Forum for Interdisciplinary Mathematics

P. V. Subrahmanyam
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Synergies in Analysis, Discrete Mathematics, Soft Computing and Modelling

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Preface

Mathematics permeates every aspect of our daily life, and it can aptly be called "God's own science". The Forum for Interdisciplinary Mathematics, Stella Maris College, and S.S.N College of Engineering jointly organized an online international conference entitled "Synergies in Computational, Mathematical, Statistical and Physical Sciences (FIM2020: SCMPS2020), from 23–27 November 2020 at Chennai, India, highlighting the interdisciplinary nature of mathematics. This conference attracted about 250 researchers, both eminent and emergent, and comprised numerous keynote addresses, plenary talks, symposia on special areas and parallel sessions of paper presentation. For the convenience of potential authors, several venues for refereed publication of submitted articles were suggested by the conference organizing committee, including the present proceedings in the Springer's *Forum for Interdisciplinary Mathematics* series.

Both the invited articles and submitted papers were broadly grouped under three parts: analysis and modelling, discrete mathematics and applications, and fuzzy set theory and applications. More than fifty papers were submitted and more than 100 referees of international standing evaluated the submissions critically despite the constraints due to the prevalent pandemic. Each paper was at least doubly refereed with the acceptance rate being one out of three. The authors of those papers which could not be accepted were given useful suggestions for improvement which will hopefully help them publish the revised versions elsewhere.

We place on record our thanks to the referees for sparing their valuable time and energy in critically appraising the articles submitted. We also acknowledge the support provided by the management of the *Forum for Interdisciplinary Mathematics* series in publishing the present proceedings.

Chennai, India — P. V. Subrahmanyam
Indore, India — V. Antony Vijesh
Kandi, India — Balasubramaniam Jayaram
Melbourne, Australia — Prakash Veeraraghavan

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About the Editors

P. V. Subrahmanyam is Former Professor and Head of the Department of Mathematics at the Indian Institute of Technology Madras (IIT Madras), India. He received his Ph.D. in Mathematics from the IIT Madras, for his dissertation on "Topics in Fixed-Point Theory" under the supervision of (late) Dr. V. Subba Rao. He received his M.Sc. degree in Mathematics from the IIT Madras and B.Sc. degree in Mathematics from Madras University. He has held several important administrative positions, such as Senior Professor and Head of the Department of Mathematics at the IIT Madras; Founder and Head of the Department of Mathematics at the Indian Institute of Technology Hyderabad (IIT Hyderabad); Executive Chairman of the Association of Mathematics Teachers of India (AMTI); President of the Forum for Interdisciplinary Mathematics (FIM); and then Chief Editor of the *Forum for Interdisciplinary Mathematics* (Springer book series). Before joining the IIT Madras, he served as Faculty Member at Loyola College, Madras University, and Hyderabad Central University. His areas of interest include classical analysis, nonlinear analysis and fixed-point theory, fuzzy set theory, functional equations, and mathematics education. He has published over 70 papers and served on the editorial board of the *Journal of Analysis* and the *Journal of Differential Equations and Dynamical Systems*. He received an award for his outstanding contributions to mathematical sciences in 2004 and the Lifetime Achievement Award from the FIM in 2016. He has given various invited talks at international conferences and completed brief visiting assignments in many countries such as Canada, Czech Republic, Germany, Greece, Japan, Slovak Republic, and the USA. He is also Life Member of the Association of Mathematics Teachers of India, FIM, Indian Mathematical Society, and the Society for Industrial and Applied Mathematics (SIAM).

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Balasubramaniam Jayaram is Professor at the Department of Mathematics at the IIT Hyderabad, India. His main areas of research interest are in approximate reasoning and high-dimensional data analysis. He is on the editorial board of many journals, including *IEEE Transactions on Fuzzy Systems*, *Fuzzy Sets and Systems* and *International Journal of Approximate Reasoning*. He has co-authored a research monograph on *Fuzzy Implications* and is author of more than 80 published papers in refereed international journals and conferences. He is also experienced Research Fellow of the Alexander von Humboldt Foundation.

Prakash Veeraraghavan is Associate Professor at the Department of Computer Science and Information Technology, La Trobe University, Melbourne, Australia. He received his Ph.D. in Mathematics from the IIT Chennai, India. He has more than 25 years of teaching, research, and industry experience. His research interest includes mathematical modeling, graph algorithms, optimization, network protocols and system security, and various areas of cybersecurity. He has published more than 120 papers in various top international conferences and reputable journals. He held a patent on software engineering.

Chapter 1
The Second- and Third-Order Hermitian Toeplitz Determinants for Some Subclasses of Analytic Functions Associated with Exponential Function

P. Gurusamy, R. Jayasankar, and S. Sivasubramanian

2010 AMS Subject Classification: 30C45

1.1 Introduction and Motivations

Let $\mathcal{H}$ denote the class of functions that are analytic in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. If $f \in \mathcal{A}$, then f is analytic in $\mathbb{U}$, and

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots . \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent function. We state that a function f is said to be subordinate to a function g, written as $f \prec g$, if there exists a function ω with $\omega(0) = 0$ and $|\omega(z)| < 1$. If g is univalent in $\mathbb{U}$, then $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$. Let

$$\mathcal{S}^*(\beta) = \left\{ f \in \mathcal{A} : \Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, 0 \le \beta < 1, z \in \mathbb{U} \right\},$$

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and

$$C(\beta)=\left\{h\in\mathcal{A}:\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\beta,0\le\beta<1,z\in\mathbb{U}\right\}$$

denote the renowned classes of starlike functions of order β, and convex functions of order β. Make a note of that $\mathcal{S}^*=\mathcal{S}^*(0)$ is called the class of starlike functions (with respect to the origin). The classes $\mathcal{S}^*(\beta)$ and $C(\beta)$ were introduced and discussed in detail by Robertson [18]. It is to be prominent that in the terminology of subordination, the class $\mathcal{S}^*=\mathcal{S}^*(0)$ can be denoted by $\frac{zf'(z)}{f(z)}\prec\frac{1+z}{1-z}$. Mendiratta et al. [15] considered the class of starlike functions $\mathcal{S}_e^*=\mathcal{S}^*(e^z)$ defined by $\frac{zf'(z)}{f(z)}\prec e^z \quad z\in\mathbb{U}$. In [7], the Hermitian Toeplitz matrix $T_{q,n}(f), q,n\in\mathbb{N}$ of a function $f\in\mathcal{A}$ of the form (1.1) is defined by

$$T_{q,n}(f)=\begin{bmatrix} a_n & a_{n+1} & \dots & a_{n+q-1}\\ \overline{a}_{n+1} & a_n & \dots & a_{n+q-2}\\ . & . & \dots & .\\ \overline{a}_{n+q-1} & \overline{a}_{n+q-2} & \dots & a_n\\ , & & & \end{bmatrix}$$

where $\overline{a}_k=\overline{a_k}$. Let $\left|T_{q,n}(f)\right|$ denote the determinant of $T_{q,n}(f)$.

Recently, Ali et al. [1] introduced the concept of the symmetric Toeplitz determinant $T_q(n)$ for $f\in\mathcal{A}$, defined as follows:

$$T_q(n)[f]=\begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1}\\ a_{n+1} & a_n & \dots & a_{n+q-2}\\ . & . & \dots & .\\ a_{n+q-1} & a_{n+q-2} & \dots & a_n \end{vmatrix}$$

and obtained the estimates of $T_2[n]$, $T_3[1]$, $T_3[2]$ and $T_2[3]$ over few subclasses of $\mathcal{A}$. In recent times, many articles have been published in finding bounds of determinants, whose elements are coefficients of functions in $f\in\mathcal{A}$ or its subclasses, to name a few Hankel matrices, i.e., square matrices which have constant entries along the reverse diagonal and the generalized Zalcman functional $J_{m,n}(f)=a_{m+n-1}-a_m a_n,\ \ m,n\in\mathbb{N}$, are of particular interest. From the collection of the number of articles available in this direction, one may refer to [2–6, 9–14, 17, 19] where the authors have discussed the bounds of second- and third-order Hankel determinants. Cudna et al. [7] established some elementary properties of the Hermitian Toeplitz determinant $\left|T_{q,1}(f)\right|$.

Given a subclass $\mathcal{G}$ of $\mathcal{A}$, let $A_2(\mathcal{G})=\max\{|a_2|:f\in\mathcal{G}\}$. Thus, for $f\in\mathcal{A}$,

$$\left|T_{2,1}(f)\right|=1-|a_2|^2,$$

and therefore, the results are mentioned by a function in $\mathcal{G}$ which is extremal and for the upper bound when f is the identity function.

In the present article, we will get the estimates $\left|T_{2,1}(f)\right|$ and $\left|T_{3,1}(f)\right|$, for few new subclasses $\mathcal{S}_s^*(e^z)$, $\mathcal{C}_s(e^z)$, $\mathcal{S}^*(e^z)$ and $\mathcal{C}(e^z)$,

where $\left|T_{2,1}(f)\right| = 1 - |a_2|^2$ and

$$\left|T_{3,1}(f)\right| = \begin{vmatrix} 1 & a_2 & a_3 \\ \overline{a_2} & 1 & a_2 \\ \overline{a_3} & \overline{a_2} & 2 \end{vmatrix} = 2\Re\left(a_2^2\overline{a}_3\right) - 2|a_2|^2 - |a_3|^2 + 1. \tag{1.2}$$

To prove the main theorems, we need the following Lemmas which we state now. Let $\mathcal{P}$ denote the class of functions p analytic in $\mathbb{U}$ for which $\Re\{p(z)\} > 0$,

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad z \in \mathbb{U}. \tag{1.3}$$

Lemma 1.1 ([16]) *If $p \in \mathcal{P}$ and is of the form (1.3), then*

$$2p_2 = p_1^2 + \left(4 - p_1^2\right)\zeta. \tag{1.4}$$

for some $\zeta \in \overline{\mathbb{U}}$

Lemma 1.2 ([7]) *Let $\mathcal{G}$ be a subclass of the class $\mathcal{A}$, where $A_2(\mathcal{G})$ exists. If the identity is an element of $\mathcal{G}$, then*

$$1 - A_2^2(\mathcal{G}) \le \left|T_{2,1}(f)\right| \le 1. \tag{1.5}$$

Both inequalities are sharp. □

Define the function p by

$$p(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + p_1 z + p_2 z^2 + \ldots,$$

analytic in $\mathbb{U}$ with $p(0) = 1$ and maps $\mathbb{U}$ onto the right half of the ω-plane.

Computing $\omega(z)$ in terms of $p(z)$, we get

$$\omega(z) = \frac{p(z)-1}{p(z)+1}.$$

This implies that

$$\omega(z) = \frac{p_1 z}{2} + \left(\frac{p_2}{2} - \frac{p_1^2}{4}\right)z^2 + \left(\frac{p_3}{2} - \frac{p_1 p_2}{2} + \frac{p_1^3}{8}\right)z^3 + \cdots. \tag{1.6}$$

Now,

$$e^{\omega(z)} = 1 + \omega(z) + \frac{(\omega(z))^2}{2} + \frac{(\omega(z))^3}{6} + \cdots. \tag{1.7}$$

From (1.6) and (1.7), we have

$$e^{\omega(z)} = 1 + \frac{p_1 z}{2} + \left(\frac{p_2}{2} - \frac{p_1^2}{8}\right) z^2 + \left(\frac{p_3}{2} - \frac{p_1 p_2}{4} + \frac{p_1^3}{48}\right) z^3 + \cdots. \tag{1.8}$$

1.2 The Second- and Third-Order Hermitian Toeplitz Determinants for $\mathcal{S}^*(e^z)$, $\mathcal{C}(e^z)$, $\mathcal{S}_s^*(e^z)$ and $\mathcal{C}_s(e^z)$

Definition 1.1 A function $f(z)$, given by (1.1), is said to be in the class $\mathcal{S}^*(e^z)$ if the following condition is satisfied:

$$\frac{zf'(z)}{f(z)} \prec e^z \quad z \in \mathbb{U}. \tag{1.1}$$

If $f \in \mathcal{S}^*(e^z)$, then it follows from (1.1) and using the principle of subordination that

$$\frac{zf'(z)}{f(z)} = e^{\omega(z)} \tag{1.2}$$

From (1.2) and (1.8), equating corresponding coefficients, we get

$$a_2 = \frac{1}{2} p_1 \tag{1.3}$$

and

$$a_3 = \frac{1}{4} p_2 + \frac{1}{16} p_1^2. \tag{1.4}$$

Since

$$\mathcal{A}_2\left(\mathcal{S}^*(e^z)\right) = 1,$$

using Lemma 1.2, we have the following theorem.

Theorem 1.1 *If $f \in \mathcal{S}^*(e^z)$, then*

$$0 \le \left|T_{2,1}(f)\right| \le 1. \tag{1.5}$$

Both inequalities are sharp. Let Ψ_e be a holomorphic function which is the solution of the differential equation:

$$\frac{\varsigma\Psi_e'(\varsigma)}{\Psi_e(\varsigma)} = e^{\varsigma}, \quad \varsigma \in \mathbb{D},\ \Psi_e(0) = 0,\ \Psi_e'(0) = 1,$$

i.e.,

$$\Psi_e(\varsigma) = \varsigma \exp\left(\int_0^{\varsigma} \frac{e^{\zeta}-1}{z} dz\right) = z + z^2 + \frac{3}{4}z^3 + \frac{17}{36}z^4 + \cdots, \quad z \in \mathbb{D}. \quad (1.6)$$

This function $\Psi_e(\varsigma)$ acts as the extremal function for the class $\mathcal{S}^*(e^z)$. □

Now we find the bounds for $\left|T_{3,1}(f)\right|$.

Theorem 1.2 *If $f \in \mathcal{S}^*(e^z)$, then*

$$-\frac{1}{16} \leq \left|T_{3,1}(f)\right| \leq 1. \quad (1.7)$$

Both inequalities are sharp. The result is sharp for the function defined in (1.6). □

Proof Since the class $f \in \mathcal{S}^*(e^z)$ and $\left|T_{3,1}(f)\right|$ are rotationally invariantly, we may assume that $p = p_1 \in [0, 2]$. Thus, we infer from (1.2), (1.3) and (1.4), that

$$\left|T_{3,1}(f)\right| = 1 - \frac{1}{2}p^2 + \frac{7}{256}p^4 - \frac{1}{16}|p_2|^2 + \frac{3}{32}p^2\Re(p_2).$$

Using (1.4), we have

$$\left|T_{3,1}(f)\right| = 1 - \frac{1}{2}p^2 + \frac{15}{256}p^4 - \frac{\left(4-p^2\right)^2|\zeta|^2}{64} - \frac{\left(4-p^2\right)p^2|\zeta|}{32} + \frac{3\left(4-p^2\right)p^2\Re(\zeta)}{64}. \quad (1.8)$$

$$= \psi(p, |\zeta|, \Re(\zeta)) \qquad \zeta \in \overline{\mathbb{U}}.$$

We consider the following several cases.

Case I : Now to show the right-hand side of (1.7).

Using (1.8), we have

$$\left|T_{3,1}(f)\right| \leq \psi(p, |\zeta|) = G\left(p^2, |\zeta|\right), \quad (p, |\zeta|) \in [0, 2] \times [0, 1], \quad (1.9)$$

where $G : [0, 4] \times [0, 1] \to \mathbb{R}$ is defined by

$$G(x, y) = 1 - \frac{x}{2} + \frac{15x^2}{256} - \frac{(4-x)^2y^2}{64} + \frac{(4-x)xy}{64}. \quad (1.10)$$

Case I(i): For $x = 0$ in (1.10), we obtain

$$G(0, y) = 1 - \frac{y^2}{4} \leq 1, \quad y \in [0, 1].$$

Case I (ii): For $x = 4$ in (1.10), we obtain

$$G\left(4, y\right) = -\frac{1}{16} < 1, \quad y \in [0, 1] .$$

Case I (iii): For $y = 0$ in (1.10), we obtain

$$G\left(x, 0\right) = 1 - \frac{x}{2} + \frac{15x^2}{256} = \gamma\left(x\right), \quad x \in [0, 4] .$$

Since

$$\gamma'\left(0\right) = -\frac{1}{2} < 0$$

and

$$\gamma'\left(4\right) = -\frac{1}{32} < 0$$

So γ is decreasing function on [0,4]. Therefore,

$$G\left(x, 0\right) = \gamma\left(x\right) \leq \gamma\left(0\right) = 1, \qquad x \in [0, 4] \ .$$

Case I(iv): For $y = 1$ in (1.10), we obtain

$$G\left(x, 1\right) = \frac{3}{4} - \frac{5x}{16} + \frac{7x^2}{256} = \varphi\left(x\right) \quad x \in [0, 4] \ .$$

Since

$$\varphi'\left(0\right) = -\frac{5}{16} < 0$$

and

$$\gamma'\left(4\right) = -\frac{24}{256} < 0$$

So, φ is decreasing function on [0,4]. Therefore,

$$G\left(x, 1\right) = \varphi\left(x\right) \leq \varphi\left(0\right) = \frac{3}{4} < 1, \qquad x \in [0, 4] \ .$$

Case I(v): Let $(x, y) \in (0, 4) \times (0, 1)$. Then

$$\frac{\partial G}{\partial y} = -\frac{(4-x)^2 2y}{64} + \frac{(4-x)\,x}{64} = 0$$

if and only if

$$y = \frac{x}{2\,(x-4)}.$$

Solving the equation

$$\frac{\partial G}{\partial x}\left(x, \frac{x}{2(x-4)}\right) = 0,$$

the solution does not belong to (0,4). Therefore G has no critical points in $(0, 4) \times (0, 1)$.

Hence, from cases from V(i) to V(v), it follows that

$$G(x, y) \leq 1 \qquad (x, y) \in [0, 4] \times [0, 1].$$

which proves the upper bound in (1.7).
Case II: Now we prove the lower bound in (1.7).

$$\psi(p, |\zeta|, \Re(\zeta)) \geq \psi(p, |\zeta|, |\zeta|) \geq \psi(p, 1, 1) = \psi\left(p^2\right), \tag{1.11}$$

for $p \in [0, 2]$ and $\zeta \in \overline{\mathbb{U}}$, where

$$\psi(x) = \frac{3}{4} - \frac{5x}{16} + \frac{7x^2}{256} \quad x \in [0, 4].$$

Since

$$\psi'(0) = -\frac{5}{16} < 0$$

and

$$\psi'(4) = -\frac{3}{32} < 0.$$

So, ψ is decreasing function on [0,4]. Therefore,

$$\psi(x) \geq \psi(4) = -\frac{1}{16}, \qquad x \in [0, 4]. \tag{1.12}$$

This completes the proof of Theorem 1.2. □

Definition 1.2 A function $f(z)$, given by (1.1), is said to be in the class $C(e^z)$ if the following condition is satisfied:

$$1 + \frac{zf''(z)}{f'(z)} \prec e^z \quad z \in \mathbb{U}. \tag{1.13}$$

Since

$$\mathcal{A}_2\left(C(e^z)\right) = \frac{1}{2}.$$

Using the Lemma1.2, we have the following theorem.

Theorem 1.3 *If $f \in C(e^z)$, then*

$$\frac{3}{4} \leq \left|T_{2,1}(f)\right| \leq 1.$$

Both inequalities are sharp.

Now we find the bounds for $\left|T_{3,1}(f)\right|$.

Theorem 1.4 *If $f \in C(e^z)$, then*

$$\frac{9}{16} \leq \left|T_{3,1}(f)\right| \leq 1. \tag{1.14}$$

Both inequalities are sharp.

Proof Let $f \in C(e^z)$, then it follows from (1.13) and using the principle of subordination that

$$1 + \frac{zf''(z)}{f'(z)} = e^{\omega(z)} \tag{1.15}$$

From (1.8) and (1.15), equating coefficients, we obtain

$$a_2 = \frac{1}{4}p_1 \tag{1.16}$$

and

$$a_3 = \frac{1}{12}p_2 + \frac{1}{48}p_1^2. \tag{1.17}$$

As explained earlier, as the class $C(e^z)$and $\left|T_{3,1}(f)\right|$ are rotationally invariantly, we may assume that
$p = p_1 \in [0, 2]$. Thus from (1.2), (1.16) and (1.17), gives

$$\left|T_{3,1}(f)\right| = 1 - \frac{1}{8}p^2 + \frac{5}{2304}p^4 - \frac{1}{144}|p_2|^2 + \frac{1}{144}p^2\text{Re}(p_2).$$

By virtue of (1.4), we have

$$\left|T_{3,1}(f)\right| = 1 - \frac{p^2}{8} + \frac{p^4}{256} - \frac{\left(4-p^2\right)^2|\zeta|^2}{576} - \frac{p^2\left(4-p^2\right)|\zeta|}{288} + \frac{p^2\left(4-p^2\right)\Re(\zeta)}{288}. \tag{1.18}$$

$$= \psi(p, |\zeta|, \Re(\zeta)) \qquad \zeta \in \overline{\mathbb{U}}.$$

We consider the following several cases.
Case III: Now to prove R H S of (1.14).
Using (1.18), we have

$$\left|T_{3,1}(f)\right| \leq \psi(p, |\zeta|, -|\zeta|) = G\left(p^2, |\zeta|\right), \quad (p, |\zeta|) \in [0, 2] \times [0, 1], \tag{1.19}$$

where $G : [0, 4] \times [0, 1] \to \mathbb{R}$ is defined by

$$G(x, y) = 1 - \frac{x}{8} + \frac{x^2}{256} - \frac{(4-x)^2 y^2}{576} - \frac{(4-x)\, xy}{144}. \tag{1.20}$$

Case III(i): For$x = 0$ in (1.20), we obtain

$$G(0, y) = 1 - \frac{y^2}{36} \leq 1, \quad y \in [0, 1].$$

Case III(ii): For$x = 4$ in (1.20), we obtain

$$G(4, y) = \frac{9}{16} < 1, \quad y \in [0, 1].$$

Case III(iii): For$y = 0$ in (1.20), we obtain

$$G(x, 0) = 1 - \frac{x}{8} + \frac{x^2}{256} = \gamma(x), \quad x \in [0, 4].$$

Since

$$\gamma'(0) = -\frac{1}{8} < 0$$

and

$$\gamma'(4) = -\frac{3}{32} < 0$$

So, γ is decreasing function on [0,4]. Therefore,

$$G(x, 0) = \gamma(x) \leq \gamma(0) = 1, \quad x \in [0, 4] .$$

Case III(iv): For $y = 1$ in (1.20), we obtain

$$G(x, 1) = \frac{35}{36} - \frac{5x}{36} + \frac{21x^2}{2304} = \varphi(x) \quad x \in [0, 4] .$$

Since

$$\varphi'(0) = -\frac{5}{36} < 0$$

and

$$\varphi'(4) = -\frac{19}{288} < 0$$

So, φ is decreasing function on [0,4]. Therefore,

$$G(x,1)=\varphi(x)\le\varphi(0)=\frac{35}{36}<1,\qquad x\in[0,4]\ .$$

Case III(v): Let $(x,y)\in(0,4)\times(0,1)$. Then

$$\frac{\partial G}{\partial y}=-\frac{(4-x)^2 2y}{576}-\frac{(4-x)\,x}{144}=0$$

if and only if

$$y=\frac{2x}{(x-4)}.$$

Solving the equation

$$\frac{\partial G}{\partial x}\left(x,\frac{2x}{x-4}\right)=0,$$

the solution is not in (0, 4). Thus, G has no critical points in $(0,4)\times(0,1)$.

Hence, from cases from VII(i) to VII(v), it follows that

$$G(x,y)\le 1\qquad (x,y)\in[0,4]\times[0,1]\,.$$

which in view of (1.19), proves the upper bound in (1.14).
Case IV: Now we prove the lower bound in (1.14).

$$\psi(p,|\zeta|,\mathrm{Re}(\zeta))\ge\psi(p,|\zeta|,|\zeta|)\ge\psi(p,1,1)=\psi\left(p^2\right),\tag{1.21}$$

for $p\in[0,2]$ and $\zeta\in\overline{\mathbb{U}}$, where

$$\psi(x)=\frac{35}{36}-\frac{x}{9}+\frac{5x^2}{2304}\quad x\in[0,4]\ .$$

Since

$$\psi'(0)=-\frac{1}{9}<0$$

and

$$\psi'(4)=-\frac{3}{32}<0.$$

So, ψ is decreasing function on [0, 4]. Therefore,

$$\psi(x)\ge\psi(4)=\frac{9}{16},\qquad x\in[0,4]\,.\tag{1.22}$$

Let Ψ_e be a holomorphic function which is the solution of the differential equation

$$1+\frac{\varsigma\Psi_e''(\varsigma)}{\Psi_e'(\varsigma)}=\mathrm{e}^{\varsigma},\quad \varsigma\in\mathbb{D},\ \Psi_e(0)=0,\ \Psi_e'(0)=1. \tag{1.23}$$

This function $\Psi_e(\varsigma)$ satisfying (1.23) acts as the extremal function for the class $C^*(e^z)$. This completes the proof of Theorem 1.4. □

Definition 1.3 A function $f(z)$, given by (1.1), is said to be in the class $\mathcal{S}_s^*(e^z)$ if the following condition is satisfied:

$$\frac{2\left[zf'(z)\right]}{f(z)-f(-z)}\prec e^z \quad z\in\mathbb{U}. \tag{1.24}$$

If $f\in\mathcal{S}_s^*(e^z)$, then it follows from (1.24) and using the principle of subordination that

$$\frac{2\left[zf'(z)\right]}{f(z)-f(-z)}=e^{\omega(z)} \tag{1.25}$$

From (1.6) and (1.7) and (1.25), (1.8) by equating corresponding coefficients, we obtain

$$a_2=\frac{1}{4}p_1 \tag{1.26}$$

and

$$a_3=\frac{1}{4}p_2-\frac{1}{16}p_1^2. \tag{1.27}$$

Since

$$\mathcal{A}_2\left(\mathcal{S}_s^*(e^z)\right)=\frac{1}{2}$$

Using the Lemma 1.2, we have the following theorem.

Theorem 1.5 *If $f\in\mathcal{S}_s^*(e^z)$, then*

$$\frac{3}{4}\le\left|T_{2,1}(f)\right|\le 1. \tag{1.28}$$

Both inequalities are sharp. □

Now we find the bounds for $\left|T_{3,1}(f)\right|$.

Theorem 1.6 *If $f\in\mathcal{S}_s^*(e^z)$, then*

$$\frac{9}{16}\le\left|T_{3,1}(f)\right|\le 1. \tag{1.29}$$

Both inequalities are sharp.

Proof Since the class $f\in\mathcal{S}_s^*(e^z)$ and $\left|T_{3,1}(f)\right|$ are rotated invariantly, we may assume that $p=p_1\in[0,2]$. Thus, from (1.2), (1.26), and (1.27), gives

$$\left|T_{3,1}(f)\right| = 1 - \frac{1}{8}p^2 - \frac{3}{256}p^4 - \frac{1}{16}|p_2|^2 + \frac{1}{16}p^2\mathrm{Re}(p_2).$$

Using (1.4), we have

$$\left|T_{3,1}(f)\right| = 1 - \frac{1}{8}p^2 + \frac{1}{256}p^4 - \frac{\left(4-p^2\right)^2|\zeta|^2}{64} + \frac{p^2\left(4-p^2\right)|\zeta|}{32} + \frac{p^2\left(4-p^2\right)\Re(\zeta)}{32}. \tag{1.30}$$

$$= \psi(p, |\zeta|, \Re(\zeta)) \qquad \zeta \in \overline{\mathbb{U}}.$$

We consider the following several cases.
Case V: Now to prove right-hand side of (1.28).

Using (1.29), we have

$$\left|T_{3,1}(f)\right| \le \psi(p, |\zeta|) = G\left(p^2, |\zeta|\right), \quad (p, |\zeta|) \in [0,2] \times [0,1], \tag{1.31}$$

where $G : [0,4] \times [0,1] \to \mathbb{R}$ is defined by

$$G(x, y) = 1 - \frac{x}{8} + \frac{x^2}{256} - \frac{(4-x)^2 y^2}{64}. \tag{1.32}$$

Case V (i): For $x = 0$ in (1.32), we obtain

$$G(0, y) = 1 - \frac{y^2}{4} \le 1, \quad y \in [0,1].$$

Case V (ii): For $x = 4$ in (1.32), we obtain

$$G(4, y) = \frac{9}{16} < 1, \quad y \in [0,1].$$

Case V (iii): For $y = 0$ in (1.32), we obtain

$$G(x, 0) = 1 - \frac{x}{8} + \frac{x^2}{256} = \gamma(x), \quad x \in [0,4].$$

Since

$$\gamma'(0) = -\frac{1}{8} < 0$$

and

$$\gamma'(4) = -\frac{3}{32} < 0$$

So, γ is decreasing function on [0,4]. Therefore,

$$G(x, 0) = \gamma(x) \le \gamma(0) = 1, \qquad x \in [0,4].$$

Case V (iv): For $y = 1$ in (1.32), we obtain

$$G(x,1) = \frac{3}{4} - \frac{3x^2}{256} = \varphi(x) \quad x \in [0,4] \ .$$

Since

$$\varphi'(0) \le 0$$

and

$$\gamma'(4) = -\frac{3}{32} < 0$$

So, φ is decreasing function on $[0, 4]$. Therefore,

$$G(x,1) = \varphi(x) \le \varphi(0) = \frac{3}{4} < 1, \qquad x \in [0,4] \ .$$

Case V (v): Let $(x, y) \in (0, 4) \times (0, 1)$. Then

$$\frac{\partial G}{\partial y} = -\frac{(4-x)^2 y}{32} = 0$$

if and only if

$$y = 0.$$

Hence,

$$\frac{\partial G}{\partial x}(x,0) = 0.$$

if and only if

$$x = 16 \tag{1.33}$$

However, the solution of (1.33) in $\mathbb{R}$ is $x = 16 \notin (0, 4)$
Thus, G has no critical points in $(0, 4) \times (0, 1)$.
Hence, from cases from I(i) to I(v), it follows that

$$G(x,y) \le 1 \qquad (x,y) \in [0,4] \times [0,1] \,.$$

which establishes the upper bound in (1.29).
Case VI: Now we prove the lower bound in (1.29).

$$\psi(p, |\zeta|, \mathrm{Re}(\zeta)) \ge \psi(p, |\zeta|, -|\zeta|) \ge \psi(p, 1, -1) = \psi\left(p^2\right), \tag{1.34}$$

for $p \in [0, 2]$ and $\zeta \in \overline{\mathbb{U}}$, where

$$\psi(x) = \frac{3}{4} - \frac{3x^2}{256} \quad x \in [0,4].$$

Since

$$\psi'(0) \leq 0$$

and

$$\psi'(4) = -\frac{3}{32} < 0$$

So, ψ is decreasing function on [0,4]. Therefore,

$$\psi(x) \geq \psi(4) = \frac{9}{16}, \qquad x \in [0,4]. \tag{1.35}$$

Let Ψ_e be a holomorphic function which is the solution of the differential equation

$$\frac{2\varsigma\Psi_e'(\varsigma)}{\Psi_e(\varsigma) + \Psi_e(-\varsigma)} = \mathrm{e}^{\varsigma}, \quad \varsigma \in \mathbb{D},\ \Psi_e(0) = 0,\ \Psi_e'(0) = 1. \tag{1.36}$$

This function $\Psi_e(\varsigma)$ satisfying (1.36) acts as the extremal function for the class $\mathcal{S}_s^*(e^z)$. This completes the proof of Theorem 1.4. This essentially completes the proof of Theorem 1.6. □

Definition 1.4 A function $f(z)$, given by (1.1), is said to be in the class $C_s(e^z)$ if the following condition is satisfied:

$$\frac{2\left[zf'(z)\right]'}{\left(f(z) - f(-z)\right)'} \prec e^z \quad z \in \mathbb{U}. \tag{1.37}$$

Since

$$\mathcal{A}_2\left(C_s\left(e^z\right)\right) = \frac{1}{4}.$$

By Lemma 1.2, we have the following theorem.

Theorem 1.7 *If $f \in C_s(e^z)$, then*

$$\frac{15}{16} \leq \left|T_{2,1}(f)\right| \leq 1.$$

Both inequalities are sharp. □

Now we find the bounds for $\left|T_{3,1}(f)\right|$. By adopting a similar technique as in Theorem 1.6, we obtain the following theorem.

Theorem 1.8 *If $f \in C_s(e^z)$, then*

$$\frac{253}{288} \leq \left|T_{3,1}(f)\right| \leq 1. \tag{1.38}$$

Let Ψ_e be a holomorphic function which is the solution of the differential equation

$$\frac{2(\varsigma\Psi_e'(\varsigma))'}{(\Psi_e(\varsigma)+\Psi_e(-\varsigma))'} = \mathrm{e}^{\varsigma}, \quad \varsigma \in \mathbb{D},\ \Psi_e(0)=0,\ \Psi_e'(0)=1. \tag{1.39}$$

This function $\Psi_e(\varsigma)$ satisfying (1.39) acts as the extremal function for the class $C_s\,(e^z)$.

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Chapter 2
Some Results on a Starlike Class with Respect to (j, m)-Symmetric Functions

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2.1 Introduction

We start with the following notations. The class of maps that are holomorphic on the unit open disk $\Delta = \{\zeta : \zeta \in \mathbb{C} \text{ with } |\zeta| < 1\}$ and of form

$$h(\zeta) = \zeta + \sum_{l=2}^{\infty} a_l \zeta^l \tag{2.1}$$

is denoted by $\mathfrak{A}$. The subclass of all functions of $\mathfrak{A}$ that are univalent in Δ is denoted by $\mathcal{S}$. A function $h(\zeta)$ in the class $\mathfrak{A}$ is said to be in the class S^* of starlike functions of zero order in Δ if and only if $\Re\left\{\zeta \frac{h'(\zeta)}{h(\zeta)}\right\} > 0$ for $\zeta \in \Delta$.

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Definition 2.1 Let $m \in \mathbb{Z}^+$. We say that a domain Ω is an m-fold symmetric, if a rotation of the domain Ω around the origin through an angle $\frac{2\pi}{m}$ radians takes Ω onto Ω. A mapping h is said to be an m-fold symmetric in Δ if for every ζ in Δ

$$h(e^{\frac{2\pi i}{m}}\zeta) = e^{\frac{2\pi i}{m}}h(\zeta).$$

Definition 2.2 Let $m \in \mathbb{N}$, $\varepsilon = (e^{\frac{2\pi i}{m}})$ and $j = 0, 1, \ldots, m-1, m > 2$. A mapping $h : \Delta \mapsto \mathbb{C}$ is called (j, m)-symmetrical if

$$h(\varepsilon\zeta) = \varepsilon^j h(\zeta) = (e^{\frac{2\pi ij}{m}})h(\zeta), \qquad \zeta \in \Delta.$$

Theorem 2.1 ([3]) *For any mapping $h : \Omega \mapsto \mathbb{C}$ and Ω an m-fold symmetric set, there is a sequence of (j, m)-symmetrical functions $h_{j,m}$,*

$$h(\zeta) = \sum_{j=0}^{m-1} h_{j,m}(\zeta),$$

where

$$h_{j,m}(\zeta) = \frac{1}{m}\sum_{\nu=0}^{m-1} \varepsilon^{-\nu j} h(\varepsilon^\nu \zeta), \tag{2.2}$$

$$(h \in \mathfrak{A};\ m = 1, 2, ...;\ j = 0, 1, \ldots, m-1)$$

From (2.2), we obtain

$$h_{j,m}(\zeta) = \frac{1}{m}\sum_{\nu=0}^{m-1} \varepsilon^{-\nu j} h(\varepsilon^\nu \zeta) = \frac{1}{m}\sum_{\nu=0}^{m-1} \varepsilon^{-\nu j}\left(\sum_{l=1}^{\infty} a_l(\varepsilon^\nu \zeta)^l\right),$$

then

$$h_{j,m}(\zeta) = \sum_{l=1}^{\infty} \delta_{l,j} a_l \zeta^l, \quad a_1 = 1$$

$$\delta_{l,j} = \frac{1}{m}\sum_{\nu=0}^{m-1} \varepsilon^{(l-j)\nu} = \begin{cases} 1, & l = lm + j; \\ 0, & l \neq lm + j, \end{cases} \tag{2.3}$$

Let $S^{*(j,m)}$ denote the subclass of S consisting of the mappings given by (2.1) such that $\Re\left\{\frac{\zeta h'(\zeta)}{h_{j,m}(\zeta)}\right\} > 0$, where $h_{j,m}(\zeta)$ is defined by (2.2) and $\zeta \in \Delta$. Such functions are said to be starlike w.r.to (j, m)-symmetric functions.

Definition 2.3 A function $h \in \mathfrak{A}$ of the form (2.1) is in class $S_s^{*(j,m)}(\mu, \eta)$, if it satisfies the following inequality:

$$\left|\frac{\zeta h'(\zeta)}{h_{j,m}(\zeta)} - 1\right| < \eta \left|\frac{\mu\zeta h'(\zeta)}{h_{j,m}(\zeta)} + 1\right|,$$

where $h_{j,m}(\zeta)$ is defined by (2.2), $\zeta \in \Delta$, $0 \le \mu \le 1$ and $0 < \eta \le 1$.

This class generalizes various classes studied by Sudharsan et al. [5] and EL-Ashwa and Thomas [4].

2.2 Important Results

We recall the following lemma due to Lakshminarasimhan [2].

Lemma 2.1 *If $G(\zeta)$ be holomorphic in Δ and satisfies the condition*

$$\left|\frac{1 - G(\zeta)}{1 + \mu G(\zeta)}\right| < \eta \tag{2.4}$$

$\zeta \in \Delta$, $0 \le \mu < 1$ *with* $G(0) = 1$, *then*

$$G(\zeta) = \frac{1 - \zeta\lambda(\zeta)}{1 + \mu\zeta\lambda(\zeta)}, \tag{2.5}$$

where $\lambda(\zeta)$ is holomorphic in Δ and $|\lambda(\zeta)| \le \eta$ for $\zeta \in \Omega$. Conversely, a function $G(\zeta)$ given by (2.5) is holomorphic in Δ and satisfies (2.4).

Lemma 2.2 *If h and g are in class $\mathcal{S}$ such that*

$$\left|\frac{\zeta h'(\zeta)}{g_{j,m}(\zeta)} - 1\right| < \eta \left|\frac{\mu\zeta h'(\zeta)}{g_{j,m}(\zeta)} + 1\right|, \tag{2.6}$$

where $0 \le \mu \le 1$ and $0 < \eta \le 1$, $\zeta \in \Delta$ and $g_{j,m}(\zeta)$ is defined by $g_{j,m}(\zeta) = \sum_{l=1}^{\infty} \delta_{l,j} b_l \zeta^l$, $a_1 = 1$. Then, for $l \ge 2$,

$$|la_l - \delta_{l,j} b_l|^2 \le 2(\mu\eta^2 + 1) \sum_{m=1}^{l-1} m\delta_{m,j} |a_m||b_m|, \quad (|a_1| = |b_1| = 1). \tag{2.7}$$

Proof We shall follow the method given by Clunie et al. [1] and Thomas [6]. By the Lemma 2.1, we have

$$\frac{\zeta h'(\zeta)}{g_{j,m}(\zeta)} = \frac{1 - \zeta\lambda(\zeta)}{1 + \mu\zeta\lambda(\zeta)},$$

where $\lambda(\zeta)$ is holomorphic in Δ and $|\lambda(\zeta)| \le \eta$ for $\zeta \in \Delta$. Then

$$\zeta h'(\zeta) = g_{j,m}(\zeta)\left[\frac{1 - \zeta\lambda(\zeta)}{1 + \mu\zeta\lambda(\zeta)}\right]$$

or

$$[\mu\zeta h'(\zeta) + g_{j,m}(\zeta)]\zeta\lambda(\zeta) = g_{j,m}(\zeta) - \zeta h'(\zeta).$$

Now, if

$$\varphi(\zeta) = \zeta\lambda(\zeta) = \sum_{l=1}^{\infty} t_l \zeta^l,$$

then

$$|\varphi(\zeta)| = \eta|\zeta| \;\; for \;\; \zeta \in \Delta.$$

Therefore,

$$\left[\sum_{l=1}^{\infty}(\mu l a_l + \delta_{l,j} b_l)\zeta^l\right]\left(\sum_{l=1}^{\infty} t_l \zeta^l\right) = \sum_{l=1}^{\infty}(\delta_{l,j} b_l - l a_l)\zeta^l. \tag{2.8}$$

By comparing the coefficients of ζ^l in (2.8), we get

$$\delta_{l,j} b_l - l a_l = (\mu + \delta_{1,j})t_{l-1} + (2\mu a_2 + \delta_{2,j} b_2)t_{l-2} + \cdots + [(l-1)\mu a_{l-1} + \delta_{(l-1),j} b_{l-1}]t_1.$$

Thus, the combination of coefficients on the r.h.s of (2.8) depends only on the combination of coefficients $(2\mu a_2 + \delta_{2,j} b_2)$, ..., $[(l-1)\mu a_{l-1} + \delta_{(l-1),j} b_{l-1}]$ on the l.h.s. Further, for $l \geq 2$,

$$\left[(\mu + \delta_{1,j})\zeta + \sum_{m=2}^{l-1}(\mu m a_m + \delta_{m,j} b_m)\zeta^m\right]\varphi(\zeta) = \sum_{m=2}^{l}(\delta_{m,j} b_m - m a_m)\zeta^m + \sum_{m=l+1}^{\infty} c_m \zeta^m \quad \text{(say)}. \tag{2.9}$$

Squaring on both sides of Eq. (2.9), integrating over $|\zeta| = r < 1$ and making use of the relation $|\varphi(\zeta)| \leq \eta|\zeta|$, we obtain

$$\sum_{m=2}^{l} |m a_m - \delta_{m,j} b_m|^2 r^{2m} + \sum_{m=l+1}^{\infty} |c_m|^2 r^{2m} < \eta^2 \left[(\mu + \delta_{1,j}^2)r^2 + \sum_{m=2}^{l-1} |\mu m a_m + \delta_{m,j} b_m|^2 r^{2m}\right].$$

Letting $r \longrightarrow 1$ in the l.h.s of the above inequality, we obtain

$$\sum_{m=2}^{l} |m a_m - \delta_{m,j} b_m|^2 < \eta^2(\mu + \delta_{1,j})^2 + \eta^2 \sum_{m=2}^{l-1} |\mu m a_m + \delta_{m,j} b_m|^2,$$

which implies

$$|la_l - \delta_{l,j}b_l| \leq \eta^2(\mu + \delta_{1,j})^2 + \eta^2 \sum_{m=2}^{l-1} |\mu m a_m + \delta_{m,j}b_m|^2 - \sum_{m=2}^{l-1} |ma_m - \delta_{m,j}b_m|^2.$$

$$\leq \eta^2(\mu + \delta_{1,j})^2 + (\mu^2\eta^2 - 1) \sum_{m=2}^{l-1} m^2|a_m|^2 + (\eta^2 - 1) \sum_{m=2}^{l-1} \delta_{m,j}|b_m|^2$$

$$+2\mu\eta^2 \sum_{m=2}^{l-1} |ma_m\delta_{m,j}b_m| + 2 \sum_{m=2}^{l-1} |ma_m\delta_{m,j}b_m|,$$

or

$$|la_l - \delta_{l,j}b_l| \leq 2\mu\eta^2 \sum_{m=1}^{l-1} m\delta_{m,j}|a_m||b_m| + 2 \sum_{m=1}^{l-1} |m\delta_{m,j}|a_m||b_m|,$$

where $(|a_1| = |b_1| = 1)$, since $0 \leq \mu \leq 1$, $0 < \eta \leq 1$. □

When $j = m = 1$, we obtain the following useful result achieved by Sudharsan et al. [5].

Corollary 2.1 *If h and g are in the class $\mathcal{S}$ and satisfy*

$$\left| \frac{\zeta h'(\zeta)}{g(\zeta)} - 1 \right| < \eta \left| \frac{\mu\zeta h'(\zeta)}{g(\zeta)} + 1 \right|,$$

where $0 \leq \mu \leq 1$ *and* $0 < \eta \leq 1$, $\zeta \in \Delta$ *and* $g(\zeta)$ *is defined by* $g(\zeta) = \sum_{l=1}^{\infty} b_l\zeta^l$, $a_1 = 1$, *then for* $l \geq 2$

$$|la_l - b_l|^2 \leq 2(\mu\eta^2 + 1) \sum_{m=1}^{l-1} m|a_m||b_m|, \quad (|a_1| = |b_1| = 1).$$

Theorem 2.2 *If h and g are in S and are such as in Lemma 2.2, then for* $l \geq 2$

$$|la_l - \delta_{l,j}b_l|^2 \leq 2(\mu\eta^2 + 1)CA(1 - \frac{1}{l}, h_{j,m})^{\frac{1}{2}} A(1 - \frac{1}{l}, g_{j,m})^{\frac{1}{2}},$$

where $A(r, h_{j,m})$ *is the area bounded by* $h(|\zeta| = r)$ *and C is a constant.*

Proof From (2.7), we have the following inequality:

$$|la_l - \delta_{l,j}b_l|^2 \leq 2(\mu\eta^2 + 1) \sum_{m=1}^{l-1} m\delta_{m,j}|a_m||b_m|, \quad (|a_1| = |b_1| = 1).$$

By the Cauchy–Schwarz inequality for $(0 < r < 1)$, we have

$$|la_l - \delta_{l,j} b_l|^2 \le 2\mu\eta^2 \left(\sum_{m=1}^{l-1} m\delta_{m,j}|a_m|^2\right)^2 \left(\sum_{m=1}^{l-1} m\delta_{m,j}|b_m|^2\right)^2$$
$$+ 2\left(\sum_{m=1}^{l-1} m\delta_{m,j}|a_m|^2\right)^2 \left(\sum_{m=1}^{l-1} m\delta_{m,j}|b_m|^2\right)^2$$

$$\le \frac{2\mu\eta^2}{r^{2l}} \left(\sum_{m=1}^{l-1} m\delta_{m,j}|a_m|^2 r^{2m}\right)^2 \left(\sum_{m=1}^{l-1} m\delta_{m,j}|b_m|^2 r^{2m}\right)^2$$

$$+\frac{2}{r^{2l}} \left(\sum_{m=1}^{l-1} m\delta_{m,j}|a_m|^2 r^{2m}\right)^2 \left(\sum_{m=1}^{l-1} m\delta_{m,j}|b_m|^2 r^{2m}\right)^2.$$

$$\le \frac{2\mu\eta^2}{\pi r^{2l}} A(r, h_{j,m})^{\frac{1}{2}} A(r, g_{j,m})^{\frac{1}{2}} + \frac{2}{\pi r^{2l}} A(r, h_{j,m})^{\frac{1}{2}} A(r, g_{j,m})^{\frac{1}{2}}.$$

Since $A(r, h_{j,m}) = \pi \sum_{m=l-1}^{\infty} m\delta_{m,j}|a_m|^2 r^{2m}$.

By choose $r = 1 - \frac{1}{l}$ for $l \ge 2$, we obtained the result. □

Remark

- When $j = m = 1$, we get Theorem 2.1 in [5].
- When $j = m = \mu = \eta = 1$, we get Theorem 1 in [4].

Theorem 2.3 *If h is in $S_s^{*(j,k)}(\mu, \eta)$ with $\mu\eta < 1$, then $a_l = O(\frac{1}{l})$ as $l \longrightarrow \infty$.*

Proof One can observe that, when $\mu\eta < 1$, for $h \in S_s^{*(j,m)}(\mu, \eta)$, $\frac{\zeta h'(\zeta)}{h_{j,m}(\zeta)}$ is bounded. Now, we shall prove that

$$(l - \delta_{l,j})^2 |a_l|^2 < 2(1 - \mu\eta^2)(\delta_{1,j} + \sum_{m=2}^{l-1} m\delta_{m,j}|a_m|^2) \quad (|a_1| = |b_1| = 1).$$

If $h \in S_s^{*(j,m)}(\mu, \eta)$ is given by (2.1), then by the Lemma 2.1

$$\frac{\zeta h'(\zeta)}{h_{j,m}(\zeta)} = \frac{1 - \zeta\lambda(\zeta)}{1 + \mu\zeta\lambda(\zeta)},$$

where $\lambda(\zeta)$ is holomorphic in Δ and $|\lambda(\zeta) < \eta|$ for $\zeta \in \Delta$. Therefore,

$$\left[\mu\zeta h'(\zeta) - h_{j,m}(\zeta)\right] \zeta\lambda(\zeta) = h_{j,m}(\zeta) - \zeta h'(\zeta).$$

Now, if

$$\zeta\varphi(\zeta)=\zeta\lambda(\zeta)=\sum_{l=1}^{\infty}t_l\zeta^l,$$

then

$$|\varphi(\zeta)|\leq\eta|\zeta| \text{ for } \zeta\in\Delta.$$

Therefore,

$$\left[\mu\zeta+\sum_{l=2}^{\infty}\mu l a_l\zeta^l-\sum_{l=1}^{\infty}\delta_{l,j}a_l\zeta^l\right]\zeta\lambda(\zeta)=\sum_{l=1}^{\infty}\delta_{l,j}a_l\zeta^l-\zeta-\sum_{l=2}^{\infty}la_l\zeta^l.$$

This implies

$$\left[\sum_{l=1}^{\infty}(\mu l-\delta_{l,j})a_l\zeta^l\right]\left(\sum_{l=1}^{\infty}t_l\zeta^l\right)=\sum_{l=1}^{\infty}(\delta_{l,j}-l)a_l\zeta^l. \tag{2.10}$$

We equate the coefficients of ζ^l on both sides in the above equation and we obtain

$$(\delta_{l,j}-l)a_l=(\mu+\delta_{1,j}t_{l-1}+(2\mu+\delta_{2,j})a_2t_{l-2}+\cdots+((l-1)\mu+\delta_{(l-1),j}a_{l-1})t_1.$$

Thus, the combination of coefficients on the r.h.s of (2.10) depends only upon the combination of coefficients

$$(2\mu+\delta_{2,j})a_2t_{l-2},\ ...,\ ((l-1)\mu+\delta_{(l-1),j}a_{l-1})t_1.$$

on the l.h.s. Hence, for $l\geq 2$, it can be written as

$$\left[(\mu-\delta_{1,j})+\sum_{m=2}^{\infty}(\mu m-\delta_{m,j})a_m\zeta^m\right]\varphi(\zeta)=\sum_{m=2}^{l}(\delta_{m,j}-m)a_m\zeta^m + \sum_{m=l+1}^{\infty}|c_m|\zeta^m, \quad \text{(say)}. \tag{2.11}$$

Upon squaring both sides of Eq. (2.11) and integrating over $|\zeta|=r<1$ and using the relation $|\varphi(\zeta)|\leq\eta|\zeta|$, we get

$$\sum_{m=2}^{l}|m-\delta_{m,j}|^2|a_m|^2r^{2m}+\sum_{m=l+1}^{\infty}|c_m|^2r^{2m} < \eta^2\left[(\mu-\delta_{1,j})^2+\sum_{m=2}^{l-1}|\mu m-\delta_{m,j}|^2|a_m|^2r^{2m}\right].$$

Letting $r \longrightarrow 1$ in the l.h.s of above inequality yields

$$\sum_{m=2}^{l} |m-\delta_{m,j}|^2|a_m|^2 \le \eta^2\left[(\mu-\delta_{1,j})^2+\sum_{m=2}^{l-1}|\mu m-\delta_{m,j}|^2|a_m|^2\right].$$

This implies

$$\left|l-\delta_{l,j}\right|^2|a_m|^2 \le \eta^2\left[(\mu-\delta_{1,j})^2+\sum_{m=2}^{l-1}|\mu m-\delta_{m,j}|^2|a_m|^2\right]-\sum_{m=2}^{l-1}|m-\delta_{m,j}|^2|a_m|^2$$

$$\le \eta^2(\mu-\delta_{1,j})^2+(\mu^2\eta^2-1)\sum_{m=2}^{l-1}k^2|a_m|^2-2(\mu\eta^2-1)\sum_{m=2}^{l-1}m\delta_{m,j}|a_m|^2 + (\eta^2-1)\sum_{m=2}^{l-1}\delta_{1,j}^2|a_m|^2,$$

or

$$\left|l-\delta_{l,j}\right|^2|a_l|^2 \le \eta^2(\mu-\delta_{1,j})^2-2(\mu\eta^2-1)\sum_{m=2}^{l-1}m\delta_{m,j}|a_m|^2$$

$$\le 2(1-\mu\eta^2)\sum_{m=2}^{l-1}m\delta_{m,j}|a_m|^2 \quad (|a_1|=|b_1|=1) \tag{2.12}$$

since $\mu\eta < 1$.

Now, we show that $a_n = 0(\frac{1}{l})$ as $l \to \infty$. From (2.12), we have

$$\left(l-\delta_{l,j}\right)^2|a_l|^2 < 2(1-\mu\eta^2)\left(\delta_{1,j}+\sum_{m=2}^{l-1}m\delta_{m,j}|a_m|^2\right). \tag{2.13}$$

Since $\zeta\frac{h'(\zeta)}{h_{j,m}(\zeta)}$ is bounded, $h_{j,m}(\zeta)$ is also bounded. Now, by following the method given by Clunie [1], it follows that the area of the image of $h_{j,k}(\zeta)$ is

$$\pi\left(\delta_{1,j}+\sum_{m=2}^{l-1}m\delta_{m,j}|a_m|^2\right), \tag{2.14}$$

and consequently, $\sum_{m=2}^{\infty} m\delta_{m,j}|a_m|^2 < \infty$ and hence $r_l = \sum_{m=2}^{\infty} k\delta_{m,j}|a_m|^2 \longrightarrow 0$ as $l \longrightarrow \infty$. Thus, we have

$$\sum_{m=2}^{l-1}m\delta_{m,j}|a_m|^2=\sum_{m=2}^{l-1}(r_m-r_{m+1})=r_2-r_l=O(1) \quad as \quad l\longrightarrow\infty \tag{2.15}$$

Using (2.13) and (2.15), we get $a_l = 0(\frac{1}{l})$ as $l \longrightarrow \infty$. □

Theorem 2.4 *Let* $h(\zeta) \in \mathfrak{A}$ *and in the form* (2.1)*. If, for* $0 \leq \mu \leq 1$*, and* $\frac{1}{2} < \eta \leq 1$*,*

$$\sum_{l=2}^{\infty}\left[\frac{(l+\delta_{l,j})}{\eta(\mu+\delta_{1,j})-(1-\delta_{1,j})}+\frac{\eta(\mu l+\delta_{l,j})}{\eta(\mu+\delta_{1,j})-(1-\delta_{1,j})}\right]|a_l| \leq 1, \tag{2.16}$$

then $h(\zeta) \in S_s^{*(j,m)}(\mu, \eta)$.

Proof Suppose that $h(\zeta) = \zeta + \sum_{l=2}^{\infty} a_l \zeta^l$. Then for $|\zeta| < 1$,

$$\left|\zeta h'(\zeta) - h_{j,m}(\zeta)\right| - \eta\left|\mu\zeta h'(\zeta) + h_{j,m}(\zeta)\right| = \left|\zeta + \sum_{l=2}^{\infty} l a_l \zeta^l - \delta_{1,j}\zeta - \sum_{l=2}^{\infty}\delta_{1,j} a_l \zeta^l\right|$$

$$-\eta\left|\mu\zeta + \mu\sum_{l=2}^{\infty} l a_l\zeta^l + \delta_{1,j}\zeta + \sum_{l=2}^{\infty}\delta_{1,j}a_l\zeta^l\right|$$

$$= \left|(1-\delta_{1,j})\zeta + \sum_{l=2}^{\infty}(l+\delta_{l,j})a_l\zeta^l\right| - \eta\left|(\mu+\delta_{1,j})\zeta + \sum_{l=2}^{\infty}(\mu l+\delta_{l,j})a_l\zeta^l\right|$$

$$\leq (1-\delta_{1,j})r + \sum_{l=2}^{\infty} l\delta_{l,j})|a_l|r^l - \eta\left[(\mu+\delta_{1,j})r + \sum_{l=2}^{\infty}(\mu l+\delta_{l,j})|a_l|r^l\right]$$

$$< \left[\sum_{l=2}^{\infty}(l+\delta_{l,j})|a_l| + (1-\delta_{1,j}) - \eta(\mu+\delta_{1,j}) + \sum_{l=2}^{\infty}\eta(\mu l+\delta_{l,j})|a_l|\right] r$$

$$< \left[\sum_{l=2}^{\infty}\left[(l+\delta_{l,j}) + \eta(\mu l+\delta_{l,j})\right]|a_l| + (1-\delta_{1,j}) - \eta(\mu+\delta_{1,j})\right] r$$

$$< \left[\sum_{l=2}^{\infty}\left[(l+\delta_{l,j}) + \eta(\mu l+\delta_{l,j})\right]|a_l| - \left[\eta(\mu+\delta_{1,j}) - (1-\delta_{1,j})\right]\right]$$

$< 0,$ by Eq. (16).

Therefore, it follows that, in $|\zeta| < 1$

$$\left|\left(\frac{\zeta h'(\zeta)}{h_{j,m}(\zeta)} - 1\right) \Big/ \left(\frac{\mu\zeta h'(\zeta)}{h_{j,m}(\zeta)} + 1\right)\right| < \eta$$

so that $h(\zeta) \in S_s^{*(j,m)}(\mu, \eta)$. Note that

$$h(\zeta) = \zeta - \frac{\eta(\mu + \delta_{1,j}) - (1 - \delta_{1,j})}{(l + \delta_{l,j}) + \eta(\mu l + \delta_{l,j})}\zeta^l$$

is an external mapping w.r.to the theorem, because

$$\left|\left(\frac{\zeta h'(\zeta)}{h_{j,m}(\zeta)} - 1\right) \Big/ \left(\frac{\mu\zeta h'(\zeta)}{h_{j,m}(\zeta)} + 1\right)\right| = \eta$$

for $\zeta = 1,\ 0 \le \mu \le 1,\ \frac{1}{2} < \eta \le 1,\ l = 1, 2, 3, ...$ □

The case of $\mu = 1$ and $\eta = 1$ in Theorem 2.4, yields the following corollary, that may provide ideas to several many problems involving coefficient techniques:

Corollary 2.2 *With the hypothesis of Theorem 2.4, with $\mu = 1$ and $\eta = 1$, we get*

$$\sum_{l=2}^{\infty}(l + \delta_{l,j})|a_l| \le \delta_{1,j}.$$

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Chapter 3
Experimental Evaluation of Four Intermediate Filters to Improve the Motion Field Estimation

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AMS Classification 65D19 · 68T45 · 65G35 · 70G75 · 34E13

3.1 Introduction

Optical Flow estimation is one of the most challenging problems in computer vision. Optical flow is defined as the per-pixel motion between two consecutive digital images. Optical flow has many applications, such as video post-production, particle velocimetry, video compression, control of autonomous vehicles, and many others. In Fig. 3.1, we show an example of these applications. A transparent plastic model full of water containing black plastic tracers is used to estimate the fluid's velocity.

In Fig. 3.1, we show velocity estimation inside a process showing an oxygen inlet and small black plastic tracers. Oxygen is injected from the left side of the plastic model, causing the fluid to move. Tracer particles move at the same velocity as the fluid. This application aims to determine the location where maximum velocity is reached to predict erosion of internal walls.

In Fig. 3.2, we show the second application of the optical flow. We show two consecutive images. In these images, we show a person riding a bike that moves to the left. We estimated the optical flow between the two images; with this optical flow,

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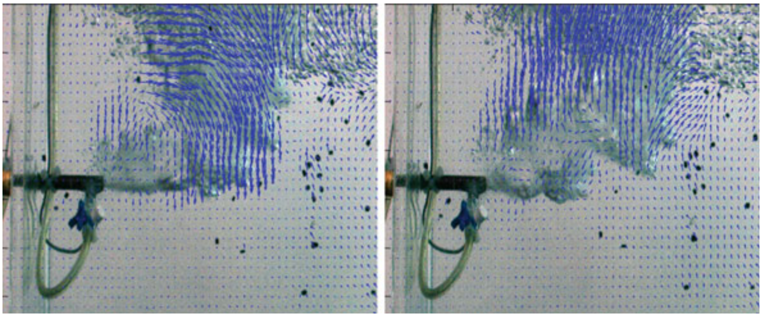

Fig. 3.1 Velocity fluid estimation inside a plastic model of a flow dynamic process. Optical flow is represented with blue arrows in these two consecutive images

Fig. 3.2 Creation of an image in between. **a** Current image. **b** Next image. **c** Image in between created based on optical estimation

we know the displacement for each pixel, so an image in between can be created. We show in Fig. 3.2c an interpolation of two images using optical flow.

3.1.1 Related Works

Since the seminal work of [1], many contributions have been made in order to improve the optical flow estimation. In that work, the authors proposed a variational model to estimate the optical flow. The proposal is an energy model to estimate the optical flow estimation error, and the argument that minimizes that energy is the optical flow of the sequence of images. The proposal is a model that uses a quadratic error, and it means that the model is susceptible to outliers and the presence of noise. Zach et al. [2] proposed another model based on the absolute value of the error. All those models in an iterative way minimize the energy error model. In each iteration, some of them filter the optical estimation to eliminate noise or outliers, avoiding noise and outliers propagating across the iterations.

Bidimensional Empirical Mode Decomposition (BEMD) presented in [3] is a method to decompose a 2D signal in its frequency modes. This work was applied to texture extraction and also 2D signal filtering. Their algorithm consists of the extraction of features at multiple scales or spatial frequencies. These features, called intrinsic mode functions, are extracted by sifting. The author performs this sifting using morphological operators to detect regional maxima and radial basis functions for surface interpolation. The author demonstrates the efficiency of their proposal with synthetic and natural images.

3.1.2 Optical Flow Filtering

In [4], the authors integrated a Median filter in the optical flow model. Their proposal formalizes the Median filter with a new model that integrates OF estimation over a local neighborhood. In [5], the Median filter of size 3×3 is used to eliminate irregularities of the optical flow and also noisy estimation. In [6], to enhance the OF estimation, a bilateral filter is used. In that work, the authors replaced the anisotropic diffusion of their proposed model with a novel multi-cue-driven bilateral filter that considers the estimated occlusion.

Dérian et al. [7] utilizes another approach, where an optical flow model based on wavelet analysis is presented. The multi-resolution approach used in the optical flow estimation is similar to the multi-resolution used in wavelets analysis. The authors constructed a scale-space representation of the optical flow; furthermore, they provide a mechanism to locally approach the optical flow using high-order polynomials by trunking wavelets at fine scales. This methodology was not evaluated in a contemporaneous dataset but evaluated video sequences of moving fluids.

Motivation

In [8] an optical flow study is presented. This study consider the optical flow estimation using image pyramid and also considers a theoretical analysis of warping, but this study does not consider the study of intermediate filters. The study in [9] varies the number of warpings, the image pyramid levels, and the influence of parameters but does not take into account the intermediate filters. Those facts motivate us to compare the optical flow estimation performance considering different intermediate filters.

3.1.3 Contribution of this Work

In this study, we use the optical flow estimation model proposed in [5]. The used model is based on the absolute value of the optical flow estimation error ($TV - L^1$)

and, thanks to other components, is robust to illumination changes and medium displacements. We present, in this work, the following contributions:

- (a) Evaluation of the OF performance estimation considering four intermediate filters: bilateral filter, median filter, weighted median filter, and a balanced weighted median filter.
- (b) We proposed an adaptive or weighted sum of the bilateral and the median filter.
- (c) We performed final evaluation in the complete MPI-Sintel test dataset and submitted the results to the MPI-Sintel benchmark web page [10].

In Sect. 3.2, we explain the principal strategies used to estimate the optical flow. We will briefly explain the linearization of the optical flow constraint, warping an image, and image pyramid. We explain these strategies in order to make our manuscript self-contained. In Sect. 3.3, we explain the filters considered in this work. In section 3.4, we explain experiments and used dataset. In Sect. 3.5, we present our obtained results and a brief discussion about other methodologies and our results. Finally, in Sect. 3.6 we present our conclusions and future work.

3.2 Preliminary

In order to state a model of the optical flow estimation, we consider two consecutive color images I_0 (reference image) and I_1 (target image), where $I_0, I_1 : \Omega \rightarrow \mathbb{R}^3$ and Ω a rectangular image domain; let $\mathbf{u} : \Omega \rightarrow \mathbb{R}^2$ be the optical flow between these two consecutive images (reference and target) where $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}))$ has two components, that is to say, $u_1, u_2 : \Omega \rightarrow \mathbb{R}$. Optical flow estimation aims to determine a motion field $\mathbf{u}(\mathbf{x})$ such that $I_0(\mathbf{x})$ and $I_1(\mathbf{x} + \mathbf{u}(\mathbf{x}))$ are measures of the same pixel $\mathbf{x}$, i.e.:

$$I_0(\mathbf{x}) - I_1(\mathbf{x} + \mathbf{u}(\mathbf{x})) = 0. \tag{3.1}$$

Equation 3.1 in the literature is called the color constancy constraint. Equation 3.1 is highly non-linear and a linearization is applied around a known optical flow $\mathbf{u}_0(\mathbf{x})$.

3.2.1 Linearized Color Constancy Constraint

Considering a known value $\mathbf{u_0}(\mathbf{x})$ of the optical flow, a Taylor expansion is valid:

$$I_0(\mathbf{x}) - I_1(\mathbf{x} + \mathbf{u}_0(\mathbf{x})) - \langle \nabla I_1(\mathbf{x} + \mathbf{u}_0(\mathbf{x})), \mathbf{u}(\mathbf{x}) - \mathbf{u}_0(\mathbf{x}) \rangle = 0, \tag{3.2}$$

with $\langle , \rangle$ being the scalar product, $I_1(\mathbf{x} + \mathbf{u}_0(\mathbf{x}))$ the warped image $(I_1(\mathbf{x}))$ by a known optical flow $\mathbf{u}_0(\mathbf{x})$, and $\nabla I_1(\mathbf{x} + \mathbf{u}_0(\mathbf{x}))$ a gradient vector of $I_1(\mathbf{x} + \mathbf{u}_0(\mathbf{x}))$.

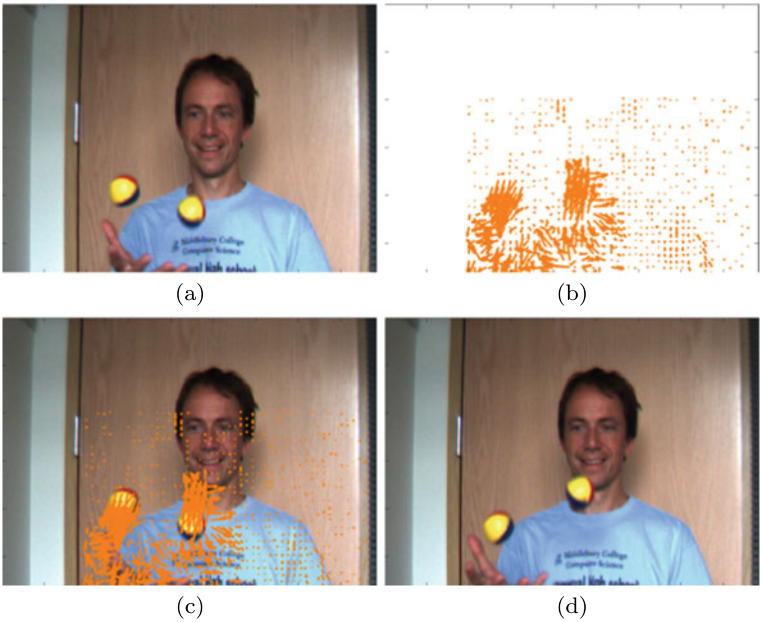

Fig. 3.3 Warping performed in a picture extracted from [11]. **a** Original image $I(\mathbf{x})$. **b** Graphic representation of the optical flow using orange arrows $\mathbf{u_0}(\mathbf{x})$. **c** Optical flow superimposed over the original image. **d** Warped image by the optical flow or compensated image $I(\mathbf{x} + \mathbf{u_0}(\mathbf{x}))$

3.2.2 Image Warping

Image warping is a process of image manipulation such that it distorts any shape contained in the image. Given an image $I(\mathbf{x})$ and an optical flow $\mathbf{u}_0(\mathbf{x})$, we can warp the image $I(\mathbf{x})$ to obtain $I(\mathbf{x} + \mathbf{u_0}(\mathbf{x}))$ as we show in Fig. 3.3.

3.2.3 Image Pyramid

In large displacements, the Taylor approximation does not hold, and we use a coarse-to-fine strategy. This strategy constructs a multi-scale pyramid down-sampling images with a factor of 2. In Fig. 3.4, a scheme is shown of the image pyramid, considering four levels.

Figure 3.4 begins with the coarsest scale, and each of them doubles the image dimension in the previous one. At each level, the optical flow is filtered with an intermediate filter to eliminate noise and outliers.

Fig. 3.4 Image pyramid using an image of the sequence bandage. At the top, we have the coarsest image and the finest at the bottom level. The computed optical flow in each level is used as an initial condition for the optical flow estimation in the consecutive finer level

3.3 Intermediate Filters

Traditionally, OF methods perform the computation in an image pyramid. In each resolution, intermediate filter processing was applied to the analysis to eliminate estimation irregularities and noisy estimation. We explain the scheme using pseudo-code in algorithm 1 where N_{levels} are the level numbers in the pyramid and N_{warpings} is the warping number in each scale once the optical flow $\mathbf{u}(\mathbf{x})$ is computed. Intermediate filtering is applied in every iteration of warping Algorithm 1.

Input : Two consecutive frames I_0, I_1
Output: Optical flow $\mathbf{u}$
down-scaled images (image pyramidal) I_0^s, I_1^s for $s = 1, \ldots, N_{\text{levels}}$;
Initialize $\mathbf{u}^{N_{\text{levels}}}$;
for $s \leftarrow N_{\text{levels}}$ **to** 1 **do**
  for $w \leftarrow 1$ **to** N_{warpings} **do**
    Warp Image $I_1^s(\mathbf{x} + \mathbf{u}(\mathbf{x}))$;
    Compute gradient $\nabla I_1^s(\mathbf{x} + \mathbf{u}(\mathbf{x}))$;
    Compute Optical flow $\mathbf{u}^s$;
    Intermediate filtering;
  end
  If $s > 1$ **t**hen up-sampling $\mathbf{u}^s$ to $\mathbf{u}^{s-1}$;
end
$u = u^1$

Algorithm 1: Pseudo-code for a traditional optical flow.

3.3.1 *Bilateral Filter*

The Bilateral filter can be represented as

$$u_{if}(\mathbf{x}) = \frac{1}{\sum_{\mathbf{y}\in\mathcal{N}(\mathbf{x})} w(\mathbf{x},\mathbf{y})} \sum_{\mathbf{y}\in\mathcal{N}(\mathbf{x})} w(\mathbf{x},\mathbf{y}) u_i(\mathbf{y}), \tag{3.3}$$

with $\mathcal{N}(\mathbf{x})$ a neighborhood around $\mathbf{x}$, $w(\mathbf{x},\mathbf{y})$ are the exponential weights, and u_i (with $i = 1, 2$) are the vertical and horizontal components of the optical flow:

$$w(\mathbf{x},\mathbf{y}) = \phi_S(\mathbf{x}-\mathbf{y})\phi_I(I_0(\mathbf{x}) - I_0(\mathbf{y})), \tag{3.4}$$

where ϕ_S is the spatial distance between $\mathbf{x}$ and $\mathbf{y}$, and ϕ_I is the photo-metric distance between the pixels in $I_0(\mathbf{x})$ and $I_0(\mathbf{y})$. ϕ_S and ϕ_I are Gaussian kernels given by $\phi_S(\mathbf{x}) = e^{-\frac{\|\mathbf{x}\|^2}{2\sigma_S^2}}$ and, $\phi_I(I_0(\mathbf{x})) = e^{-\frac{\|I_0(\mathbf{x})\|^2}{2\sigma_I^2}}$ where $\sigma_S > 0$, $\sigma_I > 0$, and $I_0(\mathbf{x})$ is the reference image.

