On the Existence and Stability Analysis for Ψ-Caputo Fractional Boundary Value Poblem

Bhagwat R. Yewale and Deepak B. Pachpatte

Abstract In this paper, we study the existence and uniqueness results of the solutions for non-linear boundary value problems involving Ψ-Caputo fractional derivative. Furthermore, we prove some stability results of the given problem. The tools used in the analysis are relies on Banach fixed point theorem and Ψ-fractional Gronwall inequality.

Keywords Fractional differential equations · Ψ-Caputo fractional derivative · Gronwall inequality · Stability · Fixed point theorem

1 Introduction

In this paper, we are concerned with the nonlinear fractional differential equations of the type

$$\mathfrak{D}_0^{\bar{\theta},\Psi}\upsilon(\mathfrak{t}) = \mathscr{G}\big(\mathfrak{t}, \upsilon(\mathfrak{t})\big), \text{ for all } \mathfrak{t} \in [0, \bar{\chi}] = \mathrm{I}, \tag{1}$$

$$\upsilon(0) + \mathrm{h}(\upsilon) = \upsilon_0, \upsilon(\bar{\chi}) = \upsilon_{\bar{\chi}}, \qquad \upsilon_0, \upsilon_{\bar{\chi}} \in \mathbb{R} \tag{2}$$

where $1 < \bar{\theta} < 2$, $\mathfrak{D}_0^{\bar{\theta},\Psi}$ is the Ψ-Caputo fractional derivative, $\mathscr{G} : [0, \bar{\chi}] \times \mathbb{R} \to \mathbb{R}$, $\mathrm{h} : \mathcal{C}(\mathrm{I}, \mathbb{R}) \times \mathbb{R} \to \mathbb{R}$ are nonlinear and continuous functions and $\upsilon \in \mathcal{C}(\mathrm{I}, \mathbb{R})$; $\mathcal{C}(\mathrm{I}, \mathbb{R})$ the space of continuous function from I to $\mathbb{R}$ with the supremum norm $\|.\|$.

Fractional order derivatives and integrals are more general cases of integer order derivatives and integrals as it provide arbitrary order derivatives and integration. It has been seen that many researchers have revealed the efficiency of fractional

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B. R. Yewale (✉) · D. B. Pachpatte
Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad 431004, Maharashtra, India
e-mail: yewale.bhagwat@gmail.com

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differential equations (FDE$_s$) in the modelling of physical phenomena in different fields of science and engineering [3, 5, 12, 15, 16], which helped fractional calculus to become a very useful and attractive research field. In the literature, there are several approaches by which authors have defined numerous fractional differential and integral operators see [9]. One such class of fractional operators is an integration and differentiation of one function with respect to another function, referred to as, Ψ-Fractional calculus. For instance, Almeida [1], presented Ψ-Caputo fractional derivative which is modified version of Caputo derivative. In [17], authors established Ψ-Hilfer fractional derivative.

On the other hand, these Ψ-fractional operators have been utilized to perform a qualitative analysis of FDE$_s$. In particular, Almeida et al. [2], investigated the existence, uniqueness, continuous dependence and stability of the Ψ-Caputo FDE$_s$ with the help of Banach fixed point theorem. Kucche et al. [10], studied existence and uniqueness of Ψ-Hilfer FDE$_s$ with the help of Schauder's fixed point theorem as well as continuous dependence of the corresponding system have been studied by employing Weissinger theorem. Recently, Pachpatte [14] have used the Banach fixed point theorem to study the existence, uniqueness and stability of the Ψ-Hilfer partial FDE$_s$. In [20], Wahash et al. proved estimate and stability of the solution involving Ψ-Caputo derivative by using Ψ-Gronwall inequality. We mention here some recent studies that focus on the qualitative properties of Ψ-fractional differential equations [4, 6, 11, 18, 19, 21].

Motivated by above work, in this paper we discuss existence, uniqueness and stability of (1)–(2). In Sect. 2, we give some preliminaries. In Sect. 3, we prove existence and uniqueness of the solution of (1)–(2) in the view of Banach fixed point theorem. In Sect. 4, we present Stability analysis of (1)–(2). In Sect. 5, an illustrative example is given to demonstrate our results.

2 Preliminaries

Here, we provide some basic definitions and important results which are used throughout this work.

Definition 2.1 ([9]) Let $\bar{\theta} > 0$ and υ be an integrable function defined on I. Let $\Psi \in \mathcal{C}^1(\mathrm{I}, \mathbb{R})$ be an increasing function such that $\Psi^{'}(\mathfrak{t}) \neq 0$, for all $\mathfrak{t} \in \mathrm{I}$. Then Ψ-Riemann Liouville fractional integral of υ of order $\bar{\theta}$ is defined as

$$\mathfrak{J}_{0+}^{\bar{\theta},\Psi}\upsilon(\mathfrak{t}) = \frac{1}{\Gamma(\bar{\theta})}\int_0^{\mathfrak{t}} \Psi^{'}(\kappa)(\Psi(\mathfrak{t}) - \Psi(\kappa)^{\bar{\theta}-1}\upsilon(\kappa)d\kappa, \ \ \mathfrak{t} > 0. \tag{3}$$

