Turbulence Without Fluctuations

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Abstract Fluctuations are very important in statistics, but their use in the study of turbulent flows, particularly in the formulation and the analysis of Large Eddy Simulations, is not so essential. In the paper some historical and personal recollections on that are recorded.

1 Introduction

Thirty years ago, in 1992, I was invited by ENEL, the Italian agency for the production and the distribution of electric power, to illustrate the recent progresses in the new emerging computational techniques applied to the Large Eddy Simulation, (LES), of turbulent flows. In Pisa, the morning of May 21, I spoke about the past and the present of LES and the related perspectives of future developments. In the afternoon some participants presented their activity in the field, and Carlo Cercignani was one of them. It was a great day, plenty of suggestions and promises for the future.

It was not my first conversation with Carlo, our friendship started in the sixties of the last century, but it was the first time that we were directly related to some joint project. Both Italians we were used to meet outside, at a congress in the United States or a meeting in Denmark. I was attracted by his way of looking at science, his passionate and disenchanted way of considering the progress, his penetrating and well-disposed judgements of new theories and people. The contacts were rapid but very significative, a dinner in a pub, a little walk in a campus or the casual encounter in a library between one session and another. For me he was like an elder brother, disposed to understand your problems and indulgent with your weaknesses.

At the time of our meeting in Pisa I recall his amused interest on my reports about my personal American Adventure, the joint gestation of the dynamic model with

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Stanford. Thirty years ago the interest for LES suddenly had a great impulse due to the new dynamic modeling approach [1] that was presented at the Summer Meeting of the CTR in 1990. This procedure, originally applied to improve dynamically the Smagorinsky model, was very general; Carlo was working with his collaborators, Antonella Abbà and Lorenzo Valdettaro, on anisotropic eddy viscosity models and a possible dynamic improvement was attracting their attention.

For me all that was a great surprise. The basic ingredient of the new modeling approach was a trivial identity between the subgrid stresses at two different filtering resolutions incidentally found in the framework of my speculations on LES. My interest on LES was relatively new, and my main problem at that time was to develop a simple multiscale approach to the analysis of turbulent flows based on a generic hierarchy of filtering operators. The starting point was a curiosity about the fundamentals, in particular, the problem of the average and the analytical formalization of LES. I was attracted by LES mainly from the point of view of a new way to represent turbulent flow, different from the statistical and the spectral ones.

The beginning was discouraging, a good friend of mine, consulted on the subject, was very strong about that: please leave that topic, it is unfitted for you. It requires big computers, you are not an expert in numerical and computational problems and there is little space left for new theoretical work. But I was attracted by some analogies between the classical Leonard formulation of LES [2] and some old ideas of Boussinesq [3] and Reynolds [4], developed in the middle of the last century by Kampé de Fériet and his school [5], concerning the rules of the mean from the algebraic point of view, and that was the starting point. Then I realized that a great obstacle to the formulation of a hierarchical filtering approach was to get rid of the fluctuations, and to remove the fluctuation-conditioned attitude peculiar of the statistical decomposition. At the time of my presentation in Pisa I was in the main of my struggle against the fluctuations, my first paper on that [6] was encountering some difficulties to be published, it was presented to JFM on July 1990 and was finally published on June 1992. That was my main interest and my major concern at that time, and the main argument of my conversation gently supported by Carlo with his particular witty smile that afternoon in Pisa. In the following I would like to recall all that, in the ideal continuity that connect all us in an endless dialogue.