3.3.2 Median Filter

The Median filter is a non-linear filter, which is used to remove noise and outliers from the optical flow estimation. Given a neighborhood around $\mathbf{x}$ (let's say $\mathcal{N}(\mathbf{x})$),

$$u_{if}(\mathbf{x}) = median_{\mathbf{y}\in\mathcal{N}(\mathbf{x})} u_i(\mathbf{y}). \tag{3.5}$$

We considered square geometries to implement the Median filter, i.e., we filter a 3×3 square around each $\mathbf{x}$ point. Each component of the optical flow is filtered with this filter.

3.3.3 Weighted Median Filter

The Weighted median filter is a non-linear filter applied to a bidimensional distribution of weights and pixel intensities. Each pixel in a neighborhood $\mathcal{N}(\mathbf{x})$ has a weight. The goal is to sort the pixel intensity values, given the distribution of weights. In this case, for each pixel $\mathbf{x}$, we use bilateral weights. The following expression represents weights in each neighborhood:

$$w(\mathbf{x},\mathbf{y}) = \phi_S(\mathbf{x}-\mathbf{y})\phi_I(I_0(\mathbf{x}) - I_0(\mathbf{y})). \tag{3.6}$$

We sorted both the weights w and also the values in $u(\mathbf{x})$. The auxiliary variables are s_w and s_u, representing weight values and optical flow values, respectively. We found in the array s_w positions p^* that hold

$$\sum_{i}^{p*} s_w(i) < \frac{1}{2}\sum_{i}^{N} s_w(i). \tag{3.7}$$

Finally, using the position p^*, we assign the weighted median filtered values of u:

$$u_{wf}(\mathbf{x}) = s_u(p^*). \tag{3.8}$$

3.3.4 Balanced Median Filter

Our proposal is a weighted combination of the bilateral and the median. This linear combination has an adaptive weight $\alpha(\mathbf{x})$. This adaptive weight balances the contribution of the bilateral and the median filter in the intermediate filtering.

Let $u_{bl}(\mathbf{x})$ and $u_m(\mathbf{x})$ be the filtered estimated optical flow by the bilateral filter and the median filter, respectively. Following the ideas in [12], we constructed a balance weight:

$$\alpha(\mathbf{x}) = \frac{1}{1 + e^{(D_{bl}(\mathbf{x}) - D_m(\mathbf{x}))}}, \tag{3.9}$$

where $D_{bl}(\mathbf{x})$ is given as

$$D_{bl}(\mathbf{x}) = |I_0(\mathbf{x}) - I_1(\mathbf{x} + \mathbf{u}_{bl}(\mathbf{x}))|, \tag{3.10}$$

and $D_m(\mathbf{x})$ is given as

$$D_m(\mathbf{x}) = |I_0(\mathbf{x}) - I_1(\mathbf{x} + \mathbf{u}_m(\mathbf{x}))|. \tag{3.11}$$

These two terms, $D_{bl}(\mathbf{x})$ and $D_m(\mathbf{x})$, represent the error of the OF in the point $\mathbf{x}$ when the bilateral filter or the median filter are applied, respectively. The convex combination gives the combined optical flow:

$$\mathbf{u}(\mathbf{x}) = (1 - \alpha(\mathbf{x}))\mathbf{u}_m(\mathbf{x}) + \alpha(\mathbf{x})\mathbf{u}_{bl}(\mathbf{x}). \tag{3.12}$$

Depending on the values of $\alpha_i(\mathbf{x})$, the flow can be more confident in the median filter or the bilateral filter. Equation 3.9 shows that when $D_{bl} >> D_m$, α_i value is almost 0 and when $D_{bl} << D_m$, α is almost 1.

3.4 Dataset and Experiments

This section presents the performed experiments and the dataset used to evaluate optical flow estimation performance.

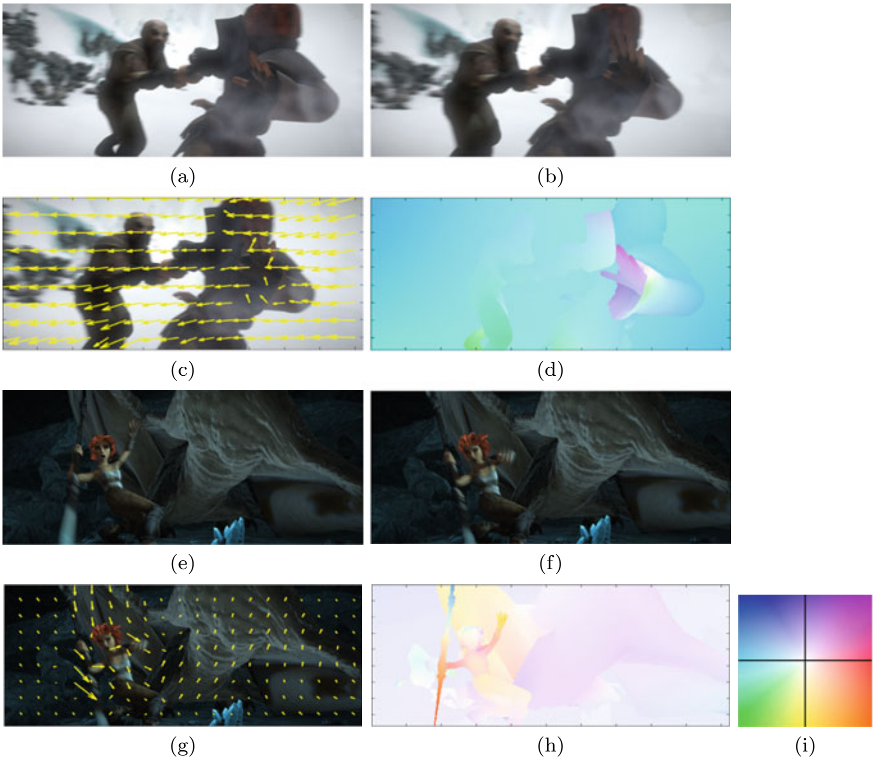

Fig. 3.5 Examples of the MPI-Sintel video dataset. **a** frame_0001 and **b** frame_0002 of video sequence ambush_2. **c** Arrow representation of the ground truth OF of the video. **d** OF color-coded representation. **e** frame_0001 and **f** frame_0002 images of the video sequence cave_4. **g** available ground truth optical flow of two consecutive images. **h** Color-coded ground truth optical flow. **i** Used optical flow color code

3.4.1 Dataset

The dataset contains different video sequences that present blur, fog, different illumination, and many scenes with large displacement and fast displacements. The dataset is divided into two subsets. One set is a training set, and the second one is a validation set. The training set is also divided into two subsets, one called clean and the other called final. The final stage considers the effects mentioned above. This set is more challenging than the clean one; therefore, we performed the experiments in this set. Figure 3.5 shows examples of video sequences in the MPI-Sintel dataset, and we offer the available ground truth optical flow with arrows and also using color code. We show the used color code in Fig. 3.5i.

Table 3.1 The number of images in each image sequence in MPI-Sintel training set

Sequence	Alley	Ambush	Bamboo	Bandage	Cave	Market	Mountain	Shaman	Sleeping	Temple
# of images	100	174	100	100	100	140	50	100	100	100

In Table 3.1, we show a numerical description of each sequence of the final MPI-Sintel training set. In the MPI-Sintel dataset, the optical flow ground truth is available as shown in Fig. 3.5. Thus, this ground truth let us compute end-point-error (EPE) and angular-average-error (AAE); these errors are giving by the following expressions:

$$\begin{aligned} EPE &= \frac{1}{n}\sum_{i=1}^{n}\sqrt{(g_{1i}-u_{1i})^2+(g_{2i}-u_{2i})^2} \\ AAE &= \frac{1}{n}\sum_{i=1}^{n} cos^{-1}\left(\frac{1+g_{1i}u_{1i}+g_{2i}u_{2i}}{\sqrt{1+g_{1i}^2+g_{2i}^2}\sqrt{1+u_{1i}^2+u_{2i}^2}}\right). \end{aligned} \tag{3.13}$$

3.4.2 *Experiments*

In the following, we explain experiments performed using the MPI-Sintel dataset.

(i) Evaluation using median filter
We evaluated the complete MPI-Sintel training set in the final version using the median filter of size 3 × 3 as an intermediate filter.

(ii) Evaluation using weighted median filter, median filter, and balanced median filter
We evaluated the complete MPI-Sintel training set in the final version using the weighted median filter considering $\sigma_I = 200$ and $\sigma_s = 200$.

(iii) Evaluation using Balanced weighted median filter also evaluated our proposed combined filter, which combines a weighted median filter and a bilateral filter.

(iv) Evaluation using a bilateral filter.

3.5 Results and Discussion

We have evaluated the filters in sequences that contain medium displacements, which are the sequences: market, mountain, shaman, sleeping, and temple. In these evaluation sets, we assessed EPE and AAE for these intermediate filters. In these experiments, we computed OF in the training set, which is around 490 images. We present in Table 3.2 second and third columns, the obtained numerical results. As a resume, we obtained an $EPE = 4.47$ and $AAE = 9.22$ and finally $EPE + AAE = 13.69$ by the Median filter. In Table 3.2 in the fourth and fifth column, we show results

Table 3.2 Results were obtained by different proposed filters, EPE and AAE. The second and third columns show results obtained by the Median filter. In the fourth and fifth columns, results were obtained by the weighted Median filter, and in the sixth and seventh columns, results by Balanced weighted median filter

Sequence Name	Median	Filter	Weighted	Median filter	Balanced	Median filter	Bilateral	Filter
	EPE	AAE	EPE	AAE	EPE	AAE	EPE	AAE
Market	8.50	13.07	8.50	13.08	8.43	12.83	10.25	15.51
Mountain	0.95	10.15	0.95	10.16	0.95	9.88	1.32	11.24
Shaman	0.37	7.11	0.37	7.10	0.35	6.58	0.40	6.99
Sleeping	0.11	1.79	0.11	1.78	0.10	1.73	0.13	2.05
Temple	9.06	12.90	9.04	12.96	9.04	12.86	11.82	16.22
Total EPE	4.47	9.22	4.47	9.23	4.44	8.99	5.56	10.71
$EPE+AAE$	13.69		13.70		13.44		16.27	

obtained in the MPI-Sintel data set using the Weighted Median filter. We obtained $EPE = 4.47$ and $AAE = 9.23$ and also $EPE + AAE = 13.70$. These results are worser than the results obtained by the Median filter.

In Table 3.2 sixth and seventh columns, that the $EPE + AAE = 13.44$. Comparing results obtained by the Median filter ($EPE + AAE = 13.69$) and Weighted Median ($EPE + AAE = 13.70$), and Balanced filter, we see that the obtained results are very similar. There are differences of 0.26 between the Weighted Median filter and the balanced median filter. We observe a small difference in the AAE between the Median filter and balanced median filter. The Balanced median filter performs better than the median filter ($AAE = 8.99$). This result indicates that the estimated optical flow is better aligned w.r.t. the ground truth.

3.5.1 Comparison with Other Methods

We submitted the obtained results in the MPI-Sintel to the MPI-Sintel webpage. Those results were ranked in a benchmark that compares different OF method results. Figure 3.6 shows the obtained performance. In Fig. 3.6, we show the performance of our proposal. Our model using the combination of bilateral filter and weighted median filter (called TVL1_BWMFilter) outperforms classic methods like Horn-Schunck [1]. Our proposal presents an $EPE = 9.034$ outperforming the $TV - L^1$ classic formulation [13] and the non-local optical flow in Classic+NL [4]. Our proposal performs similar to Motion Detail Preserving Optical [12] flow, which considers additional correspondences, giving hints to guide the OF estimation. Figure 3.7 is shown with some examples of obtained results by our method.

	EPE all	EPE matched	EPE unmatched	d0-10	d10-60	d60-140	s0-10	s10-40	s40+
GeoFlow [250]	8.459	3.908	45.553	5.805	3.550	2.899	1.398	4.667	52.653
FlowNetADF [255]	8.532	4.768	39.222	8.002	4.979	3.060	1.846	6.304	47.067
CPNFlow [256]	8.589	4.575	41.327	7.234	4.542	2.986	1.856	5.484	49.511
Back2FutureFlow_UFO [263]	8.814	5.031	39.647	7.153	4.880	3.904	1.752	5.961	50.725
TV-L1+EM [268]	8.916	4.712	43.157	7.021	4.788	3.401	1.456	5.188	54.932
TVL1_BWMFilter [269]	9.034	4.761	43.824	7.245	4.931	3.286	1.450	4.998	56.396
LDOF [270]	9.116	5.037	42.344	6.849	4.928	4.003	1.485	4.839	57.296
Classic+NL [281]	9.153	4.814	44.509	7.215	4.822	3.427	1.113	4.496	60.291
Horn+Schunck [289]	9.610	5.419	43.734	7.950	5.658	3.976	1.882	5.335	58.274

Fig. 3.6 Obtained results by our proposal in MPI-Sintel benchmark

(a) (b) (c)

(d) (e) (f)

Fig. 3.7 Examples of flow estimation in MPI-Sintel test set. **a** frame_0024 and **b** frame_0025 of sequence PERTURBED_shaman_1. **c** estimated optical flow for sequence PERTURBED_shaman_1. **d** image frame_0041 of sequence tiger1. **e** image frame_0042 of sequence tiger1. **f** estimated optical flow for sequence tiger_1

In Fig. 3.7c and f is shown obtained OF for images of the sequence PERTURBED_shaman_1 and tiger1 of the MPI-Sintel test dataset. In these two images of the sequence PERTURBED_shaman_1, we obtained a $EPE = 1.648$ (Fig. 3.7a and b), and in the two images of sequence tiger1 (Fig. 3.7d and e) we obtained $EPE = 1.678$.

3.5.2 *Bidimensional Empirical Mode Decomposition*

We have filtered the estimated optical flow using Bidimensional Empirical Mode Decomposition (BEMD) as a proof of concept. We used a MATLAB implementation available on the web. In Fig. 3.8, we show a video sequence where a dragon runs following a chicken. We estimated the optical flow, and we extracted the BEMD.

We observe in Fig. 3.8d the edges of estimated optical flow in (c), and in Figure (f) intermediate spatial frequencies are shown. In Fig. 3.8, low frequencies are shown,

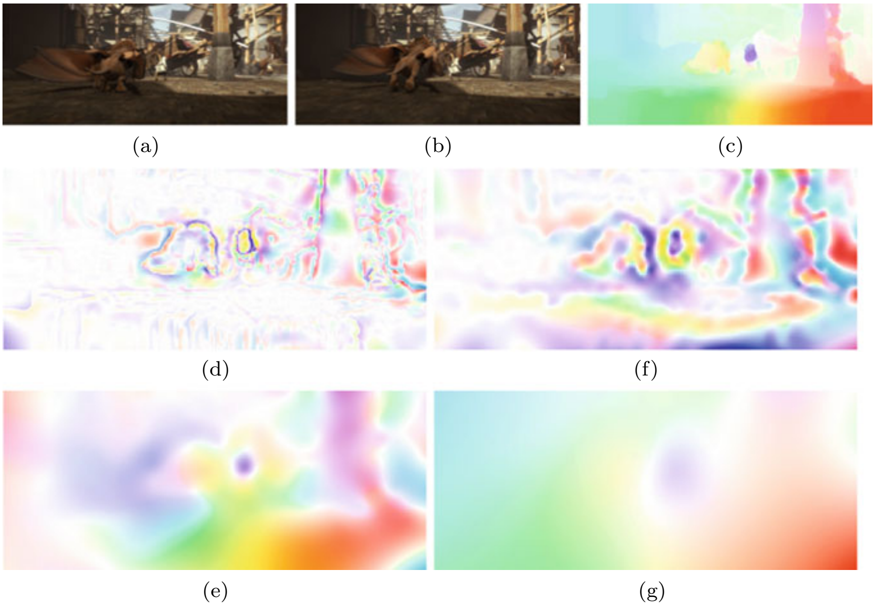

Fig. 3.8 Bidimensional Empirical Mode Decomposition. **a** and **b** two images of the video sequence market_6. In **c**, we show the color-coded estimated optical flow. **d** and **f** BEMD showing highest spatial frequencies and intermediate-high frequencies, respectively. **e** BEMD showing low frequencies. **g** lowest spatial frequencies

but the edges were not preserved, i.e., the EMD performs as an anisotropic filter. Finally, in (f), we observe the continuum component of the flow, which is very blurred and does not preserve shapes.

The MATLAB implementation of the BEMD decomposition method runs four iterations in 1435 seconds, which is not suitable for real time.

3.5.3 *Processing Time*

We measured the total processing time at each scale, which depended on the image size. That is, at different scales, we have different processing times. The measure was performed on a Laptop MSi-i7, running on a single core (please see Table 3.3).

3.5.4 *Discussion*

Regarding the BEMD method, we observe in Fig. 3.8f that the method performs similar to the Gaussian filter, i.e., it blurs the image. Edge preserving in optical flow

Table 3.3 Processing time at each scale—Balanced filter

Image size	Time processing
256 × 109	0.005186 s
512 × 218	0.016479 s
1024 × 436	0.101669 s

estimation is an essential feature that a method should have. We think that a minor modification of BEMD decomposition is possible in order to preserve edges. As a future work, we will consider modifying the BEMD method to consider anisotropic morphological operators. And then evaluate its effect on the optical flow estimation.

Concerning another approach to optical flow estimation as in [7], the author didn't evaluate their proposal in the standard dataset such as MPI-Sintel [10] or [11]. Also, there is no available code in order to compare our results with their proposal. Considering implementing the proposal in [7], as future work, we will consider assessing this proposal with ours in a standard dataset such as MPI-Sintel.

Concerning the obtained results, we were comparing the columns in Table 3.2 associated with the Median filter and Balanced median filter. We observe for sequence shaman that the average angular error dropped from 7.11 to 6.58, which is 0.43 degrees, and EPE dropped from 0.37 to 0.35 with 0.02 pixels. We observe that the most significant reduction was in the angular error, which means that the optical flow is better aligned with the ground truth for this sequence. We highlight these results because some of these sequences contain large displacement or minimal displacements, and shaman contains medium displacements. The proposal is better suited for small and medium displacements.

3.6 Conclusions

To perform our intermediate filter study, we have used a robust model for illumination changes and can handle large displacements. With this model, we assessed the performance of these filters: Median, Weighted Median, Balanced Median filter, and bilateral filter. We evaluated the performance in the MPI-Sintel dataset and submitted our results to a benchmark webpage. We obtained that the Balanced Median filter outperforms the other three filters; thus, we process the MPI-Sintel test. The obtained results show that our proposal outperforms the classical Horn-Schunck method and TV-L1 model and other models, robust to large displacements (LDOF) or non-local methods (Classic+NL). It also outperforms current methods such as GeoFlow [14] and CPNFlow [15]. In future work, we can investigate the uses of the BEMD decomposition method. However, with an anisotropic consideration to preserve edges and shapes and as future work, we will consider implementing the wavelet optical flow to assess this proposal and our proposal in a standard dataset such as MPI-Sintel to compare performance.

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Chapter 4
On the sth Derivative of a Polynomial

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Mathematics Subject Classification (2010) 30C10 · 30C15

4.1 Introduction and Statement of Results

Let $\mathbb{F}_q = \left\{ f(\omega) = \sum_{j=0}^{q} c_j \omega^j : \omega \in \mathbb{C} \ and \ c_q \neq 0 \right\}$. For $f \in \mathbb{F}_q$, Bernstein [2, 9] established that

$$\max_{|\omega|=1} |f'(\omega)| \leq q \max_{|\omega|=1} |f(\omega)|. \tag{4.1.1}$$

Inequality (4.1.1) was first proved by Riesz [8] in 1914 before Bernstein proved it in 1926. Henceforth, (4.1.1) is famously known as Bernstein's inequality.

Erdös conjectured that if $f \in \mathbb{F}_q$ not vanishing in $|\omega| < 1$, then (4.1.1) can be improved and replaced by

$$\max_{|\omega|=1} |f'(\omega)| \leq \frac{q}{2} \max_{|\omega|=1} |f(\omega)|. \tag{4.1.2}$$

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Lax [6] verified inequality (4.1.2) later in 1944. In the literature, there are many refinements and extensions of (4.1.2), for example, Malik [7], Bidkham and Dewan [3], and Jain [5].

As a generalization of (4.1.2), Malik [7] proved for a polynomial $f \in \mathbb{F}_q$ having no zeros in $|\omega| < \tau$, $\tau \geq 1$,

$$\max_{|\omega|=1} |f'(\omega)| \leq \frac{q}{1+\tau} \max_{|\omega|=1} |f(\omega)|. \tag{4.1.3}$$

Inequality (4.1.3) was further generalized by Bidkham and Dewan [3] for $f \in \mathbb{F}_q$ not vanishing in $|\omega| < \tau$, $\tau \geq 1$ and $1 \leq \Gamma \leq \tau$. They proved

$$\max_{|\omega|=\Gamma} |f'(\omega)| \leq \frac{q(\Gamma+\tau)^{q-1}}{(1+\tau)^q} \max_{|\omega|=1} |f(\omega)|. \tag{4.1.4}$$

Jain [5] extended (4.1.4) for the sth derivative.

Theorem A *If $f \in \mathbb{F}_q$ such that $f(\omega) \neq 0$ in $|\omega| < \tau$, where $\tau \geq 1$, then for $0 \leq s < q$ and $1 \leq \gamma \leq \tau$,*

$$\max_{|\omega|=\Gamma} \left|f^s(\omega)\right| \leq \frac{1}{\Gamma^s + \tau^s} \left[\left\{ \frac{d^s}{d\rho^s} \left(1+\rho^q\right) \right\}_{\rho=1} \right] \left(\frac{\Gamma+\tau}{1+\tau} \right)^q \max_{|\omega|=1} |f(\omega)|. \tag{4.1.5}$$

In (4.1.5), equality is attained for $f(\omega) = (\omega+\tau)^q$ with $s = 1$.

We now state the following theorem which improves and extends (4.1.5) by involving certain coefficients of the polynomial $f(\omega) = \sum_{j=0}^{q} c_j \omega^j$ in the inequality.

Theorem 4.1 *If $f \in \mathbb{F}_q$ such that $f(\omega) \neq 0$ in $|\omega| < \tau$, $\tau > 0$, then for $0 \leq s < q$, and $0 < \gamma \leq \Gamma \leq \tau$,*

$$\begin{aligned} \max_{|\omega|=\Gamma} \left|f^s(\omega)\right| \leq & \left[\left\{ \frac{d^s}{d\rho^s}(1+\rho^n) \right\}_{\rho=1} \right] \\ & \times \left\{ \frac{{}^qC_s \Gamma + \frac{|c_s|}{|c_0|}\tau^{s+1}}{{}^qC_s(\tau^{s+1}+\Gamma^{s+1}) + \frac{|c_s|}{|c_0|}(\tau^{s+1}\Gamma^s + \Gamma\tau^{2s})} \right\} \\ & \times A \left\{ \max_{|\omega|=\gamma} |f(\omega)| - \left(1 - \frac{1}{A}\right) m \right\}, \end{aligned} \tag{4.1.6}$$

where $m = \min_{|\omega|=\tau} |f(\omega)|$, ${}^qC_s = \dfrac{q!}{s!(q-s)!}$, and

$$A = \left(\frac{\Gamma^2 + \tau^2 + \frac{2}{q} \frac{|c_1|}{|c_0| - m} \tau^2 \Gamma}{\gamma^2 + \tau^2 + \frac{2}{q} \frac{|c_1|}{|c_0| - m} \tau^2 \gamma} \right)^{\frac{q}{2}}. \tag{4.1.7}$$

Remark 4.1 In the context of Theorem 4.1, as $f(\omega) \neq 0$ in $|\omega| < \tau$, $\tau > 0$, then $f(\theta\omega)$ has no zero in $|\omega| < \frac{\tau}{\theta}$, $\frac{\tau}{\theta} \geq 1$, where $0 < \theta \leq \tau$. Hence, applying (4.2.2) of Lemma 4.1 to $f(\theta\omega)$, we have

$$\frac{1}{{}^qC_s} \frac{|c_s|\theta^s}{|c_0|} \left(\frac{\tau}{\theta}\right)^s \leq 1,$$

which simplifies to

$$\frac{1}{{}^qC_s} \frac{|c_s|}{|c_0|} \tau^s \leq 1. \tag{4.1.8}$$

Inequality (4.1.8) is equivalent to

$$\frac{{}^qC_s\theta + \frac{|c_s|}{|c_0|}\tau^{s+1}}{{}^qC_s(\tau^{s+1} + \theta^{s+1}) + \frac{|c_s|}{|c_0|}(\tau^{s+1}\theta^s + \theta\tau^{2s})} \leq \frac{1}{\theta^s + \tau^s} \quad \text{for} \quad 0 < \theta \leq \tau. \tag{4.1.9}$$

Since $\Gamma \leq \tau$, we take $\theta = \Gamma$ in (4.1.9) and obtain

$$\frac{{}^qC_s\Gamma + \frac{|c_s|}{|c_0|}\tau^{s+1}}{{}^qC_s(\tau^{s+1} + \Gamma^{s+1}) + \frac{|c_s|}{|c_0|}(\tau^{s+1}\Gamma^s + \Gamma\tau^{2s})} \leq \frac{1}{\Gamma^s + \tau^s}. \tag{4.1.10}$$

Remark 4.2 If we take $\gamma = 1$ in Theorem 4.1, we have the following result.

Corollary 4.1 *If $f \in \mathbb{F}_q$ and $f(\omega) \neq 0$ in $|\omega| < \tau$, $\tau \geq 1$, then for $0 \leq s < q$ and $1 \leq \Gamma \leq \tau$,*

$$\begin{aligned} \max_{|\omega|=\Gamma} \left| f^s(\omega) \right| \leq & \left[\left\{ \frac{d^s}{d\rho^s}(1+\rho^n) \right\}_{\rho=1} \right] \\ & \times \left\{ \frac{{}^qC_s\Gamma + \frac{|c_s|}{|c_0|}\tau^{s+1}}{{}^qC_s(\tau^{s+1} + \Gamma^{s+1}) + \frac{|c_s|}{|c_0|}(\tau^{s+1}\Gamma^s + \Gamma\tau^{2s})} \right\} \\ & \times D \left\{ \max_{|\omega|=1} |f(\omega)| - \left(1 - \frac{1}{D}\right) m \right\}, \end{aligned} \tag{4.1.11}$$

where

$$D = \left(\frac{\Gamma^2 + \tau^2 + \frac{2}{q}\frac{|c_1|}{|c_0| - m}\tau^2\gamma}{1 + \tau^2 + \frac{2}{q}\frac{|c_1|}{|c_0| - m}\tau^2} \right)^{\frac{q}{2}}.$$

Clearly, in the light of (4.1.10) of Remark 4.1 and (4.2.7) of Lemma 4.5, Corollary 4.1 gives an improved bound over Theorem A.

Remark 4.3 For the case $s = 1$, inequality (4.1.11) of Corollary 4.1 gives a better bound than inequality (4.1.4) proved by Bidkham and Dewan [3].

Remark 4.4 If $\Gamma = 1$ and $\tau = 1$, Corollary 4.1 reduces to an extension of inequality (4.1.2) due to Lax [6] for the sth derivative.

4.2 Lemmas

This section contains auxiliary results which are used to prove our results and other claims.

Aziz and Rather [1] proved the following lemma.

Lemma 4.1 *If $f \in \mathbb{F}_q$ such that $f(\omega) \neq 0$ in $|\omega| < \tau$, $\tau \geq 1$, then for $1 \leq s < q$,*

$$\max_{|\omega|=1} \left| f^s(\omega) \right| \leq q(q-1)\dots(q-s+1) \times \left\{ \frac{{}^qC_s|c_0| + |c_s|\tau^{s+1}}{{}^qC_s|c_0|(1+\tau^{s+1}) + |c_s|(\tau^{s+1}+\tau^{2s})} \right\} \max_{|\omega|=1} |f(\omega)| \quad (4.2.1)$$

and

$$\frac{1}{{}^qC_s}\frac{|c_s|}{|c_0|}\tau^s \leq 1. \quad (4.2.2)$$

This next lemma follows easily from Lemma 4.1.

Lemma 4.2 *If $f \in \mathbb{F}_q$ and $f(\omega) \neq 0$ in $|\omega| < \tau$, $\tau \geq 1$, then for $0 \leq s < q$,*

$$\max_{|\omega|=1} |f^s(\omega)| \leq \left\{ \frac{{}^qC_s|c_0| + |c_s|\tau^{s+1}}{{}^qC_s|c_0|(1+\tau^{s+1}) + |c_s|(\tau^{s+1}+\tau^{2s})} \right\} \times \left[\left\{ \frac{d^s}{d\rho^s}(1+\rho^q) \right\}_{\rho=1} \right] \max_{|\omega|=1} |f(\omega)|. \quad (4.2.3)$$

Barchand and Dewan [4] proved the next lemma.

Lemma 4.3 *Let $f(\omega) = c_0 + \sum_{j=\sigma}^{q} c_j \omega^j$, $1 \le \sigma \le q$, be a polynomial of degree q such that $f(\omega) \neq 0$ in $|\omega| < \tau$, $\tau > 0$. Then, for $0 < \gamma \le \Gamma \le \tau$,*

$$\max_{|\omega|=\Gamma} |f(\omega)| \le \exp(B') \left[\max_{|\omega|=\gamma} |f(\omega)| - \left\{1 - \exp(-B')\right\} \right], \tag{4.2.4}$$

where $m = \min\limits_{|\omega|=\tau} |f(\omega)|$
and

$$B' = q \int_{\gamma}^{\Gamma} \frac{\frac{\sigma}{q}\left(\frac{|c_\sigma|}{|c_0| - m}\right)\tau^{\sigma+1} t^{\sigma-1} + t^{\sigma}}{t^{\sigma+1} + \tau^{\sigma+1} + \frac{\sigma}{q}\left(\frac{|c_\sigma|}{|c_0| - m}\right)(\tau^{\sigma+1} t^{\sigma} + \tau^{2\sigma} t)} dt. \tag{4.2.5}$$

Lemma 4.4 *If $f(\omega) = c_0 + \sum_{j=\sigma}^{q} c_j \omega^j$, $1 \le \sigma \le q$, is a polynomial of degree q such that $f(\omega) \neq 0$ for $|\omega| < \tau$, $\tau \ge 1$, and if $m = \min\limits_{|\omega|=\tau} |f(\omega)|$, then*

$$\frac{|c_\sigma| \tau^{\sigma}}{|c_0| - m} \le \frac{q}{\sigma}. \tag{4.2.6}$$

This lemma was proved by Gardner et al. [10, Lemma 3;Proof].

Lemma 4.5 *If $f \in \mathbb{F}_q$ such that $f(\omega) \neq 0$ in $|\omega| < \tau$, $\tau > 0$, then for $0 < \gamma \le \Gamma \le \tau$,*

$$\left(\frac{\Gamma^2 + \frac{1}{q}\frac{|c_1|}{|c_0| - m} 2\tau^2 \Gamma + \tau^2}{\gamma^2 + \frac{1}{q}\frac{|c_1|}{|c_0| - m} 2\tau^2 \gamma + \tau^2} \right)^{\frac{q}{2}} \le \left(\frac{\Gamma + \tau}{\gamma + \tau} \right)^q, \tag{4.2.7}$$

where $m = \min\limits_{|\omega|=\tau} |f(\omega)|$.

Proof Since $f(\omega) \neq 0$ in $|\omega| < \tau, \tau > 0$, then $F(\omega) = f(\theta\omega) \neq 0$ in $|\omega| < \frac{\tau}{\theta}, \frac{\tau}{\theta} \ge 1$, where $0 < \theta \le \tau$.

Applying Lemma 4.4 for $\sigma = 1$ to $F(\omega)$, we have

$$\frac{|c_1|\theta}{|c_0| - m}\left(\frac{\tau}{\theta}\right) \le q,$$

where

$$m = \min_{|\omega|=\frac{\tau}{\theta}} |F(\omega)| = \min_{|\omega|=\tau} |f(\omega)|,$$

which is equivalent to

$$q\frac{\frac{1}{q}\frac{|c_1|}{|c_0|-m}\tau^2+\theta}{\theta^2+\frac{1}{q}\frac{|c_1|}{|c_0|-m}2\tau^2\theta+\tau^2}\leq q\frac{1}{\theta+\tau}. \tag{4.2.8}$$

Now, for $0<\gamma\leq\Gamma\leq\tau$, inequality (4.2.8) on integration with respect to θ from γ to Γ gives

$$\frac{q}{2}\left|\log\left(\theta^2+\frac{1}{q}\frac{|c_1|}{|c_0|-m}2\tau^2\theta+\tau^2\right)\right|_{\gamma}^{\Gamma}\leq q\left|\log(\theta+\tau)\right|_{\gamma}^{\Gamma}.$$

That is,

$$\left(\frac{\Gamma^2+\frac{1}{q}\frac{|c_1|}{|c_0|-m}2\tau^2\Gamma+\tau^2}{\gamma^2+\frac{1}{q}\frac{|c_1|}{|c_0|-m}2\tau^2\gamma+\tau^2}\right)^{\frac{q}{2}}\leq\left(\frac{\Gamma+\tau}{\gamma+\tau}\right)^q.$$

4.3 Proof of Theorem 1

Proof Since $f(\omega)$ has no zero in $|\omega|<\tau$, $\tau>0$, and as $0<\gamma\leq\Gamma\leq\tau$, then the polynomial $f(\Gamma\omega)$ has no zero in $|\omega|<\frac{\tau}{\Gamma}$, $\frac{\tau}{\Gamma}\geq 1$. Now, applying Lemma 4.2 to $f(\Gamma\omega)$, we have for $0\leq s<q$,

$$\begin{aligned}\Gamma^s\max_{|\omega|=\Gamma}\left|f^s(\omega)\right|\leq&\left[\left\{\frac{d^{(s)}}{d\rho^s}(1+\rho^q)\right\}_{\rho=1}\right]\\&\times\left\{\frac{{}^qC_s+\frac{|c_s|\Gamma^s}{|c_0|}\left(\frac{\tau}{\Gamma}\right)^{s+1}}{{}^qC_s\left(1+\left(\frac{\tau}{\Gamma}\right)^{s+1}\right)+\frac{|c_s|\Gamma^s}{|c_0|}\left(\frac{\tau^{s+1}}{\Gamma^{s+1}}+\frac{\tau^{2s}}{\Gamma^{2s}}\right)}\right\}\\&\times\max_{|\omega|=\Gamma}|f(\omega)|,\end{aligned}$$

which is equivalent to

$$\max_{|\omega|=\Gamma}\left|f^{s}(\omega)\right| \leq \left[\left\{\frac{d^{(s)}}{d\rho^{s}}(1+\rho^{q})\right\}_{\rho=1}\right] \times \left\{\frac{{}^{q}C_{s}\Gamma+\frac{|c_s|}{|c_0|}\tau^{s+1}}{{}^{q}C_{s}(\tau^{s+1}+\Gamma^{s+1})+\frac{|c_s|}{|c_0|}(\tau^{s+1}\Gamma^{s}+\Gamma\tau^{2s})}\right\} \times \max_{|\omega|=\Gamma}|f(\omega)|. \tag{4.3.1}$$

For $0<\gamma\leq\Gamma\leq\tau$, putting $\sigma=1$, inequality (4.2.4) becomes

$$\max_{|\omega|=\gamma}|f(\omega)| \leq A\left\{\max_{|\omega|=\gamma}|f(\omega)|-\left(1-\frac{1}{A}\right)m\right\}, \tag{4.3.2}$$

where A is given by (4.1.7).

Combining (4.3.1) and (4.3.2), we have

$$\max_{|\omega|=\Gamma}\left|f^{s}(\omega)\right| \leq \left[\left\{\frac{d^{(s)}}{d\rho^{s}}(1+\rho^{q})\right\}_{\rho=1}\right] \times \left\{\frac{{}^{q}C_{s}\Gamma+\frac{|c_s|}{|c_0|}\tau^{s+1}}{{}^{q}C_{s}(\tau^{s+1}+\Gamma^{s+1})+\frac{|c_s|}{|c_0|}(\tau^{s+1}\Gamma^{s}+\Gamma\tau^{2s})}\right\} \times A\left\{\max_{|\omega|=\gamma}|f(\omega)|-\left(1-\frac{1}{A}\right)m\right\}.$$

This completes the proof of Theorem 4.1.

Conflict of interest The authors declare that they have no conflict of interest.

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Chapter 5
On the Problem of Pricing a Double Barrier Option in a Modified Black-Scholes Environment

G. Venkiteswaran and S. Udayabaskaran

5.1 Introduction

In this paper, we consider the pricing problem of a double barrier option in a modified Black-Scholes environment. Under the Black-Scholes environment, the problem of pricing different types of barrier options has been studied by several researchers (see, for example, [2, 3, 5, 7–9, 12]).

Kunitomo and Ikeda [7] have derived the probability density function of the stock price to stay in between two exponentially curved boundaries in a Black-Scholes environment and obtained the prices for double knock-out call and put options in the form of infinite series. Geman and Yor [3] consider a double barrier option in a Black-Scholes environment and obtain an expression for the Laplace transform of the option. Hui [5] considers a conventional Black-Scholes option-pricing environment and derives analytical solutions of one-touch double barrier binary options by using a separation of variables technique. Pelsser [12] has considered a Black-Scholes option-pricing environment and applied contour integration to invert analytically the Laplace transform of a double barrier option price. Hui et al. [6] have obtained the solutions of double barrier option values in terms of Fourier sine series by using both the Laplace transform and the method of separation of variables. They have also obtained the solutions in terms of the cumulative normal distribution function by employing the method of reflection. Luo [9] derives closed-form solutions based on hitting probabilities of a Brownian motion for eight types of European-style double barrier options in the Black-Scholes environment. Lo and Hui [8] have provided an analytical approach for valuing double barrier options with time-dependent parameters by the Fourier Series expansion method. Buchen and Konstandatos [2] have considered the arbitrage-free pricing of double knock-out barrier options in a Black-

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Scholes environment and derived analytical formulae for the particular class of exotic options by applying the method of images of electromagnetics.

The Black-Scholes model has the basic assumption that the underlying asset has a constant volatility parameter leading to log-normal distribution for the stock price. However, in real markets, the probability distribution of the underlying asset price may not be log-normal and to remedy this discrepancy, a volatility smile adjustment is usually made. It is also noted that the smile-effect is decreasing with time to maturity and the smiles are frequently asymmetric. To accommodate the smile-effect in the pricing problem of contingency claims, [4, 10] have considered a stochastically fluctuating type of volatility. Naki [10] has proposed and studied an asset-pricing model in which the volatility is modeled as a stochastic process which switches back and forward from one state to the other. Herzel [4] has obtained a closed-form expression for the price of a European option by considering a simple modification of the Black-Scholes model with the assumption that the volatility of the stock may jump at a random time τ from a value σ_a to a value σ_b.

In the present chapter, we accommodate the smile-effect of the volatility by proposing a modified Black-Scholes equation which includes stochastically fluctuating volatility, and analyze a pricing problem of a double barrier option. To be specific, we consider a fluctuating type of volatility driven by a Poisson process and obtain an analytical expression for the price of a barrier option. The organization of the paper is as follows: In Sect. 5.2, we describe the model of a financial market. Section 5.3 describes the volatility process and obtains its probability laws. In Sect. 5.4, we provide the existence of an equivalent martingale measure and the proof that the discounted stock price is a martingale under the equivalent martingale measure. Section 5.5 describes a double barrier option problem and provides the equation for the probability density function of the stock price to stay between the barriers. In Sect. 5.6, we evaluate the barrier densities and in Sect. 5.7, we derive an analytical expression for the pricing formula for a particular double barrier option and deduce the result of [12].

5.2 The Stochastic Market

We consider the following market which consists of a riskless bond and a risky stock. Let $B(t)$ and $S(t)$ be the prices of the bond and stock at time t, respectively. The bond $B(t)$ is described by the equation

$$dB(t) = rB(t)dt, \tag{5.1}$$

where $B(0) = 1$ and r is a positive constant which is called the riskless interest rate. The rate r is an input to the pricing problem. The stock price $S(t)$ is described by the stochastic differential equation

$$dS(t) = \mu S(t)dt + \sigma(t)S(t)dW(t), \tag{5.2}$$

where $W(t)$ is a standard Brownian motion starting at 0. Here, μ is a constant and $\{\sigma(t), t \geq 0\}$ is a stochastic process and are respectively called the drift and the volatility of the stock market. $S(0)$ may be a constant or a random variable independent of $W(t)$. From Eq. (5.1) and the initial condition, it is evident that $B(t) = e^{rt}$. Further, we assume that the volatility is a stochastic process $\{\sigma(t), t \geq 0\}$ which is described as follows. Suppose that the volatility $\{\sigma(t), t \geq 0\}$ alternates between two constant values σ_1 and σ_2 and the fluctuations occur at random time points. Then we note that volatility $\{\sigma(t), t \geq 0\}$ is a two-state stochastic process with state space $\{\sigma_1, \sigma_2\}$. We assume that $\lim_{\Delta>0,\Delta\to 0} \sigma(-\Delta) = \sigma_2$ and $\lim_{\Delta>0,\Delta\to 0} \sigma(\Delta) = \sigma_1$. That is, the volatility has just entered the level σ_1 at time $t = 0$. Let $\tau_1, \tau_2, \tau_3, \cdots$ be the successive sojourn times of the volatility $\{\sigma(t), t \geq 0\}$. We observe that the sojourn times in the state σ_1 are $\tau_1, \tau_3, \tau_5, \cdots$ and the sojourn times in the state σ_2 are $\tau_2, \tau_4, \tau_6, \cdots$. We shall assume that the sojourn times $\tau_1, \tau_3, \tau_5, \cdots$ are independent and identically distributed with probability density function $f(t)$. Similarly, we assume that the sojourn times in the state σ_2 are independent and identically distributed with probability density function $g(t)$. Let $F(t)$ and $G(t)$ be the cumulative density functions of $f(t)$ and $g(t)$, respectively. We assume that all the sojourn times are independent and define a counting process $\{\mathcal{N}(t), t \geq 0\}$ as follows:

$$\mathcal{N}(t) = \max\{n|\tau_1 + \tau_2 + \cdots + \tau_n \leq t, \tau_1 + \tau_2 + \cdots + \tau_n + \tau_{n+1} > t\}, t > 0.$$

Then, we note that $\mathcal{N}(t)$ gives the total number of fluctuations of $\{\sigma(t), t \geq 0\}$ in the interval $(0, t]$, and we have

$$\sigma(t) = \left(\frac{\sigma_1 + \sigma_2}{2}\right) + \left(\frac{\sigma_1 - \sigma_2}{2}\right) (-1)^{\mathcal{N}(t)}. \tag{5.3}$$

For simplicity, we set

$$\gamma = \frac{\sigma_1 + \sigma_2}{2}, \delta = \frac{\sigma_1 - \sigma_2}{2}.$$

Then, we have

$$\gamma^2 + \delta^2 = \frac{\sigma_1^2 + \sigma_2^2}{2}, \gamma\delta = \frac{\sigma_1^2 - \sigma_2^2}{4}.$$

Now, Eq. (5.2) becomes

$$dS(t) = \mu S(t)dt + \left(\gamma + \delta(-1)^{\mathcal{N}(t)}\right) S(t)dW(t). \tag{5.4}$$

The stock market is governed by the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Equation (5.4) can be solved by applying the Ito-differentiation formula. To this end, we observe

$$\begin{aligned} d\log_e S(t) &= \log_e \left\{1 + \frac{dS(t)}{S(t)}\right\} \\ &= \frac{\mathrm{d}S(t)}{S(t)} - \frac{1}{2}\left\{\frac{dS(t)}{S(t)}\right\}^2 + \frac{1}{3}\left\{\frac{dS(t)}{S(t)}\right\}^3 - \cdots \\ &= \left(\mu - \frac{\gamma^2+\delta^2}{2}\right)dt + \gamma dW(t) + (-1)^{\mathcal{N}(t)}\left(-\gamma\delta dt + \delta dW(t)\right). \end{aligned}$$

Hence, we obtain

$$S(t) = S(0)\exp\Bigg[\left(\mu - \frac{\gamma^2+\delta^2}{2}\right)t + \gamma W(t) + \int_0^t (-1)^{\mathcal{N}(u)}\left(-\gamma\delta du + \delta dW(u)\right)\Bigg]. \tag{5.5}$$

5.3 The Stochastic Volatility Process $\{\sigma(t), t \geq 0\}$

To study the processes $\{S(t), t \geq 0\}$, we have to know the probability law governing the volatility process $\{\sigma(t), t \geq 0\}$. For this, we shall go in steps. First, we proceed to study the point process $\{\mathcal{N}(t), t \geq 0\}$. We define

$$p_n(t) = \mathcal{P}\{\mathcal{N}(t) = n\}, n = 1, 2, 3, \ldots.$$

Then, by renewal-theoretical arguments, the probabilities $p_n(t)$ are given by

$$\begin{aligned} p_0(t) &= \bar{F}(t),\ p_1(t) = \int_0^t f(u)\bar{G}(t-u)du, \\ p_{2n}(t) &= \int_0^t (f*g)^{(n)}(u)\bar{F}(t-u)du, n = 1, 2, \ldots \\ p_{2n+1}(t) &= \int_0^t ((f*g)^{(n)} * f)(u)\bar{G}(t-u)du, n = 1, 2, \cdots, \end{aligned}$$

where

$$\begin{aligned} \bar{F}(t) &= 1 - F(t), \\ \bar{G}(t) &= 1 - G(t), \\ (f*g)^{(1)}(t) &= (f*g)(t) = \int_0^t f(u)g(t-u)du, \\ (f*g)^{(n)}(t) &= \int_0^t (f*g)^{(n-1)}(u)(f*g)(t-u)du, n = 2, 3, \ldots. \end{aligned}$$

The Laplace transforms of $p_n(t), n = 0, 1, 2, \ldots$ are given by

$$p_0{}^*(s) = \frac{1 - f^*(s)}{s},$$
$$p_1{}^*(s) = f^*(s)\left(\frac{1 - g^*(s)}{s}\right),$$
$$p_{2n}{}^*(s) = \{f^*(s)\}^n\{g^*(s)\}^n\left(\frac{1 - f^*(s)}{s}\right), n = 1, 2, 3, \ldots,$$
$$p_{2n+1}{}^*(s) = \{f^*(s)\}^{n+1}\{g^*(s)\}^n\left(\frac{1 - g^*(s)}{s}\right), n = 1, 2, 3, \ldots.$$

If $G(\theta, t) = \sum_{n=0}^{\infty} p_n(t)\theta^n$, then the Laplace transform of $G(\theta, t)$ is

$$G^*(\theta, s) = \sum_{n=0}^{\infty} p_n{}^*(s)\theta^n = \frac{1 - f^*(s) + \theta f^*(s)\{1 - g^*(s)\}}{s\{1 - \theta^2 f^*(s)g^*(s)\}}.$$

To get explicit expressions, we shall consider the following particular case

$$f(t) = \lambda_1 \mathrm{e}^{-\lambda_1 t}, g(t) = \lambda_2 \mathrm{e}^{-\lambda_2 t}.$$

Then, we have

$$G^*(\theta, s) = \frac{s + \lambda_1\theta + \lambda_2}{(s + \lambda_1)(s + \lambda_2) - \lambda_1\lambda_2\theta^2}. \tag{5.6}$$

The roots of the equation

$$(s + \lambda_1)(s + \lambda_2) - \lambda_1\lambda_2\theta^2 = 0$$

are real and distinct, since their discriminant is $(\lambda_1 - \lambda_2)^2 + 4\lambda_1\lambda_2\theta^2$ which is > 0. Let α and β be the roots. Then

$$\alpha = \frac{-(\lambda_1 + \lambda_2) + \sqrt{(\lambda_1 - \lambda_2)^2 + 4\lambda_1\lambda_2\theta^2}}{2},$$
$$\beta = \frac{-(\lambda_1 + \lambda_2) - \sqrt{(\lambda_1 - \lambda_2)^2 + 4\lambda_1\lambda_2\theta^2}}{2}.$$

Now, Eq. (5.6) can be written as

$$G^*(\theta, s) = \frac{s + \lambda_1\theta + \lambda_2}{(s - \alpha)(s - \beta)} = \frac{\alpha + \lambda_1\theta + \lambda_2}{\alpha - \beta}\frac{1}{s - \alpha} - \frac{\beta + \lambda_1\theta + \lambda_2}{\alpha - \beta}\frac{1}{s - \beta}.$$

Inverting, we obtain

$$G(\theta,t) = \frac{1}{\alpha-\beta}\left\{(\alpha+\lambda_1\theta+\lambda_2)e^{\alpha t} - (\beta+\lambda_1\theta+\lambda_2)e^{\beta t}\right\}$$
$$= \frac{e^{-\frac{(\lambda_1+\lambda_2)}{2}t}}{\kappa}\left\{\kappa\cosh\left(\frac{\kappa}{2}t\right) + (\lambda_2-\lambda_1+2\lambda_1\theta)\sinh\left(\frac{\kappa}{2}t\right)\right\}, \quad (5.7)$$

where $\kappa = \sqrt{(\lambda_1-\lambda_2)^2+4\lambda_1\lambda_2\theta^2}$.

To get $p_n(t)$, we proceed as follows:

$$\sum_{n=0}^{\infty} p_n{}^*(s)\theta^n = G^*(\theta,s) = \frac{s+\lambda_1\theta+\lambda_2}{(s+\lambda_1)(s+\lambda_2)-\lambda_1\lambda_2\theta^2}$$
$$= \left[\frac{1}{(s+\lambda_1)} + \frac{\lambda_1\theta}{(s+\lambda_1)(s+\lambda_2)}\right]\left\{1-\frac{\lambda_1\lambda_2\theta^2}{(s+\lambda_1)(s+\lambda_2)}\right\}^{-1}$$
$$= \left[\frac{1}{(s+\lambda_1)} + \frac{\lambda_1\theta}{(s+\lambda_1)(s+\lambda_2)}\right]\sum_{n=0}^{\infty}\frac{\lambda_1^n\lambda_2^n\theta^{2n}}{(s+\lambda_1)^n(s+\lambda_2)^n}$$
$$= \sum_{n=0}^{\infty}\frac{\lambda_1^n\lambda_2^n\theta^{2n}}{(s+\lambda_1)^{n+1}(s+\lambda_2)^n} + \sum_{n=0}^{\infty}\frac{\lambda_1^{n+1}\lambda_2^n\theta^{2n+1}}{(s+\lambda_1)^{n+1}(s+\lambda_2)^{n+1}}.$$

Equating the coefficients, we obtain

$$p_{2n}{}^*(s) = \frac{\lambda_1^n\lambda_2^n}{(s+\lambda_1)^{n+1}(s+\lambda_2)^n}$$
$$= \sum_{j=0}^{\infty}\frac{\lambda_1^n\lambda_2^n(-1)^j\binom{-n}{j}(\lambda_1-\lambda_2)^j}{(s+\lambda_1)^{2n+j+1}}, n = 0, 1, 2, \ldots;$$
$$p_{2n+1}{}^*(s) = \frac{\lambda_1^{n+1}\lambda_2^n}{(s+\lambda_1)^{2n+2}\left(1-\frac{\lambda_1-\lambda_2}{s+\lambda_1}\right)^{n+1}}$$
$$= \sum_{j=0}^{\infty}\frac{(-1)^j\binom{-n-1}{j}\lambda_1^{n+1}\lambda_2^n(\lambda_1-\lambda_2)^j}{(s+\lambda_1)^{2n+j+2}}, n = 0, 1, 2, \ldots.$$

Inverting the above equations, we get

$$p_{2n}(t) = \sum_{j=0}^{\infty}\frac{\lambda_1^n\lambda_2^n(-1)^j\binom{-n}{j}(\lambda_1-\lambda_2)^j e^{-\lambda_1 t}t^{2n+j}}{(2n+j)!}, \quad (5.8)$$
$$p_{2n+1}(t) = \sum_{j=0}^{\infty}\frac{\lambda_1^{n+1}\lambda_2^n(-1)^j\binom{-n-1}{j}(\lambda_1-\lambda_2)^j e^{-\lambda_1 t}t^{2n+j+1}}{(2n+j+1)!} \quad (5.9)$$

for $n = 0, 1, 2, \ldots$.

Next, we proceed to provide the probability distribution of the process $\{\sigma(t), t \geq 0\}$. For this, we define

$$f_{ij}(s,t) = \mathcal{P}\{\sigma(t) = j | \sigma(s) = i\}, s \leq t; i, j = 1, 2.$$

Then we obtain the following equations:

$$\begin{aligned}
\frac{\partial f_{11}(s,t)}{\partial t} &= -\lambda_1 f_{11}(s,t) + \lambda_2 f_{12}(s,t),\\
\frac{\partial f_{12}(s,t)}{\partial t} &= -\lambda_2 f_{12}(s,t) + \lambda_1 f_{11}(s,t),\\
\frac{\partial f_{22}(s,t)}{\partial t} &= -\lambda_2 f_{22}(s,t) + \lambda_1 f_{21}(s,t),\\
\frac{\partial f_{21}(s,t)}{\partial t} &= -\lambda_1 f_{21}(s,t) + \lambda_2 f_{22}(s,t).
\end{aligned}$$

Solving the above equations with the initial condition $f_{ij}(s,s) = \delta_{ij}$, we obtain

$$\begin{aligned}
f_{11}(s,t) &= \frac{\lambda_2}{\lambda_1+\lambda_2} + \frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)(t-s)},\\
f_{12}(s,t) &= \frac{\lambda_1}{\lambda_1+\lambda_2} - \frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)(t-s)},\\
f_{22}(s,t) &= \frac{\lambda_1}{\lambda_1+\lambda_2} + \frac{\lambda_2}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)(t-s)},\\
f_{21}(s,t) &= \frac{\lambda_2}{\lambda_1+\lambda_2} - \frac{\lambda_2}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)(t-s)}.
\end{aligned}$$

Since we have assumed the initial condition as $\sigma(0) = \sigma_1$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{2n}(t) = f_{11}(t) &= \frac{\lambda_2}{\lambda_1+\lambda_2} + \frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)t},\\
\sum_{n=0}^{\infty} p_{2n+1}(t) = f_{12}(t) &= \frac{\lambda_1}{\lambda_1+\lambda_2} - \frac{\lambda_1}{\lambda_1+\lambda_2} e^{-(\lambda_1+\lambda_2)t},
\end{aligned}$$

where $f_{1j}(t)$ is the shorthand notation for $f_{1j}(s,t)$ at $s=0$ for $j=1,2$. Now we proceed to study the stochastic process $\{M(t), t \geq 0\}$ defined by

$$M(t) = \int_0^t (-1)^{N(\tau)} d\tau.$$

For this, we define the generating functions

$$G_1(\theta, t) = \mathrm{E}\left\{e^{-\theta M(t)} | \sigma(0) = \sigma_1\right\},$$
$$G_2(\theta, t) = \mathrm{E}\left\{e^{-\theta M(t)} | \sigma(0) = \sigma_2\right\}.$$

Then we obtain the following integral equations:

$$G_1(\theta, t) = e^{-\lambda_1 t} e^{-\theta t} + \lambda_1 \int_0^t e^{-\lambda_1 \tau} e^{-\theta \tau} G_2(\theta, t-\tau) d\tau,$$
$$G_2(\theta, t) = e^{-\lambda_2 t} e^{\theta t} + \lambda_2 \int_0^t e^{-\lambda_2 \tau} e^{\theta \tau} G_1(\theta, t-\tau) d\tau.$$

Taking Laplace transforms, the above equations yield

$$G_1^*(\theta, \eta) = \frac{1}{\eta + \lambda_1 + \theta} + \frac{\lambda_1}{\eta + \lambda_1 + \theta} G_2^*(\theta, \eta),$$
$$G_2^*(\theta, \eta) = \frac{1}{\eta + \lambda_2 - \theta} + \frac{\lambda_2}{\eta + \lambda_2 - \theta} G_1^*(\theta, \eta).$$

Solving the above equations, we get

$$G_1^*(\theta, \eta) = \frac{\eta + \lambda_2 - \theta + \lambda_1}{\eta^2 + \eta(\lambda_1 + \lambda_2) - (\lambda_1 - \lambda_2)\theta - \theta^2},$$
$$G_2^*(\theta, \eta) = \frac{\eta + \lambda_2 + \theta + \lambda_1}{\eta^2 + \eta(\lambda_1 + \lambda_2) - (\lambda_1 - \lambda_2)\theta - \theta^2}.$$

Let η_1 and η_2 be the roots of the equation

$$\eta^2 + \eta(\lambda_1 + \lambda_2) - (\lambda_1 - \lambda_2)\theta - \theta^2 = 0$$

which are given by

$$\eta_1 = \frac{-(\lambda_1 + \lambda_2) + \sqrt{(\lambda_1 + \lambda_2)^2 + 4\{(\lambda_1 - \lambda_2)\theta + \theta^2\}}}{2},$$
$$\eta_2 = \frac{-(\lambda_1 + \lambda_2) - \sqrt{(\lambda_1 + \lambda_2)^2 + 4\{(\lambda_1 - \lambda_2)\theta + \theta^2\}}}{2}.$$

Then we get

$$G_1^*(\theta, \eta) = \frac{1}{(\eta_1 - \eta_2)}\left[\frac{\eta_1 + \lambda_2 - \theta + \lambda_1}{\eta - \eta_1} - \frac{\eta_2 + \lambda_2 - \theta + \lambda_1}{\eta - \eta_2}\right],$$
$$G_2^*(\theta, \eta) = \frac{1}{(\eta_1 - \eta_2)}\left[\frac{\eta_1 + \lambda_2 + \theta + \lambda_1}{\eta - \eta_1} - \frac{\eta_2 + \lambda_2 + \theta + \lambda_1}{\eta - \eta_2}\right].$$

Inverting the above, we get

$$G_1(\theta, t) = \frac{e^{-(\lambda_1+\lambda_2)t/2}}{\beta_1}\left[\beta_1 \cosh(\beta_1 t/2) + (\lambda_1 - 2\theta + \lambda_2)\sinh(\beta_1 t/2)\right], \quad (5.10)$$

$$G_2(\theta, t) = \frac{e^{-(\lambda_1+\lambda_2)t/2}}{\beta_1}\left[\beta_1 \cosh(\beta_1 t/2) + (\lambda_1 + 2\theta + \lambda_2)\sinh(\beta_1 t/2)\right], \quad (5.11)$$

where

$$\beta_1 = \sqrt{(\lambda_1 + \lambda_2)^2 + 4\{(\lambda_1 - \lambda_2)\theta + \theta^2\}}.$$

We shall now derive the conditional densities of the occupation time in the state σ_i of the volatility process $\{\sigma(t), t \geq 0\}$ in the trading interval $[0, T]$. Let $X_{ij}(t), i, j = 1, 2$ be the random variable representing the total occupation time in the state σ_i of the volatility process in the interval $[0, t]$ given that the volatility occupied the state σ_j at time 0. Then, we note that

$$X_{ij}(t) = \int_0^t \frac{1 + (-1)^{N(u)+1-\delta_{ij}}}{2} du, i, j = 1, 2,$$

where $\mathcal{N}(t)$ is the counting process representing the total number of changes made by the volatility process in the interval $[0, t]$. The conditional densities of the occupation time $X_{ij}(t), i, j = 1, 2$ of the volatility process are defined by

$$f_i(x, t|j) = \lim_{\Delta \to 0} \frac{\mathcal{P}\{x < X_{ij}(t) < x + \Delta\}}{\Delta}, i, j = 1, 2.$$

Considering the moment generating function $m_i(\theta, t|j)$ of $X_{ij}(t), i, j = 1, 2$, we have

$$m_i(\theta, t|j) = \mathrm{E}\{e^{-\theta X_{ij}(t)}\} = \int_0^t e^{-\theta x} f_i(x, t|j) dx.$$

We note that $m_i(\theta, t|j)$ is simply the Laplace transform of the probability density function $f_i(x, t|j)H(t - x)$, where $H(t - x)$ is the Heaviside function. We can derive inter-connected integral equations for the functions $m_i(\theta, t|j), i, j = 1, 2$. Arguing whether or not the volatility changes its state in the interval $(0, t)$, and using the Markov property, we get

$$m_1(\theta, t|1) = e^{-\lambda_1 t}e^{-\theta t} + \lambda_1 \int_0^t e^{-\lambda_1 u}e^{-\theta u} m_1(\theta, t-u|2)du,$$
$$m_1(\theta, t|2) = e^{-\lambda_2 t} + \lambda_2 \int_0^t e^{-\lambda_2 u} m_1(\theta, t-u|1)du,$$
$$m_2(\theta, t|1) = e^{-\lambda_1 t} + \lambda_1 \int_0^t e^{-\lambda_1 u} m_2(\theta, t-u|2)du,$$
$$m_2(\theta, t|2) = e^{-\lambda_2 t}e^{-\theta t} + \lambda_2 \int_0^t e^{-\lambda_2 u}e^{-\theta u} m_2(\theta, t-u|1)du.$$

Applying the Laplace transform, the above equations yield

$$m_1^*(\theta, s|1) = \frac{1}{s+\lambda_1+\theta} + \frac{\lambda_1}{s+\lambda_1+\theta} m_1^*(\theta, s|2),$$
$$m_1^*(\theta, s|2) = \frac{1}{s+\lambda_2} + \frac{\lambda_2}{s+\lambda_2} m_1^*(\theta, s|1),$$
$$m_2^*(\theta, s|1) = \frac{1}{s+\lambda_1} + \frac{\lambda_1}{s+\lambda_1} m_2^*(\theta, s|2),$$
$$m_2^*(\theta, s|2) = \frac{1}{s+\lambda_2+\theta} + \frac{\lambda_2}{s+\lambda_2+\theta} m_2^*(\theta, s|1).$$

Solving the above equations, we get

$$m_1^*(\theta, s|1) = \frac{s+\lambda_1+\lambda_2}{s^2+s(\lambda_1+\lambda_2+\theta)+\lambda_2\theta}, \tag{5.12}$$
$$m_1^*(\theta, s|2) = \frac{s+\lambda_1+\lambda_2+\theta}{s^2+s(\lambda_1+\lambda_2+\theta)+\lambda_2\theta}, \tag{5.13}$$
$$m_2^*(\theta, s|1) = \frac{s+\lambda_1+\lambda_2+\theta}{s^2+s(\lambda_1+\lambda_2+\theta)+\lambda_1\theta}, \tag{5.14}$$
$$m_2^*(\theta, s|2) = \frac{s+\lambda_1+\lambda_2}{s^2+s(\lambda_1+\lambda_2+\theta)+\lambda_1\theta}. \tag{5.15}$$

Taking (5.12) and inverting it with respect to θ, we get

$$f_1^*(x, s|1) = \left(1 + \frac{\lambda_1}{s+\lambda_2}\right) e^{-\lambda_1 x} e^{-sx} e^{\frac{\lambda_1\lambda_2}{s+\lambda_2}}$$
$$= e^{-\lambda_1 x} e^{\lambda_2 x} \left(e^{-(s+\lambda_2)x} e^{\frac{\lambda_1\lambda_2}{s+\lambda_2}} + e^{-(s+\lambda_2)x} \frac{\lambda_1}{s+\lambda_2} e^{\frac{\lambda_1\lambda_2}{s+\lambda_2}} \right).$$

Inverting the above equation with respect to s and using the table of Laplace transforms (refer [1]), we get

$$
\begin{aligned}
f_1(x,t|1) &= e^{-\lambda_1 x} e^{\lambda_2 x}\left[e^{-\lambda_2 t} L^{-1}\left(e^{-sx} e^{\frac{\lambda_1\lambda_2}{s}}\right) + \lambda_1 e^{-\lambda_2 t} L^{-1}\left(\frac{e^{-sx}}{s} e^{\frac{\lambda_1\lambda_2}{s}}\right)\right] \\
&= e^{-\lambda_1 x} e^{\lambda_2 x}\left[e^{-\lambda_2 t}\left\{L^{-1}\left(e^{\frac{\lambda_1\lambda_2}{s}}\right)\right\}_{t\to t-x}\right. \\
&\qquad \left. + \lambda_1 e^{-\lambda_2 t}\left\{L^{-1}\left(\frac{1}{s} e^{\frac{\lambda_1\lambda_2}{s}}\right)\right\}_{t\to t-x}\right] \\
&= e^{-\lambda_1 x} e^{-\lambda_2 (t-x)}\left[\delta(t-x) + \sqrt{\frac{\lambda_1\lambda_2}{t-x}} I_1\left(2\sqrt{\lambda_1\lambda_2(t-x)}\right)\right. \\
&\qquad \left. + \lambda_1 I_0\left(2\sqrt{\lambda_1\lambda_2(t-x)}\right)\right].
\end{aligned} \tag{5.16}
$$

Similarly, taking (5.13) and inverting it with respect to θ, we get

$$
f_1^*(x,s|2) = \frac{1}{s+\lambda_2}\delta(x) + \left(\frac{\lambda_2}{s+\lambda_2} + \frac{\lambda_1\lambda_2}{(s+\lambda_2)^2}\right) e^{(\lambda_2-\lambda_1)x} e^{-(s+\lambda_2)x} e^{\frac{\lambda_1\lambda_2}{s+\lambda_2}}.
$$

Inverting the above equation with respect to s, we get

$$
\begin{aligned}
f_1(x,t|2) &= e^{-\lambda_2 t}\delta(x) + e^{-\lambda_2 t} e^{(\lambda_2-\lambda_1)x} L^{-1}\left[e^{-sx}\left(\frac{\lambda_2}{s} + \frac{\lambda_1\lambda_2}{s^2}\right) e^{\frac{\lambda_1\lambda_2}{s}}\right] \\
&= e^{-\lambda_2 t}\delta(x) + e^{-\lambda_1 x} e^{-\lambda_2 (t-x)}\left[\lambda_2 I_0\left(2\sqrt{\lambda_1\lambda_2(t-x)}\right)\right. \\
&\qquad \left. + \sqrt{\frac{\lambda_1\lambda_2(t-x)}{x}} I_1\left(2\sqrt{\lambda_1\lambda_2(t-x)}\right)\right].
\end{aligned} \tag{5.17}
$$

Interchanging λ_1 and λ_2 in Eqs. (5.16) and (5.17), we get

$$
\begin{aligned}
f_2(x,t|2) &= e^{-\lambda_2 x} e^{-\lambda_1 (t-x)}\left[\delta(t-x) + \sqrt{\frac{\lambda_1\lambda_2}{t-x}} I_1\left(2\sqrt{\lambda_1\lambda_2(t-x)}\right)\right. \\
&\qquad \left. + \lambda_2 I_0\left(2\sqrt{\lambda_1\lambda_2(t-x)}\right)\right],
\end{aligned} \tag{5.18}
$$

$$
\begin{aligned}
f_2(x,t|1) &= e^{-\lambda_1 t}\delta(x) + e^{-\lambda_2 x} e^{-\lambda_1 (t-x)}\left[\lambda_1 I_0\left(2\sqrt{\lambda_1\lambda_2(t-x)}\right)\right. \\
&\qquad \left. + \sqrt{\frac{\lambda_1\lambda_2(t-x)}{x}} I_1\left(2\sqrt{\lambda_1\lambda_2(t-x)}\right)\right].
\end{aligned} \tag{5.19}
$$

The results (5.16), (5.17), (5.18), and (5.19) agree with [11]. Now we proceed to investigate an equivalent martingale measure Q under which the discounted asset price $\overline{S}(t) = e^{-rt} S(t)$ becomes a martingale.

