Quasilinear Wave Equations with Decaying Time-Potential

Vladimir Georgiev and Sandra Lucente

Abstract An active area of recent research is the study of global existence and blow up for nonlinear wave equations where time depending mass or damping are involved. The interaction between linear and nonlinear terms is a crucial point in determination of global evolution dynamics. When the nonlinear term depends on the derivatives of the solution, the situation is even more delicate. Indeed, even in the constant coefficients case, the null conditions strongly relate the symbol of the linear operator with the form of admissible nonlinear terms which leads to global existence. Some peculiar operators with time-dependent coefficients lead to a wave operator in which the time derivative becomes a covariant time derivative. In this paper we give a blow up result for a class of quasilinear wave equations in which the nonlinear term is a combination of powers of first and second order time derivatives and a time-dependent factor. Then we apply this result to scale invariant damped wave equations with nonlinearity involving the covariant time derivatives.

1 Introduction

We study the following Cauchy Problem:

$$\begin{cases} z_{tt} - \Delta z = (1+t)^{\gamma} A(t, x, z, z_t, z_{tt})\,, \\ z(0, x) = f(x)\,, \\ z_t(0, x) = g(x)\,, \end{cases} \tag{1}$$

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V. Georgiev et al. (eds.), *Qualitative Properties of Dispersive PDEs*,
Springer INdAM Series 52, https://doi.org/10.1007/978-981-19-6434-3_9

with $x \in \mathbb{R}^3, t \geq 0$ and $\gamma \in \mathbb{R}$. In particular we want to deal with A uniformly bounded with respect to t and $\gamma < 0$.

The importance of this quasilinear wave equation with time-dependent potential comes from the special scale invariant wave equation. Let $\mu \in \mathbb{R}$. The equation

$$z_{tt} - \Delta z = (1+t)^{-\frac{\mu}{2}(p-1)}|z|^p$$

is equivalent to

$$u_{tt} - \Delta u + \frac{\mu}{1+t}u_t + \frac{\mu(\mu-2)}{4(1+t)^2}u = |u|^p \tag{2}$$

after the transformation $u(t,x) = (1+t)^{\frac{\mu}{2}}z(t,x)$. Similarly, let $\alpha \in \mathbb{R}$, the equation

$$z_{tt} - \Delta z = (1+t)^{\alpha-\frac{\mu}{2}(p-1)}|z_t|^p$$

is equivalent to

$$u_{tt} - \Delta u + \frac{\mu}{1+t}u_t + \frac{\mu(\mu-2)}{4(1+t)^2}u = (1+t)^\alpha \left|\left(\partial_t + \frac{\mu}{2(1+t)}\right)u\right|^p . \tag{3}$$

The existence theory for initial value problems associated with (2) has been intensively studied. The case $\mu = 2$ has been firstly analyzed in [3], for $\mu \neq 2$ the interested reader can see [7] and the reference therein. Equation (3), with $\alpha = 0$, has been considered only in Girardi-Lucente [4]. The study of the quasilinear scale invariant wave equation is still incomplete, for example,

$$u_{tt} - \Delta u + \frac{\mu}{1+t}u_t + \frac{\mu(\mu-2)}{4(1+t)^2}u = \left|\left(\partial_t + \frac{\mu}{2(1+t)}\right)^2 u\right|^q \tag{4}$$

is equivalent to

$$z_{tt} - \Delta z = (1+t)^{-\frac{\mu}{2}(q-1)}|z_{tt}|^q$$

but, up to our knowledge, no result on this equation is known. This is the inspired motivation of the present paper.

While studying the more general setting (1), we want to show how a decreasing potential $(1+t)^\gamma$, with $\gamma < 0$ interacts with the growth of nonlinear term A in the variables z, z_t, z_{tt}.

On the other hand, applying such result to (3), we can describe the same phenomenon as an interaction between the potential and linear part of the equation. More precisely we will have a blow up result under a condition which relate α, p, μ. We obtain a modified Strauss exponent. In a similar way we deal with (4).