2 Fluctuation-Conditioned Turbulence

Everybody knows that an alternative, rapid, and elegant way to compute the Reynolds stress in turbulent flows is given by

$$R_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j \tag{1}$$

but for a teacher it is surprisingly scarce the number of students disposed to appreciate this simple expression. Almost all prefer the classical formulation based on the differences, the fluctuations from the mean $u'_i = u_i - \bar{u}_i$

$$R_{ij} = \overline{u'_i u'_j} \tag{2}$$

where the overline stands for the statistical operator, and this fluctuation-conditioned attitude is more evident when applied to the statistical analysis of a filtered representation of a turbulent flow. Let us indicate with the overtilde a generic filtering operator and with $\tilde{u}_i$ a filtered representation of a turbulent flow u_i. We assume as usual that

$$\overline{\widetilde{\cdots}} = \overline{\cdots} \quad ; \quad \overline{\tilde{u}_i} = \overline{u_i} \quad ; \quad \overline{\widetilde{u_i u_j}} = \overline{u_i u_j} \tag{3}$$

and we introduce the new fluctuations u''_i and u'''_i defined as

$$u''_i = \tilde{u}_i - \bar{u}_i \ ; \ u'''_i = u_i - \tilde{u}_i \tag{4}$$

Owing to the identity

$$u'_i = u''_i + u'''_i \tag{5}$$

we can write the relation

$$R_{ij} = \overline{u''_i u''_j} + \overline{u'''_i u'''_j} + \overline{u''_i u'''_j} + \overline{u'''_i u''_j} \tag{6}$$

that if the new supplementary fluctuations u''_i and u'''_i are uncorrelated reduces to the simple result

$$R_{ij} = \overline{u''_i u''_j} + \overline{u'''_i u'''_j} \tag{7}$$

I will not underestimate the importance of these relations. Triple decompositions have been studied by many authors in the past, I will cite one for all the important paper by Reynolds and Hussain [7] where a triple decomposition has been applied to the study of the mechanics of an organized wave in turbulent shear flow. Here I only remark that it is very easy and more general to realize that

$$\overline{u''_i u''_j} = \overline{\tilde{u}_i \tilde{u}_j} - \bar{\tilde{u}}_i \bar{\tilde{u}}_j \tag{8}$$

and that

$$\overline{u'''_i u'''_j} + \overline{u''_i u'''_j} + \overline{u''_i u'''_j} = \overline{\widetilde{u_j u_i}} - \overline{\tilde{u}_i \tilde{u}_j} \tag{9}$$

so that we have the alternative operational decomposition of the Reynolds stress given by

$$R_{ij} = T_{ij} + \overline{\tau}_{ij} \tag{10}$$

where T_{ij} is the resolved stress, explicitly computed

$$T_{ij} = \overline{\tilde{u}_i \tilde{u}_j} - \bar{\tilde{u}}_i \bar{\tilde{u}}_j \tag{11}$$

and τ_{ij} is the subfilter stress, usually modeled

$$\tau_{ij} = \widetilde{u_i u_j} - \tilde{u}_i \tilde{u}_j \tag{12}$$

We remark that the identity (10) is operational and only subjected to the assumption (3), so that can be also applied to two generic filtering operators, and that was the new multiscale identity that I was illustrating that day in Pisa. Here I would like to recall how important in his genesis was for me to remove the fluctuation-conditioned attitude at that time dominant in the field. A fundamental starting point was an old paper published in French in the Italian *Rendiconti del Seminario Matematico e Fisico di Milano* [5]. In that paper Kampé de Fériet poses the following question: *Quelles sont les propriétés de la moyenne, nécessaires et suffisantes pour que les équations de Reynolds se déduisent rigoureusement des équations de Navier en en prenant la moyenne?*, which are the properties of the mean that mathematically justify the RANS equations. The arguments and the developments are very interesting, but mainly two conclusions deserve in our opinion our principal attention. The first is related to the fundamental property of the statistical fluctuations, $\overline{u_i'} = 0$, their nullity when averaged, that is proved unessential: *si on attache donc une telle importance à* $\overline{u_i'} = 0$ *c'est parce qu'on la juge impliquée intuitivement dans le concept de la moyenne, mais nullement parce qu'elle intervient dans les calculs réellement effectués*[1]. The second is that the RANS equations are very simply and directly derived by the Navier-Stokes equations *si l'effet du mouvement de l'agitation turbulent sur le mouvement moyen ne s'exprime plus par le tenseur de Reynolds,* $\overline{u_i' u_j'}$, *mais par un nouveau tenseur* $R_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j$[2]. More precisely the problem that interested Kampé de Fériet in the fifties of the last century was to better understand algebraically the Reynolds rules of the mean. Given a generic linear and constant preserving filtering operator $\mathcal{F}$ that produces the quantities $\bar{u}_i$ let us introduce the fluctuations $u_i' = u_i - \bar{u}_i$. We

