Group Action on Fuzzy Ideals of Near Rings

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Abstract In this paper, we introduce the group action on a near ring $\mathcal{N}$ and with it we study group action on fuzzy ideals of $\mathcal{N}$, $\mathcal{G}$-invariant fuzzy ideals, finite products of fuzzy ideals, and $\mathcal{G}$-primeness of fuzzy ideals of $\mathcal{N}$.

Keywords Fuzzy ideals · Prime fuzzy ideals · $\mathcal{G}$-invariant fuzzy ideals · $\mathcal{G}$-prime fuzzy ideals

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1 Introduction

A set $\mathcal{N}$ with two binary operations '+' and '·' is known as left near ring if (i) $(\mathcal{N}, +)$ is a group (not necessarily abelian), (ii) $(\mathcal{N}, \cdot)$ is a semigroup, (iii) $\alpha(\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma \; \forall \; \alpha, \beta$ and γ in $\mathcal{N}$. Analogously, $\mathcal{N}$ is said to be a right near ring if $\mathcal{N}$ satisfies $(iii)^{'}$ $(\beta + \gamma)\alpha = \beta \cdot \alpha + \gamma \cdot \alpha \; \forall \; \alpha, \beta$ and γ in $\mathcal{N}$. A near ring $\mathcal{N}$ with $0x = 0$, $\forall x \in \mathcal{N}$, is known as zero symmetric if $0x = 0$, (left distributively yields that $x0 = 0$).Throughout the paper, $\mathcal{N}$ represents a zero symmetric left near ring; for simplicity, we call it a near ring. An ideal of near ring $(\mathcal{N}, +, \cdot)$ is a subset $\mathcal{M}$ of $\mathcal{N}$ such that (i) $(\mathcal{M}, +) \triangleleft (\mathcal{N}, +)$, (ii)$\mathcal{N}\mathcal{M} \subset \mathcal{M}$, (iii) $(n_1 + m)n_2 - n_1 n_2 \in \mathcal{M} \; \forall$ $m \in \mathcal{M}$ and $n_1, n_2 \in \mathcal{N}$. Note that if $\mathcal{M}$ fulfils (i) and (ii), it's referred to as a left ideal of $\mathcal{N}$. It is termed a right ideal of $\mathcal{N}$ if $\mathcal{M}$ satisfies (i) and (iii). A mapping $\phi : \mathcal{N} \to \mathcal{N}^{'}$ from near ring $\mathcal{N}$ to near ring $\mathcal{N}^{'}$ is said to be a homomorphism if (i)

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$\phi(\alpha + \beta) = \phi(\alpha) + \phi(\beta)$ (ii) $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$ $\forall\, \alpha$ and $\beta \in \mathcal{N}$. A homomorphism $\phi : \mathcal{N} \to \mathcal{N}$ which is bijective is said to be an automorphism on $\mathcal{N}$. The set of all automorphism of $\mathcal{N}$ denoted by $Aut(\mathcal{N})$ forms a group under the operation of composition of mappings.

The study of group actions on rings led to the establishment of the Galois theory for rings. Lorenz and Passman [12], Montgomery [14], and others researched the skew grouping approach in the context of the Galois theory, as well as the groupring and fixed ring. The link between the $\mathcal{G}$-prime ideals of $\mathcal{R}$ and the prime ideals of skew groupring $\mathcal{RG}$ was identified by Lorenz and Passman [12]. Montgomery [14] investigated the relationship between the prime ideals of $\mathcal{R}$ and $\mathcal{R}^{\mathcal{G}}$, leading him to broaden the scope of the action of a group to spec$\mathcal{R}$.

Fuzzy sets were introduced independently by L.A. Zadeh and Dieter Klaua in 1965 as an extension of the classical notion of set. Liu [11] studied fuzzy ideals of a ring and many researchers [4, 6, 7, 20] extended the concepts.The concept of fuzzy ideals and related features have been applied to a variety of fields, including semigroups, [8–10, 18, 19], distributive lattice [2], BCK-algebras [16], and near rings [22]. Kim and Kim [5] defined the exact analogue of fuzzy ideals for near rings.

Sharma and Sharma [19] recently investigated the action of group on the fuzzy ideals of the ring $\mathcal{R}$ and found a relationship between the $\mathcal{G}$-prime fuzzy ideals of $\mathcal{R}$ and the prime fuzzy ideals of $\mathcal{R}$. We define the action of group on a near ring $\mathcal{N}$ and investigate the action of group on fuzzy ideals and $\mathcal{G}$-invariant fuzzy ideals of $\mathcal{N}$, finite products of fuzzy ideals, and $\mathcal{G}$-primeness of fuzzy ideals of $\mathcal{N}$. As a result, we extend Sharma and Sharma's conclusions to near ring $\mathcal{N}$.

2 Preliminaries

Definition 1 ([22]) If $\mathcal{N}$ is a near ring, then a fuzzy set $\tilde{F}$ in $\mathcal{N}$ is a set of ordered pair $\tilde{F} = \{(n, \eta_{\tilde{F}}(n)) | n \in \mathcal{N}\}$, $\eta_{\tilde{F}}(n)$ is called membership function.

Definition 2 ([22]) Let η and μ be two fuzzy subsets of a near ring $\mathcal{N}$. Then $\eta \cap \mu$ and $\eta \circ \mu$ are defined as follows:

$$\eta \cap \mu(m) = min\{\eta(m), \mu(m)\}.$$

And product $\eta \circ \mu$ is defined by

$$\eta \circ \mu(m) = \begin{cases} \sup\limits_{m=m_1m_2} \{\min(\eta(m_1), \mu(m_2))\} & \text{if } m = m_1m_2 \\ 0 & \text{if } m \neq m_1m_2. \end{cases} \quad (1)$$

Definition 3 ([22]) Let $(\mathcal{G}, +)$ be a group and η be a fuzzy subset of $\mathcal{G}$. Then η is fuzzy subgroup if
(i) $\eta(g_1 + g_2) \geq min(\eta(g_1), \eta(g_2))$, $\forall\, g_1, g_2$ in $\mathcal{G}$,
(ii) $\eta(g) = \eta(-g)$, $\forall\, g$ in $\mathcal{G}$.

