Growth of Polynomials Having No Zero Inside a Circle

Khangembam Babina Devi, N Reingachan, Thangjam Birkramjit Singh, and Barchand Chanam

Abstract In this manuscript, an upper bound estimate for the maximum modulus of a general class of polynomials with restricted zeros on a circle $|z| = L$, $L \geq 1$, is obtained in terms of the maximum modulus of the same polynomials on $|z| = 1$. It is observed that a result of Hussain [J. Pure Appl. Math., (2021) (https://doi.org/10.1007/s13226-021-00169-7)] is sharpened by our result. Also, this result generalizes and sharpens some other previously proved result.

Keywords Polynomials · Zeros · Inequalities · Maximum modulus

1 Introduction

Let $b(z)$ be a polynomial of degree m and let

$$\|b\| = \max_{|z|=1} |b(z)|, \quad M(b, L) = \max_{|z|=L} |b(z)|.$$

For a polynomial $b(z)$, there is a simple deduction from the Maximum Modulus Principle [11, p. 158] that for $L \geq 1$,

$$M(b, L) \leq L^m \|b\|. \tag{1}$$

Equality is obtained in (1) for $b(z) = \lambda z^m$ with $\lambda \neq 0$, $\lambda \in \mathbb{C}$.

For a polynomial $b(z)$ having all its zeros outside $|z| < 1$, it was shown by Ankeny and Rivlin [1] that for $L \geq 1$,

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$$M(b, L) \leq \left(\frac{L^n + 1}{2}\right)\|b\|. \tag{2}$$

Equality holds in (2) for $b(z) = \alpha + \beta z^m$, where $|\alpha| = |\beta|$.

Govil [6] understood that equality in (2) holds only for polynomials $b(z) = \alpha + \beta z^m$, $|\alpha| = |\beta|$, which satisfy

$$|coefficient\ of\ z^m| = \frac{1}{2}\|b\|, \tag{3}$$

and it would be possible to refine the bound in (2) for polynomials which do not hold the condition given in (3). In an attempt to solve this problem, he [6] could obtain that for polynomial $b(z) = \sum_{v=0}^{m} w_v z^v$ having all its zeros outside $|z| < 1$ and $L \geq 1$, we have

$$\begin{aligned} M(b, L) \ &\leq \frac{(L^m + 1)}{2}\|b\| - \frac{m}{2}\left(\frac{\|b\|^2 - 4|w_m|^2}{\|b\|}\right) \\ &\times \left[\frac{(L-1)\|b\|}{\|b\| + 2|w_m|} - \ln\left\{1 + \frac{(L-1)\|b\|}{\|b\| + 2|w_m|}\right\}\right]. \end{aligned} \tag{4}$$

Recently, Hussain [8, Corollary 2] proved a generalization and extension of inequality (4) that

$$\begin{aligned} M(b, L) \ &\leq \left(\frac{L^m + s_1}{1 + s_1}\right) \| b \| - \frac{m}{1 + s_1}\left(\frac{(\| b \|)^2 - (1 + s_1)^2 |w_m|^2}{\| b \|}\right) \\ &\times \left\{\frac{(L-1) \| b \|}{\| b \| + (1 + s_1)|w_m|} - ln\left(1 + \frac{(L-1) \| b \|}{\| b \| + (1 + s_1)|w_m|}\right)\right\}, \end{aligned} \tag{5}$$

where

$$s_1 = \frac{k^{\mu+1}(\frac{\mu}{m}\frac{|w_\mu|}{|w_0|}k^{\mu-1} + 1)}{\frac{\mu}{m}\frac{|w_\mu|}{|w_0|}k^{\mu+1} + 1}, \tag{6}$$

where $b(z) = w_0 + \sum_{v=\mu}^{m} w_v z^v$, $\mu \in \{1, 2, ..., m\}$ is a polynomial such that $b(z) \neq 0$ in $|z| < k$, $k \geq 1$.

Remark 1 When $\mu = m$, the polynomial $b(z) = w_0 + \sum_{v=\mu}^{m} w_v z^v$ becomes $b(z) = w_0 + w_m z^m$. Therefore, by simple calculation, we have

$$M(b, L) = \max_{|z|=L} |w_0 + w_m z^m| = |w_0| + L^m |w_m|. \tag{7}$$

However, for $\mu = m$, inequality (5) reduces to

$$M(b,L) \le \left(\frac{L^m+s_3}{1+s_3}\right) \parallel b \parallel - \frac{m}{1+s_3}\left(\frac{(\parallel b \parallel)^2-(1+s_3)^2|w_m|^2}{\parallel b \parallel}\right) \times \left\{\frac{(L-1)\parallel b \parallel}{\parallel b \parallel +(1+s_3)|w_m|} - ln\left(1+\frac{(L-1)\parallel b \parallel}{\parallel b \parallel +(1+s_3)|w_m|}\right)\right\}, \quad (8)$$

where

$$s_3 = \frac{|\frac{w_m}{w_0}|k^{2m}+k^{m+1}}{|\frac{w_m}{w_0}|k^{m+1}+1}. \quad (9)$$

The estimate of $M(b, L)$ given by inequality (8) for $\mu = m$ is not required as we could easily get the exact value of it by a simple calculation given by (7).

2 Main Results

In this manuscript, we obtain a result which is a refinement and a generalization of inequality (5) of Hussain [8].

