Distribution of Noise in Linear Recurrent Fractal Interpolation Functions for Data Sets with α-Stable Noise

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Abstract In this study, we construct a linear recurrent fractal interpolation function (RFIF) with variable scaling parameters for data set with α-stable noise (a generalization of Gaussian noise) on its ordinate, which captures the uncertainty at any missing or unknown intermediate point. The propagation of uncertainty in this linear RFIF is investigated, and a method for estimating parameters of the uncertainty at any interpolated value is provided. Moreover, a simulation study to visualize uncertainty for interpolated values is presented.

Keywords Fractals · Random fractal interpolation function · Recurrent fractal interpolation · Stable distribution · Stable noise

1 Introduction

In 1986, Barnsley [1] introduced the notion of fractal interpolation function (FIF) based on the theory of iterated function system (IFS), which can produce nowhere differentiable self-similar continuous functions. In 1989, Barnsley et al. [3] generalized this FIF technique to recurrent FIF (RFIF) by using recurrent IFS (RIFS), which can generate even more complex locally self-similar functions. Thereafter, RFIF is widely used for obtaining missing or unknown values at any intermediate points of a prescribed deterministic data set. However, if the provided data set contains noise on its ordinate, then capturing uncertainty at these interpolated values is essential, but incapable of doing so. This motivates us to study the fractal interpolation for noisy data sets.

Over the last three decades, many researchers have constructed fractal functions for deterministic data sets in various ways (for instance, see [2, 4, 7, 12, 13]) and discussed their analytical properties. At present, fractal interpolation is an advanced

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approach to approximate and analyze a wide range of scientific data that include irregularities or self-similarities. However, fractal interpolation for data with uncertainty has received little attention from researchers (see, [5, 6]). In this study, we use data sets with α-stable noise (a generalization of Gaussian noise) on its ordinate and extend this RFIF technique to capture the uncertainty at any missing or unknown intermediate values.

The paper is organized as follows. Section 2 recalls definitions and some basic results related to RFIF and α-stable distribution. In Sect. 3, the construction of a RFIF with variable scaling for α-stable noisy data is discussed and the parameter estimation of the uncertainty at any intermediate point of this RFIF is given. Section 4 discusses numerical experiments to validate and visualize analytical results. Section 5 concludes with a brief overview of our theoretical developments.

2 Preliminaries

In this section, we briefly describe the basic notions of RIFS, RFIF, and α-stable distribution. The details are given in [2, 11, 13].

2.1 *Basics of RIFS*

Definition 1 Let (K, d) be a complete metric space and $W_i : K \to K$ $(i = 1, 2, \ldots, N)$ be contraction maps. Also, let $P = \left(p_{ij}\right)_{N\times N}$ be an $N \times N$ irreducible row-stochastic matrix. Then $\{K; P; W_i : i = 1, 2, \ldots, N\}$ is called a recurrent iterated function system.

Further, the recurrent structure of the RIFS is given by a connection matrix $C = \left(c_{ij}\right)_{N\times N}$ which is defined by

$$c_{ij} = \begin{cases} 1, & p_{ji} > 0, \\ 0, & p_{ji} = 0. \end{cases} \tag{1}$$

This C is also an irreducible matrix. Let $\mathcal{H}(K)$ be the set of all nonempty compact subsets of K, and h be the Hausdorff distance in $\mathcal{H}(K)$ defined by

$$h(A, B) = \max\{\max_{a\in A} \min_{b\in B} d(a, b), \max_{b\in B} \min_{a\in A} d(a, b)\},\ \ A, B \in \mathcal{H}(K).$$

Then $(\mathcal{H}(K), h)$ is a complete metric space. Let us denote the product space

$$\tilde{\mathcal{H}}(K) := \underbrace{\mathcal{H}(K) \times \cdots \times \mathcal{H}(K)}_{N \text{ times}} = \mathcal{H}(K)^N,$$

and define a metric $\tilde{h}$ on $\tilde{\mathcal{H}}(K)$ by

$$\tilde{h}\left((A_1, A_2, \ldots, A_N), (B_1, B_2, \ldots, B_N)\right) := \max\left\{h(A_i, B_i) : i = 1, 2, \ldots, N\right\},$$

for all $(A_1, A_2, \ldots, A_N), (B_1, B_2, \ldots, B_N) \in \tilde{\mathcal{H}}(K)$. Then $\left(\tilde{\mathcal{H}}(K), \tilde{h}\right)$ is also a complete metric space. Now, we define a transformation $W : \tilde{\mathcal{H}}(K) \to \tilde{\mathcal{H}}(K)$ by

$$W(\mathbf{B}) := \begin{pmatrix} \bigcup\limits_{j=1}^{N} c_{1j} W_1(B_j) \\ \bigcup\limits_{j=1}^{N} c_{2j} W_2(B_j) \\ \vdots \\ \bigcup\limits_{j=1}^{N} c_{Nj} W_N(B_j) \end{pmatrix} = \begin{pmatrix} \bigcup\limits_{j \in \Lambda(1)} W_1(B_j) \\ \bigcup\limits_{j \in \Lambda(2)} W_2(B_j) \\ \vdots \\ \bigcup\limits_{j \in \Lambda(N)} W_N(B_j) \end{pmatrix},$$

for all $\mathbf{B} = (B_1, B_2, \ldots, B_N) \in \tilde{\mathcal{H}}(K)$. Here we considered

$$c_{ij} W_i(B_j) = \begin{cases} W_i(B_j) & \text{if } c_{ij} = 1, \\ \emptyset & \text{if } c_{ij} = 0, \end{cases}$$

for all $i, j = 1, 2, \ldots, N$ and $\Lambda(i) = \{j : c_{ij} = 1\}$ for all $i = 1, 2, \ldots, N$. Alternatively, W can be represented in a matrix as $W = \left(c_{ij} W_i\right)_{N \times N}$, i.e.