Definition 2.2 ([1]) Let $\bar{\theta} > 0$ and $\Psi \in \mathcal{C}^n(\mathrm{I}, \mathbb{R})$, the Ψ-Caputo fractional derivative of a function $\upsilon \in \mathcal{C}^{n-1}(\mathrm{I}, \mathbb{R})$ of order $\bar{\theta}$ is defined as

$$\mathfrak{D}_{0+}^{\bar{\theta},\Psi}\upsilon(\mathfrak{t})=\mathfrak{D}_{0+}^{\bar{\theta},\Psi}\left[\upsilon(\mathfrak{t})-\sum_{m=0}^{n-1}\frac{\upsilon_{\Psi}^{[m]}(0)}{m!}(\Psi(\mathfrak{t})-\Psi(0))^{m}\right],\tag{4}$$

where $n=\lceil\bar{\theta}\rceil+1$ for $\bar{\theta}\notin\mathbb{N}$, $n=\bar{\theta}$ for $\bar{\theta}\in\mathbb{N}$.
and

$$\upsilon_{\Psi}^{[m]}(\mathfrak{t}):=\left(\frac{1}{\Psi'(\mathfrak{t})}\frac{d}{d\mathfrak{t}}\right)^{m}\vartheta(\mathfrak{t}).$$

Lemma 2.1 ([1]) *Let* $\bar{\theta}>0$. *If* $\upsilon\in\mathcal{C}^{1}(\mathrm{I},\mathbb{R})$, *then*

$$\mathfrak{D}_{0+}^{\bar{\theta},\Psi}\mathfrak{J}_{0+}^{\bar{\theta},\Psi}\upsilon(\mathfrak{t})=\upsilon(\mathfrak{t}),$$

and if $\upsilon\in\mathcal{C}^{n}(\mathrm{I},\mathbb{R})$, *then*

$$\mathfrak{J}_{0+}^{\bar{\theta},\Psi}\mathfrak{D}_{0+}^{\bar{\theta},\Psi}\upsilon(\mathfrak{t})=\upsilon(\mathfrak{t})-\sum_{m=0}^{n-1}\frac{\vartheta_{\Psi}^{[m]}(0)}{m!}(\Psi(\mathfrak{t})-\Psi(0))^{m}.\tag{5}$$

Lemma 2.2 ([9]) *For* $\bar{\theta},\ \bar{\theta}_{1}>0$ *and* $\upsilon\in\mathcal{C}^{n}(\mathrm{I})$, *we have*

$$\mathfrak{J}_{0+}^{\bar{\theta},\Psi}\mathfrak{J}_{0+}^{\bar{\theta}_{1},\Psi}\upsilon(\mathfrak{t})=\mathfrak{J}_{0+}^{\bar{\theta}+\bar{\theta}_{1},\Psi}\upsilon(\mathfrak{t}),\quad \mathfrak{t}>0.\tag{6}$$

Lemma 2.3 ([1]) *Let* $\bar{\theta}>0$. *Then*

$$\mathfrak{D}_{0+}^{\bar{\theta},\Psi}(\Psi(\kappa)-\Psi(0))^{k}=0,\ \text{for all } k=0,1,2,...,n-1,\ n\in\mathbb{N}.\tag{7}$$

Lemma 2.4 ([8]) *Let X be a Banach space and $B\subset X$ be closed. If $\zeta:B\to B$ is a contraction mapping, then ζ has a fixed point in B.*

Lemma 2.5 *Let* $1<\bar{\theta}<2$ *and* $\mathscr{G}:\mathrm{I}\times\mathbb{R}\to\mathbb{R}$ *be a continuous function. Then the problem* (1)–(2) *is equivalent to*

$$\begin{aligned}\upsilon(\mathfrak{t})=&\Big(1-\frac{\Psi(\mathfrak{t})-\Psi(0)}{\Psi(\bar{\chi})-\Psi(0)}\Big)\upsilon_{0}+\Big(\frac{\Psi(\mathfrak{t})-\Psi(0)}{\Psi(\bar{\chi})-\Psi(0)}-1\Big)\mathrm{h}(\upsilon)\\&+\Big(\frac{\Psi(\mathfrak{t})-\Psi(0)}{\Psi(\bar{\chi})-\Psi(0)}\Big)(\upsilon_{\bar{\chi}}-\mathfrak{J}_{0}^{\bar{\theta},\Psi}\mathscr{G}(\bar{\chi},\upsilon(\bar{\chi}))+\mathfrak{J}_{0}^{\bar{\theta},\Psi}\mathscr{G}(\mathfrak{t},\upsilon(\mathfrak{t})).\end{aligned}\tag{8}$$

Proof Operating $\mathfrak{J}_{0}^{\bar{\theta},\Psi}$ on both the sides of (1) and using Lemma 2.1, we get

$$\upsilon(\mathfrak{t})=c_{0}+c_{1}(\Psi(\mathfrak{t})-\Psi(0))+\mathfrak{J}_{0}^{\bar{\theta},\Psi}\mathscr{G}(\mathfrak{t},\upsilon(\mathfrak{t}))$$

Since $\upsilon(0)=\upsilon_{0}-\mathrm{h}(\upsilon)$ and $\upsilon(\bar{\chi})=\upsilon_{\bar{\chi}}$, we have

$$c_0 = \upsilon_0 - \mathrm{h}(\upsilon), \qquad c_1 = \frac{\upsilon_{\bar{\chi}} - \upsilon_0 + \mathrm{h}(\upsilon) - \mathfrak{J}_0^{\bar{\theta},\Psi}\mathscr{G}(\bar{\chi}, \upsilon(\bar{\chi}))}{\Psi(\bar{\chi}) - \Psi(0)}.$$