Theorem 1 *If* $b(z) = w_0 + \sum_{v=\mu}^{m} w_v z^v$, $\mu \in \{1, 2, ..., m-1\}$, *is a polynomial having all its zeros outside* $|z| < k$, $k \ge 1$, *then for* $L \ge 1$ *and* $N \in \mathbb{Z}^+$, $N \le m$,

$$M(b,L) \le \left(\frac{L^m+s_1}{1+s_1}\right)\|b\| - \frac{(L^m-1)s_1m^*}{(1+s_1)k^m} - m\left\{\frac{\|b\|}{1+s_1} - \frac{s_1m^*}{(1+s_1)k^m} - |w_m|\right\} f(N,s_1), \quad (10)$$

where

$$s_1 = \frac{k^{\mu+1}(\frac{\mu}{m}|\frac{w_\mu}{w_0}|k^{\mu-1}+1)}{\frac{\mu}{m}|\frac{w_\mu}{w_0}|k^{\mu+1}+1} \quad (11)$$

and

$$f(N,s_1) = \left(L-1\right) - \left\{1+\frac{(1+s_1)|w_m|}{\|b\|-\frac{s_1m^*}{k^m}}\right\} \times \ln\left\{1+\frac{(L-1)(\|b\|-\frac{s_1m^*}{k^m})}{(\|b\|-\frac{s_1m^*}{k^m})+(1+s_1)|w_m|}\right\} \quad for\ N=1, \quad (12)$$

$$f(N, s_1) = \left(\frac{L^N - 1}{N}\right) + \sum_{v=1}^{N-1}\left(\frac{L^{N-v} - 1}{N - v}\right)(-1)^v \left\{1 + \frac{(1+s_1)|w_m|}{\|b\| - \frac{s_1 m^*}{k^m}}\right\}\left\{\frac{(1+s_1)|w_m|}{\|b\| - \frac{s_1 m^*}{k^m}}\right\}^{v-1} + (-1)^N \left\{1 + \frac{(1+s_1)|w_m|}{\|b\| - \frac{s_1 m^*}{k^m}}\right\}\left\{\frac{(1+s_1)|w_m|}{\|b\| - \frac{s_1 m^*}{k^m}}\right\}^{N-1} \times \ln\left\{1 + \frac{(L-1)(\|b\| - \frac{s_1 m^*}{k^m})}{(\|b\| - \frac{s_1 m^*}{k^m}) + (1+s_1)|w_m|}\right\} \quad for\ N \geq 2 \tag{13}$$

and here and in the entire paper

$$m^* = \min_{|z|=k} |b(z)|. \tag{14}$$

Remark 2 From Lemma 3, $f(N, s_1)$ given by (12) and (13) of Theorem 1 is a monotonically increasing function of N, $N \leq m$, hence, taking $N = m$, we obtain the best bound in Theorem 1.

Further, consider $b(z)$ to be a polynomial whose degree $m = 1$. Then, by a straightforward calculation, we obtain

$$M(b, L) = \max_{|z|=L} |b(z)| = \max_{|z|=L} |w_0 + Lw_1| = |w_0| + L|w_1|. \tag{15}$$

Hence, we present the exact value of $M(b, L)$ for $m = 1$ which is given by (15).

From the preceding dicussion, Theorem 1 assumes

Corollary 1 *If* $b(z) = w_0 + \sum_{v=\mu}^{m} w_v z^v$, $\mu \in \{1, 2, ..., m-1\}$, *is a polynomial with all its zeros outside* $|z| < k$, $k \geq 1$, *then for* $L \geq 1$,

$$M(b, L) = |w_0| + L|w_1| \quad for\ m = 1 \tag{16}$$

and

$$M(b, L) \leq \left(\frac{L^m + s_1}{1 + s_1}\right)\|b\| - \frac{(L^m - 1)s_1 m^*}{(1+s_1)k^m} - m\left\{\frac{\|b\|}{1+s_1} - \frac{s_1 m^*}{(1+s_1)k^m} - |w_m|\right\} f(m, s_1), \tag{17}$$

where

$$
\begin{aligned}
f(m, s_1) = &\left(\frac{L^m - 1}{m}\right) \\
&+ \sum_{v=1}^{m-1} \left(\frac{L^{m-v} - 1}{m - v}\right)(-1)^v \left\{1 + \frac{(1 + s_1)|w_m|}{\|b\| - \frac{s_1 m^*}{k^m}}\right\} \left\{\frac{(1 + s_1)|w_m|}{\|b\| - \frac{s_1 m^*}{k^m}}\right\}^{v-1} \\
&+ (-1)^m \left\{1 + \frac{(1 + s_1)|w_m|}{\|b\| - \frac{s_1 m^*}{k^m}}\right\} \left\{\frac{(1 + s_1)|w_m|}{\|b\| - \frac{s_1 m^*}{k^m}}\right\}^{m-1} \\
&\times \ln \left\{1 + \frac{(L - 1)(\|b\| - \frac{s_1 m^*}{k^m})}{(\|b\| - \frac{s_1 m^*}{k^m}) + (1 + s_1)|w_m|}\right\} \quad for\ m \geq 2 \qquad (18)
\end{aligned}
$$

and s_1 is as defined in (11).

Remark 3 If $k = 1, s_1 = 1$, then Theorem 1 reduces to the succeeding result which refines and generalizes the result of Dewan and Bhat [3].