Definition 4 ([22]) A fuzzy subset η of a near ring $\mathcal{N}$ is said to be a fuzzy subnear ring of $\mathcal{N}$ if η is a fuzzy subgroup of $\mathcal{N}$ with respect to the addition '+' and is a fuzzy groupoid with respect to the multiplication '·', i.e.,

(i) $\eta(x - y) \geq min(\eta(x), \eta(y))$ and (ii) $\eta(xy) \geq min(\eta(x), \eta(y))$ $\forall\, x, y \in \mathcal{N}$.

Definition 5 ([22]) A fuzzy subset η of a near ring $\mathcal{N}$ is said to be a fuzzy ideal of $\mathcal{N}$ if η satisfies following conditions:

(i) η is fuzzy subnear ring,
(ii) η is normal fuzzy subgroup with respect to '+',
(iii) $\eta(rs) \geq \eta(s)$; for all r,s in $\mathcal{N}$,
(iv) $\eta((r + t)s - rs) \geq \eta(t)$; $\forall\, r, s$ and t in $\mathcal{N}$.

If η satisfies (i),(ii), and (iii), then it is called a fuzzy left ideal of $\mathcal{N}$. If η satisfies (i),(ii), and (iv), then it is called a fuzzy right ideal of $\mathcal{N}$.

Definition 6 ([1]) Let $\mathcal{G}$ be a group and $\mathcal{Z}$ be a set. Then $\mathcal{G}$ is said to act on $\mathcal{Z}$ if there is a mapping $\phi : \mathcal{G} \times \mathcal{Z} \to \mathcal{Z}$, with $\phi(a, z)$ written $a * z$, such that
(i) $a * (b * z) = (ab) * z, \quad \forall a, b \in \mathcal{G}, z \in \mathcal{Z}.$
(ii) $e * z = z. \quad e \in \mathcal{G}, z \in \mathcal{Z}$. The mapping ϕ is called the action of $\mathcal{G}$ on $\mathcal{Z}$, and $\mathcal{Z}$ is said to be a $\mathcal{G}$-set.

Definition 7 ([1]) Let $\mathcal{G}$ be a group acting on a set $\mathcal{Z}$, and let $z \in \mathcal{Z}$. Then the set

$$\mathcal{G}z = \{az | a \in \mathcal{G}\}$$

is called the orbit of $\mathcal{Z}$ in $\mathcal{G}$.

Proposition 1 *Let $\mathcal{N}$ be a near ring and $\mathcal{G} = Aut(\mathcal{N})$, group of all automorphism of $\mathcal{N}$.Then $\mathcal{G}$ acts on $\mathcal{N}$ via following map*

$$\phi : \mathcal{G} \times \mathcal{N} \to \mathcal{N} \textit{which is defined by } \phi(h, a) = h(a) \textit{ or say } \ h * a = h(a).$$

Proof Take (h_1, a_1) *and* (h_2, a_2) such that

$$(h_1, a_1) = (h_2, a_2).$$

This implies that $h_1 = h_2$ *and* $a_1 = a_2$. Thus, we have

$$h_1(a_1) = h_2(a_1)$$

or

$$\phi(h_1, a_1) = \phi(h_2, a_2).$$

Hence, ϕ is well defined. Furthermore, we show that ϕ is the action of $\mathcal{G}$ on $\mathcal{N}$. Take any $g_1, g_2 \in \mathcal{G}$ *and* $b \in \mathcal{N}$. Then

$$g_1 * (g_2 * b) = g_1 * (g_2(b)) = g_1(g_2(b)) \tag{2}$$

$$(g_1 \circ g_2) * b = (g_1 \circ g_2)(b) = g_1(g_2(b)). \tag{3}$$

From (2) and (3), we get

$$(g_1 \circ g_2) * b = g_1 * (g_2 * b).$$

Also, we have

$$e * x = x.$$

Hence, ϕ is the action of $\mathcal{G}$ on $\mathcal{N}$.

Motivated by the definition of the group action of a finite group on fuzzy ideals of a ring [19], we define a $\mathcal{G}-$fuzzy ideal of $\mathcal{N}$ as follows:

Definition 8 Let $\mathcal{G}$ be a group. Then fuzzy set η of $\mathcal{N}$ is a $\mathcal{G}-$set or $\mathcal{G}$ act on η if

$$\eta^g(r) = \eta(r^g), \quad g \in \mathcal{G}$$

where r^g denotes g acts on r, $r \in \mathcal{N}$.

Example 1 Let $\mathcal{N} = \{0, 1, 2\}$ be a set. Then under following two binary operations $\mathcal{N}$ forms a zero symmetric near ring:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

·	0	1	2
0	0	0	0
1	0	1	2
2	0	1	2

$$Aut(\mathcal{N}) = \{f | f : \mathcal{N} \to \mathcal{N} \;\; is \;\; isomorphism\}.$$

There are only two automorphisms (i) identity map and (ii) the map g defined as follows:

g(0)=0, g(1)=2, and g(2)=1.

We know that $Aut(\mathcal{N})$ forms a group. Define a map $\lambda : \mathcal{N} \to [0, 1]$ by

$$\lambda(a) = \begin{cases} 0.9 & a = 0 \\ 0.8 & a = 1, 2. \end{cases}$$

λ is a fuzzy ideal. By Definition 8, $\lambda^g : \mathcal{N} \to [0, 1]$ is defined as $\lambda^g(r) = \lambda(r^g)$, i.e.,

$$\begin{aligned} \lambda^g(0) &= \lambda(0^g) = \lambda(0) = 0.9 \\ \lambda^g(1) &= \lambda(1^g) = \lambda(2) = 0.8 \\ \lambda^g(2) &= \lambda(2^g) = \lambda(1) = 0.8. \end{aligned}$$

This implies that

$$\lambda^g = \{(0, 0.9), (1, 0.8), (2, 0.8)\} \quad \text{and} \tag{4}$$

$$\lambda^e = \lambda = \{(0, 0.9), (1, 0.8), (2, 0.8)\}. \tag{5}$$

This shows that λ^g is a fuzzy ideal of $\mathcal{N}$, since $\lambda = \lambda^g$.

3 Prime Fuzzy Ideals

Definition 9 ([19]) Let $\mathcal{Q}$ be a fuzzy ideal of $\mathcal{N}$. Then $\mathcal{Q}$ is said to be a prime ideal in $\mathcal{N}$ if $\mathcal{Q}$ is not a constant function and for any fuzzy ideals η and μ in $\mathcal{N}$, $\eta \circ \mu \subset \mathcal{Q}$ implies that either $\eta \subset \mathcal{Q}$ or $\mu \subset \mathcal{Q}$.