In this paper, we start proving a blow up result for smooth solutions of (1) in Sect. 2. Following [5] we use an averaging method. Then, in Sect. 3, we apply such result to the special scale invariant wave operator. We leave the global existence counterpart of the paper for a further coming paper, except a very simple case given in Sect. 4.

Starting from these examples we can come back to the question of the influence of the lower order terms of the linear operator on the global existence/blow up of the solution. When these terms depend on time, they may become dominant with respect to higher order terms and might cause change of the critical exponents. For this reason, the paper tries to determine null condition for wave equation with time-dependent coefficients hoping that this analysis shall be useful to obtain global existence result in the future.

2 Quasilinear Wave Equations

2.1 Statement of the Main Results

Let us consider the following 3D Cauchy Problem:

$$\begin{cases} z_{tt} - \Delta z = (1+t)^{\gamma} A(t, x, z, z_t, z_{tt}), \quad x \in \mathbb{R}^3, t \geq 0, \\ z(0, x) = f(x), \\ z_t(0, x) = g(x), \end{cases} \tag{5}$$

with $f, g \in C^2(\mathbb{R}^3)$ having compact support. In the special case, when $A = A(t, x, z_t, z_{tt})$ is independent of z we can set

$$y(t, x) = z_t(t, x),$$

so that the problem takes the form

$$\begin{cases} y_{tt} - \Delta y = \partial_t((1+t)^{\gamma} B(t, x, y, y_t)), \\ y(0, x) = g(x), \\ y_t(0, x) = h(x), \end{cases} \tag{6}$$

with suitable h and B. Some results on (6) can be found in the seminal paper by Fritz John [5]; in particular, reading that paper we can deduce the following:

Proposition 1 *If* $\gamma \geq 0$*, suppose* $B \in C^3$ *satisfies*

$$B(t, x, y, y_t) \geq (ay + by_t)^2 \text{ with } a^2 + b^2 > 0.$$

Assume in addition that $B(t, x, 0, 0) = 0$*,* g, h *are compactly supported,* $(g, h) \neq 0$ *and*

$$\int_{\mathbb{R}^3} h(x) - B(0, x, g(x), h(x))\, dx \geq 0\,. \tag{7}$$

Then the smooth maximal solution of (6) *blows up: let the* $T > 0$ *the largest value such that* $y(t, x) \in C^2([0, T) \times \mathbb{R}^3)$ *exists, then* $T < +\infty$.

Now we can explain how to relate (5)–(6) in the general case, when A depends also on z. Indeed, if z is a solution of (5), then we can set $y = z_t$, and find z as an integral operator $z(t, x) = f(x) + \int_0^t y(s, x)ds$ acting on y. In this way

$$B(t, x, y, y_t) = A\left(t, x, f(x) + \int_0^t y(s, x)\, ds, y, y_t\right) \tag{8}$$

can be interpreted as a non-local nonlinearity depending on t, x, y, y_t. The initial data $y(0, x) = g(x)$ is automatically satisfied. The other data $y_t(0, x) = h(x)$ means that we need

$$z_{tt}(0, x) = h(x)$$

so using the equation for z we get

$$h(x) = \Delta f + A(0, x, f(x), g(x), h(x))\,. \tag{9}$$

Therefore, we can make the reduction from (5) to (6) is we require that for given f, g Eq. (9) has a unique solution $h(x)$ for any $x \in \mathbb{R}^3$.

In particular if A satisfies

$$A(t, x, 0, 0, \xi) = 0 \quad \forall x \in \mathbb{R}^3\,, \xi \in \mathbb{R}\,, t \geq 0\,, \tag{10}$$

then the information on the support of initial data is preserved, indeed for any $x \in \mathbb{R}^3$ such that $f(x) = g(x) = 0$, we have $h(x) = 0$.

One can try to see how (6) is related to (5). Indeed, setting

$$\eta_0 = t, (\eta_1, \eta_2, \eta_3) = x, \ (\eta_4, \eta_5) = (y_t, y_{tt}),$$

we obtain

$$A = \gamma(1 + t)^{\gamma-1} B + \frac{\partial B}{\partial \eta_0} + \frac{\partial B}{\partial \eta_4} z_{tt}\,,$$

provided

$$\frac{\partial B}{\partial \eta_5} = 0\,.$$

Our next step is to rewrite Fritz John's result for (5).