Corollary 2 *If $b(z) = w_0 + \sum_{v=\mu}^{m} w_v z^v$, $\mu \in \{12, ..., m - 1\}$, is a polynomial with all its zeros outside $|z| < k$, $k \geq 1$, then for $L \geq 1$, and $N \in \mathbb{Z}^+$, $N \leq m$,*

$$
M(b, L) \leq \left(\frac{L^m + 1}{2}\right)\|b\| - \left(\frac{L^m - 1}{2}\right)m^* - m\left(\frac{\|b\| - m^*}{2} - |w_m|\right) f(N, 1), \qquad (19)
$$

where

$$
\begin{aligned}
f(N, 1) = &\left(\frac{L^N - 1}{N}\right) \\
&+ \sum_{v=1}^{N-1} \left(\frac{L^{N-v} - 1}{N - v}\right)(-1)^v \left\{1 + \frac{2|w_m|}{\|b\| - m^*}\right\} \left\{\frac{2|w_m|}{\|b\| - m^*}\right\}^{v-1} \\
&+ (-1)^N \left\{1 + \frac{2|w_m|}{\|b\| - m^*}\right\} \left\{\frac{2|w_m|}{\|b\| - m^*}\right\}^{N-1} \\
&\times \ln \left\{1 + \frac{(L - 1)(\|b\| - m^*)}{(\|b\| - m^*) + 2|w_m|}\right\}. \qquad (20)
\end{aligned}
$$

Remark 4 Since for $1 \leq N$, $f(1, 1) \leq f(N, 1)$ and hence, substituting the value of $f(1, 1)$, inequality (19) becomes the result of Dewan and Bhat [3].

Remark 5 For $N = 1$, Theorem 1, in particular, becomes the following interesting result.

Corollary 3 *If $b(z) = w_0 + \sum_{v=\mu}^{m} w_v z^v$, $\mu \in \{12, ..., m - 1\}$, is a polynomial with all its zeros outside $|z| < k$, $k \geq 1$, then for $L \geq 1$,*

$$M(b,L) \geq \left(\frac{L^m + s_1}{1+s_1}\right)\|b\| - \left(\frac{L^m - 1}{1+s_1}\right)\frac{s_1 m^*}{k^m}$$
$$- \frac{m}{1+s_1}\left\{\frac{(\|b\| - \frac{s_1 m^*}{k^m})^2 - |w_m|^2(1+s_1)^2}{\|b\| - \frac{s_1 m^*}{k^m}}\right\}$$
$$\times\left[\frac{(L-1)(\|b\| - \frac{s_1 m^*}{k^m})}{\|b\| - \frac{s_1 m^*}{k^m} + (1+s_1)|w_m|} - \ln\left\{1 + \frac{(L-1)(\|b\| - \frac{s_1 m^*}{k^m})}{\|b\| - \frac{s_1 m^*}{k^m} + (1+s_1)|w_m|}\right\}\right],$$

where s_1 is as defined in (11).

Remark 6 By Lemma 8, we have

$$\left(\|b\| - \frac{s_1 m^*}{k^m}\right)^2 - (1+s_1)^2|w_m|^2 \geq 0 \tag{21}$$

and $\ln(1+x) < x$ for positive values of x and hence the bound given by Corollary 3 improves and generalizes inequality (2) proved by Ankeny and Rivlin [1].

Remark 7 By Lemma 10, $k \leq s_1$ for $k \geq 1$, where s_1 is as defined in (11), therefore, we have for $m^* \geq 0$

$$\frac{m}{1+s_1}\|b\| - \frac{m s_1 m^*}{k^m(1+s_1)} \leq \frac{m}{1+s_1}\|b\|. \tag{22}$$

Applying Lemma 4 to (22), we have for $r \geq 1$,

$$r^{m-1}\left\{1 - \frac{\left(\frac{m}{1+s_1}\|b\| - \frac{m s_1 m^*}{k^m(1+s_1)} - m|w_m|\right)(r-1)}{m|w_m| + r\left(\frac{m}{1+s_1}\|b\| - \frac{m s_1 m^*}{k^m(1+s_1)}\right)}\right\}\left\{\frac{m}{1+s_1}\|b\| - \frac{m s_1 m^*}{k^m(1+s_1)}\right\}$$
$$\leq r^{m-1}\left\{1 - \frac{\left(\frac{m}{1+s_1}\|b\| - m|w_m|\right)(r-1)}{m|w_m| + \frac{rm}{1+s_1}\|b\|}\right\}\frac{m}{1+s_1}\|b\|. \tag{23}$$

On integrating (23) from both sides with respect to r from 1 to L and following similar simplification of the RHS of (73) to inequality (74) in the proof of Theorem 1, we get

$$\frac{L^m - 1}{1+s_1}\left(\|b\| - \frac{s_1 m^*}{k^m}\right) - \frac{m}{1+s_1}\left(\|b\| - \frac{s_1 m^*}{k^m}\right)(1-e)\int_1^R \frac{(r-1)r^{m-1}}{r+e}dr$$
$$\leq \frac{L^m - 1}{1+s_1}\|b\| - \frac{m}{1+s_1}\|b\|(1-g)\int_1^L \frac{(r-1)r^{m-1}}{r+g}dr, \tag{24}$$

where $e = \frac{|w_m|(1+s_1)}{\|b\| - \frac{s_1 m^*}{k^m}}$ and $g = \frac{|w_m|(1+s_1)}{\|b\|}$.