Then

$$\begin{aligned}\upsilon(\mathfrak{t}) =& \Big(1 - \frac{\Psi(\mathfrak{t}) - \Psi(0)}{\Psi(\bar{\chi}) - \Psi(0)}\Big)\upsilon_0 + \Big(\frac{\Psi(\mathfrak{t}) - \Psi(0)}{\Psi(\bar{\chi}) - \Psi(0)} - 1\Big)\mathrm{h}(\upsilon) \\ &+ \Big(\frac{\Psi(\mathfrak{t}) - \Psi(0)}{\Psi(\bar{\chi}) - \Psi(0)}\Big)(\upsilon_{\bar{\chi}} - \mathfrak{J}_0^{\bar{\theta},\Psi}\mathscr{G}(\bar{\chi}, \upsilon(\bar{\chi})) + \mathfrak{J}_0^{\bar{\theta},\Psi}\mathscr{G}(\mathfrak{t}, \upsilon(\mathfrak{t})). \qquad (9)\end{aligned}$$

Conversely, suppose that υ satisfies (8). Then from (8), for $\mathfrak{t} = 0$ and $\mathfrak{t} = \bar{\chi}$, we obtain (2). Applying $\mathfrak{D}_{0+}^{\bar{\theta},\Psi}$ on both the sides of (8) and using Lemmas 2.1, 2.3, we get (1). □

3 Existence and Uniqueness

Theorem 3.1 *Let the function* $\mathscr{G}$ *and* h *satisfying:*
[H1]: there exists $\mathscr{W}_1 > 0$ *and* $0 < \mathscr{W}_2 < 1$ *such that*

$$|\mathscr{G}(\mathfrak{t}, \upsilon) - \mathscr{G}(\mathfrak{t}, \upsilon^*)| \le \mathscr{W}_1|\upsilon - \upsilon^*|,$$

and

$$|\mathrm{h}(\upsilon) - \mathrm{h}(\upsilon^*)| \le \mathscr{W}_2|\upsilon - \upsilon^*|.$$

If

$$\mathscr{W}_2 + 2\frac{(\Psi(\bar{\chi}) - \Psi(0))^{\bar{\theta}}}{\Gamma(\bar{\theta} + 1)}\mathscr{W}_1 < 1, \qquad (10)$$

then (1)–(2) *has a unique solution.*

Proof Define $\mathscr{T} : \mathcal{C}(\mathrm{I}, \mathbb{R}) \to \mathcal{C}(\mathrm{I}, \mathbb{R})$ as follows:

$$\begin{aligned}(\mathscr{T}\upsilon)(\mathfrak{t}) =& \Big(1 - \frac{\Psi(\mathfrak{t}) - \Psi(0)}{\Psi(\bar{\chi}) - \Psi(0)}\Big)\upsilon_0 + \Big(\frac{\Psi(\mathfrak{t}) - \Psi(0)}{\Psi(\bar{\chi}) - \Psi(0)} - 1\Big)\mathrm{h}(\upsilon) \\ &+ \Big(\frac{\Psi(\mathfrak{t}) - \Psi(0)}{\Psi(\bar{\chi}) - \Psi(0)}\Big)(\upsilon_{\bar{\chi}} - \mathfrak{J}_0^{\bar{\theta},\Psi}\mathscr{G}(\bar{\chi}, \upsilon(\bar{\chi})) + \mathfrak{J}_0^{\bar{\theta},\Psi}\mathscr{G}(\mathfrak{t}, \upsilon(\mathfrak{t})). \qquad (11)\end{aligned}$$

Then for $\upsilon, \upsilon^* \in \mathcal{C}(\mathrm{I}, \mathbb{R})$, we have

$$\begin{aligned}|(\mathscr{T}\upsilon)(\mathfrak{t})-(\mathscr{T}\upsilon^*)(\mathfrak{t})| \leq & \Big(\frac{\Psi(\mathfrak{t})-\Psi(0)}{\Psi(\bar{\chi})-\Psi(0)}-1\Big)|\mathrm{h}(\upsilon)-\mathrm{h}(\upsilon^*)| \\ & + \Big(\frac{\Psi(\mathfrak{t})-\Psi(0)}{\Psi(\bar{\chi})-\Psi(0)}\Big)\mathfrak{J}_{0+}^{\bar{\theta},\Psi}|\mathscr{G}(\bar{\chi},\upsilon(\bar{\chi}))-\mathscr{G}(\bar{\chi},\upsilon^*(\bar{\chi}))| \\ & + \mathfrak{J}_{0+}^{\bar{\theta},\Psi}|\mathscr{G}(\mathfrak{t},\upsilon(\mathfrak{t}))-\mathscr{G}(\mathfrak{t},\upsilon^*(\mathrm{t}))| \\ \leq & \Big(\frac{\Psi(\mathfrak{t})-\Psi(0)}{\Psi(\bar{\chi})-\Psi(0)}-1\Big)\mathscr{W}_2|\upsilon-\upsilon^*| \\ & + \Big(\frac{\Psi(\mathfrak{t})-\Psi(0)}{\Psi(\bar{\chi})-\Psi(0)}\Big)\frac{\mathscr{W}_1}{\Gamma(\bar{\theta})}\int_0^{\bar{\chi}}\Psi'(\kappa)(\Psi(\bar{\chi})-\Psi(\kappa)^{\bar{\theta}-1}|\upsilon-\upsilon^*|d\kappa \\ & + \frac{\mathscr{W}_1}{\Gamma(\bar{\theta})}\int_0^{\mathfrak{t}}\Psi'(\kappa)(\Psi(\mathfrak{t})-\Psi(\kappa)^{\bar{\theta}-1}|\upsilon-\upsilon^*|d\kappa \\ \leq & \Big(\frac{\Psi(\mathfrak{t})-\Psi(0)}{\Psi(\bar{\chi})-\Psi(0)}\Big)\mathscr{W}_2\|\upsilon-\upsilon^*\| + \Big(\frac{\Psi(\mathfrak{t})-\Psi(0)}{\Psi(\bar{\chi})-\Psi(0)}\Big) \\ & \frac{(\Psi(\bar{\chi})-\Psi(0))^{\bar{\theta}}}{\Gamma(\bar{\theta}+1)}\mathscr{W}_1\|\upsilon-\upsilon^*\| + \frac{(\Psi(\mathfrak{t})-\Psi(0))^{\bar{\theta}}}{\Gamma(\bar{\theta}+1)}\mathscr{W}_1\|\upsilon-\upsilon^*\| \\ \leq & \Big(\mathscr{W}_2 + 2\frac{(\Psi(\bar{\chi})-\Psi(0)^{\bar{\theta}}}{\Gamma(\bar{\theta}+1)}\mathscr{W}_1\Big)\|\upsilon-\upsilon^*\|.\end{aligned}$$