Proposition 2 *Let $T \geq 0$. If $\gamma \geq 0$, suppose $A \in C^3$ satisfies*

$$A(t, x, z, z_t, z_{tt}) \geq (az_t + bz_{tt})^2 \text{ with } a^2 + b^2 > 0.$$

Assume in addition (10) *and*

$$A(t, x, f(x), 0, 0) = 0 \quad \forall x \in \mathbb{R}^3, \ t \geq 0. \tag{11}$$

Let f, g are compactly supported and $(f, g) \neq (0, 0)$ such that (9) *has unique solution $h(x)$ for any $x \in \mathbb{R}^3$. Let $z(t, x) \in C^2([0, T) \times \mathbb{R}^3)$ be the maximal smooth solution of* (5)*, then it blows up: $T < +\infty$.*

We note that it is not necessary to assume (7), indeed it reduces to $\int_{\mathbb{R}^3} \Delta f(x)\, dx = 0$ which is trivially satisfied.

In the present paper we want to deal with

$$B(t, x, y, y_t) \geq a^2|y|^p + b^2|y_t|^q, \quad p > 1, q > 1, a^2 + b^2 > 0. \tag{12}$$

Our aim is to establish that the smooth solution of (6) blows up for any $\gamma \geq \gamma_0$ with a suitable $\gamma_0 = \gamma_0(p, q) \in \mathbb{R}$. In particular we are looking for negative γ_0 not included in [5] even if $p = q = 2$.

Theorem 1 *Let $y(t, x) : [0, T) \to \mathbb{R} \times \mathbb{R}^3$ be a non-trivial C^2 solution of* (6) *with $g, h \in C^2(\mathbb{R}^3)$ compactly supported with $(g, h) \neq (0, 0)$ and*

$$\int_{\mathbb{R}^3} h(x) - B(0, x, g(x), h(x))dx \geq 0. \tag{13}$$

Suppose that $B(t, x, 0, 0) = 0$ and that (12) *is satisfied. Then we have $T < \infty$, provided one of the following:*

1. *$\gamma \geq 0$ and $p \leq 2$ or $q \leq 2$;*
2. *$1 - \frac{2}{p} < \gamma < 0$ (except the case $a = 0$);*
3. *$1 - \frac{2}{q} < \gamma < 0$ (except the case $b = 0$).*

Having in mind the relation between (5) and (6), we can deduce the following result for (5) .

Theorem 2 *Let $z(t, x) : [0, T) \to \mathbb{R} \times \mathbb{R}^3$ be a non-trivial C^2 solution of* (5) *with $f, g \in C^2(\mathbb{R}^3)$ compactly supported with $g \neq 0$. Assume* (10)*,* (11) *and that Eq.* (9) *has a unique solution $h(x)$ for any $x \in \mathbb{R}^3$. Suppose*

$$A(t, x, z, z_t, z_{tt}) \geq a^2|z_t|^p + b^2|z_{tt}|^q, \quad p > 1, q > 1, a^2 + b^2 > 0. \tag{14}$$

Then $T < \infty$ provided one of the following:

1. $\gamma \geq 0$ *and* $p \leq 2$ *or* $q \leq 2$*;*
2. $1 - \frac{2}{p} < \gamma < 0$ *(except the case* $a = 0$*);*
3. $1 - \frac{2}{q} < \gamma < 0$ *(except the case* $b = 0$*).*

Remark 1 Our results do not hold for both $p > 2$ and $q > 2$.