The expression $\int_1^L \frac{(r-1)r^{N-1}}{r+g}dr \geq 0$ and is a monotonically increasing function of N for $N \leq m$, therefore, we have

$$\int_1^L \frac{(r-1)r^{N-1}}{r+g}dr \leq \int_1^L \frac{(r-1)r^{m-1}}{r+g}dr. \tag{25}$$

Since $m^* \geq 0$, by Lemma 8, we have

$$\frac{|w_m|(1+s_1)}{\|b\|} \leq 1, \tag{26}$$

and hence

$$1-g = 1 - \frac{|w_m|(1+s_1)}{\|b\|} \geq 0. \tag{27}$$

We see that $1-g \geq 0$ and using Lemma 2 for the values of the integrals of inequality (24), we have

$$\begin{aligned}
&\left(\frac{L^m-1}{1+s_1}\right)\left(\|b\| - \frac{s_1 m^*}{k^m}\right) - \frac{m}{1+s_1}\left(\|b\| - \frac{s_1 m^*}{k^m}\right)\left\{1 - \frac{(1+s_1)|w_m|}{\|b\| - \frac{s_1 m^*}{k^m}}\right\} f(m,s_1) \\
&\leq \left(\frac{L^m-1}{1+s_1}\right)\|b\| - \frac{m\|b\|}{1+s_1}\left\{1 - \frac{(1+s_1)|w_m|}{\|b\|}\right\} h^*(N),
\end{aligned} \tag{28}$$

where $f(m,s_1)$ is as defined in (18) and

$$\begin{aligned}
h^*(N) &= (L-1) - \left\{1 + \frac{(1+s_1)|w_m|}{\|b\|}\right\} \\
&\quad \times \ln\left\{1 + \frac{(L-1)\|b\|}{\|b\| + (1+s_1)|w_m|}\right\} \quad for\ N=1,
\end{aligned} \tag{29}$$

$$\begin{aligned}
h^*(N) = &\left(\frac{L^N-1}{m}\right) \\
&+ \sum_{v=1}^{m-1}\left(\frac{L^{N-v}-1}{N-v}\right)(-1)^v\left\{1 + \frac{(1+s_1)|w_m|}{\|b\|}\right\}\left\{\frac{(1+s_1)|w_m|}{\|b\|}\right\}^{v-1} \\
&+ (-1)^N\left\{1 + \frac{(1+s_1)|w_m|}{\|b\|}\right\}\left\{\frac{(1+s_1)|w_m|}{\|b\|}\right\}^{N-1} \\
&\times \ln\left(1 + \frac{(L-1)\|b\|}{\|b\| + (1+s_1)|w_m|}\right) \quad for\ N \geq 2.
\end{aligned} \tag{30}$$

Adding $\|b\|$ on both sides of (28), we have

$$\begin{aligned}
&\left(\frac{L^m+s_1}{1+s_1}\right)\|b\| - \frac{(L^m-1)}{1+s_1}\frac{s_1 m^*}{k^m} - m\left\{\frac{\|b\|}{1+s_1} - \frac{s_1 m^*}{(1+s_1)k^m} - |w_m|\right\} f(m,s_1) \\
&\leq \left(\frac{L^m+s_1}{1+s_1}\right)\|b\| - \frac{m}{1+s_1}\{\|b\| - (1+s_1)|w_m|\} h^*(N),
\end{aligned} \tag{31}$$

which clearly shows that Corollary 1 refines the next result which further deduces to inequality (5) due to Hussain [8].

Corollary 4 *If* $b(z) = w_0 + \sum_{v=\mu}^{m} w_v z^v$, $\mu \in \{1, 2, ..., m-1\}$, *is a polynomial with all its zeros outside* $|z| < k$, $k \geq 1$, *then for* $L \geq 1$ *and* $N \in \mathbb{Z}^+$, $N \leq m$,

$$M(b,R) \leq \left(\frac{L^m + s_1}{1+s_1}\right)\|b\| - \frac{m}{1+s_1}\left\{\|b\| - (1+s_1)|w_m|\right\} h^*(N), \tag{32}$$

where

$$s_1 = \frac{k^{\mu+1}(\frac{\mu}{m}|\frac{w_\mu}{w_0}|k^{\mu-1}+1)}{\frac{\mu}{m}|\frac{w_\mu}{w_0}|k^{\mu+1}+1} \tag{33}$$

and

$$h^*(N) = \left(L-1\right) - \left\{1 + \frac{(1+s_1)|w_m|}{\|b\|}\right\} \times \ln\left\{1 + \frac{(L-1)\|b\|}{\|b\| + (1+s_1)|w_m|}\right\} \quad for\ N = 1, \tag{34}$$

$$\begin{aligned} h^*(N) = &\left(\frac{L^N - 1}{N}\right) \\ &+ \sum_{v=1}^{N-1}\left(\frac{L^{N-v}-1}{N-v}\right)(-1)^v\left\{1 + \frac{(1+s_1)|w_m|}{\|b\|}\right\}\left\{\frac{(1+s_1)|w_m|}{\|b\|}\right\}^{v-1} \\ &+ (-1)^N\left\{1 + \frac{(1+s_1)|w_m|}{\|b\|}\right\}\left\{\frac{(1+s_1)|w_m|}{\|b\|}\right\}^{N-1} \\ &\times \ln\left\{1 + \frac{(L-1)\|b\|}{\|b\| + (1+s_1)|w_m|}\right\} \quad for\ N \geq 2. \end{aligned} \tag{35}$$

Remark 8 By Lemma 3, it is noted that $h^*(N) \geq 0$ as defined in (34) and (35) of Corollary 4 and is a monotonically increasing function of N for $N \geq 1$ and therefore $h^*(1) \leq h^*(N)$. Noting this and Lemma 8 that $\{\|b\| - (1+s_1)|w_m|\} \geq 0$, Corollary 4 reduces to inequality (5) due to Hussain [8]