[1] If one attains such importance to this property is that we intuitively think it associated with the concept of mean, but it has no role at all in the computations really done

[2] If the effect of turbulence is not expressed by the Reynolds stress $\overline{u_i' u_j'}$ but by a new tensor $R_{ij} = \overline{u_i u_j} - \bar{u}_i \bar{u}_j$

can write

$$\overline{u_i u_j} = \overline{\bar{u}_i \bar{u}_j} + \overline{u'_i u'_j} + \overline{u'_i \bar{u}_j} + \overline{u'_j \bar{u}_i} \tag{13}$$

and the problem was to find the operational rules that justify the equivalence of this expression with the usual one

$$\overline{u_i u_j} = \bar{u}_i \bar{u}_j + \overline{u'_i u'_j} \tag{14}$$

Obviously this equivalence is satisfied if $\mathcal{F} = \mathcal{E}$, the statistical average, and at that time the interest to this problem was not so appreciated by the scientific community. Let us quote from Monin and Yaglom [8], volume 1, page 209: *However, all these investigations are of a formal mathematical nature and their results do not find direct application in the theory of turbulence. Furthermore, they are not even necessary, since in present day turbulence theory the question of the meaning of averaging is resolved in a completely different manner, and, moreover, in such a way that all the Reynolds conditions are evidently satisfied ....* Averages different from the statistical one, like convolutional averages in space and time, are not so interesting because *these mean values will depend on the form of the weighting function (in particular, when averaging over some time interval or region of space, it will depend on the length of the interval or the form and volume of the region)* and in conclusion *it is desirable in the theory of turbulence to avoid the use of this type of averaging altogether, and to adopt instead some other method of defining the mean value, a method that has simpler properties and is more universal. A convenient definition of this type, which we shall use throughout this book, is found in the probability-theory treatment of the fields of fluid dynamic variables in a turbulent flow as random fields.*

But things were rapidly changing due to a lot of reasons. The discovery of coherent structures and the sensation that a turbulent field at least as regards the large scales, is not so chaotic as presumed was more and more acting on the scientific imagination. The classical double decomposition in ensemble averages and statistical fluctuations was insufficient to capture the multiscale nature of turbulence, and something new had to be implemented. It was clear that in order to go on something new had to be conceived, and following the suggestions of Kampé de Fériet the time of abandoning the fluctuation-conditioned approaches was arrived. It was not so easy, and a fundamental step was to define the Generalized Central Moments.

3 Turbulence Without Fluctuations

Once removed by the help of Kampé de Fériet my fluctuation-conditioned attitude, everything became easier. Explicitly, given a generic velocity field $u_i\,(\mathbf{x}, t, \omega)$, the statistical average is given by

$$\langle u_i \rangle = \int u_i \left(\mathbf{x}', t', \omega\right) p(\omega) d\omega \tag{15}$$

and a generic space average is given by

$$\langle u_i \rangle = \int u_i \left(\mathbf{x}', t', \omega\right) \mathcal{G}\left(\mathbf{x} - \mathbf{x}'\right) d^3\mathbf{x}' \tag{16}$$