Example 2 Take $Z_4 = \{0, 1, 2, 3\}$ the zero symmetric left near ring under binary operations addition modulo 4 and for any $a, b \in Z_4$ multiplication is defined as

$$a \cdot b = \begin{cases} b & a \neq 0 \\ 0 & a = 0. \end{cases}$$

Define two maps $\eta_1, \eta_2 : Z_4 \to [0, 1]$ by $\eta_1(z_1) = \begin{cases} 0.9 & z_1 = 0 \\ 0.8 & z_1 \neq 0, \end{cases}$ and $\eta_2(z_2) = 0.9$ for all $z_1, z_2 \in Z_4$. It shows that $\eta_1 \circ \eta_2 \subseteq \eta_1$ and $\eta_1 \subseteq \eta_1$ but $\eta_2 \not\subset \eta_1$. As η_1 is non-constant function so η_1 is a prime fuzzy ideal.

Proposition 2 *If η is a fuzzy ideal of $\mathcal{N}$, then η^g is a fuzzy ideal of $\mathcal{N}$. Moreover, primeness of η as a fuzzy ideal implies the primeness of fuzzy ideal η^g of $\mathcal{N}$.*

Proof Assume that η is a fuzzy ideal of $\mathcal{N}$. Then we show that η^g is also a prime fuzzy ideal of $\mathcal{N}$, i.e., we will show that η^g satisfies following conditions:

Let $r, s \in \mathcal{N}$. Since η is a fuzzy ideal of $\mathcal{N}$, then we have

$$\eta^g(r - s) = \eta(r - s)^g = \eta(r^g - s^g) \geq min(\eta(r^g), \eta(s^g)),$$

i.e.,

$$\eta^g(r-s) \geq min(\eta^g(r), \eta^g(s)) \tag{6}$$

and

$$\eta^g(rs) = \eta(rs)^g = \eta(r^g s^g) \geq min(\eta(r^g), \eta(s^g)), \tag{7}$$

i.e.,

$$\eta^g(rs) \geq min(\eta(r^g), \eta(s^g)). \tag{8}$$

Equations (6) and (7) imply that η^g is a fuzzy subnear ring of $\mathcal{N}$.
Again $r, s \in \mathcal{N}$ and η is fuzzy ideal of $\mathcal{N}$, we have

$$\eta^g(r+s) = \eta(r+s)^g = \eta(r^g + s^g) \geq min(\eta(r^g), \eta(s^g)),$$

i.e.,

$$\eta^g(r+s) \geq min(\eta(r^g), \eta(s^g)). \tag{9}$$

Applying ([5], Lemma 2.3), we obtain

$$\eta^g(r) = \eta(r^g) = \eta(-r^g) = \eta^g(-r).$$

Also,

$$\eta^g(r) = \eta(r^g) = \eta(s^g + r^g - s^g) = \eta(s + r - s)^g,$$

i.e.,

$$\eta^g(r) = \eta^g(s + r - s). \tag{10}$$

Since η^g satisfies all conditions of normal subgroup, η^g is a normal fuzzy subgroup of $(\mathcal{N}, +)$. For $r, s \in \mathcal{N}$, we have

$$\eta^g(rs) = \eta(rs)^g = \eta(r^g s^g) \geq \eta(s^g),$$

i.e.,

$$\eta^g(rs) \geq \eta^g(s). \tag{11}$$

This implies that η^g is a fuzzy left ideal of $\mathcal{N}$. Now, for r, s *and* $t \in \mathcal{N}$, we have

$$\eta^g((r+t)s - rs) = \eta((r^g + t^g)s^g - r^g s^g) \geq \eta(t^g),$$

i.e.,

$$\eta^g((r+t)s - rs) \geq \eta^g(t). \tag{12}$$

This implies that η is a right fuzzy ideal. Thus, η is a fuzzy ideal(left fuzzy ideal as well as right fuzzy ideal) of $\mathcal{N}$.

Now we prove that η^g is a prime fuzzy ideal of $\mathcal{N}$. Let $\mathcal{A}$ and $\mathcal{B}$ be two fuzzy ideals of $\mathcal{N}$ such that $\mathcal{A} \circ \mathcal{B} \subset \eta^g$. Then $\mathcal{A}^{g^{-1}}$ and $\mathcal{B}^{g^{-1}}$ are also fuzzy ideals of $\mathcal{N}$, since $g^{-1} \in \mathcal{G}$ and as proved in η^g, we claim that $\mathcal{A}^{g^{-1}} \circ \mathcal{B}^{g^{-1}} \subset \eta$. Let $n \in \mathcal{N}$ and

$$\begin{aligned}
(\mathcal{A}^{g^{-1}} \circ \mathcal{B}^{g^{-1}})(n) &= \sup_{n=n_1n_2} \{\min(\mathcal{A}^{g^{-1}}(n_1), \mathcal{B}^{g^{-1}}(n_2))\} \\
&= \sup_{n^{g^{-1}}=n_1^{g^{-1}} n_2^{g^{-1}}} \left\{\min(\mathcal{A}(n_1^{g^{-1}}), \mathcal{B}(n_2^{g^{-1}}))\right\} \\
&= (\mathcal{A} \circ \mathcal{B})(n^{g^{-1}}) \\
&\leq \eta^g(n^{g^{-1}}) = \eta((n^{g^{-1}})^g) \\
&= \eta(n).
\end{aligned}$$

So, $\mathcal{A}^{g^{-1}} \circ \mathcal{B}^{g^{-1}} \subset \eta$. Since η is a prime fuzzy ideal, then we have $\mathcal{A}^{g^{-1}} \subset \eta$ or $\mathcal{B}^{g^{-1}} \subset \eta$. Suppose that $\mathcal{A}^{g^{-1}} \subset \eta$. Then for all $n \in \mathcal{N}$, we have

$$\mathcal{A}(n) = \mathcal{A}((n^g)^{g^{-1}}) = \mathcal{A}^{g^{-1}}(n^g) \leq \eta(n^g) = \eta^g(n).$$

Thus $\mathcal{A} \subset \eta^g$. This implies that η^g is a prime fuzzy ideal of $\mathcal{N}$.