$$W = \begin{pmatrix} c_{11} W_1 & c_{12} W_1 & \ldots & c_{1N} W_1 \\ c_{21} W_2 & c_{22} W_2 & \ldots & c_{2N} W_2 \\ \vdots & \vdots & \vdots & \vdots \\ c_{N1} W_N & c_{N2} W_N & \ldots & c_{NN} W_N \end{pmatrix}.$$

This transformation W is a contraction map on $\tilde{\mathcal{H}}(K)$ and hence there exists a unique fixed point $\mathbf{A} = (A_1, A_2, \ldots, A_N) \in \tilde{\mathcal{H}}(K)$ such that $W(\mathbf{A}) = \mathbf{A}$, which is called an invariant set or an attractor or a recurrent fractal of the RIFS. Moreover, $A_i = \bigcup_{j \in \Lambda(i)} W_i(A_j)$ for all $i = 1, 2, \ldots, N$. Usually, making a slight abuse of notation, we often call $A = \bigcup_{i=1}^{N} A_i$ as the attractor of the RIFS.

We first utilize this RIFS theory to construct a fractal function associated with a deterministic data set and then consider a noisy data set for generating a random fractal function with variable scaling based on the notion of RIFS.

2.2 RFIF with Variable Scaling for Deterministic Data Set

Let us take an initial data set $\mathcal{D} = \{(t_i, y_i) : i = 0, 1, \ldots, N\}$ in $\mathbb{R}^2$, where $t_0 < t_1 < \cdots < t_N$. We denote intervals $I := [t_0, t_N]$, and $I_i := [t_{i-1}, t_i]$ for all $i =$

$1, 2, \ldots, N$. Also, let us consider intervals $J_j := [t_{l(j)}, t_{r(j)}]$, where $l(j), r(j) \in \{0, 1, \ldots, N\}$ with $l(j) < r(j)$ for all $j = 1, 2, \ldots, N$. Now, we define homeomorphisms $L_k : J_k \to I_k$ by $L_k(t) = a_k t + b_k$ for $k = 1, 2, \ldots, N$, which map end points of J_k to end points of I_k such that $L_k(t_{l(k)}) = t_{k-1}$ and $L_k(t_{r(k)}) = t_k$. Therefore, we have

$$a_k = \frac{t_k - t_{k-1}}{t_{r(k)} - t_{l(k)}} \text{ and } b_k = \frac{t_{r(k)} t_{k-1} - t_{l(k)} t_k}{t_{r(k)} - t_{l(k)}}.$$

Also, for all $t, t^* \in J_k$, we have $|L_k(t) - L_k(t^*)| \le |a_k||t - t^*|$. If we consider the length of J_k to be greater than the length of I_k, that is $|t_k - t_{k-1}| < |t_{r(k)} - t_{l(k)}|$, then $|a_k| < 1$ and L_k becomes a contraction.

Define continuous maps $F_k : J_k \times \mathbb{R} \to \mathbb{R}$ by $F_k(t, y) = c_k t + d_k(t) y + e_k$, where d_k are real-valued continuous functions defined on I and satisfying

$$\|d_k\|_\infty := \sup\{|d_k(t)| : t \in I\} < 1. \tag{2}$$

In addition, each F_k satisfying join-up conditions $F_k\left(t_{l(k)}, y_{l(k)}\right) = y_{k-1}$ and $F_k\left(t_{r(k)}, y_{r(k)}\right) = y_k$. Therefore, we get

$$\begin{aligned} c_k &= \frac{y_k - y_{k-1}}{t_{r(k)} - t_{l(k)}} - \frac{d_k(t_{r(k)}) y_{r(k)} - d_k(t_{l(k)}) y_{l(k)}}{t_{r(k)} - t_{l(k)}}, \\ e_k &= \frac{t_{r(k)} y_{k-1} - t_{l(k)} y_k}{t_{r(k)} - t_{l(k)}} - \frac{t_{r(k)} d_k(t_{l(k)}) y_{l(k)} - t_{l(k)} d_k(t_{r(k)}) y_{r(k)}}{t_{r(k)} - t_{l(k)}}. \end{aligned}$$

Moreover, $|F_k(t, y) - F_k(t, y^*)| \le |d_k(t)||y - y^*|, \;\; t \in J_k$ and $y, y^* \in \mathbb{R}$. Hence, F_k is a contraction with respect to y-variable.