Remark 9 By Lemma 10, $k \leq s_1$ for $k \geq 1$, where s_1 is as defined in (11), therefore, by Lemma 11, we have

$$\frac{m}{1+s_1}\|b\| \leq \frac{m}{1+k}\|b\|. \tag{36}$$

Since $m \geq 0$ and $1 \leq k \leq s_1$, inequality (36) implies

$$\frac{m}{1+s_1}\|b\| - \frac{ms_1m^*}{k^m(1+s_1)} \leq \frac{m}{1+k}\|b\|. \tag{37}$$

Applying Lemma 4 to (37), we have for $r \geq 1$,

$$r^{m-1}\left\{1-\frac{\left(\frac{m}{1+s_1}\|b\|-\frac{ms_1m^*}{k^m(1+s_1)}-m|w_m|\right)(r-1)}{m|w_m|+r\left(\frac{m}{1+s_1}\|b\|-\frac{ms_1m^*}{k^m(1+s_1)}\right)}\right\}\left\{\frac{m}{1+s_1}\|b\|-\frac{ms_1m^*}{k^m(1+s_1)}\right\}$$
$$\leq r^{m-1}\left\{1-\frac{\left(\frac{m}{1+k}\|b\|-m|w_m|\right)(r-1)}{m|w_m|+r\left(\frac{m}{1+k}\|b\|\right)}\right\}\left(\frac{m}{1+k}\|b\|\right). \tag{38}$$

Inequality (38) is integrated on both sides with respect to r from 1 to L and following similar simplification of the RHS of inequality (73) to inequality (74) in the proof of Theorem 1, we get

$$\frac{L^m-1}{1+s_1}\left(\|b\|-\frac{s_1m^*}{k^m}\right)-\frac{m}{1+s_1}\left(\|b\|-\frac{s_1m^*}{k^m}\right)(1-e)\int_1^L\frac{(r-1)r^{m-1}}{r+e}dr$$
$$\leq\frac{L^m-1}{1+k}\|b\|-\frac{m\|b\|}{1+k}(1-c)\int_1^L\frac{(r-1)r^{m-1}}{r+c}dr, \tag{39}$$

where $e=\frac{|w_m|(1+s_1)}{\|b\|-\frac{s_1m^*}{k^m}}$ and $c=\frac{|w_m|(1+k)}{\|b\|}$.

The expression $\int_1^L\frac{(r-1)r^{N-1}}{r+c}dr\geq 0$ and is a monotonically increasing function of N for $N\leq m$, we have

$$\int_1^L\frac{(r-1)r^{N-1}}{r+c}dr\leq\int_1^L\frac{(r-1)r^{m-1}}{r+c}dr. \tag{40}$$

Since $m^*\geq 0$, by Lemma 9, we have

$$\frac{|w_m|(1+k)}{\|b\|}\leq 1, \tag{41}$$

and hence

$$1-c=1-\frac{|w_m|(1+k)}{\|b\|}\geq 0. \tag{42}$$

Since $1-c\geq 0$ and using Lemma 2 for the values of the integrals in (39), we get

$$\left(\frac{L^m-1}{1+s_1}\right)\left(\|b\|-\frac{s_1m^*}{k^m}\right)-\frac{m}{1+s_1}\left(\|b\|-\frac{s_1m^*}{k^m}\right)\left\{1-\frac{(1+s_1)|w_m|}{\|b\|-\frac{s_1m^*}{k^m}}\right\}f(m,s_1)$$
$$\leq\left(\frac{L^m-1}{1+k}\right)\|b\|-\frac{m\|b\|}{1+k}\left\{1-\frac{(1+k)|w_m|}{\|b\|}\right\}g^*(N), \tag{43}$$

where $f(m,s_1)$ is as defined in (18) and

$$g^*(N)=(L-1)-\left\{1+\frac{(1+k)|w_m|}{\|b\|}\right\}$$
$$\times\ln\left\{1+\frac{(L-1)\|b\|}{\|b\|+(1+k)|w_m|}\right\}\quad for\ N=1,$$

$$g^*(N) = \left(\frac{L^N-1}{m}\right) + \sum_{v=1}^{m-1}\left(\frac{L^{N-v}-1}{N-v}\right)(-1)^v\left\{1+\frac{(1+k)|w_m|}{\|b\|}\right\}\left\{\frac{(1+k)|w_m|}{\|b\|}\right\}^{v-1} + (-1)^N\left\{1+\frac{(1+k)|w_m|}{\|b\|}\right\}\left\{\frac{(1+k)|w_m|}{\|b\|}\right\}^{N-1} \times \ln\left(1+\frac{(L-1)\|b\|}{\|b\|+(1+k)|w_m|}\right) \quad for\ N \geq 2. \tag{44}$$

Adding $\|b\|$ on both sides of (43), we have

$$\left(\frac{L^m+s_1}{1+s_1}\right)\|b\| - \frac{(L^m-1)}{1+s_1}\frac{s_1m^*}{k^m} - m\left\{\frac{\|b\|}{1+s_1} - \frac{s_1m^*}{(1+s_1)k^m} - |w_m|\right\} f(m,s_1) \leq \left(\frac{L^m+k}{1+k}\right)\|b\| - \frac{m}{1+k}\{\|b\|-(1+k)|w_m|\}\, g^*(N). \tag{45}$$

Hence, it is verified that Corollary 1 improves the succeeding result.