Now we define a $\mathcal{G}$-invariant fuzzy ideal of a near ring.

Definition 10 A fuzzy ideal η of $\mathcal{N}$ is called a $\mathcal{G}$-invariant fuzzy ideal of $\mathcal{N}$ if and only if

$$\eta^g(r) = \eta(r^g) \geq \eta(r), \; \forall\, g \in \mathcal{G},\, r \in \mathcal{N}.$$

Or

$$\eta(r) = \eta((r^g)^{g^{-1}}) \geq \eta(r^g).$$

Example 3 Let $\mathcal{X}$ be a near ring. Then

$$N = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \middle| \; x, y, \;\; 0 \in X \right\}$$

is near ring with regard to matrix addition and matrix multiplication. Let

$$I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix} \middle| \; y, 0 \in X \right\}.$$

Then $\mathcal{I}$ is a fuzzy ideal of $\mathcal{N}$. Define a map $\eta : \mathcal{N} \to [0, 1]$ by

$$\eta(z) = \begin{cases} 0.9 & z = 0 \\ 0.8 & z \neq 0 \end{cases} .$$

Consider

$$\mathcal{G}(\subseteq Aut(\mathcal{N})) = \{f | f : \mathcal{N} \to \mathcal{N} \ \ is\ an\ isomorphism\}.$$

There are only two automorphisms that are identity map and the map $g : \mathcal{N} \to \mathcal{N}$ defined by

$$g\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}.$$

Since $\eta^g(r) = \eta(r^g) = \eta(r) \ \ for\ \ all\ \ g \in \mathcal{G}\ \ and\ \ r \in \mathcal{N}$, we get η is $\mathcal{G}-$invariant fuzzy ideal in $\mathcal{N}$.

Theorem 1 *Let η be a fuzzy ideal of $\mathcal{N}$ and $\eta^{\mathcal{G}} = \bigcap_{g \in \mathcal{G}} \eta^g$. Then $\eta^{\mathcal{G}}(r) = min\{\eta(r^g),$ $g \in \mathcal{G}\}$. Moreover, fuzzy ideal η contains largest $\mathcal{G}$-invariant fuzzy ideal $\eta^{\mathcal{G}}$ of $\mathcal{N}$.*

Proof Assume that

$$\begin{aligned} \eta^{\mathcal{G}}(s) &= \bigcap_{k \in \mathcal{G}} \eta^k \\ &= min\{\eta^k(s), \ \ k \in \mathcal{G}\} = min\{\eta(s^k), \ \ k \in \mathcal{G}\}. \end{aligned}$$

We prove that $\eta^{\mathcal{G}}$ is a fuzzy ideal of $\mathcal{N}$.

Let $r, s \in \mathcal{N}$. Then

$$\begin{aligned} \eta^{\mathcal{G}}(r - s) &= min\{\eta(r - s)^g, g \in \mathcal{G}\} \\ &= min\{\eta(r^g - s^g), g \in \mathcal{G}\} \\ &= min\{min(\eta(r^g), \eta(s^g)), g \in \mathcal{G}\}. \end{aligned}$$

Since η is a fuzzy ideal, we have

$$\begin{aligned} \eta^{\mathcal{G}}(r - s) &\geq min\{min(\eta(r^g), g \in \mathcal{G}), min(\eta(s^g), g \in \mathcal{G})\} \\ &= min\{\eta^{\mathcal{G}}(r), \eta^{\mathcal{G}}(s)\}. \end{aligned}$$

This implies that

$$\eta^{\mathcal{G}}(r - s) \geq \{\eta^{\mathcal{G}}(r), \eta^{\mathcal{G}}(s)\}. \tag{13}$$

Also for any $r, s \in \mathcal{N}$

$$\begin{aligned}\eta^{\mathcal{G}}(rs) &= min\{\eta(rs)^g, g \in \mathcal{G}\}\\ &= min\{\eta(r^g s^g), g \in \mathcal{G}\}\\ &= min\{min(\eta(r^g), \eta(s^g)),\ g \in \mathcal{G}\}.\end{aligned}$$

Since η is a fuzzy ideal of $\mathcal{N}$, we have

$$\begin{aligned}\eta^{\mathcal{G}}(rs) &\geq min\{min(\eta(r^g), g \in \mathcal{G}), min(\eta(s^g), g \in \mathcal{G})\}\\ &= min\{\mu^G(r), \mu^G(s)\}.\end{aligned}$$

Thus,

$$\eta^{\mathcal{G}}(rs) \geq \{\eta^{\mathcal{G}}(r), \eta^{\mathcal{G}}(s)\}. \tag{14}$$

$$\begin{aligned}\eta^{\mathcal{G}}(s + r - s) &= min\{\eta(s + r - s)^g,\ g \in \mathcal{G}\}\\ &= min\{\eta(s^g + r^g - s^g), g \in \mathcal{G}\}\\ &= min\{\eta(r^g), g \in \mathcal{G}\}\\ &= \eta^{\mathcal{G}}(r).\end{aligned}$$

Therefore,

$$\eta^{\mathcal{G}}(s + r - s) = \eta^{\mathcal{G}}(r). \tag{15}$$

Now,

$$\begin{aligned}\eta^{\mathcal{G}}(rs) &= min\{\eta(rs)^g, g \in \mathcal{G}\}\\ &= min\{\eta(r^g s^g), g \in \mathcal{G}\}.\end{aligned}$$

Again since η is fuzzy ideal, we can write for $r, s \in \mathcal{N}$

$$\begin{aligned}\eta^{\mathcal{G}}(rs) &\geq min\{\eta(s^g), g \in \mathcal{G}\}.\\ &= \eta^{\mathcal{G}}(s),\end{aligned}$$

i.e.,

$$\eta^{\mathcal{G}}(rs) \geq \eta^{\mathcal{G}}(s) \tag{16}$$

$$\begin{aligned}\eta^G((r+t)s-rs) &= min\{\eta((r+t)s-rs)^g, g \in \mathcal{G}\}\\ &= min\{\eta((r+t)^g s^g - r^g s^g), g \in \mathcal{G}\}\\ &= min\{\eta((r^g+t^g)s^g - r^g s^g), g \in \mathcal{G}\}\\ &\geq min\{\eta(t^g), g \in \mathcal{G}\}.\\ &= \eta^{\mathcal{G}}(t)\end{aligned}$$

$$\eta^{\mathcal{G}}((r+t)s-rs) \geq \eta^{\mathcal{G}}(t). \tag{17}$$

Since η^G is the left and right fuzzy ideals of $\mathcal{N}$, then $\eta^{\mathcal{G}}$ is the fuzzy ideal of $\mathcal{N}$. It is still necessary to show that it is a $\mathcal{G}$-invariant fuzzy ideal of $\mathcal{N}$.