2.2 Proof of Theorem 1

We set

$$v(t, x) = \int_0^t y(s, x) ds .$$

Let $R > 0$ such that

$$g, h \text{ are compactly supported in } B_R(0) \quad R > 0 , \tag{15}$$

hence

$$y(t, x) \text{ is compactly supported in } B_{R+t}(0) ,$$

that is

$$v(t, x) = 0 \text{ for } |x| > t + R .$$

We can deduce that

$$\begin{aligned} &\partial_t(v_{tt} - \Delta v) = y_{tt} - \Delta y = \partial_t((1 + t)^\gamma B(t, x, y, y_t)) , \\ &v(0, x) = 0 , \\ &v_t(0, x) = y(0, x) = g(x) , \\ &(v_{tt} - \Delta v)(0, x) = y_t(0, x) = h(x) . \end{aligned}$$

We gain

$$\partial_t(v_{tt} - \Delta v - (1 + t)^\gamma B(t, x, y, y_t)) = 0 .$$

So that, for any $t > 0$ we have

$$\begin{aligned} &v_{tt} - \Delta v - (1 + t)^\gamma B(t, x, y, y_t) = \\ &= (v_{tt} - \Delta v)(0, x) - B(0, x, g(x), h(x)) = h(x) - B(0, x, g(x), h(x)) . \end{aligned}$$

Summarizing we have the Cauchy Problem

$$\begin{cases} v_{tt} - \Delta v = (1+t)^{\gamma} B(t, x, v_t, v_{tt}) + h(x) - B(0, x, g, h)\,, \\ v(0, x) = 0\,, \\ v_t(0, x) = g(x)\,. \end{cases} \tag{16}$$

Then we arrive at

$$v_{tt} - \Delta v \geq (1+t)^{\gamma} (a^2 |v_t|^p + b^2 |v_{tt}|^q) + h(x) - B(0, x, g, h)\,. \tag{17}$$

Let

$$w(t, x) = v_{tt}(t, x) - \Delta v(t, x)\,,$$

so that

$$w(t, x) \geq (1+t)^{\gamma} (a^2 |v_t(t, x)|^p + b^2 |v_{tt}(t, x)|^q) + h(x) - B(0, x, g(x), h(x))\,. \tag{18}$$

We consider the spherical means

$$\bar{w}(t, r) = \frac{1}{4\pi} \int_{|\xi|=1} w(t, r\xi) d\sigma_{\xi} \quad r > 0\,.$$

We have

$$\bar{w} \geq (1+t)^{\gamma} \bar{B} + \overline{h(x) - B(0, x, g, h)}$$

and hence

$$\bar{v}(t, r) = \frac{1}{2r} \int_{r-t}^{r+t} \rho \bar{g}(\rho) d\rho + \int \int_{\mathbf{T_{r,t}}} \frac{\rho}{2r} \bar{w}(\rho, \tau) d\rho d\tau, \tag{19}$$

where (see triangle ABC in Fig. 1)

$$\mathbf{T_{r,t}} = \{(\rho, \tau) \,|\, \tau + \rho \leq t + r\,;\, \tau - \rho \leq t - r\,;\, \tau \geq 0\}.$$

Consider

$$\Sigma = \{(r, t) \,|\, r + R < t < 2r\}\,.$$

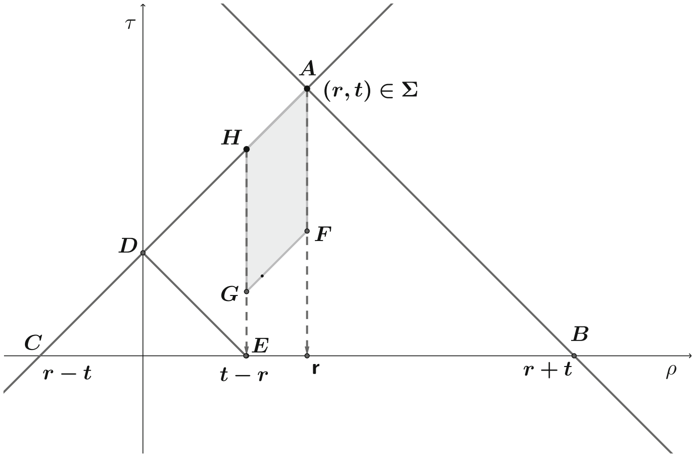

Fig. 1 Domains of integration for $(r, t) \in \Sigma$

Since we are assuming (15), the first term in (19) is zero in Σ, since $\rho\bar{g}(\rho)$ is odd. For a similar reason we can restrict the integration domain of the second term to the trapezoid $ABED$ on Fig. 1:

$$\mathbf{T}^*_{\mathbf{r},\mathbf{t}} = \{(\rho, \tau) \,|\, t - r \leq \tau + \rho \leq t + r\,; \tau - \rho \leq t - r\,; \tau \geq 0\}.$$

