i\|_\infty |y - y^*|, \quad \forall x \in I,\ y, y^* \in [k_1, k_2],
\end{aligned} \tag{7}$$

Now define mappings $W_i : H \to I_i \times \mathbb{R}, i = 1, 2, \ldots, N-1$ by

$$W_i(x, y) = (L_i(x), F_i(x, y)), \quad \forall (x, y) \in H.$$

Therefore, $\{H; W_i : i \in \mathbb{N}_{N-1}\}$ is a zipper with vertices $((x_1, y_1), (x_2, y_2), \ldots, (x_N, y_N))$ and signature $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1})$. For each $i \in \mathbb{N}_{N-1}$, $\alpha_i(x)$ is called the variable scaling function corresponding to the map W_i and B_i is called a base function. Now we will construct a C^1-RQS ZFIF using the zipper $\{H; W_i : i \in \mathbb{N}_{N-1}\}$ for the given Hermite data $\{(x_i, y_i, d_i) : i \in \mathbb{N}_N\}$.

Theorem 1 *Let $\{(x_i, y_i, d_i) : i \in \mathbb{N}_N\}$ be a given set of interpolation data such that $x_1 < x_2 < \cdots < x_N$. Let the signature $\epsilon \in \{0, 1\}^{N-1}$ be fixed. For $i \in \mathbb{N}_{N-1}$, let $L_i(x) = a_i x + b_i$ satisfies (1), and $F_i(x, y) = \alpha_i(x)y + P_\epsilon(L_i(x)) - \alpha_i(x)B_i(x)$, where P_ϵ and B_i are as defined in (4) and (6) respectively. If $\alpha_i \in C^1(I)$ and $\|\alpha_i\|_1 < \frac{|a_i|}{2}$ for all $i \in \mathbb{N}_{N-1}$, then the zipper $\{H; (L_i(x), F_i(x, y)) : i \in \mathbb{N}_{N-1}\}$ determines a rational quartic spline zipper fractal interpolation function $P_\epsilon^\alpha \in C^1(I)$.*

Proof Let $D(I) := \{g \in C^1(I) : g(x_1) = y_1, g(x_N) = y_N, g'(x_1) = d_1,$ and $g'(x_N) = d_N\}$. Then $D(I)$ is a complete metric space with respect to norm $\|.\|_1$. Now, define the Read-Bajraktarević operator $T^\alpha : D \to D$ such that

$$T^\alpha g(L_i(x)) = P_\epsilon(L_i(x)) + \alpha_i(x)(g(x) - B_i(x)), \ x \in I, \ i = 1, 2, \ldots, N-1. \tag{8}$$

Since the functions P_ϵ, B_i, and α_i belong to $C^1(I)$, $T^\alpha g \in C^1(x_i, x_{i+1})$ for each $i \in \mathbb{N}_{N-1}$. We know, for $i \in \mathbb{N}_{N-2}$, $x_{i+1} \in I_j$ for $j = i, i+1$. Since L_i and L_{i+1} satisfy (1), therefore we have

$$x_{i+1} = \begin{cases} L_i(x_N) & \epsilon_i = 0 \\ L_i(x_1) & \epsilon_i = 1, \end{cases} \text{ and } x_{i+1} = \begin{cases} L_{i+1}(x_1) & \epsilon_{i+1} = 0 \\ L_{i+1}(x_N) & \epsilon_{i+1} = 1. \end{cases} \tag{9}$$

By putting (9) in (8), we have

$$T^\alpha g(x_{i+1}) = \begin{cases} P_\epsilon(L_i(x_N)) & \epsilon_i = 0 \\ P_\epsilon(L_i(x_1)) & \epsilon_i = 1, \end{cases} \text{ and } T^\alpha g(x_{i+1}) = \begin{cases} P_\epsilon(L_{i+1}(x_1)) & \epsilon_{i+1} = 0 \\ P_\epsilon(L_{i+1}(x_N)) & \epsilon_{i+1} = 1. \end{cases} \tag{10}$$

$$\implies \lim_{x \to x_{i+1}^-} (T^\alpha g)(x) = \lim_{x \to x_{i+1}^+} (T^\alpha g)(x) = y_{i+1}$$

Similarly, after differentiating (8) once and using (9), we can obtain

$$(T^\alpha g)'(x_{i+1}) = \begin{cases} P'_\epsilon(L_i(x_N)) & \epsilon_i = 0 \\ P'_\epsilon(L_i(x_1)) & \epsilon_i = 1, \end{cases} \text{ and } (T^\alpha g)'(x_{i+1}) = \begin{cases} P'_\epsilon(L_{i+1}(x_1)) & \epsilon_{i+1} = 0 \\ P'_\epsilon(L_{i+1}(x_N)) & \epsilon_{i+1} = 1. \end{cases} \tag{11}$$

$$\implies \lim_{x \to x_{i+1}^-} (T^\alpha g)'(x) = \lim_{x \to x_{i+1}^+} (T^\alpha g)'(x) = d_{i+1}. \tag{12}$$