In view of (10), $\mathscr{T}$ is contraction mapping. By Lemma 2.4, υ is a unique solution of the problem (1)–(2). □

4 Stability Analysis

In this section, by using Ψ-fractional Gronwall inequality, we analysis the Ulam-Hyers (UH), Generalized Ulam-Hyers (GHU), Ulam-Hyers-Rassias (UHR) and Generalized Ulam-Hyers-Rassias (GUHR) of the problem (1)–(2).

Let $\varepsilon > 0$ and $f : \mathrm{I} \to \mathbb{R}$ be a continuous function. We consider following inequalities:

$$|\mathfrak{D}_{0+}^{\bar{\theta},\Psi}\omega(\mathfrak{t}) - \mathscr{G}(\mathfrak{t},\omega(\mathfrak{t}))| \leq \varepsilon; \quad \mathfrak{t} \in [0,\bar{\chi}] \tag{12}$$

and

$$|\mathfrak{D}_{0+}^{\bar{\theta},\Psi}\omega(\mathfrak{t}) - \mathscr{G}(\mathfrak{t},\omega(\mathfrak{t}))| \leq \varepsilon f(\mathfrak{t}); \quad \mathfrak{t} \in [0,\bar{\chi}]. \tag{13}$$

Definition 4.1 The Eqs. (1)–(2) is said to be UH stable if there exists a real number $\delta > 0$ such that for each $\varepsilon > 0$ and for each solution $\omega \in \mathcal{C}(\mathrm{I},\mathbb{R})$ of the inequality (12), there exists a solution $\upsilon \in \mathcal{C}(\mathrm{I},\mathbb{R})$ satisfying

$$\mathfrak{D}_a^{\bar{\theta},\Psi}\upsilon(\mathfrak{t}) = \mathscr{G}\big(\mathfrak{t},\upsilon(\mathfrak{t})\big), \text{ for all } \mathfrak{t} \in \mathrm{I},\ 1 < \bar{\theta} < 2, \tag{14}$$

$$\upsilon(0) = \omega(0), \upsilon(\bar{\chi}) = \omega(\bar{\chi}) \tag{15}$$

with

$$|\omega(\mathfrak{t}) - \upsilon(\mathfrak{t})| \leq \delta\varepsilon, \quad \mathfrak{t} \in \mathrm{I}. \tag{16}$$

Definition 4.2 The Eqs. (1)–(2) is said to be GUH stable if there exists a continuous function $\varphi : \mathrm{I} \to \mathrm{I}$ with $\varphi(0) = 0$ such that for every $\varepsilon > 0$ and for each solution $\omega \in \mathcal{C}(\mathrm{I}, \mathbb{R})$ of (12), there exist a solution $\upsilon \in \mathcal{C}(\mathrm{I}, \mathbb{R})$ of (1)–(2) with

$$|\omega(\mathfrak{t}) - \upsilon(\mathfrak{t})| \leq \varphi(\varepsilon), \quad \mathfrak{t} \in \mathrm{I}. \tag{17}$$

Definition 4.3 The Eqs. (1)–(2) is said to be UHR stable with respect to the function f if there exists a real number $\delta > 0$ such that for every $\varepsilon > 0$ and for each solution $\omega \in \mathcal{C}(\mathrm{I}, \mathbb{R})$ of (13), there exist a solution $\upsilon \in \mathcal{C}(\mathrm{I}, \mathbb{R})$ of (1)–(2) with

$$|\omega(\mathfrak{t}) - \upsilon(\mathfrak{t})| \leq \delta\varepsilon f(\mathfrak{t}), \quad \mathfrak{t} \in \mathrm{I}. \tag{18}$$

Definition 4.4 The Eqs. (1)–(2) is GUHR stable with respect to the function f if there exists a real number $\delta > 0$ such that for each solution $\omega \in \mathcal{C}(\mathrm{I}, \mathbb{R})$ of (13), there exist a solution $\upsilon \in \mathcal{C}(\mathrm{I}, \mathbb{R})$ of (1)–(2) with

$$|\omega(\mathfrak{t}) - \upsilon(\mathfrak{t})| \leq \delta f(\mathfrak{t}), \quad \mathfrak{t} \in \mathrm{I}. \tag{19}$$