Next, we consider $W_k : J_k \times \mathbb{R} \to I_k \times \mathbb{R}$ by $W_k(t, y) = (L_k(t), F_k(t, y))$ for all $k = 1, 2, \ldots, N$. We can easily check that $W_k(t_{l(k)}, y_{l(k)}) = (t_{k-1}, y_{k-1})$ and $W_k(t_{r(k)}, y_{r(k)}) = (t_k, y_k)$. Moreover, all W_k are contractions with respect to some metric, equivalent to the Euclidean metric in $\mathbb{R}^2$. Let us define a row-stochastic matrix $P = \left(p_{ij}\right)_{N \times N}$ by

$$p_{ij} = \begin{cases} \frac{1}{N_i}, & I_i \subset J_j, \\ 0, & \text{otherwise}, \end{cases}$$

where N_i denotes the number of j such that $I_i \subset J_j$ for $i = 1, 2, \ldots, N$. We can make P an irreducible matrix by selecting J_k's appropriately. Therefore, we can construct a RIFS $\{I \times \mathbb{R}; P; W_k : k = 1, 2, \ldots, N\}$ associated with $\mathcal{D}$.

Remark 1 In this RIFS, we employed function contractivity factors (or variable scaling parameters) d_k, which describe fractal objects better than constant contractivity factors and provide more flexibility to fractal functions. For detailed information, see [13].

Using (1), we obtain the connection matrix $C = \left(c_{ij}\right)_{N\times N}$, where

$$c_{ij} = \begin{cases} 1, & I_j \subset J_i, \\ 0, & \text{otherwise}. \end{cases} \tag{3}$$

Let $\mathcal{C}(I)$ be the collection of real-valued continuous functions defined on I. Define a metric d_∞ on $\mathcal{C}(I)$ by $d_\infty(f, g) :=\| f - g \|_\infty = \sup\{|f(t) - g(t)| : t \in I\}$. Then $(\mathcal{C}(I), d_\infty)$ is a complete metric space. Further, let us define

$$\mathcal{C}^*(I) := \{f \in \mathcal{C}(I) : f(t_i) = y_i,\ i = 0, 1, \dots, N\}.$$

Then $(\mathcal{C}^*(I), d_\infty)$ is also a complete metric space. Now, we define an operator $T : \mathcal{C}^*(I) \to \mathcal{C}^*(I)$ by

$$Tg(t) := F_k\left(L_k^{-1}(t), g\left(L_k^{-1}(t)\right)\right),\ t \in I_k \text{ and } k = 1, 2, \dots, N.$$

Here T is known as the Read-Bajraktarević operator, which is a contraction on $(\mathcal{C}^*(I), d_\infty)$. Therefore, T has a unique fixed point $f_{\mathcal{D}} \in \mathcal{C}^*(I)$ such that

$$f_{\mathcal{D}}(t) = Tf_{\mathcal{D}}(t) = F_k\left(L_k^{-1}(t), f_{\mathcal{D}}\left(L_k^{-1}(t)\right)\right),\ t \in I_k \text{ and } k = 1, 2, \dots, N. \tag{4}$$

This $f_{\mathcal{D}}$ is called a linear RFIF with variable scaling parameters associated with $\mathcal{D}$. Let $A := \{(t, f_{\mathcal{D}}(t)) : t \in I\}$, and $A_i := \{(t, f_{\mathcal{D}}(t)) : t \in I_i\}$ for all $i = 1, 2, \dots, N$. Then $A = \bigcup_{i=1}^{N} A_i$. Moreover,

$$\begin{aligned} A_i &= \{(t, f_{\mathcal{D}}(t)) : t \in I_i\} = \left\{\left(t, F_i\left(L_i^{-1}(t), f_{\mathcal{D}}\left(L_i^{-1}(t)\right)\right)\right) : t \in I_i\right\} \\ &= \{(L_i(t), F_i(t, f_{\mathcal{D}}(t))) : t \in J_i\} = \{W_i(t, f_{\mathcal{D}}(t)) : t \in J_i\} \\ &= \bigcup_{j \in \Lambda(i)} W_i\left(A_j\right). \end{aligned}$$

Thus, $\mathbf{A} = (A_1, A_2, \dots, A_N)$ is an attractor of the RIFS $\{I \times \mathbb{R}; P; W_i : i = 1, 2, \dots, N\}$ associated with $\mathcal{D}$.

In the subsequent section, we define α-stable distributions and some of its properties required for further study.

2.3 α-Stable Distribution

An α-stable distribution, also known as stable distribution, belongs to the family of heavy-tailed distributions and is a generalization of Gaussian distribution. A complete description of a stable distribution requires the following four parameters: an index of stability or tail index $\alpha \in (0, 2]$, a skewness parameter $\beta \in [-1, 1]$, a scale parameter

$\sigma > 0$, and a location parameter $\mu \in \mathbb{R}$. Generally, a stable distribution does not have closed form formulae for its probability density function (PDF) or cumulative distribution function (CDF) [11]. However, it can be described by its characteristic function.