Now, for $i = 1, N-1$, from (8) we can easily get that $T^\alpha g(x_1) = y_1$, $T^\alpha g(x_N) = y_N$, $(T^\alpha g)'(x_1) = d_1$, and $(T^\alpha g)'(x_N) = d_N$. Therefore, the operator T^α is well-defined, i.e. $T^\alpha g \in D$. Now, for $x \in I_i$,

$$(T^\alpha g)(x) - (T^\alpha g^*)(x) = \alpha_i(L_i^{-1}(x))(g - g^*)(L_i^{-1}(x)),$$

which implies

$$|(T^\alpha g)(x) - (T^\alpha g^*)(x)| \leq \|\alpha_i\|_\infty \|g - g^*\|_\infty \leq \|\alpha_i\|_1 \|g - g^*\|_1.$$

Similarly,

$$\begin{aligned}|(T^\alpha g)'(x) - (T^\alpha g^*)'(x)| &\leq |a_i^{-1}|(\|\alpha_i'\|_\infty \|g - g^*\|_\infty + \|\alpha_i\|_\infty \|g' - g^{*\prime}\|_\infty) \\ &\leq 2|a_i^{-1}|\|\alpha_i\|_1 \|g - g^*\|_1.\end{aligned}$$

So, if for all $i \in \mathbb{N}_{N-1}$, $\|\alpha_i\|_1 < s\frac{|a_i|}{2}$ for some $0 \leq s < 1$, then we have $\|T^\alpha g - T^\alpha g^*\|_1 < s\|g - g^*\|_1$, i.e. T^α is a contraction map on D. Therefore, by Banach fixed point theorem T^α has a unique fixed point say $P_\epsilon^\alpha \in C^1(I)$, and P_ϵ^α satisfies the recurrence relation

$$P_\epsilon^\alpha(L_i(x)) = P_\epsilon(L_i(x)) + \alpha_i(x)(P_\epsilon^\alpha(x) - B_i(x)), \; x \in I, \; i = 1, 2, \ldots, N-1. \tag{13}$$

P_ϵ^α is the desired rational quartic spline zipper α-fractal function corresponding to the function P_ϵ. For more details on α-fractal functions, see [5, 21].

Remark 2 (i) If $\alpha_i(x) = 0$, for all $x \in I$ and for all $i \in \mathbb{N}_{N-1}$, then our RQS ZFIF P_ϵ^α reduces to the RQS P_ϵ defined in (4).
(ii) If $\alpha_i(x) = 0$ and $\epsilon_i = 0$, for all $x \in I$ and for all $i \in \mathbb{N}_{N-1}$, then the proposed RQS ZFIF P_ϵ^α reduces to the rational quartic spline $R(x)$ defined in [33].
(iii) For the fixed shape control parameters and the fixed non-zero variable scaling functions, we can get 2^{N-1} different RQS ZFIFs using different values of signature for the N numbers of data points.

3 Convergence Analysis

In this section, we will derive an upper bound for the uniform error of the RQS ZFIF with a C^3 data generating function, and we will show that our RQS ZFIF has $O(h^2)$ convergence and can be increased to $O(h^3)$ under certain conditions.

We fix these notation for this section: $\Delta_i := \frac{y_{i+1}-y_i}{h_i}$, $t := \frac{x-x_i}{h_i}$, $h := \max\{h_i : i \in \mathbb{N}_{N-1}\}$, $|y|_\infty := \max\{|y_i| : i \in \mathbb{N}_N\}$, $|d|_\infty := \max\{|d_i| : i \in \mathbb{N}_N\}$, $w_{i*} := \min\{w_i, w_{i+1}\}$, $u_{i*} := \min\{u_i, u_{i+1}\}$, $w_i^* := \max\{w_i, w_{i+1}\}$, $u_i^* := \max\{u_i, u_{i+1}\}$, $w_* := \min\{w_i : i \in \mathbb{N}_N\}$, $w^* := \max\{w_i : i \in \mathbb{N}_N\}$, $u_* := \min\{u_i : i \in \mathbb{N}_N\}$, $u^* := \max\{u_i : i \in \mathbb{N}_N\}$,

$B^* := \max\{\|B_i\|_\infty : i \in \mathbb{N}_{N-1}\}$, $\alpha(x) := (\alpha_1(x), \alpha_2(x), \ldots, \alpha_{N-1}(x))$, $\|\alpha\|_\infty := \max\{\|\alpha_i\|_\infty : i \in \mathbb{N}_{N-1}\}$, and $\|\alpha\|_1 := \max\{\|\alpha_i\|_1 : i \in \mathbb{N}_{N-1}\}$.

Let $\Phi \in C^3(I)$ be a data generating function, i.e. $\Phi(x_i) = y_i, \forall i \in \mathbb{N}_N$. Let d_i's are chosen derivatives at x_i, for all $i \in \mathbb{N}_N$. Now, for $\theta = \frac{x-x_1}{x_N-x_1}$, let $t^* := L_i^{-1}(\theta)$, i.e. $t^* = \frac{x-x_{i+\epsilon_i}}{x_{i+1-\epsilon_i}-x_{i+\epsilon_i}}$. Therefore, for $x \in I_i$,

$$P_\epsilon(x) = \frac{1}{Q_i(t^*)} \sum_{k=0}^{4} A_{ik}(1-t^*)^{4-k} t^{*k}, \tag{14}$$

where $t^* = \begin{cases} t & \epsilon_i = 0 \\ 1-t & \epsilon_i = 1, \end{cases}$ and A_{ik}'s are as defined in (5).