5.4 An Equivalent Martingale Measure

We observe from (5.4) that

$$\begin{aligned}\frac{d\overline{S}(t)}{\overline{S}(t)} &= (\mu - r)dt + \left(\gamma + \delta(-1)^{\mathcal{N}(t)}\right) dW(t) \\ &= (\mu - r)dt + \gamma dW(t) + \delta(-1)^{\mathcal{N}(t)} dW(t) \\ &= \sigma_1 \left(\frac{1 + (-1)^{\mathcal{N}(t)}}{2}\right) d\left(\frac{(\mu - r)t}{\sigma_1} + W(t)\right) \\ &\quad + \sigma_2 \left(\frac{1 - (-1)^{\mathcal{N}(t)}}{2}\right) d\left(\frac{(\mu - r)t}{\sigma_2} + W(t)\right). \end{aligned} \tag{5.20}$$

We set

$$\begin{aligned} W_1(t) &= \frac{(\mu - r)t}{\sigma_1} + W(t), \\ W_2(t) &= \frac{(\mu - r)t}{\sigma_2} + W(t). \end{aligned}$$

By Girsanov's theorem, there exist probability measures $Q_i, i = 1, 2$ equivalent to the market measure $\mathcal{P}$ such that $W_1(t)$ and $W_2(t)$ are standard Brownian motions with respect to Q_1 and Q_2, respectively. The Radon-Nikodym derivative of Q_i with respect to $\mathcal{P}$ is given by

$$\frac{dQ_i}{d\mathcal{P}} = L_i(t) = \exp\left[-\left(\frac{\mu - r}{\sigma_i}\right) W_i(t) - \frac{1}{2}\left(\frac{\mu - r}{\sigma_i}\right)^2 t\right].$$

We note that $\mathrm{E}^{\mathcal{P}}[L_i(t)] = 1$ and $\mathcal{N}(t)$ is independent of $L_i(t), i = 1, 2$. Suppose that

$$L(t) = \left(\frac{1 + (-1)^{\mathcal{N}(t)}}{2}\right) L_1(t) + \left(\frac{1 - (-1)^{\mathcal{N}(t)}}{2}\right) L_2(t). \tag{5.21}$$

Then, we have

$$\begin{aligned} \mathrm{E}^{\mathcal{P}}[L(t)] &= \mathrm{E}^{\mathcal{P}}\left\{\left(\frac{1 + (-1)^{\mathcal{N}(t)}}{2}\right)\right\} \mathrm{E}^{\mathcal{P}}[L_1(t)] \\ &\quad + \mathrm{E}^{\mathcal{P}}\left\{\left(\frac{1 - (-1)^{\mathcal{N}(t)}}{2}\right)\right\} \mathrm{E}^{\mathcal{P}}[L_2(t)] \\ &= \left[\frac{\lambda_2}{\lambda_1 + \lambda_2} + \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2)t}\right] \times 1 \\ &\quad + \left[\frac{\lambda_1}{\lambda_1 + \lambda_2} - \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-(\lambda_1 + \lambda_2)t}\right] \times 1 = 1. \end{aligned}$$

Consequently, we can define a probability measure Q by the equation

$$Q[A] = \mathrm{E}^{\mathcal{P}}\left[1_A L(t)\right],$$

where A is any market event. Suppose that we define

$$W^*(t) = \left(\frac{1 + (-1)^{\mathcal{N}(t)}}{2}\right) W_1(t) + \left(\frac{1 - (-1)^{\mathcal{N}(t)}}{2}\right) W_2(t).$$

Then Eq. (4.1) becomes

$$\frac{d\overline{S}(t)}{\overline{S}(t)} = \sigma(t) dW^*(t). \tag{5.22}$$

To know about $W^*(t)$, we note that

$$d[L(t)W^*(t)] = \left(\frac{1 + (-1)^{\mathcal{N}(t)}}{2}\right) d[L_1(t)W_1(t)] + \left(\frac{1 - (-1)^{\mathcal{N}(t)}}{2}\right) d[L_2(t)W_2(t)]. \tag{5.23}$$

It is easy to see that

$$dL_i(t) = -\left(\frac{\mu - r}{\sigma_i}\right) L_i(t) dW(t),\ i = 1, 2;$$

$$dW_i(t) = \left(\frac{\mu - r}{\sigma_i}\right) dt + dW(t),\ i = 1, 2.$$

Consequently, we get

$$d[L_i(t)W_i(t)] = L_i(t)\left[1 - \left(\frac{\mu - r}{\sigma_i}\right) W(t)\right] dW(t),\ i = 1, 2.$$

From the above, we get

$$\mathrm{E}^{\mathcal{P}}[d\{L_i(t)W_i(t)\}] = 0,\ i = 1, 2.$$

Then, from (5.23), we get

$$\mathrm{E}^{\mathcal{P}}[d\{L(t)W^*(t)\}] = 0. \tag{5.24}$$

Hence, $L(t)W^*(t)$ is a $\mathcal{P}-$martingale. Now, we get

$$\mathrm{E}^{\mathcal{Q}}[W^*(t)|\mathcal{F}_s] = \mathrm{E}^{\mathcal{P}}\left[\frac{L(t)}{L(s)}W^*(t)|\mathcal{F}_s\right] = \frac{1}{L(s)}L(s)W^*(s) = W^*(s).$$

Accordingly, $W^*(t)$ is a $\mathcal{Q}-$martingale. Further, we have

$$\begin{aligned}
\mathrm{E}^{\mathcal{Q}}[W^*(t)] &= \mathrm{E}^{\mathcal{P}}\left[L(t)W^*(t)\right] \\
&= \mathrm{E}^{\mathcal{P}}\left[L(t)\left\{\left(\frac{1+(-1)^{\mathcal{N}(t)}}{2}\right)W_1(t) + \left(\frac{1-(-1)^{\mathcal{N}(t)}}{2}\right)W_2(t)\right\}\right] \\
&= \mathrm{E}^{\mathcal{P}}\left[\left(\frac{1+(-1)^{\mathcal{N}(t)}}{2}\right)L_1(t)W_1(t) + \left(\frac{1-(-1)^{\mathcal{N}(t)}}{2}\right)L_2(t)W_2(t)\right] \\
&= \mathrm{E}^{\mathcal{P}}\left[\frac{1+(-1)^{\mathcal{N}(t)}}{2}\right]\mathrm{E}^{\mathcal{P}}\left[L_1(t)W_1(t)\right] \\
&\quad + \mathrm{E}^{\mathcal{P}}\left[\frac{1-(-1)^{\mathcal{N}(t)}}{2}\right]\mathrm{E}^{\mathcal{P}}\left[L_2(t)W_2(t)\right].
\end{aligned} \tag{5.25}$$

But, we have

$$\begin{aligned}
\mathrm{E}^{\mathcal{P}}\left[L_i(t)W_i(t)\right] &= \mathrm{E}^{\mathcal{P}}\left[e^{-\theta_i W(t) - \frac{1}{2}\theta_i^2 t}\left\{\theta_i t + W(t)\right\}\right] \\
&= e^{-\frac{1}{2}\theta_i^2 t}\mathrm{E}^{\mathcal{P}}\left[e^{-\theta_i W(t)}\left\{\theta_i t + W(t)\right\}\right] \\
&= e^{-\frac{1}{2}\theta_i^2 t}\left[\theta_i t \mathrm{E}^{\mathcal{P}}\left\{e^{-\theta_i W(t)}\right\} + \mathrm{E}^{\mathcal{P}}\left\{e^{-\theta_i W(t)}W(t)\right\}\right] \\
&= e^{-\frac{1}{2}\theta_i^2 t}\left[\theta_i t e^{\frac{1}{2}\theta_i^2 t} - e^{\frac{1}{2}\theta_i^2 t}\theta_i t\right] = 0.
\end{aligned}$$

Then Eq. (5.25) becomes

$$\mathrm{E}^{\mathcal{Q}}[W^*(t)] = 0. \tag{5.26}$$

Hence, we get

$$\mathrm{E}^{\mathcal{Q}}[dW^*(t)] = 0. \tag{5.27}$$

We also note that

$$\begin{aligned}
\left[dW^*(t)\right]^2 &= \left(\frac{1+(-1)^{\mathcal{N}(t)}}{2}\right)[dW_1(t)]^2 + \left(\frac{1-(-1)^{\mathcal{N}(t)}}{2}\right)[dW_2(t)]^2 \\
&= \left(\frac{1+(-1)^{\mathcal{N}(t)}}{2}\right)\left[\left(\frac{\mu - r}{\sigma_1}\right)dt + dW(t)\right]^2 \\
&\quad + \left(\frac{1-(-1)^{\mathcal{N}(t)}}{2}\right)\left[\left(\frac{\mu - r}{\sigma_2}\right)dt + dW(t)\right]^2 \\
&= \left(\frac{1+(-1)^{\mathcal{N}(t)}}{2}\right)[dW(t)]^2 + \left(\frac{1-(-1)^{\mathcal{N}(t)}}{2}\right)[dW(t)]^2 \\
&= [dW(t)]^2.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\mathrm{E}^{Q}[dW^*(t)]^2 &= \mathrm{E}^{\mathcal{P}}[L(t)\{dW^*(t)\}^2] = \mathrm{E}^{\mathcal{P}}[L(t)\{dW(t)\}^2]\\
&= \mathrm{E}^{\mathcal{P}}\left[\left\{\left(\frac{1+(-1)^{\mathcal{N}(t)}}{2}\right)L_1(t)\right.\right.\\
&\quad \left.\left.+\left(\frac{1-(-1)^{\mathcal{N}(t)}}{2}\right)L_2(t)\right\}\{dW(t)\}^2\right]\\
&= \mathrm{E}^{\mathcal{P}}\left[\frac{1+(-1)^{\mathcal{N}(t)}}{2}\right]\mathrm{E}^{\mathcal{P}}\left[L_1(t)\{dW(t)\}^2\right]\\
&\quad + \mathrm{E}^{\mathcal{P}}\left[\frac{1-(-1)^{\mathcal{N}(t)}}{2}\right]\mathrm{E}^{\mathcal{P}}\left[L_2(t)\{dW(t)\}^2\right]\\
&= \left[\frac{\lambda_2}{\lambda_1+\lambda_2}+\frac{\lambda_1}{\lambda_1+\lambda_2}e^{-(\lambda_1+\lambda_2)t}\right]\mathrm{E}^{\mathcal{P}}\left[L_1(t)\right]\mathrm{E}^{\mathcal{P}}\{dW(t)\}^2\\
&\quad + \left[\frac{\lambda_1}{\lambda_1+\lambda_2}-\frac{\lambda_1}{\lambda_1+\lambda_2}e^{-(\lambda_1+\lambda_2)t}\right]\mathrm{E}^{\mathcal{P}}\left[L_2(t)\right]\mathrm{E}^{\mathcal{P}}\{dW(t)\}^2\\
&= \left[\frac{\lambda_2}{\lambda_1+\lambda_2}+\frac{\lambda_1}{\lambda_1+\lambda_2}e^{-(\lambda_1+\lambda_2)t}\right]\times 1dt\\
&\quad + \left[\frac{\lambda_1}{\lambda_1+\lambda_2}-\frac{\lambda_1}{\lambda_1+\lambda_2}e^{-(\lambda_1+\lambda_2)t}\right]\times 1dt\\
&= dt.
\end{aligned}$$

Now, for the exponential process

$$M(t, W^*(t)) = e^{\alpha W^*(t)-\alpha^2 t/2},$$

we note that

$$\begin{aligned}
dM(t, W^*(t)) &= -\frac{\alpha^2}{2}M(t, W^*(t))dt + M(t, W^*(t))\alpha dW^*(t) + \frac{1}{2}\alpha^2 M(t, W^*(t))dt\\
&= M(t, W^*(t))\alpha dW^*(t).
\end{aligned}$$

Then, we have

$$\mathrm{E}^{Q}[dM(t, W^*(t))] = \alpha\mathrm{E}^{Q}[M(t, W^*(t))]\mathrm{E}^{Q}[dW^*(t)] = 0.$$

This shows that $M(t, W^*(t))$ is a $Q-$martingale. Consequently, whenever $s < t$, we get

$$\mathrm{E}^{Q}\left[M(t, W^*(t))|\mathcal{F}(s)\right] = M(s, W^*(s)).$$

From the above, we get

$$E^Q\left[\frac{M(t, W^*(t))}{M(s, W^*(s))}|\mathcal{F}(s)\right] = 1.$$

Using the definition of $M(t, W^*(t))$ in the above equation, we get

$$E^Q\left[\frac{e^{\alpha W^*(t)-\alpha^2 t/2}}{e^{\alpha W^*(s)-\alpha^2 s/2}}|\mathcal{F}(s)\right] = 1.$$

The above equation gives

$$E^Q\left[e^{\alpha\{W^*(t)-W^*(s)\}-\alpha^2(t-s)/2}|\mathcal{F}(s)\right] = 1.$$

Rewriting the equation, we get

$$E^Q\left[e^{\alpha\{W^*(t)-W^*(s)\}}|\mathcal{F}(s)\right] = e^{\alpha^2(t-s)/2}.$$

The right-hand side of the above equation is identified with the moment-generating function of a Gaussian distribution. Hence, by using Levy's characterization of Brownian motion, we observe that $\{W^*(t), t \geq 0\}$ is a standard Brownian motion. We are now comfortably placed to go ahead to the stochastic analysis of the logarithm of the stock price $S(t)$ under the $Q-$measure. First, we write

$$Z(t) = \log_e S(t) = \log_e\{e^{rt}\overline{S(t)}\} = rt + \log_e \overline{S(t)}.$$

Then we find

$$\begin{aligned} dZ(t) &= rdt + \log_e\left(1 + \frac{d\overline{S(t)}}{\overline{S(t)}}\right) \\ &= rdt + \frac{d\overline{S(t)}}{\overline{S(t)}} - \frac{1}{2}\left\{\frac{d\overline{S(t)}}{\overline{S(t)}}\right\}^2 + \cdots . \end{aligned} \tag{5.28}$$

From the definition of Q and using (5.21), we get

$$E^Q\left[\frac{d\overline{S}(t)}{\overline{S}(t)}\right] = E^Q\left[\sigma(t)dW^*(t)\right] = E^{\mathcal{P}}\left[L(t)\sigma(t)dW^*(t)\right].$$

But we have

$$L(t)\sigma(t) = \left(\frac{1+(-1)^{\mathcal{N}(t)}}{2}\right)\sigma_1 L_1(t) + \left(\frac{1-(-1)^{\mathcal{N}(t)}}{2}\right)\sigma_2 L_2(t).$$

Hence, we get

$$L(t)\sigma(t)dW^*(t) = \sigma_1\left(\frac{1+(-1)^{\mathcal{N}(t)}}{2}\right)L_1(t)dW_1(t) + \sigma_2\left(\frac{1-(-1)^{\mathcal{N}(t)}}{2}\right)L_2(t)dW_2(t).$$

Then, applying the independence of $\mathcal{N}(t)$ and $W(t)$, we get

$$\mathrm{E}^{\mathcal{P}}\left[L(t)\sigma(t)dW^*(t)\right] = \mathrm{E}^{\mathcal{P}}\left[\sigma_1\left(\frac{1+(-1)^{\mathcal{N}(t)}}{2}\right)\right]\mathrm{E}^{\mathcal{P}}\left[L_1(t)dW_1(t)\right] + \mathrm{E}^{\mathcal{P}}\left[\sigma_2\left(\frac{1-(-1)^{\mathcal{N}(t)}}{2}\right)\right]\mathrm{E}^{\mathcal{P}}\left[L_2(t)dW_2(t)\right] = 0.$$

But we have

$$\begin{aligned}\mathrm{E}^{\mathcal{P}}\left[L_i(t)dW_i(t)\right] &= \mathrm{E}^{\mathcal{P}}\left[e^{-\theta_i W(t)-\theta_i^2 t/2}dW_i(t)\right]\\ &= e^{-\theta_i^2 t/}\mathrm{E}^{\mathcal{P}}\left[e^{-\theta_i W(t)}dW_i(t)\right] = 0, i=1,2.\end{aligned}$$

Hence, we get

$$\mathrm{E}^{Q}\left[\frac{d\overline{S}(t)}{\overline{S}(t)}\right] = 0. \tag{5.29}$$

Next, we have

$$\begin{aligned}\mathrm{E}^{Q}[\sigma(t)dW^*(t)]^2 &= \mathrm{E}^{Q}\left[\left\{\left(\tfrac{1+(-1)^{\mathcal{N}(t)}}{2}\right)\sigma_1^2 + \left(\tfrac{1-(-1)^{\mathcal{N}(t)}}{2}\right)\sigma_2^2\right\}\{dW^*(t)\}^2\right]\\
&= \mathrm{E}^{Q}\left[\left\{\left(\tfrac{1+(-1)^{\mathcal{N}(t)}}{2}\right)\sigma_1^2 + \left(\tfrac{1-(-1)^{\mathcal{N}(t)}}{2}\right)\sigma_2^2\right\}\{dW(t)\}^2\right]\\
&= \mathrm{E}^{\mathcal{P}}\left[L(t)\left\{\left(\tfrac{1+(-1)^{\mathcal{N}(t)}}{2}\right)\sigma_1^2\right\}\{dW(t)\}^2\right]\\
&\quad + \mathrm{E}^{\mathcal{P}}\left[L(t)\left\{\left(\tfrac{1-(-1)^{\mathcal{N}(t)}}{2}\right)\sigma_2^2\right\}\{dW(t)\}^2\right]\\
&= \mathrm{E}^{\mathcal{P}}\left[\left\{\left(\tfrac{1+(-1)^{\mathcal{N}(t)}}{2}\right)\sigma_1^2\right\}L_1(t)\{dW(t)\}^2\right]\\
&\quad + \mathrm{E}^{\mathcal{P}}\left[\left\{\left(\tfrac{1-(-1)^{\mathcal{N}(t)}}{2}\right)\sigma_2^2\right\}L_2(t)\{dW(t)\}^2\right]\\
&= \mathrm{E}^{\mathcal{P}}\left[\left(\tfrac{1+(-1)^{\mathcal{N}(t)}}{2}\right)\sigma_1^2\right]\mathrm{E}^{\mathcal{P}}\left[L_1(t)\right]\mathrm{E}^{\mathcal{P}}\{dW(t)\}^2\\
&\quad + \mathrm{E}^{\mathcal{P}}\left[\left(\tfrac{1-(-1)^{\mathcal{N}(t)}}{2}\right)\sigma_2^2\right]\mathrm{E}^{\mathcal{P}}\left[L_2(t)\right]\mathrm{E}^{\mathcal{P}}\{dW(t)\}^2\\
&= \left[\frac{\lambda_2}{\lambda_1+\lambda_2} + \frac{\lambda_1}{\lambda_1+\lambda_2}e^{-(\lambda_1+\lambda_2)t}\right]\sigma_1^2 dt\\
&\quad + \left[\frac{\lambda_1}{\lambda_1+\lambda_2} - \frac{\lambda_1}{\lambda_1+\lambda_2}e^{-(\lambda_1+\lambda_2)t}\right]\sigma_2^2 dt\\
&= \left[f_{11}(t)\sigma_1^2 + f_{12}(t)\sigma_2^2\right]dt.\end{aligned}$$

Hence, we get

$$\mathrm{E}^{Q}\left[\left\{\frac{d\overline{S}(t)}{\overline{S}(t)}\right\}^{2}\right] = \left[f_{11}(t)\sigma_1^2 + f_{12}(t)\sigma_2^2\right]dt. \tag{5.30}$$

Equation (5.30) implies that the process $\dfrac{d\overline{S}(t)}{\overline{S}(t)}$ has quadratic variation. Applying (5.29) and (5.16) in (5.28), we get

$$\mathrm{E}^{Q}[dZ(t)] = rdt - \frac{1}{2}\left[f_{11}(t)\sigma_1^2 + f_{12}(t)\sigma_2^2\right]dt.$$

Let us put

$$\rho(t) = \sqrt{f_{11}(t)\sigma_1^2 + f_{12}(t)\sigma_2^2}.$$

Then, under the measure Q, the logarithm of the stock price can be 0 to satisfy the stochastic differential equation

$$dZ(t) = \left(r - \frac{1}{2}\rho^2(t)\right)dt + \rho(t)dW^*(t). \tag{5.31}$$

5.5 The Conditional Transition Probability Density Function

Our aim in this chapter is to value a double barrier option which is traded in the market model described in Sect. 5.2. Let L and U be the lower and upper barriers of the stock price $S(t)$. We assume that the barriers are moving and we take Le^{rt} and Le^{l+rt} as the lower and upper barriers respectively. Then, it is clear that the option remains alive at time t if the stock price has touched neither L nor H up to time t starting at time 0. That is,

$$Le^{rt} < S(u) < Le^{l+rt} \text{ when } 0 \le u \le t.$$

Normalizing the above condition and taking the logarithm, we get

$$rt < \log\frac{S(t)}{L} < l + rt.$$

This observation suggests that the normalized price $Z(t) = \frac{S(t)}{L}$ can be taken without any loss of generality as the stock price $S(t)$ at time t and the lower barrier and upper barrier for the logarithmic process $Z(t)$ as rt and $l + rt$, respectively. In other words, we assume that the process $Z(t)$ is absorbed at 0 as soon as it hits either of the barriers starting from a value different from 0 and l. Thus, the process

$\{Z(t), t \geq 0\}$ has two absorbing barriers at rt and $l + rt$. To study this process, we consider the probability density function $p(t, x; s, y)$ which is defined by

$$p(t, x; s, y) = \lim_{\Delta \to 0} \frac{\mathrm{P}\{y \leq Z(s) \leq y + \Delta, \text{ for all } u \in (t, s) | Z(t) = x\}}{\Delta},$$

where $0 \leq x, y \leq l + rT; t \leq s$. We shall now derive the backward equation satisfied by the function $p(t, x; s, y)$. Using the Markov property, we get

$$\frac{\partial p}{\partial t} + \left(r - \frac{\rho^2(t)}{2}\right) \frac{\partial p}{\partial x} + \frac{\rho^2(t)}{2} \frac{\partial^2 p}{\partial x^2} = 0. \tag{5.32}$$

The above equation is subject to the conditions

$$p(t, rt; ., .) = 0, \tag{5.33}$$

$$p(t, l + rt; ., .) = 0, \tag{5.34}$$

$$p(s, x; s, y) = \delta(y - x). \tag{5.35}$$

For simplicity, we put $a(t) = r - \frac{\rho^2(t)}{2}$ and $b(t) = \frac{\rho^2(t)}{2}$. Then Eq. (5.32) becomes

$$\frac{\partial p}{\partial t} + a(t) \frac{\partial p}{\partial x} + b(t) \frac{\partial^2 p}{\partial x^2} = 0. \tag{5.36}$$

Let the following transformation be made:

$$\xi = x - \int_0^t a(u) du - \int_0^t b(u) du,$$
$$\eta = \int_0^t b(u) du.$$

Then we obtain

$$\frac{\partial p}{\partial t} = \frac{\partial p}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial p}{\partial \eta} \frac{\partial \eta}{\partial t} = \frac{\partial p}{\partial \xi} [-a(t) - b(t)] + \frac{\partial p}{\partial \eta} b(t),$$
$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial \xi},$$
$$\frac{\partial^2 p}{\partial x^2} = \frac{\partial^2 p}{\partial \xi^2} \frac{\partial \xi}{\partial x} + \frac{\partial^2 p}{\partial \eta \partial \xi} \frac{\partial \eta}{\partial x} = \frac{\partial^2 p}{\partial \xi^2}.$$

Consequently, Eq. (5.35) becomes

$$\frac{\partial p}{\partial \xi}[-a(t)-b(t)]+\frac{\partial p}{\partial \eta}b(t)+a(t)\frac{\partial p}{\partial \xi}+b(t)\frac{\partial^2 p}{\partial \xi^2}=0.$$

Simplifying, we get

$$\frac{\partial p}{\partial \eta}-\frac{\partial p}{\partial \xi}+\frac{\partial^2 p}{\partial \xi^2}=0. \tag{5.37}$$

The Jacobian of the transformation is given by

$$\frac{\partial(\xi,\eta)}{\partial(t,x)}=\begin{vmatrix}-a(t)-b(t) & 1\\ b(t) & 0\end{vmatrix}=-b(t)\neq 0.$$

Setting $p=e^{\xi/2}q$, we get

$$\begin{aligned}\frac{\partial p}{\partial \eta}&=e^{\xi/2}\frac{\partial q}{\partial \eta},\\ \frac{\partial p}{\partial \xi}&=e^{\xi/2}\frac{\partial q}{\partial \xi}+\frac{1}{2}e^{\xi/2}q,\\ \frac{\partial^2 p}{\partial \xi^2}&=e^{\xi/2}\frac{\partial^2 q}{\partial \xi^2}+e^{\xi/2}\frac{\partial q}{\partial \xi}+\frac{1}{4}e^{\xi/2}q.\end{aligned}$$

Substituting the above in (5.37) and canceling the common factor, we obtain

$$\frac{\partial q}{\partial \eta}+\frac{\partial^2 q}{\partial \xi^2}=\frac{1}{4}q. \tag{5.38}$$

Putting $q=e^{\eta/4}V$, we get

$$\begin{aligned}\frac{\partial q}{\partial \eta}&=e^{\eta/4}\frac{\partial V}{\partial \eta}+\frac{1}{4}e^{\eta/4}V,\\ \frac{\partial q}{\partial \xi}&=e^{\eta/4}\frac{\partial V}{\partial \xi},\\ \frac{\partial^2 q}{\partial \xi^2}&=e^{\eta/4}\frac{\partial^2 V}{\partial \xi^2}.\end{aligned}$$

So Eq. (5.37) becomes

$$\frac{\partial V}{\partial \eta}+\frac{\partial^2 V}{\partial \xi^2}=0. \tag{5.39}$$

Solving Eq. (5.39) by the method of separation of variables, we put $q(\xi,\eta)=A(\xi)B(\eta)$. Then we obtain

$$A(\xi)B^{'}(\eta)+A^{''}(\xi)B(\eta)=0.$$

Dividing by $A(\xi)B(\eta)$, we get

$$\frac{B^{'}(\eta)}{B(\eta)} + \frac{A^{''}(\xi)}{A(\xi)} = 0.$$

Then we get

$$-\frac{B^{'}(\eta)}{B(\eta)} = \frac{A^{''}(\xi)}{A(\xi)} = -\omega^2.$$

Taking the equation

$$\frac{B^{'}(\eta)}{B(\eta)} = \omega^2,$$

and integrating, we obtain

$$B(\eta) = C_1 e^{\omega^2 \eta}.$$

Similarly, taking the equation

$$\frac{A^{''}(\xi)}{A(\xi)} = -\omega^2,$$

we obtain $A^{''}(\xi) + \omega^2 A(\xi) = 0$ which yields $A(\xi) = C_2 \cos \omega\xi + C_3 \sin \omega\xi$.
Consequently, we obtain

$$p(t, x; s, y) = e^{\left(\omega^2+\frac{1}{4}\right)\int_0^t b(u)du} e^{\frac{1}{2}(x-\psi(t))} \left[D_1(t) \cos \omega x + D_2(t) \sin \omega x\right], \quad (5.40)$$

where

$$\begin{aligned} D_1(t) &= A_1 \cos \omega\psi(t) - A_2 \sin \omega\psi(t), \\ D_2(t) &= A_1 \sin \omega\psi(t) + A_2 \cos \omega\psi(t), \\ \psi(t) &= \int_0^t a(u)du + \int_0^t b(u)du. \\ D_1^{'}(t) &= -\omega[a(t) + b(t)]D_2(t), \quad D_2^{'}(t) = \omega[a(t) + b(t)]D_1(t). \end{aligned}$$

In our problem, we have

$$a(t) = r - \frac{\sigma^2(t)}{2}, b(t) = \frac{\sigma^2(t)}{2}.$$

So we have $a(t) + b(t) = r$. Hence, we get $\psi(t) = rt$. Substituting this in (5.40), we get

$$p(t,x;s,y) = e^{\left(\omega^2+\frac{1}{4}\right)\int_0^t b(u)du} e^{\frac{1}{2}(x-rt)} \left[A_1 \cos\omega(x-rt) + A_2 \sin\omega(x-rt)\right]. \tag{5.41}$$

Applying the condition (5.33) in (5.40), we get $A_1 = 0$. Consequently, Eq. (5.41) becomes

$$p(t,x;s,y) = A_2 e^{\left(\omega^2+\frac{1}{4}\right)\int_0^t b(u)du} e^{\frac{1}{2}(x-rt)} \sin\omega(x-rt). \tag{5.42}$$

Applying the condition (5.34) in (5.42), we get $\sin\omega l = 0$ which yields

$$\omega = \frac{k\pi}{l}, k = 1, 2, \cdots.$$

Hence, we obtain the possible solution as

$$p(t,x;s,y) = \sum_{k=1}^{\infty} B_k e^{\left(\frac{k^2\pi^2}{l^2}+\frac{1}{4}\right)\int_0^t b(u)du} e^{\frac{1}{2}(x-rt)} \sin\frac{k\pi\,(x-rt)}{l}. \tag{5.43}$$

Applying the condition (5.35) in (5.43), we get

$$\delta(y-x) = \sum_{k=1}^{\infty} B_k e^{\left(\frac{k^2\pi^2}{l^2}+\frac{1}{4}\right)\int_0^s b(u)du} e^{\frac{1}{2}(x-rs)} \sin\frac{k\pi(x-rs)}{l}. \tag{5.44}$$

Identifying (5.44) as a Fourier sine series, we get

$$B_k = \frac{2}{l} e^{-\left(\frac{k^2\pi^2}{l^2}+\frac{1}{4}\right)\int_0^s b(u)du} e^{-\frac{1}{2}(y-rs)} \sin\frac{k\pi\,(y-rs)}{l}.$$

Hence, the explicit solution is given by

$$p(t,x;s,y) = \frac{2}{l} e^{\frac{1}{2}(x-y+r(s-t))} \sum_{k=1}^{\infty} e^{-\left(\frac{k^2\pi^2}{l^2}+\frac{1}{4}\right)\int_t^s b(u)du} \times \sin\frac{k\pi\,(x-rt)}{l} \sin\frac{k\pi\,(y-rs)}{l}. \tag{5.45}$$

Putting $\sigma_1 = \sigma_2 = \sigma$, we get

$$a(t) = r - \frac{\sigma^2}{2}, b(t) = \frac{\sigma^2}{2}$$

and hence Eq. (5.45) gives

$$p(t,x;s,y)=\frac{2}{l}e^{-\frac{1}{2}(y-x)}\sum_{k=1}^{\infty}e^{-\left(\frac{k^2\pi^2}{l^2}+\frac{1}{4}-\frac{r}{\sigma^2}\right)\frac{\sigma^2(s-t)}{2}}\times\sin\frac{k\pi\,(x-rt)}{l}\sin\frac{k\pi\,(y-rs)}{l}. \tag{5.46}$$

Equation (5.46) is the result of [12] corresponding to the case of moving boundaries. Using the probability density function $p(t,x;s,y)$ in (5.46), we now proceed to obtain the pricing formula for a double barrier option which pays a constant amount K_U if the upper barrier is hit first before reaching the maturity date T, a constant amount K_L if the lower barrier is hit first before reaching the maturity date T, and an amount K if neither barrier is hit before reaching the maturity date T. In order to obtain the formula, we need the barrier probability density functions $g^{\pm}(t,x;s)$. We fulfill this task in the next section.

5.6 The Barrier Densities $g^{\pm}(t,x;s)$

The density $g^{+}(t,x;s)ds$ represents the probability that the logarithm of the stock level knocks the upper barrier value l in the interval $(s,s+ds)$ without knocking the lower barrier level 0 in the interval (t,s), given that it is at level x at time t. Similarly, the density $g^{-}(t,x;s)ds$ represents the probability that the logarithm of the stock level knocks the lower barrier value 0 in the interval $(s,s+ds)$ without knocking the upper barrier level l in the interval (t,s), given that it is at level x at time t. With probability 1, the logarithm Z of the stock price either knocks the upper barrier level l, or the lower barrier level 0, or stays within $(0,l)$ till the maturity time T, given that Z is at level x at time $t<T$ and prior to t the process Z has hit neither 0 nor $l+rT$. Then, we have

$$\int_t^T g^{+}(t,x;s)ds+\int_t^T g^{-}(t,x;s)ds+\int_{rT}^{l+rT} p(t,x;T,y)dy=1. \tag{5.47}$$

Differentiating (5.47) with respect to T and using (5.45),

$$
\begin{aligned}
g^{+}(t,x;T)+g^{-}(t,x;T) &= -\frac{\partial}{\partial T}\int_{rT}^{l+rT} p(t,x;T,y)dy.\\
&= \frac{2e^{\frac{1}{2}(x-rt)}}{l^2}\sum_{k=1}^{\infty}\Bigg[e^{-\left(\frac{k^2\pi^2}{l^2}+\frac{1}{4}\right)\int_t^T b(u)du}k\pi b(T)\\
&\qquad \sin\frac{k\pi\,(x-rt)}{l}\left\{1-(-1)^k e^{-l/2}\right\}\Bigg]\\
&= \frac{2e^{\frac{1}{2}(x-rt)}}{l^2}\Bigg[\sum_{k=1}^{\infty}e^{-\left(\frac{k^2\pi^2}{l^2}+\frac{1}{4}\right)\int_t^T b(u)du}k\pi b(T)\\
&\qquad \sin\frac{k\pi\,(x-rt)}{l}\Bigg]+\Bigg[\frac{2e^{-\frac{1}{2}(l+rt-x)}}{l^2}\\
&\quad \sum_{k=1}^{\infty}e^{-\left(\frac{k^2\pi^2}{l^2}+\frac{1}{4}\right)\int_t^T b(u)du}k\pi b(T)\sin\frac{k\pi\,(l+rt-x)}{l}\Bigg].
\end{aligned}
\tag{5.48}
$$

Replacing T by s in (5.48), we get

$$
g^{+}(t,x;s)+g^{-}(t,x;s) = \frac{2b(s)\pi e^{\frac{1}{2}(x-rt)}}{l^2}\sum_{k=1}^{\infty}e^{-\left(\frac{k^2\pi^2}{l^2}+\frac{1}{4}\right)\int_t^s b(u)du} \tag{5.49}
$$

$$
\times\, k\sin\frac{k\pi\,(x-rt)}{l}+\frac{2b(s)\pi e^{-\frac{1}{2}(l+rt-x)}}{l^2} \tag{5.50}
$$

$$
\sum_{k=1}^{\infty}e^{-\left(\frac{k^2\pi^2}{l^2}+\frac{1}{4}\right)\int_t^s b(u)du}k\sin\frac{k\pi\,(l+rt-x)}{l}. \tag{5.51}
$$

The above equation provides the sum of the barrier densities. Now we proceed to obtain the individual densities $g^{+}(t,x;s)$ and $g^{-}(t,x;s)$. By definition, we have

$$
\begin{aligned}
g^{+}(t+\Delta,x;s) &= g^{+}(t,x-a(t)\Delta-\sigma(t)\sqrt{\Delta};s)\\
&= g^{+}(t,x;s)-a(t)\frac{\partial g^{+}(t,x;s)}{\partial x}\Delta-b(t)\frac{\partial^2 g^{+}(t,x;s)}{\partial x^2}\Delta.
\end{aligned}
$$

The above equation yields

$$
\frac{\partial g^{+}(t,x;s)}{\partial t}+a(t)\frac{\partial g^{+}(t,x;s)}{\partial x}+b(t)\frac{\partial^2 g^{+}(t,x;s)}{\partial x^2}=0. \tag{5.52}
$$

Equation (5.52) is subject to the following conditions:

$$g^+(t, rt; s) = 0; \tag{5.53}$$

$$g^+(t, l + rt; s) = \delta(s - t); \tag{5.54}$$

$$g^+(s, x; s) = \delta(l + rs - x). \tag{5.55}$$

As before, we make the transformation

$$\xi = x - \int_0^t a(u)du - \int_0^t b(u)du = x - rt,$$

$$\eta = \int_s^t b(u)du = -\int_t^s b(u)du,$$

$$g^+(t, x; s) = e^{\xi/2} e^{\eta/4} V(\xi, \eta).$$

Then we obtain

$$\frac{\partial V}{\partial \eta} + \frac{\partial^2 V}{\partial \xi^2} = 0. \tag{5.56}$$

We note that $\eta = 0 \iff t = s$. Taking Laplace transform on both sides of (5.56) with respect to η, we obtain

$$\theta V^* + \frac{\partial^2 V^*}{\partial \xi^2} = 0. \tag{5.57}$$

Solving Eq. (5.57), we get

$$V^*(\xi, \theta) = A_+ \cos \sqrt{\theta}\xi + B_+ \sin \sqrt{\theta}\xi. \tag{5.58}$$

From (5.53), we get $V^*(0, \theta) = 0$. Applying this condition in (5.58), we get $A_+ = 0$ and so Eq. (5.58) becomes

$$V^*(\xi, \theta) = B_+ \sin \sqrt{\theta}\xi. \tag{5.59}$$

From (5.54), we get $V(l, \eta) = -e^{-l/2} e^{-\eta/4} b(s)\delta(\eta)$ and so $V^*(l, \theta) = -e^{-l/2} b(s)$. Applying this condition in (5.59), we get

$$B_+ \sin \sqrt{\theta} l = -e^{-l/2} b(s).$$

Consequently, we get

$$V^*(\xi, \theta) = -\frac{e^{-l/2} b(s) \sin \sqrt{\theta}\xi}{\sin \sqrt{\theta} l}. \tag{5.60}$$

We shall now invert $V^*(\xi, \theta)$ in (5.60). For this, we have to apply Bromwich's integral formula for the inverse Laplace transform:

$$V(\xi,\eta)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\eta\theta}V^*(\xi,\theta)d\theta,$$

where all the singularities of $V^*(\xi,\theta)$ in the $\theta-$plane lies to the right side of the line $\Re(\theta)=\gamma$. Hence by applying Cauchy's residue theorem, we get

$$V(\xi,\eta)=\text{the sum of the residues of } e^{\eta\theta}V^*(\xi,\theta) \text{ at the poles of } V^*(\xi,\theta). \tag{5.61}$$

The poles of $V^*(\xi,\theta)$ (as a function of θ) are given by $\sin\sqrt{\theta}l=0$. This is possible if and only if $\sqrt{\theta}l=k\pi$, $k=0,1,2,\cdots$. Squaring, we get $\theta l^2=k^2\pi^2$, $k=0,1,2,\cdots$. Hence, the poles are simple and they are

$$\theta_k=\frac{k^2\pi^2}{l^2},\, k=0,1,2,\cdots.$$

The residue at θ_k is given by

$$\begin{aligned}R_k&=\lim_{\theta\to\theta_k}(\theta-\theta_k)\,e^{\eta\theta}\frac{\sin\sqrt{\theta}\xi}{\sin\sqrt{\theta}l}\\&=\lim_{\theta\to\theta_k}\frac{2\sqrt{\theta}e^{\eta\theta}\sin\sqrt{\theta}\xi}{l\cos\sqrt{\theta}l}\\&=e^{\frac{k^2\pi^2\eta}{l^2}}\frac{2k\pi}{l^2}(-1)^k\sin\frac{k\pi\xi}{l}.\end{aligned}$$

Hence Eq. (5.61) gives

$$V(\xi,\eta)=\sum_{k=0}^{\infty}e^{-l/2}e^{\frac{k^2\pi^2\eta}{l^2}}\frac{-2kb(s)\pi}{l^2}(-1)^k\sin\frac{k\pi\xi}{l}$$

$$=\sum_{k=1}^{\infty}e^{-l/2}e^{\frac{k^2\pi^2\eta}{l^2}}\frac{2kb(s)\pi}{l^2}(-1)^k\sin\frac{k\pi\xi}{l}.$$

Consequently, we get

$$\begin{aligned}g^+(t,x;s)&=e^{\xi/2}e^{-l/2}e^{\eta/4}\sum_{k=1}^{\infty}e^{\frac{k^2\pi^2\eta}{l^2}}\frac{-2kb(s)\pi}{l^2}(-1)^k\sin\frac{k\pi\xi}{l}\\&=\frac{2b(s)\pi e^{-(l+rt-x)/2}}{l^2}\sum_{k=1}^{\infty}e^{-\left(\frac{k^2\pi^2}{l^2}+\frac{1}{4}\right)\int_t^s b(u)du}k\sin\frac{k\pi(l+rt-x)}{l}.\end{aligned} \tag{5.62}$$

Next, we proceed to find the density $g^-(t,x;s)$. By definition, we have

$$g^-(t+\Delta, x; s) = g^-(t, x - a(t)\Delta - \sigma(t)\sqrt{\Delta}; s)$$
$$= g^-(t, x; s) - a(t)\tfrac{\partial g^-(t,x;s)}{\partial x}\Delta - b(t)\tfrac{\partial^2 g^-(t,x;s)}{\partial x^2}\Delta.$$

The above equation yields

$$\frac{\partial g^-(t,x;s)}{\partial t} + a(t)\frac{\partial g^-(t,x;s)}{\partial x} + b(t)\frac{\partial^2 g^-(t,x;s)}{\partial x^2} = 0. \tag{5.63}$$

Equation (5.63) is subject to the following conditions:

$$g^-(t, l + rt; s) = 0; \tag{5.64}$$
$$g^-(t, rt; s) = \delta(s - t); \tag{5.65}$$
$$g^-(s, x; s) = \delta(rs - x). \tag{5.66}$$

As before, we make the transformation

$$\xi = x - \int_0^t a(u)du - \int_0^t b(u)du = x - rt,$$
$$\eta = \int_s^t b(u)du = -\int_t^s b(u)du,$$
$$g^-(t,x;s) = e^{\xi/2}e^{\eta/4}V^-(\xi,\eta).$$

Then we obtain

$$\frac{\partial V^-}{\partial \eta} + \frac{\partial^2 V^-}{\partial \xi^2} = 0. \tag{5.67}$$

We note that $\eta = 0 \iff t = s$. Taking Laplace transform on both sides of (5.67) with respect to η, we obtain

$$\theta V^{-*} + \frac{\partial^2 V^{-*}}{\partial \xi^2} = 0. \tag{5.68}$$

Solving Eq. (5.68), we get

$$V^{-*}(\xi,\theta) = A_- \cos\sqrt{\theta}\xi + B_- \sin\sqrt{\theta}\xi. \tag{5.69}$$

From (5.64), we get

$$V^{-*}(l,\theta) = 0.$$

Applying this condition in (5.69), we get

$$A_- \cos\sqrt{\theta}l + B_- \sin\sqrt{\theta}l = 0$$

and so Eq. (5.69) becomes

$$V^{-*}(\xi,\theta) = A_{-}\left\{\cos\sqrt{\theta}\xi - \frac{\cos\sqrt{\theta}l}{\sin\sqrt{\theta}l}\sin\sqrt{\theta}\xi +\right\} = A_{-}\frac{\sin\sqrt{\theta}(l-\xi)}{\sin\sqrt{\theta}l}. \quad (5.70)$$

From (5.65), we get

$$V^{-}(0,\eta) = -e^{-\eta/4}b(s)\delta(\eta)$$

and so $V^{-*}(0,\theta) = -b(s)$. Applying this condition in (5.70), we get

$$A_{-} = -b(s).$$

Consequently, we get

$$V^{-*}(\xi,\theta) = -b(s)\frac{\sin\sqrt{\theta}(l-\xi)}{\sin\sqrt{\theta}l}. \quad (5.71)$$

We shall now invert $V^{-*}(\xi,\theta)$ in (5.71). For this, we have to apply Bromwich's integral formula for the inverse Laplace transform:

$$V^{-}(\xi,\eta) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\eta\theta}V^{-*}(\xi,\theta)d\theta,$$

where all the singularities of $V^{-*}(\xi,\theta)$ in the $\theta-$plane lies to the right side of the line $\Re(\theta) = \gamma$. Hence by applying Cauchy's residue theorem, we get

$$V^{-}(\xi,\eta) = \text{the sum of the residues of } e^{\eta\theta}V^{-*}(\xi,\theta) \text{ at the poles of } V^{-*}(\xi,\theta). \quad (5.72)$$

The poles of $V^{-*}(\xi,\theta)$ (as a function of θ) are given by $\sin\sqrt{\theta}l = 0$. This is possible if and only if $\sqrt{\theta}l = k\pi, k = 0, 1, 2, \cdots$. Squaring, we get $\theta l^2 = k^2\pi^2, k = 0, 1, 2, \cdots$. Hence, the poles are simple and they are

$$\theta_k = \frac{k^2\pi^2}{l^2}, k = 0, 1, 2, \cdots.$$

The residue at θ_k is given by

$$R_k = \lim_{\theta\to\theta_k}(\theta-\theta_k)\,e^{\eta\theta}\frac{\sin\sqrt{\theta}(l-\xi)}{\sin\sqrt{\theta}l} = \lim_{\theta\to\theta_k}\frac{2\sqrt{\theta}e^{\eta\theta}\sin\sqrt{\theta}(l-\xi)}{l\cos\sqrt{\theta}l}$$

$$= e^{\frac{k^2\pi^2\eta}{l^2}}\frac{2k\pi}{l^2}(-1)^k\sin\left(k\pi - \frac{k\pi\xi}{l}\right)$$

$$= -e^{\frac{k^2\pi^2\eta}{l^2}}\frac{2k\pi}{l^2}(-1)^k(-1)^k\sin\frac{k\pi\xi}{l} = -e^{\frac{k^2\pi^2\eta}{l^2}}\frac{2k\pi}{l^2}\sin\frac{k\pi\xi}{l}.$$

Hence Eq. (5.72) gives

$$V^-(\xi,\eta) = \sum_{k=0}^{\infty} e^{\frac{k^2\pi^2\eta}{l^2}}\frac{2kb(s)\pi}{l^2}\sin\frac{k\pi\xi}{l}$$

$$= \sum_{k=1}^{\infty} e^{\frac{k^2\pi^2\eta}{l^2}}\frac{-2kb(s)\pi}{l^2}\sin\frac{k\pi\xi}{l}.$$

Consequently, we get

$$g^-(t,x;s) = e^{\xi/2}e^{\eta/4}\sum_{k=1}^{\infty} e^{\frac{k^2\pi^2\eta}{l^2}}\frac{2kb(s)\pi}{l^2}\sin\frac{k\pi\xi}{l}$$

$$= \frac{2b(s)\pi e^{(x-rt)/2}}{l^2}\sum_{k=1}^{\infty} e^{-\left(\frac{k^2\pi^2}{l^2}+\frac{1}{4}\right)\int_t^s b(u)du} k\sin\frac{k\pi(x-rt)}{l}. \tag{5.73}$$

Clearly, the sum of $g^+(t,x;s)$ and $g^-(t,x;s)$ obtained by using (5.62) and (5.73) is the same as the expression given by (5.51).

5.7 Option Value at Maturity

Using the densities $p(t,x;s,y)$ and $g^{\pm}(t,x;s)$, we now proceed to obtain the value of a double barrier option at time $t < T$ which pays a constant amount at the maturity time T given that no barrier is hit before t according to the following rule:

$$V(t) = \begin{cases} K^+ & \text{if the upper barrier is hit first;} \\ K^- & \text{if the lower barrier is hit first;} \\ K & \text{if both the barriers are not hit up to T.} \end{cases}$$

Let $P^+(T)$ be the probability that the upper barrier is hit first in the interval (t,T) and $P^-(T)$ be the probability that the lower barrier is hit first in the interval (t,T). Then we get

$$P^+(T) = \int_t^T g^+(t,x;s)ds, \quad P^-(T) = \int_t^T g^-(t,x;s)ds.$$

Suppose that $V(t)$ is the value of the option at time $t < T$ and δ is the constant discount factor of money value. Then we obtain

$$V(t) = e^{-\delta(T-t)} \left\{ K^+ P^+(T) + K^- P^-(T) + K \left(1 - P^+(T) - P^-(T)\right) \right\},$$

where we have by direct integration

$$\begin{aligned} P^+(T) &= \int_t^T g^+(t, x; s) ds \\ &= \frac{2\pi e^{-(l+rt-x)/2}}{l^2} \sum_{k=1}^{\infty} k \sin \frac{k\pi(l+rt-x)}{l} \\ &\qquad \times \left(\frac{1 - e^{-\left(\frac{k^2\pi^2}{l^2} + \frac{1}{4}\right) \int_t^T b(u) du}}{\frac{k^2\pi^2}{l^2} + \frac{1}{4}} \right); \\ P^-(T) &= \int_t^T g^-(t, x; s) ds \\ &= \frac{2\pi e^{(x-rt)/2}}{l^2} \sum_{k=1}^{\infty} k \sin \frac{k\pi(x-rt)}{l} \left(\frac{1 - e^{-\left(\frac{k^2\pi^2}{l^2} + \frac{1}{4}\right) \int_t^T b(u) du}}{\frac{k^2\pi^2}{l^2} + \frac{1}{4}} \right). \end{aligned}$$

5.8 Conclusion

In this article, we considered the pricing of a double barrier option in a Black-Scholes environment with volatility which has random jumps at random times, and derived a closed-form solution. The uniqueness lies in the fact that while the volatility has traditionally been handled as a diffusion process, our approach has been to generalize it to have random jumps at random times. For a special case of a double barrier option, we deduced from our computations, the results of [12], reinforcing the generalization. Finally, we have derived the hitting probabilities for both barriers explicitly in closed form.

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Chapter 6
Caputo Sequential Fractional Differential Equations with Applications

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6.1 Introduction

Many scientists and mathematicians felt the need to exploit dynamic equations of fractional order in their mathematical modeling, since these equations have been proven to be more fruitful and cost-effective. In terms of Fick's law of diffusion, Oldham, a chemist, and Keith, a mathematician, found that using half-order derivatives and integrals leads to a formulation of certain electrochemical issues that is more economical and useful than the conventional approach. See [36] for details. See [5, 12, 13, 21, 25, 26, 29–32, 36, 38, 42, 47, 56] for details on the study of fractional differential equations. Also, see [3, 9, 15, 16, 18, 19, 22, 24, 33, 35, 37, 39, 40, 48, 50, 55] for some more applications.

However, the result of the first order differential equation will be a special case of the Caputo dynamic equations of order q as $q \to 1$. In most of these applications, the order of the fractional derivative is q with $0 < q < 1$. In addition, by definition, the fractional derivative has a global nature, whereas the integer derivative has a local nature. This is one of the main advantages of using the fractional derivative in place of the integer derivative. See [3], where they have demonstrated that an appropriate order fractional differential equation results were closer to the real data than the integer model results. As a result, the authors were able to enhance the model to fit real-world data by using 'q' as a parameter, where q is the order of the derivative. The solution of linear Caputo fractional differential equations with lower order fractional derivative is difficult to compute. See example 4.2 from [38] presented below,

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$$ {}^cD^Qu + b^cD^qu = h(t), \tag{6.1}$$

with appropriate initial conditions. Note that this linear problem cannot be solved by Laplace transform method. If Q and q are integers, then we can easily solve the problem. One of the main reasons for that is, the fractional derivative is not sequential, whereas the integer derivative is sequential. In the present work, we will develop results for nq order Caputo sequential fractional differential equation which is sequential of order q and $nq \in (n-1, n),\ n \in \mathbb{N}$. In the current literature, majority of the work done on Caputo fractional initial and boundary value problem, the initial and boundary conditions has been taken to be the same as that of the closest integer derivative. In order to apply Laplace transform method, we require the initial conditions should involve all fractional derivative of order kq with $k < n$. See [44, 51–53] for Caputo sequential initial value problems and [46, 52, 53] for Caputo sequential boundary value problems. Also, see [25, 28, 38, 44–46, 49, 51–53] for more results on sequential fractional differential equations. See [6, 7, 43, 44, 46] for numerical results on sequential fractional differential equations. It is to be noted that, the basis solution for the nonsequential Caputo derivative of order nq with $(n-1) < nq < n, n \in \mathbb{N}$, is $1, t, t^2, \ldots t^{(n-1)}$, whereas the basis solution for Caputo sequential derivative of order nq which is sequential of order q is $1, t^q, t^{2q}, \ldots t^{(n-1)q}$. If $Q = nq$ is sequential of order q in the Eq. (6.1), then the solution of the corresponding homogeneous equation will be in terms of Mittag-Leffler functions with q as a parameter. In fact, we can solve a linear Caputo sequential fractional differential equations with fractional initial conditions by assuming the solution to be in the form of $E_{q,1}(rt^q)$, just the way we use e^{rt} in the integer case. The advantage of this approach, is the nature of the solution of Mittag-Leffler function with $q < 1$ can be compared with the integer solution which is an exponential function. However, the Laplace transform method is the most suitable method for solving linear non-homogeneous Caputo sequential fractional differential equations with fractional initial conditions. In this work, we have developed Laplace transform table which will be useful in solving linear non-homogeneous Caputo sequential fractional differential equations with fractional initial conditions. We have discussed the possible solutions for a general $2q$ order Caputo sequential fractional differential equation with fractional initial conditions when $0.5 < q < 1$. To compare our solution with the integer second order solution, we have provided graphs by numerical computation of fractional trigonometric functions. One of the interesting outcome of this is that the fractional trigonometric functions exhibits damping behavior without a damping term in the model. Thus, the value of q can be chosen which agrees with the real data compared with the results with $q = 1$.

6.2 Preliminaries

In this section, we will go over some definitions and well-known outcomes that will help us with our main results.

Definition 6.1 The $nq > 0$ order Caputo derivative of $u(t)$, where $u(t) : (0, \infty) \longrightarrow \mathbb{R}$ is given by

$$^{c}D^{nq}u(t) = \frac{1}{\Gamma(n - nq)} \int_0^t (t - s)^{-nq+n-1} u^{(n)}(s)ds,$$

where $n \in \mathbb{N}$, such that $nq \in (n - 1, n)$. In particular, if $q = 1$ (then $nq = n \in \mathbb{N}$), then, by definition $^{c}D^{nq}u = \frac{d^n u}{dt^n}$ and if $q = 1$, $^{c}D^{q}u = \frac{du}{dt}$.

Definition 6.2 Let $q > 0$, and $u(t) : (0, \infty) \longrightarrow \mathbb{R}$. Then the Riemann-Liouville derivative of $u(t)$ of order nq is given by

$$D^{nq}u(t) = \frac{1}{\Gamma(n - nq)} (\frac{d}{dt})^n \int_0^t (t - s)^{-nq+n-1} u(s)ds.$$

Here $n \in \mathbb{N}$ such that $nq \in (n - 1, n)$.

The definition of qth order Riemann–Liouville fractional integral, where $0 < q < 1$ is given below.

Definition 6.3 The qth order Riemann–Liouville fractional integral of $u(t)$, where $0 < q \leq 1$, is given by

$$D^{-q}u(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} u(s)ds. \tag{6.2}$$

We note that the Caputo integral of order q and the Riemann–Liouville fractional integral of order q are one in the same when $0 < q < 1$.

Next, we define Mittag-Leffler function. It is a generalization of the exponential function that serves the same purpose for fractional differential equations as it does for integer derivative dynamic equations, especially when $0 < q < 1$.

Definition 6.4 Mittag-Leffler function of two parameters q and r is given by

$$E_{q,r}(\lambda(t - t_0)^q) = \sum_{k=0}^{\infty} \frac{(\lambda(t - t_0)^q)^k}{\Gamma(qk + r)},$$

where $q, r > 0$. Also, for $t_0 = 0$ and $r = 1$, we get

$$E_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk + 1)},$$

where $q > 0$.

See [21, 25, 38] for more details on the Mittag-Leffler functions.

If $q = r = 1$,, then the Mittag-Leffler (ML for short) function is the usual exponential function. In this work, we use ML function when $0 < q \leq 1$. In addition, Mittag-Leffler functions with $r = 1$ and $r = q$ with $0 < q < 1$, will be useful when sequential derivatives are involved. In the integer case, we have the solutions of linear equations with constant coefficients which depend on trigonometric functions of sine and cosine, which are defined in terms of the exponential function. In this work, we need fractional sine and cosine functions which depend on the value of q. For that purpose, we define the following $\sin_{q,1}(\lambda t^q)$ and $\cos_{q,1}(\lambda t^q)$ functions.

Definition 6.5 The fractional trigonometric $\sin_{q,1}(\lambda t^q)$ and $\cos_{q,1}(\lambda t^q)$, functions are given by

$$\sin_{q,1}(\lambda t^q) = \frac{1}{2i}[E_{q,1}(i\lambda t^q) - E_{q,1}(-i\lambda t^q)]$$

and

$$\cos_{q,1}(\lambda t^q) = \frac{1}{2}[E_{q,1}(i\lambda t^q) + E_{q,1}(-i\lambda t^q)]$$

respectively.

Using the above definitions, we can also define $\sin_{q,q}(\lambda t^q)$ and $\cos_{q,q}(\lambda t^q)$ in a similar way using $E_{q,q}(\lambda t^q)$ in place of $E_{q,1}(\lambda t^q)$.

Another functions, which will be useful in fractional trigonometric function involving complex numbers of the form $\lambda + i\mu$, are the Generalized fractional trigonometric functions $G\sin_{q,1}((\lambda + i\mu)t^q)$ and $G\cos_{q,1}((\lambda + i\mu)t^q)$, which are defined as follows:

Definition 6.6 The Generalized trigonometric $G\sin_{q,1}((\lambda + i\mu)t^q)$ and $G\cos_{q,1}((\lambda + i\mu)t^q)$ functions are given by

$$G\sin_{q,1}((\lambda + i\mu)t^q) = \frac{1}{2i}[E_{q,1}((\lambda + i\mu)t^q) - E_{q,1}((\lambda - i\mu)t^q)]$$

and

$$G\cos_{q,1}((\lambda + i\mu)t^q) = \frac{1}{2}[E_{q,1}((\lambda + i\mu)t^q) + E_{q,1}((\lambda - i\mu)t^q)]$$

respectively.

Similarly, $G\sin_{q,q}((\lambda + i\mu)t^q)$ and $G\cos_{q,q}((\lambda + i\mu)t^q)$ is also define by replacing $E_{q,1}((\lambda + i\mu)t^q)$ and $E_{q,1}((\lambda - i\mu)t^q)$ by $E_{q,q}((\lambda + i\mu)t^q)$ and $E_{q,q}((\lambda - i\mu)t^q)$ respectively.

Remark 6.1 If $q = 1$, in the above definition, then the functions $G\sin_{q,1}((\lambda + i\mu)t^q)$ and $G\sin_{q,q}((\lambda + i\mu)t^q)$ reduces to $e^{\lambda t}\sin(\mu t)$. Similarly, $G\cos_{q,1}((\lambda + i\mu)t^q)$ and $G\cos_{q,q}((\lambda + i\mu)t^q)$ also reduces to $e^{\lambda t}\cos(\mu t)$. Note that this simplification cannot be done when Mittag-Leffler function is involved, mainly because the Mittag-Leffler function does not enjoy the nice properties of the exponential function.

Next, we consider the q^{th} order linear Caputo fractional differential equation of the following form:

$$ {}^{c}D^{q}u(t) = \lambda u(t) + f(t), \quad u(0) = u_0, \quad on \quad J. \tag{6.3} $$

Here $J = [0, T]$, λ is a constant, $f(t) \in C[J, \mathbb{R}]$ and $0 < q < 1$. The solution of (6.3) exists and is unique. The explicit solution of (6.3) is given by

$$ u(t) = u_0 E_{q,1}(\lambda t^q) + \int_0^t (t-s)^{q-1} E_{q,q}(\lambda (t-s)^q) f(s) ds. \tag{6.4} $$

Note that the solution of (6.3) has been obtained in [25, 29, 38] by mathematical induction using the initial approximation as the initial condition in the integral representation of (6.3). Note that we can also obtain the solution (6.4) using the Laplace transform method as in [12, 36]. It is easy to observe that, the variation of parameter method cannot be used to solve (6.3), since the Mittag-Leffler function does not enjoy the exponential property of the natural exponential function. In addition, the product rule which is true for the integer derivative does not hold true for Caputo fractional derivative.

Next, consider the linear non-homogeneous nq order Caputo fractional differential equation with initial conditions, with $(n-1) < nq < n, n \in \mathbb{N}$,

$$ {}^{c}D^{nq}u(t) = \lambda u + f(t), \ t > 0, \tag{6.5} $$

$$ u^{kq}(0) = b_k \ (b_k \in R; \ k = 0, 1, \ldots n-1), \tag{6.6} $$

where $f(t)$ is continuous on $[0, \infty)$. Using the Laplace transform method, the solution of (6.5) and (6.6) can be obtained as

$$ u(t) = \sum_{j=0}^{n-1} b_j t^j E_{nq,j+1}(\lambda t^{nq}) + \int_0^t (t-s)^{nq-1} E_{nq,nq}(\lambda (t-s)^{nq}) f(s) ds. \tag{6.7} $$

In particular, for $0 < nq < 1$, the analytical solution of (6.5) and (6.6) is the same as the solution given by (6.3) with $u_0 = b_0$.

Note that the analytical solution of the above form can also be obtained by writing the equivalent of (6.5) and (6.6) in the space of $n-1$ continuously differentiable function on $[0, \infty)$ in the Volterra integral equation. In order to find the solution, the basis solution is used as $1, t, t^2, \ldots t^{(n-1)}$. In short, the first approximation to the solution by this approach is

$$ \sum_{j=0}^{n-1} b_j t^j. $$

See section 4.1.3 of [25] monograph for more details.

If $q = 1$,, then the Eq. (6.5) is an nth order linear differential equations which is sequential. The standard method to solve nth order linear homogeneous differential equation is assuming the solution to be of the form e^{rt}, where r would be the n roots of the nth degree polynomial $r^n = \lambda$. Using variation of parameter method together with n linearly independent solutions that was determined, we can obtain the particular solutions of the corresponding non-homogeneous equation.

Can we apply a similar method to a linear homogeneous Caputo fractional differential equation of order nq with $(n-1) < nq < n, n \in \mathbb{N}$? In short, can we seek a solution of the form $E_{q,1}(rt^q)$, where the values of r would be a root of the polynomial of degree n, for linear nq order Caputo fractional differential equations with constant coefficients of the form

$$\sum_{k=0}^{n} a_k \, {}^cD^{kq}u(t) = 0. \tag{6.8}$$

The answer is affirmative, if the Caputo fractional derivative of order nq is sequential of order q, which we define next. In general, the Caputo fractional derivative is not sequential, whereas the integer derivative is always sequential.

Recall the definition of Caputo derivative of order nq when $(n-1) < nq < n, n \in \mathbb{N}$, we have

$$ {}^cD^{2q}(u) = \frac{1}{\Gamma(2-2q)} \int_0^t \frac{u^{(2)}(s)}{(t-s)^{2q-1}} ds.$$

Let $u(t) = t^{\omega}$ in the above definition where $1 < 2q < 2$, then,

$$ {}^cD^{2q}(t^{\omega}) = \frac{\Gamma(\omega+1))t^{\omega-2q}}{\Gamma(1-2q+\omega)}.$$

In particular, if $\omega = q$,, we get

$$ {}^cD^{2q}(t^{q}) = \frac{\Gamma(q+1))t^{-q}}{\Gamma(1-q)}.$$

On the other hand, if we compute first ${}^cD^q(t^q) = \Gamma(1+q)$ and ${}^cD^q({}^cD^q(t^q)) = 0$.

In short,

$$ {}^cD^{2q}(E_{q,1}(\lambda t^q)) \neq (\lambda)^2 E_{q,1}(\lambda t^q),$$

when ${}^cD^{2q}(u)$ is not sequential.

If ${}^cD^{2q}(u)$ is sequential, then we have

$$ {}^cD^{2q}(E_{q,1}(\lambda t^q)) = (\lambda)^2 E_{q,1}(\lambda t^q).$$

Definition 6.7 The nq order Caputo fractional derivative, where $nq \in (n-1, n)$, $n \in \mathbb{N}$ is said to be sequential of order q if the relation

$$ {}^{c}D_{0+}^{nq}u(t) = {}^{c}D_{0+}^{q}({}^{c}D_{0+}^{(n-1)q})u(t), \tag{6.9} $$

holds for $n \geq 2,\ n \in \mathbb{N}$.

In short, we can rewrite (6.9) as follows:

$$ ({}^{c}D_{0+}^{nq})u(t) = {}^{c}\ D_{0+}^{q}({}^{c}\ D_{0+}^{q}({}^{c}\ D_{0+}^{q} \ldots n - times \ldots ({}^{c}D_{0+}^{q})u)). $$

Although we can find the general solution of the homogeneous sequential Caputo fractional differential equation (6.8), using $E_{q,1}(rt^q)$, we also require the initial conditions should be in terms of the Caputo fractional derivative of lower order evaluated at $t = 0$. Basically, the form of the initial conditions will be

$$ ({}^{c}D_{0+}^{kq})u(t)|_{t=0} = b_k \ (b_k \in R;\ k = 0, 1, \ldots n-1). \tag{6.10} $$

Once we have the initial conditions of the above form, then one can also use the Laplace transform method to solve the linear homogeneous Caputo sequential fractional differential equation (6.8) with initial conditions. The Laplace transform method is very useful to find the solution of the linear non-homogeneous Caputo sequential fractional differential equation with fractional initial conditions, since variation of parameter method will not be useful.

The next result is related to the Laplace transform of nq order Caputo sequential derivative which is sequential of order q. This will be useful in solving nq order constant coefficients linear sequential Caputo fractional differential equation which is sequential of order q. Now onwards, we will use the notation ${}^{sc}D^{nq}u(t)$, for a Caputo sequential derivative of order nq which is sequential of order q. The next known result is related to Caputo sequential derivative of order nq.

Theorem 6.1 *The Laplace transform of a Caputo sequential fractional differentiable function $u(t)$ of order nq on $[0, \infty)$ such that $(n-1) < nq < n$, is given by*

$$ \begin{aligned} \mathcal{L}({}^{sc}D^{nq}u(t)) =& s^{nq}U(s) - s^{(nq-1)}u(0) - s^{(n-1)q-1}({}^{sc}D^{q}u(t)|_{t=0}) \\ & - s^{(n-2)q-1}({}^{sc}D^{2q}u(t)|_{t=0}) \cdots - s^{q-1}({}^{sc}D^{(n-1)q})u(t)|_{t=0}), \end{aligned} \tag{6.11} $$

where $U(s) = \mathcal{L}(u(t))$.

See [51], for proof and other details.

Next, we provide Laplace transform tables which is useful in solving nq order linear non-homogeneous sequential Caputo fractional differential equation (which is sequential of order q with $0 < q < 1$) with Caputo fractional initial conditions. In fact for all practical purposes and computational purposes, we choose the value of q

such that $0.5 \leq q < 1$, when $n = 2$. This way, we use the value of q such that $q < 1$ and provides a solution with least error with the available data.

See [3] as an example.

Laplace transform table

S.N	$f(t) = \mathcal{L}^{-1}[F(s)]$	$F(s) = \mathcal{L}f(t)$	
1.	t^q	$\frac{\Gamma(q+1)}{s^{q+1}}$	$s > 0, q > -1$
2.	$E_{q,1}(\pm\lambda t^q)$	$\frac{s^{q-1}}{s^q \mp \lambda}$	$s^q > \lambda, q > -1$
3.	$t^{q-1}E_{q,q}(\pm\lambda t^q)$	$\frac{1}{s^q \mp \lambda}$	$s^q > \lambda, q > -1$
4.	$\frac{t^q}{q}E_{q,q}(\pm\lambda t^q)$	$\frac{s^{q-1}}{(s^q \mp \lambda)^2}$	$s^q > \lambda, q > -1$
5.	$sin_{q,1}(\lambda t^q)$	$\frac{\lambda s^{q-1}}{s^{2q}+\lambda^2}$	$s > 0$
6.	$cos_{q,1}(\lambda t^q)$	$\frac{s^{2q-1}}{s^{2q}+\lambda^2}$	$s > 0$
7.	$sinh_{q,1}(\lambda t^q)$	$\frac{\lambda s^{q-1}}{s^{2q}-\lambda^2}$	$s > 0$
8.	$cosh_{q,1}(\lambda t^q)$	$\frac{s^{2q-1}}{s^{2q}-\lambda^2}$	$s > 0$
9.	$t^{q-1}sin_{q,q}(\lambda t^q)$	$\frac{\lambda}{s^{2q}+\lambda^2}$	$s > 0$
10.	$t^{q-1}cos_{q,q}(\lambda t^q)$	$\frac{s^q}{s^{2q}+\lambda^2}$	$s > 0$
11.	$t^{q-1}sinh_{q,q}(\lambda t^q)$	$\frac{\lambda}{s^{2q}-\lambda^2}$	$s > 0$
12.	$t^{q-1}cosh_{q,q}(\lambda t^q)$	$\frac{s^q}{s^{2q}-\lambda^2}$	$s > 0$
13.	$E_{q,1}(\lambda t^q) + \frac{\lambda t^q}{q}E_{q,q}(\lambda t^q)$	$\frac{s^{2q-1}}{(s^q-\lambda)^2}$	
14.	$t^{q-1}\sum_{k=0}^{\infty}\frac{(k+1)\lambda^k t^{qk}}{\Gamma(qk+q)}$	$\frac{s^q}{(s^q-\lambda)^2}$	
15.	$t^{2q-1}\sum_{k=0}^{\infty}\frac{(k+1)\lambda^k t^{qk}}{\Gamma(qk+2q)}$	$\frac{1}{(s^q-\lambda)^2}$	
16.	$\sum_{k=0}^{\infty}\frac{k(k+1)}{2}\frac{(\lambda t^q)^{k-1}}{\Gamma(q(k-1)+1)}$	$\frac{s^{3q-1}}{(s^q-\lambda)^3}$	
17.	$t^q\sum_{k=0}^{\infty}\frac{k(k+1)}{2}\frac{(\lambda t^q)^{k-1}}{\Gamma(qk+1)}$	$\frac{s^{2q-1}}{(s^q-\lambda)^3}$	
18.	$t^{2q}\sum_{k=0}^{\infty}\frac{k(k+1)}{2}\frac{(\lambda t^q)^{k-1}}{\Gamma(q(k+1)+1)}$	$\frac{s^{q-1}}{(s^q-\lambda)^3}$	
19.	$t^{q-1}E_{q,q}(\lambda t^q) + \sum_{k=0}^{\infty}\frac{(k+1)(k+4}{2}\frac{\lambda^{k+1}t^{q(k+2)-1}}{\Gamma(qk+2q)}$	$\frac{s^{2q}}{(s^q-\lambda)^3}$	
20.	$\sum_{k=0}^{\infty}\frac{(k+1)(k+2)}{2}\frac{\lambda^k t^{q(k+2)-1}}{\Gamma(qk+2q)}$	$\frac{s^q}{(s^q-\lambda)^3}$	
21.	$\sum_{k=0}^{\infty}\frac{k(k+1)}{2}\frac{\lambda^{k-1}t^{q(k+2)-1}}{\Gamma(qk+2q)}$	$\frac{1}{(s^q-\lambda)^3}$	
22.	$Gcos_{q,1}\{(\lambda + i\mu)t^q\}$	$\frac{s^{q-1}(s^q-\lambda)}{(s^q-\lambda)^2+\mu^2}$	
23.	$Gsin_{q,1}\{(\lambda + i\mu)t^q\}$	$\frac{\mu s^{q-1}}{(s^q-\lambda)^2+\mu^2}$	
24.	$t^{q-1}Gcos_{q,q}\{(\lambda + i\mu)t^q\}$	$\frac{s^q-\lambda}{(s^q-\lambda)^2+\mu^2}$	
25.	$t^{q-1}Gsin_{q,q}\{(\lambda + i\mu)t^q\}$	$\frac{\mu}{(s^q-\lambda)^2+\mu^2}$	

6.3 Main Results

6.3.1 *Solution of Linear Sequential Caputo Fractional Differential Equations with Fractional Initial Conditions*

In this section, we will solve a $2q$ order linear non-homogeneous Caputo sequential fractional differential equations with fractional initial conditions. Throughout this section, we use Laplace transform method.

For that purpose, we will consider the $2q$ order linear non-homogeneous Caputo sequential fractional differential equation with constant coefficients of the form

$$ {}^{sc}D^{2q}u + b^{sc}D^{q}u + cu = f(t),\ t \in (0, \infty), \tag{6.12} $$

with initial conditions

$$ u(0) = A,\ \ {}^{sc}D^{q}(u(t))|_{t=0} = B. \tag{6.13} $$

We assume that $0.5 < q < 1$ and $f(t)$ is a continuous function on $[0, \infty)$, which is bounded by an exponential function. Now applying Laplace transform on (6.12) and (6.13) using Theorem 6.1, we get

$$ \begin{aligned} s^{2q}U(s) - s^{(2q-1)}u(0) - s^{(q-1)}({}^{sc}D^{q})(u(t))|_{t=0} + b(s^{q}U(s) - s^{(q-1)}u(0)) \\ + cU(s) = F(s), \end{aligned} \tag{6.14} $$

where $U(s) = \mathcal{L}(u(t))$ and $F(s) = \mathcal{L}(f(t))$.

Our aim here is to show that when $q = 1$, our results yield the results of the second order linear non-homogeneous differential equation with initial conditions as a special case.

Now solving for $U(s)$ from Eq. (6.14) and substituting the initial conditions from Eq. (6.13), we get

$$ U(s) = \frac{As^{(2q-1)} + (B + bA)s^{(q-1)}}{s^{2q} + bs^{q} + c} + \frac{F(s)}{s^{2q} + bs^{q} + c}. \tag{6.15} $$

For convenience, we will denote $As^{(2q-1)} + (B + bA)s^{(q-1)} = s^{q-1}G(s)$.

Now if we can take the inverse Laplace transform on both sides of equation (6.15), we get the solution of the sequential Caputo initial value problem (6.12) and (6.13) of order $2q$.

Next, we present an example 4.2 from [38] of Caputo fractional differential equation which cannot be solved by Laplace transform method. Consider the following fractional differential equation

$$ {}^cD^Q u + b{}^cD^q u = h(t), \tag{6.16} $$

with appropriate initial conditions. In this example, they have used Riemann–Liouville derivative instead of Caputo derivative, but the problem they encounter with Caputo derivative will be very similar. In [38], they claim that Laplace transform method encounters great difficulties in solving equation (6.16) when $q - Q$ is not an integer or half integer. However, if we assume Q is an integer multiple of q and the Caputo fractional differential equation (6.16) is sequential, the Laplace transform method is very useful. Also, if $q < 1$, it involves Mittag-Leffler function whose behavior is almost similar to that of the exponential function and for $q = 1$, it produces an integer result.

We can compute the solution of (6.12) and (6.13) based on the roots of the quadratic equation $s^{2q} + bs^q + c = 0$, in terms of s^q.