$$\begin{aligned}\eta^{\mathcal{G}}(r^g) &= \min\{\eta((r^g)^k), k \in \mathcal{G}\}\\ &= \min\{\eta(r^{gk}), k \in \mathcal{G}\}\\ &= \min\{\eta(r^{g'}), g' \in \mathcal{G}\}\\ &= \eta^{\mathcal{G}}(r).\end{aligned}$$

Now we prove that η^G is the largest. Assume that μ is any $\mathcal{G}$-invariant fuzzy ideal of $\mathcal{N}$ such that $\mu \subseteq \eta$. Then for any $g \in \mathcal{G}$

$$\mu(r^g) = \mu(r) \leq \eta(r).$$

Also,

$$\mu(r^g) = \mu(r) = \mu((r^g)^{g^{-1}}) \leq \eta(r^g).$$

This implies that

$$\mu(r) \leq min\{\eta(r^g), g \in \mathcal{G}\} = \eta^{\mathcal{G}}(r).$$

Thus,

$$\mu \subseteq \eta^{\mathcal{G}}.$$

Hence, $\eta^{\mathcal{G}}$ contained in η as the largest $\mathcal{G}$-invariant fuzzy ideal of $\mathcal{N}$.

Remark 1 If a fuzzy ideal η of $\mathcal{N}$ satisfies $\eta = \eta^{\mathcal{G}}$. Then η is called as $\mathcal{G}$-invariant fuzzy ideal of $\mathcal{N}$ and vice versa.

4 Union of Fuzzy Ideals of Near Ring

The following example demonstrates that the union of fuzzy ideals of a near ring $\mathcal{N}$ need not be a fuzzy ideal in $\mathcal{N}$.

Example 4 Let $\mathcal{Q}$ be a near ring. Then

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & p \\ 0 & q \end{pmatrix} \middle| \; p, q \;\; 0 \in \mathcal{Q} \right\}$$

is a near ring with regard to matrix addition and matrix multiplication. Let

$$\mathcal{I}_1 = \left\{ \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} \middle| \; p, \; 0 \in \mathcal{Q} \right\}$$

and

$$\mathcal{I}_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \middle| \; q, \; 0 \in \mathcal{Q} \right\}.$$

We can check that $\mathcal{I}_1$ and $\mathcal{I}_2$ are ideals of $\mathcal{N}$. Define maps

$$\eta_1 : \mathcal{N} \to [0, 1] \quad and \quad \eta_2 : \mathcal{N} \to [0, 1]$$

by

$$\eta_1(x) = \begin{cases} 0.5 & x \in \mathcal{I}_1 \\ 0 & x \notin \mathcal{I}_1 \end{cases}$$

and

$$\eta_2(x) = \begin{cases} 0.6, & x \in \mathcal{I}_2 \\ 0, & x \notin \mathcal{I}_2. \end{cases}$$

Then η_1 and η_2 are fuzzy ideals of $\mathcal{N}$. However

$$(\eta_1 \cup \eta_2)(x) = \begin{cases} max\{0.5, 0.6\}, & x \in \mathcal{I}_1 \cup \mathcal{I}_2 \\ 0, & x \notin \mathcal{I}_1 \cup \mathcal{I}_2 \end{cases}$$

is not a fuzzy ideal of $\mathcal{N}$, since for $m = \begin{pmatrix} 0 & p \\ 0 & 0 \end{pmatrix} n = \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix}$, $m - n = \begin{pmatrix} 0 & p \\ 0 & -q \end{pmatrix} \notin \mathcal{I}_1 \cup \mathcal{I}_2$. We see that $\eta_1 \cup \eta_2(m - n) = 0$, $\eta_1 \cup \eta_2(m) = 0.6$, and $\eta_1 \cup \eta_2(n) = 0.5$. Thus,

$$\begin{aligned} \eta_1 \cup \eta_2(m - n) = 0 &\ngeq \max\{\eta_1 \cup \eta_2(m), \eta_1 \cup \eta_2(n)\} \\ &\ngeq \max\{0.6, 0.5\} \\ &\ngeq 0.6. \end{aligned}$$

Hence, $\eta_1 \cup \eta_2$ is not a fuzzy ideal of $\mathcal{N}$.

Proposition 3 *Let $\mathcal{C} = \{\eta_k\}$ be a chain of fuzzy ideals of $\mathcal{N}$. Then for any $m, n \in \mathcal{N}$*

$$min(\sup_k\{\eta_k(m)\}, \sup_k\{\eta_k(n)\}) = \sup_k\{min(\eta_k(m), \eta_k(n))\}.$$

Proof We can easily see that

$$\sup_k\{\min(\eta_k(m), \eta_k(n))\} \leq \min(\sup_k\{\eta_k(m)\}, \sup_k\{\eta_k(n)\}).$$

Now, assume that

$$\sup_k\{\min(\eta_k(m), \eta_k(n))\} = I.$$

And

$$I < \min(\sup_k\{\eta_k(m)\}, \sup_k\{\eta_k(n)\}).$$

Then

$$\sup_k\{\eta_k(m)\} > I, \quad or \quad \sup_k\{\eta_k(n)\} > I.$$

η_r and η_s exist in such a way that

$$\eta_r(m) > I, \quad \& \quad \eta_s(n) > I$$

or

$$\eta_r(m) > I \geq \min(\eta_r(m), \eta_r(n)) \tag{18}$$

and

$$\eta_r(n) > I \geq \min(\eta_s(m), \eta_s(n)). \tag{19}$$

Since, $\eta_r, \eta_s \in \mathcal{C}$, so without loss of generality, we may assume that $\eta_r \subseteq \eta_s$ and $\eta_s(n) \geq \eta_s(m)$ Therefore, from (18) and (19), we get

$$I < \eta_r(m) \leq \eta_s(m) = \min(\eta_s(m), \eta_s(n)).$$

This contradicts the fact that

$$I = \sup_k\{\min(\eta_k(m), \eta_k(n))\}.$$

Hence,

$$min(\sup_k\{\eta_k(m)\}, \sup_k\{\eta_k(n)\}) = \sup_k\{min(\eta_k(m), \eta_k(n))\}.$$