Definition 2 A random variable X follows a stable distribution, denoted by $X \sim S_\alpha(\beta, \sigma, \mu)$, if its characteristic function has the form

$$\begin{aligned}\phi_X(t) &= \mathbb{E}\left[e^{itX}\right] \\ &= \begin{cases}\exp\left(it\mu - |\sigma t|^\alpha \left\{1 + i\beta \operatorname{sign}(t) \tan\left(\frac{\pi\alpha}{2}\right)\left[|\sigma t|^{1-\alpha} - 1\right]\right\}\right), & \alpha \neq 1, \\ \exp\left(it\mu - |\sigma t| \left\{1 + i\beta \operatorname{sign}(t) \frac{2}{\pi} \ln |\sigma t|\right\}\right), & \alpha = 1,\end{cases}\end{aligned}$$

for $t \in \mathbb{R}$, where $\operatorname{sign}(t) = \begin{cases}\frac{t}{|t|}, & t \neq 0, \\ 0, & t = 0.\end{cases}$

Remark 2 Several parameterizations for α-stable distributions are available in the literature, but Nolan's [8] parameterization is used here for numerical reasons.

For $\alpha = 2$, the Gaussian distribution is obtained, i.e. $X \sim \mathcal{N}(\mu, 2\sigma^2)$. The nth moment of a non-Gaussian ($\alpha \neq 2$) stable random variable X is finite iff $n < \alpha$. When $\beta = 0$, the distribution is symmetric about its location parameter μ.

Property 1 If $X \sim S_\alpha(\beta, \sigma, \mu)$ and $0 \neq a, b \in \mathbb{R}$, then

$$aX + b \sim S_\alpha(\operatorname{sign}(a)\beta, |a|\sigma, a\mu + b).$$

Property 2 For all $i = 0, 1, 2, \ldots, N$, if $X_i \sim S_\alpha(\beta_i, \sigma_i, \mu_i)$ are independent and $\omega_i \in \mathbb{R}$, then $\sum_{i=0}^N \omega_i X_i \sim S_\alpha(\beta, \sigma, \mu)$, where

$$\sigma^\alpha = \sum_{i=0}^N |\omega_i \sigma_i|^\alpha, \qquad \beta\sigma^\alpha = \sum_{i=0}^N \operatorname{sign}(\omega_i)\beta_i |\omega_i \sigma_i|^\alpha,$$
$$\mu = \begin{cases}\sum_{i=0}^N \omega_i \mu_i + \tan\left(\frac{\pi\alpha}{2}\right)\left(\beta\sigma - \sum_{i=0}^N \omega_i \beta_i \sigma_i\right) & \alpha \neq 1, \\ \sum_{i=0}^N \omega_i \mu_i + \frac{\pi}{2}\left(\beta\sigma \ln \sigma - \sum_{i=0}^N \omega_i \beta_i \sigma_i \ln |\omega_i \sigma_i|\right) & \alpha = 1.\end{cases}$$

For more detailed information, the reader can see [9–11].

In the following section, we construct a linear RFIF with variable scaling for any given α-stable noisy data set and determine the probability distribution of any interpolated value of this RFIF.

3 RFIF for Noisy Data Set

Consider a data set $\Delta = \{(t_i, y_i, \epsilon_i) : i = 0, 1, \ldots, N\}$, where $t_0 < t_1 < \cdots < t_N$ and $\epsilon_i \sim S_\alpha(\beta_i, \sigma_i, 0)$ is the α-stable noise in the value of y_i. We assume that these ϵ_i's are independent. First, we construct RIFS for this noisy data set. Let $Y_i := y_i + \epsilon_i$, using Property 1, we have $Y_i \sim S_\alpha(\beta_i, \sigma_i, y_i)$ for all $i = 0, 1, \ldots, N$. These Y_i's are also independent. Let Y be a real-valued continuous random variable. Define $\mathcal{F}_k : J_k \times \mathbb{R} \to \mathbb{R}$ (a random analog of F_k) by $\mathcal{F}_k(t, Y) = C_k t + d_k(t)Y + E_k$ satisfying $\mathcal{F}_k\left(t_{l(k)}, Y_{l(k)}\right) = Y_{k-1}$ and $\mathcal{F}_k\left(t_{r(k)}, Y_{r(k)}\right) = Y_k$ for all $k = 1, 2, \ldots, N$. Therefore,

$$\begin{aligned} C_k &= \frac{Y_k - Y_{k-1}}{t_{r(k)} - t_{l(k)}} - \frac{d_k(t_{r(k)})Y_{r(k)} - d_k(t_{l(k)})Y_{l(k)}}{t_{r(k)} - t_{l(k)}}, \\ E_k &= \frac{t_{r(k)}Y_{k-1} - t_{l(k)}Y_k}{t_{r(k)} - t_{l(k)}} - \frac{t_{r(k)}d_k(t_{l(k)})Y_{l(k)} - t_{l(k)}d_k(t_{r(k)})Y_{r(k)}}{t_{r(k)} - t_{l(k)}}. \end{aligned} \tag{5}$$