1. Let $b \neq 0,\ b^2 - 4c > 0$. In this case, the quadratic equation will have real and distinct roots. So, $s^{2q} + bs^q + c = (s^q - \lambda_1)(s^q - \lambda_2)$.
   Clearly $\lambda_1 \neq \lambda_2$. Using partial fraction method, we can write

   $$ \frac{s^{q-1}G(s)}{s^{2q} + bs^q + c} = \frac{c_1 s^{q-1}}{(s^q - \lambda_1)} + \frac{c_2 s^{q-1}}{(s^q - \lambda_2)} $$

   and

   $$ \frac{1}{s^{2q} + bs^q + c} = \frac{c_3}{(s^q - \lambda_1)} + \frac{c_4}{(s^q - \lambda_2)}. $$

   The constants c_i for $i = 1, 2, 3$ and 4 depends on A, B, b, c, λ_1 and λ_2. Using the above relation in the Eq. (6.15) and taking the inverse Laplace transform, we can write the solution of (6.12) as

   $$ u(t) = c_1 E_{q,1}(\lambda_1 t^q) + c_2 E_{q,1}(\lambda_2 t^q) + c_3 \int_0^t (t-s)^{q-1} E_{q,q}(\lambda_1 (t-s)^q) f(s) ds $$
   $$ + c_4 \int_0^t (t-s)^{q-1} E_{q,q}(\lambda_2 (t-s)^q) f(s) ds. $$

   We will have two real and distinct roots when $b = 0$ and $c < 0$. The solution, in this case, can be obtained on the same lines as above.
2. Let $b \neq 0,\ b^2 - 4c = 0$. In this case, the quadratic equation will have real and coincident roots, and hence $s^{2q} + bs^q + c = (s^q - \lambda)^2$.
   By algebraic manipulation, we can write

   $$ \frac{s^{q-1}G(s)}{s^{2q} + bs^q + c} = \frac{As^{q-1}}{(s^q - \lambda)} + \frac{(\lambda + B + bA)s^{q-1}}{(s^q - \lambda)^2} $$

and

$$\frac{1}{s^{2q}+bs^q+c}=\frac{1}{(s^q-\lambda)^2}.$$

Using the Laplace transform table, the above relation and the relation (6.15), we can write the solution of (6.12) as

$$u(t)=AE_{q,1}(\lambda t^q)+\frac{(\lambda A+B+bA)t^q}{q}E_{q,q}(\lambda t^q)$$
$$+\int_0^t (t-s)^{2q-1}\sum_{k=0}^{\infty}\frac{(k+1)(\lambda)^k(t-s)^{qk})}{\Gamma((k+2)q)}f(s)ds.$$

3. Let $b\neq 0,\ b^2-4c<0$. In this case, the quadratic equation will have complex roots of the form $\lambda\pm i\mu$ and hence $s^{2q}+bs^q+c=(s^q-\lambda)^2+\mu^2$.
   By algebraic manipulation, we can write

$$\frac{s^{q-1}G(s)}{s^{2q}+bs^q+c}=\frac{g_1s^{q-1}(s^q-\lambda)}{(s^q-\lambda)^2+\mu^2}+\frac{\mu g_2}{(s^q-\lambda)^2+\mu^2}$$

and

$$\frac{1}{s^{2q}+bs^q+c}=\frac{1}{\mu}\frac{\mu}{((s^q-\lambda)^2+\mu^2)},$$

where g_1 and g_2 are constants that can be determined. Now using the Laplace transform table and the above relation, we can take the inverse Laplace transform of relation (6.15) to obtain the solution of (6.12) as

$$u(t)=g_1Gcos_{q,1}\{(\lambda+i\mu)t^q\}+g_2Gsin_{q,1}\{(\lambda+i\mu)t^q\}$$
$$+\frac{1}{\mu}\int_0^t (t-s)^{q-1}Gsin_{q,q}\{(\lambda+i\mu)(t-s)^q\}f(s)ds.$$

Note that if $\lambda=0$, in which case $s^{2q}+c=s^{2q}+\mu^2$, where $\mu=\sqrt{-c}$. In this case, the solution of (6.12) will be

$$u(t)=g_1cos_{q,1}\{\mu t^q\}+g_2sin_{q,1}\{\mu t^q\}$$
$$+\frac{1}{\mu}\int_0^t (t-s)^{q-1}sin_{q,q}\{\mu(t-s)^q\}f(s)ds.$$

Also note that if $q = 1$, one can easily observe that the integer results can be obtained as a special case even when the roots are complex numbers with the real part not zero.

We have used the following fractional trigonometric functions given by

$$\sin_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda t^q)^{2k+1}}{\Gamma((2k+1)q+1)} \qquad \cos_{q,1}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda t^q)^{2k}}{\Gamma(2kq+1)}$$

$$\sin_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda t^q)^{2k+1}}{\Gamma((2k+1)q+q)} \qquad \cos_{q,q}(\lambda t^q) = \sum_{k=0}^{\infty} \frac{(-1)^k (\lambda t^q)^{2k}}{\Gamma(2kq+q)}$$

$$G\sin_{q,1}((\lambda + i\mu)t^q) = \sum_{k=0}^{\infty} \frac{Im(\lambda + i\mu)^k t^{qk}}{\Gamma(qk+1)}$$
$$G\cos_{q,1}((\lambda + i\mu)t^q) = \sum_{k=0}^{\infty} \frac{Re(\lambda + i\mu)^k t^{qk}}{\Gamma(qk+1)}$$

$$G\sin_{q,q}((\lambda + i\mu)t^q) = \sum_{k=0}^{\infty} \frac{Im(\lambda + i\mu)^k t^{qk}}{\Gamma(qk+q)}$$
$$G\cos_{q,q}((\lambda + i\mu)t^q) = \sum_{k=0}^{\infty} \frac{Re(\lambda + i\mu)^k t^{qk}}{\Gamma(qk+q)}$$

Next, as an application, we present an example of prey and predator model.
Example: Let u_1 and u_2 be the prey and predator densities respectively. Now consider the prey and predator model

$$^cD^q_{0+}u_1(t) = u_1\left(\frac{2}{3} - \frac{1}{3}u_1 - \frac{1}{3}u_2\right) \quad \textit{and} \quad ^cD^q_{0+}u_2(t) = u_2\left(-\frac{8}{3} + 3u_1 - \frac{1}{3}u_2\right),$$

where $0 < q < 1$.

The only positive equilibrium solution of the above system is $(1, 1)$ and we reduce the above problem to the corresponding linear system in the neighborhood of equilibrium. Then the linear system will be in the form

$$^cD^q_{0+}v(t) = Bv(t), \quad u(0) = u_0, \quad 0 < q < 1,$$

where $B = \begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} \\ 3 & -\frac{1}{3} \end{bmatrix}$ and $v(t) = \begin{bmatrix} u_1(t) - 1 \\ u_2(t) - 1 \end{bmatrix}$.

Then matrix B has eigenvalues $-\frac{1}{3} \pm i$ and we will plot the figure of $Gsin_{q,1}(-\frac{1}{3} + i)t^q$ and $Gcos_{q,1}(-\frac{1}{3} + i)t^q$.

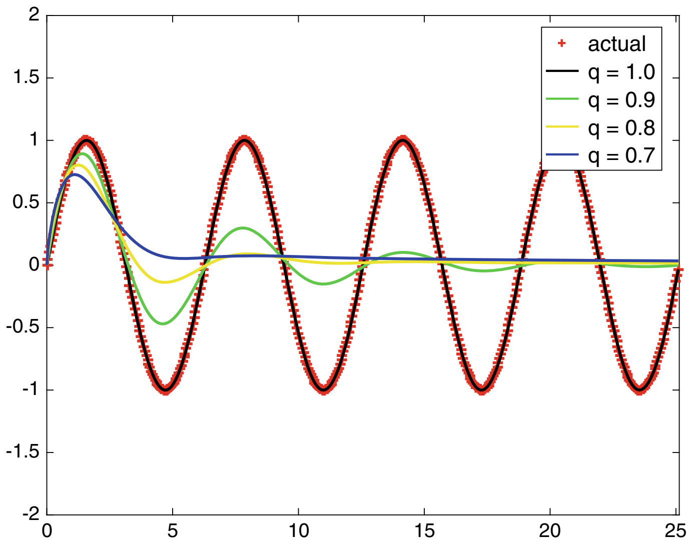

Fig. 6.1 $\sin_{q,1}(t^q)$ graph

Next, we present graphs of several fractional trigonometric functions for different values of q with $q < 1$, and its relation with the original trigonometric function with $q = 1$. All of these graphs have been plotted using MATLAB. Figures 6.1 and 6.2 are fractional trigonometric functions $\sin_{q,1}(t^q)$ and $\cos_{q,1}(t^q)$ when the real part of the complex roots are zero and imaginary part as one. Figures 6.3 and 6.4 are graphs of $\sin_{q,q}(t^q)$ and $\cos_{q,q}(t^q)$ functions that will be needed when we need to find the convolution integral with the Laplace transform of the non-homogeneous term. Figures 6.5, 6.6, 6.7 and 6.8 are graphs of the Generalized fractional trigonometric functions $Gsin_{q,1}((-\frac{1}{3}+i)t^q)$, $Gcos_{q,1}((-\frac{1}{3}+i)t^q)$, $Gsin_{q,q}((-\frac{1}{3}+i)t^q)$ and $Gcos_{q,q}((-\frac{1}{3}+i)t^q)$ when the real part of the complex roots of the quadratic equations involved is not zero. In each of these graphs, we have drawn the graph for $q = 0.7, 0.8, 0.9, 1$. It is to be noted that the fractional trigonometric functions exhibit damping behavior without a damping term in the model.

Remarks: In Figs. 6.1, 6.2, 6.3, and 6.4, when the value of q is closer to 1, the fractional sine and cosine graph is closer to the graph of original sine and cosine functions, respectively. In Figs. 6.5, 6.6, 6.7, and 6.8, the generalized fractional sine and cosine graph is getting closer to the original graph of $e^{-\frac{t}{3}}sint$ and $e^{-\frac{t}{3}}cost$ respectively. When the value of q is smaller, the graph oscillates faster and have faster convergence but for higher values, it oscillates in a slower pace with slower convergence. For the modeling point of view, if the physical model has damping and there is no damping terms in the integer derivative, then it is best to use Caputo sequential fractional derivative. An appropriate order fractional differential equation's results were more closer to the real data than the integer model results.

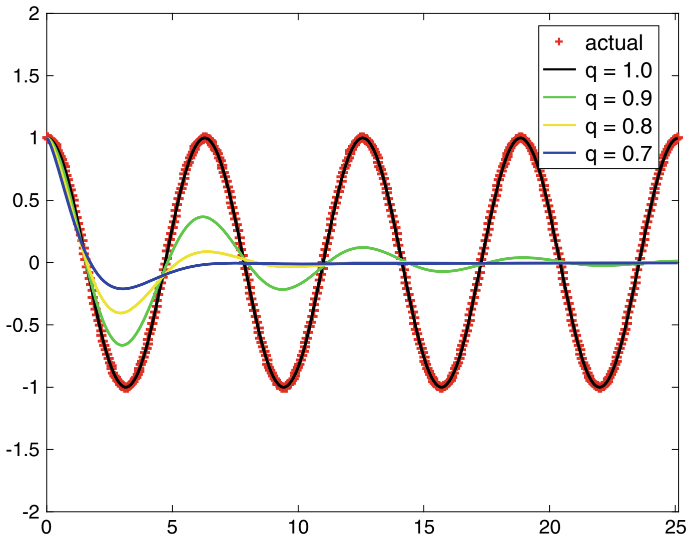

Fig. 6.2 $\cos_{q,1}(t^q)$ graph

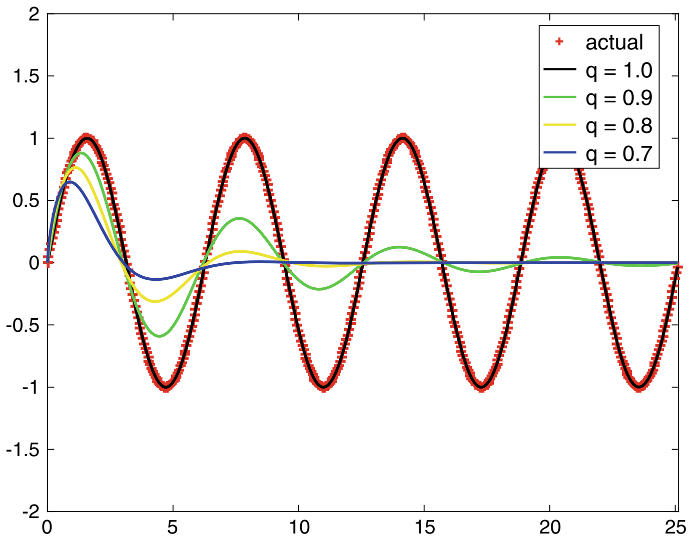

Fig. 6.3 $\sin_{q,q}(t^q)$ graph

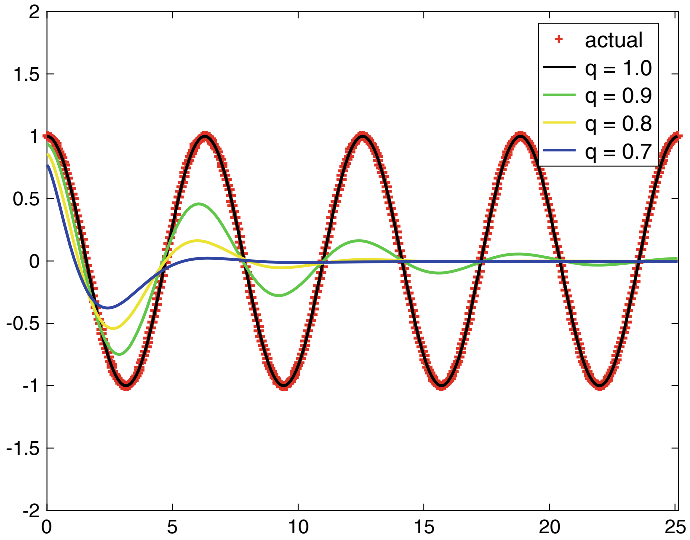

Fig. 6.4 $\cos_{q,q}(t^q)$ graph

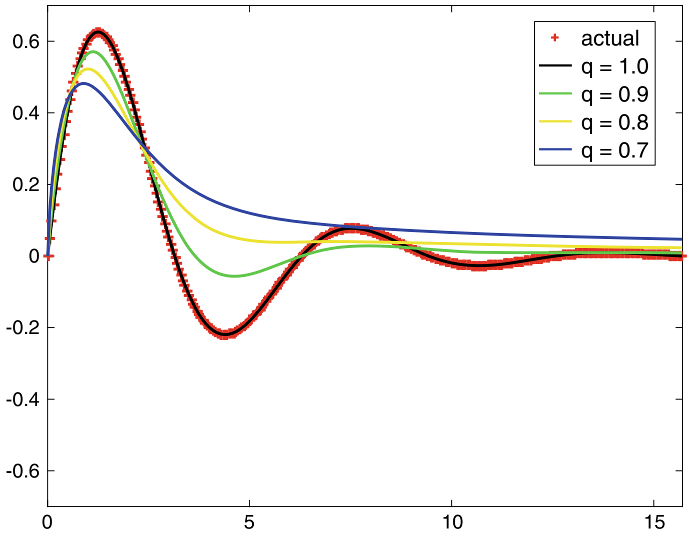

Fig. 6.5 $Gsin_{q,1}((-\frac{1}{3}+i)t^q)$ graph

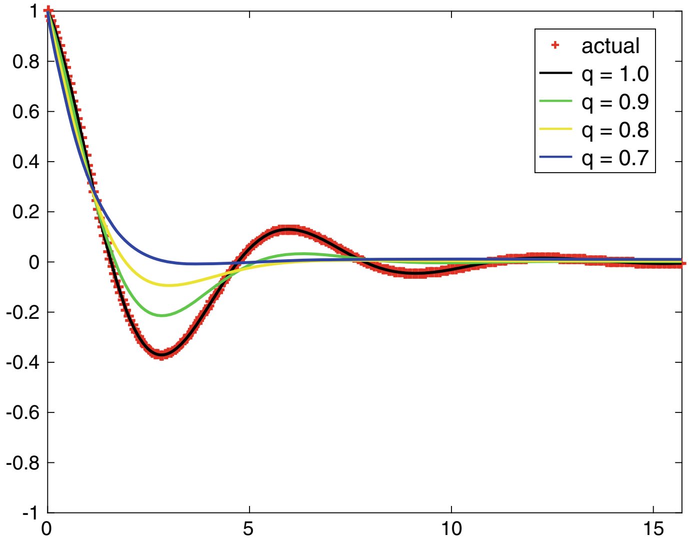

Fig. 6.6 $Gcos_{q,1}((-\frac{1}{3}+i)t^q)$ graph

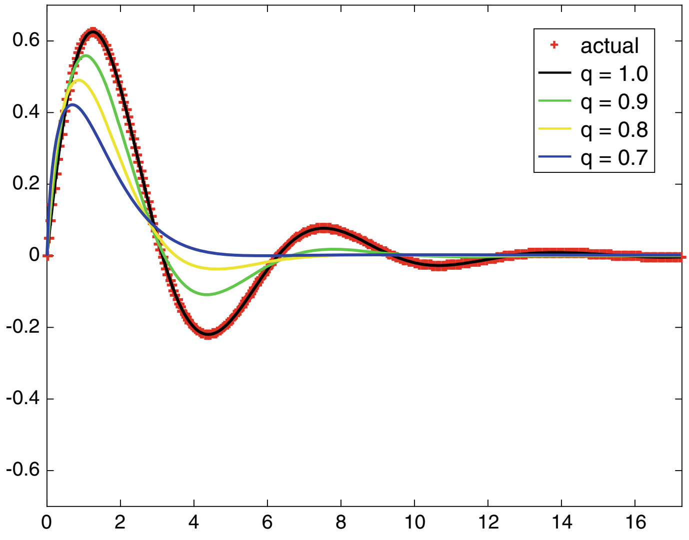

Fig. 6.7 $Gsin_{q,q}((-\frac{1}{3}+i)t^q)$ graph

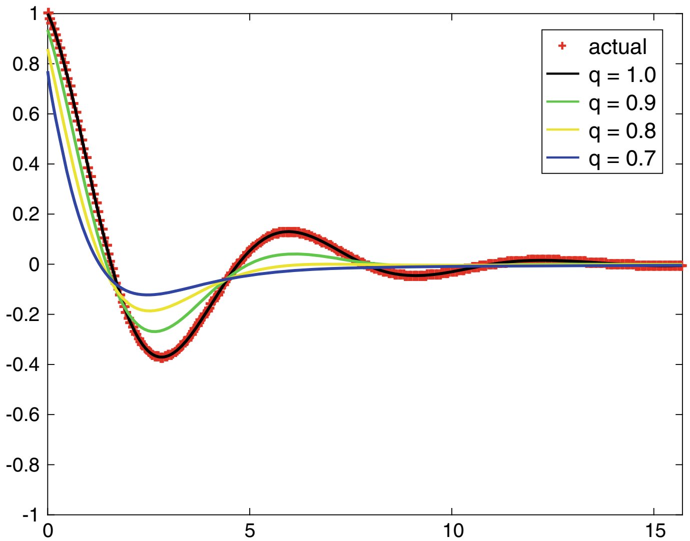

Fig. 6.8 $Gcos_{q,q}((-\frac{1}{3}+i)t^q)$ graph

Conclusion: In this work, we have presented a method to compute the solution of a linear non-homogeneous $2q$ order Caputo sequential fractional differential equation which is sequential of order q with fractional initial conditions. Our results demonstrates that q can be used as a parameter to enhance the model compared with the integer order derivative. One can extend our method to solve nq order linear Caputo sequential fractional differential equations which is sequential of order q of the following form:

$$\sum_{k=0}^{n} a_k \, {}^cD^{kq}u(t) = f(t),$$

where a_k are given constants for $k = 0, 1, \ldots n$ with fractional initial conditions. This can achieved using Theorem 6.1 and Laplace transform method.

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Chapter 7
Herscovici's Conjecture on Product of Some Complete Bipartite Graphs

A. Lourdusamy and S. Saratha Nellainayaki

7.1 Introduction

Graph Pebbling is a game played on pebbles. In finding more intuitive evidence for a variety of theoretical conjectures, graph pebbling was first proposed by Lagarias and Sags. In literature, Chung implemented graph pebbling [1].

Graph pebbling is the model for network optimization for transporting resources consumed during transit. Electricity, heat or any other energy may be lost during transmission from one place to another. This model finds the solution to the question that how many pebbles are necessary to reach any node of the system for any distribution of the pebbles. A survey paper on graph pebbling was published by Hulbert [5].

Let G be a simple and connected graph. We distribute a finite number of pebbles on the vertices of the graph G. A Pebbling move is removing two pebbles from one vertex and adding another pebble to its adjacent vertex. In a graph G, the t-pebbling number of any vertex v, is the minimum number of pebbles required to move t pebbles to the vertex v by moving pebbles in a sequential manner. It is denoted by $f_t(v, G)$. The t-pebbling number, is the minimum number of pebbles required to move t pebbles to any vertex of the graph G by moving pebbles in a sequential manner. It is denoted by $f_t(G)$.

In [1], Pebbling on graphs, the following conjecture is probably the most convincing open question called as Graham's Conjecture was suggested by Chung.

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Conjecture 7.1 (Graham [1]) $f(G \times H) \leq f(G).f(H)$, where G and H are connected graphs. □

There are two important properties used in the literature for solving these open questions. They are the 2-pebbling property and the odd 2-pebbling property. They are as follows.

A graph G is said to have two pebbling property, if we can move 2 pebbles to any target vertex, whenever the following inequality holds. Let p and q denotes the number of pebbles and the number of occupied vertices, respectively, then $p + q > 2f(G)$. Also, a graph G is said to have the $2t$-pebbling property, if we can move $2t$ pebbles to any target vertex, whenever the following inequality $p + q > 2f_t(G)$ holds. Lourdusamy extended Graham's conjecture as follows:

Conjecture 7.2 (Lourdusamy [7]) $f_t(G \times H) \leq f(G).f_t(H)$, where G and H are connected graphs. □

There are many articles [2–4, 6–10] have given evidences supporting Conjectures 7.1 and 7.2. In [10], the authors proved that the Conjecture 7.2 is true for the product of complete r- partite graphs with a graph which has the $2t$ pebbling property.

Lourdusamy's conjecture was extended by Herscovici as follows:

Conjecture 7.3 (Herscovici [7]) $f_{st}(G \times H) \leq f_s(G).f_t(H)$, where G and H are any connected graphs.

In this paper, we prove that Herscovici's conjecture is true, for the product of complete bipartite graph $K_{m,2}$, $m \geq 2$ with a graph which has the $2t$-pebbling property.

The second section of this paper consists of the preliminary results and notations which we use in the proof of the main results. And in the third section, the main result is proved. We conclude this paper by giving some corollaries from the main result.

7.2 Preliminaries

We have the following findings with respect to the t-pebbling numbers.

Theorem 7.1 ([8]) *The t-pebbling number for the complete graph with n vertices is* $f_t(K_n) = 2t + n - 2$

Theorem 7.2 ([8]) *The t-pebbling number for the star graph is* $f_t(K_{1,n}) = 4t + n - 2$.

Theorem 7.3 ([10]) *Let* $G = K_{s_1,s_2,\ldots s_r}$, *then* $f_t(G) = \begin{cases} 2t + n - 2, if\ 2t \leq n - s_1 \\ 4t + s_1 - 2, if\ 2t \geq n - s_1 \end{cases}$

Theorem 7.4 ([4]) *The t-pebbling number for the cycle C_{2k} vertices is $f_t(C_{2k}) = t2^k$ and the $t-$ pebbling number for the cycle C_{2k+1} is $f_t(C_{2k+1}) = \frac{2^{k+2}-(-1)^{k+2}}{3} + (t-1)2^k$.*

Theorem 7.5 *The t-pebbling number for a path P_n with n vertices is $f_t(P_n) = t2^{n-1}$.*

We have the following findings with respect to the $2t$-pebbling property.

Theorem 7.6 ([7]) *The $2t$-pebbling property holds for the path graphs P_n with n vertices.*

Theorem 7.7 ([8]) *The $2t$-pebbling property holds for the complete graphs with n vertices.*

Theorem 7.8 ([4]) *The $2t$-pebbling property holds for all the cycles with n vertices.*

Theorem 7.9 ([10]) *The $2t$-pebbling property holds for the star graph $K_{1,n}$, where $n > 1$.*

Theorem 7.10 *[9] The $2t$-pebbling property holds for the fan graph with n vertices.*

Theorem 7.11 ([10]) *The product of complete bipartite graph $K_{m,n}$ with a graph which is having the $2t$-pebbling property satisfies the t-pebbling conjecture. In other words, $f_t(K_{m,n} \times G) \leq f(K_{m,n}) f_t(G)$, where G satisfies the $2t$-pebbling property.*

Theorem 7.12 ([7]) *The product of path graph P_n with a graph which is having the $2t$-pebbling property satisfies the t-pebbling conjecture. In other words, $f_t(P_n \times G) \leq f(P_n) f_t(G)$, where G satisfies the $2t$-pebbling property.*

Let the vertex set of the graph $K_{m,2}$,($m \geq 2$) m vertices on the first partite and two vertices on the second partite be notated as $V(K_{m,2}) = U_1 \cup U_2$, where $U_1 = \{u_{11}, u_{12}, ..., u_{1m}\}$ and $U_2 = \{u_{21}, u_{22}\}$. For any placement of pebbles on the product graph $K_{m,2} \times G$, let G_{ij} denotes $\{u_{ij}\} \times G$ and p_{ij} be the number of pebbles distributed on G_{ij}, where $i = 1, 2, ..., m$ and $j = 1, 2$.

7.3 Main Result

In this section, we prove that Conjecture 7.3 is true, when G is a complete bipartite graph $K_{m,2}$. First, let us prove that the conjecture is true, when $m \leq 5$.

Theorem 7.13 *For any distribution of pebbles on the product graph $K_{m,2} \times G$, where $m \leq 5$ and G having the $2t$-pebbling property, $f_{st}(K_{m,2} \times G) \leq f_s(K_{m,2}) f_t(G)$.*

Proof We prove the result by induction on s. For $s = 1$, the results follows from Theorem 7.11. We assume that the result is true for $2 \leq s' < s$.

Let $|D|$ be any placement of $f_s(K_{m,2})f_t(G)$ pebbles on the graph $K_{m,2} \times G$. Since $s > 1$, $f_s(K_{m,2}) = 4s + m - 2$. Thus, the number of pebbles distributed is $(4s + m - 2)f_t(G)$.

Case 1: Let (u_{11}, y), where $y \in G$ be the target vertex.

Suppose $p_{11} \geq f_t(G)$. Then we can move t pebbles to the target(u_{11}, y) by using $f_t(G)$ pebbles. Thus, we are left with at least

$$\begin{aligned} f_s(K_{m,2})f_t(G) - f_t(G) &= (4s + m - 2)f_t(G) - f_t(G) \\ &\geq (4(s-1) + m - 2)f_t(G) \end{aligned}$$

pebbles on the vertices of the graph $K_{m,2} \times G$. By induction hypothesis, we can move $(s-1)t$ additional pebbles to the target. Thus, the target vertex receives $t + (s-1)t = st$ pebbles. Hence, assume $p_{11} < f_t(G)$.

We claim that either $p_{2j} \geq 2f_t(G)$, for some j or $p_{1i} \geq 4f_t(G)$, for some i or there exists an i and j such that $p_{1i} \geq 2f_t(G)$ and $p_{2j} \geq f_t(G)$. Otherwise the vertices of the graph $K_{m,2} \times G$ contains fewer than $2(m+1)f_t(G)$ pebbles. This gives us a contradiction as $s > 1$, $|D| \geq (m+6)f_t(G)$. Thus, our claim follows.

Subcase 1.1: Suppose $p_{2j} \geq 2f_t(G)$. Then we can transfer $f_t(G)$ pebbles from $2f_t(G)$ pebbles to $\{u_{11}\} \times G$. Thus, we can make the vertex (u_{11}, y) to have t pebbles. Thus, we are left with at least

$$\begin{aligned} (4s + m - 2)f_t(G) - 2f_t(G) &\geq (4(s-1) + m - 2)f_t(G) \\ &= f_{(s-1)}(K_{m,2})f_t(G) \end{aligned}$$

pebbles on the vertices of the graph $K_{m,2} \times G$. By induction hyposthesis, we can move $(s-1)t$ additional pebbles to the target. Thus, the target vertex receives $t + (s-1)t = st$ pebbles.

Subcase 1.2: Suppose there exists an $i, i \neq 1$ and j such that $p_{1i} \geq 2f_t(G)$ and $p_{2j} \geq f_t(G)$. Then we can move $f_t(G)$ pebbles from $p_{1i}, i \neq 1$ to p_{2j} using $2f_t(G)$ pebbles. Now p_{2j} will have more than $2f_t(G)$ pebbles. Thus, we can transfer $f_t(G)$ pebbles to the target copy. Therefore, t pebbles can be moved to the target vertex. Now we are left with at least

$$\begin{aligned} (4s + m - 2)f_t(G) - 3f_t(G) &\geq (4(s-1) + m - 2)f_t(G) \\ &= f_{(s-1)}(K_{m,2})f_t(G) \end{aligned}$$

pebbles on the graph $K_{m,2} \times G$. By induction, we can move $(s-1)t$ additional pebbles to the target vertex. Thus, we are done.

Subcase 1.3: Suppose $p_{1i} \geq 4f_t(G)$, for some i. Since G posses the $2t$-pebbling property and by Theorem 7.12, we can move t pebbles using $4f_t(G)$ pebbles. Then there are remaining $(4(s-1) + m - 2)f_t(G)$ pebbles placed on the graph $K_{m,2} \times G$.

By induction, we can move $(s-1)f_t(G)$ additional pebbles to the target vertex. Thus, (u_{11}, y) receives $t + (s-1)t = st$ pebbles.

Case 2: Let (u_{21}, y) be the target vertex.

Suppose $p_{21} \geq f_t(G)$. Then we make t pebbles to be moved to the target vertex (u_{21}, y). Thus, we are left with at least

$$\begin{aligned} f_s(K_{m,2})f_t(G) - f_t(G) &= (4s + m - 2)f_t(G) - f_t(G) \\ &\geq (4(s-1) + m - 2)f_t(G) \\ &= f_{(s-1)}(K_{m,2})f_t(G) \end{aligned}$$

pebbles on the graph $K_{m,2} \times G$. By induction hyposthesis, we can move $(s-1)t$ additional pebbles to the target. Thus, the target vertex receives $t + (s-1)t = st$ pebbles. Hence, assume $p_{21} < f_t(G)$.

We claim that either $p_{1j} \geq 2f_t(G)$, for some j or $p_{22} \geq 4f_t(G)$ or there exists an i such that $p_{1i} \geq 2f_t(G)$ and $p_{22} \geq f_t(G)$. Otherwise, the vertices on the graph $K_{m,2} \times G$ contains fewer than $(m+3)f_t(G)$ pebbles. This gives us a contradiction as $s > 1$, $|D| \geq (m+6)f_t(G)$. Thus, either one of these possibilities occur.

Case 2.1: Suppose $p_{1j} \geq 2f_t(G)$. Then we can transfer $f_t(G)$ pebbles to $\{u_{21}\} \times G$ from $2f_t(G)$ pebbles. Therefore, the vertex (u_{21}, y) can be made to have t pebbles. Thus, we are left with at least

$$(4s + m - 2)f_t(G) - 2f_t(G) \geq f_{(s-1)}(K_{m,2})f_t(G)$$

pebbles. By induction hyposthesis, we can move $(s-1)t$ additional pebbles to the target. Thus, the target vertex receives $t + (s-1)t = st$ pebbles.

Case 2.2: Suppose there exists an i such that $p_{1i} \geq f_t(G)$ and $p_{22} \geq 2f_t(G)$. Then we can move $f_t(G)$ pebbles from p_{22} to p_{1i} using $2f_t(G)$ pebbles. Now p_{1i} will have more than $2f_t(G)$ pebbles. Thus, we can transfer $f_t(G)$ pebbles to the target copy and hence we put t pebbles to the target vertex. Now we are left with at least

$$(4s + m - 2)f_t(G) - 3f_t(G) \geq (4(s-1) + m - 2)f_t(G)$$

pebbles on the graph $K_{m,2} \times G$. By induction we can move $(s-1)t$ additional pebbles to the target vertex. Thus, we are done.

Subcase 2.3: Suppose $p_{22} \geq 4f_t(G)$. Since G is having the $2t-$pebbling property and by Theorem 7.12, we can move t pebbles using $4f_t(G)$ pebbles. Then there are $(4(s-1) + m - 2)f_t(G)$ pebbles distributed on the graph $K_{m,2} \times G$. By induction we can transfer $(s-1)t$ additional pebbles to the target vertex. Thus, (u_{11}, y) receives $t + (s-1)t = st$ pebbles. □

Theorem 7.14 *For any distribution of pebbles on the product graph $K_{m,2} \times G$, where $m \geq 6$ and G having the $2t$-pebbling property, $f_{st}(K_{m,2} \times G) \leq f_s(K_{m,2})f_t(G)$.*

Proof We prove the result by induction on s. If $s = 1$, the results follows from Theorem 7.11. Let $s \geq 2$. We assume that the result is for less s.

Let $|D|$ be any placement of $f_s(K_{m,2})f_t(G)$ pebbles on the graph $K_{m,2} \times G$. Since $s > 1$, $f_s(K_{m,2}) = 4s + m - 2$. Hence we assume that the number of pebbles placed on the graph $K_{m,2} \times G$ is $(4s + m - 2)f_t(G)$.

Without loss of generality, let us assume that (u_{11}, y), where $y \in G$ is the target vertex.

Suppose $p_{11} + p_{21} + p_{1j} + p_{22} \geq 4f_t(G)$ for some j. Since the subgraph induced by $< \{v_{11}\} \times G, \{v_{21}\} \times G, \{v_{1j}\} \times G, \{v_{22}\} \times G >$ is isomorphic to $K_{2,2} \times G$, where G is a graph having the $2t$-pebbling property, by Theorem 7.11, we can transfer $f_t(G)$ pebbles to the $\{u_{11}\} \times G$, and hence t pebble to the vertex (u_{11}, y). Now we are left with at least

$$\begin{aligned}(4s + m - 2)f_t(G) - 4f_t(G) &= (4(s - 1) + m - 2)f_t(G)\\ &= f_{(s-1)}(K_{m,2})f_t(G)\end{aligned}$$

pebbles on the graph. This suffices to transfer $(s - 1)t$ additional pebbles to the vertex (u_{11}, y). Hence we have moved st pebbles to the target vertex. Otherwise, the vertices of the graph $K_{m,2} \times G- < G_{11}, G_{21}, G_{22}, G_{1j} >$ contains at least $(4s + m - 2)f_t(G) - 4f_t(G) = (4s + m - 6)f_t(G)$. Thus, $(4s + m - 6)f_t(G)$ pebbles are distributed in $m - 2$ copies of $\{u_i \times G\}$. Since $s \geq 2$ at least two $\{u_i\} \times G$ contains more than $2f_t(G)$ pebbles or one $\{u_i\} \times G$ contains more than $4f_t(G)$ pebbles, $2f_t(G)$ pebbles can be moved from those copies. Thus, we can put t pebbles to (u_{11}, y) using $4f_t(G)$ pebbles. Now using the remaining number of $(4s + m - 2)f_t(G) - 4f_t(G) = (4(s - 1) + m - 2)f_t(G)$ pebbles we transfer $(s - 1)f_t(G)$ pebbles to the target copy and hence $(s - 1)t$ pebbles to the target vertex. Thus, we have moved $t + (s - 1)t$ pebbles to the target vertex. □

Corollary 7.1 *For the product of $K_{m,2}$ with $m + 2$ vertices and path graph P_m with m vertices,*

$$f_{st}(K_{m,2} \times P_m) \leq f_s(K_{m,2})f_t(P_m).$$

Corollary 7.2 *For the product of $K_{m,2}$ with $m + 2$ vertices and complete graph K_m with m vertices*

$$f_{st}(K_{m,2} \times K_m) \leq f_s(K_{m,2})f_t(K_m).$$ □

Corollary 7.3 *For the product of $K_{m,2}$ with $m + 2$ vertices and cycle C_m with m vertices*

$$f_{st}(K_{m,2} \times C_m) \leq f_s(K_{m,2})f_t(C_m)$$ □

Corollary 7.4 *For the product of $K_{m,2}$ with $m + 2$ vertices and star graph $K_{1,m}$, $m > 1$ with $m + 1$ vertices*

$$f_{st}(K_{m,2} \times K_{1,m}) \leq f_s(K_{m,2})f_t(K_{1,m}).$$ □

Corollary 7.5 *For the product of $K_{m,2}$ with $m+2$ vertices and fan graph F_m with m vertices*

$$f_{st}(K_{m,2} \times F_m) \leq f_s(K_{m,2}) f_t(F_m). \qquad \square$$

Corollary 7.6 *For the product of $K_{m,2}$ with $m+2$ vertices and another $K_{m,2}$ with $m+2$ vertices,*

$$f_{st}(K_{m,2} \times P_m^2) \leq f_s(K_{m,2}) f_t(D_2(P_n)). \qquad \square$$

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Chapter 8
On Fault-Tolerant Metric Dimension of Heptagonal Circular Ladder and Its Related Graphs

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8.1 Introduction

The concept of *metric dimension* was first independently introduced by Slater [16] and then by Harary and Melter [5]. Since then, it has been extensively studied in a number of articles, see [11, 12, 14]. The diverseness and emergence of metric dimension applications have prevailed in a wide range of logical and scientific areas. For instance, computational chemistry [3], the computation of the set of landmark nodes required with the minimum cardinality in robot navigation [7], determining the protocols of routing geographically [8], network discovery and telecommunication [2], etc.

Suppose $\mathbb{F} = \mathbb{F}(V, E)$ be a simple (i.e., $\mathbb{F}$ has no loops and parallel edges), finite (i.e., the cardinality of the vertex set in $\mathbb{F}$ is finite), connected (i.e., there must exist a path between every pair of distinct vertices in $\mathbb{F}$), and an undirected (i.e., the edges in $\mathbb{F}$ are bidirectional) graph. E and V denote the set of edges and vertices in $\mathbb{F}$ respectively. The topological distance (geodesic) between two vertices l and m in a simple connected graph $\mathbb{F}$, denoted by $d(l, m)$, is the length of a shortest $l - m$ path between the vertices l and m.

If for any three vertices $a, b, c \in V(\mathbb{F})$, we have $d_{\mathbb{F}}(a, b) \neq d_{\mathbb{F}}(a, c)$, then the vertex a is said to distinguish (resolve) the pair of vertices b, c ($b \neq c$) in $V(\mathbb{F})$. If this condition of resolvability is fulfilled by some vertices comprising a subset

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$D \subseteq V(\mathbb{F})$ i.e., every pair of different vertices in the given undirected graph $\mathbb{F}$ is distinguished by at least one element of D, then D is said to be a *metric generator* (*resolving set*). The concept behind this aforementioned definition goes back to Melter and Harary [5], who demonstrated that this idea inherently uprises from communication networks. The *metric dimension* of the given graph $\mathbb{F}$ is just the minimum cardinality of the metric generator D, and is usually denoted by $\beta(\mathbb{F})$. The metric generator D with minimum cardinality is the metric basis for $\mathbb{F}$. For a subset $D = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, ..., \varepsilon_k\}$ of an ordered vertices in $\mathbb{F}$, the k-coordinate (representation) of vertex c in $V(\mathbb{F})$ is

$$\varphi(c|D) = (d_{\mathbb{F}}(\varepsilon_1, c), d_{\mathbb{F}}(\varepsilon_2, c), d_{\mathbb{F}}(\varepsilon_3, c), ..., d_{\mathbb{F}}(\varepsilon_k, c))$$

Following this, the set D is a resolving set for $\mathbb{F}$, if $\varphi(a|D) \neq \varphi(y|D)$, for any $a, y \in V(\mathbb{F})$ with $a \neq y$.

In [1], Avizienis defined the concept of fault tolerance in computer networks as *the ability to execute specified algorithms irrespective of program errors and hardware failures*. Fault tolerance in all components of the system is a cost-effective way to increase reliability. This notion of fault-tolerant computing gives rise to three main subcategories: (a) The analysis and design of fault-tolerant systems and circuits, (b) the validation of programs, and (c) the testing and diagnosis of digital systems and circuits. Similarly, in metric dimension the concept of fault tolerance was introduced by Hernando et al. [6] and called it as the fault-tolerant metric dimension (FTMD). Members of metric basis were made reference to as censors in an application presented in [4]. Even if one of the censors is not operating, the fault-tolerant resolving set (FTRS) provides accurate information. As FTRS is also a resolving set, so the FTMD has applications in all of the areas where the metric dimension does [10].

A FTRS is one that continues to resolve a graph even when one of its element is removed. Formally, a resolving set D^* of any connected graph $\mathbb{F}$ is said to be a FTRS if $D^* - \{\varpi\}$ for all $\varpi \in D^*$ is also a resolving set of $\mathbb{F}$ [6]. The fault-tolerant metric dimension (FTMD), fault-tolerant metric basis (FTMB), and the fault-tolerant metric co-ordinates (i.e., $\varphi_f(c|D^*)$, for a vertex c and the FTRS D^*, for short FTMC) are defined similarly as metric dimension. By $\beta^*(\mathbb{F})$, we denote the FTMD of $\mathbb{F}$. From the definition of metric dimension and that of FTMD, we obtain the following relation:

$$\beta(\mathbb{F}) + 1 \leq \beta^*(\mathbb{F}) \tag{8.1}$$

The concept of a FTMD is important and interesting from both applicative and mathematical perspectives, and it has been studied by many authors [9, 12, 13]. The graphs of convex polytopes [11], which are rotationally symmetric, are essential in intelligent networks due to the uniform rate of data transformation to all vertices. In Chapter 18 of *Parallel and Distributed Computing Handbook*, Stojmenovi [15] posed several open problems for various interconnection networks. One of them involves the conception of new architectural structures. Toward this, we intend to construct and define new architectures. We start our exploration in the field of planar geometry. The metric dimension and FTMD problems have been studied in recent years for various families of convex polytope graphs. We discovered several types of convex polytope graphs with undefined distance-based graph parameters such as FTMD, edge metric dimension (EMD), and metric dimension during our analysis of the literature. As a result, we commence this research by considering the three infinite families of convex polytope graphs (viz., Γ_n, Σ_n, and Υ_n) and study their FTMD.

This article is organized as follows. In Sect. 8.2, we recall some existing results related to the metric dimension and the FTMD of graphs. In Sect. 8.3, we compute the FTMD of a heptagonal circular ladder. In Sects. 8.4 and 8.5, we study the FTMD of Σ_n and Υ_n. Finally, the conclusion and future work of this paper is presented in Sect. 8.6.

8.2 Preliminaries

In this section, we provide some basic results in the literature which are used in order to obtain our findings in subsequent sections.

For an arbitrary graph $\mathbb{F}$, the following lemma represents a connection between a resolving set and FTRS. For a vertex $w \in V(\mathbb{F})$, assume that $N(w)$ $(N[w])$ represents an open (a close) neighborhood, where $N(w) = \{y \in V(\mathbb{F}) | wy \in E(\mathbb{F})\}$ $(N[w] = N(w) \cup \{w\})$.

Lemma 8.1 ([6]) *Let D be a resolving set for a non-trivial graph $\mathbb{F}$. Then, for any $w \in D$, let $T(w) = \{a \in V(\mathbb{F}) : N(w) \subseteq N(a)\}$. Then $D^* = \cup_{w \in D}(N[w] \cup T(w))$ is a FTRS for $\mathbb{F}$.*

Let $\lambda(a, z)$ represent the set consisting of the common neighbors of the vertices a and z in $\mathbb{F}$. Similarly, for $K \subseteq V(\mathbb{F})$, $\lambda(K)$ denotes the set of the common neighbors of each vertex in K.

Lemma 8.2 ([9]) *Let D be a resolving set for a non-trivial graph $\mathbb{F}$. Then $D^* = \cup_{w \in D}(N[w] \cup \lambda(N(w)))$ is a FTRS for $\mathbb{F}$.*

Sharma and Bhat [11], considered three families of plane graphs Γ_n, Σ_n, and Υ_n. For these three families of graphs, they obtained the following results:

Theorem 8.1 ([11]) *For $n \geq 6$, we have $\beta(\Gamma_n) = 3$.*

Theorem 8.2 ([11]) *For $n \geq 6$, we have $\beta(\Sigma_n) = 3$.*

Theorem 8.3 ([11]) *For $n \geq 6$, we have $\beta(\Upsilon_n) = 3$.*

8.3 Fault-Tolerant Metric Number for Γ_n

The plane graph Γ_n [11] comprises of $n + 2$ number of faces, $5n$ number of edges, and $4n$ number of vertices (see Fig. 8.1). The edge set and the vertex set of Γ_n are denoted by $E(\Gamma_n)$ and $V(\Gamma_n)$, respectively. Therefore, we have

$$V(\Gamma_n) = \{p_{\bar{y}}, q_{\bar{y}}, r_{\bar{y}}, s_{\bar{y}} : 1 \leq \bar{y} \leq n\}$$

and

$$E(\Gamma_n) = \{p_{\bar{y}}q_{\bar{y}}, q_{\bar{y}}r_{\bar{y}}, r_{\bar{y}}s_{\bar{y}}, s_{\bar{y}}r_{\bar{y}+1}, p_{\bar{y}}p_{\bar{y}+1} : 1 \leq \bar{y} \leq n\}$$

We call the vertices $\{p_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ in the graph, Γ_n as the vertices of p-cycle, the vertices $\{q_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ in the graph, Γ_n as the q-vertices, and the vertices $\{r_{\bar{y}}, s_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ in the graph, Γ_n as the vertices of rs-cycle. For our convenience, we take $p_1 = p_{n+1}, q_1 = q_{n+1}, r_1 = r_{n+1}$, and $s_1 = s_{n+1}$. In the accompanying result, we determine the accurate FTMD for the plane graph Γ_n.

Theorem 8.4 *For $n \geq 6$, we have $\beta^*(\Gamma_n) = 4$.*

Proof From Theorem 8.1, we obtain that $\beta(\Gamma_n) = 3$ i.e., the graph Γ_n has a minimum resolving set having cardinality three for every $n \geq 6$. Then for Γ_n with $n \geq 6$, the lower bound for FTMD is obtained by using Eq. (8.1) and is as follows:

$$\beta^*(\Gamma_n) \geq 4 \tag{8.2}$$

Next, in order to prove that $\beta^*(\Gamma_n) \leq 4$, we take the following two cases depending upon the natural n i.e., $n \equiv 1$ $(mod\ 2)$ and $n \equiv 0$ $(mod\ 2)$.

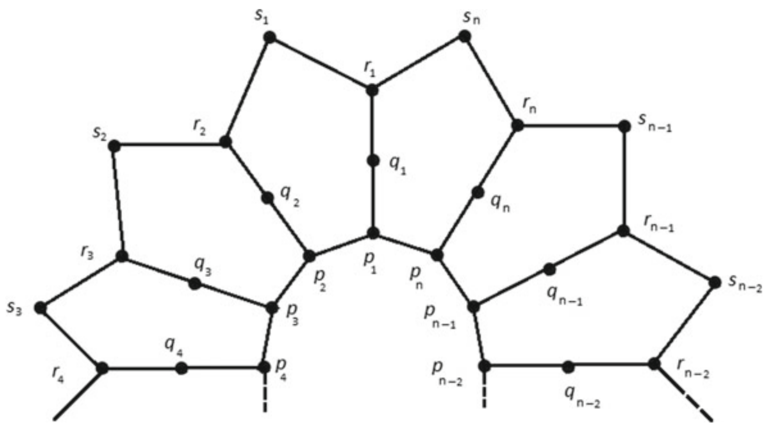

Fig. 8.1 The graph Γ_n

Case (I) $n \equiv 0\ (mod\ 2)$
From this, we have $n = 2b, b \in \mathbb{N}$ and $b \geq 3$. Suppose $\mathfrak{V}^* = \{p_1, p_2, p_{b+1}, p_{b+2}\} \subseteq V(\Gamma_n)$. Next, we give fault-tolerant metric codes to every node of $\Gamma_n \setminus \mathfrak{V}^*$ with respect to the set $\mathfrak{V}^*$.

For p-cycle vertices $\{p_{\bar{y}} : 1 \leq \bar{y} \leq n\}$, the fault-tolerant metric codes are as follows

$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$	p_1	p_2	p_{b+1}	p_{b+2}
$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$:$(\bar{y} = 1)$	$\bar{y} - 1$	1	$b - \bar{y} + 1$	$b - 1$
$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$:$(2 \leq \bar{y} \leq b + 1)$	$\bar{y} - 1$	$\bar{y} - 2$	$b - \bar{y} + 1$	$b - \bar{y} + 2$
$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$:$(\bar{y} = b + 2)$	$2b - \bar{y} + 1$	$\bar{y} - 2$	$\bar{y} - b - 1$	$b - \bar{y} + 2$
$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$:$(b + 3 \leq \bar{y} \leq 2b)$	$2b - \bar{y} + 1$	$2b - \bar{y} + 2$	$\bar{y} - b - 1$	$\bar{y} - b - 2$

The fault-tolerant metric codes for q-vertices $\{q_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ are $\varphi(q_{\bar{y}}|\mathfrak{V}^*) = \varphi_f(p_{\bar{y}}|\mathfrak{V}^*) + (1, 1, 1, 1)$ for $1 \leq \bar{y} \leq 2b$. Next, the fault-tolerant metric codes for rs-cycle vertices $\{r_{\bar{y}}, s_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ are $\varphi_f(r_{\bar{y}}|\mathfrak{V}^*) = \varphi_f(q_{\bar{y}}|\mathfrak{V}^*) + (1, 1, 1, 1)$ for $1 \leq \bar{y} \leq 2b$ and From these codes, we see that no two elements in $V(\Gamma_n)$ have the

$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$	p_1	p_2	p_{b+1}	p_{b+2}
$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$:$(\bar{y} = 1)$	$\bar{y} + 2$	3	$b - \bar{y} + 3$	$b + 2$
$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$:$(2 \leq \bar{y} \leq b)$	$\bar{y} + 2$	$\bar{y} + 1$	$b - \bar{y} + 3$	$b - \bar{y} + 4$
$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$:$(\bar{y} = b + 1)$	$2b - \bar{y} + 3$	$\bar{y} + 1$	$\bar{y} - b + 2$	$b - \bar{y} + 1$
$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$:$(b + 2 \leq \bar{y} \leq 2b)$	$2b - \bar{y} + 3$	$2b - \bar{y} + 4$	$\bar{y} - b + 2$	$\bar{y} - b + 1$

same fault-tolerant metric codes, suggesting $\mathfrak{V}^*$ to be resolving set for Γ_n. Since, by definition of FTRS, the subsets $\mathfrak{V}^* \setminus \{a\}, \forall\ a \in \mathfrak{V}^*$ are $V_1 = \{p_1, p_2, p_{b+1}\}$, $V_2 = \{p_1, p_2, p_{b+2}\}$, $V_3 = \{p_1, p_{b+1}, p_{b+2}\}$, and $V_4 = \{p_2, p_{b+1}, p_{b+2}\}$. Now, to unveil that the set $\mathfrak{V}^*$ is the FTRS for the graph Γ_n, we have to prove that the sets V_1, V_2, V_3, and V_4 are the resolving sets for Γ_n. Then, effortlessly one can find from the fault-tolerant metric codes, as shown above, that the sets V_1, V_2, V_3, and V_4 are also resolving sets for Γ_n, as the metric representation for every different pair of vertices of Γ_n are distinct with respect to the sets V_1, V_2, V_3, and V_4. This implies $\beta^*(\Gamma_n) \leq 4$. Thus, from these lines and Eq. (8.2), we have $\beta^*(\Gamma_n) = 4$, in this case.
Case (II) $n \equiv 1\ (mod\ 2)$
From this, we have $n = 2b + 1$, $b \in \mathbb{N}$ and $b \geq 3$. Suppose $\mathfrak{V}^* = \{p_1, p_2, p_{b+1}, p_{b+2}\} \subseteq V(\Gamma_n)$. Next, we give fault-tolerant metric codes to every node of $\Gamma_n \setminus \mathfrak{V}^*$ with respect to the set $\mathfrak{V}^*$.

For p-cycle vertices $\{p_{\bar{y}} : 1 \leq \bar{y} \leq n\}$, the fault-tolerant metric codes are as follows The fault-tolerant metric codes for q-vertices $\{q_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ are $\varphi(q_{\bar{y}}|\mathfrak{V}^*) = \varphi_f(p_{\bar{y}}|\mathfrak{V}^*) + (1, 1, 1, 1)$ for $1 \leq \bar{y} \leq 2b + 1$. Next, the fault-tolerant metric codes for rs-cycle vertices $\{r_{\bar{y}}, s_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ are $\varphi_f(r_{\bar{y}}|\mathfrak{V}^*) = \varphi_f(q_{\bar{y}}|\mathfrak{V}^*) + (1, 1, 1, 1)$ for $1 \leq \bar{y} \leq 2b + 1$ and

$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$	p_1	p_2	p_{b+1}	p_{b+2}
$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$:$(\bar{y}=1)$	$\bar{y}-1$	1	$b-\bar{y}+1$	$b-1$
$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$:$(2\leq\bar{y}\leq b+1)$	$\bar{y}-1$	$\bar{y}-2$	$b-\bar{y}+1$	$b-\bar{y}+2$
$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$:$(\bar{y}=b+2)$	$2b-\bar{y}+2$	$\bar{y}-2$	$\bar{y}-b-1$	$b-\bar{y}+2$
$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$:$(b+3\leq\bar{y}\leq 2b+1)$	$2b-\bar{y}+2$	$2b-\bar{y}+3$	$\bar{y}-b-1$	$\bar{y}-b-2$

$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$	p_1	p_2	p_{b+1}	p_{b+2}
$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$:$(\bar{y}=1)$	$\bar{y}+2$	3	$b-\bar{y}+3$	$b-\bar{y}+4$
$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$:$(2\leq\bar{y}\leq b)$	$\bar{y}+2$	$\bar{y}+1$	$b-\bar{y}+3$	$b-\bar{y}+4$
$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$:$(\bar{y}=b+1)$	$2b-\bar{y}+4$	$\bar{y}+1$	$\bar{y}-b+2$	$b-\bar{y}+1$
$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$:$(b+2\leq\bar{y}\leq 2b+1)$	$2b-\bar{y}+4$	$2b-\bar{y}+5$	$\bar{y}-b+2$	$\bar{y}-b+1$

Again, from these codes, we see that no two elements in $V(\Gamma_n)$ have the same fault-tolerant metric codes, suggesting $\mathfrak{V}^*$ to be resolving set for Γ_n. Since, by definition of FTRS, the subsets $\mathfrak{V}^* \setminus \{a\}, \forall a \in \mathfrak{V}^*$ are $V_1 = \{p_1, p_2, p_{b+1}\}$, $V_2 = \{p_1, p_2, p_{b+2}\}$, $V_3 = \{p_1, p_{b+1}, p_{b+2}\}$, and $V_4 = \{p_2, p_{b+1}, p_{b+2}\}$. Now, to unveil that the set $\mathfrak{V}^*$ is the FTRS for the graph Γ_n, we have to prove that the sets V_1, V_2, V_3, and V_4 are the resolving sets for Γ_n. Then, effortlessly one can find from the fault-tolerant metric codes, as shown above, that the sets V_1, V_2, V_3, and V_4 are also resolving sets for Γ_n, as the metric representation for every different pair of vertices of Γ_n are distinct with respect to the sets V_1, V_2, V_3, and V_4. This implies $\beta^*(\Gamma_n) \leq 4$. Thus, from these lines and Eq. (8.2), we have $\beta^*(\Gamma_n) = 4$, in this case also and hence the theorem. □

Now, from Theorem 8.4, we have the following corollary (Fig. 8.2).

Corollary 8.1 *The FTMD is constant, for the graph* Γ_n.

In the next section, we determine the accurate FTMD for the plane graph Σ_n.

8.4 Fault-Tolerant Metric Number for Σ_n

The plane graph Σ_n [11] consists of $6n$ number of edges, $2n+2$ number of faces, and $4n$ number of vertices (see Fig. 8.2). The edge set and the vertex set of Σ_n are denoted by $E(\Sigma_n)$ and $V(\Sigma_n)$, respectively. Therefore, we have

$$V(\Sigma_n) = \{p_{\bar{y}}, q_{\bar{y}}, r_{\bar{y}}, s_{\bar{y}} : 1 \leq \bar{y} \leq n\}$$

and

$$E(\Sigma_n) = \{p_{\bar{y}}q_{\bar{y}}, q_{\bar{y}}r_{\bar{y}}, r_{\bar{y}}s_{\bar{y}}, s_{\bar{y}}r_{\bar{y}+1}, p_{\bar{y}}p_{\bar{y}+1}, r_{\bar{y}}r_{\bar{y}+1} : 1 \leq \bar{y} \leq n\}$$

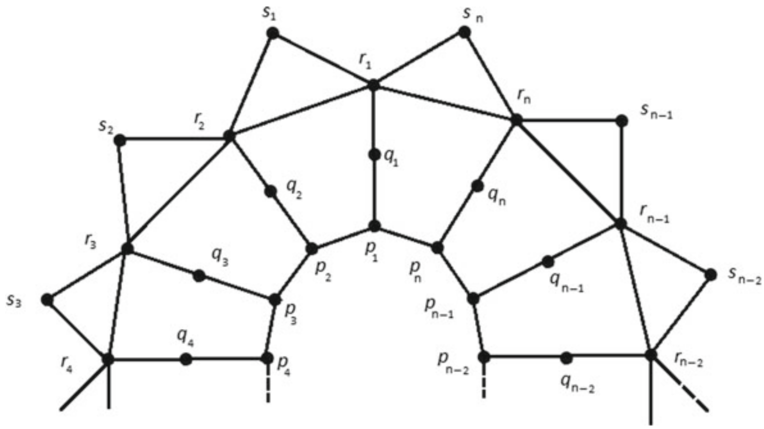

Fig. 8.2 The graph Σ_n

We call the vertices $\{p_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ in the graph, Σ_n as the vertices of p-cycle, the vertices $\{q_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ in the graph, Σ_n as the q-vertices, the vertices $\{r_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ in the graph, Σ_n as the vertices of r-cycle, and the vertices $\{s_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ in the graph, Σ_n as the s-vertices. For our convenience, we take $p_1 = p_{n+1}, q_1 = q_{n+1}$, $r_1 = r_{n+1}$, and $s_1 = s_{n+1}$. Again, Sharma and Bhat [11] proved that the families of plane graphs Σ_n and Υ_n have a metric dimension equal to three and so, they constitute the families of the plane graphs having the constant metric dimension. In the accompanying result, we determine the accurate FTMD for the plane graph Σ_n.

Theorem 8.5 *For $n \geq 6$, we have $\beta^*(\Sigma_n) = 4$.*

Proof From Theorem 8.2, we obtain that $\beta(\Sigma_n) = 3$ i.e., the graph Σ_n has a minimum resolving set having cardinality three for every $n \geq 6$. Then for Σ_n with $n \geq 6$, the lower bound for FTMD is obtained by using Eq. (8.1) and is as follows:

$$\beta^*(\Sigma_n) \geq 4 \tag{8.3}$$

Next, in order to prove that $\beta^*(\Sigma_n) \leq 4$, we take the following two cases depending upon the natural n i.e., $n \equiv 1 \ (mod\ 2)$ and $n \equiv 0 \ (mod\ 2)$.

Case (I) $n \equiv 0 \ (mod\ 2)$
From this, we have $n = 2b, b \in \mathbb{N}$ and $b \geq 3$. Suppose $\mathfrak{V}^* = \{p_1, p_2, p_{b+1}, p_{b+2}\} \subseteq V(\Sigma_n)$. Next, we give fault-tolerant metric codes to every node of $\Sigma_n \setminus \mathfrak{V}^*$ with respect to the set $\mathfrak{V}^*$.

For p-cycle vertices $\{p_{\bar{y}} : 1 \leq \bar{y} \leq n\}$, the fault-tolerant metric codes are as follows

The fault-tolerant metric codes for q-vertices $\{q_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ are $\varphi(q_{\bar{y}}|\mathfrak{V}^*) = \varphi_f(p_{\bar{y}}|\mathfrak{V}^*) + (1, 1, 1, 1)$ for $1 \leq \bar{y} \leq 2b$. Next, the fault-tolerant metric codes for

$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$	p_1	p_2	p_{b+1}	p_{b+2}
$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$:$(\bar{y}=1)$	$\bar{y}-1$	1	$b-\bar{y}+1$	$b-1$
$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$:$(2\leq\bar{y}\leq b+1)$	$\bar{y}-1$	$\bar{y}-2$	$b-\bar{y}+1$	$b-\bar{y}+2$
$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$:$(\bar{y}=b+2)$	$2b-\bar{y}+1$	$\bar{y}-2$	$\bar{y}-b-1$	$b-\bar{y}+2$
$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$:$(b+3\leq\bar{y}\leq 2b)$	$2b-\bar{y}+1$	$2b-\bar{y}+2$	$\bar{y}-b-1$	$\bar{y}-b-2$

rs-cycle vertices $\{r_{\bar{y}}, s_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ are $\varphi_f(r_{\bar{y}}|\mathfrak{V}^*) = \varphi_f(q_{\bar{y}}|\mathfrak{V}^*) + (1, 1, 1, 1)$ for $1 \leq \bar{y} \leq 2b$ and

$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$	p_1	p_2	p_{b+1}	p_{b+2}
$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$:$(\bar{y}=1)$	$\bar{y}+2$	3	$b-\bar{y}+3$	$b+2$
$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$:$(2\leq\bar{y}\leq b)$	$\bar{y}+2$	$\bar{y}+1$	$b-\bar{y}+3$	$b-\bar{y}+4$
$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$:$(\bar{y}=b+1)$	$2b-\bar{y}+3$	$\bar{y}+1$	$\bar{y}-b+2$	$b-\bar{y}+1$
$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$:$(b+2\leq\bar{y}\leq 2b)$	$2b-\bar{y}+3$	$2b-\bar{y}+4$	$\bar{y}-b+2$	$\bar{y}-b+1$

From these codes, we see that no two elements in $V(\Sigma_n)$ have the same fault-tolerant metric codes, suggesting $\mathfrak{V}^*$ to be resolving set for Σ_n. Since, by definition of FTRS, the subsets $\mathfrak{V}^* \setminus \{a\}, \forall a \in \mathfrak{V}^*$ are $V_1 = \{p_1, p_2, p_{b+1}\}$, $V_2 = \{p_1, p_2, p_{b+2}\}$, $V_3 = \{p_1, p_{b+1}, p_{b+2}\}$, and $V_4 = \{p_2, p_{b+1}, p_{b+2}\}$. Now, to unveil that the set $\mathfrak{V}^*$ is the FTRS for the graph Σ_n, we have to prove that the sets V_1, V_2, V_3, and V_4 are the resolving sets for Σ_n. Then, effortlessly one can find from the fault-tolerant metric codes, as shown above, that the sets V_1, V_2, V_3, and V_4 are also resolving sets for Σ_n, as the metric representation for every different pair of vertices of Σ_n are distinct with respect to the sets V_1, V_2, V_3, and V_4. This implies $\beta^*(\Sigma_n) \leq 4$. Thus, from these lines and Eq. (8.3), we have $\beta^*(\Sigma_n) = 4$, in this case.

Case (II) $n \equiv 1\ (mod\ 2)$

From this, we have $n = 2b + 1, b \in \mathbb{N}$ and $b \geq 3$. Suppose $\mathfrak{V}^* = \{p_1, p_2, p_{b+1}, p_{b+2}\} \subseteq V(\Sigma_n)$. Next, we give fault-tolerant metric codes to every node of $\Sigma_n \setminus \mathfrak{V}^*$ with respect to the set $\mathfrak{V}^*$.

For p-cycle vertices $\{p_{\bar{y}} : 1 \leq \bar{y} \leq n\}$, the fault-tolerant metric codes are as follows

$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$	p_1	p_2	p_{b+1}	p_{b+2}
$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$:$(\bar{y}=1)$	$\bar{y}-1$	1	$b-\bar{y}+1$	$b-1$
$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$:$(2\leq\bar{y}\leq b+1)$	$\bar{y}-1$	$\bar{y}-2$	$b-\bar{y}+1$	$b-\bar{y}+2$
$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$:$(\bar{y}=b+2)$	$2b-\bar{y}+2$	$\bar{y}-2$	$\bar{y}-b-1$	$b-\bar{y}+2$
$\varphi_f(p_{\bar{y}}\|\mathfrak{V}^*)$:$(b+3\leq\bar{y}\leq 2b+1)$	$2b-\bar{y}+2$	$2b-\bar{y}+3$	$\bar{y}-b-1$	$\bar{y}-b-2$

The fault-tolerant metric representations for q-vertices $\{q_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ are $\varphi(q_{\bar{y}}|\mathfrak{V}^*) = \varphi_f(p_{\bar{y}}|\mathfrak{V}^*) + (1, 1, 1, 1)$ for $1 \leq \bar{y} \leq 2b + 1$. Next, the fault-tolerant metric representations for rs-cycle vertices $\{r_{\bar{y}}, s_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ are $\varphi_f(r_{\bar{y}}|\mathfrak{V}^*) = \varphi_f(q_{\bar{y}}|\mathfrak{V}^*) + (1, 1, 1, 1)$ for $1 \leq \bar{y} \leq 2b + 1$ and

$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$	p_1	p_2	p_{b+1}	p_{b+2}
$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$:$(\bar{y} = 1)$	$\bar{y} + 2$	3	$b - \bar{y} + 3$	$b - \bar{y} + 4$
$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$:$(2 \leq \bar{y} \leq b)$	$\bar{y} + 2$	$\bar{y} + 1$	$b - \bar{y} + 3$	$b - \bar{y} + 4$
$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$:$(\bar{y} = b + 1)$	$2b - \bar{y} + 4$	$\bar{y} + 1$	$\bar{y} - b + 2$	$b - \bar{y} + 1$
$\varphi_f(s_{\bar{y}}\|\mathfrak{V}^*)$:$(b + 2 \leq \bar{y} \leq 2b + 1)$	$2b - \bar{y} + 4$	$2b - \bar{y} + 5$	$\bar{y} - b + 2$	$\bar{y} - b + 1$

Again, from these codes, we see that no two elements in $V(\Sigma_n)$ have the same fault-tolerant metric codes, suggesting $\mathfrak{V}^*$ to be resolving set for Σ_n. Since, by definition of FTRS, the subsets $\mathfrak{V}^* \setminus \{a\}, \forall\, a \in \mathfrak{V}^*$ are $V_1 = \{p_1, p_2, p_{b+1}\}$, $V_2 = \{p_1, p_2, p_{b+2}\}$, $V_3 = \{p_1, p_{b+1}, p_{b+2}\}$, and $V_4 = \{p_2, p_{b+1}, p_{b+2}\}$. Now, to unveil that the set $\mathfrak{V}^*$ is the FTRS for the graph Σ_n, we have to prove that the sets V_1, V_2, V_3, and V_4 are the resolving sets for Σ_n. Then, effortlessly one can find from the fault-tolerant metric codes, as shown above, that the sets V_1, V_2, V_3, and V_4 are also resolving sets for Σ_n, as the metric representation for every different pair of vertices of Σ_n are distinct with respect to the sets V_1, V_2, V_3, and V_4. This implies $\beta^*(\Sigma_n) \leq 4$. Thus, from these lines and Eq. (8.3), we have $\beta^*(\Sigma_n) = 4$, in this case also and hence the theorem. □

Now, from Theorem 8.5, we have the following corollary.

Corollary 8.2 *The FTMD is constant, for the graph* Σ_n.

In the next section, we determine the bounds (lower and upper) for the FTMD of Σ_n.

8.5 Bounds on FTMD for the Plane Graph Υ_n

The plane graph Υ_n [11] comprises of $3n + 2$ number of faces, $6n$ number of edges, and $4n$ number of vertices (see Fig. 8.3). The edge set and the vertex set of Υ_n are denoted by $E(\Upsilon_n)$ and $V(\Upsilon_n)$, respectively. Therefore, we have

$$V(\Upsilon_n) = \{p_{\bar{y}}, q_{\bar{y}}, r_{\bar{y}}, s_{\bar{y}} : 1 \leq \bar{y} \leq n\}$$

and

$$E(\Upsilon_n) = \{p_{\bar{y}}q_{\bar{y}}, q_{\bar{y}}r_{\bar{y}}, r_{\bar{y}}s_{\bar{y}}, s_{\bar{y}}r_{\bar{y}+1}, r_{\bar{y}}p_{\bar{y}+1}, p_{\bar{y}}p_{\bar{y}+1} : 1 \leq \bar{y} \leq n\}$$

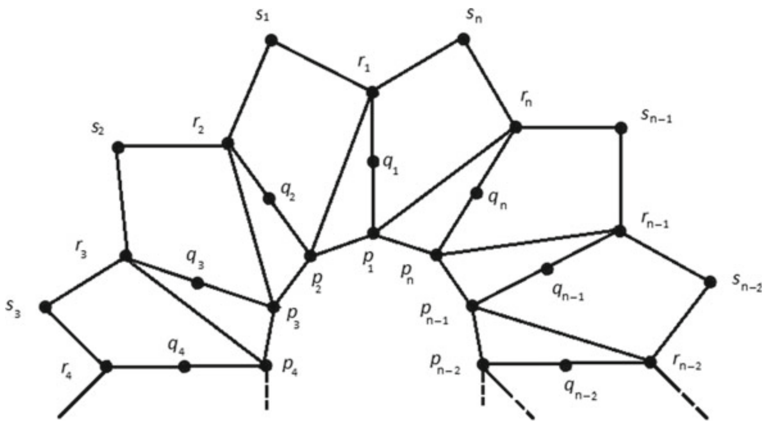

Fig. 8.3 The graph Υ_n

We call the vertices $\{p_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ in the graph, Υ_n as the vertices of p-cycle, the vertices $\{q_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ in the graph, Υ_n as the q-vertices, and the vertices $\{r_{\bar{y}}, s_{\bar{y}} : 1 \leq \bar{y} \leq n\}$ in the graph, Υ_n as the vertices of rs-cycle. For our convenience, we take $p_1 = p_{n+1}$, $q_1 = q_{n+1}$, $r_1 = r_{n+1}$, and $s_1 = s_{n+1}$. In the accompanying outcome, we demonstrate some upper and lower bounds on the FTMD of the rotationally symmetrical plane graph Υ_n, when the minimum resolving set is independent.

Theorem 8.6 *For* $n \geq 6$, *we have* $4 \leq \beta^*(\Upsilon_n)$
and

$$\beta^*(\Upsilon_n) \leq \begin{cases} 10, & if n = 6; \\ 11, & if\ n \geq 7. \end{cases}$$

Proof From Theorem 8.3, we obtain that $\beta(\Upsilon_n) = 3$ i.e., the graph Υ_n has a minimum resolving set having cardinality three for every $n \geq 6$. It was proved that the set $D = \{s_1, s_3, p_{b+2}\}$ and $D = \{s_1, s_3, p_{b+3}\}$ are the basis set for the plane graph Υ_n [11], when the positive natural is even ($n = 2b\ b \in \mathbb{N}$ and $b \geq 3$)and odd ($n = 2b + 1\ b \in \mathbb{N}$ and $b \geq 3$) respectively. Then for Υ_n with $n \geq 6$, the lower bound for FTMD is obtained by using Eq. (8.1) and is as follows $\beta^*(\Upsilon_n) \geq 4$. Clearly, for $n = 6$ and Υ_n, we have $\beta^*(\Upsilon_6) \leq 10$ (by using Lemma 8.2). Next, in order to finish the proof, we have to determine the upper bounds for the FTMD of the rotationally symmetric plane graph Υ_n, for each $n \geq 7$.

Claim: The plane graph Υ_n has a FTRS of cardinality 11, for $n \geq 8$.

To prove this, we take the following two cases depending upon the natural n i.e., when $n \equiv 1\ (mod\ 2)$ and $n \equiv 0\ (mod\ 2)$.

Case (I) When $n \equiv 0$ $(mod\ 2)$.
Then, $D = \{s_1, s_3, p_{b+2}\} \subset V(\Upsilon_n)$ is a minimum resolving set for Υ_n [11], for even natural n. Next, We will prove that Υ_n consists of a FTRS having cardinality 11. Clearly, $N[s_1] = \{s_1, r_1, r_2\}$, $N[s_3] = \{s_3, r_3, r_4\}$, and $N[p_{b+2}] = \{p_{b+1}, p_{b+2}, p_{b+3}, q_{b+2}, r_{b+1}\}$ (one can obtain these close neighborhoods easily from Fig. 8.3). We also determine that $\lambda(N(s_1)) = \lambda(N(p_{b+2})) = \lambda(N(s_3)) = \phi$. From this fact and by Lemma 8.2, we obtain that $D^* = \{s_1, s_3, r_1, r_2, r_3, r_4, p_{b+1}, p_{b+2}, p_{b+3}, q_{b+2}, r_{b+1}\}$ is a FTRS of Υ_n. Therefore, we conclude that, the graph Υ_n has FTRSs with cardinality 11, if n is even.