Corollary 1 *Assume that $\mathcal{C} = \{\eta_k\}$ is a chain of fuzzy ideals of $\mathcal{N}$. Then for each $x_1, x_2, ..., x_m \in \mathcal{N}$,*

$$min(\sup_k\{\eta_k(x_1)\}, \sup_k\{\eta_k(x_2)\}, ..., \sup_k\{\eta_k(x_m)\}) = \sup_k\{min(\eta_k(x_1), \eta_k(x_2), ..., \eta_k(x_m))\}.$$

Theorem 1 *Let $\mathcal{C} = \{\eta_k\}$ be a chain of fuzzy ideals of $\mathcal{N}$. Then $\bigcup_k \eta_k$ is a fuzzy ideal of $\mathcal{N}$.*

Proof Let $r, s \in \mathcal{N}$, and η_k be a fuzzy ideal of $\mathcal{N}$, where k is a natural number. Then

$$\begin{aligned}(\bigcup_k \eta_k)(r - s) &= \sup_k(\eta_k(r - s)) \\ &\geq \sup_k\{\min(\eta_k(r), \eta_k(s))\}.\end{aligned}$$

Using Corollary 1, we get

$$(\bigcup_k)(r - s) \geq \min\{\sup_k(\eta_k(r)), \sup_k(\eta_k(s))\},$$

i.e.,

$$(\bigcup_k \eta_k)(r - s) \geq \min\{(\bigcup_k \eta_k)(r), (\bigcup_k \eta_k)(s)\}. \quad (20)$$

Also,

$$\begin{aligned}(\bigcup_k \eta_k)(rs) &= \sup_k(\bigcup_k(rs)) \\ &\geq \sup_k\{\min(\eta_k(r), \eta_r(s))\}.\end{aligned}$$

Again from Corollary 1, we have

$$(\bigcup_k \eta_k)(rs) \geq \min\{\sup_k(\eta_k(r)), \sup_k(\eta_k(s))\},$$

i.e.,

$$(\bigcup_k \eta_k)(rs) \geq \min\{(\bigcup_k \eta_k)(r), (\bigcup_k \eta_k)(s)\}. \quad (21)$$

Now

$$\begin{aligned}(\bigcup_k \eta_k)(s+r-s) &= \sup_k(\eta_k(s+r-s)) \\ &= \sup_k\{\eta_k(r)\}.\end{aligned}$$

Since η_k is a fuzzy ideal in $\mathcal{N}$, we obtain

$$(\bigcup_k \eta_k)(s+r-s) = (\bigcup_k \eta_k)(r),$$

i.e.,

$$(\bigcup_k \eta_k)(s+r-s) = (\bigcup_k \eta_k)(r). \tag{22}$$

$$\begin{aligned}(\bigcup_k \eta_k)(rs) &= \sup_k(\eta_k(rs)) \\ &\geq \sup_k\{\eta_k(s)\}.\end{aligned}$$

Again using the fact that η_k is fuzzy ideal, we get

$$(\bigcup_k \eta_k)(rs) \geq (\bigcup_k \eta_k)(s) \tag{23}$$

$$\begin{aligned}(\bigcup_k \eta_k)((r+t)s-rs) &= \sup_k(\eta_k((r+t)s-rs)) \\ &\geq \sup_k\{\eta_k(t)\}.\end{aligned}$$

Also,

$$(\bigcup_k \eta_k)((r+t)s-rs) \geq (\bigcup_k \eta_k)(t). \tag{24}$$

Hence, $(\bigcup_k \eta_k)$ is a fuzzy ideal of $\mathcal{N}$.

5 G-Prime Fuzzy Ideals of a Near Ring

Motivated by the definition of $\mathcal{G}$-prime fuzzy ideals of the rings [19], we define $\mathcal{G}-$prime fuzzy ideals in a near ring as follows.

Definition 11 Let the fuzzy ideal η of $\mathcal{N}$ be $\mathcal{G}$-invariant and non-constant. If $\mu \circ \lambda \subseteq \eta$ implies that either $\mu \subseteq \eta$ or $\lambda \subseteq \eta$ for any two $\mathcal{G}$-invariant fuzzy ideals μ and λ of $\mathcal{N}$, then η is a $\mathcal{G}$-prime fuzzy ideal.

Example 5 Take $Z_3 = \{0, 1, 2\}$ which is a zero symmetric left near ring under binary operations addition modulo 3 and for any $r, s \in Z_3$ multiplication is defined as follows:

$$r \cdot s = \begin{cases} s & r \neq 0 \\ 0 & r = 0. \end{cases}$$

$$Aut(Z_3) = \{f | f : Z_3 \to Z_3 \ is \ isomorphism\}.$$

We can check that there are only two automorphisms on Z_3; one is the identity map and the other is the map g defined by

g(0)=0, g(1)=2 and g(2)=1.

$Aut(Z_3)$ forms a group under the composition of mappings. Now we define two maps $\eta_1, \eta_2 : Z_3 \to [0, 1]$ by $\eta_1(r) = \begin{cases} 0.9 & r = 0 \\ 0.8 & r \neq 0, \end{cases}$ and $\eta_2(s) = 0.9$ for all $r, s \in Z_3$. By Definition 8, $\eta_1^g : Z_3 \to [0, 1]$ is defined as $\eta_1^g(r) = \eta_1(r^g)$, i.e.,

$$\begin{aligned} \eta_1^g(0) &= \eta_1(0^g) = \eta_1(0) = 0.9 \\ \eta_1^g(1) &= \eta_1(1^g) = \eta_1(2) = 0.8 \\ \eta_1^g(2) &= \eta_1(2^g) = \eta_1(1) = 0.8. \end{aligned}$$

This implies that

$$\eta_1^g = \{(0, 0.9), (1, 0.8), (2, 0.8)\} \tag{25}$$

and

$$\eta_1^e = \eta_1 = \{(0, 0.9), (1, 0.8), (2, 0.8)\}. \tag{26}$$

Also, we can see that η_2 is a $\mathcal{G}$-invariant fuzzy ideal of Z_3. Since $\eta_1 \circ \eta_2 \subseteq \eta_1$ and $\eta_1 \subseteq \eta_1$ but $\eta_2 \not\subseteq \eta_1$, so it follows that η_1 is $\mathcal{G}$-prime fuzzy ideal as η_1 is non-constant function.