where $p(\omega)$ is a probability density function and $\mathcal{G}\left(\mathbf{x} - \mathbf{x}'\right)$ a normalized space filtering kernel, but operationally the filtered, averaged, Reynolds, LES equations are all equivalent. In terms of the generalized Reynolds stress introduced by Kampé de Fériet the Navier-Stokes equations are invariant to the filtering process, and this *averaging invariance* is claimed in my paper [6], but essentially it is due to my connection to the past. To generalize all that was simple. Let us consider a generic filtering operator $\mathcal{F}$ representative of our Large Eddy Simulation, the explicit or implicit *LES filter*, and given the generic turbulent quantities, $a, b, c, \cdots$, let us introduce the Generalized Central Moments (GCM) [6, 9, 10] associated with $\mathcal{F}$

$$\begin{aligned}
\tau_f(a, b) &\equiv \langle ab \rangle_f - \langle a \rangle_f \langle b \rangle_f \\
\tau_f(a, b, c) &\equiv \langle abc \rangle_f - \langle a \rangle_f \tau_f(b, c) - \langle b \rangle_f \tau_f(c, a) - \langle c \rangle_f \tau_f(a, b) - \langle a \rangle_f \langle b \rangle_f \langle c \rangle_f
\end{aligned} \tag{17}$$

Let us now define a *test filter* $\mathcal{G}$ that we only assume linear and constant preserving and let us consider the quantities extracted by this test filter $\mathcal{G}$ when applied to the LES $\mathcal{F}$-filtered quantities. In other words we are interested to the *filter product* $\mathcal{P} = \mathcal{G}\mathcal{F}$. We assume that $\mathcal{G}\mathcal{F} = \mathcal{F}\mathcal{G}$, and our notation is the following

$$\begin{aligned}
\mathcal{P}[\cdots] &\equiv \mathcal{G}[\mathcal{F}[\cdots]] \equiv \mathcal{G}\mathcal{F}[\cdots] \equiv \langle \cdots \rangle_{gf} \equiv \langle \langle \cdots \rangle_f \rangle_g \equiv \langle \cdots \rangle_p \\
&\equiv \mathcal{F}[\mathcal{G}[\cdots]] \equiv \mathcal{F}\mathcal{G}[\cdots] \equiv \langle \cdots \rangle_{fg} \equiv \langle \langle \cdots \rangle_g \rangle_f \equiv \langle \cdots \rangle_p
\end{aligned} \tag{18}$$

It is easy to see that we can write the following identities

$$\begin{aligned}
\tau_p(a, b) &= \tau_g(\langle a \rangle_f, \langle b \rangle_f) + \langle \tau_f(a, b) \rangle_g = \tau_f(\langle a \rangle_g, \langle b \rangle_g) + \langle \tau_g(a, b) \rangle_f \\
\tau_p(a, b, c) &= \tau_g(\langle a \rangle_f, \langle b \rangle_f, \langle c \rangle_f) + \langle \tau_f(a, b, c) \rangle_g \\
&\quad + \tau_g(\langle a \rangle_f, \tau_f(b, c)) + \tau_g(\langle b \rangle_f, \tau_f(c, a)) + \tau_g(\langle c \rangle_f, \tau_f(a, b)) \\
&= \tau_f(\langle a \rangle_g, \langle b \rangle_g, \langle c \rangle_g) + \langle \tau_g(a, b, c) \rangle_f \\
&\quad + \tau_f(\langle a \rangle_g, \tau_g(b, c)) + \tau_f(\langle b \rangle_g, \tau_g(c, a)) + \tau_f(\langle c \rangle_g, \tau_g(a, b))
\end{aligned} \tag{19}$$