Define $\mathcal{W}_k : J_k \times \mathbb{R} \to I_k \times \mathbb{R}$ by $\mathcal{W}_k(t, Y) = (L_k(t), \mathcal{F}_k(t, Y))$ for all $k = 1, 2, \ldots, N$, and construct RIFS $\{I \times \mathbb{R}; P; \mathcal{W}_k : k = 1, 2, \ldots, N\}$ associated with Δ, which is a random analog to the RIFS $\{I \times \mathbb{R}; P; W_k : k \in \mathbb{N}_N\}$ associated with $\mathcal{D}$. There exists a unique [up to distribution] RFIF $f_\Delta : I \to \mathbb{R}$ such that

$$\begin{aligned} f_\Delta(t) &= \mathcal{F}_k\left(L_k^{-1}(t), f_\Delta\left(L_k^{-1}(t)\right)\right) \\ &= C_k L_k^{-1}(t) + d_k\left(L_k^{-1}(t)\right) f_\Delta\left(L_k^{-1}(t)\right) + E_k,\ t \in I_k, k = 1, \ldots, N. \end{aligned} \tag{6}$$

Apparently, this f_Δ is a random analog of $f_\mathcal{D}$. Next, we write f_Δ in explicit form to find its distribution. We can see that $I = \bigcup_{k=1}^N I_k$ and $I_k = L_k(J_k) = \bigcup_{j \in \Lambda(k)} L_k(I_j)$. Therefore, I is the attractor of RIFS $\{I; P; L_k : k = 1, 2, \ldots, N\}$. Hence, for any given point $t \in I$, there exists a sequence $\{k_n\}_{n \in \mathbb{N}}$, where each $k_n \in \{1, 2, \ldots, N\}$, such that

$$\lim_{n \to \infty} L_{k_1} \circ L_{k_2} \circ \cdots \circ L_{k_n}(s) = t,\ \text{for } s \in I. \tag{7}$$

By recursively applying (6), we can easily obtain the following expression:

$$f_\Delta(T_0(s)) = D_n(s) f_\Delta(s) + \sum_{j=1}^{n} D_{j-1}(s)\left(C_{k_j} T_j(s) + E_{k_j}\right), \tag{8}$$

where

$$T_j(s) = \begin{cases} L_{k_{j+1}} \circ \cdots \circ L_{k_n}(s) & \text{for } j = 0, 1, \ldots, n-1, \\ s & \text{for } j = n, \end{cases}$$

and

$$D_j(s) = \begin{cases} 1 & \text{for } j = 0, \\ \prod_{i=1}^{j} d_{k_i}\,(T_i(s)) & \text{for } j = 1, 2, \dots, n. \end{cases}$$

We can rewrite (7) as $\lim_{n\to\infty} T_0(s) = t$. Also, we get $\lim_{n\to\infty} D_n(s) = 0$ by using (2). Since f_Δ is a continuous function, as n approaches ∞ in (8), we obtain

$$f_\Delta(t) = \sum_{j=1}^{\infty} D_{j-1}(s)\left(C_{k_j} T_j(s) + E_{k_j}\right),\ s \in I. \tag{9}$$

Using (5), we can rewrite (9) as

$$\begin{aligned} f_\Delta(t) = \sum_{j=1}^{\infty} D_{j-1}(s)\Bigg[\left(\frac{t_{r(k_j)} - T_j(s)}{t_{r(k_j)} - t_{l(k_j)}}\right) Y_{k_j-1} + \left(\frac{T_j(s) - t_{l(k_j)}}{t_{r(k_j)} - t_{l(k_j)}}\right) Y_{k_j} - \\ \left(\frac{t_{r(k_j)} - T_j(s)}{t_{r(k_j)} - t_{l(k_j)}}\right) d_{k_j}(t_{l(k_j)}) Y_{l(k_j)} - \left(\frac{T_j(s) - t_{l(k_j)}}{t_{r(k_j)} - t_{l(k_j)}}\right) d_{k_j}(t_{r(k_j)}) Y_{r(k_j)}\Bigg]. \end{aligned} \tag{10}$$

For each $k_j \in \{1, 2, \dots, N\}$, we have $Y_{k_j-1}, Y_{k_j}, Y_{l(k_j)}, Y_{r(k_j)} \in \{Y_0, Y_1, \dots, Y_N\}$. Therefore, by equating coefficients of each Y_i in (10), we get

$$f_\Delta(t) = \sum_{i=0}^{N} \omega_i Y_i,\ t \in I, \tag{11}$$

where ω_i depends on the sequence $\{k_j\}$ of t. We can easily see that the linear RFIF $f_\Delta(t)$ is a random variable for each $t \in I$. Now, we determine the probability distribution of $f_\Delta(t)$. By using Property 2 in (11), we get

$$f_\Delta(t) \sim S_\alpha(\beta, \sigma, \mu),$$

where

$$\sigma = \left(\sum_{i=0}^{N} |\omega_i \sigma_i|^\alpha\right)^{1/\alpha}, \qquad \beta = \frac{\sum_{i=0}^{N} \operatorname{sign}(\omega_i)\beta_i |\omega_i \sigma_i|^\alpha}{\sigma^\alpha},$$

$$\mu = \begin{cases} \sum_{i=0}^{N} \omega_i y_i + \tan\left(\frac{\pi\alpha}{2}\right)\left(\beta\sigma - \sum_{i=0}^{N} \omega_i \beta_i \sigma_i\right) & \alpha \neq 1, \\ \sum_{i=0}^{N} \omega_i y_i + \frac{\pi}{2}\left(\beta\sigma \ln\sigma - \sum_{i=0}^{N} \omega_i \beta_i \sigma_i \ln|\omega_i \sigma_i|\right) & \alpha = 1. \end{cases}$$