Corollary 5 *If* $b(z) = w_0 + \sum_{v=\mu}^{m} w_v z^v,\ \mu \in \{1, 2, ..., m-1\}$, *is a polynomial with all its zeros outside* $|z| < k$, $k \geq 1$, *then for* $L \geq 1$ *and* $N \in \mathbb{Z}^+$, $N \leq m$,

$$M(b, L) \leq \left(\frac{L^m+k}{1+k}\right)\|b\| - \frac{m}{1+k}\left(\|b\|-(1+k)|w_m|\right)g^*(N), \tag{46}$$

where

$$g^*(N) = (L-1) - \left\{1+\frac{(1+k)|w_m|}{\|b\|}\right\} \times \ln\left\{1+\frac{(L-1)\|b\|}{\|b\|+(1+k)|w_m|}\right\} \quad for\ N = 1, \tag{47}$$

$$g^*(N) = \left(\frac{L^N-1}{N}\right) + \sum_{v=1}^{N-1}\left(\frac{L^{N-v}-1}{N-v}\right)(-1)^v\left\{1+\frac{(1+k)|w_m|}{\|b\|}\right\}\left\{\frac{(1+k)|w_m|}{\|b\|}\right\}^{v-1} + (-1)^N\left\{1+\frac{(1+k)|w_m|}{\|b\|}\right\}\left\{\frac{(1+k)|w_m|}{\|b\|}\right\}^{N-1} \times \ln\left\{1+\frac{(L-1)\|b\|}{\|b\|+(1+k)|w_m|}\right\} \quad for\ N \geq 2. \tag{48}$$

Remark 10 For $N = m$, it can be easily verified that the result of Mir et al. [10, Corollary 1] is obtained from Corollary 5.

Remark 11 By Lemma 3, it is observed that $g^*(N) \geq 0$ as defined in (47) and (48) of Corollary 5 and is a monotonically increasing function of N for $N \geq 1$ and hence $g^*(1) \leq g^*(N)$. With this fact and Lemma 9, Corollary 5 gives a result which is a generalization of inequality (4) of Govil [6].

Corollary 6 *If* $b(z) = w_0 + \sum_{v=\mu}^{m} w_v z^v,\ \mu \in \{1, 2, ..., m-1\}$*, is a polynomial with all its zeros outside* $|z| < k$*,* $k \geq 1$*, then for* $L \geq 1$*,*

$$M(b, L) \leq \left(\frac{L^m + k}{1+k}\right)\|b\| - \frac{m}{1+k}\left\{\frac{\|b\|^2 - (1+k)^2|w_m|^2}{\|b\|}\right\} \times \left[\frac{(L-1)\|b\|}{\|b\| + (1+k)|w_m|} - \ln\left\{1 + \frac{(L-1)\|b\|}{\|b\| + (1+k)|w_m|}\right\}\right]. \quad (49)$$

Remark 12 Also for $k = 1$, inequality (49) of Corollary 6 reduces to inequality (4) of Govil [6].

3 Lemmas

We require the following lemmas.

Lemma 1 *Let* $b(z) = \sum_{v=0}^{m} w_m z^m$ *be a polynomial. Then for* $|z| = L \geq 1$*,*

$$|b(z)| \leq L^m \left\{1 - \frac{(\|b\| - |w_m|)(L-1)}{|w_m| + L\|b\|}\right\} \|b\|. \quad (50)$$

Lemma 1 is due to Govil [6].

Lemma 2 *Let*

$$J(N) = \int_1^L \frac{(r-1)r^{N-1}}{r+x} dr, \quad x > 0 \ . \quad (51)$$

Then for $N \geq 2$*,*

$$J(N) = \left(\frac{L^N - 1}{N}\right) + \sum_{v=1}^{N-1}\left(\frac{L^{N-v} - 1}{N-v}\right)(-1)^v (x+1)x^{v-1} + (-1)^N (x+1)x^{N-1} \ln\left(\frac{L+x}{1+x}\right), \quad (52)$$

and for $N = 1$*,*

$$J(1) = (L-1) - (1+x)\ln\left(1 + \frac{L-1}{1+x}\right). \quad (53)$$

Lemma 2 is due to Dalal and Govil [2, Lemma 3.6].

Lemma 3 *$J(N)$ defined in Lemma 2 is a non-negative increasing function of N for $N \geq 1$.*

Proof (*Proof of Lemma* 3) Dalal and Govil [2, Lemma 3.7] has done this proof, but, we present another proof of it using the method of differentiation under the integral sign.

By the method of differentiation under the integral sign, we obtain

$$\frac{d}{dN} J(N) = \int_1^L \frac{(r-1)(r^{N-1})}{r+x} \ln r dr. \tag{54}$$

Since, for $r \in [1, L]$, $\frac{(r-1)r^{N-1}}{r+x} \ln r \geq 0$, therefore, we have

$$\int_1^L \frac{(r-1)r^{N-1}}{r+x} \ln r dr \geq 0. \tag{55}$$

From equality (54),

$$\frac{d}{dN} J(N) \geq 0, \quad for\ N \geq 1. \tag{56}$$

Hence, $J(N)$ is an increasing function of N for $N \geq 1$.