where, consistently with the definition of the GCM,

$$
\begin{aligned}
\tau_p(a,b) &\equiv \tau_{gf}(a,b) = \langle ab\rangle_{gf} - \langle a\rangle_{gf}\langle b\rangle_{gf} \\
&\equiv \tau_{fg}(a,b) = \langle ab\rangle_{fg} - \langle a\rangle_{fg}\langle b\rangle_{fg} \\
\tau_p(a,b,c) \equiv \tau_{gf}(a,b,c) &= \langle abc\rangle_{gf} - \langle a\rangle_{gf}\tau_{gf}(b,c) - \langle b\rangle_{gf}\tau_{gf}(c,a) \\
&\quad - \langle c\rangle_{gf}\tau_{gf}(a,b) - \langle a\rangle_{gf}\langle b\rangle_{gf}\langle c\rangle_{gf} \\
\equiv \tau_{fg}(a,b,c) &= \langle abc\rangle_{fg} - \langle a\rangle_{fg}\tau_{fg}(b,c) - \langle b\rangle_{fg}\tau_{fg}(c,a) \\
&\quad - \langle c\rangle_{fg}\tau_{fg}(a,b) - \langle a\rangle_{fg}\langle b\rangle_{fg}\langle c\rangle_{fg} \\
\tau_g(\langle a\rangle_f, \langle b\rangle_f) &\equiv \langle\langle a\rangle_f\langle b\rangle_f\rangle_g - \langle a\rangle_{gf}\langle b\rangle_{gf} \\
\tau_g(\langle a\rangle_f, \tau_f(b,c)) &\equiv \langle\langle a\rangle_f\tau_f(b,c)\rangle_g - \langle a\rangle_{gf}\langle\tau_f(b,c)\rangle_g
\end{aligned}
\tag{20}
$$

A particularly important case is $\mathcal{G} = \mathcal{E}$. In this case, if we also assume that $\mathcal{EF} = \mathcal{E}$, we have

$$
\begin{aligned}
\tau_e(a,b) &= \tau_e(\langle a\rangle_f, \langle b\rangle_f) + \langle\tau_f(a,b)\rangle_e \\
\tau_e(a,b,c) &= \tau_e(\langle a\rangle_f, \langle b\rangle_f, \langle c\rangle_f) + \langle\tau_f(a,b,c)\rangle_e + \tau_e(\langle a\rangle_f, \tau_f(b,c)) \\
&\quad + \tau_e(\langle b\rangle_f, \tau_f(c,a)) + \tau_e(\langle c\rangle_f, \tau_f(a,b))
\end{aligned}
\tag{21}
$$

and we remark that the *total* turbulence represented by the statistical central moments $\tau_e(a,b)$ and $\tau_e(a,b,c)$ is operationally decomposed in the *resolved* large scale turbulence $\tau_e(\langle a\rangle_f, \langle b\rangle_f)$ and $\tau_e(\langle a\rangle_f, \langle b\rangle_f, \langle c\rangle_f)$, and in the *subfilter* small scale turbulence given by $\langle\tau_f(a,b)\rangle_e$, $\langle\tau_f(a,b,c)\rangle_e$ and $\tau_e(\langle a\rangle_f, \tau_f(b,c))$.

This multiscale operational technique can be easily extended to an ensemble of filtering operators hierarchically organized

$$
\mathcal{F} \ , \ \mathcal{GF} \ , \ \mathcal{HGF} \ , \ \cdots \tag{22}
$$

and has a lot of different applications. At the time of our meeting in Pisa the main interest was in subgrid modeling, and the first application of this multiscale filtering approach was dedicated to the optimization of the Smagorinsky model. In this case the filter $\mathcal{G}$ is a test filter that applied to the $\mathcal{F}$-filtered quantities should provide a different representation at a different resolution level, useful to compare and optimize the subgrid model at different grid resolution. We refer for more detail on that to [10], and we also remark an interesting review and generalization of this modeling procedure recently provided by Meneveau [11].