The following proposition is an extension of Lemma 2.6 of [22] in case of near rings:

Proposition 4 *If $\mathcal{N}$ is near ring and $\lambda_1, \lambda_2, ..., \lambda_n$ are fuzzy ideals of $\mathcal{N}$, then*

$$\lambda_1 \circ \lambda_2 \circ \cdots \circ \lambda_n \subset \lambda_1 \bigcap \lambda_2 \bigcap \cdots \bigcap \lambda_n.$$

Proof Let $\lambda_1 \circ \lambda_2 \circ \cdots \circ \lambda_n(x) = 0$. Then, there is nothing to demonstrate. Otherwise

$$\lambda_1 \circ \lambda_2 \circ \cdots \circ \lambda_n(x) = \sup_{x=x_1x_2\cdots x_n} \{\min(\lambda_1(x_1), \lambda_2(x_2), ..., \lambda_n(x_n))\}.$$

Since λ_i is a fuzzy ideal of $\mathcal{N}$, we get

$$\lambda_i((x+z)y - xy) \geq \lambda_i(z).$$

Since $\mathcal{N}$ is zero symmetric, we have

$$\begin{aligned} \lambda_1(x) = \lambda_1(x_1x_2\cdots x_n) &= \lambda_1((0+x_1)x_2\cdots x_n - 0 \cdot x_1x_2\cdots x_n). \\ &\geq \lambda_1(x_1), \end{aligned}$$

i.e.,

$$\lambda_1(x) \geq \lambda_1(x_1).$$

Also, λ_2 is a fuzzy ideal; hence,

$$\begin{aligned} \lambda_2(x) = \lambda_2(x_1x_2\cdots x_n) \geq \lambda_2(x_2x_3\cdots x_n) &= \lambda_2((0+x_2)x_3\cdots x_n - 0 \cdot x_2x_3\cdots x_n). \\ &\geq \lambda_2(x_2), \end{aligned}$$

i.e.,

$$\lambda_2(x) \geq \lambda_2(x_2).$$

In a similar manner, we can prove that

$$\lambda_3(x) \geq \lambda_3(x_3),$$

$$\lambda_4(x) \geq \lambda_4(x_4),$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$$\lambda_{n-1}(x) \geq \lambda_{n-1}(x_{n-1}).$$

Since λ_n is a fuzzy ideal in $\mathcal{N}$, we get

$$\lambda_n(x) \geq \lambda_n(x_n).$$

Therefore,

$$\lambda_1 \circ \lambda_2 \circ \cdots \circ \lambda_n(x) = \min(\lambda_1(x_1), \lambda_2(x_2), ..., \lambda_n(x_n))$$

or

$$\underset{1\leq i\leq n}{\circ} \lambda_i(x) \leq (\bigcap_{1\leq i\leq n} \lambda_i)(x)$$

or

$$\underset{1\leq i\leq n}{\circ} \lambda_i \subset \bigcap_{1\leq i\leq n} \lambda_i.$$

Now we will prove the main result.

Theorem 2 *If η is a prime fuzzy ideal of $\mathcal{N}$. Then η^G is a $\mathcal{G}$-prime fuzzy ideal of $\mathcal{N}$. Conversely, if λ is a $\mathcal{G}$-prime fuzzy ideal of $\mathcal{N}$, then there exists a prime fuzzy ideal η of $\mathcal{N}$ such that $\eta^{\mathcal{G}} = \lambda$, η is unique up to its $\mathcal{G}$-orbit.*

Proof Assume that η is a prime fuzzy ideal of $\mathcal{N}$ and $\mathcal{P}$, $\mathcal{Q}$ are two $\mathcal{G}$-invariant fuzzy ideals of $\mathcal{N}$ such that $\mathcal{P} \circ \mathcal{Q} \subseteq \eta^{\mathcal{G}}$. Since $\eta^{\mathcal{G}}$ is the largest $\mathcal{G}$-invariant fuzzy ideal contained in η, then $\mathcal{P} \circ \mathcal{Q} \subseteq \eta$. Also primeness of η implies that either $\mathcal{P} \subseteq \eta$ or $\mathcal{Q} \subseteq \eta$. Therefore, by Theorem 1 either $\mathcal{P} \subseteq \eta^{\mathcal{G}}$ or $\mathcal{Q} \subseteq \eta^{\mathcal{G}}$. Thus, $\eta^{\mathcal{G}}$ is a $\mathcal{G}$-prime fuzzy ideal.

Conversely, suppose that λ is a $\mathcal{G}$-prime fuzzy ideal of $\mathcal{N}$ and consider

$$\mathcal{S} = \{\eta, \text{a fuzzy ideal of N} |\ \eta^{\mathcal{G}} \subseteq \lambda\}.$$

Before using Zorn's lemma on $\mathcal{S}$ to get the maximal element(i.e., maximal ideal), we have to show that if $\mathcal{C} = \{\eta_k\} \subset \mathcal{S}$ is a chain in $\mathcal{S}$, then $\bigcup_k \eta_k \in \mathcal{S}$.