Moreover, initial data set $\mathcal{D}$ is a realization of the noisy data set Δ. Therefore, by using (11), we get

$$f_{\mathcal{D}}(t) = \sum_{i=0}^{N} \omega_i y_i.$$

Hence, the location parameter μ of $f_\Delta(t)$ becomes

$$\mu = \begin{cases} f_{\mathcal{D}}(t) + \tan\left(\frac{\pi\alpha}{2}\right)\left(\beta\sigma - \sum_{i=0}^{N}\omega_i\beta_i\sigma_i\right) & \alpha \neq 1, \\ f_{\mathcal{D}}(t) + \frac{\pi}{2}\left(\beta\sigma\ln\sigma - \sum_{i=0}^{N}\omega_i\beta_i\sigma_i\ln|\omega_i\sigma_i|\right) & \alpha = 1. \end{cases}$$

Thus, $f_\Delta(t)$ is an α-stable random variable for each $t \in I$.

Remark 3 If α-stable noise in the data set Δ is symmetric, i.e. $\epsilon_i \sim S_\alpha(0, \sigma_i, 0)$ for all $i = 0, 1, \ldots, N$, then $f_\Delta(t)$ is also a symmetric α-stable variate and its location parameter is $f_{\mathcal{D}}(t)$ that is $f_\Delta(t) \sim S_\alpha(0, \sigma, f_{\mathcal{D}}(t))$ for all $t \in I$, where $\sigma = \left(\sum_{i=0}^{N}|\omega_i\sigma_i|^\alpha\right)^{1/\alpha}$. Moreover, if $\alpha = 2$, then $\epsilon_i \sim \mathcal{N}(0, 2\sigma_i^2)$ for $i = 0, 1, \ldots, N$ and $f_\Delta(t) \sim \mathcal{N}(f_{\mathcal{D}}(t), \sigma^2)$, where $\sigma^2 = \frac{1}{2}\sum_{i=0}^{N}\omega_i^2\sigma_i^2$.

4 Simulation

In this section, we present a simulation study through a numerical example to illustrate the propagation of uncertainty in a linear RFIF with variable scaling parameters for a given α-stable noisy data set.

Let $\Delta = \{(t_0, y_0, \epsilon_0), (t_1, y_1, \epsilon_1), (t_2, y_2, \epsilon_2), (t_3, y_3, \epsilon_3), (t_4, y_4, \epsilon_4)\}$ be a given data set, where

$$t_0 = 0,\ t_1 = 0.3,\ t_2 = 0.5,\ t_3 = 0.7,\ t_4 = 1;$$
$$y_0 = 2.3,\ y_1 = 1.6,\ y_2 = 3.8,\ y_3 = 2.9,\ y_4 = 1.2;$$

and

$$\epsilon_0 \sim S_{1.8}(0.3, 0.4, 0),\ \epsilon_1 \sim S_{1.8}(-0.3, 0.5, 0),\ \epsilon_2 \sim S_{1.8}(0.5, 0.7, 0),$$
$$\epsilon_3 \sim S_{1.8}(0.7, 0.6, 0),\ \epsilon_4 \sim S_{1.8}(-0.2, 0.3, 0).$$

For this data set, we have $N = 4$; $I = [0, 1]$; and

$$I_1 = [0, 0.3],\ I_2 = [0.3, 0.5],\ I_3 = [0.5, 0.7],\ I_4 = [0.7, 1].$$

Now, let us take $J_1 = [0.3, 0.7],\ J_2 = [0.5, 1.0],\ J_3 = [0, 0.5],\ J_4 = [0, 0.5]$. Then, by using (3), we can form the connection matrix

$$C = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

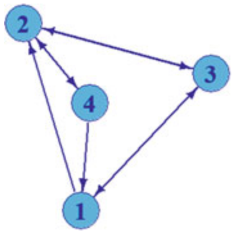

Fig. 1 The directed graph of C

From Fig. 1, we can observe that the directed graph of C is strongly connected, implying that C and therefore P is irreducible.

By using the given data set Δ, we can form the deterministic data set

$$\mathcal{D} = \{(0, 2.3), (0.3, 1.6), (0.5, 3.8), (0.7, 2.9), (1, 1.2)\},$$

and for this data set, we can construct the RIFS $\{I \times \mathbb{R}; P; W_1, W_2, W_3, W_4\}$. If we consider the variable scaling factors:

$$d_1(t) = \frac{1}{3}e^{-5t} + 0.5, \; d_2(t) = \frac{1}{2}\sin(3t) + 0.4,$$
$$d_3(t) = \frac{1}{8}e^{2t}\cos(3t) + 0.6, \; d_4(t) = \frac{1}{2}e^{-5t} + 0.3.$$

Then, we can calculate other parameters of the above RIFS:

$$a_1 = 0.75, \; a_2 = 0.4, \; a_3 = 0.4, \; a_4 = 0.6;$$
$$b_1 = -0.225, \; b_2 = 0.1, \; b_3 = 0.5, \; b_4 = 0.7;$$
$$c_1 = -3.1505, \; c_2 = 10.1011, \; c_3 = -3.2077, \; c_4 = -2.3119;$$
$$e_1 = 2.3261, \; e_2 = -6.8658, \; e_3 = 2.1325, \; e_4 = 1.06.$$

Further, by using (4), we can calculate the values of RFIF $f_{\mathcal{D}}$, whose graph is shown in Fig. 2. In this figure, the red colored dots represent the data points of $\mathcal{D}$, and the RFIF $f_{\mathcal{D}}$ passing through these points is shown in the blue curve. Moreover, we also represent the 95% lower and upper quantile bands of the linear RFIF f_{Δ} in Fig. 2, which imply that any realization of the RFIF f_{Δ} will lie between these bands with a probability of 0.95.