Here we would like to also recall the application of this multiscale filtering technique to the analysis of LES data, in particular, the extraction of statistical data from a LES database. Let us first of all consider a constant density turbulent field and let us assume that $\mathcal{EF} = \mathcal{E}$. We explicitly define

$$
\begin{aligned}
R_{ij} &\equiv \tau_e(u_i, u_j) \\
R_{ijk} &\equiv \tau_e(u_i, u_j, u_k) \\
\tau_{ij} &\equiv \tau_f(u_i, u_j) \\
\tau_{ijk} &\equiv \tau_f(u_i, u_j, u_k) \\
T_{ij} &\equiv \tau_e(\langle u_i \rangle_f, \langle u_j \rangle_f) \\
T_{ijk} &\equiv \tau_e(\langle u_i \rangle_f, \langle u_j \rangle_f, \langle u_k \rangle_f)
\end{aligned}
\tag{23}
$$

where u_i are the components of the velocity field at a given time and location, and we have

$$
\begin{aligned}
R_{ij} &= \langle \tau_{ij} \rangle_e + T_{ij} \\
R_{ijk} &= \langle \tau_{ijk} \rangle_e + T_{ijk} + \tau_e(\langle u_i \rangle_f, \tau_{jk}) + \tau_e(\langle u_j \rangle_f, \tau_{ki}) + \tau_e(\langle u_k \rangle_f, \tau_{ij})
\end{aligned}
\tag{24}
$$

where

$$
\tau_e(\langle u_i \rangle_f, \tau_{jk}) \equiv \langle \langle u_i \rangle_f \tau_{jk} \rangle_e - \langle \langle u_i \rangle_f \rangle_e \langle \langle \tau_{jk} \rangle_f \rangle_e \tag{25}
$$

Let us now only assume that $\mathcal{EF} = \mathcal{FE}$. We define the following additive GCM

$$
\begin{aligned}
\vartheta_{ij} &\equiv \tau_f(\langle u_i \rangle_e, \langle u_j \rangle_e) \\
\vartheta_{ijk} &\equiv \tau_f(\langle u_i \rangle_e, \langle u_j \rangle_e, \langle u_k \rangle_e)
\end{aligned}
\tag{26}
$$

and we have finally

$$
\langle R_{ij} \rangle_f + \vartheta_{ij} = \langle \tau_{ij} \rangle_e + T_{ij} \tag{27}
$$

$$
\begin{aligned}
&\langle R_{ijk} \rangle_f + \vartheta_{ijk} + \tau_f(\langle u_i \rangle_e, R_{jk}) + \tau_f(\langle u_j \rangle_e, R_{ki}) + \tau_f(\langle u_k \rangle_e, R_{ij}) = \\
&\langle \tau_{ijk} \rangle_e + T_{ijk} + \tau_e(\langle u_i \rangle_f, \tau_{jk}) + \tau_e(\langle u_j \rangle_f, \tau_{ki}) + \tau_e(\langle u_k \rangle_f, \tau_{ij})
\end{aligned}
\tag{28}
$$

Another important application of this fluctuation-free multiscale approach is to Variable Density Turbulent Fields ϱ, u_i. In this case we also remove the mass-weighted averages [12], and in terms of the GCM we explicitly write

$$
\begin{aligned}
R_{\varrho i} &\equiv \tau_e(\varrho, u_i) \\
R_{ij} &\equiv \tau_e(u_i, u_j) \\
R_{\varrho ij} &\equiv \tau_e(\varrho, u_i, u_j) \\
\\
\tau_{\varrho i} &\equiv \tau_f(\varrho, u_i) \\
\tau_{ij} &\equiv \tau_f(u_i, u_j) \\
\tau_{\varrho ij} &\equiv \tau_f(\varrho, u_i, u_j) \\
\\
T_{\varrho i} &\equiv \tau_e(\langle \varrho \rangle_f, \langle u_i \rangle_f) \\
T_{ij} &\equiv \tau_e(\langle u_i \rangle_f, \langle u_j \rangle_f) \\
T_{\varrho ij} &\equiv \tau_e(\langle \varrho \rangle_f, \langle u_i \rangle_f, \langle u_j \rangle_f)
\end{aligned}
\tag{29}
$$

where ϱ is the density and u_i are the components of the velocity field at a given time and location. We have

$$
\begin{aligned}
R_{\varrho i} &= \langle \tau_{\varrho i} \rangle_e + T_{\varrho i} \\
R_{ij} &= \langle \tau_{ij} \rangle_e + T_{ij}
\end{aligned}
\tag{30}
$$

$$
R_{\varrho ij} = \langle \tau_{\varrho ij} \rangle_e + T_{\varrho ij} + \tau_e(\langle \varrho \rangle_f, \tau_{ij}) + \tau_e(\langle u_i \rangle_f, \tau_{\varrho j}) + \tau_e(\langle u_j \rangle_f, \tau_{\varrho i})
\tag{31}
$$