Now, from Theorem 1, $\bigcup_k \eta_k$ is a fuzzy ideal of $\mathcal{N}$. Since $\eta_k \in S$, we get $\eta_k^{\mathcal{G}} \subseteq \lambda$, and we can take any $r \in \mathcal{N}$ and $\eta_k \in \mathcal{C}$ such that

$$\eta_k^g(r) = \eta_k(r^g) \quad \text{and} \quad \eta_k^g \subseteq \lambda.$$

Then

$$\eta_k(r^g) = \eta_k^g(r) \leq \lambda(r),$$

or

$$\min(\eta_k(r^g), g \in \mathcal{G}) \leq \lambda(r).$$

This implies that

$$\sup\{\min(\eta_k(r^g), g \in \mathcal{G})\} \leq \lambda(r). \tag{27}$$

Since $\mathcal{G}$ is finite, by Corollary 1, we obtain

$$\min\{\sup(\eta_k(r^g), g \in \mathcal{G})\} = \sup_k\{\min(\eta_k(r^g), g \in \mathcal{G})\}. \tag{28}$$

From (27) and (28), we have

$$\min\{\sup_k(\eta_k(r^g), g \in \mathcal{G})\} \leq \lambda(r)$$

or

$$\min\{(\bigcup_k \eta_k)(r^g), g \in \mathcal{G}\} \leq \lambda(r).$$

Now by Theorem 1, we get

$$(\bigcup_k \eta_k)^{\mathcal{G}}(r) \leq \lambda(r).$$

Thus, we obtain

$$(\bigcup_k \eta_k)^{\mathcal{G}} \subseteq \lambda.$$

This shows that $(\bigcup_k \eta_l) \in \mathcal{S}$, i.e., $\mathcal{S}$ has upper bound. Now we use Zorn's lemma on $\mathcal{S}$ to choose a maximal fuzzy ideal say η. Let $\mathcal{P},\ \mathcal{Q}$ be fuzzy ideals of $\mathcal{N}$ such that $\mathcal{P} \circ \mathcal{Q} \subseteq \eta$. Then

$$(\mathcal{P} \circ \mathcal{Q})^{\mathcal{G}} \subseteq \eta^{\mathcal{G}} \subseteq \lambda. \tag{29}$$

Since $\mathcal{P}^{\mathcal{G}}$ and $\mathcal{Q}^{\mathcal{G}}$ are the largest fuzzy ideals contained in $\mathcal{P}$ and $\mathcal{Q}$, respectively.

Now we prove that $\mathcal{P}^{\mathcal{G}} \circ \mathcal{Q}^{\mathcal{G}} \subseteq \mathcal{P} \circ \mathcal{Q}$ is a $\mathcal{G}$-invariant,

$$
\begin{aligned}
(\mathcal{P}^{\mathcal{G}} \circ \mathcal{Q}^{\mathcal{G}})(r^g) &= \sup_{r^g=ab} \{\min(\mathcal{P}^{\mathcal{G}}(a), \mathcal{Q}^{\mathcal{G}}(b))\} \\
&= \sup_{r=a^{g^{-1}}b^{g^{-1}}} \{\min(\mathcal{P}^{\mathcal{G}}(a^{g^{-1}}), \mathcal{Q}^{\mathcal{G}}(b^{g^{-1}}))\} \\
&= \mathcal{P}^{\mathcal{G}} \circ \mathcal{Q}^{\mathcal{G}}(r).
\end{aligned}
$$

Hence, by Theorem 1, $(\mathcal{P}^{\mathcal{G}} \circ \mathcal{Q}^{\mathcal{G}}) \subseteq (\mathcal{P} \circ \mathcal{Q})^{\mathcal{G}} \subseteq \lambda$. Since λ is $\mathcal{G}$-prime, then we have either $\mathcal{P}^{\mathcal{G}} \subseteq \lambda$ or $\mathcal{Q}^{\mathcal{G}} \subseteq \lambda$. By maximality of η either $\mathcal{P} \subseteq \eta$ or $\mathcal{Q} \subseteq \eta$. This implies that η is prime fuzzy ideal of $\mathcal{N}$. As $\lambda^{\mathcal{G}} = \lambda$, we have $\lambda \in \mathcal{S}$. But maximality of η gives that $\lambda \subseteq \eta$. Since λ and $\eta^{\mathcal{G}}$ are $\mathcal{G}$-invariant ideal and $\eta^{\mathcal{G}}$ is largest in η, we get

$$\lambda \subseteq \eta^{\mathcal{G}}. \tag{30}$$

Thus, from (29) and (30), we obtain

$$\eta^{\mathcal{G}} = \lambda.$$

Let there exist another prime fuzzy ideal σ of $\mathcal{N}$ such that $\sigma^{\mathcal{G}} = \lambda$. Then

$$\bigcap_{g\in\mathcal{G}} \eta^g = \eta^{\mathcal{G}} = \sigma^G \subseteq \sigma.$$

Since $\mathcal{G}$ is finite, so from Proposition 4, we get

$$\mathop{\circ}_{g\in\mathcal{G}} \eta^g \subseteq \bigcap_{g\in\mathcal{G}} \eta^g.$$

Or for any $h(\neq g) \in \mathcal{G}$, we have

$$\eta^h \circ (\bigcap_{\substack{g\in\mathcal{G}\\ g\neq h}} \eta^g) \subseteq \bigcap_{g\in\mathcal{G}} \eta^g \subseteq \sigma.$$

By fuzzy primeness either $\eta^h \subseteq \sigma$ or $\bigcap_{\substack{g\in\mathcal{G}\\ g\neq h}} \eta^g \subseteq \sigma$. If $\eta^h \subseteq \sigma$, then $\eta \subseteq \sigma^{h^{-1}}$ and maximality of η with $(\sigma^{h^{-1}})^{\mathcal{G}} \subseteq \lambda$ implies that

$$\eta = \sigma^{h^{-1}}. \tag{31}$$

On the other hand, if $\eta^h \nsubseteq \sigma$, we get $\bigcap_{\substack{g\in\mathcal{G}\\ g\neq h}} \eta^g \subseteq \sigma$. Thus, there exists some $(h \neq) g \in \mathcal{G}$ such that $\eta^g \subseteq \sigma$ and hence $\eta \subseteq \sigma^{g^{-1}}$. Again maximality of η with $(\sigma^{g^{-1}})^{\mathcal{G}} \subseteq \lambda$ yields that

$$\eta = \sigma^{g^{-1}}. \quad (32)$$

Equations (31) and (32) show that η is unique up to its $\mathcal{G}$-orbit.

Conclusion: In the future, we plan to study partial group action (the existence of $g * (h * x)$ implies the existence of $(gh) * x$, but not necessarily conversely) on fuzzy ideals of near rings. The theorems that we prove are the following which are generalizations of Theorems 1 and 2.

Open Problem 1. Can we establish relation between $\mathcal{G}-$invariant fuzzy ideal and largest $\mathcal{G}-$invariant fuzzy ideal of $\mathcal{N}$ under partial group action?

Open Problem 2. Can we investigate relationship between primeness and $\mathcal{G}$-primeness of fuzzy ideal if a group $\mathcal{G}$ partially acts on a fuzzy ideal?

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