Case I: Let $x \in I_i$ and $\epsilon_i = 0$, then

$$\begin{aligned}
&P_\epsilon(x) = \frac{1}{Q_i(t)} \sum_{k=0}^{4} A_{ik}(1-t)^{4-k} t^k, \\
&A_{i0} = w_i y_i, \quad A_{i1} = u_i y_i + w_i(2y_i + h_i d_i), \\
&A_{i2} = (u_i + w_i) y_{i+1} + (u_{i+1} + w_{i+1}) y_i, \\
&A_{i3} = u_{i+1} y_{i+1} + w_{i+1}(2y_{i+1} - h_i d_{i+1}), \quad A_{i4} = w_{i+1} y_{i+1}, \\
&Q_i(t) = w_i(1-t)^3 + (w_i + u_i)(1-t)^2 t + (w_{i+1} + u_{i+1})(1-t)t^2 + w_{i+1} t^3.
\end{aligned} \tag{15}$$

Now from [13, 33], for $x \in I_i$ and $\epsilon_i = 0$, choosing $w_i, w_{i+1} > 0$ and $u_i, u_{i+1} \geq 0$, we have

$$\begin{aligned}
|\Phi(x) - P_\epsilon(x)| \leq\ & \frac{h_i^3}{96}\|\Phi^{(3)}\|_\infty + \frac{h_i}{4}\max\left\{|\Phi'(x_i) - d_i|, |\Phi'(x_{i+1}) - d_{i+1}|\right\} \\
&+ \frac{h_i}{2\sqrt{w_i w_{i+1}} + \min\{u_i, u_{i+1}\}}\Big[\frac{27}{256}|u_i(\Delta_i - d_i) - w_i(2\Delta_i - d_i - d_{i+1})| \\
&+ \frac{1}{16}|(w_{i+1} - w_i)(2\Delta_i - d_i - d_{i+1}) + u_{i+1}(\Delta_i - d_i) + u_i(d_{i+1} - \Delta_i)| \\
&+ \frac{27}{256}|w_{i+1}(2\Delta_i - d_i - d_{i+1}) + u_{i+1}(d_{i+1} - \Delta_i)|\Big].
\end{aligned} \tag{16}$$

Case II: Let $x \in I_i$ and $\epsilon_i = 1$, then

$$
\begin{aligned}
&P_\epsilon(x) = \frac{1}{Q_i(1-t)}\sum_{k=0}^{4} A_{ik}t^{4-k}(1-t)^k,\\
&A_{i0} = w_i y_{i+1}, \quad A_{i1} = u_i y_{i+1} + w_i(2y_i - h_i d_{i+1}),\\
&A_{i2} = (u_i + w_i)y_i + (u_{i+1} + w_{i+1})y_{i+1},\\
&A_{i3} = u_{i+1}y_i + w_{i+1}(2y_i + h_i d_i), \quad A_{i4} = w_{i+1}y_i,\\
&Q_i(1-t) = w_i t^3 + (w_i + u_i)t^2(1-t) + (w_{i+1} + u_{i+1})t(1-t)^2 + w_{i+1}(1-t)^3.
\end{aligned}
\tag{17}
$$

After interchanging w_i and w_{i+1}, u_i and u_{i+1}, (17) becomes equivalent to (15). Therefore, using similar analysis, for $x \in I_i$ and $\epsilon_i = 1$, choosing $w_i, w_{i+1} > 0$ and $u_i, u_{i+1} \geq 0$, we have

$$
\begin{aligned}
|\Phi(x) - P_\epsilon(x)| \leq\ & \frac{h_i^3}{96}\|\Phi^{(3)}\|_\infty + \frac{h_i}{4}\max\left\{|\Phi'(x_i) - d_i|, |\Phi'(x_{i+1}) - d_{i+1}|\right\}\\
& + \frac{h_i}{2\sqrt{w_{i+1}w_i} + \min\{u_{i+1}, u_i\}}\Big[\frac{27}{256}|u_{i+1}(\Delta_i - d_i) - w_{i+1}(2\Delta_i - d_i - d_{i+1})|\\
& + \frac{1}{16}|(w_i - w_{i+1})(2\Delta_i - d_i - d_{i+1}) + u_i(\Delta_i - d_i) + u_{i+1}(d_{i+1} - \Delta_i)|\\
& + \frac{27}{256}|w_i(2\Delta_i - d_i - d_{i+1}) + u_i(d_{i+1} - \Delta_i)|\Big].
\end{aligned}
\tag{18}
$$