Now, we consider an arbitrarily point $t = 0.58$ in I. If we select $s = 0.3$, then we can obtain a sequence $\{k_n\}$ of t such that

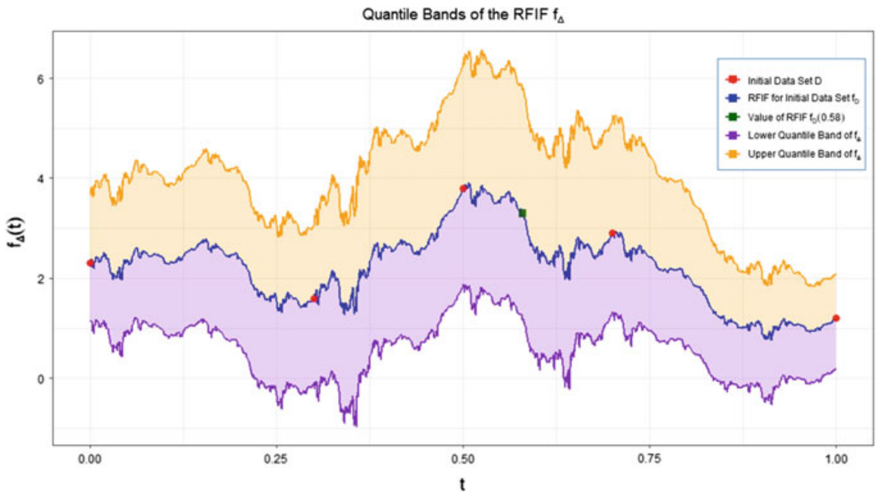

Fig. 2 95% Quantile band of the RFIF f_Δ and graph of the RFIF $f_{\mathcal{D}}$ along with the points of the data set $\mathcal{D}$

$$\{\ 3, 1, 3, 1, 3, 1, 2, 3, 2, 4, 2, 3, 1, 3, 2, 4, 1, 3, 2, 4, 2, 4, 2, 4, 1, 2, 4, 2, 4, 2, 3, 2, 4, 1, 3, 2, 4, 1, 2, 3, 2, 3, 1, 3, 1, 3, 1, 3, 2, 4, 1, 2, 4, 2, 4, 2, 4, 1, 2, 4, 2, 4, 2, 4, 2, 4, 1, 2, 4, 1, 2, 3, 2, 3, 1, 2, 3, 2, 4, 2, 4, 2, 3, 2, 3, 1, 3, 2, 4, 1, 2, 4, 1, 2, 3, 1, 3, 2, 4, 2, 4, 2, 3, 1, 3, 1, 3, 2, 4, 2, 4, 1, 3, 2, 4, 2, 3, 1, 3, 1, 2, 4, 2, 4, 1, 2, 3, 1, 2, 3, 2, 3, 2, 3, 2, 4, 2, 4, 1, 3, 1, 3, 1, 3, 1, 3, 2, 4, 2, 3, 2, 3\ \},$$

with a maximum tolerance error of 0.001 in (7). This sequence is called a fractal code of t. By utilizing (10) and (11), we can compute the coefficients of Y_i as follows:

$$\omega_0 = -0.259217,\ \omega_1 = 0.472842,\ \omega_2 = 0.631665,$$
$$\omega_3 = 0.257078,\ \omega_4 = -0.010617.$$

Hence, the distribution of RFIF f_Δ at point $t = 0.58$ is given as follows:

$$f_\Delta(0.58) \sim S_{1.8}(0.31393, 0.56363, 3.3099). \tag{12}$$

Next, we consider 8000 random samples of the data set Δ. For each realization, we form a RFIF with variable scaling (as we have constructed for the data set $\mathcal{D}$). Therefore, we have 8000 realizations of the RFIF f_Δ and thus we have 8000 realizations of $f_\Delta(0.58)$.