Case (II) When $n \equiv 1$ $(mod\ 2)$.
Then, $D = \{s_1, s_3, p_{b+3}\} \subset V(\Upsilon_n)$ is a minimum resolving set for Υ_n [11], for odd natural n. Next, We will prove that Υ_n consists of a FTRS having cardinality 11. Clearly, $N[s_1] = \{s_1, r_1, r_2\}$, $N[s_3] = \{s_3, r_3, r_4\}$, and $N[p_{b+3}] = \{p_{b+2}, p_{b+3}, p_{b+4}, q_{b+3}, r_{b+2}\}$ (one can obtain these close neighborhoods easily from Fig. 8.3). We also determine that $\lambda(N(s_1)) = \lambda(N(s_3)) = \lambda(N(p_{b+3})) = \phi$. From this fact and by Lemma 8.2, we obtain that $D^* = \{s_1, s_3, r_1, r_2, r_3, r_4, p_{b+2}, p_{b+3}, p_{b+4}, q_{b+3}, r_{b+2}\}$ is a FTRS of Υ_n. Therefore, we conclude that, the graph Υ_n has FTRSs with cardinality 11, if n is odd.

Therefore, we conclude that Υ_n for each n, has FTRSs of cardinality 11, and thus, claim 2. □

Now, from Theorem 8.6, we have the following corollary.

Corollary 8.3 *The FTMD is constant, for the graph* Σ_n.

8.6 Conclusion

In this article, we have obtained the exact FTMD of Γ_n and Σ_n. For these two graphs, we have also proved that $\beta^*(\Gamma_n) = \beta^*(\Sigma_n) = 4$. For the third graph Υ_n, we have found the bounds (upper and lower) for the FTMD. In future, we will try to obtain the exact FTMD for the plane graph Υ_n.

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Chapter 9
BKS Fuzzy Inference Employing h-Implications

Sayantan Mandal and Balasubramaniam Jayaram

9.1 Introduction

In approximate reasoning (AR), inference mechanisms are functions that produce meaningful outcomes using imprecise input data [1]. To deal with the imprecision in the data, fuzzy sets are applied. Inference mechanisms that use fuzzy set theory for specific purposes are referred to as fuzzy inference mechanisms. The fuzzy inference mechanisms such as (i) Fuzzy Relational Inference (FRI), (ii) Similarity Based Reasoning (SBR) [2–5], and (iii) Takagi–Sugeno (TS) fuzzy system [6] are very well known. Two of the well-known FRIs are the Compositional Rule of Inference (CRI) [7, 8], and the Bandler–Kohout Subproduct (BKS) [9] based on the works of Bandler and Kohout [10]. As part of this study, we primarily focus on BKS.

The fuzzy logic operations employed in a fuzzy inference mechanism (FIM) can be chosen from plenty of choices available in the literature. The arbitrary choice of these operators in an FIM may lead the inference mechanism to a less than satisfactory output. There are various parameters in the literature which measure the 'goodness' of an FIM. In the literature, some such parameters are as follows: (i) Interpolativity [1, 11–15], (ii) Continuity [1, 14, 16, 17], (iii) Robustness [1, 14, 18, 19], (iv) Approximation capability [20–23], (v) Efficiency [24, 25] and (v) Monotonicity [26–28].

Most of the works done so far use fuzzy logic operations from a rich residuated lattice structure and the operations, being from a residuated lattice structure, result

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in a lot of advantages in the proofs of the results. The BKS inference with operations not coming from a residuated lattice structure as well has been studied in [1, 27].

(S, N)-implications are a well-known class of fuzzy implications. Even though the intersection of the set of (S, N)-implications and residual implications is non-empty, there are some sub-classes of (S, N)-implications that are not residual implications. One such class is the class of h-implications.

Similar to the work of [1], we study the goodness of the BKS inference mechanism with the h-implication operators [29]. The modified BKS inference is called BKS-h inference mechanism. Note that the suitability of this inference mechanism has not been studied—which gives the main motivation for this work. We rigorously study the interpolativity, continuity and robustness of this inference mechanism.

Section 9.2 outlines some preliminaries. We present the modified BKS inference with h-implications in Sect. 9.3, which introduces the context of this work. Our main findings in Sect. 9.4 describe how BKS with h-implications is suitable as an inference process, and the results are presented in that section. An example of the work in Sect. 9.4 is provided in Sect. 9.5. Lastly, we conclude our work in Sect. 9.6.

9.2 Preliminaries

As a means of ensuring the comprehensiveness of this paper, we provide definitions of fuzzy logic operations that are of significance in the sequel.

Definition 9.1 ([30]) A *t-norm* (*t-cornorm*, resp.) is defined as a function $T : [0,1]^2 \rightarrow [0,1]$ ($S : [0,1]^2 \rightarrow [0,1]$, resp.) that increases in both variables, is commutative, associative, and takes 1 (0, resp.) as the neutral element.

Definition 9.2 ([31]) A fuzzy implication is a function $I : [0,1]^2 \rightarrow [0,1]$ that satisfies the following conditions for all $x, x_1, x_2, y, y_1, y_2 \in [0,1]$:

$$\text{if } x_1 \leq x_2, \text{ then } I(x_1, y) \geq I(x_2, y), \text{ i.e. } I(\cdot, y) \text{ is decreasing },$$
$$\text{if } y_1 \leq y_2, \text{ then } I(x, y_1) \leq I(x, y_2), \text{ i.e. } I(x, \cdot) \text{ is increasing },$$
$$I(0,0) = 1 , I(1,1) = 1 , I(1,0) = 0 .$$

Definition 9.3 ([31]) An (S, N)- implication is a function $I : [0,1]^2 \rightarrow [0,1]$ such that there exists a t-conorm S and a fuzzy negation N for which

$$I(x, y) = S(N(x), y) , \quad x, y \in [0,1] .$$

Definition 9.4 ([31]) A fuzzy implication I satisfies *neutrality property*, if

$$I(1, y) = y , \qquad y \in [0,1] . \tag{NP}$$

Definition 9.5 ([31]) A fuzzy implication I satisfies the *law of importation*, if

$$I(x, I(y,z)) = I(T(x,y), z) , \qquad x, y, z \in [0,1] . \tag{LI}$$

for a t-norm T.

Definition 9.6 ([29]) Let $h\colon [0,1] \to [0,1]$ be a continuous function that is strictly decreasing with $h(0) = 1$. The binary function I_h on $[0,1]$ defined by

$$I_h(x, y) = \max\left\{h^{-1}\left(x \cdot h(y)\right), h(1)\right\}, \qquad x, y \in [0,1], \tag{9.1}$$

is called *h-implication*, The function h is multiplicative generator of a continuous Archimedean t-conorm.

We write I_h as $\longrightarrow_h$ wherever suitable. For examples of h-implications; see [29].

Proposition 9.1 ([32]) *If h is an h-generator, then I_h is an (S, N)-implication generated from some t-conorm S and continuous negation N.*

Remark 9.1 It can be easily verified that I_h satisfies (NP) as well as (LI) with respect to the product t-norm, $T_{\mathbf{P}}(x, y) = xy$.

Proposition 9.2 ([1]) *For any finite index set $\mathcal{I}$ and for any h-implication $\longrightarrow_h$, the following are true:*

$$x \longrightarrow_h \bigwedge_{i\in\mathcal{I}}(y_i) = \bigwedge_{i\in\mathcal{I}}(x \longrightarrow_h y_i), \tag{9.2}$$

$$\bigvee_{i\in\mathcal{I}}(x_i) \longrightarrow_h y = \bigwedge_{i\in\mathcal{I}}(x_i \longrightarrow_h y). \tag{9.3}$$

Proposition 9.3 *For any arbitrary index set $\mathcal{I}$ and for any h-implication $\longrightarrow_h$, (9.3) is true*

Proof

$$\begin{aligned}
\text{L. H. S.} = h^{-1}\left(\max\left\{\bigvee_{i\in\mathcal{I}}\left(x_i \cdot h(y)\right), h(1)\right\}\right) &= \min\left\{h^{-1}\left(\bigvee_{i\in\mathcal{I}}\left(x_i \cdot h(y)\right)\right), 1\right\} \\
= h^{-1}\left(\bigvee_{i\in\mathcal{I}}\left(x_i \cdot h(y)\right)\right) &= \bigwedge_{i\in\mathcal{I}} h^{-1}\left(\left(x_i \cdot h(y)\right)\right) \\
= \bigwedge_{i\in\mathcal{I}} h^{-1}\left(\max\left\{x_i \cdot h(y), h(1)\right\}\right) &= \bigwedge_{i\in\mathcal{I}}(x_i \longrightarrow_h y) = \text{R. H. S.}
\end{aligned}$$

This proves the proposition. □

Definition 9.7 We denote $I_{\mathbf{G}}$ by $\longrightarrow_{\mathbf{G}}$ for simplicity. The bi-implication [33] obtained from the Goguen implication,

$$I_{\mathbf{G}}(x, y) = \begin{cases} 1, & \text{if } x \leq y \\ \dfrac{y}{x}, & \text{if } x > y \end{cases}, \qquad x, y \in [0,1],$$

defined and denoted as follows:

$$\begin{aligned} x \longleftrightarrow_{\mathbf{G}} y &= I_{\mathbf{G}}(x, y) \wedge I_{\mathbf{G}}(y, x) \\ &= \min \left\{ \frac{x}{y}, \frac{y}{x} \right\}, \quad x, y \in [0, 1] \\ &= \frac{\min\{x, y\}}{\max\{x, y\}}, \quad x, y \in [0, 1]\,. \end{aligned}$$

assuming $\frac{1}{0} = \infty$, $0 \cdot \infty = \infty$ and $\frac{0}{0} = \infty$.

We will now outline some of the important properties of Goguen bi-implication that will be required later in proving our results.

Proposition 9.4 ([33]) *The following holds true:*

$$(a \longleftrightarrow_{\mathbf{G}} b) \cdot (c \longleftrightarrow_{\mathbf{G}} d) \leq (a \cdot c) \longleftrightarrow_{\mathbf{G}} (b \cdot d)\ , a, b, c, d \in [0, 1] \qquad (9.4)$$

where '·' is the product t-norm.

Proposition 9.5 ([33]) *For an index set $\mathcal{I}$ and $i \in \mathcal{I}$, the following inequality holds*

$$\left(\bigvee_{i \in \mathcal{I}} a_i \right) \longleftrightarrow_{\mathbf{G}} \left(\bigvee_{i \in \mathcal{I}} b_i \right) \geq \bigwedge_{i \in \mathcal{I}} (a_i \longleftrightarrow_{\mathbf{G}} b_i)\ ,\ a_i, b_i \in [0, 1]. \qquad (9.5)$$

The above propositions play an important role in the sequel in giving crisp expressions to many results and properties.

9.3 Bandler–Kohout Subproduct with *h*-Implication

Let $\phi \neq X, Y \subseteq \mathbb{R}$ and $\mathcal{F}(X), \mathcal{F}(Y)$ be the space of all fuzzy sets on X and Y, respectively. A Single-input-single-output (SISO) rule base is of the form:

$$\textbf{IF}\ \tilde{x}\ is\ A_i\ \textbf{THEN}\ \tilde{y}\ is\ B_i\ ,\ i = 1, 2, \ldots, n\ , \qquad (9.6)$$

where $\tilde{x}$,$\tilde{y}$ are the linguistic variables and $A_i \in \mathcal{F}(X)$, $B_i \in \mathcal{F}(Y)$. An inference mechanism enables us to find out an output B' for an input A'. For a given rule base (9.6), and an input $A' \in \mathcal{F}(X)$, the output $B' \in \mathcal{F}(Y)$ obtained through BKS inference mechanism [9] is as follows:

$$B'(y) = (A' \lhd R)(y) = \bigwedge_{x \in X} [A'(x) \longrightarrow R(x, y)], \qquad y \in Y, \qquad \text{(BKS-}R\text{)}$$

where $\lhd$ is a function from $\mathcal{F}(X) \times \mathcal{F}(X \times Y)$ to $\mathcal{F}(Y)$, $R \in \mathcal{F}(X \times Y)$ represents the rule base (9.6) and $\longrightarrow$ is a fuzzy implication.

In this work, we consider $\longrightarrow$ as h-implication, i.e. $\longrightarrow = \longrightarrow_h$ and the corresponding BKS inference is called as BKS with h-Implications. The inference in BKS with h-Implications is as follows:

$$B'(y) = (A' \lhd_h R)(y) = \bigwedge_{x \in X} [A'(x) \longrightarrow_h R(x, y)], \qquad y \in Y, \tag{9.7}$$

Furthermore, our study focuses only on implicative rules with an h-implication, i.e. the relation R in (9.7) is

$$\hat{R}_h(x, y) = \bigwedge_{i=1}^{n} (A_i(x) \longrightarrow_h B_i(y)), \quad x \in X, y \in Y\ . \tag{Imp-$\hat{R}_h$}$$

9.4 BKS with *h*-Implications: Its Suitability

We study interpolativity, continuity, and robustness properties of a BKS inference mechanism that employs h-implication.

9.4.1 Interpolativity of BKS with h-Implications

As part of the inference procedure, an inference mechanism is supposed to have interpolativity and a system is interpolative if the input is one of the antecedents in the rule base then the output must be the corresponding consequent, i.e.

$$B_i = A_i \lhd R, \qquad i = 1, 2 \ldots, n., \quad A_i \in \mathcal{F}(X), \ R \in \mathcal{F}(X \times Y).$$

For works on interpolativity of BKS with residuated implications or Yager's classes of fuzzy implications; see [1, 19, 34, 35]. A necessary and sufficient condition for the interpolativity of a BKS inference mechanism with an h-implication is established in the following result.

Theorem 9.1 *Let A_i for $i = 1 \ldots n$ be normal, "$\longleftrightarrow_{\mathbf{G}}$" is the Goguen bi-implication and h is the generator function of the corresponding h-implication. The fuzzy relation $\hat{R}_h$ is a solution of $B_i = A_i \lhd_h R$ iff*

$$\bigvee_{x \in X} \left(A_i(x) \cdot A_j(x)\right) \le \bigwedge_{y \in Y} \left(h\left(B_i(y)\right) \longleftrightarrow_{\mathbf{G}} h\left(B_j(y)\right)\right)\ , \tag{9.8}$$

for any $i, j \in \{1, 2, \ldots, n\}$.

Proof ($\Longrightarrow$): Since $\hat{R}_h$ is a solution of $B_i = A_i \lhd_h R$, we have, for any $y \in Y$, $i \in \{1, 2, \ldots, n\}$,

$$\begin{aligned}
&\left(A_i \lhd_h \hat{R}_h\right)(y) = B_i(y),\\
&\Longrightarrow \bigwedge_{x\in X}\left(A_i(x) \longrightarrow_h \bigwedge_j \left(A_j(x) \longrightarrow_h B_j(y)\right)\right) = B_i(y),\\
&\Longrightarrow A_i(x) \longrightarrow_h \left(A_j(x) \longrightarrow_h B_j(y)\right) \geq B_i(y), && (\forall j, \forall x),\\
&\Longrightarrow \left(A_i(x)\cdot A_j(x)\right) \longrightarrow_h B_j(y) \geq B_i(y), \quad (\forall j, \forall x), && (\text{by } (LI)),\\
&\Longrightarrow h^{-1}\left(\max\left\{A_i(x)\cdot A_j(x)\cdot h\left(B_j(y)\right), h(1)\right\}\right) \geq B_i(y), && (\forall j, \forall x),\\
&\Longrightarrow \max\left\{A_i(x)\cdot A_j(x)\cdot h\left(B_j(y)\right), h(1)\right\} \leq h\left(B_i(y)\right), && (\forall j, \forall x),\\
&\Longrightarrow A_i(x)\cdot A_j(x)\cdot h\left(B_j(y)\right) \leq h\left(B_i(y)\right), && (\forall j, \forall x),\\
&\Longrightarrow A_i(x)\cdot A_j(x) \leq \frac{h(B_i(y))}{h(B_j(y))}, && (\forall j, \forall x).
\end{aligned}$$

Due to arbitrariness, interchanging i, j in the above inequality, we obtain

$$A_j(x)\cdot A_i(x) \leq \frac{h(B_j(y))}{h(B_i(y))}\ .$$

Now, combining the two inequalities obtained above, we have, for all i, j,

$$\begin{aligned}
&A_i(x)\cdot A_j(x) \leq \min\left\{\frac{h(B_i(y))}{h(B_j(y))}, \frac{h(B_j(y))}{h(B_i(y))}\right\} \quad (\forall x, y),\\
&\Longrightarrow \bigvee_{x\in X}\left(A_i(x)\cdot A_j(x)\right) \leq \bigwedge_{y\in Y}\min\left\{\frac{h(B_i(y))}{h(B_j(y))}, \frac{h(B_j(y))}{h(B_i(y))}\right\}\\
&\Longrightarrow \bigvee_{x\in X}\left(A_i(x)\cdot A_j(x)\right) \leq \bigwedge_{y\in Y}\left(h\left(B_i(y)\right) \longleftrightarrow_{\mathbf{G}} h\left(B_j(y)\right)\right).
\end{aligned}$$

($\Longleftarrow$): Now, let us assume that (9.8) is true. Firstly, it is observed that the following is always true:

$$\left(A_i \lhd_h \hat{R}_h\right)(y) \leq B_i(y)\ , \qquad (\forall i, \forall y). \tag{9.9}$$

The validity of the above can be seen from the following:

$$\begin{aligned}\left(A_i \lhd_h \hat{R}_h\right)(y) &= \bigwedge_{x\in X}\left(A_i(x) \longrightarrow_h \bigwedge_j \left(A_j(x) \longrightarrow_h B_j(y)\right)\right)\\ &\le \left(A_i(x_0) \longrightarrow_h \bigwedge_j \left(A_j(x_0) \longrightarrow_h B_j(y)\right)\right)\\ &\qquad \text{(Assuming } A_i \text{ attains normality at } x_0\text{)}\\ &= \bigwedge_j \left(A_j(x_0) \longrightarrow_h B_j(y)\right) \qquad \text{(Using (NP))}\\ &\le A_i(x_0) \longrightarrow_h B_i(y) = B_i(y) \qquad \text{(Using (NP))}.\end{aligned}$$

Thus, it only remains to show that

$$\left(A_i \lhd_h \hat{R}_h\right)(y) \ge B_i(y), \qquad (\forall i, \forall y). \tag{9.10}$$

From (9.8), we have

$$\begin{aligned}&A_i(x)\cdot A_j(x) \le \min\left\{\frac{h(B_i(y))}{h(B_j(y))}, \frac{h(B_j(y))}{h(B_i(y))}\right\}, && (\forall i, j)(\forall x, y),\\ \Longrightarrow\, &A_i(x)\cdot A_j(x) \le \frac{h(B_i(y))}{h(B_j(y))}, && (\forall i, j, \forall x, y),\\ \Longrightarrow\, &A_i(x)\cdot A_j(x)\cdot h\left(B_j(y)\right) \le h\left(B_i(y)\right), && (\forall i, j, \forall x, y),\\ \Longrightarrow\, &\max\left\{A_i(x)\cdot A_j(x)\cdot h\left(B_j(y)\right), h(1)\right\} \le h\left(B_i(y)\right), && (\forall i, j, \forall x, y),\\ \Longrightarrow\, &h^{-1}\left(\max\left\{A_i(x)\cdot A_j(x)\cdot h\left(B_j(y)\right), h(1)\right\}\right) \ge B_i(y), && (\forall i, j, \forall x, y),\\ \Longrightarrow\, &\left(A_i(x)\cdot A_j(x)\right) \longrightarrow_h B_j(y) \ge B_i(y), && (\forall i, j, \forall x, y),\\ \Longrightarrow\, &A_i(x) \longrightarrow_h \left(A_j(x) \longrightarrow_h B_j(y)\right) \ge B_i(y), && (\forall i, j, \forall x, y), \text{(Using } (LI)),\\ \Longrightarrow\, &\bigwedge_j \left(A_i(x) \longrightarrow_h \left(A_j(x) \longrightarrow_h B_j(y)\right)\right) \ge B_i(y), && (\forall i, \forall x, y),\\ \Longrightarrow\, &\left(A_i(x) \longrightarrow_h \bigwedge_j \left(A_j(x) \longrightarrow_h B_j(y)\right)\right) \ge B_i(y), && (\forall i, \forall x, y), \text{(Using (2))},\\ \Longrightarrow\, &\bigwedge_{x\in X}\left(A_i(x) \longrightarrow_h \bigwedge_j \left(A_j(x) \longrightarrow_h B_j(y)\right)\right) \ge B_i(y), && (\forall i, \forall y),\\ \Longrightarrow\, &\left(A_i \lhd_h \hat{R}_h\right)(y) \ge B_i(y), && (\forall i, \forall y).\end{aligned}$$

Now, from (9.9) and (9.10), it follows that

$$\left(A_i \lhd_h \hat{R}_h\right)(y) = B_i(y).$$

This completes the proof. □

9.4.2 Continuity of BKS with h-Implications

Štěpnička and Jayaram [19] have discussed continuity of a BKS inference mechanism under a residual structure. Definition of continuity has been suitably proposed in [1] when the considered fuzzy implications are non-residual fuzzy implications, specifically, the Yager's f- and g-implications. Since we are again dealing with a non-residual fuzzy implication, we define continuity accordingly and discuss a necessary and sufficient condition for continuity.

Definition 9.8 A fuzzy relation $R \in \mathcal{F}(X \times Y)$ is a continuous model of fuzzy rule base (9.6) in a BKS inference (9.7) employing h-implications if

$$\bigwedge_{y \in Y} \left[h\left(B_i(y)\right) \longleftrightarrow_{\mathbf{G}} h\left(\left(A \lhd_h R\right)(y)\right) \right] \geq \bigwedge_{x \in X} \left[A_i(x) \longleftrightarrow_{\mathbf{G}} A(x) \right] \tag{9.11}$$

for each $i \in I$ and for each $A \in \mathcal{F}(X)$.

Theorem 9.2 *The fuzzy relation $\hat{R}_h \in \mathcal{F}(X \times Y)$ representing the rule base* (9.6) *is a solution of* (9.7)*, i.e. $B_i = A_i \lhd_h R$ if and only if it is a continuous model of the rule base.*

Proof Let $\hat{R}_h$ be a continuous model of (9.6). By **Definition** 9.8, the inequality (9.11) is valid for all $i = 1, 2, \ldots, n$ and arbitrary $A \in \mathcal{F}(X)$. Putting $A = A_i$ in (9.11), we have

$$\begin{aligned}
&\bigwedge_{y \in Y} \left[h\left(B_i(y)\right) \longleftrightarrow_{\mathbf{G}} h\left(\left(A_i \lhd_h \hat{R}_h\right)(y)\right) \right] \geq 1, && (\forall i),\\
&\Longrightarrow h\left(B_i(y)\right) \longleftrightarrow_{\mathbf{G}} h\left(\left(A_i \lhd_h \hat{R}_h\right)(y)\right) = 1, && (\forall i, y),\\
&\Longrightarrow h\left(B_i(y)\right) = h\left(\left(A_i \lhd_h \hat{R}_h\right)(y)\right), && (\forall i, y),\\
&\Longrightarrow \left(A_i \lhd_h \hat{R}_h\right)(y) = B_i(y), && (\forall i, y).
\end{aligned}$$

Hence, the fuzzy relation $\hat{R}_h$ representing the rule base (9.6) is a solution of $B_i = A_i \lhd_h R$.
Now, let us assume that $\hat{R}_h$ is a solution of (9.7), i. e.

$$\left(A_i \lhd_h \hat{R}_h\right)(y) = B_i(y), \ \forall\, i,\ y.$$

Towards proving continuity of $\hat{R}_h$, we show that $\hat{R}_h$ satisfies (9.11). For arbitrary $y \in Y$, and $i = 1, 2, \ldots, n$, we observe that the following holds true:

$$
\begin{aligned}
& h\left(\left(A \lhd_h \hat{R}_h\right)(y)\right) \longleftrightarrow_{\mathbf{G}} h\left(B_i(y)\right) \\
&= h\left(\left(A \lhd_h \hat{R}_h\right)(y)\right) \longleftrightarrow_{\mathbf{G}} h\left(\left(A_i \lhd_h \hat{R}_h\right)(y)\right), \\
&\qquad\qquad\qquad\qquad \text{(Since } (A_i \lhd_h \hat{R}_h)(y) = B_i(y)) \\
&= h\left(\bigwedge_{x\in X}\left[A(x) \longrightarrow_h \hat{R}_h(x,y)\right]\right) \longleftrightarrow_{\mathbf{G}} h\left(\bigwedge_{x\in X}\left[A_i(x) \longrightarrow_h \hat{R}_h(x,y)\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
&= h\left\{\bigwedge_{x\in X} h^{-1}\left[\max\left\{A(x)\cdot h\left(\hat{R}_h(x,y), h(1)\right\}\right)\right]\right\} \\
&\qquad \longleftrightarrow_{\mathbf{G}} h\left\{\bigwedge_{x\in X} h^{-1}\left[\max\left\{A_i(x)\cdot h\left(\hat{R}_h(x,y), h(1)\right\}\right)\right]\right\} \\
&= \bigvee_{x\in X}\left[\max\left\{A(x)\cdot h\left(\hat{R}_h(x,y), h(1)\right\}\right)\right] \\
&\qquad \longleftrightarrow_{\mathbf{G}} \bigvee_{x\in X}\left[\max\left\{A_i(x)\cdot h\left(\hat{R}_h(x,y), h(1)\right\}\right)\right] \\
&\geq \bigwedge_{x\in X}\left[\max\left\{A(x)\cdot h\left(\hat{R}_h(x,y)\right), h(1)\right\}\right. \\
&\qquad \left.\longleftrightarrow_{\mathbf{G}} \max\left\{A_i(x)\cdot h\left(\hat{R}_h(x,y)\right), h(1)\right\}\right] \quad \text{(Using (5))} \\
&\geq \bigwedge_{x\in X}\left[A(x)\cdot h\left(\hat{R}_h(x,y)\right) \longleftrightarrow_{\mathbf{G}} A_i(x)\cdot h\left(\hat{R}_h(x,y)\right)\right] \wedge [h(1) \longleftrightarrow_{\mathbf{G}} h(1)] \\
&\qquad\qquad\qquad\qquad \text{(Using (5))} \\
&\geq \bigwedge_{x\in X}\left([A(x) \longleftrightarrow_{\mathbf{G}} A_i(x)]\cdot\left[h(\hat{R}_h(x,y)) \longleftrightarrow_{\mathbf{G}} h(\hat{R}_h(x,y))\right]\right) \quad \text{(Using (4))} \\
&= \bigwedge_{x\in X}[A(x) \longleftrightarrow_{\mathbf{G}} A_i(x)]\ .
\end{aligned}
$$

Hence we conclude that $\hat{R}_h$ is a continuous model of (9.6). □

9.4.3 Robustness of BKS with h-Implications

t-equivalence relation plays the similar role in fuzzy set theory as equivalence relation plays in classical set theory. In a fuzzy system, indistinguishability has been studied using t-equivalence relations.

Definition 9.9 ([18]) A t-equivalence relation $(E, \star)$ with $E \in \mathcal{F}(X \times X)$ and $\star$ being a t-norm, is a fuzzy relation that satisfies the following conditions:

(i) $E(x, x) = 1$,
(ii) $E(x, y) = E(y, x)$,
(iii) $E(x, y) \star E(y, z) \leq E(x, z)$.

Definition 9.10 ([18]) A fuzzy set $A \in \mathcal{F}(X)$ is extensional with respect to $(E, \star)$ on X if,

$$A(x) \star E(x, y) \leq A(y), \qquad x, y \in X .$$

Definition 9.11 ([18]) The extensional hull of $A \in \mathcal{F}(X)$ for a t-equivalence relation $(E, \star)$ is a fuzzy set

$$\hat{A}(x) = \bigvee \{C : A \leq C \text{ and } C \text{ is extensional w.r.t. } E\}$$

where $\leq$ is inclusion ordering on $\mathcal{F}(X)$.

Proposition 9.6 ([18]) *For $A \in \mathcal{F}(X)$ and $(E, \star)$ being a t-equivalence relation on X,*

$$\hat{A}(x) = \bigvee \{A(y) \star E(x, y) \,|\, y \in X\} .$$

The robustness of BKS inference mechanism with residual implications can be found in [19]. Furthermore, Mandal and Jayaram [1] have investigated robustness of BKS inference mechanism with non-residual Yager's classes of f- and g-implications. Here, we discuss the robustness of BKS employing h-implications.

Theorem 9.3 *Let '$\cdot$' be the product t-norm and $(E, \cdot)$ be a t-equivalence relation on X. For the rule base (9.6) in which each A_i is extensional w.r.t. $(E, \cdot)$,*

$$A' \lhd_h \hat{R}_h = \hat{A}' \lhd_h \hat{R}_h \quad \textit{for any} \;\; A' \in \mathcal{F}(X).$$

Proof From definition of $\hat{A}'$, for a fuzzy relation $\hat{R}_h$, we have

$$\begin{aligned} \hat{A}' \geq A' &\Longrightarrow \hat{A}' \longrightarrow_h \hat{R}_h \leq A' \longrightarrow_h \hat{R}_h \\ &\Longrightarrow \hat{A}' \lhd_h \hat{R}_h \leq A' \lhd_h \hat{R}_h \ . \end{aligned}$$

Using the fuzzy relation $\hat{R}_h$ given by (Imp-$\hat{R}_h$), we have

$$\hat{A}' \lhd_h \hat{R}_h = \bigwedge_{x \in X} \left[\hat{A}'(x) \longrightarrow_h \bigwedge_{i=1}^{n} \big(A_i(x) \longrightarrow_h B_i(y)\big) \right], \quad y \in Y \ .$$

For any $x, x' \in X$ and for any $i = 1, 2, \ldots n$, by extensionality A_i with respect to $(E, \cdot)$, we have

$$A_i(x') \geq A_i(x) \cdot E(x, x') \\ \Longrightarrow A_i(x') \longrightarrow_h B_i(y) \leq \left[A_i(x) \cdot E(x, x')\right] \longrightarrow_h B_i(y)\,, \quad y \in Y\,. \tag{9.12}$$

Now we have the following: for any $x \in X$,

$$\hat{A}'(x) \longrightarrow_h \bigwedge_{i=1}^{n} (A_i(x) \longrightarrow_h B_i(y))$$

$$= \left(\bigvee_{x' \in X} \left[A'(x') \cdot E(x, x')\right]\right) \longrightarrow_h \bigwedge_{i=1}^{n} (A_i(x) \longrightarrow_h B_i(y))\,,$$

(Using Proposition 6)

$$= \bigwedge_{x' \in X} \left(\left[A'(x') \cdot E(x, x')\right] \longrightarrow_h \bigwedge_{i=1}^{n} (A_i(x) \longrightarrow_h B_i(y))\right),$$

(Using Proposition 3)

$$= \bigwedge_{i=1}^{n} \bigwedge_{x' \in X} \left(\left[A'(x') \cdot E(x, x')\right] \longrightarrow_h (A_i(x) \longrightarrow_h B_i(y))\right), \qquad \text{(Using (2))}$$

$$= \bigwedge_{i=1}^{n} \bigwedge_{x' \in X} \left(A'(x') \longrightarrow_h \left[E(x, x') \longrightarrow_h (A_i(x) \longrightarrow_h B_i(y))\right]\right), \qquad \text{(By (LI))}$$

$$= \bigwedge_{i=1}^{n} \bigwedge_{x' \in X} \left(A'(x') \longrightarrow_h \left[\left(E(x, x') \cdot A_i(x)\right) \longrightarrow_h B_i(y)\right]\right), \qquad \text{(By (LI))}$$

$$\geq \bigwedge_{i=1}^{n} \bigwedge_{x' \in X} \left(A'(x') \longrightarrow_h \left[A_i(x') \longrightarrow_h B_i(y)\right]\right), \qquad \text{(Using (12))}$$

$$= (A' \lhd_h \hat{R}_h)(y)\,.$$

Thus, $\hat{A}' \lhd_h \hat{R}_h \geq A' \lhd_h \hat{R}_h$ and the result follows.

9.5 An Illustrative Example

Let the input and output space be $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$, respectively. Consider the following rule base:

$$\textbf{IF } \tilde{x} \textit{ is } A_i \textbf{ THEN } \tilde{y} \textit{ is } B_i, \; i = 1, 2, 3\,, \tag{9.13}$$

where $A_i, B_i, i = 1, 2, 3$ are represented by fuzzy sets in their corresponding domains, i.e. $A_i \in \mathcal{F}(X)$, $B_i \in \mathcal{F}(Y)$. Let us consider the fuzzy sets as follows:

$$A_1 = [0.20\ 1.00\ 0.90],\ \ A_2 = [0.20\ 0.90\ 1.00],\ \ A_3 = [0.20\ 1.00\ 0.80]$$

and

$$B_1 = [0.40\ 0.90\ 1.00],\ \ B_2 = [0.40\ 1.00\ 0.90],\ \ B_3 = [0.30\ 0.80\ 1.00],$$

where a triplet $[a\ b\ c]$ represents a fuzzy set on the spaces X or Y and each component of the triplets correspond to the membership value at each point of the corresponding spaces. The above rule base can be represented in the form of a fuzzy relation as in (Imp-$\hat{R}_h$) as follows:

$$\hat{R}(x, y) = \bigwedge_{i=1}^{3} (A_i(x) \longrightarrow_h B_i(y)), \quad x \in X, y \in Y\ . \tag{9.14}$$

We consider the fuzzy inference as $\lhd_h = \inf -I_h$ which is Bandler–Kohout subproduct fuzzy inference with $h-$implication. Next, we check for the availability of the desirable properties, viz., interpolativity, continuity, and robustness for this inference with particular choice of I_h as follows:

$$I_h(x, y) = x \longrightarrow_h y = 1 - x + xy,\ \ x, y \in [0, 1]\ . \tag{9.15}$$

If we consider $\longrightarrow_h$ as in (9.15), using (9.14), we have

$$\hat{R} = \begin{bmatrix} 0.86\ 0.96\ 0.98 \\ 0.30\ 0.80\ 0.91 \\ 0.40\ 0.84\ 0.90 \end{bmatrix}\ .$$

Check for Interpolativity:
We can easily check that

$$A_2 \lhd_h \hat{R} = \begin{bmatrix} 0.20\ 0.90\ 0.91 \end{bmatrix} \lhd_h \begin{bmatrix} 0.86\ 0.96\ 0.98 \\ 0.30\ 0.80\ 0.91 \\ 0.40\ 0.84\ 0.90 \end{bmatrix}$$
$$= \begin{bmatrix} 0.37\ 0.82\ 0.90 \end{bmatrix} \neq \begin{bmatrix} 0.40\ 1.00\ 0.90 \end{bmatrix} = B_2\ .$$

Hence, the fuzzy inference $\lhd_h$ is not interpolative. We can also check for interpolativity directly by checking the necessary and sufficient condition as proposed in (9.8). For instance, considering $i = 1$, $j = 2$, we have the following:

$$\text{L.H.S. of (8)} = \bigvee_{x \in X} (A_1(x) \cdot A_2(x)) = \max\{0.04, 0.90, 0.90\} = 0.90\ ,$$

$$
\begin{aligned}
\text{R.H.S. of (8)} &= \bigwedge_{y \in Y} \left(h\left(B_1(y)\right) \overset{*}{\longleftrightarrow}_{\mathbf{G}} h\left(B_2(y)\right)\right) \\
&= \bigwedge_{y \in Y} \min\left\{1, \frac{h(B_1(y))}{h(B_2(y))}, \frac{h(B_2(y))}{h(B_1(y))}\right\} \\
&= \min\left\{1, \frac{h(B_1(y_1))}{h(B_2(y_1))}, \frac{h(B_2(y_1))}{h(B_1(y_1))}\right\} \\
&\quad \bigwedge \min\left\{1, \frac{h(B_1(y_2))}{h(B_2(y_2))}, \frac{h(B_2(y_2))}{h(B_1(y_2))}\right\} \\
&\quad \bigwedge \min\left\{1, \frac{h(B_1(y_3))}{h(B_2(y_3))}, \frac{h(B_2(y_3))}{h(B_1(y_3))}\right\} \\
&= \min\left\{1, \frac{0.60}{0.60}, \frac{0.60}{0.60}\right\} \bigwedge \min\left\{1, \frac{0.10}{0}, \frac{0}{0.10}\right\} \\
&\quad \bigwedge \min\left\{1, \frac{0}{0.10}, \frac{0.10}{0}\right\} = 0
\end{aligned}
$$

$$
\text{So,} \quad \bigvee_{x \in X} (A_1(x) \cdot A_2(x)) \nleq \bigwedge_{y \in Y} \left(h\left(B_1(y)\right) \longleftrightarrow_{\mathbf{G}} h\left(B_2(y)\right)\right) .
$$

Hence, we conclude using Theorem 9.1 that the corresponding system is not interpolative.

Check for Continuity:

Since the system is not interpolative, it is not a correct model too. Again, since the system is not correct, using Theorem 9.2, we can conclude that the chosen fuzzy relation $\hat{R}$ is not a continuous model of fuzzy rules (9.13). Without checking for interpolativity, we can identify whether the chosen fuzzy relation $\hat{R}$ is a continuous model of fuzzy rules or not only by checking whether condition (9.11) holds or not.

Let $A = [0.30\ 0.50\ 1.00]$ be any arbitrary input. Then, the corresponding output B can be found as follows:

$$
B = A \lhd_h \hat{R} = \begin{bmatrix}0.30 & 0.50 & 1.00\end{bmatrix} \lhd_h \begin{bmatrix}0.86 & 0.96 & 0.98 \\ 0.30 & 0.80 & 0.91 \\ 0.40 & 0.84 & 0.90\end{bmatrix} = \begin{bmatrix}0.40 & 0.84 & 0.90\end{bmatrix} .
$$

Now, for the input $A = [0.30\ 0.50\ 1.00]$ and $i = 1$, we have the following:

$$
\begin{aligned}
\text{L.H.S. of (11)} &= \bigwedge_{y \in Y} \left[h\left(B_1(y)\right) \longleftrightarrow_{\mathbf{G}} h\left((A \lhd_h R)(y)\right)\right] \\
&= \min\left\{1, \frac{0.10}{0.16}, 0\right\} = 0 ,
\end{aligned}
$$

$$\text{R.H.S. of (11)} = \bigwedge_{x\in X} [A_1(x) \longleftrightarrow_{\mathbf{G}} A(x)]$$
$$= \min\left\{\frac{0.2}{0.3}, \frac{0.5}{1}, \frac{0.9}{1}\right\} = 0.5\,.$$

So,

$$\bigwedge_{y\in Y} \left[h\left(B_1(y)\right) \longleftrightarrow_{\mathbf{G}} h\left(\left(A \lhd_h R\right)(y)\right)\right] \not\geq \bigwedge_{x\in X} [A_1(x) \longleftrightarrow_{\mathbf{G}} A(x)]\,.$$

Hence, using Definition 9.8, we can conclude that the chosen fuzzy relation R given in (9.14) is not a continuous model of the fuzzy rules (9.13).

Check for Robustness:

For the input $A = [0.30\ 0.50\ 1]$, the corresponding output is as follows:

$$B = A \lhd_h \hat{R} = \begin{bmatrix}0.30\ 0.50\ 1\end{bmatrix} \lhd_h \begin{bmatrix}0.86\ 0.96\ 0.98\\ 0.30\ 0.80\ 0.91\\ 0.40\ 0.84\ 0.90\end{bmatrix} = \begin{bmatrix}0.40\ 0.84\ 0.90\end{bmatrix}\,.$$

Let us consider a t-equivalence relation $(E, \cdot)$ on X, which is as follows:

$$E = \begin{bmatrix}1.00\ 0.12\ 0.41\\ 0.12\ 1.00\ 0.12\\ 0.41\ 0.12\ 1.00\end{bmatrix}\,.$$

Then, the corresponding extensional hull of A can be found using Proposition 9.6 as follows:

$$\hat{A}(x) = \bigvee\{A(y) \cdot E(x, y) \mid y \in X\}$$
$$= \bigvee \begin{bmatrix}0.30\ 0.50\ 1\end{bmatrix} \cdot \begin{bmatrix}1.00\ 0.12\ 0.41\\ 0.12\ 1.00\ 0.12\\ 0.41\ 0.12\ 1.00\end{bmatrix} = \begin{bmatrix}0.40\ 0.50\ 1.00\end{bmatrix}\,.$$

Note that $A \neq \hat{A}$. Now for $\hat{A}$ as an input, the corresponding output is given by

$$B' = \hat{A} \lhd_h \hat{R} = \begin{bmatrix}0.40\ 0.50\ 1\end{bmatrix} \lhd_h \begin{bmatrix}0.86\ 0.96\ 0.98\\ 0.30\ 0.80\ 0.91\\ 0.40\ 0.84\ 0.90\end{bmatrix}$$
$$= \begin{bmatrix}0.40\ 0.84\ 0.90\end{bmatrix} = B\,.$$

Hence, using Theorem 9.3, we conclude that the system is robust.

9.6 Conclusions

With this study, we have demonstrated that the properties, viz. interpolativity, continuity, and robustness are retained for the BKS inference when we employ h-implications generated from multiplicative generator h of a continuous Archimedean t-conorm. Note that this h-implication class is a subset of (S, N)-implications, and hence this work should be seen as a first attempt at exploring the employability of the family of (S, N)-implications in Fuzzy Relational Inference Schemes. Thus, our work will give the practitioners more flexibility in choosing the fuzzy logic operators while employing the BKS fuzzy inference mechanism.

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Chapter 10
Note on Distributivity of Different String Operations Over Language Sets

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10.1 Introduction

Duplication operation has been studied extensively in the area of DNA computing and Formal Language Theory [1–7, 11, 12, 14, 15] in the last few years. Such operations are inspired by DNA replication. Duplication is one of the basic phenomena that occur in the molecular evolution of a biological sequence. During DNA replication, the subsequence of a strand of DNA is copied several times which leads to the iterated duplication operation. Given a string $w = w_1w_2w_3$ and a substring w_2, a duplication of w is $w_1w_2w_2w_3$. The research on duplication is motivated by errors that occur during DNA replication. Tandem repeats occur in DNA when a pattern of one or more nucleotides is repeated and the repetitions are directly adjacent to each other. Several protein domains also form tandem repeats within their amino acid primary structure. The occurrence of tandem repeats can occur through different mechanisms. For example, slipped strand mispairing, also known as replication slippage, is a mutation that occurs during DNA replication [13].

Tandem repeats describe a pattern that helps to determine an individual's inherited traits. Tandem repeats can be very useful in determining parentage. In the field of Computer Science, tandem repeats in strings (e.g., DNA sequences) can be efficiently detected using suffix trees or suffix arrays.

DNA strand can be presented as a word over the alphabet of four complementary pairs of nucleotides (A, T), (T, A), (G, C), (C, G). Thus, DNA may be viewed as a

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language for specifying the structures and processes of life. As an operation on words and languages, duplication operation and its abstract part like reverse-duplication, pseudo-duplication, bounded duplication, uniformly bounded duplication, prefix-suffix duplication, bounded prefix-suffix duplication, etc. have been well studied in the literature of Formal Language Theory [1–7, 11, 12, 14, 15].

This paper is a study from a theoretical aspect only. In this paper, we see the distributivity of duplication, reverse-duplication and pseudo-duplication operations over language operations. We have investigated the effect of duplication operations on language operations: union, intersection, concatenation, bi-concatenation, Kleene star, complement, homomorphism, reversal, shuffle, insertion, deletion and quotient.

This paper is divided into four sections. In Sect. 10.2, we give some basic notions, notations and the definitions of duplication, reverse-duplication and pseudo-duplication. Section 10.3 describes the effect of duplication operators on language operations. We have given conclusions in Sect. 10.4.

10.2 Preliminaries

In this section, we recall some basic notions, notations and definitions. An alphabet set is a finite set Σ and its elements are called letters or symbols. A word $w = a_1a_2\cdots a_n$ is a finite sequence of symbols, where $a_i \in \Sigma$. By Σ^* we denote the set of all words over the alphabet Σ and by λ- the empty word. A language L is a subset of Σ^* and $L^c = \Sigma^* \setminus L$ denotes the complement of L. The symbol $\emptyset$ represents the empty language. For a given word $w = a_1a_2\cdots a_n$, the reversal of w is denoted by w^R which is equal to $a_n\cdots a_2a_1$. For an arbitrary word $w \in \Sigma^*$, we denote its length, or the number of letters in it, by $|w|$. Note that $\Sigma^+ = \Sigma^* \setminus \{\lambda\}$ and $|\lambda| = 0$. Two sets A and B are incomparable if and only if neither $A \subseteq B$ nor $B \subseteq A$. For the basic definition of language operations (like union, intersection, etc.) the reader is referred to [10, 17].

We recall the following definitions of duplication operation and reverse-duplication from [5]:

Definition 10.1 Let $w \in \Sigma^*$ be a string. The duplication of $w \neq \lambda$ is defined as follows: $D(w) = \{x_1x_2x_2x_3 : w = x_1x_2x_3 \text{ and } x_1, x_3 \in \Sigma^*, x_2 \in \Sigma^+\}$ and $D(\lambda) = \{\lambda\}$. The definition is extended canonically to a language L as follows: $D(L) = \bigcup_{w\in L} D(w)$.

Definition 10.2 Let $w \in \Sigma^*$ be a string. The reverse-duplication of $w \neq \lambda$ is defined as follows: $RD(w) = \{x_1x_2x_2^Rx_3 : w = x_1x_2x_3 \text{ and } x_1, x_3 \in \Sigma^*, x_2 \in \Sigma^+\}$ and $RD(\lambda) = \{\lambda\}$. The definition is extended canonically to a language L as follows: $RD(L) = \bigcup_{w\in L} RD(w)$.

Note that in literature, the substring x_2 of w that is duplicated can either be empty or non-empty. However, throughout this paper, we only consider non-empty substrings to be duplicated.

The edit-distance between two strings x and y (defined in [16]) is the smallest number of basic edit operations that transforms x to y. The basic edit operations are insertion, deletion and substitution. Given an alphabet set Σ,

1. Insertion operation that inserts $a \in \Sigma$ is denoted as $(\lambda \to a)$,
2. Deletion operation that deletes $a \in \Sigma$ is denoted as $(a \to \lambda)$,
3. Substitution operation that substitutes $b \in \Sigma$ for $a \in \Sigma$ is denoted as $(a \to b)$.

We denote the edit-distance between x and y by $d(x, y)$. Note that, $d(x, x) = 0$ for all $x \in \Sigma^*$.

Now we recall the following definition of pseudo-duplication from [1] with a slight modification:

Definition 10.3 Let $w \in \Sigma^*$ be a string and $k \geq 0$ be a non-negative integer. The $k-$pseudo-duplication of $w \neq \lambda$ is defined as follows:

$$PD_k(w) = \{x_1x_2x_2'x_3 : w = x_1x_2x_3, d(x_2, x_2') \leq k \text{ and } x_1, x_3 \in \Sigma^*,\ x_2 \in \Sigma^+\}$$

and $PD_k(\lambda) = \{\lambda\}$. The definition is extended canonically to a language L as follows: $PD_k(L) = \bigcup\limits_{w \in L} PD_k(w)$.

In [1], the authors are allowing x_2 to be empty, also, in the definition of pseudo-duplication, but we are considering only non-empty x_2, as we have considered in the definition of duplication and reverse-duplication.

10.3 Duplication Operations and Language Operations

In literature, duplication (and its variants) and duplication closure (and its variants) of a word or language are compared with the Chomsky hierarchy. Decidability of membership, inclusion, equivalence and regularity for unbounded duplication also has been given [12]. The complexity of the prefix-suffix duplication closure of a word can be seen in [14]. An algorithm deciding whether a regular language is a finite $k-$prefix-suffix duplication is given in [4]. In [1], the authors define a new type of duplication called Pseudo-duplication. In this paper, closure properties of the families of the Chomsky hierarchy under duplication, reverse-duplication and Pseudo-duplication are discussed.

In this section, we analyse whether the operations duplication, reverse-duplication and pseudo-duplication are distributive over various language operations such as union, intersection, concatenation, bi-catenation, Kleene star, shuffle, insertion, deletion and quotient. If not distributive, we prove the inclusion (under certain conditions). We also check the commutative nature of these operations with complement, reversal and homomorphism.

10.3.1 Distributive Nature of Duplication Operations over Set Operations

By definitions of duplication, reverse-duplication and $k-$pseudo-duplication of a language, we have the following proposition for union operation:

Proposition 10.1 *For any two languages $L_1, L_2 \subseteq \Sigma^*$:*

1. $D(L_1 \cup L_2) = D(L_1) \cup D(L_2)$.
2. $RD(L_1 \cup L_2) = RD(L_1) \cup RD(L_2)$.
3. $PD_k(L_1 \cup L_2) = PD_k(L_1) \cup PD_k(L_2)$.

Though these operations are distributive over the union, we see that they are not, in the case of intersection.

Proposition 10.2 *For any two languages $L_1, L_2 \subseteq \Sigma^*$:*

1. $D(L_1 \cap L_2) \subseteq D(L_1) \cap D(L_2)$.
2. $RD(L_1 \cap L_2) \subseteq RD(L_1) \cap RD(L_2)$.
3. $PD_k(L_1 \cap L_2) \subseteq PD_k(L_1) \cap PD_k(L_2)$.

Proof We only prove the first statement, the remaining can be proved similarly. Take $w \in D(L_1 \cap L_2)$, then $w = xyyz$, where $xyz \in L_1 \cap L_2$. Then $xyz \in L_1$ and $xyz \in L_2$, therefore $xyyz \in D(L_1)$ and $xyyz \in D(L_2)$ and hence $xyyz \in D(L_1) \cap D(L_2)$ that is $w \in D(L_1) \cap D(L_2)$. Hence $D(L_1 \cap L_2) \subseteq D(L_1) \cap D(L_2)$. □

Observation 10.1 *We note that the reverse inclusions $X(L_1) \cap X(L_2) \subseteq X(L_1 \cap L_2)$ are not true always, where $X \in \{D, RD, PD_k\}$.*

Take $L_1 = \{aa, b\}$ and $L_2 = \{aaa, b\}$. Then $L_1 \cap L_2 = \{b\}$, $D(L_1) = \{aaa, aaaa, bb\}$, $D(L_2) = \{aaaa, aaaaa, aaaaaa, bb\}$, $D(L_1 \cap L_2) = \{bb\}$. Clearly, $D(L_1) \cap D(L_2) \nsubseteq D(L_1 \cap L_2)$.

To show $RD(L_1) \cap RD(L_2) \nsubseteq RD(L_1 \cap L_2)$ and $PD_1(L_1) \cap PD_1(L_2) \nsubseteq PD_1(L_1 \cap L_2)$, the same L_1, L_2 can be used.

In the case of complement operation, $X(L^c)$ and $(X(L))^c$ are incomparable, where $X \in \{D, RD\}$, and we also have some inclusion results under certain conditions.

Remark 10.1 Consider the language $L = \{abbb\}$.

a. Then $abb \in L^c$, therefore $abbbb \in D(L^c)$ but $abbbb \notin (D(L))^c$ since $abbbb \in D(L)$. Hence $D(L^c) \nsubseteq (D(L))^c$.
   Also, since $a \in (D(L))^c$ and $a \notin D(L^c)$, therefore $(D(L))^c \nsubseteq D(L^c)$.
b. Similarly, by taking the same language, we can show that $(RD(L))^c$ and $RD(L^c)$ are incomparable.

Proposition 10.3 *Let $L \subseteq \Sigma^*$ be a language, then $(PD_k(L))^c \subseteq PD_k(L^c)$ for $k \geq$* 1.

Proof Since $d(a, \lambda) = 1$, for any $a \in \Sigma$, therefore $L \subseteq PD_k(L)$ for $k \geq 1$. Hence $(PD_k(L))^c \subseteq L^c \subseteq PD_k(L^c)$ for $k \geq 1$. Hence $(PD_k(L))^c \subseteq PD_k(L^c)$. □

Observation 10.2 *Note that in the case of* $k = 0$*,* $PD_0(L)$ *is in fact* $D(L)$ *and by Remark 10.1* $D(L^c)$ *and* $(D(L))^c$ *are incomparable.*

In Remark 10.1, we observe that $X(L^c)$ and $(X(L))^c$ are incomparable for $X \in \{D, RD\}$. But under certain conditions they become comparable as shown in the following proposition.

Proposition 10.4 *For any two languages* $L_1, L_2 \subseteq \Sigma^*$*, the followings are true:*

1. *If* $D(w) \subseteq (D(L))^c$ *for* $w \notin L$*, then* $D(L^c) \subseteq (D(L))^c$*.*
2. *If* $RD(w) \subseteq (RD(L))^c$ *for* $w \notin L$*, then* $RD(L^c) \subseteq (RD(L))^c$*.*
3. *If* $PD_k(w) \subseteq (PD_k(L))^c$ *for* $w \notin L$*, then* $PD_k(L^c) \subseteq (PD_k(L))^c$*.*

Proof Let $w \in D(L^c)$, then $w = xyyz$ where $xyz \in L^c$. Then $xyz \notin L$, therefore $xyyz \in D(L)^c$, i.e., $w \in D(L)^c$. Hence $D(L^c) \subseteq D(L)^c$. □

Remark 10.2 From Propositions 10.3 and 10.4, we have $PD_k(L^c) = (PD_k(L))^c$ for $k \geq 1$, when $PD_k(w) \subseteq (PD_k(L))^c$ for $w \notin L$.

10.3.2 Distributive Nature of Duplication Operations over Concatenation

We now concentrate on the operations concatenation, bi-catenation and Kleene star. We prove some inclusion results under some conditions.

We show that $X(L_1)X(L_2)$ and $X(L_1L_2)$ are incomparable, where $X \in \{D, RD, PD_k\}$.

Remark 10.3 Take $L_1 = \{a\}$ and $L_2 = \{b\}$. Then $L_1L_2 = \{ab\}$. Clearly, $D(L_1) = \{aa\}$ and $D(L_2) = \{bb\}$ and $D(L_1L_2) = \{aab, abb, abab\}$. Here, neither $D(L_1L_2) \subseteq D(L_1)D(L_2)$ nor $D(L_1)D(L_2) \subseteq D(L_1L_2)$.

To show $X(L_1)X(L_2)$ and $X(L_1L_2)$, where $X \in \{RD, PD_k\}$, are incomparable, the same languages L_1, L_2 can be used.

Proposition 10.5 *For any two languages* $L_1, L_2 \subseteq \Sigma^*$*, the followings are true:*

1. *If* $D(L_1) \subseteq L_1$ *or* $D(L_2) \subseteq L_2$ *, then* $D(L_1)D(L_2) \subseteq D(L_1L_2)$*.*
2. *If* $RD(L_1) \subseteq L_1$ *or* $RD(L_2) \subseteq L_2$ *, then* $RD(L_1)RD(L_2) \subseteq RD(L_1L_2)$*.*
3. *If* $PD_k(L_1) \subseteq L_1$ *or* $PD_k(L_2) \subseteq L_2$ *, then* $PD_k(L_1)PD_k(L_2) \subseteq PD_k(L_1L_2)$*.*

Proof We only prove the first statement, the remaining can be proved similarly. Let $D(L_1) \subseteq L_1$. Take $w \in D(L_1)D(L_2)$, then $w = w_1w_2$, where $w_1 \in D(L_1)$ and $w_2 = x_2y_2y_2z_2 \in D(L_2)$ for some $x_2y_2z_2 \in L_2$. Since $D(L_1) \subseteq L_1$, therefore $w_1 \in L_1$ and hence $w_1x_2y_2z_2 \in L_1L_2$. Therefore $w_1x_2y_2y_2x_2 \in D(L_1L_2)$ that is $w = w_1w_2 \in D(L_1L_2)$. Hence $D(L_1)D(L_2) \subseteq D(L_1L_2)$. □

In the case of unary language, we have the following proposition for concatenation. Also, note that in the case of unary language $RD(L)$ will become $D(L)$ and the following proposition holds for $RD(L)$ also.

Proposition 10.6 *If* $L_1 = \{a^{n_i} \mid i \geq 1\}$ *and* $L_2 = \{a^{m_j} \mid j \geq 1\}$*, then* $D(L_1L_2) = D(L_1)D(L_2) \cup \{a^{n_i+m_j+1} \mid i, j \geq 1\}$.

Proof Let $w \in D(L_1L_2)$. Then $w = w_1w_2w_2w_3$, where $w_1w_2w_3 \in L_1L_2$. Since $w_1w_2w_3$ has only a's so we can write $w_1w_2w_3 = x_1x_2$, for some $x_1 \in L_1$ and $x_2 \in L_2$. If $|w_2| = 1$, then $w \in \{a^{n_i+m_j+1} \mid i, j \geq 1\}$, otherwise we can write $w_2 = w_2'w_2''$ for some $w_2', w_2'' \in \{a\}^+$ such that $|w_2'| \leq |x_1|$ and $|w_2''| \leq |x_2|$ and hence $w \in D(L_1)D(L_2)$. Hence $D(L_1L_2) \subseteq D(L_1)D(L_2) \cup \{a^{n_i+m_j+1} \mid i, j \geq 1\}$. Conversely, let $w \in D(L_1)D(L_2) \cup \{a^{n_i+m_j+1} \mid i, j \geq 1\}$. If $w \in D(L_1)D(L_2)$, then $w = w_1w_2$,where $w_1 \in D(L_1)$ and $w_2 \in D(L_2)$. Then $w_i = w_i'w_i''w_i''w_i''' = w_i'w_i''w_i'''w_i''$, where $w_i'w_i''w_i''' \in L_i$ for $i = 1, 2$. Then $w = w_1'w_1''w_1'''w_2'w_2''w_2'''w_1''w_2''$ and hence $w \in D(L_1L_2)$. If $w \in \{a^{n_i+m_j+1} \mid i, j \geq 1\}$, then clearly $w \in D(L_1L_2)$. Hence $D(L_1)D(L_2) \cup \{a^{n_i+m_j+1} \mid i, j \geq 1\} \subseteq D(L_1L_2)$. □

Observation 10.3 *Note that, in the case of unary language,* $L_1 \sqcup\!\sqcup L_2$ *and* $L_1 \leftarrow L_2$ *are the same as* L_1L_2 *and hence the above proposition holds true for* $L_1 \sqcup\!\sqcup L_2$ *and* $L_1 \leftarrow L_2$ *also.*

We now analyse the distribution of the duplication operations over the bi-catenation operation which is defined as follows:

Definition 10.4 ([17]) The Bi-catenation of $L_1, L_2 \subseteq \Sigma^*$ is defined as

$$L_1 \cdot L_2 = \{uv, vu \mid u \in L_1,\ v \in L_2\}$$

Clearly, $L_1 \cdot L_2 = L_1L_2 \cup L_2L_1$..

Observation 10.4 *Note that, by the Proposition 10.1 for* $X \in \{D, RD, PD_k\}$*,* $X(L_1 \cdot L_2) = X(L_1L_2 \cup L_2L_1) = X(L_1L_2) \cup X(L_2L_1)$ *and hence the Proposition 10.5 can be proved for bi-catenation also.*

Now, we concentrate on the familiar Kleene star operation. The following remark shows that $(X(L))^*$ and $X(L^*)$ are incomparable, where $X \in \{D, RD, PD_k\}$.

Remark 10.4 Consider the language $L = \{ab, ba\}$.

a. The duplication language is $D(L) = \{aab, abb, abab, bba, baa, baba\}$. Since $abb, baa \in D(L)$, therefore $abbbaa \in (D(L))^*$ and since $abbbaa$ has length 6 and L^* contains only words of even length, so $abbbaa$ can only be obtained by duplicating 4 length word of L^* but there is no word of length 4 in L^* whose duplication is $abbbaa$ so $abbbaa \notin D(L^*)$. This shows that $(D(L))^* \nsubseteq D(L^*)$. Now, since $abab \in L^*$, therefore $aabab \in D(L^*)$ and since $|aabab| = 5$, therefore $aabab \notin (D(L))^*$ and hence $D(L^*) \nsubseteq (D(L))^*$.
b. Similarly, by taking the same language, we can show that $(RD(L))^*$ and $RD(L^*)$ are incomparable.

In fact, $(X(L))^*$ and $X(L^*)$, where $X \in \{D, RD, PD_k\}$, become comparable under certain conditions. We have the following results:

Proposition 10.7 *The followings are true:*

1. *If $D(L) \subseteq L$, then $(D(L))^* \subseteq D(L^*)$.*
2. *If $RD(L) \subseteq L$, then $(RD(L))^* \subseteq RD(L^*)$.*
3. *If $PD_k(L) \subseteq L$, then $(PD_k(L))^* \subseteq PD_k(L^*)$.*

Proof We only prove the first statement, the remaining can be proved similarly. Let $D(L) \subseteq L$. Take $w \in (D(L))^*$, then $w = w_1 w_2 \dots w_n$, where $w_i \in D(L)$. Since $w_1 \in D(L)$, therefore $w_1 = x_1 y_1 y_1 z_1$, where $x_1 y_1 z_1 \in L$ and $y_1 \in \Sigma^+$. Also since $D(L) \subseteq L$, therefore $w_i \in L$ and hence $x_1 y_1 z_1 w_2 \dots w_n \in L^*$. Thus $x_1 y_1 y_1 z_1 w_2 \dots w_n \in D(L^*)$ that is $w \in D(L^*)$ and hence $(D(L))^* \subseteq D(L^*)$. □

In the case of unary language, we have the following proposition for Kleene star.

Proposition 10.8 *If $L \subseteq \{a\}^*$, then $(D(L))^* \subseteq D(L^*)$.*

Proof Let $w \in (D(L))^*$. Then $w = w_1 w_2 \cdots w_n$, where $w_i \in D(L)$ for $1 \leq i \leq n$. Since $w_i \in D(L)$, therefore $w_i = w_i' w_i'' w_i'' w_i'''$ where $w_i' w_i'' w_i''' \in L$. Hence $w = w_1' w_1'' w_1'' w_1''' \cdots w_n' w_n'' w_n'' w_n''' = w_1' w_1'' w_1''' \cdots w_n' w_n'' w_n''' w_1'' \cdots w_n''$. Thus $w \in D(L^*)$ and hence $(D(L))^* \subseteq D(L^*)$. □

But $D(L^*) \subseteq (D(L))^*$ is not true even in the case of unary language. For example, consider the language $L = \{a^3, a^7\}$.

10.3.3 Distributive Nature of Duplication Operations over Other Language Operations

In this section, we discuss operations homomorphism, reversal, shuffle, insertion, deletion and quotient (left and right). We prove some equalities and inclusions and give some examples for incomparability.

The following proposition shows the inclusion $\Phi(D(L)) \subseteq D(\Phi(L))$.

Proposition 10.9 *Let Σ_1, Σ_2 be two alphabet sets and $\Phi : \Sigma_1 \to \Sigma_2$ be a homomorphism and let $L \subseteq \Sigma_1^*$ be a language. Then $\Phi(D(L)) \subseteq D(\Phi(L))$.*

Proof Let $w \in \Phi(D(L))$, then $w = \Phi(xyyz)$, where $xyz \in L$. Then $\Phi(xyz) \in \Phi(L)$ that is $\Phi(x)\Phi(y)\Phi(z) \in \Phi(L)$, therefore $\Phi(x)\Phi(y)\Phi(y)\Phi(z) = \Phi(xyyz) \in D(\Phi(L))$. Hence $\Phi(D(L)) \subseteq D(\Phi(L))$. □

The following remark shows $D(\Phi(L)) \nsubseteq \Phi(D(L))$ as well as $X(\Phi(L))$ and $\Phi(X(L))$ are incomparable, where Φ is a homomorphism and $X \in \{RD, PD_k\}$.

Remark 10.5 Consider the homomorphism $\Phi : \{a, b\}^* \to \{a, b\}^*$ defined by $\Phi(a) = ab$ and $\Phi(b) = aaa$ and the language $L = \{a\}^*$. Then $\Phi(L) = \Phi(\{a\}^*) = \{ab\}^*$.

a. Clearly $\Phi(D(L)) = \{ab\}^* \setminus \{ab\}$ and $D(\Phi(L)) = D(\{ab\}^*)$. We can see that $aab \in D(\Phi(L))$ but $aab \notin \Phi(D(L))$ because $\Phi(D(L))$ does not contain a word of length 3 and hence $D(\Phi(L)) \nsubseteq \Phi(D(L))$.
b. Clearly $\Phi(RD(L)) = \{ab\}^* \setminus \{ab\}$ and $RD(\Phi(L)) = RD(\{ab\}^*)$. Since $ab \in \Phi(L)$, therefore $aab \in RD(\Phi(L))$ but $aab \notin \Phi(RD(L))$. Hence $RD(\Phi(L)) \nsubseteq \Phi(RD(L))$.
Now $abab \in \Phi(RD(L))$ but $abab \notin RD(\Phi(L))$ because $\Phi(L)$ contains only even length words so only reverse-duplication of words of length two of $\Phi(L)$ can generate $abab$ but $ab \in \Phi(L)$ is the only length two words and clearly $abab$ is not any reverse-duplication of ab. Hence $\Phi(RD(L) \nsubseteq RD(\Phi(L))$.
c. Since $ab \in \Phi(L)$, therefore $abb \in PD_1(\Phi(L)$ but $abb \notin \Phi(PD_1(L))$ because $\Phi(PD_1(L))$ doesn't contain any word of length 3 and hence $PD_1(\Phi(L)) \nsubseteq \Phi(PD_1(L))$.
Now since $a \in L$ and $d(a, b) = 1$, therefore $ab \in PD_1(L)$. Thus $\Phi(ab) = abaaa \in \Phi(PD_1(L)$ but $abaaa \notin PD_1(\Phi(L))$ and hence $\Phi(PD_1(L)) \nsubseteq PD_1(\Phi(L))$.

The following proposition shows the equality between $D(L^R)$ and $(D(L))^R$.

Proposition 10.10 *For any language L, $D(L^R) = (D(L))^R$.*

Proof Let $w \in D(L^R)$, then $w = xyyz$, where $xyz \in L^R$. Then $z^R y^R x^R \in L$ and hence $z^R y^R y^R x^R \in D(L)$, i.e., $xyyz \in D(L)^R$. Hence $D(L^R) \subseteq D(L)^R$.

Now take $w \in D(L)^R$, then $w = (xyyz)^R$, where $xyz \in L$. Then $z^R y^R x^R \in L^R$, therefore $z^R y^R y^R x^R \in D(L^R)$, i.e., $(xyyz)^R \in D(L^R)$. Hence $D(L)^R \subseteq D(L^R)$. □

The following remark shows that $(X(L))^R$ and $X(L^R)$ are incomparable, where $X \in \{RD, PD_k\}$.

Remark 10.6 Consider the language $L = \{ab\}$, then $L^R = \{ba\}$.

a. Since $ab \in L$, therefore $abba \in RD(L)$ and hence $abba \in (RD(L))^R$ but $abba \notin RD(L^R)$. Hence $(RD(L))^R \nsubseteq RD(L^R)$.
Now, Since $ba \in L^R$, therefore $baab \in RD(L^R)$ but $baab \notin (RD(L))^R$. Hence $RD(L^R) \nsubseteq (RD(L))^R$.
b. Since $ab \in L$ and $d(b, a) = 1$, therefore $aba \in PD_1(L)$. Thus $aba \in (PD_1(L))^R$ but $aba \notin PD_1(L^R)$ because all elements of $PD_1(L^R)$ will start from b. Hence $(PD_1(L))^R \nsubseteq PD_1(L^R)$.
Now since $ba \in L^R$, therefore $bab \in PD_1(L^R)$ but $bab \notin (PD_1(L))^R$ because all elements of $(PD_1(L))^R$ will end with a. Hence $PD_1(L^R) \nsubseteq (PD_1(L))^R$.

Definition 10.5 ([17]) The shuffle of $L_1, L_2 \subseteq \Sigma^*$ is defined as

$$L_1 \sqcup\!\sqcup L_2 = \{u_1 v_1 \cdots u_n v_n \mid n \geq 1, u = u_1 \cdots u_n, v = v_1 \cdots v_n,\ u_i \in \Sigma^*,\ v_i \in \Sigma^*,\ 1 \leq i \leq n,\ u \in L_1,\ v \in L_2\}.$$

The following remark shows that $X(L_1 \sqcup\!\sqcup L_2)$ and $X(L_1) \sqcup\!\sqcup X(L_2)$ are incomparable, where $X \in \{D, RD, PD_k\}$.

Remark 10.7 Let $L_1 = \{ab\}$ and $L_2 = \{ba\}$ be two languages. Then $L_1 \sqcup\!\sqcup L_2 = \{abba, baba, abab, baab\}$, $D(L_1) = \{aab, abb, abab\}$ and $D(L_2) = \{bba, baa, baba\}$.

Since $abba \in L_1 \sqcup\!\sqcup L_2$, therefore $aabba \in D(L_1 \sqcup\!\sqcup L_2)$ but $aabba \notin D(L_1) \sqcup\!\sqcup D(L_2)$ because $aabba$ has length 5 but $D(L_1) \sqcup\!\sqcup D(L_2)$ will not contain any word of length 5. Hence $D(L_1 \sqcup\!\sqcup L_2) \nsubseteq D(L_1) \sqcup\!\sqcup D(L_2)$. Now, since $aab \in D(L_1)$ and $bba \in D(L_2)$, therefore $aabbba \in D(L_1) \sqcup\!\sqcup D(L_2)$ but $aabbba \notin D(L_1 \sqcup\!\sqcup L_2)$. Hence $D(L_1) \sqcup\!\sqcup D(L_2) \nsubseteq D(L_1 \sqcup\!\sqcup L_2)$.

Using the same languages, we can show that $RD(L_1 \sqcup\!\sqcup L_2)$ and $RD(L_1) \sqcup\!\sqcup RD(L_2)$ are incomparable.

We recall the following definitions of insertion and deletion from [8] to study the distributivity of duplication operations over insertion and deletion operations.

Definition 10.6 The insertion of $L_2 \subseteq \Sigma^*$ in $L_1 \subseteq \Sigma^*$ is defined as

$$L_1 \leftarrow L_2 = \{u_1 v u_2 \mid u = u_1 u_2,\ u_1, u_2 \in \Sigma^*,\ u \in L_1,\ v \in L_2\}.$$

Definition 10.7 The deletion of $L_2 \subseteq \Sigma^*$ from $L_1 \subseteq \Sigma^*$ is defined as

$$L_1 \rightarrow L_2 = \{u_1 u_2 \mid u = u_1 v u_2,\ u_1, u_2 \in \Sigma^*,\ u \in L_1,\ v \in L_2\}.$$

The following remark shows that $X(L_1 \leftarrow L_2)$ and $X(L_1) \leftarrow X(L_2)$ are incomparable, where $X \in \{D, RD, PD_k\}$.

Remark 10.8 Let $L_1 = \{ab\}$ and $L_2 = \{ba\}$. Then $D(L_1) = \{aab, abb, abab\}$, $D(L_2) = \{bba, baa, baba\}$ and $L_1 \leftarrow L_2 = \{baab, abab, abba\}$.

Since $baab \in L_1 \leftarrow L_2$, therefore $bbaab \in D(L_1 \leftarrow L_2)$ but $bbaab \notin D(L_1) \leftarrow D(L_2)$ because it will not contain any word of length 5. Hence $D(L_1 \leftarrow L_2) \nsubseteq D(L_1) \leftarrow D(L_2)$.

Since $aab \in D(L_1)$ and $bba \in D(L_2)$, therefore $bbaaab \in D(L_1) \leftarrow D(L_2)$ but $bbaaab \notin D(L_1 \leftarrow L_2)$. Hence $D(L_1) \leftarrow D(L_2) \nsubseteq D(L_1 \leftarrow L_2)$.