Further, we see that $\frac{(r-1)r^{N-1}}{r+x}$ is non-negative for $N \geq 1$ which implies that $J(N) \geq 0$ for $N \geq 1$,
and hence Lemma 3 is proved. □

Lemma 4 *For polynomial $b(z) = w_0 + \sum_{v=\mu}^{m} w_v z^v$, $\mu \in \{1, 2, ..., m\}$ and $r \geq 1$, the function*

$$t(y) = \left\{1 - \frac{(y - m|w_m|)(r-1)}{m|w_m| + ry}\right\} y \tag{57}$$

is an increasing function of y for $y > 0$.

Proof of Lemma 4. The proof simply follows by using the derivative test and we omit it.

The next lemma is due to Qazi [12, Remark 1].

Lemma 5 *If $b(z) = w_0 + \sum_{v=\mu}^{m} w_v z^v$, $\mu \in \{1, 2, ..., m\}$, is a polynomial with all its zeros outside $|z| < k$, $k \geq 1$, then*

$$\frac{\mu}{m} \left| \frac{w_\mu}{w_0} \right| k^\mu \leq 1. \tag{58}$$

Lemma 6 *If* $b(z) = \sum_{v=0}^{m} w_v z^v$ *is a polynomial with all its zeros outside* $|z| < k$, $k \geq 1$, *then*

$$\|b'\| \leq \frac{m}{1+k}\|b\|. \tag{59}$$

Lemma 6 is due to Malik [9].

Lemma 7 *If* $b(z) = w_0 + \sum_{v=\mu}^{m} w_v z^v$, $\mu \in \{1, 2, ..., m-1\}$, *is a polynomial with no zero in* $|z| < k$, $k \geq 1$, *then*

$$\|b'\| \leq \frac{m}{1+s_1}\|b\| - \frac{m}{k^m}\left(1 - \frac{1}{1+s_1}\right)m^*, \tag{60}$$

where s_1 *is as defined in (11).*

Lemma 7 is due to Dewan et al. [4].

Lemma 8 *If* $b(z) = w_0 + \sum_{v=\mu}^{m} w_v z^v$, $\mu \in \{1, 2, ..., m-1\}$, *is a polynomial with no zero in* $|z| < k$, $k \geq 1$, *then*

$$|w_m| \leq \frac{1}{1+s_1}\left(\|b\| - \frac{m^* s_1}{k^m}\right), \tag{61}$$

where s_1 *is as defined in (11).*

Proof (*Proof of Lemma 8*)
For a polynomial $b(z) = w_0 + \sum_{v=\mu}^{m} w_v z^v$, $\mu \in \{1, 2, ..., m-1\}$, then we get $b'(z) = \sum_{v=\mu}^{m} v w_v z^{v-1}$.

Using Cauchy's inequality to $b'(z)$ on $|z| = 1$, we have

$$\left|\frac{d^{m-1}}{dz^{m-1}} b'(z)\right|_{z=0} \leq (m-1)! \max_{|z|=1} |b'(z)|. \tag{62}$$

That is,

$$|m w_m| \leq \|b'\|. \tag{63}$$

Combining inequality (60) of Lemma 7 and (63), we have inequality (61) of Lemma 8 and this completes the proof of Lemma 8. □

Lemma 9 *If* $b(z) = \sum_{v=0}^{m} w_v z^v$ *is a polynomial with no zero in* $|z| < k$, $k \geq 1$, *then*

$$|w_m| \leq \frac{1}{1+k}\|b\|. \tag{64}$$

Proof (*Proof of Lemma* 9) This lemma is proved in similar ways as that of Lemma 8, but we apply inequality (59) of Lemma 6 in place of (60) of Lemma 7 and we omit the details □.

Lemma 10 *If* $b(z) = w_0 + \sum_{v=\mu}^{m} w_v z^v$, $\mu \in \{1, 2, ..., m\}$, *is a polynomial with no zero in* $|z| < k$, $k \geq 1$, *then*

$$s_1 \geq k, \tag{65}$$

where s_1 *is as defined in (11).*

Proof (*Proof of Lemma* 10) Let $b(z) = w_0 + \sum_{v=\mu}^{m} w_v z^v$, $\mu \in \{1, 2, ..., m\}$, is a polynomial with no zero in $|z| < k$, $k \geq 1$.
From inequality (58) of Lemma 5, we have

$$0 \leq \frac{\mu}{m}\left|\frac{w_\mu}{w_0}\right|k^\mu \leq 1. \tag{66}$$

Since $k \geq 1$ and $\mu = 1, 2, \cdots$, we have

$$k - k^{\mu-1} \leq k^\mu - 1. \tag{67}$$

Multiplying (66) and (67) sidewise, we have

$$k^\mu\left\{\frac{\mu}{m}\left|\frac{w_\mu}{w_0}\right|k^{\mu-1} + 1\right\} \geq \frac{\mu}{m}\left|\frac{w_\mu}{w_0}\right|k^{\mu+1} + 1, \tag{68}$$

which is equivalent to

$$s_1 \geq k,$$

and hence, Lemma 10 is obtained. □

Lemma 11 *If* $b(z) = w_0 + \sum_{v=\mu}^{m} w_v z^v$, $\mu \in \{1, 2, ..., m\}$, *is a polynomial having no zero in* $|z| < k$, $k \geq 1$, *then*

$$\frac{m}{1 + s_1}\|b\| \leq \frac{m}{1 + k}\|b\|, \tag{69}$$

where s_1 *is as defined in (11).*

Lemma 11 is due to Qazi [12].