Now, if the derivative parameters d_i's are chosen such that

$$
\begin{aligned}
d_1 &= \Delta_1 - \frac{h_1}{h_1 + h_2}(\Delta_2 - \Delta_1),\\
d_N &= \Delta_{N-1} + \frac{h_{N-1}}{h_{N-1} + h_{N-2}}(\Delta_{N-1} - \Delta_{N-2}),\\
d_i &= \frac{h_i}{h_{i-1} + h_i}\Delta_{i-1} + \frac{h_{i-1}}{h_{i-1} + h_i}\Delta_i, \quad i = 2, 3, \ldots, N-1,
\end{aligned}
\tag{19}
$$

then by using Peano kernel analysis, we can easily get following results:

$$
\begin{aligned}
&\Phi'(x_1) - d_1 = \frac{1}{6}h_1(h_1 + h_2)\Phi^{(3)}(\zeta_1), \quad d_i - \Phi'(x_i) = \frac{1}{6}h_{i-1}h_i\Phi^{(3)}(\zeta_i),\\
&\Phi'(x_N) - d_N = \frac{1}{6}h_{N-1}(h_{N-1} + h_{N-2})\Phi^{(3)}(\zeta_N), \quad \Delta_1 - d_1 = \frac{1}{2}h_1\Phi^{(2)}(\chi_2),\\
&\Delta_i - d_i = \frac{1}{2}h_i\Phi^{(2)}(\chi_i), \quad d_{i+1} - \Delta_i = \frac{1}{2}h_i\Phi^{(2)}(\chi_{i+1}),\\
&d_N - \Delta_{N-1} = \frac{1}{2}h_{N-1}\Phi^{(2)}(\chi_{N-1}), \quad 2\Delta_1 - d_1 - d_2 = 0,\\
&d_{N-1} + d_N - 2\Delta_{N-1} = 0, \quad d_i + d_{i+1} - 2\Delta_i = \frac{1}{6}h_i(h_{i-1} + h_i + h_{i+1})\Phi^{(3)}(\chi_i^*),
\end{aligned}
\tag{20}
$$

where $\zeta_1 \in (x_1, x_3)$, $\zeta_i \in (x_{i-1}, x_{i+1})$, $\zeta_N \in (x_{N-2}, x_N)$, $\chi_i \in (x_{i-1}, x_{i+1})$, $i = 2, 3, \ldots, N-1$ and $\chi_i^* \in (x_{i-1}, x_{i+2})$, $i = 2, 3, \ldots, N-2$. Therefore, using (20) in

(16) and (18), we have

$$|\Phi(x) - P_\epsilon(x)| \leq \frac{h^3}{96}\|\Phi^{(3)}\|_\infty + \frac{h^3}{12}\|\Phi^{(3)}\|_\infty + \frac{h^2}{2w_{i*}+u_{i*}}\Big[\frac{43}{256}\big(u_i^*\|\Phi^{(2)}\|_\infty + hw_i^*\|\Phi^{(3)}\|_\infty\big)\Big],$$

for $x \in I_i$ and $\epsilon_i \in \{0, 1\}$.

Now we summarize the above discussions in the following as a theorem:

Theorem 2 *Let $\Phi \in C^3(I)$ be a data generating function such that $\Phi(x_i) = y_i$, $i \in \mathbb{N}_N$. For a fixed signature $\epsilon \in \{0, 1\}^{N-1}$, let P_ϵ be the rational quartic spline defined in (4). If for all $i \in \mathbb{N}_N$, we choose our shape control parameters such that $w_i > 0$, $u_i \geq 0$ and the derivative parameters as prescribed in (19), then*

$$\|\Phi - P_\epsilon\|_\infty \leq \frac{9h^3}{96}\|\Phi^{(3)}\|_\infty + \frac{h^2}{2w_*+u_*}\Big[\frac{43}{256}\big(u^*\|\Phi^{(2)}\|_\infty + hw^*\|\Phi^{(3)}\|_\infty\big)\Big]. \tag{21}$$

Now we will try to find the upper bound for the difference between RQS P_ϵ defined in (4) and RQS ZFIF P_ϵ^α defined in (13). If $\alpha \not\equiv 0$, then $P_\epsilon \neq P_\epsilon^\alpha$, and the interpolants P_ϵ^α and P_ϵ are the fixed points of T^α defined in (8) with $\alpha \not\equiv 0$ and $\alpha(x) = (0, 0, \ldots, 0)$ respectively.

For $i \in \mathbb{N}_{N-1}$ and $x \in I$,

$$\begin{aligned}
|P_\epsilon^\alpha(L_i(x)) - P_\epsilon(L_i(x))| &= |T^\alpha P_\epsilon^\alpha(L_i(x)) - T^0 P_\epsilon(L_i(x))| \\
&= |P_\epsilon(L_i(x)) + \alpha_i(x)(P_\epsilon^\alpha(x) - B_i(x)) - P_\epsilon(L_i(x))| \\
&= |\alpha_i(x)(P_\epsilon^\alpha(x) - B_i(x))| \\
&\leq \|\alpha_i\|_\infty\|P_\epsilon^\alpha - B_i\|_\infty \\
&\leq \|\alpha_i\|_\infty\|P_\epsilon^\alpha - P_\epsilon\|_\infty + \|\alpha_i\|_\infty\|P_\epsilon - B_i\|_\infty \\
&\leq \|\alpha_i\|_\infty\|P_\epsilon^\alpha - P_\epsilon\|_\infty + \|\alpha_i\|_\infty(\|P_\epsilon\|_\infty + B^*).
\end{aligned}$$