Using the same languages, we can show that $X(L_1 \leftarrow L_2)$ and $X(L_1) \leftarrow X(L_2)$ are incomparable, $X \in \{RD, PD_k\}$.

The following remark shows that $X(L_1 \rightarrow L_2)$ and $X(L_1) \rightarrow X(L_2)$ are incomparable, where $X \in \{D, RD, PD_k\}$.

Remark 10.9 Let $L_1 = \{ab\}$ and $L_2 = \{a\}$. Then $D(L_1) = \{aab, abb, abab\}$, $D(L_2) = \{aa\}$ and $L_1 \rightarrow L_2 = \{b\}$. Also, $D(L_1 \rightarrow L_2) = \{bb\}$ and $D(L_1) \rightarrow D(L_2) = \{b\}$. Clearly, neither $D(L_1 \rightarrow L_2) \subseteq D(L_1) \rightarrow D(L_2)$ nor $D(L_1) \rightarrow D(L_2) \subseteq D(L_1 \rightarrow L_2)$.

Using the same languages, we can show that $RD(L_1 \rightarrow L_2)$ and $RD(L_1) \rightarrow RD(L_2)$ are incomparable.

Definition 10.8 ([17]) For $L_1, L_2 \subseteq \Sigma^*$, the left quotient and the right quotient of L_2 by L_1 are defined as

$$L_1^{-1}L_2 = \{v \in \Sigma^* \mid \exists\, u \in L_1 \textit{ such that } uv \in L_2\}$$

$$L_2L_1^{-1} = \{v \in \Sigma^* \mid \exists\, u \in L_1 \textit{ such that } vu \in L_2\}.$$

The following remark shows that $X(L_1)^{-1}X(L_2)$ and $X(L_1^{-1}L_2)$ are incomparable, where $X \in \{D, RD\}$ and $PD_1(L_1)^{-1}PD_1(L_2) \nsubseteq PD_1(L_1^{-1}L_2)$.

Remark 10.10 Let $L_1 = \{a\}$ and $L_2 = \{ab\}$. Then $D(L_1) = \{aa\}$, $D(L_2) = \{aab, abb, abab\}$, $L_1^{-1}L_2 = \{b\}$, $D(L_1^{-1}L_2) = \{bb\}$ and $D(L_1)^{-1}D(L_2) = \{b\}$. Clearly neither $D(L_1)^{-1}D(L_2) \subseteq D(L_1^{-1}L_2)$ nor $D(L_1^{-1}L_2) \subseteq D(L_1)^{-1}D(L_2)$.

Using the same languages, we can show that $RD(L_1)^{-1}RD(L_2)$ and $RD(L_1^{-1}L_2)$ are incomparable. Also, we can see, using the same languages, $PD_1(L_1)^{-1}PD_1(L_2) \nsubseteq PD_1(L_1^{-1}L_2)$ as $a \in PD_1(L_1)^{-1}PD_1(L_2)$ but $a \notin PD_1(L_1^{-1}L_2)$.

We show in the following Proposition that $PD_k(L_1^{-1}L_2) \subseteq PD_k(L_1)^{-1}PD_k(L_2)$.

Proposition 10.11 *For* $k \geq 1$, $PD_k(L_1^{-1}L_2) \subseteq PD_k(L_1)^{-1}PD_k(L_2)$.

Proof Let $w \in PD_k(L_1^{-1}L_2)$. Then $w = w_1w_2w_2'w_3$, where $w_1w_2w_3 \in L_1^{-1}L_2$ and $d(w_2w_2') \leq k$. Since $w_1w_2w_3 \in L_1^{-1}L_2$, therefore there exists $u \in L_1$ such that $uw_1w_2w_3 \in L_2$ and hence $uw_1w_2w_2'w_3 \in PD_k(L_2)$ since $d(w_2w_2') \leq k$. Also for $k \geq 1$, $L \subseteq PD_k(L)$, therefore $u \in PD_k(L_1)$. Hence $w_1w_2w_2'w_3 \in PD_k(L_1)^{-1}PD_k(L_2)$, i.e., $w \in PD_k(L_1)^{-1}PD_k(L_2)$. □

10.4 Conclusions

In this paper, we have investigated the relations between $X(L_1 * L_2)$ and $X(L_1) * X(L_2)$ as well as $X(\circ(L))$ and $\circ(X(L))$, where $*$ is a binary language operation, $\circ$ is a unary language operation and $X \in \{D, RD, PD_k\}$. We found that equality holds only for a few cases and one-way inclusion also holds in some cases. We also observed incomparability in many cases. As a future work, it will be interesting to look for some descriptional complexity issues.

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Chapter 11
A Generalization of χ-Binding Functions

M. A. Shalu and T. P. Sandhya

11.1 Introduction

Let $\mathcal{G}$ be an induced hereditary class of graphs. We say $\mathcal{G}$ admits a χ-*binding function* if there exists a function $f : \mathbb{N} \to \mathbb{R}^+$ such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{G}$ [5]. For example, the class of perfect graphs is χ-bounded by the identity function and the class of $2K_2$-free graphs is χ-bounded by the function $f(x) = \binom{x+1}{2}$ [11]. The study of χ-binding functions is useful because it may lead us to a polynomial time approximation algorithm with a constant performance ratio for the NP-complete vertex coloring problem. If the proof of the existence of a linear χ-binding function f of $\mathcal{G}$ provides a polynomial time algorithm to find an $f(\omega(G))$-coloring for every $G \in \mathcal{G}$, then the vertex coloring problem on $\mathcal{G}$ admits a polynomial time approximation algorithm with constant performance ratio (see page no. 7 of [5]).

But not all the classes of graphs admit a χ-binding function. For example, the class of all triangle-free graphs does not admit a χ-binding function [1, 7]. This motivates us to ask the following question.

Question 11.1 Let $\mathcal{G}$ be an induced hereditary class of graphs. Does there exist a function $g : \mathbb{N} \to \mathbb{R}^+$ such that $\chi(G) \leq g(\alpha(G))$ for all $G \in \mathcal{G}$ (we call g as a (χ, α)-*binding function*).

Note that for a triangle-free graph G, $\Delta(G) \leq \alpha(G)$ and $\chi(G) \leq \alpha(G) + 1$ and hence, it admits a linear (χ, α)-binding function $g(x) = x + 1$. Let $\mathcal{C}$ be the class of all complete graphs. If $\mathcal{C}$ admits a (χ, α)-binding function g, then $n = \chi(K_n) \leq$

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$g(\alpha(K_n)) = g(1)$ (a constant), for every $K_n \in C$, a contradiction. Hence, C does not admit a (χ, α)-binding function. Thus, there exist classes of graphs that do not admit a χ-binding function and classes of graphs that do not admit a (χ, α)-binding function. This leads us to the following question.

Question 11.2 Let $\mathcal{G}$ be an induced hereditary class of graphs. Does there exist a function $h : \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ such that $\chi(G) \leq h(\alpha(G), \omega(G))$ for all $G \in \mathcal{G}$ (we call h as a (χ, α, ω)-*binding function*).

It is easy to see that the graph class that admits either a χ-binding function or a (χ, α)-binding function admits a (χ, α, ω)-binding function. In this paper, we initiate the study of (χ, α, ω)-binding functions with a focus on the class of $\{C_4, K_1 + 2K_2\}$-free graphs that admits neither a χ nor a (χ, α)-binding functions.

Erdös proved that for any two integers k and g ($k, g \geq 2$), there exist graphs with girth g and chromatic number k [1]. It implies that for a finite set of graphs $\mathcal{F}$, the class of $\mathcal{F}$-free graphs admits a χ-binding function only if at least one member of $\mathcal{F}$ is a forest (an acyclic graph). So the class of $\{C_4, K_1 + 2K_2\}$-free graphs does not admit a χ-binding function. We also note that the class of all complete graphs C is a subclass of the class of $\{C_4, K_1 + 2K_2\}$-free graphs and since C does not admit a (χ, α)-binding function, the class of $\{C_4, K_1 + 2K_2\}$-free graphs also does not admit a (χ, α)-binding function. We also derive an exponential (χ, α, ω)-binding function for an arbitrary class of graphs.

All graphs considered in this paper are finite, simple, and undirected. For graph terminologies, we refer [12]. A *clique (independent set)* is a subset of vertices of a graph G which are pairwise adjacent (respectively, non-adjacent) in G. The size of a maximum clique (independent set) in G is denoted by $\omega(G)$ (respectively, $\alpha(G)$). A *k-vertex coloring* of a graph G is a function $f : V(G) \to \{1, 2, \ldots, k\}$ such that $f(x) \neq f(y)$ whenever $xy \in E(G)$. The *chromatic number*, $\chi(G)$, of a graph G is the minimum k for which G admits a k-vertex coloring. A *clique cover* of a graph G is a set of cliques in G such that every vertex of G is a member of at least one clique, and $\theta(G)$ denotes the size of a minimum clique cover of G. Let K_n and C_n, respectively, denote the complete graph and the cycle on n vertices. Let H be a graph. We say a graph G is H-free if G contains no induced subgraph isomorphic to H. Let $\mathcal{F} = \{H_1, H_2, \ldots, H_k\}$. A graph G is $\mathcal{F}$-free if G is H_i-free for all i, $1 \leq i \leq k$. For a set $X \subseteq V(G)$, $[X]$ denotes the graph induced by X in G. For a vertex v of a graph G, $N(v) = \{u \in V(G) : uv \in E(G)\}$ and $N[v] = \{v\} \cup N(v)$. Let $\Delta(G) = \max\{|N(v)| : v \in V(G)\}$. The *union* $G_1 \cup G_2 \cup \ldots \cup G_k$ of pairwise vertex disjoint graphs $G_1, G_2, \ldots, G_k$ is a graph with vertex set $V(G_1) \cup V(G_2) \cup \ldots \cup V(G_k)$ and edge set $E(G_1) \cup E(G_2) \cup \ldots \cup E(G_k)$. Let A and B be two disjoint subsets of $V(G)$. We denote $[A, B] = \{\{a, b\} : a \in A, b \in B\}$, where A and B are two non-empty disjoint sets. The *join* $G_1 + G_2$ of two vertex disjoint graphs G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup [V(G_1), V(G_2)]$. For convenience, we denote $A \oplus B$ in G if $ab \in E(G)$ for all $a \in A$ and for all $b \in B$. Let $V_1, V_2, \ldots, V_t$ be disjoint subsets of the vertex

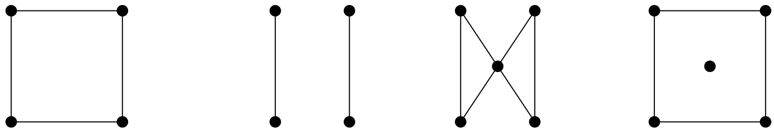


Fig. 11.1 The graphs C_4, $2K_2$, $K_1 + 2K_2$, and $C_4 \cup K_1$

set of a graph G. Then we say $[V_1 \cup V_2 \cup \ldots \cup V_t]$ is a complete multipartite graph if (1) V_i is an independent set in G and (2) $V_i \oplus V_j$ in G for all $1 \le i < j \le k$.

The structure of the paper is as follows. In Sect. 11.2, we study the structure of $\{2K_2, C_4 \cup K_1\}$-free graphs and derive an upper bound of the clique cover number of such a graph in terms of its clique number and independence number. In Sect. 11.3, by generalizing the result in Sect. 11.2, we find an upper bound of the clique cover number of an arbitrary graph in terms of its clique number and independence number (Fig. 11.1).

11.2 $\{2K_2, C_4 \cup K_1\}$-Free Graphs

First, we study the structure of a $\{2K_2, C_4 \cup K_1\}$-free graph.

Lemma 11.1 *Let G be a $\{2K_2, C_4 \cup K_1\}$-free graph. Then $V(G)$ can be partitioned as $V(G) = S \cup A \cup B \cup C$ (see Fig. 11.2) such that*

1. *S is a maximum independent set in G,*
2. *B and C are cliques in G,*
3. *$S \oplus A$ in G, and*
4. *every vertex of $B \cup C$ has at least one non-neighbor in S.*

Proof For a maximum independent set S in G, let $A = \{x \in V(G) \setminus S : \{x\} \oplus S \text{ in } G\}$, $B = \{y \in V(G) \setminus (S \cup A) : |N(y) \cap S| = 1\}$, and $C = V(G) \setminus (A \cup B)$ (see Fig. 11.2. In Fig. 11.2, a double straight line segment between two sets represents $\oplus$ operator between them in G and the zigzag line represents no restriction on edges between respective sets.). Then $S \oplus A$ in G and every vertex of $B \cup C$ has a non-neighbor in S. Hence, conditions 3 and 4 hold.

Claim B is a clique in G.

If not, let $x, y \in B$ such that $xy \notin E(G)$. By definition, let $\{u_1\} = N(x) \cap S$ and $\{u_2\} = N(y) \cap S$. Then there are two cases.

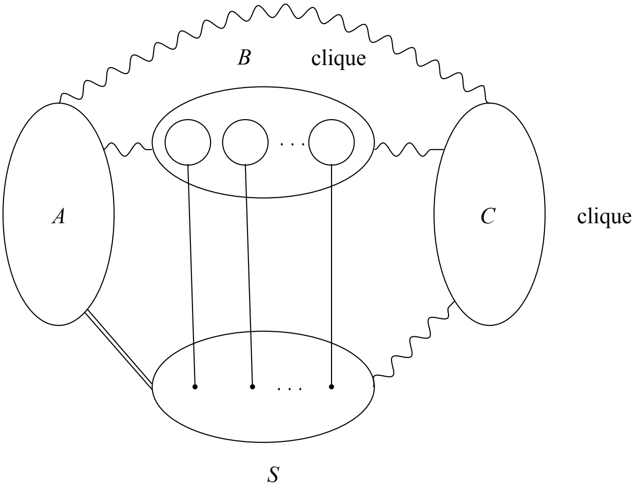
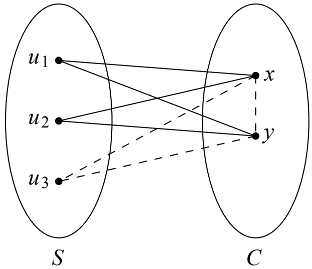


Fig. 11.2 The partition of $V(G)$ in Lemma 11.1

Fig. 11.3 *Case 1* of Claim 11.2



Case 1: $u_1 = u_2$. Then $(S \setminus \{u_1\}) \cup \{x, y\}$ is an independent set of size $|S| + 1$ in G, a contradiction.

Case 2: $u_1 \neq u_2$. Then $[\{x, u_1, y, u_2\}] \cong 2K_2$, a contradiction. Hence, B is a clique in G.

Claim C is a clique in G.

If not, let $x, y \in C$ such that $xy \notin E(G)$. Then there are two cases.
Case 1: $N(x) \cap S \subseteq N(y) \cap S$ or $N(y) \cap S \subseteq N(x) \cap S$. W. l. o. g., assume that $N(x) \cap S \subseteq N(y) \cap S$. Since $x \in C$ and $x \notin B$, x has at least two neighbors in S, say u_1 and u_2. Since $N(x) \cap S \subseteq N(y) \cap S$, $yu_1, yu_2 \in E(G)$. By Condition (4), y has at least one non-neighbor in S, say u_3. Since $N(x) \cap S \subseteq N(y) \cap S$, $xu_3 \notin E(G)$. So $[\{u_1, u_2, x, y, u_3\}] \cong C_4 \cup K_1$ in G (see Fig. 11.3), a contradiction.
Case 2: $N(x) \cap S \nsubseteq N(y) \cap S$ and $N(y) \cap S \nsubseteq N(x) \cap S$. Then there exist $s_1, s_2 \in S$ such that $xs_1, ys_2 \in E(G)$ and $xs_2, ys_1 \notin E(G)$. Then $[\{x, s_1, y, s_2\}] \cong 2K_2$ in G, a contradiction. Hence, C is a clique in G. □

Next, we employ Lemma 11.1 to find a clique cover of a $\{2K_2, C_4 \cup K_1\}$-free graph G using two phases.

Phase I: If $\alpha(G) \geq 2$, then partition $V(G) = S \cup A \cup B \cup C$ such that $[B \cup C]$ admits a clique cover of size at most two and let $G := [A]$. Go to Phase 2.

Phase 2: If $\alpha(G) \geq 2$, go to Phase I, else STOP.

Lemma 11.2 *If G is a $\{2K_2, C_4 \cup K_1\}$-free graph, then (1) $V(G)$ can be partitioned as $V(G) = X \cup Y$, where $[X]$ is a complete multipartite graph and $\theta([Y]) \leq 2\omega(G)$, and (2) $\theta(G) \leq \alpha(G) + 2\omega(G)$. In addition, a clique cover of G with at most $\alpha(G) + 2\omega(G)$ cliques can be found in $O(n^5)$ time, where $n = |V(G)|$.*

Proof Let $G(V, E)$ be a $\{2K_2, C_4 \cup K_1\}$-free graph. The following Algorithm 1 (Phases 1 and 2) partitions V into $V_0, V_1, V_2, \ldots, V_t$ such that $\alpha([V_i]) \geq 2$, for $i \in \{1, 2, \ldots, t\}$ and $\alpha([V_0]) \leq 1$. By Lemma 11.1, V_i can be partitioned as $V_i = S_i \cup A_i \cup B_i \cup C_i$, where (i) S_i is a maximum independent set in $[V_i]$, (ii) $S_i \oplus A_i$ in G, (iii) every vertex of $B_i \cup C_i$ has at least one non-neighbor in S_i, and (iv) B_i and C_i are cliques (see Fig. 11.4).

Algorithm 1

Input: A $\{2K_2, C_4 \cup K_1\}$-free graph $G(V, E)$ and $t = 0$.
Output: A partition of V, $V = V_0 \cup V_1 \cup V_2 \cup \ldots \cup V_t$ such that $\alpha([V_i]) \geq 2$, for $i \in \{1, 2, \ldots, t\}$ and $\alpha([V_0]) \leq 1$.

```
while (α(G) ≥ 2)
   {
    t := t + 1;
    S_t ; a maximum independent set in G.
    A_t := {y ∈ V(G)\S_t : {y} ⊕ S_t in G};
    B_t := {y ∈ V(G)\(S_t ∪ A_t) : |S_t ∩ N(y)| = 1};
    C_t := V(G) \ (S_t ∪ A_t ∪ B_t);
    V_t := S_t ∪ B_t ∪ C_t;
    G := [A_t]; .
   }
V_0 := V(G);
```

By Algorithm 1, V can be partitioned as $V = V_1 \cup V_2 \ldots \cup V_t \cup V_0$, where $\alpha([V_i]) \geq 2$, for $i \in \{1, 2, \ldots, t\}$ and $\alpha([V_0]) \leq 1$. If $\alpha(G) \leq 1$, then let $X = V(G)$ and $Y = \emptyset$ and hence conditions of the lemma hold. So we consider the case when $\alpha(G) \geq 2$.

Claim $1 \leq t \leq \omega = \omega(G)$.

Since $\alpha(G) \geq 2$, $t \geq 1$. Consider the ith iteration of Algorithm 1. We have $S_i \oplus A_i$ and $S_j \subseteq A_i$, for $j > i$ and $i, j \in \{1, 2, \ldots, t\}$. So, $S_i \oplus S_j$ in $G(V, E)$ for $i, j \in \{1, 2, \ldots, t\}$. Hence, $S_1 \cup S_2 \cup \ldots \cup S_t$ induces a complete multipartite graph in $G(V, E)$. Therefore, $t = \omega([S_1 \cup S_2 \cup \ldots \cup S_t]) \leq \omega(G)$.

Let $X = V_0 \cup (\bigcup_{i=1}^{t} S_i)$ and $Y = \bigcup_{i=1}^{t}(B_i \cup C_i)$. Clearly, $V(G) = X \cup Y$. Since S_i is an independent set, V_0 is a clique, $S_i \oplus S_j$ in G, and $S_i \oplus V_0$ in G for all

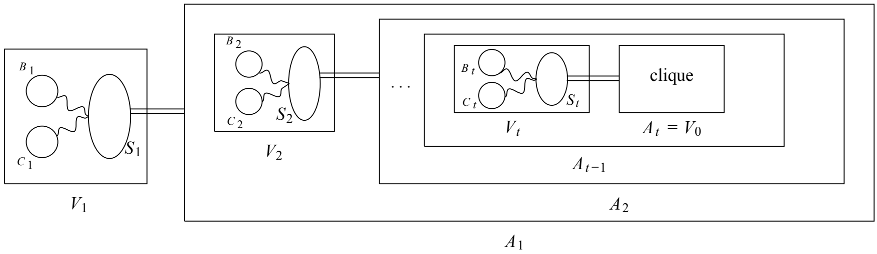


Fig. 11.4 The partition of $V(G)$ in Algorithm 1

$1 \leq i < j \leq t$, X induces a complete multipartite graph (see Fig. 11.4). So $[X]$ is a perfect graph and hence $\theta([X]) = \alpha([X]) = |S_1| = \alpha(G)$. By Lemma 11.1, B_i and C_i are cliques in G and hence $\theta([B_i \cup C_i]) \leq 2$. In addition, by 5, $t \leq \omega(G)$. So $\theta([Y]) \leq 2t \leq 2\omega(G)$ and thus $\theta(G) \leq \theta([X]) + \theta([Y]) \leq \alpha(G) + 2\omega(G)$.

We note that a maximum independent set S of a $2K_2$-free graph can be found in $O(n^4)$ time [2] and it takes $O(n^2)$ time to find A_t, B_t, and C_t in Algorithm 1, where $n = |V(G)|$. So the time required for an iteration of Algorithm 1 is $O(n^4 + n^2) = O(n^4)$. In addition, the algorithm terminates in at most n iterations. So the time complexity of Algorithm 1 is $O(n^5)$. □

Note that a graph G is $\{2K_2, C_4 \cup K_1\}$-free if and only if G^c is $\{C_4, K_1 + 2K_2\}$-free and hence by Lemma 11.2, we have the following theorem.

Theorem A. *If G is a $\{C_4, K_1 + 2K_2\}$-free graph, then $\chi(G) \leq \omega(G) + 2\alpha(G)$. In addition, a coloring of G with at most $\omega(G) + 2\alpha(G)$ colors can be found in $O(n^5)$ time, where $n = |V(G)|$.*

Next, by extending Theorem A to an arbitrary graph, we prove the existence of an upper bound of the chromatic number of a graph in terms of its clique and independence numbers.

11.3 A (χ, α, ω)-Binding Function

First, we derive an upper bound of the clique number of a graph with at least one vertex.

Observation For $x, y, p \in \mathbb{N} = \{1, 2, 3, \ldots\}$ and $x \leq y$, $xy^p + (y-x)^p \leq y^{p+1}$.

Proof If $x = y$, $xy^p + (y-x)^p = y^{p+1}$. So $x < y$. For $x, y, p \in \mathbb{N}$, $(y-x)^{p-1} \leq y^{p-1} \leq y^p$. Hence, $(y-x)^p \leq y^p(y-x) = y^{p+1} - xy^p$ which implies $xy^p + (y-x)^p \leq y^{p+1}$. □

Lemma 11.3 *Let S be a maximum independent set of a graph $G(V, E)$ and let $\mathcal{F} = \mathbb{P}(S) \backslash \{S\}$, where $\mathbb{P}(S)$ denote the power set of S. For $T \subseteq S$, let $N_T = \{y \in$*

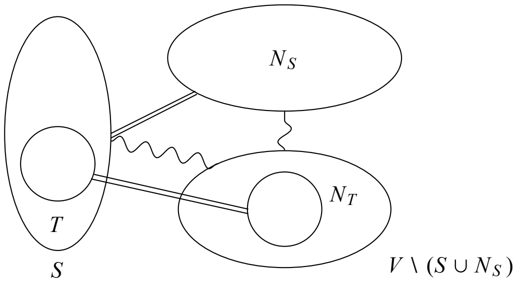


Fig. 11.5 The partition of $V(G)$ in Lemma 11.3

$V \backslash S : N(y) \cap S = T\}$. Then V can be partitioned as $V = S \cup N_S \cup \bigcup_{T \in \mathcal{F}} N_T$ (see Fig. 11.5) such that $\alpha([N_T]) \leq |T| \leq \alpha(G) - 1$, for every $T \in \mathcal{F}$.

Proof If $u \in N_{T_1} \cap N_{T_2}$, then $N(u) \cap S = T_1$ and $N(u) \cap S = T_2$ (by definition), and hence $T_1 = T_2$. So $V = S \cup N_S \cup \big(\bigcup_{T \in \mathcal{F}} N_T \big)$ is a partition of the vertex set of G. Since $T \in \mathcal{F}$, $|T| \leq |S| - 1 = \alpha(G) - 1$. In addition, $\alpha([N_T]) \leq |T|$ for $T \in \mathcal{F}$, else $\alpha([N_T \cup (S \backslash T)]) \geq \alpha(G) + 1$, a contradiction. Hence $\alpha([N_T]) \leq |T| \leq \alpha(G) - 1$ for all $T \in \mathcal{F}$. □

Lemma 11.4 *For a graph G with at least one vertex, $\theta(G) \leq 2^{\frac{\alpha(\alpha+1)}{2}} \omega^{\alpha-1}$, where $\omega = \omega(G)$ and $\alpha = \alpha(G)$.*

Proof We prove this lemma by induction on $\alpha(G)$. If $\alpha(G) = 1$, then G is a clique and $\theta(G) = 1 \leq 2^1 \omega^0 = 2$. If $\alpha(G) = 2$, then the complement of G, G^c, is triangle-free and $\Delta(G^c) \leq \alpha(G^c)$. So,

$$\begin{aligned}\theta(G) &= \chi(G^c) \leq \Delta(G^c) + 1 \\ &\leq \alpha(G^c) + 1 = \omega(G) + 1 \\ &\leq 8\omega(G) = 2^{\frac{2(2+1)}{2}} \omega(G)^{2-1}.\end{aligned}$$

Let $\alpha(G) = p \geq 3$ be an integer. Next, assume that the result is true for all graphs H with $\alpha(H) \leq p - 1$. Let $G(V, E)$ be a graph with $\alpha(G) = p \geq 3$. Next, we partition V using Lemma 11.3 in Phases 1 and 2 as follows.

Phase I: If $\alpha(G) = p$, then partition $V(G)$ as $V(G) = S \cup N_S \cup (V(G) \setminus (S \cup N_S))$ such that S is a maximum independent set in G and $S \oplus N_S$ in G (see Fig. 11.5). Let $G := [N_S]$ and go to Phase 2.

Phase 2: If $\alpha(G) = p$, go to Phase I, else STOP.

Algorithm 2 captures Phases 1 and 2. It partitions V into $V_0, V_1, V_2, \ldots, V_t$ such that $\alpha([V_i]) = p = \alpha(G)$, for $i \in \{1, 2, \ldots, t\}$ and $\alpha([V_0]) \leq p - 1 < \alpha(G)$ (see Fig. 11.6).

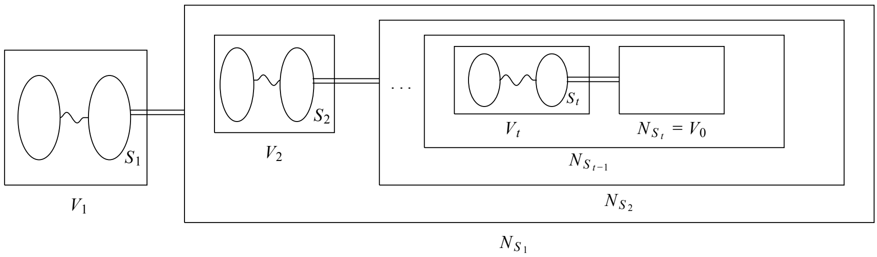


Fig. 11.6 The partition of $V(G)$ in Algorithm 2

Algorithm 2

Input: $G(V, E)$, $\alpha(G) = p$, and $t = 0$.
Output: A partition of V, $V = V_0 \cup V_1 \cup V_2 \cup \ldots \cup V_t$ such that $\alpha([V_i]) = p$, for $i \in \{1, 2, \ldots, t\}$ and $\alpha([V_0]) < \alpha(G) = p$.
while $(\alpha(G) = p)$
{
$t := t + 1$;
$H_t := G$;
S_t ; a maximum independent set in H_t.
$N_{S_t} := \{y \in V(H_t)\backslash S_t : S_t \subseteq N(y)\}$;
$V_t := V(H_t)\backslash N_{S_t}$;
$G := [N_{S_t}]$; the graph induced by N_{S_t} in H_t.
}
$V_0 := V(G)$;

Claim $1 \leq t \leq \omega = \omega(G)$ and $\omega([V_0]) \leq \omega - t$.

Since $\alpha(G) = p$, $t \geq 1$. Consider the ith iteration of the Algorithm 2 (see Fig. 11.6). We have $S_i \oplus N_{S_i}$ and $S_j \subseteq N_{S_i}$, for $j > i$ and $i, j \in \{1, 2, \ldots, t\}$. So, $S_i \oplus S_j$ in $G(V, E)$ for $i, j \in \{1, 2, \ldots, t\}$. Hence, $S_1 \cup S_2 \cup \ldots \cup S_t$ induces a complete multipartite graph in $G(V, E)$. Therefore, $t = \omega([S_1 \cup S_2 \cup \ldots \cup S_t]) \leq \omega(G)$. Moreover, $S_1 \oplus S_2 \oplus \ldots \oplus S_t \oplus V_0$ in $G(V, E)$. So, $\omega = \omega(G) \geq \omega([S_1 \cup S_2 \cup \ldots \cup S_t \cup V_0]) = t + \omega([V_0])$ and $\omega([V_0]) \leq \omega - t$.

Since $\alpha([V_0]) \leq p - 1$ and $\omega([V_0]) \leq \omega - t$, by induction, $\theta([V_0]) \leq 2^{\frac{\alpha([V_0])(\alpha([V_0])+1)}{2}} \omega([V_0])^{\alpha([V_0])-1} \leq 2^{\frac{(p-1)p}{2}} (\omega - t)^{p-2}$. Next, for $i = 1, 2, \ldots, t$, we find a clique cover of $[V_i]$. Clearly, $V(H_i) = S_i \cup N_{S_i} \cup \bigcup_{T \in \mathcal{F}_i} N_T$, where $\mathcal{F}_i = \mathbb{P}(S_i)\backslash\{S_i\}$ and $N_T = \{y \in V(H_i)\backslash S_i : N(y) \cap S_i = T\}$. Note that $|\mathcal{F}_i| = 2^p - 1$. By Lemma 11.3, $\alpha([N_T]) \leq \alpha(H_i) - 1 \leq \alpha(G) - 1 = p - 1$, for $T \in \mathcal{F}_i$.

By induction, $\theta([N_T]) \leq 2^{\frac{\alpha([N_T])(\alpha([N_T])+1)}{2}} \omega([N_T])^{\alpha([N_T])-1} \leq 2^{\frac{(p-1)p}{2}} \omega^{p-2}$, for $T \in \mathcal{F}_i$.

In addition, $V_i = V(H_i)\backslash N_{S_i} = S_i \cup \bigcup_{T \in \mathcal{F}_i} N_T$. So, for $1 \leq i \leq t$,

$$\begin{aligned}\theta([V_i]) &= \theta\big([S_i \cup (\bigcup_{T\in\mathcal{F}_i} N_T)]\big)\\ &\le \theta([S_i]) + |\mathcal{F}_i|\,\theta([N_T])\\ &\le p + (2^p - 1)2^{\frac{(p-1)p}{2}}\omega^{p-2}\\ &= p + 2^{\frac{p(p+1)}{2}}\omega^{p-2} - 2^{\frac{(p-1)p}{2}}\omega^{p-2}\\ &\le 2^{\frac{p(p+1)}{2}}\omega^{p-2},\ \text{since } p - 2^{\frac{(p-1)p}{2}}\omega^{p-2} \le 0,\ \text{for } p \ge 3 \text{ and } \omega \ge 1.\end{aligned}$$

Next, we prove that result is true for graphs G with $\alpha(G) = p$.

$$\begin{aligned}\theta(G) &= \theta([V_1 \cup V_2 \ldots \cup V_t \cup V_0])\\ &\le \sum_{i=1}^{t}\theta([V_i]) + \theta([V_0])\\ &\le t2^{\frac{p(p+1)}{2}}\omega^{p-2} + 2^{\frac{(p-1)p}{2}}(\omega - t)^{p-2}\\ &\le 2^{\frac{p(p+1)}{2}}\big(t\omega^{p-2} + (\omega - t)^{p-2}\big)\\ &\le 2^{\frac{p(p+1)}{2}}\omega^{p-1},\ \text{by Observation 1},\ \text{since } t, \omega, p-2 \ge 1,\ \text{and } t \le \omega\\ &= 2^{\frac{\alpha(\alpha+1)}{2}}\omega^{\alpha-1}.\end{aligned}$$

Due to Lemma 11.4, we have the following theorem. □

Theorem B. *For a graph G with at least one vertex, $\chi(G) \le 2^{\frac{\omega(\omega+1)}{2}}\alpha^{\omega-1}$, where $\omega = \omega(G)$ and $\alpha = \alpha(G)$.*

11.3.1 Applications of Theorem B

11.3.1.1 A χ-Binding Function for a Subclass of Odd-Hole-Free Graphs

Recently, Scott and Seymour proved that if G is an odd-hole-free graph, then $\chi(G) \le f(\omega) = 2^{2^{\omega+2}}$ [8] and this answers a famous conjecture by Gyárfás [5] that odd-hole-free graphs are χ-bounded. Note that $f(\omega) = 2^{2^{\omega+2}}$ is a super exponential function in ω. Though Theorem B does not exclusively deal with the structure of odd-hole-free graphs, it helps us to derive an exponential χ-binding function for the class of odd-hole-free graphs with bounded independence number as follows.

Corollary 11.1 *Let $\mathcal{G}$ be a class of odd-hole-free graphs such that there exists a fixed positive integer k, where $\alpha(G) \le k$ for all $G \in \mathcal{G}$. Then $\chi(G) \le 2^{\frac{\omega(G)(\omega(G)+1)}{2}}k^{\omega(G)-1}$.*
□

Note that the above corollary is true for the class of all graphs with independence number at most k.

11.3.1.2 Existence of a (χ, α)-Binding Function

Since an arbitrary class of graphs does not admit a χ-binding function in general, Gyárfás and Sumner [4, 10] independently proposed the following beautiful conjecture and it is open for the last four decades.

Gyárfás-Sumner Conjecture: Let F be fixed forest. If $\mathcal{G}$ is F-free, then $\mathcal{G}$ admits a χ-binding function.

Kierstead and Penrice [6] proved that the above conjecture is true for trees with diameter two. Let T be a fixed tree. A topological version of the above Conjecture is proved by Scott [9] by showing that if G contains no subdivision of T as an induced subgraph for all $G \in \mathcal{G}$, then $\mathcal{G}$ admits a χ-binding function. Next, we deduce an analogous of Gyárfás-Sumner conjecture for (χ, α)-binding functions.

Theorem 11.1 *Let $\mathcal{G}$ be an induced hereditary class of graphs. Then $\mathcal{G}$ admits a (χ, α)-binding function if and only if there exists a fixed positive integer k such that $\omega(G) \leq k$ for all $G \in \mathcal{G}$.*

Proof Assume that $\mathcal{G}$ admits a (χ, α)-binding function. We prove that there exists a fixed positive integer k such that $\omega(G) \leq k$ for all $G \in \mathcal{G}$. If not, for any positive integer k, there exists $G \in \mathcal{G}$ such that $\omega(G) > k$. Then the class of complete graphs, $C \subseteq \mathcal{G}$. Note that C does not admit a (χ, α)-binding function and hence $\mathcal{G}$ does not admit a (χ, α)-binding function, a contradiction. Conversely, assume that $\omega(G) \leq k$, $\forall\, G \in \mathcal{G}$ for a fixed positive integer k. Then by Theorem B, $\chi(G) \leq 2^{\frac{k(k+1)}{2}} \alpha^{k-1}$. So $g(x) = 2^{\frac{k(k+1)}{2}} x^{k-1}$ is a (χ, α)-binding function of $\mathcal{G}$, where k is a fixed positive integer. □

11.4 Conclusion

We note that triangle-free graphs do not admit a χ-binding function [1], but it admits a linear (χ, α)-binding function. Next, we consider a sequence of odd-hole-free graphs constructed by Scott and Seymour [8]. Let $G_0 = K_1$ and let G_k be obtained by replacing every vertex of G_{k-1} by a seven vertex anti hole, for $k \geq 1$. In addition, for $k \geq 1$, $\alpha(G_k) = 2^k$ and $\chi(G_k) \geq \frac{|V(G_k)|}{\alpha(G_k)} = (\frac{7}{2})^k$. So $\chi(G_k) \geq \alpha(G_k)^l$, where $l =$ ($\log_2 7$)-1 ≈ 1.807. Hence, any induced hereditary class of graphs $\mathcal{G}$ that contains $\{G_k : k \geq 0\}$ does not admit a linear (χ, α)-binding function. The class of perfect graphs admits a χ-binding function (identity function), but it contains the class of complete graphs C and hence it does not admit a (χ, α)-binding function. Note that the class of $\{C_4, K_1 + 2K_2\}$-free graphs admits neither a χ-binding function nor a (χ, α)-binding function, but $\chi \leq \omega + 2\alpha$ for every graph in that class.

In fact we proved that, for an arbitrary graph, there exists an upper bound of χ in terms of its α and ω. But this upper bound may not be the best possible. So we ask the following question.

Question 11.3 Let $\mathcal{G}$ be an induced hereditary class of graphs. Does there exist a function $f : \mathbb{N} \to \mathbb{R}^+$ such that $\chi(G) \leq f(\alpha(G) + \omega(G))$, for all $G \in \mathcal{G}$?

Next, we provide a class of graphs $\mathcal{G}$ that answers Question 11.3 negatively when we restrict f to be a linear function. That is, there is no constant c such that $\chi(G) \leq c(\alpha(G) + \omega(G))$ for every $G \in \mathcal{G}$. Consider the sequence of graphs $\{G_n\}$, where $G_1 = C_5$ and G_{i+1} is obtained by replacing every vertex of C_5 by G_i, for $i \geq 1$ [3]. For $n \geq 1$, $\omega(G_n) = \alpha(G_n) = 2^n$ and $\chi(G_n) \geq \frac{|V(G_n)|}{\alpha(G_n)} = (\frac{5}{2})^n$. We claim that there doesn't exist a constant c such that $\chi(G_n) \leq c(\alpha(G_n) + \omega(G_n))$ for every $n \in \mathbb{N}$. On the contrary, let there be a real number M such that $\chi(G_n) \leq M(\alpha(G_n) + \omega(G_n))$ for every $n \in \mathbb{N}$. Then $(\frac{5}{2})^n \leq M2^{n+1}$. This implies $(\frac{5}{4})^n \leq 2M$, for every $n \in \mathbb{N}$, a contradiction since $\{x^n\}$ is not bounded above when $x > 1$. Let $\mathcal{G}$ be an induced hereditary class of graphs such that $\{G_n : n \in \mathbb{N}\} \subseteq \mathcal{G}$. Then $\mathcal{G}$ does not admit a linear function f as in Question 11.3.

It is an interesting problem to find an improved upper bound of χ in terms of α and ω.

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Chapter 12
A Short Proof of Ore's f-Factor Theorem Using Flows

Sriraman Sridharan and Patrick Vilamajó

12.1 Introduction

We generally follow the notation and terminology of Berge [2]. Let G be a directed graph with vertex set X and arc set U. The number of vertices is denoted by n and the number of arcs (directed edges) by m. We allow at most one directed arc between any two vertices in the same direction. A multigraph G is an *undirected* graph without *loops* and it will be denoted by $G = (X, E)$ where E is the multiset of edges (Note that a bipartite graph cannot have a loop). A multiset is a set in which elements can be repeated and the number of repetitions of an element in a multiset is called its *multiplicity*. For a vertex x, the neighborhood of the vertex x is $\Gamma(x) = \{y \in X | xy \in E\}$ and $\Gamma(S) = \cup_{x \in S}\Gamma(x)$ for $S \subset X$.

Consider a multigraph $G = (X, E)$ and a function f defined on the vertex set with values into the set of integers $\mathbf{Z}$. An *f-factor* of G is a spanning subgraph H of G such that $d_H(x) = f(x)$ for each vertex x of G. Ore [5] characterized the existence of an f-factor in a bipartite graph. It is a *good* characterization of bipartite graphs *without* an f-factor. In any multigraph (loops admitted), Tutte [6] characterized the existence of an f-factor. In this paper, we give a short proof of Ore's f-factor theorem using the flows in transportation networks (see [2]).

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12.2 Notation and Terminology

Consider a bipartite graph (undirected) $G = (X, Y; E)$ with bipartition X and Y and an integer-valued function f defined on the vertex set of G. The function f induces its *dual* function f' on the vertex set of G where $f'(x) = d_G(x) - f(x)$ for each vertex x. Consider two sets S and T which are subsets of X and Y, respectively. $m(S, T)$ is the number of edges of G having one end in X and the other end in Y. The deficiency $\delta(S, T)$ is defined as one of the numbers

$$\delta(S, T) = m(S, T) - \sum_{s \in S} f(s) - \sum_{t \in T} f'(t) = \sum_{t \in T} f(t) - \sum_{s \in S} f(s) - m(X \setminus S, T).$$

The function f is *balanced* with respect to (X, Y) if

$$\sum_{x \in X} f(x) = \sum_{y \in Y} f(y).$$

We observe that if f is *not* balanced, say, for example, $\sum_{x \in X} f(x) < \sum_{y \in Y} f(y)$, then the graph G has *no* f-factor, because, if H were an f-factor of G, then on the one hand the number of edges of H is $\sum_{x \in X} f(x)$, and on the other, it is $\sum_{y \in Y} f(y)$ (since H is bipartite) which is impossible. Further, it is seen that $\delta(X, Y) > 0$. Henceforth, we suppose that the function f is *balanced.*

We are now in a position to state Ore's f-factor theorem.

Theorem 12.1 (Ore [5]) *Let $G = (X, Y; E)$ be a bipartite graph with an integer-valued balanced function f defined on $X \cup Y$. The G has no f-factor if and only if there are subsets $S \subset X$ and $T \subset Y$ such that*

$$\delta(S, T) = m(S, T) - \sum_{s \in S} f(s) - \sum_{t \in T} f'(t) > 0.$$

12.3 New Proof of Ore's Theorem

We use a theorem proved by Gale [4]. To state the theorem of Gale, we need some preliminaries. Let $G = (X, Y; E)$ be a bipartite graph. Consider a bipartite transportation (oriented) network (see Berge [2], p.79) $G = (X \cup \{ b \}, Y \cup \{ a \}; U)$ where X and Y are disjoint sets, with a source vertex a and a sink vertex b with the collection of directed edges as follows:

type 1 (x, y) with $x \in X$ et $y \in Y$, $xy \in E$;
type 2 (a, x) with $x \in X$;
type 3 (y, b) with $y \in Y$;
type 4 (b, a), the return arc.

The function c is the capacity function defined on the arc set U with values in the set of non-negative integers.

Set

$$d(y) = \begin{cases} c(y, b) & \text{if } y \in Y, \\ 0 & \text{if not.} \end{cases}$$

Here, $c(y, b)$ is the capacity of the arc (y, b) and $d(y)$ is the demand at the vertex y. If $B \subset Y$, we call *the demand of the set* B, the quantity

$$d(B) = \sum_{y \in B} d(y).$$

If $B \subset Y$, we denote on $F(B)$ *the maximum quantity of flow that can be sent to the set* B, that is, the value of a maximum flow of the network G' obtained from G obtained by changing the capacities as follows (c' is the new capacity function of G'.) :

- $c'(y, b) = \infty$ if $y \in Y$;
- $c'(y, b) = 0$ if $y \in Y \setminus B$;
- $c'(x, y) = c(x, y)$ for all other arcs (x, y).

We can now state Gale's theorem.

Theorem 12.2 (Gale [4]) *A bipartite network $G = (X, Y; E)$ has a flow saturating all the arcs with heads in the sink b if and only if*

$$F(T) \geq d(T) \quad \textit{for all} \quad T \subset Y.$$

We shall now prove Ore's f-factor theorem using the theorem of Gale.

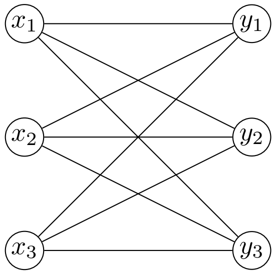
Theorem 12.3 (Ore [5]) *Let $G = (X, Y; E)$ be a bipartite graph with an integer-valued balanced function f defined on $X \cup Y$. The G has no f-factor if and only if there are subsets $S \subset X$ and $T \subset Y$ such that*

$$\delta(S, T) = m(S, T) - \sum_{s \in S} f(s) - \sum_{t \in T} f'(t) > 0.$$

Proof First of all, we orient the edges of the bipartite graph in the following way: each edge is directed from X to Y, that is, if xy is an edge with $x \in X$ and $y \in Y$, then we orient the edge xy as the directed edge (x, y). If there are several multiple edges between the vertices x and y in G, then draw *only* one arc (x, y) in the orientation. Then we add a source vertex a and a sink vertex b with the arcs (a, x) for each $x \in X$ and the arcs (y, b) for each $y \in Y$. Call the directed graph thus obtained as G'. Now define the capacity function c' on the arc set of G' as follows: $c'(a, x) = f(x)$ for each $x \in X$, $c'(y, b) = f(y)$ for each $y \in Y$, $c'(x, y) = m(x, y)$ if (x, y) is an arc of G'. Here, $m(x, y)$ denotes the number of edges having x as the initial vertex and y as the final vertex *in* G. (see Figs. 12.1 and 12.2 for illustration of the construction.)

Now there is a one to one correspondence between an f-factor H *in* G and a *maximum* flow *saturating* the arcs going out of the source vertex a and into the sink

Fig. 12.1 A bipartite graph G with $f(x_1) = f(x_3) = f(y_1) = f(y_2) = 3$ and $f(x_2) = f(y_3) = 2$



vertex b *in* G'. For, if H is an f-factor H in G, then we can construct a maximum flow f' in G' as follows: $f'(a, x) = f(x)$ for each $x \in X$, $f'(y, b) = f(y)$ for each $y \in Y$, $f'(x, y) = m_H(x, y)$ where $m_H(x, y)$ is the number of edges in H having one end in x and the other in y. One can easily verify that the function f' is a *flow* in G' which saturates the directed edges going out of the source a and entering into the sink b. Conversely, if f' is a maximum flow in G' saturating the directed edges going out of a and entering into b in G', then we define an $f-$factor of G using the flow f' of G' in the following manner: We add as many multiple edges between x and y in H as the value of the flow $f'(x, y)$ along the arc (x, y). It can be easily verified that the spanning subgraph H defines an f-factor of G (see Figs. 12.3 and 12.4).

By Gale's theorem [4], G' has a flow saturating the arcs entering into the sink vertex b (and hence the flow saturates the arcs going out of a as well) if and only if $F(T) \geq d(T)$ for all $T \subset Y$, where $F(T)$ is the value of a maximum flow that can be sent to the set T and $d(T)$ is the demand of the set T, we would like to satisfy.

But then $F(T) = \sum_{s \in \Gamma(T)} \min(f(s), m(s, T))$ and since there are *no* arcs between $X \setminus \Gamma(T)$ and T, we can write $F(T) = \sum_{x \in X} \min(f(x), m(x, T))$. Now by definition, $d(T) = \sum_{y \in T} d(y) = \sum_{y \in T} c'(y, b) = \sum_{y \in T} f(y)$.

Hence, $F(T) \geq d(T)$ is equivalent to $\sum_{y \in T} f(y) \leq \sum_{x \in X} \min(f(x), m(x, T))$.

We shall show that the above inequality is equivalent to

$$\delta(S, T) = \sum_{t \in T} f(t) - \sum_{s \in S} f(s) - m(X \setminus S, T) \leq 0.$$

If $\sum_{t \in T} f(t) - \sum_{s \in S} f(s) - m(X \setminus S, T) \leq 0$ is true, then $\sum_{t \in T} f(t) \leq \sum_{s \in S} f(s) + m(X - S, T) \leq \sum_{x \in S} f(s) + \sum_{x \in X \setminus S} m(x, T) \leq \sum_{x \in X} \min(f(x), m(x, T))$ (since $S \cup X \setminus S = X$.)

Conversely, if $\sum_{y \in T} f(y) \leq \sum_{x \in X} \min(f(x), m(x, T))$ is true, then set $S = \{s \in X | f(s) < m(s, T)\}$. Then $\sum_{y \in T} f(y) \leq \sum_{s \in S} f(s) + m(X \setminus S, T)$ which means the deficiency $\delta(S, T) \leq 0$. Hence the proof. ∎

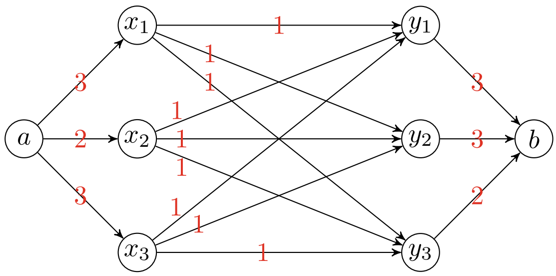


Fig. 12.2 Network G' associated with the graph G with capacities indicated on arcs

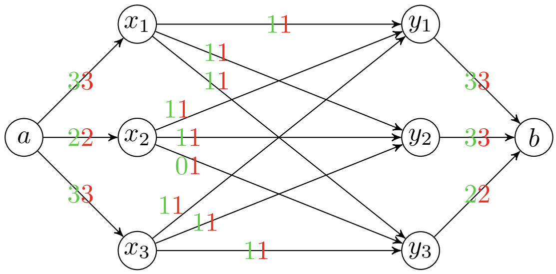


Fig. 12.3 A flow saturating the arcs going into b is indicated in green (the first number on the arcs). Capacities are red (the second number on the arcs.)

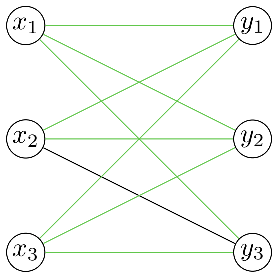


Fig. 12.4 An f-factor in G induced by the flow in G', all green edges, that is, all edges of G except x_2y_3

We now derive an algorithm to find an f-factor in a bipartite graph if it exists, or else we find a *succinct* certificate to prove the non-existence of an f-factor.

Algorithm

Input: A bipartite graph $G = (X, Y; E)$ with an integer-valued balanced function f defined on the vertex set of G.
Output: An f-factor if it exists or else a succinct certificate to show its non-existence.
Algorithm: We suppose that $0 \leq f(v) \leq d_G(v)$ for each vertex v of G. Otherwise, G has clearly no f-factor. If $f(x) < 0$x for some $x \in X$, then it is seen that $S = \{x\}$ and $T = \emptyset$ is a succinct certificate as $\delta(S, T) > 0$. Similarly, if $f(y) < 0$ for some $y \in Y$, then we can consider the bipartition (Y, X) instead of (X, Y) and b as the source vertex and a as the sink vertex to obtain a succinct certificate $S = \{y\}$ and $T = \emptyset$. Dually, if $f(y) > d_G(y)$, with $y \in Y$, then for $S = \emptyset$ and $T = \{y\}$, $\delta(S, T) > 0$. The case $f(x) > d_G(x)$ can be treated analogously.

Construct a network G' with the help of G and the function f as indicated in the course of the proof of the above theorem. Then apply the Ford-Fulkerson algorithm (for example see [2]) to find a maximum flow f' *in* G'. If the flow *saturates* all arcs going into the sink vertex b, then this flow induces an f-factor as in the proof of the above theorem. Otherwise, using the maximum flow f' construct a *search tree* for f' (see [3]). Since f' is a maximum flow, this search tree does not contain the sink vertex b. Let T be the set of all vertices of Y *not* in the tree. Then, it is not difficult to verify that $\sum_{t\in T} f(t) \leq \sum_{x\in X} \min(f(x), m(x, T))$. Hence, T is a succinct certificate.
The complexity of the above algorithm: Clearly, the complexity of the above algorithm is the same as the complexity of the Ford-Fulkerson algorithm [2] (by using the breadth-first search [1]) is $O(m+n)$.

12.4 A Deficiency Version of Ore's Theorem

The following extension of Gale's theorem is proved in [2].

Theorem 12.4 *In a bipartite transportation network* $G = (X, Y; U)$ *with the source a and sink b, the value of a maximum flow is*

$$d(Y) + \min_{B\subset Y}(F(B) - d(B)).$$

Let G = (X, E) be multigraph. For a function f defined on X with $0 \leq f(x) \leq d_G(x)$ for each $x \in X$, an f*-matching* is a spanning subgraph H of G such that $d_H(x) \leq f(x)$ for all $x \in X$. An f-matching is an f-factor if and only if $d_H(x) = f(x)$ for all $x \in X$.

Theorem 12.5 (The deficiency version) *Let* $G = (X, Y; E)$ *be a bipartite graph with a balanced integer-valued function* f *defined on the entire vertex set of* G *with* $0 \leq d_H(x) \leq f(x)$. *Then the maximum number edges in an* f*-matching is* $d(Y) + \min_{B\subset Y}(F(B) - d(B))$ *where* $F(B)$ *and* $d(B)$ *are calculated in the associated transportation network* G' *of* G.

We raise the following question: Can we prove Tutte's f-factor theorem using the Ford-Fulkerson theorem [2] of max flow-min cut?

Acknowledgements The author thanks the referee for suggestions.

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Chapter 13
A Decision-Making Problem Involving Soft Fuzzy Number Valued Information System: Energy-Efficient Light-Emitting Diode Blubs

Felbin C. Kennedy, S. Masilla Moses Kennedy, S. Arul Roselet Meryline, and M. Jayachandiran

13.1 Introduction

In real-life situations, we come across problems that comprise imprecise or uncertain, simple or complex information that needs to be analyzed for various requirements. To tackle this situation, Zadeh [25] in 1965 formulated the concept of a fuzzy set, which in an imprecise environment that captured the inexactness present in a system. Later, Zadeh [28] elucidated the concept of linguistic variables to handle situations that involved less preciseness in humanistic systems. This was further studied by several researchers using appropriate quantification of fuzziness on kinds of fuzzy numbers. To process and analyze the features connected with entities in such a scenario, fuzzy information systems were studied in the literature (to cite a few, [3, 6, 7]).

On the other hand, the concept of a soft set as a mathematical tool for dealing with uncertainty was introduced by Molodtsov [15] in 1999. In soft set theory, the parameterization tool involved in the concepts played a major role and had drawn the attention of many researchers over the years which led to the rapid development of the theory. A combination of soft sets with fuzzy sets was noticed by some researchers as more rewarding to capture the nature of entities in the problem in hand viewed as an

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information system (to cite a few, [17–20]). Two of the authors [21, 22] introduced and developed $\widetilde{IS}$ involving fuzzy numbers in combination with soft set theory in information systems to process an analysis of the involved data in a system. Experimental studies have been considered and explained using a fuzzy rule-based inference system (to cite a few [1, 2, 11]).

So far in the literature, there is no reference incorporating the experimental study and soft fuzzy number valued information system, which is considered in the current study. Several companies are manufacturing low-wattage efficient Light-Emitting Diode (LED) based bulbs for household usage or beautification. Our problem in such a situation is to select a suitable bulb for the desired purpose. It is interesting to conduct an experimental study and compare the outcomes with that of the information given by the manufacturer to make better decisions. The expressiveness of soft fuzzy numbers plays an important role in modeling the experimental outcomes as a fuzzy information system that is studied in the present paper.

Voltage variations are in general between 180 and 240 V, which may vary from place to place, and time to time. Hence, experiments are conducted on bulbs, from different brands, to study the various characteristic properties. For this purpose, Seven brands of low-wattage bulbs, say, 7 W manufactured by 7 different companies (here after mentioned as 'seven bulbs') are purchased, and an experimental study was conducted on them under the same conditions. One experiment was conducted to measure the spectral light output of the chosen bulb at an applied voltage from 60 to 240 V. Also, the same was measured by fixing the applied voltage at 220 V. Another experiment was conducted to measure the current at this applied voltage.

The output was converted into a fuzzy analogue of a soft fuzzy number valued information system for the purpose of decision-making which would lead to a better choice of bulb for appropriate applications. Here, the consideration of various parameters (relative intensity at 220 V, total light output between 140 and 240 V, maximum light output at low voltage and parameter related to lower voltage at which the output of a bulb is the same as that of the one which yields at higher voltage) and the influencing attributes are incorporated as information. Information from three situations (Catalog, experimental results at standard 220 V and varying voltage) are considered and analyzed using an algorithm which is proposed.

The companies that make the bulbs for sale can use or apply our methodology to measure the efficiency of the bulbs with respect to various attributes and incorporate the same as information on the catalog available to consumers, buying a single bulb or in bulk that would quench the thirst for the knowledge for that particular information. This new approach would enhance awareness among the customers to make the best choice from the available products in the market based on various attributes involved or characteristic features. By choosing appropriate parameters, experimental study could be conducted for any other products and incorporated as $\widehat{IS}$ to make an optimal choice of the product.

The paper is systematized as follows: In Sect. 13.2, we provide the needed prerequisites. In Sect. 13.3, we state the problem under consideration, and the basic information obtained from the LED bulb manufacturers' catalog is presented with values normalized to the same scale. In Sect. 13.4, an experimental study on various

LED bulbs, with wattage 7 W, measuring the relative intensity and voltage-current characteristics, is carried out. Relative intensity (light output) per watt and CRI values are computed. In Sect. 13.5, the variations in power consumption of bulb brand through experiment are captured and the same is incorporated along with cost, warranty, the experimental observations corresponding to each bulb at 220 V and over a range of voltages are framed as fuzzy analogues of soft fuzzy number valued information systems. Also, the catalog information is modeled as a soft fuzzy number valued information system. In Sect. 13.6, an algorithm is proposed to measure the efficiency of the bulbs, and the solution set obtained for catalog and experimental information is discussed.

13.2 Preliminaries

In this section, for the sake of completeness we record the required definitions.

Definition 13.1 ([25]) A *fuzzy set* $\tilde{A}$ on a universal set U is characterized by a membership function $\mu_{\tilde{A}}$ which associates with each u in U a real number $\mu_{\tilde{A}}(u)$ in the interval [0, 1] that represents 'the grade of membership' of u in U, i.e., $\tilde{A}$ is $\mu_{\tilde{A}}(u) : U \to [0, 1]$.

Definition 13.2 ([8]) The $\alpha-$ *cut* of a fuzzy set $\tilde{A}$, $0 \leq \alpha \leq 1$, denoted by $[\tilde{A}]_\alpha$, is defined as $[\tilde{A}]_\alpha = \left\{u \in U | \tilde{A}(u) \geq \alpha\right\}$.

Definition 13.3 ([16]) A *fuzzy number* $\tilde{A}$ is defined by a mapping $\tilde{A} : \mathbb{R} \to [0, 1]$ which is

1. Upper semi-continuous, i.e., for all $t \in \mathbb{R}$ and $c > 0$ with $\tilde{A}(t) = a$, there is $c > 0$ such that $|s - t| < c$.
2. Convex, i.e., for $s \leq t \leq r$, $\tilde{A}(t) \geq \tilde{A}(s) \wedge \tilde{A}(r) \equiv min(\tilde{A}(s), \tilde{A}(r))$.
3. Normal, i.e., there exists a $t_o \in \mathbb{R}$ such that $\tilde{A}(t_o) = 1$.

The collection of such fuzzy numbers is denoted by $\mathcal{F}(\mathbb{R})$.

Definition 13.4 ([16]) A fuzzy number $\tilde{A} \in \mathcal{F}(\mathbb{R})$ is said to be non-negative if $\tilde{A}(t) = 0, \quad \text{for} \quad t < 0$. The collection of all non-negative fuzzy numbers is denoted as $\mathcal{F}^*(\mathbb{R})$.

For the rest of the paper, we consider only $\mathcal{F}^*(\mathbb{R})$.

Proposition 13.1 ([16]) *Let* $[a^\alpha, b^\alpha]$, $0 < \alpha \leq 1$ *be a given family of intervals. Suppose*

(a) for all $0 < \alpha_1 \leq \alpha_2$, $[a_1^\alpha, b_1^\alpha] \supset [a_2^\alpha, b_2^\alpha]$*;*

(b) for any increasing sequence $\{\alpha_k\}$ *in* $(0, 1]$ *converging to* α*,*

$$\left[\lim_{k\to\infty} a_k^{\alpha}, \lim_{k\to\infty} b^{\alpha}\right] = [a^{\alpha}, b^{\alpha}].$$

Then the family $[a^{\alpha}, b^{\alpha}]$ *represents the* $\alpha-$ *cut sets of a fuzzy number* $\tilde{A} \in \mathcal{F}^*(\mathbb{R})$*.*

Conversely, if $[a^{\alpha}, b^{\alpha}]$*,* $0 < \alpha \leq 1$ *are* $\alpha-$ *cuts of a fuzzy number* $\tilde{A} \in \mathcal{F}^*(\mathbb{R})$*, then the conditions (a) and (b) are satisfied.*

Remark 13.1 In [16], it was shown that $\alpha-$ cut of a fuzzy number $\tilde{A}$ in $\mathcal{F}^*(\mathbb{R})$, for $\alpha \in (0, 1]$, to be a closed interval $[a^{\alpha}, b^{\alpha}]$ (say) and for any two fuzzy numbers $\tilde{A}_1, \tilde{A}_2$ in $\mathcal{F}^*(\mathbb{R})$, their $\alpha-$ cuts $[\tilde{A}_1]_{\alpha} = [a_1^{\alpha}, b_1^{\alpha}]$, $[\tilde{A}_2]_{\alpha} = [a_2^{\alpha}, b_2^{\alpha}]$ satisfied the following:

$$\left[\tilde{A}_1\right]_{\alpha} + \left[\tilde{A}_2\right]_{\alpha} = [a_1^{\alpha} + a_2^{\alpha}, b_1^{\alpha} + b_2^{\alpha}]$$

Definition 13.5 ([14, 16]) For any two fuzzy numbers $\tilde{A}_1$, $\tilde{A}_2$ in $\mathcal{F}^*(\mathbb{R})$, their sum yields fuzzy numbers as follows:

$$\tilde{A}_1 \oplus \tilde{A}_2 = \underset{\alpha\in(0,1]}{\cup} \alpha \left(\left[\tilde{A}_1\right]_{\alpha} + \left[\tilde{A}_2\right]_{\alpha}\right)$$

Definition 13.6 ([9]) The scalar multiplication of any $\tilde{A} \in \mathcal{F}^*(\mathbb{R})$ by a non-negative real number λ is defined by $\lambda\tilde{A} = \cup_{\alpha\in(0,1]}\alpha[\lambda\tilde{A}]_{\alpha}$ and $\lambda\tilde{A} \in \mathcal{F}^*(\mathbb{R})$, $[\lambda\tilde{A}]_{\alpha} = [\lambda a_1^{\alpha}, \lambda b_1^{\alpha}]$.

Types of Fuzzy Numbers

Zadeh [26] in 1969 defined fuzzy numbers that are approximately equal to $x \in \mathbb{R}$. In 1971, Zadeh [27] stated that the membership function $\tilde{A}$ for fuzzy numbers could be defined in different ways for example by a formula, a table and an algorithm, in terms of other membership functions available in the literature, and he considered the linguistic terms as labels for fuzzy subsets of real numbers.

In 1973, Kauffmann [8] had listed a variety of shapes for membership functions defining terms such as large and small. Also, in 1991 Kauffmann and Gupta [9] in their book have discussed other kinds of fuzzy numbers like hybrid numbers, uncertain numbers, random fuzzy numbers and fuzzy numbers of type 2. Zadeh [28] in 1975 expounded linguistic variables, and following Zadeh several mathematicians, statisticians and engineers interpreted linguistic concepts by using different membership functions either linear or non-linear and the details were recorded by Dubois and Hendri Prade [5].

Fuzzy numbers that were very often used in the literature were triangular and trapezoidal fuzzy numbers as represented in Fig. 13.1.

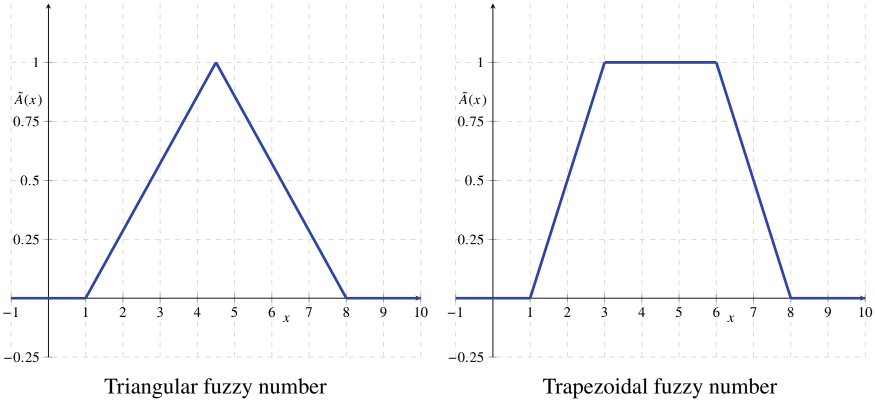

Fig. 13.1 Linear fuzzy numbers

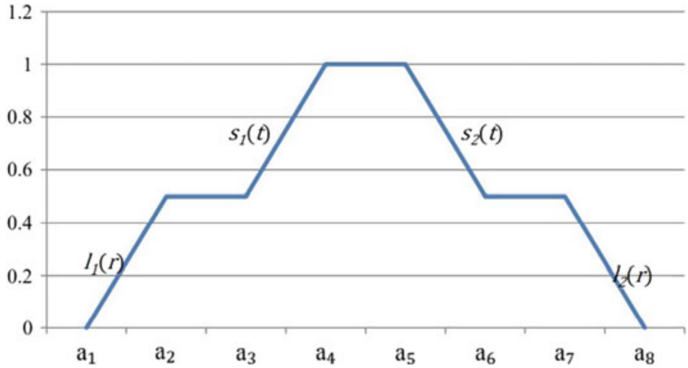

Fig. 13.2 General octagonal fuzzy number

In 2013, the concept of octagonal fuzzy number was introduced by Malini and Kennedy [12] and used some special class of octagonal fuzzy numbers for solving real-life problems [12, 23, 24].

Definition 13.7 ([12]) A general octagonal fuzzy number is defined to be a quadruple $\tilde{A} = (l_1(r), s_1(t), s_2(t), l_2(r))$, for $r \in [0, k]$ and $t \in [k, 1]$ where $l_1(r)$ is a bounded right continuous non-decreasing function over $[0, w_1]$, $0 \leq w_1 \leq k$, $s_1(t)$ is a bounded right continuous non-decreasing function over $[k, w_2]$, $k \leq w_2 \leq 1$, $s_2(t)$ is a bounded right continuous non-increasing function over $[k, w_2]$, $k \leq w_2 \leq 1$ and $l_2(r)$ is a bounded right continuous non-increasing function over $[0, w_1]$, $0 \leq w_1 \leq k$.

The Graphical representation of a general octagonal fuzzy number is given in Fig. 13.2.

Suppose $l_1(r), s_1(t), s_2(t), l_2(r)$ are linear in nature, then we have the following definition.

Definition 13.8 A fuzzy number $\tilde{A}$ is said to be a *linear octagonal fuzzy number* denoted by $(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8; k)$ where $a_1 \le a_2 \le a_3 \le a_4 \le a_5 \le a_6 \le a_7 \le a_8 \in \mathbb{R}$ with membership function $\tilde{A}(x)$ given by

$$\tilde{A}(x) = \begin{cases} k(\frac{x-a_1}{a_2-a_1}) & a_1 \le x \le a_2 \\ k & a_2 \le x \le a_3 \\ k + (1-k)(\dfrac{x-a_3}{a_4-a_3}) & a_3 \le x \le a_4 \\ 1 & a_4 \le x \le a_5 \\ k + (1-k)(\dfrac{a_6-x}{a_6-a_5}) & a_5 \le x \le a_6 \\ k & a_6 \le x \le a_7 \\ k(\dfrac{a_8-x}{a_8-a_7}) & a_7 \le x \le a_8 \\ 0 & \text{otherwise} \end{cases}$$

where $0 \le k \le 1$.

Example: A linear octagonal fuzzy number would look like (Fig. 13.3).

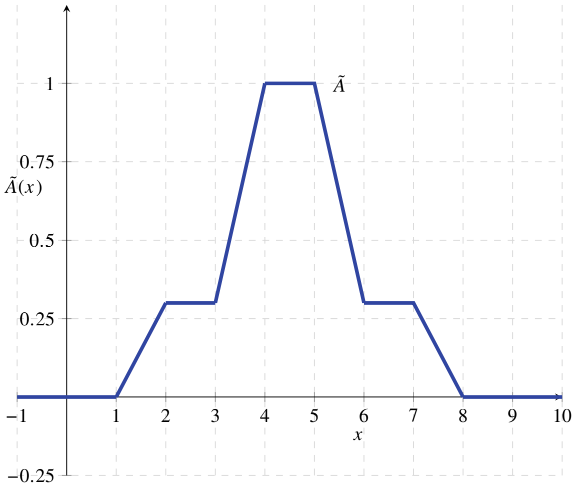

Fig. 13.3 $\tilde{A} = (1, 2, 3, 4, 5, 6, 7, 8; 0.3)$

The $\alpha-$ cut of a linear octagonal fuzzy number was computed as follows:

$$[\tilde{A}_\alpha] = \begin{cases} \left[a_1 + \frac{\alpha}{k}(a_2 - a_1), a_8 - \frac{\alpha}{k}(a_8 - a_7)\right] & \alpha \in [0, k] \\ \left[a_3 + \frac{\alpha-k}{1-k}(a_4 - a_3), a_6 - \frac{\alpha-1}{1-k}(a_6 - a_5)\right] & \alpha \in (k, 1] \end{cases}$$

Definition 13.9 ([12]) The *measure* on a linear octagonal fuzzy number $\tilde{A}$ is defined by $M^{Oct}(\tilde{A}) = \frac{1}{4}[(a_1 + a_2 + a_7 + a_8)k + (a_3 + a_4 + a_5 + a_6)(1 - k)]$.

Remark 13.2 ([12]) Any two linear octagonal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are compared using the following:

1. $\tilde{A} \preceq \tilde{B} \iff M^{Oct}(\tilde{A}) \leq M^{Oct}(\tilde{B})$
2. $\tilde{A} \approx \tilde{B} \iff M^{Oct}(\tilde{A}) = M^{Oct}(\tilde{B})$
3. $\tilde{A} \succeq \tilde{B} \iff M^{Oct}(\tilde{A}) \geq M^{Oct}(\tilde{B})$.

Remark 13.3 Linear octagonal fuzzy numbers yield better results for the choices of $k < 0.5$ ([13, 24]) in solving transportation and decision-making problems. For our study (in a decision-making problem), we consider linear octagonal fuzzy numbers with $k = 0.3$.

Definition 13.10 ([10]) Let $\tilde{A} \approx (a_1, a_2, \ldots, a_8; k, 1)$ and $\tilde{B} \approx (b_1, b_2, b_3, \ldots, b_8; k, 1)$ be two linear octagonal fuzzy numbers, then addition and scalar multiplication are defined by

1. $\tilde{A} + \tilde{B} \approx (a_1 + b_1, a_2 + b_2, \ldots, a_8 + b_8; k)$
2. $\lambda\tilde{A} \approx (\lambda a_1, \lambda a_2, \ldots, \lambda a_8; k, 1)$ for any non-negative real number λ.

Remark 13.4 It was shown in 2017 by Dhanalakshmi and Kennedy [10] that the arithmetic operations addition and scalar multiplication by a non-negative scalar on octagonal fuzzy numbers yielded the same output when calculated using α—cut- and co-ordinate-wise approach.