For any $(r, t) \in \Sigma$ we get

$$\begin{aligned}\bar{v}(t, r) &= \iint_{\mathbf{T}^*_{\mathbf{r},\mathbf{t}}} \frac{\rho}{2r} \bar{w}(\rho, \tau) d\rho d\tau \\ &\geq \iint_{\mathbf{T}^*_{\mathbf{r},\mathbf{t}}} \frac{\rho}{2r} (1 + \tau)^\gamma \bar{B} d\rho d\tau + 2 \int_0^R \frac{\rho^2}{2r} \overline{h - B(0, x, g, h)}(\rho) d\rho\,.\end{aligned}$$

Due to (13), we conclude

$$\bar{v}(t, r) \geq \iint_{\mathbf{T}^*_{\mathbf{r},\mathbf{t}}} \frac{\rho}{2r} (1 + \tau)^\gamma \bar{B} d\rho d\tau \quad (r, t) \in \Sigma\,. \tag{20}$$

Here we used (18). For any $(r, t) \in \Sigma$, we restrict the integration domain to the parallelogram $AFGH$ on Fig. 1:

$$\mathbf{S}_{\mathbf{r},\mathbf{t}} = \{(\rho, \tau) \,|\, t - r < \rho < r\,; \rho - R < \tau < \rho + t - r\} \subset \mathbf{T}^*_{\mathbf{r},\mathbf{t}}\,.$$

Applying Jensen inequality to (17), we arrive at

$$\bar{v}(t,r) \geq \frac{1}{2r}\int_{t-r}^{r} \rho d\rho \int_{\rho-R}^{\rho+t-r} (1+\tau)^{\gamma}(a^2|\bar{v}_\tau|^p + b^2|\bar{v}_{\tau\tau}|^q)d\tau \,. \tag{21}$$

Having in mind the location of the support of $\bar{v}(t,r)$, we can write

$$\bar{v}(\rho+t-r,\rho) = \int_{\rho-R}^{\rho+t-r} \bar{v}_\tau(\tau,\rho)d\tau \tag{22}$$

and also

$$\bar{v}(\rho+t-r,\rho) = \int_{\rho-R}^{\rho+t-r} (\rho+t-r-\tau)\bar{v}_{\tau\tau}(\tau,\rho)d\tau \,. \tag{23}$$

The idea is now to slice Σ into half-lines:

$$\sigma_c = \{(r,t)\,|\,t = c+r\,;\, r > c\}\,, \quad \Sigma = \bigcup_{c>R} \sigma_c \,.$$

Let us denote by α the restriction of $\bar{v}$ on these half-lines:

$$\alpha(r) = |\bar{v}(r+c,r)| \quad r > c > R \,.$$

Our aim is to prove that

$$\alpha(r) = 0 \text{ for } r > c > R \,, \tag{24}$$

so that

$$\bar{v}(t,r) = 0 \text{ on } \Sigma \,. \tag{25}$$

Let

$$\beta(r) = \int_c^r \rho\alpha^p(\rho)\,d\rho + \int_c^r \rho\alpha^q(\rho)\,d\rho := \beta_1(r) + \beta_2(r)\,.$$

If $\beta(r) = 0$, then we get (24).

Assume by contradiction that there exists $r_0 > 0$ such that $\beta(r_0) \neq 0$.