Let us now only assume that $\mathcal{EF} = \mathcal{FE}$. We define the following additive GCM

$$
\begin{aligned}
\vartheta_{\varrho i} &\equiv \tau_f(\langle \varrho \rangle_e, \langle u_i \rangle_e) \\
\vartheta_{ij} &\equiv \tau_f(\langle u_i \rangle_e, \langle u_j \rangle_e) \\
\vartheta_{\varrho ij} &\equiv \tau_f(\langle \varrho \rangle_e, \langle u_i \rangle_e, \langle u_j \rangle_e)
\end{aligned}
\tag{32}
$$

and we have finally

$$
\begin{aligned}
\langle R_{\varrho i} \rangle_f + \vartheta_{\varrho i} &= \langle \tau_{\varrho i} \rangle_e + T_{\varrho i} \\
\langle R_{ij} \rangle_f + \vartheta_{ij} &= \langle \tau_{ij} \rangle_e + T_{ij}
\end{aligned}
\tag{33}
$$

and

$$\langle R_{\varrho ij}\rangle_f + \vartheta_{\varrho ij} + \tau_f(\langle\varrho\rangle_e, R_{ij}) + \tau_f(\langle u_i\rangle_e, R_{\varrho j}) + \tau_f(\langle u_j\rangle_e, R_{\varrho i}) =$$
$$\langle\tau_{\varrho ij}\rangle_e + T_{\varrho ij} + \tau_e(\langle\varrho\rangle_f, \tau_{ij}) + \tau_e(\langle u_i\rangle_f, \tau_{\varrho j}) + \tau_e(\langle u_j\rangle_f, \tau_{\varrho i}) \tag{34}$$

These last decompositions of the Reynolds stresses in the case of compressible turbulent flow are new, simple, and fluctuation-free. Moreover they are also mass weighted-free, and the author really hope that they could be appreciated more and more in the future in the analysis of LES databases. We recall finally that some recent applications of the filtering approach concerning the subfilter stress [13], the decomposition of the Reynolds stresses [14, 15], the dynamic modeling of the Shock Driven Turbulent Mixing [16] and the definition of statistical homogeneity indices [17] have been recently reviewed in [18].

4 Conclusions

But let us return to that afternoon in Pisa and to the conversation with Carlo. Everything went on well: two months later my paper was finally published, and in the following years the application of the new dynamic modeling procedure to the anisotropic eddy viscosity model proposed by Carlo, Antonella, and Lorenzo was developed and worked successfully [19]. Fluctuations continue to be applied in LES, but some final lessons deserve to be remarked. The first is related to our starting point, the identity (10). Why this simple and, in our opinion, interesting decomposition of the Reynolds stress was appreciated so lately in turbulence? We remark that in applied statistics the identities (3) and (10) are better known as the *Adam and Eve's laws* [20, 21]: given a partition of the probability space, the total statistical covariance R_{ij} of u_i and u_j is the average $\overline{\tau}_{ij}$ of the partial covariance τ_{ij} plus the statistical covariance T_{ij} of the partial mean values. Science as every human activity is often conditioned by great ideas that in some cases obscure other different possibilities. Obviously the fluctuations remain very important, but not always they are essential, and in the case of LES their importance has been overvalued. Individual deviations from the average are intuitively very appealing, but not always you are obliged to decline them: differences are not so relevant in turbulence.

Another important lesson regards finally our ideal endless dialogue with the past: old papers are often plenty of useful observations and suggestions.

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