Remark 4.1 A function $\omega \in \mathcal{C}(\mathrm{I}, \mathbb{R})$ is a solution of (12) if and only if there exists a function $\mathrm{g} \in \mathcal{C}(\mathrm{I}, \mathbb{R})$ (where g depends on ω) such that

(1) $|\mathrm{g}(\mathfrak{t})| < \varepsilon$

(2) $\mathfrak{D}^{\bar{\theta},\Psi}_{0+}\omega(\mathfrak{t}) = \mathscr{G}(\mathfrak{t}, \omega(\mathfrak{t})) + \mathrm{g}(\mathfrak{t}),\ \mathfrak{t} \in \mathrm{I}.$

Remark 4.2 A function $\omega \in \mathcal{C}(\mathrm{I}, \mathbb{R})$ is a solution (13) if and only if there exists function $\mathrm{g}, f \in \mathcal{C}(\mathrm{I}, \mathbb{R})$ (where g depends on ω) such that

(1) $|\mathrm{g}(\mathfrak{t})| < \varepsilon f(\mathfrak{t})$

(2) $\mathfrak{D}^{\bar{\theta},\Psi}_{0+}\omega(\mathfrak{t}) = \mathscr{G}(\mathfrak{t}, \omega(\mathfrak{t})) + \mathrm{g}(\mathfrak{t}),\ \mathfrak{t} \in \mathrm{I}.$

Lemma 4.1 ([18]) *Ψ-Gronwall inequality:*
Assume that υ and u are nonnegative integrable functions on I. *Let ρ be a nonnegative continuous function on* I *such that ρ is nondecreasing. If*

$$\upsilon(\mathfrak{t}) \leq \mathrm{u}(\mathfrak{t}) + \rho(\mathfrak{t}) \int_0^{\mathfrak{t}} \Psi'(\kappa)(\Psi(\mathfrak{t}) - \Psi(\kappa))^{\bar{\theta}-1}\upsilon(\kappa)d\kappa, \tag{20}$$

then

$$\upsilon(\mathfrak{k}) \leq \mathrm{u}(\mathfrak{k}) \int_0^{\mathfrak{k}} \sum_{m=1}^{\infty} \frac{[\rho(\mathfrak{k})\Gamma(\bar{\theta})]^m}{\Gamma(\bar{\theta}m)} \Psi'(\kappa)(\Psi(\mathfrak{k}) - \Psi(\kappa))^{\bar{\theta}-1} \mathrm{u}(\kappa) d\kappa, \tag{21}$$

for $\mathfrak{k} \in \mathrm{I}$.

Remark 4.3 ([18]) Under the assumptions of Lemma 4.1, let $\upsilon(\mathfrak{k})$ be a nondecreasing function on I. Then we have

$$\upsilon(\mathfrak{k}) \leq \mathrm{u}(\mathfrak{k}) E_{\bar{\theta}}(\rho(\mathfrak{k})\Gamma(\bar{\theta})(\Psi(\mathfrak{k}) - \Psi(0))^{\bar{\theta}}),$$

where $E_{\bar{\theta}}(\mathfrak{k}) = \sum_{m=0}^{\infty} \frac{\mathfrak{k}^m}{\Gamma(\bar{\theta}+1)}$.

In the next theorem, we discuss the UH stability of the problem (1)–(2) with the help of Ψ-Gronwall inequality.

Theorem 4.1 *Suppose that [H1] hold and inequality* (12) *is satisfied, then the problem* (1)–(2) *is UH stable.*

Proof Let $\varepsilon > 0$. Assume that υ be a solution of (1)–(2). Then

$$\upsilon(\mathfrak{k}) = \Phi_{\upsilon} + \mathfrak{J}_0^{\bar{\theta},\Psi} \mathscr{G}(\mathfrak{k}, \upsilon(\mathfrak{k})), \tag{22}$$

where

$$\begin{aligned} \Phi_{\upsilon} =& \Big(1 - \frac{\Psi(\mathfrak{k}) - \Psi(0)}{\Psi(\bar{\chi}) - \Psi(0)}\Big)\upsilon_0 + \Big(\frac{\Psi(\mathfrak{k}) - \Psi(0)}{\Psi(\bar{\chi}) - \Psi(0)} - 1\Big)\mathrm{h}(\upsilon) \\ &+ \Big(\frac{\Psi(\mathfrak{k}) - \Psi(0)}{\Psi(\bar{\chi}) - \Psi(0)}\Big)(\upsilon_{\bar{\chi}} - \mathfrak{J}_0^{\bar{\theta},\Psi} \mathscr{G}(\bar{\chi}, \upsilon(\bar{\chi})). \end{aligned} \tag{23}$$

From (15), we can write

$$\upsilon(\mathfrak{k}) = \Phi_{\omega} + \mathfrak{J}_0^{\bar{\theta},\Psi} \mathscr{G}(\mathfrak{k}, \upsilon(\mathfrak{k})), \tag{24}$$

where

$$\begin{aligned} \Phi_{\omega} =& \Big(1 - \frac{\Psi(\mathfrak{k}) - \Psi(0)}{\Psi(\bar{\chi}) - \Psi(0)}\Big)\omega_0 + \Big(\frac{\Psi(\mathfrak{k}) - \Psi(0)}{\Psi(\bar{\chi}) - \Psi(0)} - 1\Big)\mathrm{h}(\omega) \\ &+ \Big(\frac{\Psi(\mathfrak{k}) - \Psi(0)}{\Psi(\bar{\chi}) - \Psi(0)}\Big)\omega_{\bar{\chi}} - \mathfrak{J}_0^{\bar{\theta},\Psi} \mathscr{G}(\bar{\chi}, \omega(\bar{\chi})). \end{aligned} \tag{25}$$