By using (22), we have

$$a^2\beta_1(r) \leq a^2 \int_c^r \rho \left| \int_{\rho-R}^{\rho+c} a^{-\frac{2}{p}} a^{\frac{2}{p}} \bar{v}_\tau(\rho,\tau)(1+\tau)^{\gamma/p}(1+\tau)^{-\gamma/p} d\tau \right|^p d\rho$$

$$\leq \int_c^r \rho \left(\int_{\rho-R}^{\rho+c} (1+\tau)^{-\gamma p'/p} d\tau \right)^{p'/p} \left(\int_{\rho-R}^{\rho+c} (a^2 \bar{|} v_\tau|^p (1+\tau)^{\gamma} d\tau \right) d\rho \,.$$

Let us recall that $p/p' = p-1$. Setting

$$\Gamma_1(r) = \sup_{\rho\in[c,r]} \left(\int_{\rho-R}^{\rho+c} (1+\tau)^{-\frac{\gamma}{p-1}} d\tau\right)^{p-1},$$

Having in mind (21), we can conclude that

$$a^2\beta_1(r) \le 2r\Gamma_1(r)\alpha(r).$$

Similarly, we can estimate

$$\begin{aligned} b^2\beta_2(r) &\le \int_c^r \rho \left|\int_{\rho-R}^{\rho+c} (\rho+c-\tau) b^{\frac{2}{q}} \bar{v}_{\tau\tau}(\rho,\tau)(1+\tau)^{\gamma/q}(1+\tau)^{-\gamma/q} d\tau\right|^q d\rho \\ &\le \int_c^r \rho \left(\int_{\rho-R}^{\rho+c} (\rho+c-\tau)^{q'}(1+\tau)^{-\gamma q'/p} d\tau\right)^{q'/q} \\ &\qquad \left(\int_{\rho-R}^{\rho+c} b^2 |\bar{v}_{\tau\tau}|^q (1+\tau)^{\gamma} d\tau\right) d\rho. \end{aligned}$$

We can conclude that

$$b^2\beta_2(r) \le 2r\Gamma_2(r)\alpha(r)$$

with

$$\Gamma_2(r) = \sup_{\rho\in[c,r]} \left(\int_{\rho-R}^{\rho+c} (\rho+c-\tau)^{q'}(1+\tau)^{-\frac{\gamma}{q-1}} d\tau\right)^{q-1}.$$

On the other hand

$$\beta'(r) = r\alpha^p(r) + r\alpha^q(r) \ge \frac{a^{2p}\beta^p(r)}{2^p r^{p-1}\Gamma_1^p(r)} + \frac{b^{2q}\beta^q(r)}{2^q r^{q-1}\Gamma_1^q(r)}.$$

We can deduce that β is increasing and for $r > r_0$ we get

$$(\beta(r_0))^{1-p} \ge \frac{(p-1)a^{2p}}{2^p} \int_{r_0}^r \frac{1}{\xi^{p-1}\Gamma_1^p(\xi)} d\xi$$

and

$$(\beta(r_0))^{1-q} \ge \frac{(q-1)b^{2q}}{2^q} \int_{r_0}^r \frac{1}{\xi^{q-1}\Gamma_2^q(\xi)} d\xi.$$

In order to have contradiction, it remains to find (p, q, γ) such that

$$\int_{r_0}^{+\infty} \frac{1}{\xi^{p-1}\Gamma_1^p(\xi)}\, d\xi = +\infty \quad \text{or} \quad \int_{r_0}^{+\infty} \frac{1}{\xi^{q-1}\Gamma_2^q(\xi)}\, d\xi = +\infty\,. \tag{26}$$

In the case $b = 0$ and $a \neq 0$ we only require that the first integral is divergent. In the case $a = 0$ and $b \neq 0$ we only require that the second integral is divergent.