As for each $i \in \mathbb{N}_{N-1}$, the above inequality holds, hence

$$\|P_\epsilon^\alpha - P_\epsilon\|_\infty \leq \|\alpha\|_\infty\|P_\epsilon^\alpha - P_\epsilon\|_\infty + \|\alpha\|_\infty(\|P_\epsilon\|_\infty + B^*),$$

i.e.

$$\|P_\epsilon^\alpha - P_\epsilon\|_\infty \leq \frac{\|\alpha\|_\infty(\|P_\epsilon\|_\infty + B^*)}{1 - \|\alpha\|_\infty}. \tag{22}$$

Now, let us deduce upper bounds for $\|P_\epsilon\|_\infty$ and B^*. From (4), for $i \in \mathbb{N}_{N-1}$ and $x \in I$,

$$|P_\epsilon(L_i(x))| \leq \frac{\max\{|P_i(\theta)| : 0 \leq \theta \leq 1\}}{\min\{|Q_i(\theta)| : 0 \leq \theta \leq 1\}}.$$

Using inequalities $(1-\theta)^3\theta \leq \frac{27}{256}$, $(1-\theta)^2\theta^2 \leq \frac{1}{16}$, $(1-\theta)\theta^3 \leq \frac{27}{256}$, and $(1-\theta)^4+\theta^4 \leq 1$, we can easily deduce that

$$\begin{aligned}|P_i(\theta)| &\leq w_i^*(\max\{|y_i|,|y_{i+1}|\})\\ &\quad+\frac{27}{128}\Big[(u_i^*+2w_i^*)(\max\{|y_i|,|y_{i+1}|\})+w_i^*h_i(\max\{|d_i|,|d_{i+1}|\})\Big]\\ &\quad+\frac{1}{8}\Big[(u_i^*+w_i^*)(\max\{|y_i|,|y_{i+1}|\})\Big],\\ \text{i.e. } |P_i(\theta)| &\leq \Big(\frac{99}{64}w^*+\frac{43}{128}u^*\Big)|y|_\infty+w^*h|d|_\infty,\end{aligned}$$

and

$$\begin{aligned}|Q_i(\theta)| &\geq w_i(1-\theta)^2+u_i(1-\theta)^2\theta+u_{i+1}(1-\theta)\theta^2+w_{i+1}\theta^2\\ &\geq w_i(1-\theta)^2+w_{i+1}\theta^2 \geq \frac{1}{2}w_{i*} \geq \frac{1}{2}w_*.\end{aligned}$$

Hence,

$$\|P_\epsilon\|_\infty \leq 2\frac{\Big(\frac{99}{64}w^*+\frac{43}{128}u^*\Big)|y|_\infty+w^*h|d|_\infty}{w_*}.$$

Similarly,

$$\|B_i\|_\infty \leq 2\frac{\Big(\frac{99}{64}w_i^*+\frac{43}{128}u_i^*\Big)\max\{|y_1|,|y_N|\}+w_i^*|I|(\max\{|d_1|,|d_N|\})}{w_i^*}.$$

Therefore,

$$B^* \leq 2\frac{\Big(\frac{99}{64}w^*+\frac{43}{128}u^*\Big)\max\{|y_1|,|y_N|\}+w^*|I|(\max\{|d_1|,|d_N|\})}{w_*}.$$

Now we will present the main theorem of this section.

Theorem 3 *Let $\Phi \in C^3(I)$ be a data generating function such that $\Phi(x_i)=y_i$, $i \in \mathbb{N}_N$. For a fixed signature $\epsilon \in \{0,1\}^{N-1}$, let P_ϵ be the rational quartic spline defined in (4) and P_ϵ^α be the proposed rational quartic spline zipper fractal interpolation function defined in (13). If for all $i \in \mathbb{N}_N$, we choose our shape control points such that $w_i>0$, $u_i \geq 0$ and the derivative parameters as given in (19), then*

$$\begin{aligned}\|\Phi-P_\epsilon^\alpha\|_\infty &\leq \frac{9h^3}{96}\|\Phi^{(3)}\|_\infty+\frac{h^2}{2w_*+u_*}\Big[\frac{43}{256}\big(u^*\|\Phi^{(2)}\|_\infty+hw^*\|\Phi^{(3)}\|_\infty\big)\Big]\\ &\quad+\frac{\|\alpha\|_\infty(\|P_\epsilon\|_\infty+B^*)}{1-\|\alpha\|_\infty}.\end{aligned}$$

Proof We know that

$$\|\Phi - P_\epsilon^\alpha\|_\infty \le \|\Phi - P_\epsilon\|_\infty + \|P_\epsilon^\alpha - P_\epsilon\|_\infty. \tag{23}$$

Therefore, using (21) and (22) in (23), we can easily get our desired result.