Since $\omega \in \mathcal{C}(\mathrm{I}, \mathbb{R})$ is a solution of inequality (12). By Remark 4.1, we have

$$|\mathfrak{D}_0^{\bar{\theta},\Psi} \omega(\mathfrak{k}) - \mathscr{G}\big(\mathfrak{k}, \omega(\mathfrak{k})\big)| \leq \varepsilon, \quad \text{for all } \mathfrak{k} \in \mathrm{I}. \tag{26}$$

Operating $\mathfrak{J}_0^{\bar{\theta},\Psi}$ on both the sides of (26), we obtain

$$|\omega(\mathfrak{t}) - \Phi_\omega - \frac{1}{\Gamma(\bar{\theta})}\int_0^{\mathfrak{t}} \Psi^{'}(\kappa)(\Psi(\mathfrak{t}) - \Psi(\kappa))^{\bar{\theta}-1} \mathscr{G}\big(\kappa, \omega(\kappa)d\kappa| \le \frac{(\Psi(\bar{\chi}) - \Psi(0))^{\bar{\theta}}}{\Gamma(\bar{\theta}+1)}\varepsilon. \tag{27}$$

By our assumption and from (24) and (27), we obtain

$$\begin{aligned} |\omega(\mathfrak{t}) - \upsilon(\mathfrak{t})| = & \ \Big|\omega(\mathfrak{t}) - \Phi_\omega - \frac{1}{\Gamma(\bar{\theta})}\int_0^{\mathfrak{t}} \Psi^{'}(\kappa)(\Psi(\mathfrak{t}) - \Psi(\kappa))^{\bar{\theta}-1}\mathscr{G}(\kappa, \upsilon(\kappa))d\kappa\Big| \\ \le & \ \Big|\omega(\mathfrak{t}) - \Phi_\omega - \frac{1}{\Gamma(\bar{\theta})}\int_0^{\mathfrak{t}} \Psi^{'}(\kappa)(\Psi(\mathfrak{t}) - \Psi(\kappa))^{\bar{\theta}-1}\mathscr{G}(\kappa, \omega(\kappa))d\kappa\Big| \\ & + \frac{1}{\Gamma(\bar{\theta})}\int_0^{\mathfrak{t}} \Psi^{'}(\kappa)(\Psi(\mathfrak{t}) - \Psi(\kappa))^{\bar{\theta}-1}|\mathscr{G}(\kappa, \omega(\kappa)) - \mathscr{G}(\kappa, \upsilon(\kappa))|d\kappa \\ \le & \ \frac{(\Psi(\bar{\chi}) - \Psi(0))^{\bar{\theta}}}{\Gamma(\bar{\theta}+1)}\varepsilon + \frac{\mathscr{W}_1}{\Gamma(\bar{\theta})}\int_0^{\mathfrak{t}} \Psi^{'}(\kappa)(\Psi(\mathfrak{t}) - \Psi(\kappa))^{\bar{\theta}-1}|\omega(\kappa) - \upsilon(\kappa)|d\kappa. \end{aligned} \tag{28}$$

Applying Lemma 4.1 to (28), we get

$$\begin{aligned} & |\omega(\mathfrak{t}) - \upsilon(\mathfrak{t})| \\ & \le \frac{(\Psi(\bar{\chi}) - \Psi(0))^{\bar{\theta}}}{\Gamma(\bar{\theta}+1)}\varepsilon\Big[1 + \int_0^{\mathfrak{t}}\sum_{m=1}^{\infty}\frac{\mathscr{W}_1^m}{\Gamma(\bar{\theta}m)}\Psi^{'}(\kappa)(\Psi(\mathfrak{t}) - \Psi(\kappa))^{\bar{\theta}m-1}d\kappa\Big] \\ & = \frac{(\Psi(\bar{\chi}) - \Psi(0))^{\bar{\theta}}}{\Gamma(\bar{\theta}+1)}\varepsilon\Big[1 + \sum_{m=1}^{\infty}\frac{\mathscr{W}_1^m}{\Gamma(\bar{\theta}m)}\int_0^{\mathfrak{t}}\Psi^{'}(\kappa)(\Psi(\mathfrak{t}) - \Psi(\kappa))^{\bar{\theta}m-1}d\kappa\Big] \\ & \le \frac{(\Psi(\bar{\chi}) - \Psi(0))^{\bar{\theta}}}{\Gamma(\bar{\theta}+1)}\varepsilon\Big[1 + \sum_{m=1}^{\infty}\frac{\mathscr{W}_1^m}{\Gamma(\bar{\theta}m+1)}(\Psi(\bar{\chi})) - \Psi(0))^{\bar{\theta}m}\Big] \\ & = \frac{\varepsilon(\Psi(\bar{\chi})) - \Psi(0))^{\bar{\theta}}}{\Gamma(\bar{\theta}+1)}E_{\bar{\theta}}(\mathscr{W}_1(\Psi(\bar{\chi}) - \Psi(0))^{\bar{\theta}}). \end{aligned} \tag{29}$$