We observe that there exist $s_1 \in [-R, c]$ and $s_2 \in [-R, c]$ such that

$$\left(\int_{\rho-R}^{\rho+c} (1+\tau)^{-\frac{\gamma}{p-1}}\, d\tau\right)^{p-1} = (1 + s_1 + \rho)^{-\gamma}$$

and

$$\left(\int_{\rho-R}^{\rho+c} (\rho + c - \tau)^{q'} (1+\tau)^{-\frac{\gamma}{q-1}}\, d\tau\right)^{q-1} = (c - s_2)^{q'} (1 + s_2 + \rho)^{-\gamma}\,.$$

For $\gamma > 0$ we find $\Gamma_1(r) \leq 1$. It follows that (26) is satisfied for any $p \leq 2$. Similarly, for $\gamma > 0$ we find $\Gamma_2(r) \leq 1$ and (26) is satisfied for any $q \leq 2$. If $\gamma < 0$, then (26) is equivalent to

$$\int_{r_0}^{+\infty} \frac{1}{\xi^{p-1-\gamma p}}\, d\xi = +\infty \text{ or } \int_{r_0}^{+\infty} \frac{1}{\xi^{q-1-\gamma q}}\, d\xi = +\infty\,.$$

Again for $b = 0$ and $a \neq 0$ we only require that the first integral is divergent. In the case $a = 0$ and $b \neq 0$ we only require that the second integral is divergent. We can conclude that (26) is satisfied in one of the following cases

1. $\gamma > 0$ and $p \leq 2$ or $q \leq 2$;
2. $1 - \frac{2}{p} < \gamma < 0$ (except the case $a = 0$);
3. $1 - \frac{2}{q} < \gamma < 0$ (except the case $b = 0$).

Coming back to the proof of the blow up of the solution of (5), through the solution of (16) and (6), from $\bar{v} = 0$ on Σ we can deduce that

$$y(x, t) = 0 \qquad \text{for } x \in \mathbb{R}^3, t > R\,.$$

First we notice that combining (25) with (20) we have $\bar{B} = 0$ on $T^*_{r,t}$ with $(r, t) \in \Sigma$. But this trapezoids cover the region $\{(\rho, t) \mid \rho + t > R\}$, hence, being $B \geq 0$ we have $B = 0$ in the region $|x| + t > 0$. This implies

$$a^2|v_t|^p + b^2|v_{tt}|^q = 0 \text{ for } |x| + t > 0\,, t > 0\,.$$

We can deduce

$$v_t(x,t) = 0\,,\, v_{tt}(x,t) = 0\,,\ \text{for } |x| + t > 0\,, t > 0\,.$$

Recalling that $y(x,t) = v_t(x,t)$ we get

$$y_t(x,t) = 0\,,\ \text{for } |x| + t > 0\,, t > 0\,. \tag{27}$$

In turn this implies that

$$y(x,t) = y(x,t+R) = 0\,,\ \text{for } x \in \mathbb{R}^3\,, t > 0\,.$$

This gives the conclusion. Indeed, this and (27) are impossible for $(g,h) \neq (0,0)$.

2.3 Proof of Theorem 2

We assume that $z(t,x) : [0,T) \to \mathbb{R} \times \mathbb{R}^3$ is a non-trivial C^2 solution of (5) with $f, g \in C^2(\mathbb{R}^3)$ compactly supported with $g \neq 0$. Make the substitution $v(x,t) = z(t,x) - f(x)$. The function h is determined as the unique solution of (9). Then we can write $h(x) = v_{tt}(0,x)$ and moreover we have

$$\begin{cases} v_{tt} - \Delta v = (1+t)^\gamma B(t,x,v_t,v_{tt}) + \Delta f\,, \\ v(0,x) = 0\,, \\ v_t(0,x) = g(x)\,, \end{cases} \tag{28}$$

with $B(t,x,v_t,v_{tt}) = A(t,x,v+f,v_t,v_{tt})$. It is not necessary to assume (13) since Δf has vanishing mean. Assumption (14) guarantees that v satisfies (21) and we arrive at the same absurd as before. We can conclude that blow up holds.