Remark 13.5 ([12]) Triangular fuzzy numbers and Trapezoidal fuzzy numbers can be obtained as a particular case of octagonal fuzzy numbers by considering the following:

- If $k = 0$, an octagonal fuzzy number reduces to a trapezoidal fuzzy number (a_3, a_4, a_5, a_6).
- If $k = 1$, it reduces to the trapezoidal fuzzy number (a_1, a_4, a_5, a_8).
- A degenerate form of an octagonal fuzzy number will be given by $\tilde{r} = (r, r, r, r, r, r, r, r)$.

Definition 13.11 ([15]) Let U be a universal set and E be a set of parameters. A *soft set* is defined as a mapping F from E to the set of all subsets of U, denoted by(F, E).

Definition 13.12 ([4]) A soft real set (F, E) is defined as a mapping $F : \mathbb{R} \to \mathcal{P}(\mathbb{R})$, where $\mathcal{P}(\mathbb{R})$ is the collection of bounded subsets of $\mathbb{R}$.

A soft real number denoted by (F, E) is defined as a particular soft real set which is a singleton soft real set that has been identified with the corresponding soft element.

Definition 13.13 ([22]) A real measure on soft real number (F, E) with $E = \{e_j\}_{j=1}^{l}$ denoted by $\tilde{M}(F, E)$ is defined by $\tilde{M}(F, E) = \sum_{j=1}^{l} w_j F(e_j)$, where w_j are the weights assigned to the parameters e_j such that $\sum_{j=1}^{l} w_j = 1$.

Definition 13.14 ([21]) A soft fuzzy number is defined as a mapping $\tilde{\tilde{f}} : E \to \mathcal{F}^*(\mathbb{R})$, and E the parameter set. The collection of soft fuzzy numbers is denoted as $\tilde{\mathcal{F}}^*(\mathbb{R})(E)$.

Remark 13.6 If the fuzzy number associated with the parameter set E is trapezoidal, then the corresponding soft fuzzy number will be called a soft trapezoidal fuzzy number. Similarly, if the fuzzy number associated with E is a linear octagonal fuzzy number, then the soft fuzzy number will be called a soft linear octagonal fuzzy number.

Definition 13.15 ([21]) The sum of any two soft fuzzy numbers $(\tilde{\tilde{f}}, E), (\tilde{\tilde{g}}, E) \in \tilde{\mathcal{F}}^*(\mathbb{R})(E)$ is defined by the function:

$$\tilde{\tilde{f}} \tilde{+} \tilde{\tilde{g}} : E \to \tilde{\mathcal{F}}^*(\mathbb{R})$$

where $(\tilde{\tilde{f}} \tilde{+} \tilde{\tilde{g}})(e) = \tilde{\tilde{f}}(e) \oplus \tilde{\tilde{g}}(e),\ \ e \in E$ and $\oplus$ represents the sum of two fuzzy numbers. The scalar multiplication is defined by $\lambda(\tilde{\tilde{f}}, E) = \{\lambda \tilde{\tilde{f}}(e),\ \ e \in E\}$ for any non-negative real number λ.

Definition 13.16 ([21]) Let $(\tilde{\tilde{f}}, E) \in \tilde{\mathcal{F}}^*(\mathbb{R})(E)$ with $E = \{e_j\}_{j=1}^l$. A fuzzy number valued measure on $(\tilde{\tilde{f}}, E)$ is defined by

$\tilde{\boldsymbol{M}}\left[(\tilde{\tilde{f}}, E)\right] = \oplus\sum_{j=1}^{l}\left[w_j \tilde{\tilde{f}}(e_j)\right]$ where $w_j \geq 0$ are weights of the parameters in E with $\sum_{j=1}^{l} w_j = 1$. Here, note that $\oplus\sum$ represents the sum of fuzzy numbers.

Definition 13.17 ([21]) Let $\boldsymbol{M}(\tilde{A})$ denote the defuzzified value (real numbers) of a fuzzy number $\tilde{A} \in \tilde{\mathcal{F}}^*(\mathbb{R})$ for any suitable defuzzification method. Then any two soft fuzzy numbers $(\tilde{\tilde{f}}, E), (\tilde{\tilde{g}}, E) \in \tilde{\mathcal{F}}^*(\mathbb{R}(E))$ are related by the relation '$<$' given by

$$(\tilde{\tilde{f}}, E) \tilde{\prec} (\tilde{\approx}, \tilde{\preceq} \text{or} \tilde{\succ}) (\tilde{\tilde{g}}, E) \quad \text{if} \quad \tilde{\boldsymbol{M}}[(\tilde{\tilde{f}}, E)] \prec (\approx, \preceq \text{ or } \succ) \tilde{\boldsymbol{M}}\left[(\tilde{\tilde{g}}, E)\right]$$

and

$$\boldsymbol{M}\left(\tilde{\boldsymbol{M}}[(\tilde{\tilde{f}}, E)]\right) < (=, \leq \text{ or } >) \boldsymbol{M}\left(\tilde{\boldsymbol{M}}\left[(\tilde{\tilde{g}}, E)\right]\right)$$

Note that $(\tilde{\tilde{f}}, E) \tilde{\prec} (\tilde{\approx}\ , \tilde{\preceq} \text{or } \tilde{\succ}) (\tilde{\tilde{g}}, E)$ only if
$\boldsymbol{M}\left(\tilde{\boldsymbol{M}}[(\tilde{\tilde{f}}, E)]\right) < (=, \leq \text{ or } >) \boldsymbol{M}\left(\tilde{\boldsymbol{M}}\left[(\tilde{\tilde{g}}, E)\right]\right)$.

Remark 13.7 $(\tilde{\mathcal{F}}^*(\mathbb{R}), \preceq)$ is a partially ordered set.

Note that though we have considered element $(\tilde{\tilde{f}}, E)$, we do not distinguish between two elements $(\tilde{\tilde{f}}_1, E)$, $(\tilde{\tilde{f}}_2, E)$ which have the same measure $\tilde{\boldsymbol{M}}$, rather we consider instead of individual element's $(\tilde{\tilde{f}}, E)$ equivalence classes of elements of the same measure $\tilde{\boldsymbol{M}}$ without explicit mention.

Definition 13.18 ([21]) A soft fuzzy number valued information system is a quadruple
$\widetilde{IS} = (U, A, \tilde{\mathcal{F}}^*(\mathbb{R})(\mathcal{E}), \tilde{\tilde{I}})$ where
$U = \{u_i\}_{i=1}^m$ is the set of objects under consideration,
$A = \{a_j\}_{j=1}^n$ is the attribute set,
$\mathcal{E} = \{E_1, E_2, ..., E_n\}$ and $E_j = \{e_{jk}\}_{k=1}^{l_j}$ is the parameter set associated with attribute a_j, l_j representing the number of parameters in E_j and
if $\tilde{\tilde{I}} : U \times A \to \tilde{\mathcal{F}}^*(\mathbb{R})(\mathcal{E})$ is a mapping such that $\tilde{\tilde{I}}(u_i, a_j) = (\tilde{\tilde{f}}_{ij}, E_j) \in \tilde{\mathcal{F}}^*(\mathbb{R})(\mathcal{E})$ for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$ where $(\tilde{\tilde{f}}_{ij}, E_j)$ is a soft fuzzy number.

Definition 13.19 Let $\tilde{U}$ be a finite universal set with members being fuzzy in nature and representing some characteristic feature. A soft fuzzy number valued fuzzy information system is a quadruple $\widetilde{FIS} = (\tilde{U}, A, \tilde{\mathcal{F}}^*(\mathbb{R})(\mathcal{E}), \tilde{\tilde{I}})$, where $\tilde{U}$ is a fuzzy set on U, $\tilde{\tilde{I}} : \tilde{U} \times A \to \tilde{\mathcal{F}}^*(\mathbb{R})(\mathcal{E})$.

Catalog Data:
The various facts about Light-Emitting Diode (LED) bulbs given by the manufacturers are collected on a random purchase of one bulb of 7 W from 7 companies. The information available from the catalog for most of the bulb brands showed similar characteristic properties. The CRI values (>80) and the standard voltage (220 V) specification were the same for all the bulb brands. The lumen value is 700 for 5 bulb brands and 665 and 630 for the other two, respectively. The bulbs had a few variations in the other two features' cost (ranging from Rs. 129, Rs. 130, Rs. 140, Rs. 160, Rs. 199) and warranty (1 year and 2 years). The information is tabulated in Table 13.1.

Table 13.1 Catalog information

Bulb brands	Lumen	CRI	Voltage	Cost	Warranty
1	665	>80	220	130	2
2	700	>80	220	140	1
3	700	>80	220	199	1
4	630	>80	220	140	1
5	700	>80	220	160	1
6	700	>80	220	129	2
7	700	>80	220	160	2

13.3 Problem of Choosing a LED Bulb Based on Various Characteristic Properties for Different Usages

The need for selecting low-wattage, highly efficient bulbs for household usage or beautification at low cost and reduced power consumption is fulfilled by LED bulbs. We consider the problem of measuring the efficiency level of the bulbs and choose the optimum bulb for household usage and beautification. We analyze the bulbs based on various characteristic properties to choose the appropriate bulb for different purposes.

The problem is to

1. build a suitable model using the information obtained from the manufacturers' catalogs and experimental study;
2. develop and follow procedure according to the nature of the model;
3. select a suitable low-wattage LED bulb that is most efficient in terms of the attributes such as high Color Rendering Index (CRI), better light output, low cost, better replacement warranty and low power consumption for household and commercial applications.

Characteristic Properties of LED bulbs: To find the efficiency (with respect to quality and quantity of light output, power consumption) of the bulbs and make appropriate decisions for longer life with less cost, we have considered a few characteristic properties of the LED bulbs. Cost and warranty are taken from the catalog and a couple of experiments are conducted to study CRI, Light Output and 'Voltage and Current characteristics'.

1. **CRI:** It is a relative measure of how an object looks under a light source when compared with sunlight at noon. The index is measured from 0–100, with a perfect 100 indicating that colors under the light source appear the same as they would under natural sunlight. The higher the CRI rating is, the better is its color rendering ability.
2. **Light Output (lumen):** It describes the brightness of the light.
3. **Voltage:** In India, 220 V is considered as the standard voltage for electrical appliances though there might be low voltages due to natural (lightning, power switching on the lines, household appliances drawing too much power in either same building or neighbor's building, strong winds causing lines to clash, trees touching the line, vehicle accidents involving powerlines, or birds or other animals on the lines) or artificial (loose or corroded connections either at the building or on the powerlines, overloading on the network or too thin a conductor wire) situations.

Remark 13.8 The study at lower voltage is necessary and the same is taken care of later in Sect. 13.5 using more parameters in modeling the experimental outputs.

1. **Current:** Electric current is the flow of electric charge. The SI unit of electric current is ampere (A), which is equal to a flow of one coulomb of charge per second.

The information corresponding to some characteristic features is gathered from the manufacturers' catalog and experimental results.

Catalog information:

The information (see Table 13.1) obtained for all bulbs corresponding to the characteristic features are normalized to the same scale by dividing each value by the maximum value (700 for lumen, 2 for warranty), and for cost the minimum value (129) is divided by corresponding cost value of each bulb. The computed values are shown in Table 13.2.

Table 13.2 Catalog information—normalized values

Bulbs	Lumen	CRI	Voltage	Cost	Warranty
1	0.950	1	1	0.992	1
2	1	1	1	0.921	0.5
3	1	1	1	0.648	0.5
4	0.9	1	1	0.921	0.5
5	1	1	1	0.806	0.5
6	1	1	1	1	1
7	1	1	1	0.806	1

13.4 Experimental Studies

One LED bulb from each brand is chosen at random, and experiments are conducted under rigorous conditions.

Experiment 1: Each bulb was mounted on a lamp housing with a hole to let out light for detection. The spectral light output (corrected) was measured using a UV-Visible mini-spectrometer (USB 4000-OceanOptics). The distance between the light bulb and the mini-spectrometer was maintained constant (93 cm). The voltage applied to the bulb was varied from 60 to 240 V, and the spectral light output was recorded along with the CRI values. From the obtained spectra, the area under each spectrum was computed (see Table 13.3) and the integrated light output (relative intensity) with voltage is shown in Fig. 13.4.

Also fixing the voltage at 220 V, the spectral light output is measured for each bulb and is shown in Fig. 13.5.

From the spectral distribution, the following facts are obtained:

- All the bulbs use the same principle, i.e., they employ phosphor-converted white LEDs (pcWLEDs).

Table 13.3 Area under the Spectral output for varying voltages

Bulbs	60 V	80 V	100 V	120 V	140 V	160 V	180 V
B_1	1304.34	3908.4	6714.14	9565.78	12222.26	13030.36	12931.16
B_2	808.7	2294.61	3877.11	5590.93	7192.69	7818.10	7780.66
B_3	977.09	2653.29	4621.48	6772.42	8893.4	10366.03	10384.77
B_4	1726.36	3639.71	5846.49	8075.05	9790.48	9815.69	9772.67
B_5	634.68	1883.1	3064.35	4215.08	5317.97	6387.24	7377.86
B_6	2056.27	4461.42	7109.35	9644.16	10465.56	10401.09	10082.23
B_7	1335.09	3629.79	5993.89	8571.6	10937.16	11629.71	11586.24
Bulbs	190 V	200 V	210 V	220 V	230 V	240 V	
B_1	12870.78	12812.40	12739.9	12798.93	12665.26	12617.99	
B_2	7716.39	7683.44	7638.6	7612.63	7580.7	7500.7	
B_3	10342	10305.43	10262.12	10229.46	10197.48	10153.78	
B_4	9711.26	9676.85	9655.44	9632.36	9622.65	9587.99	
B_5	7497.21	7470.7	7446.17	7419.47	7381.59	7366.02	
B_6	10354.38	10273.72	10236.74	10192.27	10142.81	10120.07	
B_7	11529.09	11471.63	11400.64	11352.31	11306.57	11266.31	

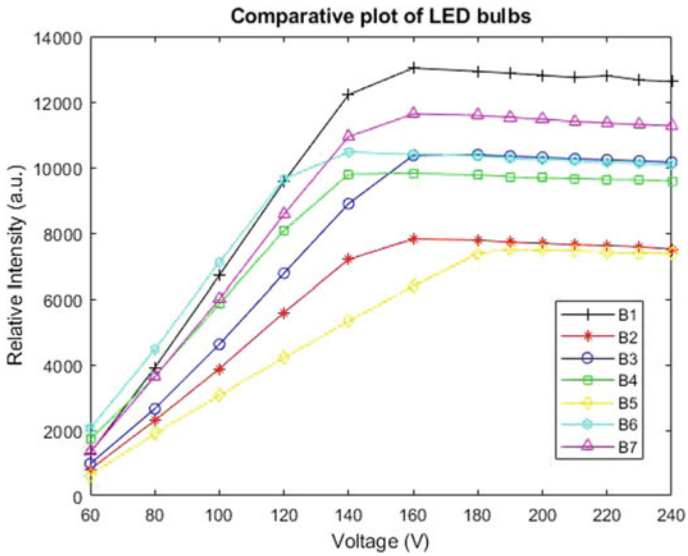

Fig. 13.4 Relative intensity with respect to applied voltage

- They use Indium gallium nitride (InGaN) blue LED chip to render blue light. Part of the blue light is used to excite the phosphor to yield yellow light and the rest comes out.

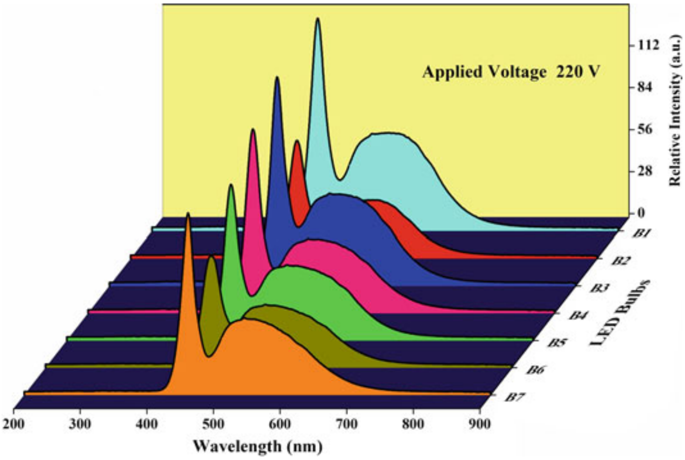

Fig. 13.5 Spectral light output with respect to 220 voltage

Table 13.4 Relative intensity and its normalized values at 220 applied voltage

Bulbs	B_1	B_2	B_3	B_4	B_5	B_6	B_7
Area	13516.15	7774.25	10730.06	10329.18	8600.51	10485.14	12618.56
Normalized area	1	0.575	0.794	0.794	0.636	0.776	0.930

- The phosphor used is YAG:Ce (Yttrium Aluminum Garnet doped with cerium, $Y_3Al_5O_12$:Ce), and this absorbs blue light and emits yellow light.
- The blue and the yellow light combined give near-white light.
- The variation in the light output from one bulb to another can be attributed to different origins of the starting materials and the method of synthesis.

The area under each spectrum is given in the second row of Table 13.4, and the normalized values are computed by dividing the relative intensity corresponding to each bulb by the maximum relative intensity (13516.15 obtained for the bulb B_1) and tabulated in the last row of Table 13.4.

Experiment 2: Using a milli-ammeter, the current was measured (see Table 13.5) while varying the voltage for all the bulbs and the data are shown in Fig. 13.6.

We note that at 220 V, the bulb B_4 utilizes minimum current compared to the other bulbs. The normalized values are computed by dividing the minimum value by the corresponding current values of bulbs and are given in Table 13.6.

Using the data from Table 13.5, at 220 V the actual power consumption (W) of the bulbs is computed and tabulated in Table 13.7 along with the respective CRI values.

The normalized values are computed by dividing the optimum value by the corresponding value of the bulb and are recorded in Table 13.8.

Table 13.5 Current (mA)

Bulbs	60 V	80 V	100 V	120 V	140 V	160 V	180 V	190 V	200 V	210 V	220 V	230 V	240 V
B_1	8.7	18.8	27.2	35.5	44.1	42	38	36.3	34.7	33.3	32	31	30
B_2	9.2	18.9	27.4	35.3	42.2	42.4	37.9	36.2	34.5	33.1	31.8	30.7	29.6
B_3	8.6	16.4	23.5	30.5	37	39.9	35.9	34.2	32.7	31.3	30.1	29	28
B_4	12.5	20.9	29.2	36.7	41.5	37.1	33.4	31.7	30.3	29.2	28	27	26.1
B_5	7.8	15.7	21.6	26.3	30.2	33.7	36.6	36.2	34.6	33.4	32.2	31.2	30.2
B_6	14.4	25	35	43.9	43.3	38.2	34.4	32.8	31.3	30.1	28.9	27.9	27
B_7	8.5	18.1	26.1	34	41.2	40.1	35.9	34.3	32.7	31.2	29.8	28.7	27.6

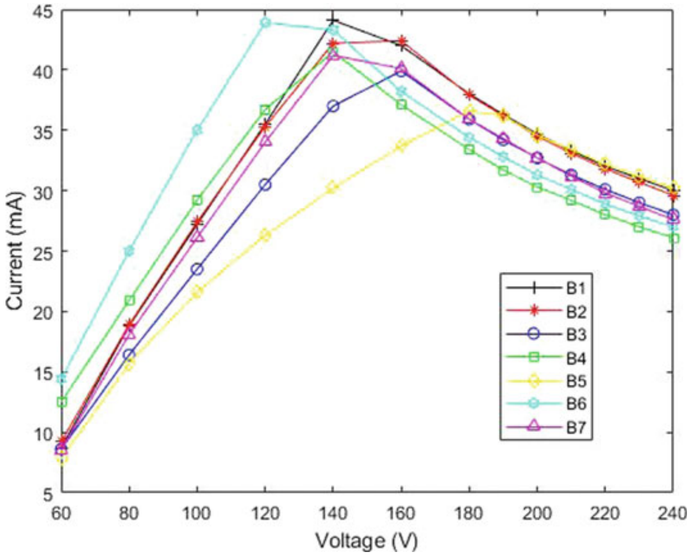

Fig. 13.6 The variation of current with voltage in the LED bulbs

Table 13.6 Normalized current at 220 applied voltage

Bulbs	B_1	B_2	B_3	B_4	B_5	B_6	B_7
Normalized current	0.875	0.881	0.930	1.000	0.870	0.969	0.940

Table 13.7 Wattage and CRI at 220 applied voltage

Bulbs	B_1	B_2	B_3	B_4	B_5	B_6	B_7
Wattage	7.040	6.996	6.622	6.160	7.084	6.358	6.556
CRI	74.44	73.48	80.4	78.46	79.17	76.29	75.97

Table 13.8 Normalized Wattage and CRI at 220 applied voltage

Bulbs	B_1	B_2	B_3	B_4	B_5	B_6	B_7
Normalized wattage	0.875	0.881	0.930	1.000	0.870	0.969	0.940
Normalized CRI	0.926	0.914	1.000	0.975	0.985	0.949	0.945

Remark 13.9 In the experimental study, it was found that the wattage was not the same for all the LED bulbs (refer to Table 13.7). This bore an effect on the relative intensity. Hence, there is a need for computing the relative intensity per watt, and the problem is then formulated mathematically.

Remark 13.10 All the normalized values are computed using R-script code (see Appendix I).

13.5 Mathematical Modeling of the Experimental Studies and Catalog Information

In this section, the results obtained from the experiments and information gathered from the catalog are formulated mathematically.

Nature of the Bulbs:
In view of Remark 13.9, the universal set U consisting of 7 W bulbs is now considered as a collection of *around 7 W bulbs* $\tilde{U}$ represented by $\tilde{U} = \frac{B_1}{0.875} + \frac{B_2}{0.881} + \frac{B_3}{0.930} + \frac{B_4}{1.000} + \frac{B_5}{0.870} + \frac{B_6}{0.969} + \frac{B_7}{0.940}$ (the normalized wattage values, from Table 13.8).

Experimental outcomes at 220 V:
The characteristic properties of the bulbs at 220 V such as light output, CRI and current are considered as the attributes a_1, a_2 and a_3 respectively, and their experimental observations (see Sect. 13.4) are described using linguistic states.

The bulbs are categorized as Best Light Output (BLO), Better Light Output (BTLO), Good Light Output (GLO), Very Fair Light Output (VFLO), Fair Light Output (FLO), Unfair Light Output (UFLO), Poor Light Output (PLO), Very Poor Light Output (VPLO) and Worst Light Output (WLO) depending on their brightness (Pictorial representation, Fig. 13.7).

The whiteness of the light (with respect to sunlight at noon) emitted by the bulbs is evaluated using linguistic terms such as perfectly near to white light (PNW), very much near to white light (VNW), near to white light (NW), slightly near to white light (SNW), neutral (NE), slightly far from white light (SFW), far from white light (FW), not near to white light (NNW) and very much away from white light (VAW) corresponding to the attribute CRI (pictorial representation, Fig. 13.7).

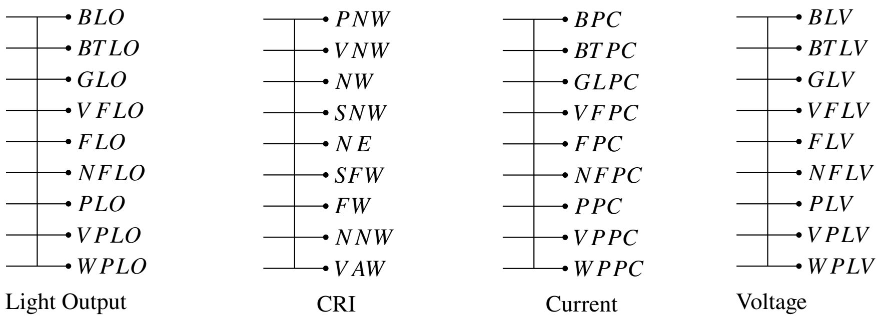

Fig. 13.7 Pictorial representation of the linguistic states (not to scale)

The capability of the bulbs in consuming less power is evaluated subjectively using linguistic states such as best in low power consumption (BPC), good in low power consumption (GPC), very fair in low power consumption (VFPC), fair in low power consumption (FPC), unfair in low power consumption (UFPC), poor in low power consumption (PPC), very poor in low power consumption (VPPC) and worst in low power consumption (WPC) (pictorial representation, Fig. 13.7).

Based on the normalized values for the light output, CRI and power consumption at 220 V with respect to corresponding values (relative), the bulbs with a light output less than 50 percent/CRI less than 65/power consumption of more than 9 W may not be preferred. Therefore, based on the problem we require to use only linguistic states listed in Table 13.9 which are quantified using linear octagonal fuzzy numbers whose membership function is defined on $[0, 1] \in \mathbb{R}$.

The diagrammatic representation of the linguistic values corresponding to the light output, CRI and current are shown in Fig. 13.8.

The subjective evaluations of the 7 bulbs (with assigned weights to the attributes) are shown in Table 13.10.

For the attributes a_j, $j = 1, 2, 3$ with the associated parameter sets $E_j = \{e_{j,1}\}$ where $e_{1,1}$ = Relative intensity at 220 V; $e_{2,1} = a_2$; $e_{3,1} = a_3$, the evaluations are framed as soft linear octagonal fuzzy numbers $(\tilde{\tilde{f}}_{ij}, E_j) = \left\{ \tilde{\tilde{f}}_{ij}(e_{j,1}) \right\}, i = 1, \ldots, 7,$ with

$\tilde{\tilde{f}}_{11}(e_{1,1}) = (0.970, 0.980, 0.990, 0.990, 1.000, 1.000, 1.000, 1.000; 0.3);$

$\tilde{\tilde{f}}_{21}(e_{1,1}) = (0.500, 0.500, 0.500, 0.500, 0.630, 0.640, 0.660, 0.680; 0.3);$

$\tilde{\tilde{f}}_{31}(e_{1,1}) = (0.770, 0.780, 0.800, 0.820, 0.840, 0.850, 0.870, 0.890; 0.3);$

$\tilde{\tilde{f}}_{41}(e_{1,1}) = (0.700, 0.710, 0.730, 0.750, 0.770, 0.780, 0.800, 0.820; 0.3);$

$\tilde{\tilde{f}}_{51}(e_{1,1}) = (0.630, 0.640, 0.660, 0.680, 0.700, 0.710, 0.730, 0.750; 0.3);$

$\tilde{\tilde{f}}_{61}(e_{1,1}) = (0.770, 0.780, 0.800, 0.820, 0.840, 0.850, 0.870, 0.890; 0.3);$

$\tilde{\tilde{f}}_{71}(e_{1,1}) = (0.840, 0.850, 0.870, 0.890, 0.910, 0.920, 0.940, 0.950; 0.3);$

$\tilde{\tilde{f}}_{12}(e_{2,1}) = (0.915, 0.920, 0.925, 0.930, 0.935, 0.940, 0.945, 0.950; 0.3);$

Table 13.9 Fuzzy numbers corresponding to linguistic terms

Linguistic states	Octagonal fuzzy number
BLO, BLV	(0.970, 0.980, 0.990, 0.990, 1.000, 1.000, 1.000, 1.000;0.3)
BTLO, BTLVT	(0.910, 0.920, 0.940, 0.950, 0.970, 0.980, 0.990, 1.000;0.3)
GLO, GLV	(0.840, 0.850, 0.870, 0.890, 0.910, 0.920, 0.940, 0.950;0.3)
VFLO, VFLV	(0.770, 0.780, 0.800, 0.820, 0.840, 0.850, 0.870, 0.890;0.3)
FLO, FLV	(0.700, 0.710, 0.730, 0.750, 0.770, 0.780, 0.800, 0.820;0.3)
UFLO, UFLV	(0.630, 0.640, 0.660, 0.680, 0.700, 0.710, 0.730, 0.750;0.3)
PLO, PLV	(0.540, 0.560, 0.580, 0.600, 0.620, 0.640, 0.660, 0.680;0.3)
VPLO, VPLV	(0.460, 0.480, 0.500, 0.520, 0.540, 0.560, 0.580, 0.600;0.3)
WLO, WLV	(0.440, 0.440, 0.440, 0.440, 0.460, 0.480, 0.500, 0.520;0.3)
PNW	(0.975, 0.98, 0.985, 0.990, 1, 1, 1, 1;0.3)
VNW	(0.955, 0.960, 0.965, 0.970, 0.975, 0.980, 0.985, 0.990;0.3)
NW	(0.935, 0.940, 0.945, 0.950, 0.955, 0.960, 0.965, 0.970;0.3)
SNW	(0.915, 0.920, 0.925, 0.930, 0.935, 0.940, 0.945, 0.950;0.3)
NE	(0.895, 0.900, 0.905, 0.910, 0.915, 0.920, 0.925, 0.93;0.3)
SFW	(0.875, 0.880, 0.885, 0.890, 0.895, 0.90, 0.905, 0.910;0.3)
FW	(0.855, 0.860, 0.865, 0.870, 0.875, 0.880, 0.885, 0.890;0.3)
NNW	(0.835, 0.840, 0.845, 0.850, 0.855, 0.860, 0.865, 0.870;0.3)
VAW	(0.830, 0.830, 0.830, 0.830, 0.835, 0.840, 0.845, 0.850;0.3)
BPC	(0.970, 0.975, 0.980, 0.990, 1.000, 1.000, 1.000, 1.000;0.3)
BTPC	(0.930, 0.940, 0.950, 0.960, 0.970, 0.975, 0.980, 0.990;0.3)
GPC	(0.890, 0.900, 0.910, 0.920, 0.930, 0.940, 0.950, 0.960;0.3)
VFPC	(0.850, 0.860, 0.870, 0.880, 0.890, 0.90, 0.910, 0.920;0.3)
FPC	(0.810, 0.820, 0.830, 0.840, 0.850, 0.860, 0.870, 0.880;0.3)
UFPC	(0.770, 0.780, 0.790, 0.800, 0.810, 0.820, 0.830, 0.840;0.3)
PPC	(0.730, 0.740, 0.750, 0.760, 0.770, 0.780, 0.790, 0.800;0.3)
VPPC	(0.690, 0.700, 0.710, 0.720, 0.730, 0.740, 0.750, 0.760;0.3)
WPC	(0.680, 0.680, 0.680, 0.680, 0.690, 0.700, 0.710, 0.720;0.3)

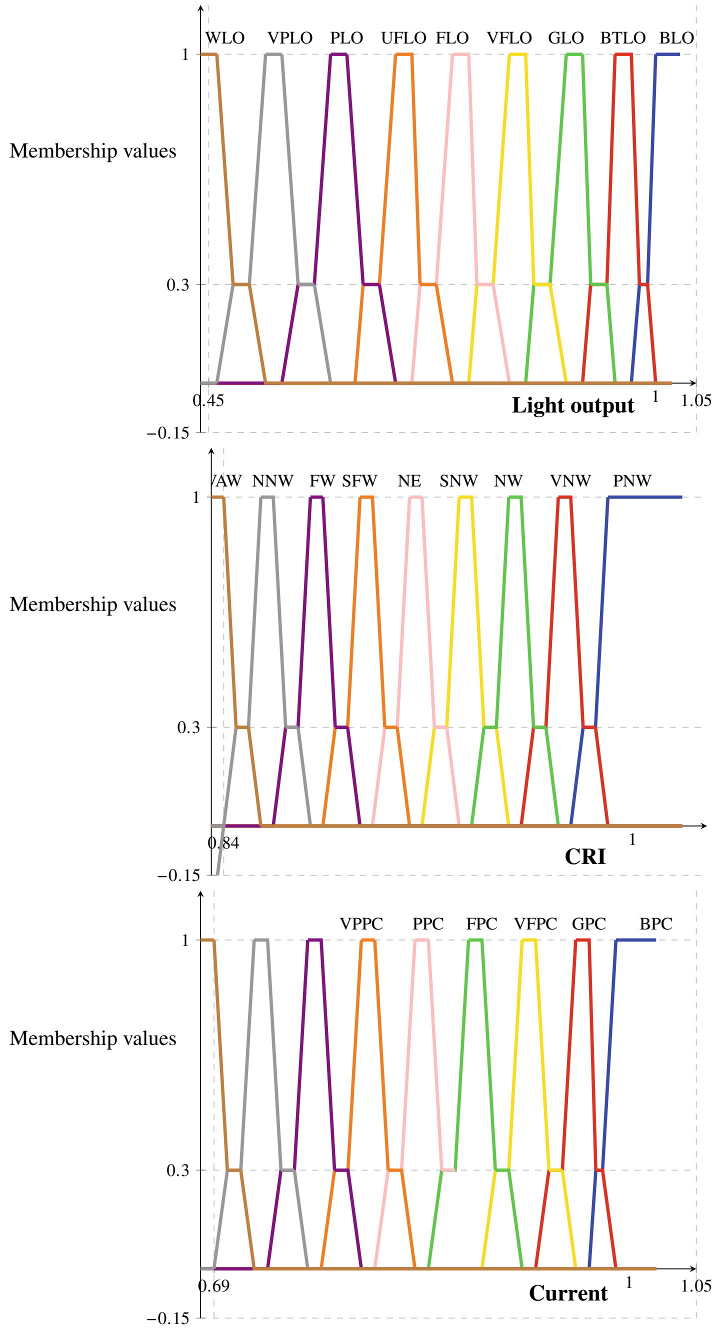

Fig. 13.8 Linear octagonal fuzzy numbers representing the linguistic states

Table 13.10 Linguistic evaluation—experimental outputs at 220 V

$\tilde{U}$	$a_1(0.3)$	$a_2(0.05)$	$a_3(0.2)$
	Relative intensity at 220 V		
$(B_1, 0.875)$	BLO	NE	FPC
$(B_2, 0.881)$	PLO	SFW	FPC
$(B_3, 0.930)$	VFLO	PNW	VFPC
$(B_4, 1.000)$	FLO	VNW	BPC
$(B_5, 0.870)$	UFLO	PNW	FPC
$(B_6, 0.969)$	VFLO	NW	GPC
$(B_7, 0.940)$	GLO	NW	VFPC

$\tilde{\tilde{f}}_{22}(e_{2,1}) = (0.895, 0.900, 0.905, 0.910, 0.915, 0.920, 0.925, 0.930; 0.3);$
$\tilde{\tilde{f}}_{32}(e_{2,1}) = (0.975, 0.980, 0.985, 0.990, 1.000, 1.000, 1.000, 1.000; 0.3);$
$\tilde{\tilde{f}}_{42}(e_{2,1}) = (0.955, 0.960, 0.965, 0.970, 0.975, 0.980, 0.985, 0.990; 0.3);$
$\tilde{\tilde{f}}_{52}(e_{2,1}) = (0.975, 0.980, 0.985, 0.990, 1.000, 1.000, 1.000, 1.000; 0.3);$
$\tilde{\tilde{f}}_{62}(e_{2,1}) = (0.935, 0.940, 0.945, 0.950, 0.955, 0.960, 0.965, 0.970; 0.3);$
$\tilde{\tilde{f}}_{72}(e_{2,1}) = (0.935, 0.940, 0.945, 0.950, 0.955, 0.960, 0.965, 0.970; 0.3);$
$\tilde{\tilde{f}}_{13}(e_{3,1}) = (0.850, 0.860, 0.870, 0.880, 0.890, 0.900, 0.910, 0.920; 0.3);$
$\tilde{\tilde{f}}_{23}(e_{3,1}) = (0.850, 0.860, 0.870, 0.880, 0.890, 0.900, 0.910, 0.920; 0.3);$
$\tilde{\tilde{f}}_{33}(e_{3,1}) = (0.890, 0.900, 0.910, 0.920, 0.930, 0.940, 0.950, 0.960; 0.3);$
$\tilde{\tilde{f}}_{43}(e_{3,1}) = (0.970, 0.975, 0.980, 0.990, 1.000, 1.000, 1.000, 1.000; 0.3);$
$\tilde{\tilde{f}}_{53}(e_{3,1}) = (0.850, 0.860, 0.870, 0.880, 0.890, 0.900, 0.910, 0.920; 0.3);$
$\tilde{\tilde{f}}_{63}(e_{3,1}) = (0.930, 0.940, 0.950, 0.960, 0.970, 0.975, 0.980, 0.990; 0.3);$
$\tilde{\tilde{f}}_{73}(e_{3,1}) = (0.890, 0.900, 0.910, 0.920, 0.930, 0.940, 0.950, 0.960; 0.3).$

Here, the universal set considered is fuzzy in nature (with respect to wattage). The evaluation of some attributes (light intensity) is influenced by this nature. We model the current scenario as a soft fuzzy number valued fuzzy information system (see Definition 13.19).

To measure the efficiency of the various bulbs of different brands, the set $\{a_1, a_2, a_3\}$ of attributes from experimental results at 220 V, together with cost (a_4) and warranty (a_5) from Table 13.2 (the normalized values), are considered and modeled as a soft linear octagonal fuzzy number valued fuzzy information system, $\widetilde{FIS} = (\tilde{U}, A, \mathcal{F}^*(\mathbb{R})(\mathcal{E}), \tilde{\tilde{I}})$, where
$A = \{a_1, a_2, a_3, a_4, a_5\}$, $\mathcal{E} = \{E_1, E_2, E_3, E_4, E_5\}$ the associated parameter sets $E_j = \{e_{j,1}\}$ with $e_{j,1} = a_j$, $j = 4, 5$, $\tilde{\tilde{I}} : \tilde{U} \times B \to \mathcal{F}^*(\mathbb{R})(\mathcal{E})$ such that

Table 13.11 $\widetilde{FIS}$: experimental outcome at 220 V

$\tilde{U}$	A				
	a_1	a_2	a_3	a_4	a_5
$(B_1, 0.875)$	$(\tilde{\tilde{f}}_{11}, E_1)$	$(\tilde{\tilde{f}}_{12}, E_2)$	$(\tilde{\tilde{f}}_{13}, E_3)$	$(\tilde{\tilde{f}}_{14}, E_4)$	$(\tilde{\tilde{f}}_{15}, E_5)$
$(B_2, 0.881)$	$(\tilde{\tilde{f}}_{21}, E_1)$	$(\tilde{\tilde{f}}_{22}, E_2)$	$(\tilde{\tilde{f}}_{23}, E_3)$	$(\tilde{\tilde{f}}_{24}, E_4)$	$(\tilde{\tilde{f}}_{25}, E_5)$
$(B_3, 0.930)$	$(\tilde{\tilde{f}}_{31}, E_1)$	$(\tilde{\tilde{f}}_{32}, E_2)$	$(\tilde{\tilde{f}}_{33}, E_3)$	$(\tilde{\tilde{f}}_{34}, E_4)$	$(\tilde{\tilde{f}}_{35}, E_5)$
$(B_4, 1.000)$	$(\tilde{\tilde{f}}_{41}, E_1)$	$(\tilde{\tilde{f}}_{42}, E_2)$	$(\tilde{\tilde{f}}_{43}, E_3)$	$(\tilde{\tilde{f}}_{44}, E_4)$	$(\tilde{\tilde{f}}_{45}, E_5)$
$(B_5, 0.870)$	$(\tilde{\tilde{f}}_{51}, E_1)$	$(\tilde{\tilde{f}}_{52}, E_2)$	$(\tilde{\tilde{f}}_{53}, E_3)$	$(\tilde{\tilde{f}}_{54}, E_4)$	$(\tilde{\tilde{f}}_{55}, E_5)$
$(B_6, 0.969)$	$(\tilde{\tilde{f}}_{61}, E_1)$	$(\tilde{\tilde{f}}_{62}, E_2)$	$(\tilde{\tilde{f}}_{63}, E_3)$	$(\tilde{\tilde{f}}_{64}, E_4)$	$(\tilde{\tilde{f}}_{65}, E_5)$
$(B_7, 0.940)$	$(\tilde{\tilde{f}}_{71}, E_1)$	$(\tilde{\tilde{f}}_{72}, E_2)$	$(\tilde{\tilde{f}}_{73}, E_3)$	$(\tilde{\tilde{f}}_{74}, E_4)$	$(\tilde{\tilde{f}}_{75}, E_5)$

$$\tilde{\tilde{I}}(\tilde{U}(B_i), a_j) = \left\{ (\tilde{\tilde{f}}_{ij}, E_j), \; i = 1, \ldots, 7, \; j = 1, \ldots, 5 \right.$$

and soft linear octagonal fuzzy numbers describing the cost and warranty for the 7 bulbs are given by $(\tilde{\tilde{f}}_{ij}, E_j) = \left\{ \tilde{\tilde{f}}_{ij}(e_{j,1}) \right\}$ for $j = 4, 5$ and $i = 1, \ldots, 7$ where
$\tilde{\tilde{f}}_{14}(e_{4,1}) = \widetilde{0.992}$; $\tilde{\tilde{f}}_{24}(e_{4,1}) = \widetilde{0.921}$; $\tilde{\tilde{f}}_{34}(e_{4,1}) = \widetilde{0.648}$; $\tilde{\tilde{f}}_{44}(e_{4,1}) = \widetilde{0.921}$; $\tilde{\tilde{f}}_{54}(e_{4,1}) = \widetilde{0.806}$; $\tilde{\tilde{f}}_{64}(e_{4,1}) = \widetilde{1.000}$; $\tilde{\tilde{f}}_{74}(e_{4,1}) = \widetilde{0.806}$; $\tilde{\tilde{f}}_{15}(e_{5,1}) = \widetilde{1.000}$; $\tilde{\tilde{f}}_{25}(e_{5,1}) = \widetilde{0.500}$; $\tilde{\tilde{f}}_{35}(e_{5,1}) = \widetilde{0.500}$; $\tilde{\tilde{f}}_{45}(e_{5,1}) = \widetilde{0.500}$; $\tilde{\tilde{f}}_{55}(e_{5,1}) = \widetilde{0.500}$; $\tilde{\tilde{f}}_{65}(e_{5,1}) = \widetilde{1.000}$; $\tilde{\tilde{f}}_{75}(e_{5,1}) = \widetilde{1.000}$;
and tabulating these, we have Table 13.11.

In our problem, we notice that the attribute (a_1 = light output) is fuzzy in nature due to the Wattage variation appearing in $\tilde{U}$. Then the required $\widetilde{FIS}$ is given by $\widetilde{FIS} = (\tilde{U}, B, \mathcal{F}^*(\mathbb{R})(\mathcal{E}), \tilde{\tilde{I}}))$ where $\tilde{U}$ = the bulbs with wattage around 7, $B = (a_1 \cup (A \setminus \{a_1\})$ $\tilde{\tilde{I}} : \tilde{U} \times B \to \mathcal{F}^*(\mathbb{R})(\mathcal{E})$ satisfying

$$\tilde{\tilde{I}}(\tilde{U}(B_i), b) = \begin{cases} \tilde{\tilde{I}}(\tilde{U}(B_i), b)_{\frac{1}{\tilde{U}(B_i)}}, & b = a_1 \\ \tilde{\tilde{I}}(\tilde{U}(B_i), b), & b \neq a_1 \end{cases}$$

Experimental outcomes at low voltage involving more parameters:
At 140 V all the bulbs give nearly 80 percentage of their maximum light output. We infer from Fig. 13.4 that the relative intensity values of various bulbs are not the same at different voltage. This information is incorporated by characterizing the light output using more parameters

$e_{1,2}$ = Total light output between 140 V to 240 V
$e_{1,3}$ = maximum light output per power consumption

and considering a new attribute a_6 = 'Voltage' with associated parameter set $E_6 = \{e_{6,1}\}$ where $e_{6,1}$ = At lower voltage which bulb gives equal light output with respect tothe bulb which yields the least output at high voltage

Table 13.12 Normalized total relative intensity

Bulbs	B_1	B_2	B_3	B_4	B_5	B_6	B_7
Normalized total area	1	0.598	0.798	0.759	0.561	0.802	0.893

Table 13.13 Voltage—Maximum light output

Bulbs	B_1	B_2	B_3	B_4	B_5	B_6	B_7
Voltage	160	160	180	160	190	140	160
Current	42	42.4	35.9	37.1	30.2	43.3	40.1
Power consumption	6.720	6.784	6.462	5.936	5.738	6.062	6.416

Remark 13.11 Using the information based on parameter $e_{6,1}$, during low voltage situations, the bulb that gives better light output can be identified (see Remark 13.8).

Evaluation of light output over a range of voltage:
The total relative intensity is computed for the observations considered from Fig. 13.4 by finding the area under the curve (between 140 and 220 V) using MATLAB 2016a function '*trapz*' and is given below

```
The area of the bulbs between 140 V and 240 V are:
bulbs           Area
B_1                     1.0227e+05
B_2           61177
B_3           81611
B_4           77576
B_5           57322
B_6           81995
B_7           91378
```

The maximum total relative intensity is attained by the bulb B_1. The normalized total relative intensity is computed by dividing the maximum value by the total relative intensity corresponding to each bulb and is given in Table 13.12.

From Table 13.3, we note that the maximum light output is attained by the bulbs at different voltages. The power consumption of the bulb at their respective voltage is computed as shown in Table 13.13.

The maximum yield of light output per watt and the normalized values are computed as given in Table 13.14.

Based on these values, the bulbs are subjectively evaluated using the linguistic terms as given in Table 13.15.

The attribute a_1 = light output of the 7 bulbs with respect to standard voltage and at low voltage are described by representing it as the soft linear octagonal fuzzy numbers $(\tilde{\tilde{f}}^*_{i1}, E^*_1) = \left\{\tilde{\tilde{f}}^*_{i1}(e_{1,j})\right\}$ with parameter set $E^*_1 = \{e_{1,j}\}$, $j = 1, 2, 3$ where

Table 13.14 Maximum light output—low voltage

Bulbs	B_1	B_2	B_3	B_4	B_5	B_6	B_7
Maximum Light intensity	13030.36	7818.10	10384.77	9815.69	7497.21	10465.09	11629.71
Maximum Light intensity/watt	1939.042	71152.432	1607.0527	1653.587	1306.589	1726.343	1812.611
Normalized value	1	0.594	0.829	0.853	0.674	0.890	0.935

Table 13.15 Subjective evaluation—more parameters

$\tilde{U}$	$e_{1,2}$	$e_{1,3}$
$(B_1, 0.875)$	BLO	BLO
$(B_2, 0.881)$	PLO	PLO
$(B_3, 0.930)$	VFLO	VFLO
$(B_4, 1.000)$	FLO	VFLO
$(B_5, 0.870)$	PLO	PLO
$(B_6, 0.969)$	VFLO	GLO
$(B_7, 0.940)$	GLO	GLO

$$\tilde{\tilde{f}}_{11}^*(e_{1,1}) = (0.970, 0.980, 0.990, 0.990, 1.000, 1.000, 1.000, 1.000; 0.3);$$
$$\tilde{\tilde{f}}_{11}^*(e_{1,2}) = (0.970, 0.980, 0.990, 0.990, 1.000, 1.000, 1.000, 1.000; 0.3);$$
$$\tilde{\tilde{f}}_{11}^*(e_{1,3}) = (0.970, 0.980, 0.990, 0.990, 1.000, 1.000, 1.000, 1.000; 0.3);$$
$$\tilde{\tilde{f}}_{21}^*(e_{1,1}) = (0.500, 0.500, 0.500, 0.500, 0.630, 0.640, 0.660, 0.680; 0.3);$$
$$\tilde{\tilde{f}}_{21}^*(e_{1,2}) = (0.500, 0.500, 0.500, 0.500, 0.630, 0.640, 0.660, 0.680; 0.3);$$
$$\tilde{\tilde{f}}_{21}^*(e_{1,3}) = (0.500, 0.500, 0.500, 0.500, 0.630, 0.640, 0.660, 0.680; 0.3);$$
$$\tilde{\tilde{f}}_{31}^*(e_{1,1}) = (0.720, 0.740, 0.760, 0.780, 0.820, 0.840, 0.860, 0.880; 0.3);$$
$$\tilde{\tilde{f}}_{31}^*(e_{1,2}) = (0.720, 0.740, 0.760, 0.780, 0.820, 0.840, 0.860, 0.880; 0.3);$$
$$\tilde{\tilde{f}}_{31}^*(e_{1,3}) = (0.720, 0.740, 0.760, 0.780, 0.820, 0.840, 0.860, 0.880; 0.3);$$
$$\tilde{\tilde{f}}_{41}^*(e_{1,1}) = (0.700, 0.710, 0.730, 0.750, 0.770, 0.780, 0.800, 0.820; 0.3);$$
$$\tilde{f}_{41}^*(e_{1,2}) = (0.700, 0.710, 0.730, 0.750, 0.770, 0.780, 0.800, 0.820; 0.3);$$
$$\tilde{\tilde{f}}_{41}^*(e_{1,3}) = (0.720, 0.740, 0.760, 0.780, 0.820, 0.840, 0.860, 0.880; 0.3);$$
$$\tilde{\tilde{f}}_{51}^*(e_{1,1}) = (0.520, 0.540, 0.560, 0.580, 0.620, 0.640, 0.660, 0.680; 0.3);$$
$$\tilde{\tilde{f}}_{51}^*(e_{1,2}) = (0.500, 0.500, 0.500, 0.500, 0.630, 0.640, 0.660, 0.680; 0.3);$$
$$\tilde{\tilde{f}}_{51}^*(e_{1,3}) = (0.500, 0.500, 0.500, 0.500, 0.630, 0.640, 0.660, 0.680; 0.3);$$
$$\tilde{\tilde{f}}_{61}^*(e_{1,1}) = (0.770, 0.780, 0.800, 0.820, 0.840, 0.850, 0.870, 0.890; 0.3);$$
$$\tilde{\tilde{f}}_{61}^*(e_{1,2}) = (0.770, 0.780, 0.800, 0.820, 0.840, 0.850, 0.870, 0.890; 0.3);$$
$$\tilde{\tilde{f}}_{61}^*(e_{1,3}) = (0.840, 0.850, 0.870, 0.890, 0.910, 0.920, 0.940, 0.950; 0.3);$$
$$\tilde{\tilde{f}}_{71}^*(e_{1,1}) = (0.840, 0.850, 0.870, 0.890, 0.910, 0.920, 0.940, 0.950; 0.3);$$

Table 13.16 Constant light output—voltage relation

Bulbs	B_1	B_2	B_3	B_4	B_5	B_6	B_7
Voltage	105.49	149.74	126.84	114.81	190	103.06	111.66

Table 13.17 Percentage of voltage drop allowed to maintain light output of B_5

Bulbs	B_1	B_2	B_3	B_4	B_5	B_6	B_7
% with respect to 220 V	47.95%	68.06%	57.65%	52.19%	86.36%	46.85%	50.75%
% of voltage drop	52.05%	31.94%	42.35%	47.81%	13.64%	53.15%	49.25%

Table 13.18 Normalized voltage values—constant light output

Bulbs	B_1	B_2	B_3	B_4	B_5	B_6	B_7
Voltage	0.977	0.688	0.813	0.898	0.542	1.000	0.923

$$\tilde{\tilde{f}}^{*}_{71}(e_{1,2}) = (0.840, 0.850, 0.870, 0.890, 0.910, 0.920, 0.940, 0.950; 0.3);$$
$$\tilde{\tilde{f}}^{*}_{71}(e_{1,3}) = (0.840, 0.850, 0.870, 0.890, 0.910, 0.920, 0.940, 0.950; 0.3).$$

Evaluation of voltage with parameters:
From Table 13.3 and Fig. 13.4, we note that the relative light intensity for the bulb B_5 is the least compared to that of the other bulbs, and the same is attained at 190 V.

The Voltage at which the other bulbs give the same light output with respect to B_5 is calculated using coded instructions in 'R-Script environment' (see Appendix I) and tabulated in Table 13.16.

Remark 13.12 Using the voltage shown in Table 13.16, the percentage of the same with respect to 220 V and the percentage of voltage drop allowed to maintain the light output of B_5 (at 190 V) are computed and given in Table 13.17.

Remark 13.13 Note that bulbs B_1 and B_6 give light output equivalent to that of bulb B_5 even at 50% of voltage drop.

The bulb B_6 gives the light output equivalent to that of B_5 at the minimum voltage 103.06. The normalized values for the voltages with respect to constant light output for the bulbs are obtained by dividing 103.06 by its corresponding voltage values and are given in Table 13.18.

The capability of bulbs in providing better light output during low voltage situations is subjectively evaluated using linguistic states ranging from best at low voltage (BLV), better at low voltage (BTLV), good at low voltage (GLV), very fair at

Table 13.19 Linguistic evaluation—voltage-related parameters

$\tilde{U}$	a_6
	$e_{6,1}$
$(B_1, 0.875)$	BTLV
$(B_2, 0.881)$	UFLV
$(B_3, 0.930)$	VFLV
$(B_4, 1.000)$	GLV
$(B_5, 0.870)$	PLV
$(B_6, 0.969)$	BLV
$(B_7, 0.940)$	GLV

low voltage (VFLV), fair at low voltage (FLV), unfair at low voltage (UFLV), poor at low voltage (PVF), very poor at low voltage (VPLV) to worst at low voltage (WLV) (pictorial representation, Fig. 13.7).

The values of these linguistic states are described as linear octagonal fuzzy numbers given in Table 13.9. The linguistic evaluation for the bulbs based on the obtained values (see Table 13.18) are tabulated in Table 13.19.

For the attribute $a_6 =$ Voltage, its associated parameter set $E_6 = \{e_{6,1}\}$ with respect to the better light output during low voltage, and the evaluation is described by modeling it as soft linear octagonal fuzzy numbers $(\tilde{\tilde{f}}_{i6}, E_6) = \left\{\tilde{\tilde{f}}_{i6}(e_{6,1})\right\}$, for $i = 1, \ldots, 7$ with

$\tilde{\tilde{f}}_{16}(e_{6,1}) = (0.910, 0.920, 0.940, 0.950, 0.970, 0.980, 0.990, 1.000; 0.3)$;
$\tilde{\tilde{f}}_{26}(e_{6,1}) = (0.610, 0.620, 0.640, 0.660, 0.680, 0.690, 0.710, 0.730; 0.3)$;
$\tilde{\tilde{f}}_{36}(e_{6,1}) = (0.770, 0.780, 0.800, 0.820, 0.840, 0.850, 0.870, 0.890; 0.3)$;
$\tilde{\tilde{f}}_{46}(e_{6,1}) = (0.840, 0.850, 0.870, 0.890, 0.910, 0.920, 0.940, 0.950; 0.3)$;
$\tilde{\tilde{f}}_{56}(e_{6,1}) = (0.540, 0.550, 0.570, 0.590, 0.610, 0.620, 0.640, 0.660; 0.3)$;
$\tilde{\tilde{f}}_{66}(e_{6,1}) = (0.980, 0.985, 0.990, 1.000, 1.000, 1.000, 1.000, 1.000; 0.3)$;
$\tilde{\tilde{f}}_{76}(e_{6,1}) = (0.840, 0.850, 0.870, 0.890, 0.910, 0.920, 0.940, 0.950; 0.3)$.

Remark 13.14 The observations from the experimental data are represented using parameters as soft fuzzy numbers, and the complete information required for measuring the efficiency of all the 7 bulbs is modeled as a soft linear octagonal fuzzy number valued fuzzy information system, denoted as $\widetilde{FIS}_1$ given by
$\widetilde{FIS}_1 = (\tilde{U}, A_1 = A \cup a_6, \mathcal{F}^*(\mathbb{R})(\mathcal{E}_1), \tilde{\tilde{I}}_1)$, where $\mathcal{E}_1 = \{E_1^*, E_2, \ldots, E_7\}$, and
$\tilde{\tilde{I}}_1 : \tilde{U} \times A_1 \to \mathcal{F}^*(\mathbb{R})(\mathcal{E}_1)$ such that

$$\tilde{\tilde{I}}_1(\tilde{U}(B_i), a_j) = \begin{cases} (\tilde{\tilde{f}}_{ij}, E_j), \ i = 1, \ldots, 7, \ j = 2, \ldots, 6 \\ (\tilde{\tilde{f}}_{i1}^*, E_1^*), \qquad\qquad j = 1 \end{cases}$$

Table 13.20 Catalog information

Bulbs	Lumen	CRI	Voltage
1	BTLO	VNW	BLV
2	BLO	PNW	BLV
3	BLO	PNW	BLV
4	GLO	PNW	BLV
5	BLO	PNW	BLV
6	BLO	PNW	BLV
7	BLO	PNW	BLV

Mathematical formulation of Catalog Information:
The subjective evaluation of the bulbs for lumen, CRI and Voltage in catalog information (see Table 13.2) is recorded as shown in Table 13.20.

And is modeled as a soft linear octagonal fuzzy number valued information system
$\widetilde{IS} = (U, C, \mathcal{F}^*(\mathbb{R})(\mathcal{E}_2), \tilde{\tilde{I}}_2)$, where
$U = \{B_1, \ldots, B_7\}$, the collection of 7 W bulbs under consideration,
$C = \{c_1, c_2, c_3, c_4, c_5\}$, $c_1 =$ lumen, $c_2 =$ CRI, $c_3 = a_6, c_4 = a_4, c_5 = a_5$ the attribute set, $\mathcal{E}_2 = \{E_1^{**}, E_2^{**}, E_3^{**}, E_4, E_5\}$ with $E_j^{**} = \{e_{j,1}\}$, where $e_{j,1} = c_j$, $j = 1, 2, 3$,
$\tilde{\tilde{I}}_2 : U \times C \to \mathcal{F}^*(\mathbb{R})(\mathcal{E}_2)$ such that $\tilde{\tilde{I}}_2(B_i, c_j) = (\tilde{\tilde{f}}_{ij}, E_j^{**})$, $j = 1, 2, 3$ and $\tilde{\tilde{I}}_2(B_i, c_j) = (\tilde{\tilde{f}}_{ij}, E_j)$, for $i = 1, \ldots, 7$, $j = 4, 5$ are soft linear octagonal fuzzy numbers (similar sense as discussed in experimental studies).

13.6 Measuring the Efficiency of the Bulb in Different Brands: Procedure and Inference

The evaluation of the light output is influenced by the nature of the bulbs. In this section, an algorithm is proposed to measure the efficiency of the bulbs described, wherein we distinguish between influencing and non-influencing attributes. The computations and inferences for information gathered from the catalog, experimental result at 220 V and experimental observations at low voltage are obtained. Also, the need for a mathematical model involving a soft linear octagonal fuzzy number valued information system to incorporate the intricacies is discussed.

Algorithm to measure the efficiency of bulbs denoted by $\mathcal{Q}_{B_i}$

Step i: Input $\widetilde{FIS}$.
Step ii: Identify the attributes influenced by the nature of the entities and obtain the related $\widetilde{FIS}$.

Table 13.21 Ranking of bulbs corresponding to attributes—At 220 V

$\tilde{U}$	A				
	$a_1(0.3)$	$a_2(0.05)$	$a_3(0.2)$	$a_4(0.4)$	$a_5(0.05)$
$(B_1, 0.875)$	1	6	6	2	1
$(B_2, 0.881)$	7	7	7	3	2
$(B_3, 0.930)$	3	1	4	5	2
$(B_4, 1.000)$	5	3	1	3	2
$(B_5, 0.870)$	6	2	5	4	2
$(B_6, 0.969)$	4	4	2	1	1
$(B_7, 0.940)$	2	5	3	4	1

Step iii: For each $B_i \in U$, determine $\tilde{\boldsymbol{M}}\left[\tilde{\tilde{I}}(\tilde{U}(B_i), b)\right], b \in B$.

Step iv: For each $B_i \in U$ compute $\tilde{\boldsymbol{M}}\left[(\tilde{\tilde{f}}_i, B)\right] \in \tilde{\mathbb{R}}$ and $\mathscr{Q}_{B_i} = \boldsymbol{M}\left[\tilde{\boldsymbol{M}}\left[(\tilde{\tilde{f}}_i, B)\right]\right]$.

Step v: Choose B_i for which $\mathscr{Q}_{B_i}$ is optimum.

Remark 13.15 To measure the efficiency of the bulbs corresponding to each attribute, compute the following: Execute the above Algorithm till step iii. Compute $\mathscr{Q}^b_{B_i} = \boldsymbol{M}\left[\tilde{\boldsymbol{M}}\left[\tilde{\tilde{I}}(\tilde{U}(B_i), b)\right]\right], i = 1, \ldots, 7,$ for each $b \in B$. Choose B_i for optimum efficiency.

Remark 13.16 In the procedures discussed above, step iii is computed using Definition 13.16 and $\mathscr{Q}_{B_i}$, $\mathscr{Q}^b_{B_i}$ are calculated using Definition 13.9.

Computation of efficiency of bulbs corresponding to each attribute:
Using the procedure given in Remark 13.15, the ranking order of the bulbs corresponding to each attribute at 220 V is computed and shown in Table 13.21.
From Table 13.21, the ranking order of the bulbs is found to vary with respect to varying attributes. Henceforth, the information can be utilized for various purposes for example when people are not bothered about light output but are more concerned about less power consumption, just a light is required. In such a situation, one would like to have a bulb which emits some light with minimum power consumption. In such a case, the bulb B_4 would be their choice.

In some commercial applications, one may be interested in the quality of light (CRI values) and accordingly would choose the suitable bulb brand and in this case one would opt for bulb B_6.

Computation of efficiency with all attributes: Execution of the Algorithm using a program in MATLAB 2016a (see Appendix II) yields the efficiency of the various bulbs for all the attributes and information considered in Sect. 13.5. The output is recorded in Table 13.22.

Remark 13.17 Column 6 gives the cumulative efficiency of the various bulbs; note that B_6 is the best one. Comparing column 2, column 4 and column 6, we find that

Table 13.22 Efficiency of the bulbs with all parameters

Bulbs	Catalog information		Experimental studies			
			At 220 V		At lower voltage	
	Measure	Rank	Measure	Rank	Measure	Rank
B_1	0.9821	2	1.0108	1	0.9648	2
B_2	0.9547	3	0.8111	6	0.7512	6
B_3	0.8455	7	0.7857	7	0.8048	5
B_4	0.9259	4	0.8674	3	0.8843	4
B_5	0.9087	6	0.8113	5	0.6969	7
B_6	0.9924	1	0.9463	2	0.9695	1
B_7	0.9180	5	0.8912	4	0.8935	3

there are variations. Based on the need, one could make an appropriate choice of bulbs. Using octagonal fuzzy numbers with $0 \leq k < 0.5$ and $0.5 < k < 1$, the value of cumulative efficiency of a bulb at maximum level and minimum level are obtained. In such cases, the bulbs are ranked with better accuracy.

Need for Octagonal fuzzy numbers to explain the situation:

Remark 13.18 In this problem, if we consider computation with soft trapezoidal fuzzy numbers in the place of soft linear octagonal fuzzy numbers, we notice that there is better accuracy in the latter case. In fact, it also causes a difference in the ranking order.

- The measure values of bulbs B_1 and B_6 with respect to information at lower voltage are very close when computed using trapezoidal fuzzy numbers than linear octagonal fuzzy numbers, though the rankings are the same.
- However, the measure values of bulbs B_2 and B_5 with respect to information at 220 V showed small variations causing a change in the ranking order.
- With respect to catalog information, measure values are different for all bulbs causing the same ranking order.

We present in the following the need for a soft linear octagonal fuzzy number valued information system for the experimental information at 220 V.

Remark 13.19 Need for Soft fuzzy environment:
Suppose the experimental output at 220 V for the attribute light output is evaluated on a 9-point scale (non-fuzzy setup) as given in Table 13.23.

Table 13.23 9-point scale

Linguistic states	Points
BLO	9
BTLO	8
GLO	7
VFLO	6
FLO	5
UFLO	4
PLO	3
VPLO	2
WPLO	1

Table 13.24 Ranking corresponding to light output

Bulbs	B_1	B_2	B_3	B_4	B_5	B_6	B_7
Ranking (soft only)	1	5	3	4	3	3	2
Ranking (fuzzy only)	2	1	1	3	1	1	1

Then the light output is described by soft real numbers $(F_{i1}, E_1) = \{F_{i1}(e_{1,1})\}$, $i = 1, \ldots, 7$ where $F_{11}(e_{1,1}) = 7$; $F_{21}(e_{1,1}) = 3$; $F_{31}(e_{1,1}) = 5$; $F_{41}(e_{1,1}) = 5$; $F_{51}(e_{1,1}) = 4$; $F_{61}(e_{1,1}) = 5$; $F_{71}(e_{1,1}) = 6$.

The ranking order of bulbs in this case is computed using Definition 13.13 and is recorded in the second row of Table 13.24, and we infer that the ranks of few bulbs are repeated.

Considering the catalog information (see Table 13.20, only fuzzy) corresponding to the attribute light output, using Definition 13.9 the ranking order of bulbs is computed and shown in the third row of Table 13.24.

From column 2 of Table 13.21, we note that the bulbs are uniquely ranked when the information obtained from experiments are modeled as a soft linear octagonal fuzzy number valued information system.

Along the lines, attributes CRI and current are evaluated. The efficiency of the bulbs with respect to these attributes are computed, and their ranking order corresponding to the different models are given in Table 13.25.

We note that the ranking order has improved when the information is modeled as a soft fuzzy number valued fuzzy information system involving soft linear octagonal fuzzy numbers.

Remark 13.20 Various varieties of fuzzy numbers are available in the literature. Comparing the problem for all possible fuzzy numbers to give a comparative study will be unwieldy and out of direction with respect to the paper.

Table 13.25 Ranking corresponding to attributes light output, CRI and current

Bulbs	B_1	B_2	B_3	B_4	B_5	B_6	B_7
Ranking (soft fuzzy)	1	7	3	4	6	5	2
Ranking (fuzzy only)	1	5	3	4	3	3	2
Ranking (soft only)	3	5	2	1	4	3	1

Scope of the Study: Here, the experimental study was undertaken with one bulb from 7 brands. Based on the manufacturers' data, light output, CRI values and power consumption, the bulbs may be ranked with respect to the desired attribute and interpreted. This study does not restrict the size of the collection of bulbs/brands.

13.7 Conclusion

In this paper, the outcome of experiments conducted on different bulbs (7 W/220 V) from various brands to study the relative light output, current and CRI values with voltage is discussed. The experimental data along with catalog information are used to analyze the bulb's brand based on efficiency. Also, the experimental outcomes related to the attributes are considered to choose the bulb's brand for appropriate applications. The mathematical model $\widetilde{FIS}$ and the method proposed are applied to frame the in-depth information obtained from the experiment. Finally, we infer that the experimental study gives more scope for a better choice than decision based on the catalog information, and modeling the outcome of experimental studies using a soft linear octagonal fuzzy number valued information system yields better results and also the selection of bulb depends on the need for appropriate situations that would satisfy customer's expectation.

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Appendix I

We present the computations involved in Sect. 13.4 using R-scripted codes.

R-Programming Code:

```
#####Computations of experimental observations:#####
##Calculation of normalized values involved in the problem###
##Relative intensity at 220##
intensity=c(13516.15,7774.25,10730.06,10329.18,8600.51,10485.14,12618.56)
normalintensity=intensity/13516.15
#current at 220 V
minimumcurrent=c(28,28,28,28,28,28,28)
current=c(32,31.8,30.1,28,32.2,28.9,29.8)
print("The normalised current")
normalcurrent=minimumcurrent/current
print(normalcurrent)
##wattage Calculation##
Am=current*0.001
print(Am)
W=220*Am
print(W)
normalW=6.160/W
print(normalW)
##CRI at 220##
CRI=c(74.44,73.48,80.4,78.46,79.17,76.29,75.97)
normalCRI=CRI/80.4
print("Normal CRI")
print(normalCRI)
##Warranty##
warranty=c(2,1,1,1,1,2,2)
normalwarranty=warranty/2
print(normalwarranty)
##Cost##
mincost=c(129,129,129,129,129,129,129)
cost=c(130,140,199,140,160,129,160)
print("Normal Cost")
normalcost=mincost/cost
print(normalcost)
#More Parameters-e12,e13 -Total intensity between area (140 V-240 V)#
print("Normalised area under the intensity curve
      between the voltages 140 and 220")
ARintensity=c(102269,61177,81611,77576,57322,81995,91377)
normalARintensity=ARintensity/102269
print(normalARintensity)
#Calculation of maximum Voltage
mv=c(160,160,180,160,190,140,160)
Respectivecurrent=c(42,42.4,35.9,37.1,30.2,43.3,40.1)
##Power consumption of bulbs corresponding to respective voltag##
PC=mv*Respectivecurrent*0.001
#Maximum light intensity per watt##
MaxLI=c(13030.36, 7818.10, 10384.77, 9815.69, 7497.21, 10465.09, 11629.71)
MaxLIwatt=MaxLI/PC
nMaxLIwatt=MaxLIwatt/max(MaxLIwatt)
print(nMaxLIwatt)
##Voltage characterised by more parameters##
##V that gives 90 percent light output and V that gives equal output ##
##Calculation in volts with respect to least relative intensity##
## light output##
y1=c(6714.14,7192.69,6772.42,5846.49,7377.86,7109.35,5993.89)
y2=c(9565.78,7818.10,8893.4,8075.05,7497.21,9644.26,8571.6)
x1=c(100,140,120,100,180,100,100)
x2=c(120,160,140,120,190,120,120)
y=c(7497.21,7497.21,7497.21,7497.21,7497.21,7497.21,7497.21)
n=7
```

```
for(i in seq(1,n,1))
  {
  x=((20/(y2-y1))*(y-y2))+x2
  }
for(i in seq(1,n,1))
  {
  normal_x=103.0601/x
  }
print("the voltage at which the bulbs gives 90 percent light intensity")
for(i in seq(1,n,1))
  {
  x[i]=((20/(y2[i]-y1[i]))*(y[i]-y2[i]))+x2[i]
  print(x[i])
  }
for(i in seq(1,n,1))
  {
  normal_x=103.0601/x[i]
  print(normal_x)
  }
print("The normalised values of VOltage")
normalv=c(0.977,0.688,0.813,0.898,0.542,1.000,0.923)
print(normalv)
```

Appendix II

We present a program developed for the Algorithm proposed in Sect. 13.6, using MATLAB 2016a to measure the efficiency of the LED bulbs with various forms of information. Appropriate inputs corresponding to the information (see Sect. 13.5) gathered from experimental observation at 220 V, at low voltage and catalog are formatted in matrix form using soft linear octagonal fuzzy numbers. Also with the plot function, a tool from MATLAB 2016a is coded with experimental observation to obtain Figs. 13.5, 13.6 in Sect. 13.4.