Put

$$\delta = \frac{(\Psi(\bar{\chi}) - \Psi(0))^{\bar{\theta}}}{\Gamma(\bar{\theta}+1)}E_{\bar{\theta}}(\mathscr{W}_1(\Psi(\bar{\chi}) - \Psi(0))^{\bar{\theta}}). \tag{30}$$

Therefore

$$|\omega(\mathfrak{t}) - \upsilon(\mathfrak{t})| \le \delta\varepsilon. \tag{31}$$

Hence, the problem (1)–(2) is UH stable. □

Theorem 4.2 *If there exists a function continuous function* $\varphi : \mathrm{I} \to \mathrm{I}$ *with* $\varphi(0) = 0$. *Then under the assumption of Theorem 4.1, the problem* (1)–(2) *is GUH stable*

Proof In a same fashion similar to Theorem 4.1, setting $\varphi(\varepsilon) = \delta\varepsilon$ with $\varphi(0) = 0$, we get

$$|\omega(\mathfrak{t}) - \upsilon(\mathfrak{t})| \leq \varphi(\varepsilon). \tag{32}$$

□

In order to prove UHR and GUHR stability, the following hypothesis must be satisfied:
[H2]: There exist an increasing function $f \in \mathcal{C}(\mathrm{I}, \mathbb{R})$ and $\gamma > 0$ such that

$$\mathfrak{J}_{0+}^{\bar{\theta},\Psi} f(\mathfrak{t}) \leq \gamma f(\mathfrak{t}), \quad \mathfrak{t} \in \mathrm{I}.$$

Lemma 4.2 *Let* $\varepsilon > 0$ *and* $\omega(\mathfrak{t}) \in \mathcal{C}(\mathrm{I}, \mathbb{R})$ *be a solution* (13)*. Then*

$$\left|\omega(\mathfrak{t}) - \Phi_\omega - \mathfrak{J}_{0+}^{\bar{\theta},\Psi}\mathscr{G}(\mathfrak{t}, \omega(\mathfrak{t}))\right| \leq \varepsilon\gamma f(\mathfrak{t}). \tag{33}$$

Proof By Remark 4.2, g, $f \in \mathcal{C}(\mathrm{I}, \mathbb{R})$ such that

$$|\mathfrak{D}_{0+}^{\bar{\theta},\Psi}\omega(\mathfrak{t}) - \mathscr{G}(\mathfrak{t}, \omega(\mathfrak{t}))| = |\mathrm{g}(\mathfrak{t})| \leq \varepsilon f(\mathfrak{t}). \tag{34}$$

Operating $\mathfrak{J}_{0+}^{\bar{\theta},\Psi}$ and using the hypothesis [H2], we deduce that

$$|\omega(\mathfrak{t}) - \Phi_\omega - \mathfrak{J}_{0+}^{\bar{\theta},\Psi}\mathscr{G}(\mathfrak{t}, \omega(\mathfrak{t}))| \leq \varepsilon\mathfrak{J}_{0+}^{\bar{\theta},\Psi} f(\mathfrak{t}) \leq \gamma\varepsilon f(\mathfrak{t}). \tag{35}$$

□

Theorem 4.3 *Let* $\varepsilon > 0$ *and* $\omega \in \mathcal{C}(J, \mathbb{R})$ *be a solution* (13) *and* $\mathscr{W}_1\gamma \neq 1$, *then* (1)–(2) *is UHR stable.*

Proof Let $\upsilon(\mathfrak{t})$ be a solution of (1)–(2) and using $\Phi_\upsilon = \Phi_\omega$ Then

$$\upsilon(\mathfrak{t}) = \Phi_\omega + \mathfrak{J}_{0+}^{\bar{\theta},\Psi}\mathscr{G}(\mathfrak{t}, \omega(\mathfrak{t})). \tag{36}$$

By hypothesis [H1] and Lemma 4.2, we get

$$\begin{aligned} |\omega(\mathfrak{t}) - \upsilon(\mathfrak{t})| \leq & \left|\omega(\mathfrak{t}) - \Phi_\omega - \frac{1}{\Gamma(\bar{\theta})}\int_0^{\mathfrak{t}} \Psi^{'}(\kappa)(\Psi(\mathfrak{t}) - \Psi(\kappa))^{\bar{\theta}-1}\mathscr{G}(\kappa, \omega(\kappa))\right| d\kappa \\ & + \frac{1}{\Gamma(\bar{\theta})}\int_0^{\mathfrak{t}} \Psi^{'}(\kappa)(\Psi(\mathfrak{t}) - \Psi(\kappa))^{\bar{\theta}-1}|\mathscr{G}(\kappa, \omega(\kappa)) - \mathscr{G}(\kappa, \upsilon(\kappa))| d\kappa \\ \leq & \gamma\varepsilon f(\mathfrak{t}) + \frac{\mathscr{W}_1}{\Gamma(\bar{\theta})}\int_0^{\mathfrak{t}} \Psi^{'}(\kappa)(\Psi(\mathfrak{t}) - \Psi(\kappa))^{\bar{\theta}-1}|\omega(\kappa) - \upsilon(\kappa)| d\kappa. \end{aligned} \tag{37}$$