Remark 3 (i) If we choose $\|\alpha\|_1 < \min\{h^2, \frac{h}{2|I|}\}$, then from Theorem 3, we can deduce that our proposed zipper fractal interpolant P_ϵ^α converges to a C^3-data generating function Φ with the order $O(h^2)$ as $h \to 0$ on I.
(ii) If we choose $u_i = 0$ for all $i \in \mathbb{N}_N$ and $\|\alpha\|_1 < \min\{h^3, \frac{h}{2|I|}\}$, then $u^* = 0$, and hence from Theorem 3 we can conclude that our proposed zipper fractal interpolant P_ϵ^α converges to a C^3-data generating function Φ with the order $O(h^3)$ as $h \to 0$ on I.

4 Positivity-Preserving RQS ZFIFs

Many real-life problems like monthly rainfall amounts, the half-life of a radioactive substance, probability distribution functions, speed of winds and the numbers of covid-19 patients at different intervals of time are based on positivity. So, the problem is to find a positive interpolant for a given positive data set. In this section, we are going to construct positive RQS ZFIFs for the given positive data by restricting our shape control parameters and variable scaling functions.

Theorem 4 *Let $\{(x_i, y_i) : i = 1, 2, \ldots, N\}$ be a given set of strictly positive data with increasing abscissae. Suppose d_i's are chosen derivative values at the knots x_i's. For the fixed value of signature $\epsilon \in \{0, 1\}^{N-1}$, if the non-negative variable scaling functions and shape control points are chosen as*

$$\begin{aligned}
&\|\alpha_i\|_1 < \frac{|a_i|}{2}, \ \|\alpha_i\|_\infty < \min\left\{\frac{y_{i+\epsilon_i}}{y_1}, \frac{y_{i+1-\epsilon_i}}{y_N}\right\},\\
&w_i > 0, \quad w_{i+1} > 0,\\
&u_i \ge \max\left\{0, -w_i\left(2 + \frac{h_i^* d_{i+\epsilon_i} - \alpha_i(x)|I|d_1}{y_{i+\epsilon_i} - \alpha_i(x)y_1}\right)\right\},\\
&u_{i+1} \ge \max\left\{0, -w_{i+1}\left(2 - \frac{h_i^* d_{i+1-\epsilon_i} - \alpha_i(x)|I|d_N}{y_{i+1-\epsilon_i} - \alpha_i(x)y_N}\right)\right\}, \quad \forall x \in I, \quad \forall i \in \mathbb{N}_{N-1},
\end{aligned}$$

then the corresponding C^1-rational quartic spline zipper fractal interpolation function P_ϵ^α defined in (13) will be strictly positive on I.

Proof Since $\|\alpha_i\|_1 < \frac{|a_i|}{2}$, and α_i, P_ϵ, $B_i \in C^1(I)$, therefore according to Theorem 1 the proposed RQS ZFIF $P_\epsilon^\alpha \in C^1(I)$, and it satisfies (13). We can rewrite (13) as

$$P_\epsilon^\alpha(L_i(x)) = \alpha_i(x)P_\epsilon^\alpha(x) + (P_\epsilon(L_i(x)) - \alpha_i(x)B_i(x)), \quad i \in \mathbb{N}_{N-1}, \ x \in I. \tag{24}$$

Equation (24) is equivalent to

$$P_\epsilon^\alpha(L_i(x)) = \alpha_i(x)P_\epsilon^\alpha(x) + \frac{P_i^{**}(\theta)}{Q_i(\theta)}, \quad P_i^{**}(\theta) = \sum_{k=0}^{4} B_{ik}(1-\theta)^{4-k}\theta^k,$$
$$B_{i0} = w_i(y_{i+\epsilon_i} - \alpha_i(x)y_1),$$
$$B_{i1} = u_i(y_{i+\epsilon_i} - \alpha_i(x)y_1) + w_i(2(y_{i+\epsilon_i} - \alpha_i(x)y_1) + h_i^* d_{i+\epsilon_i} - \alpha_i(x)|I|d_1),$$
$$B_{i2} = [(u_i + w_i)(y_{i+1-\epsilon_i} - \alpha_i(x)y_N)] + [(u_{i+1} + w_{i+1})(y_{i+\epsilon_i} - \alpha_i(x)y_1)],$$
$$B_{i3} = u_{i+1}(y_{i+1-\epsilon_i} - \alpha_i(x)y_N) + w_{i+1}(2(y_{i+1-\epsilon_i} - \alpha_i(x)y_N) - h_i^* d_{i+1-\epsilon_i} + \alpha_i(x)|I|d_N),$$
$$B_{i4} = w_{i+1}(y_{i+1-\epsilon_i} - \alpha_i(x)y_N). \tag{25}$$