4 Proof of the Theorem

Proof (*Proof of Theorem* 1) For each θ, $0 \le \theta < 2\pi$ and $1 \le r \le L$, we have

$$b(Le^{i\theta}) - b(e^{i\theta}) = \int_1^L e^{i\theta} b'(re^{i\theta}) dr, \tag{70}$$

which implies

$$|b(Le^{i\theta}) - b(e^{i\theta})| \le \int_1^L |b'(re^{i\theta})| dr. \tag{71}$$

Now, applying Lemma 1 to the polynomial $b'(z)$ which is of degree $m-1$, we get

$$|b(Le^{i\theta}) - b(e^{i\theta})| \le \int_1^L r^{m-1} \left\{1 - \frac{(\|b'\| - m|w_m|)(r-1)}{m|w_m| + r\|b'\|}\right\} \|b'\| dr. \tag{72}$$

Since by Lemma 4, in the integrand of (72), the quantity $\left\{1 - \frac{(\|b'\|-m|w_m|)(r-1)}{m|w_m|+r\|b'\|}\right\} \|b'\|$ is a monotonically increasing function of $\|b'\|$, hence using Lemma 7, we have for $0 \le \theta < 2\pi$,

$$\begin{aligned}
&|b(Le^{i\theta}) - b(e^{i\theta})| \\
&\le \int_1^L r^{m-1} \left[1 - \frac{\left\{\frac{m}{1+s_1}\|b\| - \frac{m}{k^m}\left(1 - \frac{1}{1+s_1}\right)m^* - m|w_m|\right\}(r-1)}{m|w_m| + r\left\{(\frac{m}{1+s_1})\|b\| - \frac{m}{k^m}\left(1 - \frac{1}{1+s_1}\right)m^*\right\}}\right] \\
&\times \left\{\frac{m}{1+s_1}\|b\| - \frac{m}{k^m}\left(1 - \frac{1}{1+s_1}\right)m^*\right\} dr \\
&= \left\{\frac{m}{1+s_1}\|b\| - \frac{mm^* s_1}{k^m(1+s_1)}\right\} \int_1^L r^{m-1} dr - \left\{\frac{m}{1+s_1}\|b\| - \frac{mm^* s_1}{k^m(1+s_1)}\right\} \\
&\times \int_1^L r^{m-1} \left[\frac{\|b\| - \frac{m^* s_1}{k^m} - (1+s_1)|w_m|}{(1+s_1)|w_m| + r\left\{\|b\| - \frac{m^* s_1}{k^m}\right\}}\right](r-1) dr \\
&= \frac{L^m - 1}{1+s_1}\left\{\|b\| - \frac{m^* s_1}{k^m}\right\} - \left\{\frac{m}{1+s_1}\|b\| - \frac{mm^* s_1}{k^m(1+s_1)}\right\} \\
&\times (1-e) \int_1^L \frac{(r-1)r^{m-1}}{r+e} dr,
\end{aligned} \tag{73, 74}$$

where s_1 is as defined in (11) and $e = \frac{|w_m|(1+s_1)}{\|b\| - \frac{m^* s_1}{k^m}}$.

It is observed that $\int_1^L \frac{(r-1)r^{N-1}}{r+e}dr \geq 0$ and is a monotonically increasing function of N for $N \leq m$, therefore, we have

$$\int_1^L \frac{(r-1)r^{N-1}}{r+e}dr \leq \int_1^L \frac{(r-1)r^{m-1}}{r+e}dr. \tag{75}$$

We see from Lemma 8 that $(1-e) \geq 0$ and using inequality (75) to (74), we get for every N, $N \leq m$,

$$|b(Le^{i\theta}) - b(e^{i\theta})| \leq \frac{L^m - 1}{1+s_1}\left\{\|b\| - \frac{m^* s_1}{k^m}\right\} - \left\{\frac{m}{1+s_1}\|b\| - \frac{mm^* s_1}{k^m(1+s_1)}\right\}$$
$$\times\ (1-e)\int_1^L \frac{(r-1)r^{N-1}}{r+e}dr. \tag{76}$$

Using Lemma 2 (on replacing x by e) for the value of the integral in (76), we have,

$$|b(Le^{i\theta}) - b(e^{i\theta})| \leq \frac{L^m - 1}{1+s_1}\left\{\|b\| - \frac{m^* s_1}{k^m}\right\}$$
$$- \left\{\frac{m}{1+s_1}\|b\| - \frac{mm^* s_1}{k^m(1+s_1)}\right\}(1-e)f(N, s_1), \tag{77}$$

where $f(N, s_1)$ is as defined in (12) and (13).

Now, putting the value of e and using the relation

$$|b(Le^{i\theta})| \leq |b(Le^{i\theta}) - b(e^{i\theta})| + |b(e^{i\theta})|$$
$$\leq |b(Le^{i\theta}) - b(e^{i\theta})| + \|b\| \tag{78}$$

in (77), we get

$$|b(Le^{i\theta})| \leq \left(\frac{L^m + s_1}{1+s_1}\right)\|b\| - \frac{(L^m - 1)}{1+s_1}\frac{s_1 m^*}{k^m}$$
$$- m\left\{\frac{\|b\|}{1+s_1} - \frac{s_1 m^*}{(1+s_1)k^m} - |w_m|\right\}f(N, s_1), \tag{79}$$

which is equivalent to inequality (10) and hence, Theorem 1 is obtained. □

5 Conclusions

We have improved and generalized inequality (5) proved by Hussain [8] by involving $\min_{|z|=k}|b(z)|$. Moreover, through Remarks and Corollaries, we have discussed the implications of Theorem 1 on other well-known results .

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