Applying Lemma 4.1 to (37) and using hypothesis [H2], we obtain

$$\begin{aligned}|\omega(\mathfrak{t})-\upsilon(\mathfrak{t})| &\leq \gamma\varepsilon f(\mathfrak{t})+\gamma\varepsilon\int_0^{\mathfrak{t}}\sum_{k=1}^{\infty}\frac{\mathscr{W}_1^k}{\Gamma(\bar{\theta}m)}\Psi^{'}(\kappa)(\Psi(\mathfrak{t})-\Psi(\kappa))^{\bar{\theta}k-1}f(\kappa)d\kappa \\ &= \gamma\varepsilon f(\mathfrak{t})+\gamma\varepsilon\Big[\int_0^{\mathfrak{t}}\frac{\mathscr{W}_1}{\Gamma(\bar{\theta})}\Psi^{'}(\kappa)(\Psi(\mathfrak{t})-\Psi(\kappa))^{\bar{\theta}-1}f(\kappa)d\kappa \\ &+ \int_0^{\mathfrak{t}}\frac{\mathscr{W}_1^2}{\Gamma(2\bar{\theta})}\Psi^{'}(\kappa)(\Psi(\mathfrak{t})-\Psi(\kappa))^{2\bar{\theta}-1}f(\kappa)d\kappa+....\Big] \\ &= \gamma\varepsilon f(\mathfrak{t})+\gamma\varepsilon\big[\mathscr{W}_1\mathfrak{I}_{0+}^{\bar{\theta},\Psi}f(\mathfrak{t})+\mathscr{W}_1^2\mathfrak{I}_{0+}^{2\bar{\theta},\Psi}f(\mathfrak{t})+....\big] \\ &\leq \gamma\varepsilon f(\mathfrak{t})+\gamma\varepsilon\big[\mathscr{W}_1\gamma f(\mathfrak{t})+(\mathscr{W}_1\gamma)^2f(\mathfrak{t})+....\big] \\ &= \gamma\varepsilon f(\mathfrak{t})\sum_{k=0}^{\infty}(\mathscr{W}_1\gamma)^k \\ &= \frac{\gamma}{1-\mathscr{W}_1\gamma}\varepsilon f(\mathfrak{t}). \end{aligned} \tag{38}$$

Setting

$$\delta=\frac{\gamma}{1-\mathscr{W}_1\gamma}. \tag{39}$$

From (38) and (39), we have

$$|\omega(\mathfrak{t})-\upsilon(\mathfrak{t})|\leq\delta\varepsilon\rho(\mathfrak{t}).$$

□

Theorem 4.4 *Under the assumption of Theorem 4.3, problem (1)–(2) is GUHR stable.*

Proof In a same fashion similar to Theorem 4.3, setting $\varepsilon=1$, we get

$$|\omega(\mathfrak{t})-\upsilon(\mathfrak{t})|\leq\delta f(\mathfrak{t}). \tag{40}$$

□

5 Example

Example 5.1 Consider the following fractional differential equation involving Ψ-Caputo derivative

$$\mathfrak{D}_0^{\frac{3}{2},\Psi}\upsilon(\mathfrak{t})=\mathfrak{t}+\frac{1}{6}sin\upsilon(\mathfrak{t}),\ \text{for all } \mathfrak{t}\in[0,1], \tag{41}$$

$$\upsilon(0) + \frac{1}{4}\upsilon(\frac{1}{3}) = 0, \upsilon(1) = \frac{1}{2}. \tag{42}$$

Here, $\bar{\theta} = \frac{3}{2}$, $\mathscr{G}\big(\mathfrak{t}, \upsilon(\mathfrak{t})\big) = \mathfrak{t} + \frac{1}{6} sin \upsilon(\mathfrak{t})$, $\mathrm{h}(\upsilon) = \frac{1}{4}\upsilon(\frac{1}{3})$. Then for $\mathfrak{t} \in [0, 1]$,

$$|\mathscr{G}(\mathfrak{t}, \upsilon) - \mathscr{G}(\mathfrak{t}, \upsilon^*)| \leq \frac{1}{6}|\upsilon - \upsilon^*| \text{ and } |\mathrm{h}(\upsilon) - \mathrm{h}(\upsilon^*)| \leq \frac{1}{4}|\upsilon - \upsilon^*|.$$

Therefore $\mathscr{W}_1 = \frac{1}{6}$ and $\mathscr{W}_2 = \frac{1}{4}$. For $\Psi(\mathfrak{t}) = \mathfrak{t}$, we have

$$\mathscr{W}_2 + 2\frac{(\Psi(\bar{\chi}) - \Psi(0))^{\bar{\theta}}}{\Gamma(\bar{\theta} + 1)}\mathscr{W}_1 = \frac{1}{4} + \frac{2(1-0)^{\frac{3}{2}}}{6\Gamma(\frac{5}{2})} = \frac{1}{4} + \frac{4}{9\sqrt{\pi}} < 1.$$

Hence, all the conditions of Theorem 3.1 are satisfied. Thus, by the Theorem 3.1, problem (41)–(42) has unique solution.

6 Concluding Remark

In this research work, the existence and uniqueness of the proposed system have been successfully examined using Banach fixed point theorem under some specific assumptions and conditions. Along with the existence and uniqueness, we established stability results such as UH, GUH, UHR and GUHR in the sense of Ψ-Gronwall inequality. It should be noted that, for different values of Ψ, the Ψ-Caputo fractional derivative reduces to many classical fractional operators such as Caputo [9], Caputo-Hadamard [7], Caputo-Erdélyi-Kober [13] fractional derivative. Thus, we believe that the results derived in this article are general in character and contributes in the theory of fractional differential equations.

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