After choosing $w_i > 0$, $w_{i+1} > 0$, $u_i \geq 0$ and $u_{i+1} \geq 0$, our cubic denominator $Q_i(\theta)$ in (25) becomes strictly positive on I. Since P_ϵ^α is the attractor of the zipper $\{H; (L_i(x) = a_i x + b_i, F_i(x, y) = \alpha_i(x)y + P_\epsilon(L_i(x)) - \alpha_i(x)B_i(x)) : i \in \mathbb{N}_{N-1}\}$ and defined recursively by (25), therefore to show $P_\epsilon^\alpha(x) > 0$ on I, enough to prove that for all $x \in I$, $P_\epsilon^\alpha(L_i(x)) > 0\ \forall i \in \mathbb{N}_{N-1}$, whenever $P_\epsilon^\alpha(x) > 0$. Now, let $x \in I$, $P_\epsilon^\alpha(x) > 0$, then choosing non-negative scaling functions we have $\alpha_i(x)P_\epsilon^\alpha(x) \geq 0$. Therefore, after these assumptions on the shape control parameters and the variable scaling functions, the positivity of $P_\epsilon^\alpha(L_i(x))$ reduces to the positivity of $P_i^{**}(\theta)$, $\forall \theta \in [0, 1]$. Now, if $B_{ij} \geq 0$, $\forall j \in \{0, 1, 2, 3, 4\}$ and $B_{ij} > 0$ for $j \in \{0, 4\}$, then we have $P_i^{**}(\theta) > 0$. Now,

$$w_i > 0, \text{ and } \alpha_i(x) < \frac{y_{i+\epsilon_i}}{y_1} \implies B_{i0} > 0,$$
$$u_i > \max\left\{0, -w_i\left(2 + \frac{h_i^* d_{i+\epsilon_i} - \alpha_i(x)|I|d_1}{y_{i+\epsilon_i} - \alpha_i(x)y_1}\right)\right\}, \text{ and } \alpha_i(x) < \frac{y_{i+\epsilon_i}}{y_1} \implies B_{i1} \geq 0,$$
$$u_i + w_i > 0, \ u_{i+1} + w_{i+1} > 0, \text{ and } \alpha_i(x) < \left\{\frac{y_{i+\epsilon_i}}{y_1}, \frac{y_{i+1-\epsilon_i}}{y_N}\right\} \implies B_{i2} > 0,$$
$$u_{i+1} \geq \max\left\{0, -w_{i+1}\left(2 - \frac{h_i^* d_{i+1-\epsilon_i} - \alpha_i(x)|I|d_N}{y_{i+1-\epsilon_i} - \alpha_i(x)y_N}\right)\right\}, \text{ and } \alpha_i(x) < \frac{y_{i+1-\epsilon_i}}{y_N} \implies B_{i3} \geq 0,$$
$$w_{i+1} > 0, \text{ and } \alpha_i(x) < \frac{y_{i+1-\epsilon_i}}{y_N} \implies B_{i4} > 0.$$

Hence, coupling these above restrictions on the shape control parameters and the scaling functions, we have the desired sufficient conditions for this theorem.

Remark 4 (i) For all $i \in \mathbb{N}_{N-1}$ and $x \in I$, if we choose $\alpha_i(x) = 0$ and $\epsilon_i = 0$, then Theorem 4 gives sufficient conditions on the shape control parameters such that the RQS function R defined in [33] becomes positive for a given positive data set $\{(x_i, y_i) : i = 1, 2, \ldots, N\}$.
(ii) Let the given data set be strictly positive and $\alpha_i(x) = 0$ for all $x \in I$ and for all $i \in \mathbb{N}_{N-1}$. If we choose our shape control parameters such that

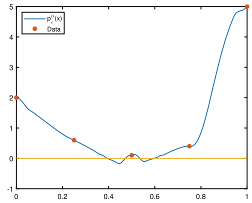
$$w_i > 0, \quad w_{i+1} > 0, \quad u_i \geq \max\left\{0, -w_i\left(2 + \frac{h_i^* d_{i+\epsilon_i}}{y_{i+\epsilon_i}}\right)\right\},$$
$$u_{i+1} \geq \max\left\{0, -w_{i+1}\left(2 - \frac{h_i^* d_{i+1-\epsilon_i}}{y_{i+1-\epsilon_i}}\right)\right\}, \quad \forall i \in \mathbb{N}_{N-1}. \tag{26}$$

Then, Theorem 4 instructs that our corresponding rational quartic spline P_ϵ defined in (4) satisfies $P_\epsilon(x) > 0$ for all $x \in I$.
(iii) For $N(> 2)$ number of positive data points and the fixed non-zero variable scaling functions, we can get total numbers of 2^{N-1} different C^1-rational quartic spline zipper fractal interpolation functions depending on the different values of signature ϵ.

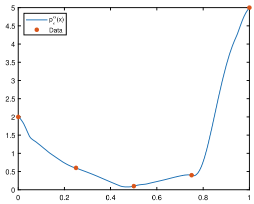
Example 1 In Theorem 4, we have provided sufficient conditions on the shape control parameters and the variable scaling functions such that our corresponding RQS ZFIF becomes positive on I, whenever our given data set is positive. It can happen that if we do not choose our parameters as prescribed in Theorem 4, then our corresponding RQS ZFIF P_ϵ^α may not be positive on I for a given positive data set, but after restricting our shape control parameters and variable scaling functions as prescribed in Theorem 4 our corresponding RQS ZFIF becomes positive on I.