Program to Compute the Efficiency of LED Bulbs:

```
%Computation of eficiency of LED bulbs:
unit=[1 1 1 1 1 1 1];
n=7;
q=input('enter the total number of attributes');
fprintf('enter the weight for the attributes of size %d',q);
v=input('');
entitynature=input('Type Y to evaluate the attributes influenced by fuzzy
nature of entities','s');
switch entitynature,
    case 'Y',
        fprintf('Enter the number of influencing attributes');
        a11=input('');
        for s1=1:a11
        fprintf('Enter the fuzzy numbers representing the fuzzy nature of the
        entities of size %d',n);
        U=input('');
        U1=unit./U;
        fprintf('Enter the soft fuzzy numbers corresponding to influencing
        attribute');
        B=input('');
    [m1,n]=size(B);
     fprintf('Enter the weight vector of size %d ' ,m1);
    w1=input('');
```

```
        fprintf('the measure of  Attribute %d of \n',s1);
    for j=1:n
        for t=1:8
              b{j,s1}(t)=0;
              for i=1:m1
                  b{j,s1}(t)=b{j,s1}(t)+(w1(i)*B{i,j}(t));
              end
        end
         Eb{j,s1}=((b{j,s1}(1)+b{j,s1}(2)+b{j,s1}(7)+b{j,s1}(8))*(.3)+(b{j,
         s1}(3)+b{j,s1}(4)+b{j,s1}(5)+b{j,s1}(6))*(.7))/4;
        fprintf('%d entity is %f %f %f %f %f %f %f %f crisp value is %f \n',
        j,b{j,s1},Eb{j,s1});
    end
end
    B1=(b);
    for j=1:n
        for t=1:8

        SB{j,s1}(t)=B1{j,s1}(t)*U1(j);

        end
    end
    otherwise,
end;
fprintf('Enter the number of non-influencing atributes');
a=input('');
for s=1:a
    fprintf('Enter the soft fuzzy numbers for attribute %d',s);
    E=input('' );
    [m,n]=size(E);
    C{s}=(E);
    fprintf('Enter the weight vector of size %d ' ,m);
    w=input('');
    D{s}=(w);
    fprintf('the measure of  Attribute %d of \n',s);
    for j=1:n
        for t=1:8
              k{j,s}(t)=0;
              for i=1:m
                  k{j,s}(t)=k{j,s}(t)+(w(i)*E{i,j}(t));
              end
        end
        Ek{j,s}=((k{j,s}(1)+k{j,s}(2)+k{j,s}(7)+k{j,s}(8))*(.3)+(k{j,s}(3)+k
        {j,s}(4)+k{j,s}(5)+k{j,s}(6))*(.7))/4;
        fprintf('%d entity is %f %f %f %f %f %f %f %f crisp value is %f \n',
        j,k{j,s},Ek{j,s});
    end
end
%catenation of influencing and non influencing attributes
T=cat(2,SB,k);
[n,m2]=size(T);
I=(T);
%Computation of Efficiency
 fprintf('The efficiency of the entity of\n');
 for j=1:n
     for t=1:8
         A{j}(t)=0;
         for s2=1:m2
             A{j}(t)=A{j}(t)+v(s2)*T{j,s2}(t);
         end
     end
   EM{j}=((A{j}(1)+A{j}(2)+A{j}(7)+A{j}(8))*(.3)+(A{j}(3)+A{j}(4)+A{j}(5)+A
   {j}(6))*(.7))/4;
   ETM{j}=(A{j}(1)+A{j}(4)+A{j}(5)+A{j}(8))/4;
   fprintf(' %d entity is %f %f %f %f %f %f %f %f crisp value is %f and %f
   \n',j,A{j},EM{j},ETM{j});
    end
```

MATLAB 2016a Code: To Plot Figs. 13.4 and 13.6:

To plot relative intensity with respect to varying voltages:

```
x=[60 80 100 120 140 160 180 190 200  210  220 230 240]
B1=[1304.34 3908.4 6714.14 9565.78 12222.26 13030.36
12931.16 12870.78 12812.40 12739.9 12798.93 12665.26 12617.99];
B2=[808.7 2294.61 3877.11 5590.93 7192.69 7818.10 7780.66
7716.39 7683.44 7638.6 7612.63 7580.7 7500.7];
B3=[977.09 2653.29 4621.48 6772.42 8893.4 10366.03 10384.77
10342 10305.43 10262.1217 10229.46 10197.48 10153.78];
B4=[1726.36 3639.71 5846.49 8075.05 9790.48 9815.69 9772.67
9711.26 9676.85 9655.44 9632.36 9622.65 9587.99];
B5=[634.68 1883.1 3064.35 4215.08 5317.97 6387.24 7377.86
7497.21 7470.7 7446.17 7419.47 7381.59 7366.02];
B6=[2056.27 4461.42 7109.35 9644.16 10465.56 10401.09
10354.38 10273.72 10236.74 10192.27 10142.81 10120.07 10082.23];
B7=[1335.09 3629.79 5993.89 8571.6 10937.16 11629.71
11586.24 11529.09 11471.63 11400.64 11352.31 11306.57 11266.31];
plot(x,B1,’k-+’,x,B2,’r-*’,x,B3,’b-o’,x,B4,’g-s’,x,B5,
’y-d’,x,B6, c-h’,x,B7,’m-^’)
%title(’Comparative plot of LED bulbs’)
xlabel(’Voltage’)
ylabel(’Relative Intensity’)
    legend(’B1’,’B2’,’B3’,’B4’,’B5’,’B6’,’B7’)
```

To plot current with respect to varying voltages:

```
%Computation of Current and Voltage:
x=[60 80 100 120 140 160 180 190 200 210 220 230 240];
B1=[8.7 18.8 27.2 35.5 44.1 42 38 36.3 34.7 33.3 32 31 30];
B2=[9.2 18.9 27.4 35.3 42.2 42.4 37.9 36.2 34.5 33.1 31.8
30.7 29.6];
B3=[8.6 16.4 23.5 30.5 37 39.9 35.9 34.2 32.7 31.3 30.1 29 28];
B4=[12.5 20.9 29.2 36.7 41.5 37.1 33.4 31.7 30.3 29.2 28 27
26.1];
B5=[7.8 15.7 21.6 26.3 30.2 33.7 36.6 36.2 34.6 33.4 32.2 31.2
30.2];
B6=[14.4 25 35 43.9 43.3 38.2 34.4 32.8 31.3 30.1 28.9 27.9
27];
B7=[8.5 18.1 26.1 34 41.2 40.1 35.9 34.3 32.7 31.2 29.8 28.7
27.6];
plot(x,B1,’k-+’,x,B2,’r-*’,x,B3,’b-o’,x,B4,’g-s’,x,B5,’y-d’,
x,B6,’c-h’,x,B7,’m-^’)
%title(’Comparative plot of LED bulbs in consumption of
current’)
xlabel(’Voltages’)
ylabel(’Current (mA)’)
legend(’B1’,’B2’,’B3’,’B4’,’B5’,’B6’,’B7’)
```

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Chapter 14
Role of Single Valued Linear Octagonal Neutrosophic Numbers in Multi-attribute Decision-Making Problems

S. Subasri, S. Arul Roselet Meryline, and Felbin C. Kennedy

14.1 Introduction

L. A. Zadeh in 1965 [14] set forth the exceptional concept of fuzzy sets, that have Brobdingnagian applications in several fields of study. In 1986, a generalization of fuzzy sets was created by Atanassov [2], which is understood as Intuitionistic fuzzy sets (IFS). In IFS, in addition to one membership grade, there will additionally be another grade called non-membership grade that's hooked up to every part. To boot, there's a restriction that the total of those two grades at most be unity. A new theory was introduced by Smarandache [12] in 1999, called neutrosophic sets and logic. Uncertainty describes a lack of knowledge in one's knowledge but whereas ambiguity describes the ability to entertain more than in one interpretation. Thus, Neutrosophic set is used to deal with incomplete, indeterminate, and inconsistent information present in the real world. A neutrosophic set (NS) is employed to tackle uncertainty using the truth membership, indeterminacy membership, and falsity membership grades which are considered to be independent. The generalization of IFS is the neutrosophic sets since there is no restriction between the degree of truth, indeterminacy, falsity, and these degrees can individually vary within $]0^{-}, 1^{+}[$. Applications of Neutrosophic set can be found in the field of medicine, information technology, information system, decision support system, etc [8, 9].

From scientific or engineering purpose of reading, the neutrosophic set and set theory-based operators got to be specific. Hence, Wang et al. in 2005 [13], presented

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an associate case of the neutrosophic set called single valued neutrosophic sets that were intended from the sensible purpose of reading wherein the degree of truth, indeterminateness, and falsity takes the value from the unit interval [0, 1]. To capture the imprecise information in the truth membership degree, indeterminate degree, falsity degree, Deli et al. [7] in 2016 introduced Single Valued Neutrosophic Numbers (SVNNs) which is a single valued neutrosophic subset of a real line that satisfies normality, convexity, and upper semicontinuity for truth degree, lower semicontinuity for indeterminate and falsity degree with bounded support. Several researchers studied different types of single valued neutrosophic numbers (to cite few [4, 11]). Various ranking methods were developed for SVNNs to make the optimal decision in real-life problems involving indeterminate data. Ranking using the value and ambiguity for the membership grades captures the ill-defined magnitude which underlies any fuzzy number and using the same ranking of the Single Valued Trapezoidal Neutrosophic number (SVTrNN) was studied by Biswas et al. [3]. In this paper, we introduce Single Valued Linear Octagonal Neutrosophic Numbers (SVLONNs) where the truth membership, indeterminacy membership, falsity membership functions are exhibited as Linear Octagonal Fuzzy Numbers. Ranking technique on SVLONNs plays a crucial role in higher cognitive process issues that involve indeterminate data in ordering and comparing the same.

The remaining paper is structured as follows: The definition of SVLONNs, (α_o,β_o,γ_o)-cuts of SVLONNs and arithmetic operations on SVLONNs are proposed in Sect. 14.2. Section 14.3 is dedicated to discussing the value and ambiguity indices of SVLONNs and a ranking technique of SVLONNs is introduced for defuzzification processes. In Sect. 14.4, we deal with the formulation of a Multi-Attribute Decision-Making Problem. In Sect. 14.5, a hypothetical problem is conferred for SVLONNs. In Sect. 14.6, we record closing remarks and some applications of the planned technique are put forth for future study.

14.2 Single Valued Linear Octagonal Neutrosophic Number

In this section, we define SVLONNs, cuts of SVLONNs, and arithmetic operations on SVLONNs.

Definition 14.1 A Single Valued Linear Octagonal Neutrosophic Number

(SVLONN) $\tilde{A}_{\mathcal{N}}$ denoted by $\langle p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8; q_1, q_2, q_3, q_4, q_5, q_6, q_7, q_8; r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8; k\rangle$ where $p_1 \leq p_2 \leq \cdots \leq p_8;\ q_1 \leq q_2 \leq \cdots \leq q_8;$ $r_1 \leq r_2 \leq \cdots \leq r_8$ are real numbers, its truth membership function $T_{\tilde{A}_{\mathcal{N}}}$, indeterminacy membership function $I_{\tilde{A}_{\mathcal{N}}}$, falsity membership function $F_{\tilde{A}_{\mathcal{N}}}$ are defined as follows: For $0 < k < 1$,

$$T_{\tilde{A}_{\mathcal{N}}}(x)=\begin{cases}0 & for \quad x<p_1;\\ k\left(\dfrac{x-p_1}{p_2-p_1}\right) & for \quad p_1\le x\le p_2;\\ k & for \quad p_2\le x\le p_3;\\ k+(1-k)\left(\dfrac{x-p_3}{p_4-p_3}\right) & for \quad p_3\le x\le p_4;\\ 1 & for \quad p_4\le x\le p_5;\\ k+(1-k)\left(\dfrac{p_6-x}{p_6-p_5}\right) & for \quad p_5\le x\le p_6;\\ k & for \quad p_6\le x\le p_7;\\ k\left(\dfrac{p_8-x}{p_8-p_7}\right) & for \quad p_7\le x\le p_8;\\ 0 & for \quad x>p_8;\end{cases}$$

$$I_{\tilde{A}_{\mathcal{N}}}(x)=\begin{cases}1 & for \quad x<q_1;\\ 1+(1-k)\left(\dfrac{q_1-x}{q_2-q_1}\right) & for \quad q_1\le x\le q_2;\\ k & for \quad q_2\le x\le q_3;\\ k+k\left(\dfrac{q_3-x}{q_4-q_3}\right) & for \quad q_3\le x\le q_4;\\ 0 & for \quad q_4\le x\le q_5;\\ k\left(\dfrac{x-q_5}{q_6-q_5}\right) & for \quad q_5\le x\le q_6;\\ k & for \quad q_6\le x\le q_7;\\ k+(1-k)\left(\dfrac{x-q_7}{q_8-q_7}\right) & for \quad q_7\le x\le q_8;\\ 1 & for \quad x>q_8;\end{cases}$$

$$F_{\tilde{A}_{\mathcal{N}}}(x)=\begin{cases}1 & for \quad x<r_1;\\ 1+(1-k)\left(\dfrac{r_1-x}{r_2-r_1}\right) & for \quad r_1\le x\le r_2;\\ k & for \quad r_2\le x\le r_3;\\ k+k\left(\dfrac{r_3-x}{r_4-r_3}\right) & for \quad r_3\le x\le r_4;\\ 0 & for \quad r_4\le x\le r_5;\\ k\left(\dfrac{x-r_5}{r_6-r_5}\right) & for \quad r_5\le x\le r_6;\\ k & for \quad r_6\le x\le r_7;\\ k+(1-k)\left(\dfrac{x-r_7}{r_8-r_7}\right) & for \quad r_7\le x\le r_8;\\ 1 & for \quad x>r_8;\end{cases}$$

Remark 14.1 The diagrammatic representation of SVLONNs for different values of k (Figs. 14.1, 14.2 and 14.3).

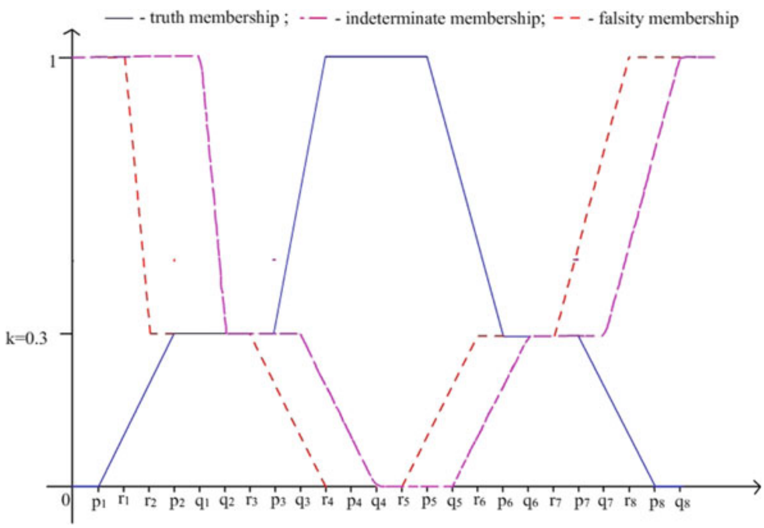

Fig. 14.1 SVLONN for k = 0.3

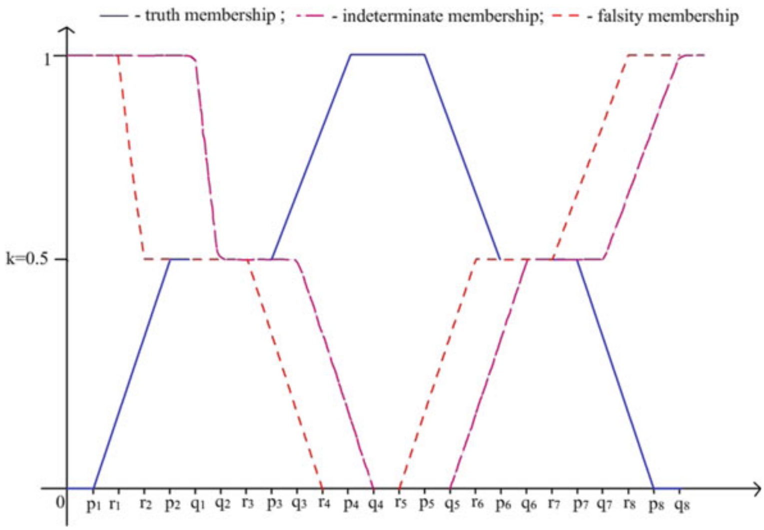

Fig. 14.2 SVLONN for k = 0.5

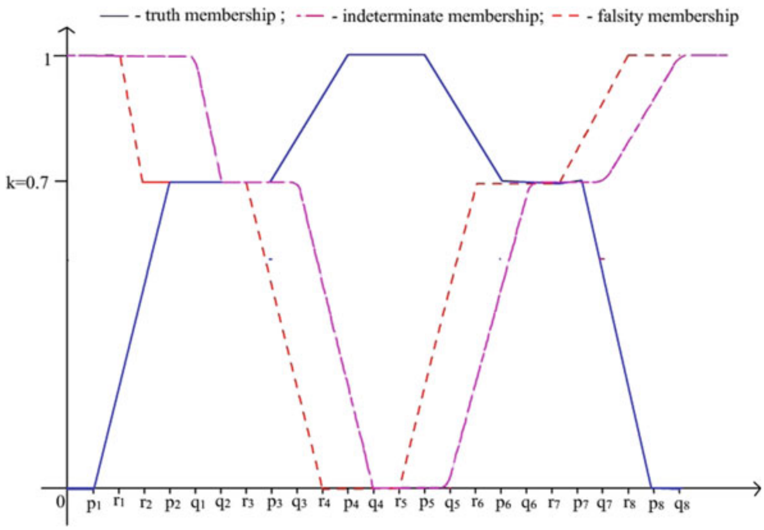

Fig. 14.3 SVLONN for k = 0.7

Remark 14.2 When k = 0 and k = 1, SVLONN reduces to the Single Valued Trapezoidal Neutrosophic number (SVTrNN) denoted $< p_3, p_4, p_5, p_6; q_1, q_2, q_7, q_8; r_1, r_2, r_7, r_8 >$ and $< p_1, p_2, p_7, p_8; q_3, q_4, q_5, q_6; r_3, r_4, r_5, r_6 >$ respectively (Figs. 14.4 and 14.5).

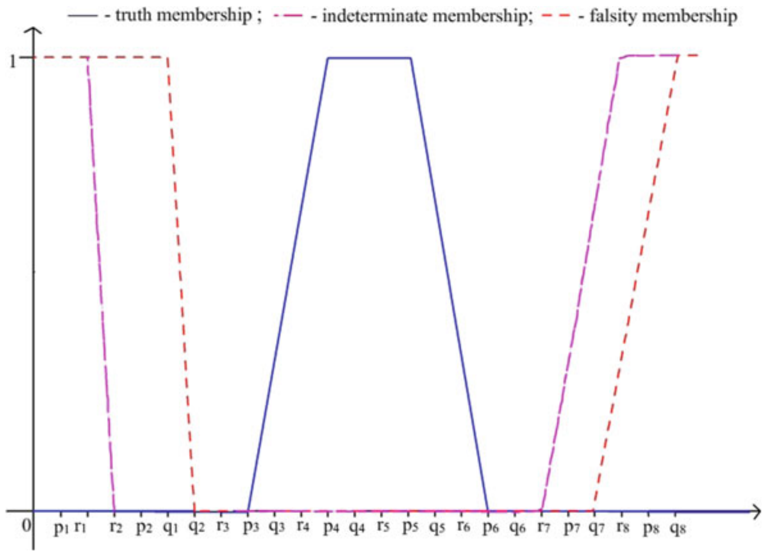

Fig. 14.4 SVLONN for k = 0

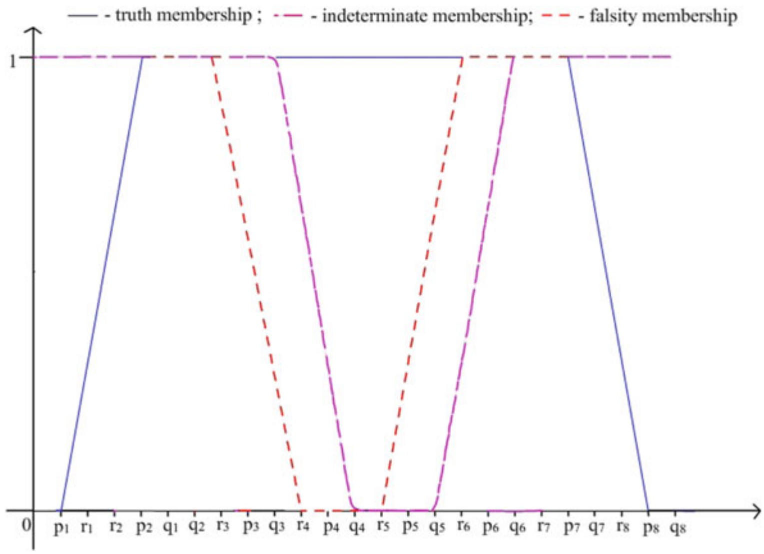

Fig. 14.5 SVLONN for k = 1

Definition 14.2 An $(\alpha_o,\beta_o,\gamma_o)$-cut set of SVLONN $\tilde{A}_{\mathcal{N}}$ is given as

$$(\tilde{A}_{\mathcal{N}})_{\alpha_o,\beta_o,\gamma_o} = \{x : T_{\tilde{A}_{\mathcal{N}}}(x) \geq \alpha_o, I_{\tilde{A}_{\mathcal{N}}}(x) \leq \beta_o, F_{\tilde{A}_{\mathcal{N}}}(x) \leq \gamma_o\} = ((\tilde{A}_{\mathcal{N}})_{\alpha_o}, (\tilde{A}_{\mathcal{N}})_{\beta_o}, (\tilde{A}_{\mathcal{N}})_{\gamma_o})$$

where

$$(\tilde{A}_{\mathcal{N}})_{\alpha_o} = [(\tilde{A}_{\mathcal{N}})^L_{\alpha_o}, (\tilde{A}_{\mathcal{N}})^R_{\alpha_o}] = \begin{cases} [((\tilde{A}_{\mathcal{N}})^L_{\alpha_o})_1, ((\tilde{A}_{\mathcal{N}})^R_{\alpha_o})_1], & for\ \alpha_o \in [0,k] \\ [((\tilde{A}_{\mathcal{N}})^L_{\alpha_o})_2, ((\tilde{A}_{\mathcal{N}})^R_{\alpha_o})_2], & for\ \alpha_o \in (k,1] \end{cases}$$

$$(\tilde{A}_{\mathcal{N}})_{\beta_o} = [(\tilde{A}_{\mathcal{N}})^L_{\beta_o}, (\tilde{A}_{\mathcal{N}})^R_{\beta_o}] = \begin{cases} [((\tilde{A}_{\mathcal{N}})^L_{\beta_o})_1, ((\tilde{A}_{\mathcal{N}})^R_{\beta_o})_1], & for\ \beta_o \in [0,k] \\ [((\tilde{A}_{\mathcal{N}})^L_{\beta_o})_2, ((\tilde{A}_{\mathcal{N}})^R_{\beta_o})_2], & for\ \beta_o \in (k,1] \end{cases}$$

$$(\tilde{A}_{\mathcal{N}})_{\gamma_o} = [(\tilde{A}_{\mathcal{N}})^L_{\gamma_o}, (\tilde{A}_{\mathcal{N}})^R_{\gamma_o}] = \begin{cases} [((\tilde{A}_{\mathcal{N}})^L_{\gamma_o})_1, ((\tilde{A}_{\mathcal{N}})^R_{\gamma_0})_1], & for\ \gamma_o \in [0,k] \\ [((\tilde{A}_{\mathcal{N}})^L_{\gamma_o})_2, ((\tilde{A}_{\mathcal{N}})^R_{\gamma_o})_2], & for\ \gamma_o \in (k,1] \end{cases}$$

Computing the $(\tilde{A}_{\mathcal{N}})_{\alpha_o,\beta_o,\gamma_o}$ of $\tilde{A}_{\mathcal{N}}$ in Definition 14.1, we have
$[((\tilde{A}_{\mathcal{N}})^L_{\alpha_o})_1, ((\tilde{A}_{\mathcal{N}})^R_{\alpha_o})_1] = [p_1 + \frac{\alpha_0}{k}(p_2 - p_1), p_8 - \frac{\alpha_o}{k}(p_8 - p_7)]$
$[((\tilde{A}_{\mathcal{N}})^L_{\alpha_o})_2, ((\tilde{A}_{\mathcal{N}})^R_{\alpha_o})_2] = [p_3 + \frac{\alpha_o - k}{1-k}(p_4 - p_3), p_6 - \frac{\alpha_o - k}{1-k}(p_6 - p_5)]$
$[((\tilde{A}_{\mathcal{N}})^L_{\beta_o})_1, ((\tilde{A}_{\mathcal{N}})^R_{\beta_o})_1] = [q_3 - \frac{\beta_o - k}{k}(q_4 - q_3), q_5 + \frac{\beta_o}{k}(q_6 - q_5)]$
$[((\tilde{A}_{\mathcal{N}})^L_{\beta_o})_2, ((\tilde{A}_{\mathcal{N}})^R_{\beta_o})_2] = [q_1 - \frac{\beta_o - 1}{1-k}(q_2 - q_1), q_7 + \frac{\beta_o - k}{1-k}(q_8 - q_7)]$
$[((\tilde{A}_{\mathcal{N}})^L_{\gamma_o})_1, ((\tilde{A}_{\mathcal{N}})^R_{\gamma_o})_1] = [r_3 - \frac{\gamma_o - k}{k}(r_4 - r_3), r_5 + \frac{\gamma_o}{k}(r_6 - r_5)]$
$[((\tilde{A}_{\mathcal{N}})^L_{\gamma_o})_2, ((\tilde{A}_{\mathcal{N}})^R_{\gamma_o})_2] = [r_1 - \frac{\gamma_o - 1}{1-k}(r_2 - r_1), r_7 + \frac{\gamma_o - k}{1-k}(r_8 - r_7)]$

We introduce arithmetic operation on SVLONNs as follows:

Definition 14.3 Let $\tilde{A}_{\mathcal{N}} = \langle p_1, p_2, \ldots, p_8; q_1, q_2, \ldots, q_8; r_1, r_2, \ldots, r_8\rangle$ and $\tilde{B}_{\mathcal{N}} = \langle x_1, x_2, \ldots, x_8; y_1, y_2, \ldots, y_8; z_1, z_2, \ldots, z_8\rangle$ be two SVLONNs and let s be any real number, then addition and scalar multiplication by $(\alpha_o,\beta_o,\gamma_o)$-cut approach are given by:

1. Addition:

$$(\tilde{A}_{\mathcal{N}})_{\alpha_o,\beta_o,\gamma_o} + (\tilde{B}_{\mathcal{N}})_{\alpha_o,\beta_o,\gamma_o} = \big[[(\tilde{A}_{\mathcal{N}})^L_{\alpha_o}, (\tilde{A}_{\mathcal{N}})^R_{\alpha_o}] + [(\tilde{B}_{\mathcal{N}})^L_{\alpha_o}, (\tilde{B}_{\mathcal{N}})^R_{\alpha_o}]; [(\tilde{A}_{\mathcal{N}})^L_{\beta_o}, (\tilde{A}_{\mathcal{N}})^R_{\beta_o}] + [(\tilde{B}_{\mathcal{N}})^L_{\beta_o}, (\tilde{B}_{\mathcal{N}})^R_{\beta_o}]; [(\tilde{A}_{\mathcal{N}})^L_{\gamma_o}, (\tilde{A}_{\mathcal{N}})^R_{\gamma_o}] + [(\tilde{B}_{\mathcal{N}})^L_{\gamma_o}, (\tilde{B}_{\mathcal{N}})^R_{\gamma_o}]\big] \quad (14.1)$$

where

$$[(\tilde{A}_{\mathcal{N}})^L_{\alpha_o},(\tilde{A}_{\mathcal{N}})^R_{\alpha_o}]+[(\tilde{B}_{\mathcal{N}})^L_{\alpha_o},(\tilde{B}_{\mathcal{N}})^R_{\alpha_o}]=\begin{cases}[((\tilde{A}_{\mathcal{N}})^L_{\alpha_o})_1+((\tilde{B}_{\mathcal{N}})^L_{\alpha_o})_1,((\tilde{A}_{\mathcal{N}})^R_{\alpha_o})_1+((\tilde{B}_{\mathcal{N}})^R_{\alpha_o})_1] \\ \quad for\ \alpha_o\in[0,k] \\ [((\tilde{A}_{\mathcal{N}})^L_{\alpha_o})_2+((\tilde{B}_{\mathcal{N}})^L_{\alpha_o})_2,((\tilde{A}_{\mathcal{N}})^R_{\alpha_o})_2+((\tilde{B}_{\mathcal{N}})^R_{\alpha_o})_2] \\ \quad for\ \alpha_o\in(k,1]\end{cases}$$

$$=\begin{cases}[p_1+y_1+\frac{\alpha_o}{k}(p_2-p_1+y_2-y_1),p_8+y_7-\frac{\alpha_o}{k}(p_8-p_7+y_8-y_7)] & for\ \alpha_o\in[0,k]\\ [p_3+y_3+\frac{\alpha_o-k}{1-k}(p_4-p_3+y_4-y_3),p_6+y_6-\frac{\alpha_o-k}{1-k}(p_6-p_5+y_6-y_5)] & for\ \alpha_o\in(k,1]\end{cases}$$

$$[(\tilde{A}_{\mathcal{N}})^L_{\beta_o},(\tilde{A}_{\mathcal{N}})^R_{\beta_o}]+[(\tilde{B}_{\mathcal{N}})^L_{\beta_o},(\tilde{B}_{\mathcal{N}})^R_{\beta_o}]=\begin{cases}[((\tilde{A}_{\mathcal{N}})^L_{\beta_o})_1+((\tilde{B}_{\mathcal{N}})^L_{\beta_o})_1,((\tilde{A}_{\mathcal{N}})^R_{\beta_o})_1+((\tilde{B}_{\mathcal{N}})^R_{\beta_o})_1] \\ \quad for\ \beta_o\in[0,k] \\ [((\tilde{A}_{\mathcal{N}})^L_{\beta_o})_2+((\tilde{B}_{\mathcal{N}})^L_{\beta_o})_2,((\tilde{A}_{\mathcal{N}})^R_{\beta_o})_2+((\tilde{B}_{\mathcal{N}})^R_{\beta_o})_2] \\ \quad for\ \beta_o\in(k,1]\end{cases}$$

$$=\begin{cases}[q_3+z_3-\frac{\beta_o-k}{k}(q_4-q_3+z_4-z_3),q_5+z_5+\frac{\beta_o}{k}(q_6-q_5+z_6-z_5)] & for\ \beta_o\in[0,k]\\ [q_1+z_1-\frac{\beta_o-1}{1-k}(q_2-q_1+z_2-z_1),q_7+z_7+\frac{\beta_o-k}{1-k}(q_8-q_7+z_8-z_7)] & for\ \beta_o\in(k,1]\end{cases}$$

$$[(\tilde{A}_{\mathcal{N}})^L_{\gamma_o},(\tilde{A}_{\mathcal{N}})^R_{\gamma_o}]+[(\tilde{B}_{\mathcal{N}})^L_{\gamma_o},(\tilde{B}_{\mathcal{N}})^R_{\gamma_o}]=\begin{cases}[((\tilde{A}_{\mathcal{N}})^L_{\gamma_o})_1+((\tilde{B}_{\mathcal{N}})^L_{\gamma_o})_1,((\tilde{A}_{\mathcal{N}})^R_{\gamma_o})_1+((\tilde{B}_{\mathcal{N}})^R_{\gamma_o})_1] \\ \quad for\ \gamma_o\in[0,k] \\ [((\tilde{A}_{\mathcal{N}})^L_{\gamma_o})_2+((\tilde{B}_{\mathcal{N}})^L_{\gamma_o})_2,((\tilde{A}_{\mathcal{N}})^R_{\gamma_o})_2+((\tilde{B}_{\mathcal{N}})^R_{\gamma_o})_2] \\ \quad for\ \gamma_o\in(k,1]\end{cases}$$

$$=\begin{cases}[r_3+g_3-\frac{\gamma_o-k}{k}(r_4-r_3+g_4-g_3),r_5+g_5+\frac{\gamma_o}{k}(r_6-r_5+g_6-g_5)] & for\ \gamma_o\in[0,k]\\ [r_1+g_1-\frac{\gamma_o-1}{1-k}(r_2-r_1+g_2-g_1),r_7+g_7+\frac{\gamma_o-k}{1-k}(r_8-r_7+g_8-g_7)] & for\ \gamma_o\in(k,1]\end{cases}$$

2. Scalar Multiplication:

$$s(\tilde{A}_{\mathcal{N}})_{\alpha_o,\beta_o,\gamma_o}=\begin{cases}[(s\tilde{A}_{\mathcal{N}})^L_{\alpha_o},(s\tilde{A}_{\mathcal{N}})^R_{\alpha_o}];[(s\tilde{A}_{\mathcal{N}})^L_{\beta_o},(s\tilde{A}_{\mathcal{N}})^R_{\beta_o}];[(s\tilde{A}_{\mathcal{N}})^L_{\gamma_o},(s\tilde{A}_{\mathcal{N}})^R_{\gamma_o}] & for\ s\geq 0\\ [(s\tilde{A}_{\mathcal{N}})^R_{\alpha_o},(s\tilde{A}_{\mathcal{N}})^L_{\alpha_o}];[(s\tilde{A}_{\mathcal{N}})^R_{\beta_o},(s\tilde{A}_{\mathcal{N}})^L_{\beta_o}];[(s\tilde{A}_{\mathcal{N}})^R_{\gamma_o},(s\tilde{A}_{\mathcal{N}})^L_{\gamma_o}] & for\ s<0\end{cases} \tag{14.2}$$

where, for $s\geq 0$

$$[(s\tilde{A}_{\mathcal{N}})^L_{\alpha_o},(s\tilde{A}_{\mathcal{N}})^R_{\alpha_o}]=\begin{cases}[(s(\tilde{A}_{\mathcal{N}})^L_{\alpha_o})_1,(s(\tilde{A}_{\mathcal{N}})^R_{\alpha_o})_1] & \alpha_o\in[0,k]\\ [(s(\tilde{A}_{\mathcal{N}})^L_{\alpha_o})_2,(s(\tilde{A}_{\mathcal{N}})^R_{\alpha_o})_2] & \alpha_o\in(k,1]\end{cases}$$

$$=\begin{cases}[sp_1+\frac{\alpha_o}{k}(sp_2-sp_1),sp_8-\frac{\alpha_o}{k}(sp_8-sp_7)] & for\ \alpha_o\in[0,k]\\ [sp_3+\frac{\alpha_o-k}{1-k}(sp_4-sp_3),sp_6-\frac{\alpha_o-k}{1-k}(sp_6-sp_5)] & for\ \alpha_o\in(k,1]\end{cases}$$

$$[(s\tilde{A}_{\mathcal{N}})^R_{\beta_o},(s\tilde{A}_{\mathcal{N}})^L_{\beta_o}]=\begin{cases}[(s(\tilde{A}_{\mathcal{N}})^L_{\beta_o})_1,(s(\tilde{A}_{\mathcal{N}})^R_{\beta_o})_1] & \beta_o\in[0,k]\\ [(s(\tilde{A}_{\mathcal{N}})^L_{\beta_o})_2,(s(\tilde{A}_{\mathcal{N}})^R_{\beta_o})_2] & \beta_o\in(k,1]\end{cases}$$

$$=\begin{cases}[sq_3-\frac{\beta_o-k}{k}(sq_4-sq_3),sq_5+\frac{\beta_o}{k}(sq_6-sq_5)] & for\ \beta_o\in[0,k]\\ [sq_1-\frac{\beta_o-1}{1-k}(sq_2-sq_1),sq_7+\frac{\beta_o-k}{1-k}(sq_8-sq_7)] & for\ \beta_o\in(k,1]\end{cases}$$

$$[(s\tilde{A}_{\mathcal{N}})^L_{\gamma_o},(s\tilde{A}_{\mathcal{N}})^R_{\gamma_o}]=\begin{cases}[(s(\tilde{A}_{\mathcal{N}})^L_{\gamma_o})_1,(s(\tilde{A}_{\mathcal{N}})^R_{\gamma_o})_1] & \gamma_o\in[0,k]\\ [(s(\tilde{A}_{\mathcal{N}})^L_{\gamma_o})_2,(s(\tilde{A}_{\mathcal{N}})^R_{\gamma_o})_2] & \gamma_o\in(k,1]\end{cases}$$

$$=\begin{cases}[sr_3-\frac{\gamma_o-k}{k}(sr_4-sr_3),sr_5+\frac{\gamma_o}{k}(sr_6-sr_5)] & for\ \gamma_o\in[0,k]\\ [sr_1-\frac{\gamma_o-1}{1-k}(sr_2-sr_1),sr_7+\frac{\gamma_o-k}{1-k}(sr_8-sr_7)] & for\ \gamma_o\in(k,1]\end{cases}$$

and for $s<0$

$$[(s\tilde{A}_{\mathcal{N}})^R_{\alpha_o},(s\tilde{A}_{\mathcal{N}})^L_{\alpha_o}]=\begin{cases}[(s(\tilde{A}_{\mathcal{N}})^R_{\alpha_o})_1,(s(\tilde{A}_{\mathcal{N}})^L_{\alpha_o})_1] & \alpha_o\in[0,k]\\ [(s(\tilde{A}_{\mathcal{N}})^R_{\alpha_o})_2,(s(\tilde{A}_{\mathcal{N}})^L_{\alpha_o})_2] & \alpha_o\in(k,1]\end{cases}$$

$$=\begin{cases}[sp_8-\frac{\alpha_o}{k}(sp_8-sp_7),sp_1+\frac{\alpha_o}{k}(sp_2-sp_1)] & for\ \alpha_o\in[0,k]\\ [sp_6-\frac{\alpha_o-k}{1-k}(sp_6-sp_5),sp_3+\frac{\alpha_o-k}{1-k}(sp_4-sp_3)] & for\ \alpha_o\in(k,1]\end{cases}$$

$$[(s\tilde{A}_{\mathcal{N}})^R_{\beta_o}, (s\tilde{A}_{\mathcal{N}})^L_{\beta_o}] = \begin{cases} [(s(\tilde{A}_{\mathcal{N}})^R_{\beta_o})_1, (s(\tilde{A}_{\mathcal{N}})^L_{\beta_o})_1] & \beta_o \in [0, k] \\ [(s(\tilde{A}_{\mathcal{N}})^R_{\beta_o})_2, (s(\tilde{A}_{\mathcal{N}})^L_{\beta_o})_2] & \beta_o \in (k, 1] \end{cases}$$

$$= \begin{cases} [sq_5 + \frac{\beta_o}{k}(sq_6 - sq_5), sq_3 - \frac{\beta_o - k}{k}(sq_4 - sq_3)] & for\ \beta_o \in [0, k] \\ [sq_7 + \frac{\beta_o - k}{1-k}(sq_8 - sq_7), sq_1 - \frac{\beta_o - 1}{1-k}(sq_2 - sq_1)] & for\ \beta_o \in (k, 1] \end{cases}$$

$$[(s\tilde{A}_{\mathcal{N}})^R_{\gamma_o}, (s\tilde{A}_{\mathcal{N}})^L_{\gamma_o}] = \begin{cases} [(s(\tilde{A}_{\mathcal{N}})^R_{\gamma_o})_1, (s(\tilde{A}_{\mathcal{N}})^L_{\gamma_o})_1] & \gamma_o \in [0, k] \\ [(s(\tilde{A}_{\mathcal{N}})^R_{\gamma_o})_2, (s(\tilde{A}_{\mathcal{N}})^L_{\gamma_o})_2] & \gamma_o \in (k, 1] \end{cases}$$

$$= \begin{cases} [sr_5 + \frac{\gamma_o}{k}(sr_6 - sr_5), sr_3 - \frac{\gamma_o - k}{k}(sr_4 - sr_3)] & for\ \gamma_o \in [0, k] \\ [sr_7 + \frac{\gamma_o - k}{1-k}(sr_8 - sr_7), sr_1 - \frac{\gamma_o - 1}{1-k}(sr_2 - sr_1)] & for\ \gamma_o \in (k, 1] \end{cases}$$

Definition 14.4 Let $\tilde{A}_{\mathcal{N}} = \langle p_1, p_2, \ldots, p_8; q_1, q_2, \ldots, q_8; r_1, r_2, \ldots, r_8 \rangle$ and

$\tilde{B}_{\mathcal{N}} = \langle x_1, x_2, \ldots, x_8; y_1, y_2, \ldots, y_8; z_1, z_2, \ldots, z_8 \rangle$ be two SVLONNs and let s be any real number, then the coordinatewise addition and scalar multiplication are defined as follows:

1. $$\tilde{A}_{\mathcal{N}} \oplus \tilde{B}_{\mathcal{N}} = \langle p_1 + x_1, \ldots, p_8 + x_8; q_1 + y_1, \ldots, q_8 + y_8; r_1 + z_1, \ldots, r_8 + z_8 \rangle \tag{14.3}$$

2. $$s\tilde{A}_{\mathcal{N}} = \begin{cases} \langle sp_1, \ldots, sp_8; sq_1, \ldots, sq_8; sr_1, \ldots, sr_8 \rangle & for\ s \geq 0 \\ \langle sp_8, \ldots, sp_1; sq_8, \ldots, sq_1; sr_8, \ldots, sr_1 \rangle & for\ s < 0 \end{cases} \tag{14.4}$$

Theorem 14.1 *The (α_o,β_o,γ_o)-cut approach and coordinate approach of the addition and scalar multiplication of SVLONNs yields the same SVLONN.*

Proof It is enough to show that
$(\tilde{A}_{\mathcal{N}})_{\alpha_o,\beta_o,\gamma_o} + (\tilde{B}_{\mathcal{N}})_{\alpha_o,\beta_o,\gamma_o} = (\tilde{A}_{\mathcal{N}} + \tilde{B}_{\mathcal{N}})_{\alpha_o,\beta_o,\gamma_o}$ and $s(\tilde{A}_{\mathcal{N}})_{\alpha_o,\beta_o,\gamma_o} = (s\tilde{A}_{\mathcal{N}})_{\alpha_o,\beta_o,\gamma_o}$.
From Definition 14.3 and Definition 14.4,we observe that
RHS of equation (14.1)=$(\tilde{A}_{\mathcal{N}} + \tilde{B}_{\mathcal{N}})_{\alpha_o,\beta_o,\gamma_o}$ and RHS of equation (14.2)=$(s\tilde{A}_{\mathcal{N}})_{\alpha_o,\beta_o,\gamma_o}$ respectively. □

14.3 Value and Ambiguity Index-Based Ranking for SVLONNs

In this section, we define Value and Ambiguity of SVLONNs. And by using the same, we define value and ambiguity index of SVLONNs.

Definition 14.5 Let $\tilde{A}_{\mathcal{N}}$ be a SVLONN, then the value of the truth $[V_T(\tilde{A}_{\mathcal{N}})]$, indeterminacy $[V_I(\tilde{A}_{\mathcal{N}})]$ and falsity $[V_F(\tilde{A}_{\mathcal{N}})]$ membership grade of $\tilde{A}_{\mathcal{N}}$ are respectively defined using the weighting function $f(\alpha_o)$, $g(\beta_o)$ and $h(\gamma_o)$ as

$$V_T(\tilde{A}_{\mathcal{N}}) = \int_0^k [((\tilde{A}_{\mathcal{N}})^L_{\alpha_o})_1 + ((\tilde{A}_{\mathcal{N}})^R_{\alpha_o})_1] f(\alpha_o) d\alpha_o + \int_k^1 [((\tilde{A}_{\mathcal{N}})^L_{\alpha_o})_2 + ((\tilde{A}_{\mathcal{N}})^R_{\alpha_o})_2] f(\alpha_o) d\alpha_o \tag{14.5}$$

$$V_I(\tilde{A}_{\mathcal{N}}) = \int_0^k [((\tilde{A}_{\mathcal{N}})^L_{\beta_o})_1 + ((\tilde{A}_{\mathcal{N}})^R_{\beta_o})_1] g(\beta_o) d\beta_o + \int_k^1 [((\tilde{A}_{\mathcal{N}})^L_{\beta_o})_2 + ((\tilde{A}_{\mathcal{N}})^R_{\beta_o})_2] g(\beta_o) d\beta_o \tag{14.6}$$

$$V_F(\tilde{A}_{\mathcal{N}}) = \int_0^k [((\tilde{A}_{\mathcal{N}})^L_{\gamma_o})_1 + ((\tilde{A}_{\mathcal{N}})^R_{\gamma_o})_1] h(\gamma_o) d\gamma_o + \int_k^1 [((\tilde{A}_{\mathcal{N}})^L_{\gamma_o})_2 + ((\tilde{A}_{\mathcal{N}})^R_{\gamma_o})_2] h(\gamma_o) d\gamma_o \tag{14.7}$$

Remark 14.3 1. The decision maker can set the Weighting functions according to the nature of the problems in real situations.

2. For example the choice of the functions $f(\alpha_o) = \alpha_o$, $g(\beta_o) = 1 - \beta_o$ and $h(\gamma_o) = 1 - \gamma_o$ where $\alpha_o, \beta_o, \gamma_o \in [0, 1]$, give rise to different weights to elements in different α_o-cut, β_o-cut, γ_o-cut sets and make less the contribution of the lower α_o-cut sets, reduce the contribution of higher β_o-cut, γ_o-cut sets. These are acceptable as the cut sets arising from values of $T_{\tilde{A}_{\mathcal{N}}}(x)$, $I_{\tilde{A}_{\mathcal{N}}}(x)$, and $F_{\tilde{A}_{\mathcal{N}}}(x)$ deals with a considerable amount of uncertainty.
3. $V_T(\tilde{A}_{\mathcal{N}})$, $V_I(\tilde{A}_{\mathcal{N}})$, and $V_F(\tilde{A}_{\mathcal{N}})$ synthetically reflects the information on every membership degree, indeterminacy degree, and falsity degree respectively.
4. $V_T(\tilde{A}_{\mathcal{N}})$, $V_I(\tilde{A}_{\mathcal{N}})$, and $V_F(\tilde{A}_{\mathcal{N}})$ may be considered as a central value that represents the membership function, indeterminacy function, and falsity function.

Substituting $f(\alpha_o) = \alpha_o$, $g(\beta_o) = 1 - \beta_o$ and $h(\gamma_o) = 1 - \gamma_o$ in Eqs. (14.5), (14.6) and (14.7) respectively, we compute the values of truth membership function, indeterminacy membership function, and falsity membership function using MathCad 14 and are given by

$$V_T(\tilde{A}_{\mathcal{N}}) = \frac{k^2}{6}(p_1 + 2p_2 + 2p_7 + p_8) + \frac{k-1}{6}(2kp_3 + kp_4 + kp_5 + 2kp_6 + p_3 + 2p_4 + 2p_5 + p_6)$$

$$V_I(\tilde{A}_{\mathcal{N}}) = \frac{k}{2}(q_3 + q_4 + q_5 + q_6) - \frac{k^2}{6}(2q_3 + q_4 + q_5 + 2q_6) + \frac{(k-1)^2}{6}(q_1 + 2q_2 + 2q_7 + q_8)$$

$$V_F(\tilde{A}_{\mathcal{N}}) = \frac{k}{2}(r_3 + r_4 + r_5 + r_6) - \frac{k^2}{6}(2r_3 + r_4 + r_5 + 2r_6) + \frac{(k-1)^2}{6}(r_1 + 2r_2 + 2r_7 + r_8)$$

Definition 14.6 Let $\tilde{A}_{\mathcal{N}}$ be a SVLONN. Then the ambiguity of the truth $[A_T(\tilde{A}_{\mathcal{N}})]$, indeterminacy $[A_I(\tilde{A}_{\mathcal{N}})]$ and falsity $[A_F(\tilde{A}_{\mathcal{N}})]$ membership grade of $\tilde{A}_{\mathcal{N}}$ are respectively defined using the functions $f(\alpha_o)$, $g(\beta_o)$ and $h(\gamma_o)$ as

$$A_T(\tilde{A}_{\mathcal{N}}) = \int_0^k [((\tilde{A}_{\mathcal{N}})^R_{\alpha_o})_1 - ((\tilde{A}_{\mathcal{N}})^L_{\alpha_o})_1] f(\alpha_o) d\alpha_o + \int_k^1 [((\tilde{A}_{\mathcal{N}})^R_{\alpha_o})_2 - ((\tilde{A}_{\mathcal{N}})^L_{\alpha_o})_2] f(\alpha_o) d\alpha_o \tag{14.8}$$

$$A_I(\tilde{A}_{\mathcal{N}}) = \int_0^k [((\tilde{A}_{\mathcal{N}})^R_{\beta_o})_1 - ((\tilde{A}_{\mathcal{N}})^L_{\beta_o})_1] g(\beta_o) d\beta_o + \int_k^1 [((\tilde{A}_{\mathcal{N}})^R_{\beta_o})_1 - ((\tilde{A}_{\mathcal{N}})^L_{\beta_o})_2] g(\beta_o) d\beta_o \tag{14.9}$$

$$A_F(\tilde{A}_{\mathcal{N}}) = \int_0^k [((\tilde{A}_{\mathcal{N}})^R_{\gamma_o})_1 - ((\tilde{A}_{\mathcal{N}})^L_{\gamma_o})_1] h(\gamma_o) d\gamma_o + \int_k^1 [((\tilde{A}_{\mathcal{N}})^R_{\gamma_o})_2 - ((\tilde{A}_{\mathcal{N}})^L_{\gamma_o})_2] h(\gamma_o) d\gamma_o \tag{14.10}$$

Remark 14.4 In the above definition, we observe that $((\tilde{A}_{\mathcal{N}})^R_{\alpha_o})_j - ((\tilde{A}_{\mathcal{N}})^L_{\alpha_o})_j$, $((\tilde{A}_{\mathcal{N}})^R_{\beta_o})_j - ((\tilde{A}_{\mathcal{N}})^L_{\beta_o})_j$, $((\tilde{A}_{\mathcal{N}})^R_{\gamma_o})_j - ((\tilde{A}_{\mathcal{N}})^L_{\gamma_o})_j$ for j=1,2 express the length of the intervals of $(\tilde{A}_{\mathcal{N}})_{\alpha_o}, (\tilde{A}_{\mathcal{N}})_{\beta_o}, (\tilde{A}_{\mathcal{N}})_{\gamma_o}$ respectively. Thus $A_T(\tilde{A}_{\mathcal{N}}), A_I(\tilde{A}_{\mathcal{N}}), A_F(\tilde{A}_{\mathcal{N}})$ can be viewed as the global spreads of the truth membership function, indeterminacy membership function, and falsity membership function respectively. The vagueness of $\tilde{A}_{\mathcal{N}}$ is determined using the ambiguity of these three functions.

We derive the ambiguity of truth membership function, indeterminacy membership function, and falsity membership function using MathCad 14 for $f(\alpha_o) = \alpha_o$, $g(\beta_o) = 1 - \beta_o$ and $h(\gamma_o) = 1 - \gamma_o$ in Eqs. (14.8), (14.9) and (14.10) respectively yield:

$$A_T(\tilde{A}_{\mathcal{N}}) = \frac{k-1}{6}(2kp_3 + kp_4 - kp_5 - 2kp_6 + p_3 + 2p_4 - 2p_5 - p_6) + \frac{k^2}{6}(2p_7 - 2p_2 + p_8 - p_1)$$

$$A_I(\tilde{A}_{\mathcal{N}}) = \frac{k}{2}(q_5 - q_4 - q_3 + q_6) - \frac{(k-1)^2}{6}(q_1 + 2q_2 - 2q_7 - q_8) + \frac{(k)^2}{6}(2q_3 + q_4 - q_5 - q_6)$$

$$A_F(\tilde{A}_{\mathcal{N}}) = \frac{k}{2}(r_5 - r_4 - r_3 + r_6) - \frac{(k-1)^2}{6}(r_1 + 2r_2 - 2r_7 - r_8) + \frac{(k)^2}{6}(2r_3 + r_4 - r_5 - r_6)$$

By using the value and ambiguity of truth membership function, indeterminacy membership function, falsity membership function computed in Remarks 14.3, 14.4, the value and ambiguity index of SVLONNs are defined as follows:

Definition 14.7 For a SVLONN $\tilde{A}_{\mathcal{N}}$, the value index and ambiguity index are given by

$$V_{\psi,\eta,\zeta} = \psi V_T + \eta V_I + \zeta V_F \tag{14.11}$$

$$A_{\psi,\eta,\zeta} = \psi A_T + \eta A_I + \zeta A_F \tag{14.12}$$

where the co-efficient ψ, η, ζ appearing in Eqs. (14.11) *and* (14.12) *expresses respectively the preference value of the decision maker such that $\psi + \eta + \zeta = 1$.*

Remark 14.5 For $\psi \in [0, \frac{1}{3}]$ and $\eta + \zeta \in [\frac{1}{3}, 1]$, a decision maker makes a pessimistic decision in an uncertain environment. On the other hand, the decision maker may desire to make an optimistic decision in an uncertain environment for $\psi \in [\frac{1}{3}, 1]$ and $\eta + \zeta \in [0, \frac{1}{3}]$. Also, if a decision maker chooses $\psi = \eta = \zeta = \frac{1}{3}$, then there is an equal importance to truth, indeterminacy, and falsity. Therefore, the value index and ambiguity index reflect the attitude of the decision maker for SVLONN.

Theorem 14.2 *Let $\tilde{A}_{\mathcal{N}_1}$ and $\tilde{A}_{\mathcal{N}_2}$ be two SVLONNs. Then for $\psi, \eta, \zeta \in [0, 1]$ and $\phi \in \mathbb{R}$,*

$$V_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_1} + \tilde{A}_{\mathcal{N}_2}) = V_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_1}) + V_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_2}) \tag{14.13}$$

$$V_{\psi,\eta,\zeta}(\phi\tilde{A}_{\mathcal{N}_1}) = \phi V_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_1}) \tag{14.14}$$

Proof We prove this theorem by using the Definitions 14.4 and 14.7.

$$
\begin{aligned}
V_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_1}+\tilde{A}_{\mathcal{N}_2}) &= \psi V_T(\tilde{A}_{\mathcal{N}_1}+\tilde{A}_{\mathcal{N}_2}) + \eta V_I(\tilde{A}_{\mathcal{N}_1}+\tilde{A}_{\mathcal{N}_2}) + \zeta V_F(\tilde{A}_{\mathcal{N}_1}+\tilde{A}_{\mathcal{N}_2}) \\
&= \frac{\psi}{6}\Big[\frac{k^2}{6}\left((p_1+x_1)+2(p_2+x_2)+2(p_7+x_7)+(p_8+x_8)\right) \\
&\quad + \frac{k-1}{6}(2k(p_3+x_3)+k(p_4+x_4)) + k(p_5+x_5) \\
&\quad + 2k(p_6+x_6) + (p_3+x_3) + 2(p_4+x_4) + 2(p_5+x_5) + (p_6+x_6)\Big] \\
&\quad + \frac{\eta}{6}\Big[\frac{k}{2}\big((q_3+y_3)+(q_4+y_4)+(q_5+y_5)+(q_6+y_6)\big) \\
&\quad - \frac{k^2}{6}(2(q_3+y_3)+(q_4+y_4)+(q_5+y_5) \\
&\quad + 2(q_6+y_6)) + \frac{(k-1)^2}{6} \\
&\quad \times ((q_1+y_1)+2(q_2+y_2)+2(q_7+y_7)+(q_8+y_8))\Big] \\
&\quad + \frac{\zeta}{6}\Big[\frac{k}{2}\big((r_3+z_3)+(r_4+z_4)+(r_5+z_5)+(r_6+z_6)\big) \\
&\quad - \frac{k^2}{6}(2(r_3+z_3)+(r_4+z_4)+(r_5+z_5)+2(r_6+z_6)) \\
&\quad + \frac{(k-1)^2}{6}((r_1+z_1)+2(r_2+z_2)+2(r_7+z_7) \\
&\quad + (r_8+z_8))\Big] - \frac{k^2}{6}(2(r_3+z_3)+(r_4+z_4) \\
&= V_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_1}) + V_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_2})
\end{aligned}
$$

and

$$
\begin{aligned}
V_{\psi,\eta,\zeta}(\phi\tilde{A}_{\mathcal{N}_1}) &= \psi V_T(\phi\tilde{A}_{\mathcal{N}_1}) + \eta V_I(\phi\tilde{A}_{\mathcal{N}_1}) + \zeta V_F(\phi\tilde{A}_{\mathcal{N}_1}) \\
&= \frac{\psi}{6}\Big[\frac{k^2}{6}\left(\phi p_1 + 2\phi p_2 + 2\phi p_7 + \phi p_8\right) + \frac{k-1}{6}(2k\phi p_3 + k\phi p_4 + k\phi p_5 + 2k\phi p_6 + \phi p_3 + 2\phi p_4 + 2\phi p_5 \\
&\quad + \phi p_6)\Big] + \frac{\eta}{6}\Big[\frac{k}{2}\left(\phi q_3 + \phi q_4 + \phi q_5 + \phi q_6\right) - \frac{k^2}{6}(2\phi q_3 + \phi q_4 + \phi q_5 + 2\phi q_6) + \frac{(k-1)^2}{6}(\phi q_1 + 2\phi q_2 \\
&\quad + 2\phi q_7 + \phi q_8)\Big] + \frac{\zeta}{6}\Big[\frac{k}{2}\left(\phi r_3 + \phi r_4 + \phi r_5 + \phi r_6\right) - \frac{k^2}{6}(2\phi r_3 + \phi r_4 + \phi r_5 + 2\phi r_6) \\
&\quad + \frac{(k-1)^2}{6}(\phi r_1 + 2\phi r_2 + 2\phi r_7 + \phi r_8)\Big] \\
&= \phi V_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_1})
\end{aligned}
$$

which completes the proof. □

Theorem 14.3 *Let $\tilde{A}_{\mathcal{N}_1}$ and $\tilde{A}_{\mathcal{N}_2}$ be two SVLONNs. Then for $\psi, \eta, \zeta \in [0, 1]$ and $\phi \in \mathbb{R}$,*

$$A_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_1}+\tilde{A}_{\mathcal{N}_2}) = A_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_1}) + V_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_2}) \tag{14.15}$$

$$A_{\psi,\eta,\zeta}(\phi\tilde{A}_{\mathcal{N}_1}) = \phi A_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_1}) \tag{14.16}$$

Proof We prove this theorem by using the Definitions 14.4 and 14.7. The proof of this theorem is as that of theorem (14.2). □

Definition 14.8 We compare two SVLONNs $\tilde{A}_{\mathcal{N}_1}$ and $\tilde{A}_{\mathcal{N}_2}$ by using value and ambiguity indices as follows:

1. If $\tilde{A}_{\mathcal{N}_1} \preceq \tilde{A}_{\mathcal{N}_2}$ then $V_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_1}) \leq V_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_2})$.
2. If $\tilde{A}_{\mathcal{N}_1} \succeq \tilde{A}_{\mathcal{N}_2}$ then $V_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_1}) \geq V_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_2})$.
3. If $V_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_1}) = V_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_2})$, then by using ambiguity index we compare them in the following ways:
   a. If $A_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_1}) \geq A_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_2})$ then $\tilde{A}_{\mathcal{N}_1} \succeq \tilde{A}_{\mathcal{N}_2}$.
   b. If $A_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_1}) \leq A_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_2})$ then $\tilde{A}_{\mathcal{N}_1} \preceq \tilde{A}_{\mathcal{N}_2}$.
   c. If $A_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_1}) = A_{\psi,\eta,\zeta}(\tilde{A}_{\mathcal{N}_2})$ then $\tilde{A}_{\mathcal{N}_1} \approx \tilde{A}_{\mathcal{N}_2}$.

14.4 Formulation of a Multi-attribute Decision-Making Problem

In this section, we consider an Multi-Attribute Decision Making (MADM) problem where the attributes are given by SVLONNs. Let us assume that for an MADM problem, $U = \{U_1, U_2, \ldots, U_t\}$ be the set of t alternatives, and $E = \{E_1, E_2, \ldots, E_v\}$ be the set of v attributes and the weight vector provided by the decision maker for the attributes be $W = (w_1, w_2, \ldots, w_v)^T$, where $w_j \in [0, 1]$, $\sum_{j=1}^{v} w_j = 1$ and w_j is the degree of importance for the attribute E_j. Therefore, we express the alternatives U_i over the attributes E_j by SVLONN $d_{ij} = \langle p_{ij}^1, p_{ij}^2, p_{ij}^3, p_{ij}^4, p_{ij}^5, p_{ij}^6, p_{ij}^7, p_{ij}^8; q_{ij}^1, q_{ij}^2, q_{ij}^3, q_{ij}^4, q_{ij}^5, q_{ij}^6, q_{ij}^7, q_{ij}^8; r_{ij}^1, r_{ij}^2, r_{ij}^3, r_{ij}^4, r_{ij}^5, r_{ij}^6, r_{ij}^7, r_{ij}^8\rangle$ where $p_{ij}^k, q_{ij}^k, r_{ij}^k, \in \mathbb{R}$ for $i = 1, 2, 3, \ldots, t$; $j = 1, 2, 3, \ldots, v$ and $k = 1, 2, 3, \ldots, 8$ and the neutrosophic decision matrix $D = (\tilde{d}_{ij})_{m\times n}$, where

$$(\tilde{d}_{ij})_{m\times n} = \begin{array}{c|cccc} & E_1 & E_2 & \ldots & E_n \\ \hline U_1 & \tilde{d}_{11} & \tilde{d}_{12} & \ldots & \tilde{d}_{1v} \\ U_2 & \tilde{d}_{21} & \tilde{d}_{22} & \ldots & \tilde{d}_{2v} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ U_t & \tilde{d}_{t1} & \tilde{d}_{t2} & \ldots & \tilde{d}_{tv} \end{array}$$

for $i = 1, 2, \ldots, t$ and $j = 1, 2, \ldots, v$. Here we apply Value and ambiguity indices of SVLONNs to solve MADM problem.

1. **Normalization of SVLONN based on decision matrix**
   In order to eliminate the effect of different physical dimensions during the process of final decision-making, the decision matrix $(\tilde{d}_{ij})_{t\times v}$ is normalized into $(\tilde{r}_{ij})_{t\times v}$ by using the linear normalization technique.
2. **Aggregation of the weighted values of alternatives**
   The aggregated weighted values of the alternatives $U_i (i = 1, 2, \ldots, t)$ is determined by
   $$\tilde{S}_i = \sum_{j=1}^{v} w_j \tilde{r}_{ij} \tag{14.17}$$
   respectively. Here the aggregated weighted values $\tilde{S}_i (i = 1, 2, \ldots, t)$ are defined using SVLONNs.
3. **Ranking of all alternatives**
   Ranking of all alternatives of all alternatives is determined by using the value and ambiguity indices of SVLONN using $\tilde{S}_i$.

14.5 Illustration of MADM Problem

Consider the following situation experienced by a Principal of a College in selecting a candidate for the post of Assistant Professor in the college. Suppose three candidates say U_1, U_2, U_3 has been shortlisted after a written test to be appointed as an assistant professor based on the criteria such as educational qualification (E_1), past experience (E_2), and research publications (E_3). The assessment of the recruiter for each candidate based on the information corresponding to the attributes are expressed as SVLONNs using the linguistic terms:
Extremely Low-(0.0, 0.0, 0.0, 0.0, 0.0, 0.05, 0.1, 0.15;0.3);
Low-(0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4;0.3);
Neither low nor medium-(0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5, 0.55;0.3);
Medium-(0.35, 0.40, 0.45, 0.5, 0.55, 0.6, 0.65, 0.7;0.3);
Neither medium nor high-(0.5, 0.55, 0.6, 0.65, 0.7, 0.75, 0.8, 0.85;0.3);
High-(0.6, 0.65, 0.7, 0.75, 0.8, 0.85, 0.9, 0.95;0.3);
Extremely High-(0.75, 0.8, 0.85, 0.9, 0.92, 0.96, 0.98, 1.0;0.3). The assigned weight vectors of three attributes are $w = \{0.31, 0.34, 0.35\}$. The complete perception of the Principal about the individual is modeled as SVLONNs and the decision matrix is given by Table 14.1 where E_i represents the attributes, U_1, U_2, U_3 are alternatives and $\tilde{d}_{ij}$ are SVLONNs.

Table 14.1 SVLONNs-based decision matrix

$$(\tilde{d}_{ij})_{3\times 3} = \begin{array}{c|ccc} & E_1 & E_2 & E_3 \\ \hline U_1 & \tilde{d}_{11} & \tilde{d}_{12} & \tilde{d}_{13} \\ U_2 & \tilde{d}_{21} & \tilde{d}_{22} & \tilde{d}_{23} \\ U_3 & \tilde{d}_{31} & \tilde{d}_{32} & \tilde{d}_{33} \end{array}$$

Based on the individual U_1's educational qualification, we have
$\tilde{d}_{11}$= ⟨ extremely high; medium; extremely low⟩
=⟨0.75,0.80 ,0.85, 0.90, 0.92, 0.96, 0.98, 1.00;0.35, 0.40, 0.45, 0.50, 0.550.60, 0.65, 0.70;0.00, 0.00, 0.00, 0.00, 0.05, 0.1, 0.15, 0.20;0.3⟩

similarly for U_1's past experience, we have
$\tilde{d}_{12}$= ⟨medium; low; extremely low⟩
=⟨0.35, 0.40, 0.45 ,0.50 ,0.55 ,0.60 ,0.65 ,0.70;0.05 ,0.10 ,0.15 ,0.20 ,0.25 ,0.30 ,0.35 ,0.40;0.00 ,0.00 ,0.00 ,0.00 ,0.05 ,0.10 ,0.15 ,0.20;0.3⟩

and for U_1's research publications, we have
$\tilde{d}_{13}$=⟨neither low nor medium; low; neither medium nor high⟩
=⟨0.20 ,0.25 ,0.30 ,0.35 ,0.40 ,0.45 ,0.50 ,0.55;0.05 ,0.10 ,0.15 ,0.20 ,0.25 ,0.30 ,0.35 ,0.40;0.50 ,0.55 ,0.60 ,0.65, 0.70, 0.75, 0.80, 0.85;0.3⟩

Based on the individual U_2's educational qualification, we have
$\tilde{d}_{21}$=⟨neither low nor medium; high; neither medium nor high⟩
=⟨0.20, 0.25, 0.30, 0.35, 0.40, 0.45, 0.50, 0.55;0.60, 0.65, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95;0.50, 0.55, 0.60, 0.65, 070, 0.75, 0.80, 0.85;0.3⟩

for U_2's past experience, we have
$\tilde{d}_{22}$=⟨high; medium; extremely low⟩
= ⟨0.60, 0.65, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95;0.35, 0.40, 0.45, 0.50, 0.55, 0.60, 0.65, 0.70;0.00, 0.00, 0.00, 0.00, 0.05, 0.10, 0.15, 0.20;0.3⟩

and for U_2's research publications, we have
$\tilde{d}_{23}$=⟨high; medium; extremely low⟩
= ⟨0.60, 0.65, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95;0.35, 0.40, 0.45, 0.50, 0.55, 0.60, 0.65, 0.70;0.00, 0.00, 0.00, 0.00, 0.05, 0.10, 0.15, 0.20;0.3⟩

Based on the individual U_3's educational qualification, we have
$\tilde{d}_{31}$=⟨extremely low; neither low nor medium; high ⟩
=⟨0.00, 0.00, 0.00, 0.00, 0.05, 0.10, 0.15, 0.20;0.20, 0.25, 0.30, 0.35, 0.40, 0.45, 0.50, 0.55;0.60, 0.65, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95;0.3⟩
for U_3's past experience, we have
$\tilde{d}_{32}$=⟨neither low nor medium; high; extremely high⟩
= ⟨0.20, 0.25, 0.30, 0.35, 0.40, 0.45, 0.50, 0.55;0.60, 0.65, 0.70, 0.75, 0.80, 0.85, 0.90, 0.95;0.75, 0.80, 0.85, 0.90, 0.92, 0.96, 0.98, 1.00;0.3⟩

and for U_3's research publications, we have
$\tilde{d}_{33}$=⟨ extremely high; neither low nor medium; low⟩
=0.75, 0.80, 0.85, 0.90, 0.92, 0.96, 0.98, 1.00;0.20, 0.25, 0.30, 0.35, 0.40, 0.45, 0.50, 0.55;0.05, 0.10, 0.05, 0.20, 0.25, 0.30, 0.35, 0.40;0.3⟩

We rank the alternatives U_1, U_2, U_3 by examining the value index and the ambiguity index of each alternative for different values of $\psi, \eta, \zeta \in [0, 1]$ as tabulated in Table 14.2 (using MathCad 14).

Table 14.2 Ranking results for alternatives

Alternative	Value of ψ, η, ζ	Value index	Ambiguity index	Ranking order
U_1		0.30	0.11	
U_2	$\psi = 0.10, \eta = 0.40, \zeta = 0.50$	0.38	0.10	$U_3 > U_2 > U_1$
U_3		0.48	0.09	
U_1		0.10	0.09	
U_2	$\psi = \frac{1}{3}\ \eta = \frac{1}{3}\ \zeta = \frac{1}{3}$	0.15	0.09	$U_3 > U_2 > U_1$
U_3		0.25	0.08	
U_1		–0.22	0.07	
U_2	$\psi = 0.70\ \eta = 0.20\ \zeta = 0.10$	–0.21	0.07	$U_3 > U_2 > U_1$
U_3		–0.10	0.06	

For this choice of values of $\psi, \eta, \zeta \in [0, 1]$, the ranking of alternatives are obtained as follows: $U_3 > U_2 > U_1$.

Remark 14.6 When the problem is carried for the value $k = 0$, the ranking is obtained as follows (Table 14.3):

Table 14.3 Ranking results for alternatives

Alternatives	Value of ψ, η, ζ	Value Index	Ambiguity index	Ranking order
U_1		0.29	0.17	
U_2	$\psi = 0.10, \eta = 0.40, \zeta = 0.50$	0.37	0.17	$U_3 > U_2 > U_1$
U_3		0.47	0.12	
U_1		0.06	0.22	
U_2	$\psi = \frac{1}{3}\ \eta = \frac{1}{3}\ \zeta = \frac{1}{3}$	0.11	0.25	$U_3 > U_2 > U_1$
U_3		0.23	0.10	
U_1		–0.28	0.30	
U_2	$\psi = 0.70\ \eta = 0.20\ \zeta = 0.10$	–0.29	0.37	$U_3 > U_2 > U_1$
U_3		–0.15	0.06	

Remark 14.7 For a different choice of weighting functions, $f(\alpha_o) = 1 - \alpha_o$, $g(\beta_o) = \beta_o$ and $h(\gamma_o) = \gamma_o$ we have the problem worked out along line and the outcome is tabulated in Table 14.4.

Table 14.4 Ranking results for alternatives

Alternatives	Value of ψ, η, ζ	Value index	Ambiguity index	Ranking order
U_1		0.39	0.16	
U_2	$\psi = 0.10, \eta = 0.40, \zeta = 0.50$	0.49	0.15	$U_3 > U_2 > U_1$
U_3		0.55	0.14	
U_1		0.45	0.14	
U_2	$\psi = \frac{1}{3}\ \eta = \frac{1}{3}\ \zeta = \frac{1}{3}$	0.54	0.14	$U_2 > U_3 > U_1$
U_3		0.52	0.12	
U_1		0.53	0.12	
U_2	$\psi = 0.70\ \eta = 0.20\ \zeta = 0.10$	0.61	0.12	$U_2 > U_1 > U_3$
U_3		0.48	0.16	

Remark 14.8 When the problem is carried for the value $k = 0$, the ranking is obtained as follows:

Comparing Tables 14.2 and 14.4 we note that the variations in α_o, β_o and γ_o affect the ranking system. So depending on the importance given to the various criteria considered there will be variation in the ranking also. Thus the choice of the candidate will differ from recruiter to recruiter. This is one recruiter's perception for two patterns.

Table 14.5 Ranking results for alternatives

Alternative	Value of ψ, η, ζ	Value index	Ambiguity index	Ranking order
U_1		0.38	0.15	
U_2	$\psi = 0.10, \eta = 0.40, \zeta = 0.50$	0.48	0.14	$U_3 > U_2 > U_1$
U_3		0.54	0.13	
U_1		0.44	0.13	
U_2	$\psi = \frac{1}{3}\ \eta = \frac{1}{3}\ \zeta = \frac{1}{3}$	0.53	0.12	$U_2 > U_3 > U_1$
U_3		0.51	0.11	
U_1		0.52	0.09	
U_2	$\psi = 0.70\ \eta = 0.20\ \zeta = 0.10$	0.61	0.09	$U_2 > U_1 > U_3$
U_3		0.47	0.07	

From Tables 14.2, 14.3, 14.4, 14.5 we observe that for better ranking, SVLONNs are used.

14.6 Conclusion

In this paper, we introduced and studied the idea of SVLONN. Value index and Ambiguity index of SVLONNs are discussed. With the help of the same, a ranking method for SVLONNs is developed and applied to a MADM problem. Depending on the need of the person making choices with respect to α_o, β_o γ_o, ψ, η, ζ there will be variation in the output. Based on the need one can choose truth value, indeterminacy value, falsity value, coefficients of value, and ambiguity indices which cause variation in the ranking. In a similar type of setup in any other field, this model can be used (to cite a few medical diagnosis, pattern recognition, personal selection). Further, value index and ambiguity index can be used in transportation problem.

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