In Fig. 3(i), we represent the histogram of these 8000 random samples of $f_\Delta(0.58)$. In the same figure, we have fitted an empirical PDF to these observed values and also plotted the PDF of the analytically estimated distribution of $f_\Delta(0.58)$, which is given in (12). Here, we can see that the analytically estimated PDF of $f_\Delta(t)$ is very

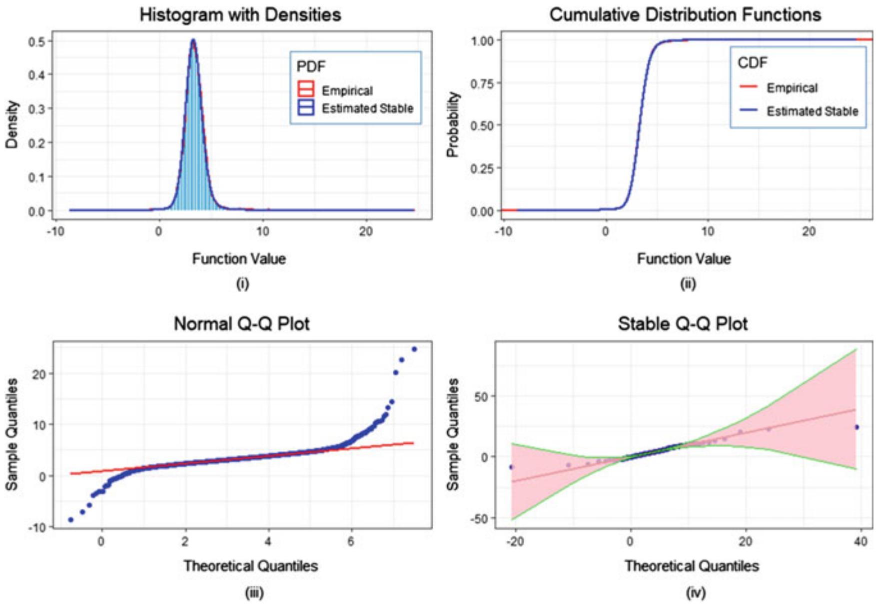

Fig. 3 (i) Histogram with Empirical & Estimated PDFs, (ii) Empirical & Estimated CDFs, (iii) Normal Q-Q Plot, and (iv) Stable Q-Q Plot with 95% Confidence Bands

close to its empirically fitted PDF. A similar conclusion can be drawn from the CDFs plot displayed in Fig. 3(ii).

Moreover, a normal quantile-quantile plot for observed samples of $f_\Delta(0.58)$ is shown in Fig. 3(iii). We can observe here that both tails deviate from the red color reference line, indicating that the distribution of $f_\Delta(0.58)$ has heavier tails than the normal distribution.

Further, a stable quantile-quantile plot is exhibited in Fig. 3(iv). In the same figure, we have displayed 95% confidence band for the simulated values of $f_\Delta(0.58)$, which represents the variation in the estimate of $f_\Delta(0.58)$ from its location based on the noisy data Δ. Here, we can see that nearly all the observed samples of $f_\Delta(0.58)$ fall along the reference line, implying that $f_\Delta(0.58)$ follows the same distribution as we specified in (12). Therefore, our analytically estimated distribution for $f_\Delta(t)$ in (12) is valid. Moreover, $t = 0.58$ is an arbitrarily chosen point in I; therefore, for any $t \in I$, we can similarly estimate and validate the distribution of $f_\Delta(t)$.

5 Concluding Remarks

A commonly used tool for analyzing uncertainty at any point is the estimation of the probability distribution at that point. If the data is collected from a process that has fractal properties and contains α-stable noise, in that case, the recurrent fractal interpolation technique efficiently determines uncertainty at any intermediate point in this noisy data set. Moreover, for any given data set with α-stable noise on its ordinate, the probability distribution of a recurrent fractal interpolation function at any interpolated value is also an α-stable. And the remaining parameters of this distribution can be estimated analytically.

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References

1. Barnsley, M.F.: Fractal functions and interpolation. Constr. Approx. **2**(1), 303–329 (1986)
2. Barnsley, M.F.: Fractals Everywhere, 2nd ed. Morgan Kaufmann (2000)
3. Barnsley, M.F., Elton, J.H., Hardin, D.P.: Recurrent iterated function systems. Constr. Approx. **5**(1), 3–31 (1989)
4. Chand, A.K.B., Kapoor, G.P.: Generalized cubic spline fractal interpolation functions. SIAM J. Numer. Anal. **44**, 655–676 (2006)
5. Kumar, M., Upadhye, N.S., Chand, A.K.B.: Distribution of linear fractal interpolation function for random dataset with stable noise. Fractals **29**(4), 1–12 (2021)
6. Luor, D.C.: Fractal interpolation functions for random data sets. Chaos Solitons Fractals **114**, 256–263 (2018)
7. Navascués, M.A., Chand, A.K.B., Viswanathan, P., Sebastián, M.V.: Fractal interpolation functions: a short survey. Appl. Math. **5**, 1834–1841 (2014)
8. Nolan, J.P.: Numerical calculation of stable densities and distribution functions. Commun. Stat. Stoch. Models **13**(4), 759–774 (1997)
9. Nolan, J.P.: Parametrizations and modes of stable distributions. Stat. Prob. Lett. **38**, 187–195 (1998)
10. Nolan, J.P.: Univariate Stable Distributions: Models for Heavy Tailed Data. Springer Nature (2020)
11. Samorodnitsky, G., Taqqu, M.: Stable non-Gaussian random processes: stochastic models with infinite variance. CRC Press (1994)
12. Vijender, N., Chand, A.K.B., Navascués, M.A., Sebastián, M.V.: Quantum Bernstein fractal functions. Comp. Math. Methods **3**, 1–13 (2021)
13. Wang, H.Y., Yu, J.S.: Fractal interpolation functions with variable parameters and their analytical properties. J. Approx. Theory **175**, 1–18 (2013)