Consider the positive data set $\{(0, 2, -1), (0.25, 0.6, -3), (0.5, 0.1, 2), (0.75, 0.4, -2), (1, 5, 6)\}$. For the fixed shape control parameters $u = (1, 2, 3, 1, 1)$ and $w = (1, 0.2, 0.5, 1, 3)$, Figs. 1(a)-(f) are the plots of RQS ZFIFs generated with scaling functions and signature $\{(\frac{x^2}{17}, \frac{e^x}{25}, \frac{e^x}{25}, \frac{-x}{10}), (1, 0, 1, 0)\}$, $\{(\frac{e^x}{25}, \frac{x}{60}, \frac{1}{100}, \frac{e^x}{35}), (1, 0, 1, 0)\}$, $\{(\frac{e^x}{25}, \frac{x}{60}, \frac{1}{100}, \frac{e^x}{35}), (0, 0, 1, 0)\}$, $\{(0, 0, 0, 0), (0, 0, 0, 0)\}$, $\{(0, 0.01 + \frac{x}{120}, 0, 0), (0, 0, 0, 0)\}$, and $\{(0, 0, 0, 0), (1, 0, 0, 1)\}$ respectively. For Fig. 1(a), we do not restrict our scaling functions as recommended by Theorem 4, and the corresponding RQS ZFIF is not positive on $I = [0, 1]$. But for the other plots, we have restricted our shape control parameters and scaling functions as recommended by Theorem 4 and hence the corresponding RQS ZFIFs are positive on I.

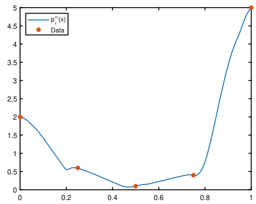
To see the effect of signature, we have plotted Fig. 1(b) and (c) with the same parameters except for ϵ_1, and we have turned up with very different RQS ZFIF on $I_1 = [0, 0.25]$. Fig. 1(d) is the plot of the classical rational quartic spline defined in [33]. To see the effect of scaling functions, we have plotted Fig. 1(e) by changing the scaling function α_2 from the parameters used for Fig. 1(d). Fig. 1(f) is the plot of RQS defined by us in (4) using the binary vector signature $\epsilon = (1, 0, 0, 1)$ and we can see that the RQS defined by us and classical RQS defined in [33] are not the same. Thus, the proposed method enlarge the class of rational quartic splines with fixed shape parameters.



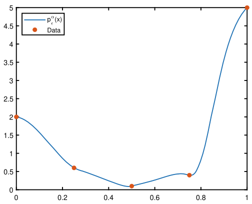
(a) RQS ZFIF with $\alpha = (\frac{x^2}{17}, \frac{e^x}{25}, \frac{e^x}{25}, \frac{-x}{10})$ and $\epsilon = (1, 0, 1, 0)$.



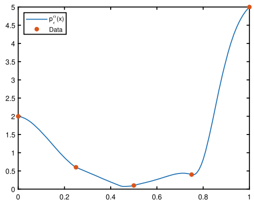
(b) RQS ZFIF with $\alpha = (\frac{e^x}{25}, \frac{x}{60}, \frac{1}{100}, \frac{e^x}{35})$ and $\epsilon = (1, 0, 1, 0)$.



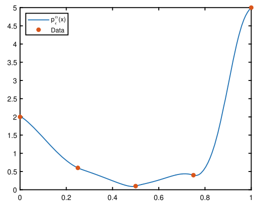
(c) RQS ZFIF with $\alpha = (\frac{e^x}{25}, \frac{x}{60}, \frac{1}{100}, \frac{e^x}{35})$ and $\epsilon = (0, 0, 1, 0)$.



(d) Classical RQS ($\alpha = (0, 0, 0, 0)$ and $\epsilon = (0, 0, 0, 0)$).



(e) RQS ZFIF with $\alpha = (0, 0.01 + \frac{x}{120}, 0, 0)$ and $\epsilon = (0, 0, 0, 0)$.



(f) RQS ZFIF with $\alpha = (0, 0, 0, 0)$ and $\epsilon = (1, 0, 0, 1)$.

Fig. 1 Positive RQS ZFIFs

5 Conclusions

We have derived a new type of C^1-rational quartic spline using the binary vector called a signature. For the fixed shape control parameters, we can generate 2^{N-1} different new C^1-rational quartic spline interpolation functions using different signatures for the N numbers of data points. Then, by using the fractal technique we have introduced rational quartic spline zipper fractal interpolation functions. It has been shown that for a data generating function $\Phi \in C^3(I)$, the proposed RQS ZFIF has the order of convergence $O(h^2)$ as $h \to 0$, and it can be increased to the next order of convergence as the classical rational quartic spline defined in [33] under suitable assumptions on the IFS parameters. We have derived sufficient conditions on the shape control parameters and the variable scaling functions so that our RQS ZFIF (consequently, the class of RQS) becomes positive for a given positive data set.

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