3 Applications

A trivial application of Theorem 1 is the blow up of compactly supported classical solution of

$$y_{tt} - \Delta y = \partial_t\left((1+t)^\gamma |y|^p\right)\,,$$

provided initial data $(g,h) \in C^3 \times C^2$ satisfies

$$\int h(x)\,dx \geq \int |g(x)|^p\,dx$$

and $p \le 2$ with $\gamma > 1 - 2/p$ or $p = 2$ and $\gamma \ge 0$. This example is deeply different from the results in [5]. Indeed the right side can be written as

$$\partial_t \left((1+t)^\gamma |y|^p\right) = \gamma(1+t)^{\gamma-1}|y|^p + p(1+t)^\gamma |y|^{p-2} y y_t$$

and we do not know any sign assumption on y and y_t, so that John's result is not directly available.

Now we turn to other applications of Theorem 2. Our starting point is a scale invariant damping wave equations that can be reduced to (5) by means of a suitable transformation. Let us consider the covariant time derivative

$$\partial_{(\mu),t} = \partial_t + \frac{\mu}{2(1+t)} \quad \mu \ge 0.$$

We can write

$$u_{tt} + \frac{\mu}{1+t}u_t + \frac{\mu(\mu-2)}{4(1+t)^2}u = \partial_{(\mu),t}\partial_{(\mu),t}u - \Delta u \tag{29}$$

and hence a meaningful nonlinear term for this equation is $\left|\partial_{(\mu),t}u\right|^p$. On the other hand, the relation between this covariant derivative and the standard derivative is given by the transformation $u = (1+t)^{-\frac{\mu}{2}}z$, indeed $\partial_{(\mu),t}u = (1+t)^{-\frac{\mu}{2}}\partial_t z$. For this reason the equation

$$u_{tt} - \Delta u + \frac{\mu}{1+t}u_t + \frac{\mu(\mu-2)}{4(1+t)^2}u = (1+t)^\alpha \left(a^2\left|\partial_{(\mu),t}u\right|^p + b^2\left|\partial_{(\mu),t}\partial_{(\mu),t}u\right|^q\right) \tag{30}$$

becomes

$$z_{tt} - \Delta z = a^2(1+t)^{\alpha-\frac{\mu}{2}(p-1)}|z_t|^p + b^2(1+t)^{\alpha-\frac{\mu}{2}(q-1)}|z_{tt}|^q \tag{31}$$

and this is a special case of (5) with

$$\gamma = \alpha - \frac{\mu}{2}(\max\{p,q\} - 1).$$

In (30) the linear zero-order term can be seen as a positive time-dependent mass only for $\mu \ge 2$. As seen in the Introduction, many papers deal with the scale invariant damping wave equation

$$u_{tt} + \frac{\mu}{1+t}u_t + \frac{\mu(\mu-2)}{4(1+t)^2}u = F(t,u,u_t,u_{tt}) \tag{32}$$

for $F = |u|^p$. The case $F(u_t) = |u_t|$ has been analyzed in [8]. With the choice of a different nonlinear term in (30), we add another step to understand the interplay between the lower order terms of the wave equation and some *admissible* nonlinear terms.

Let us start considering (30) with $b = 0$.

Corollary 1 *Let us consider the Cauchy problem*

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t} u_t + \frac{\mu(\mu-2)}{4(1+t)^2} u = (1+t)^\alpha \left| \left(\partial_t + \frac{\mu}{2(1+t)} \right) u \right|^p, & x \in \mathbb{R}^3, t \geq 0, \\ u(0, x) = f(x), \\ u_t(0, x) = g(x). \end{cases}$$

Let $u(t, x) : [0, T) \to \mathbb{R}$ *be the corresponding maximal solution with* $f, g \in C^2(\mathbb{R}^3)$ *compactly supported. Let* $1 < p < 2$ *and*

$$\alpha > 1 - \frac{2}{p} + \frac{\mu}{2}(p - 1), \tag{33}$$

or $p = 2$ *and* $\alpha \geq \frac{\mu}{2}$. *Then* $T < +\infty$.

Proof After the transformation $z = (1 + t)^{-\frac{\mu}{2}} u$ the previous Cauchy problem becomes

$$\begin{cases} z_{tt} - \Delta z = (1+t)^{\alpha - \frac{\mu}{2}(p-1)} |z_t|^p, & x \in \mathbb{R}^3, t \geq 0, \\ z(0, x) = f(x), \\ z_t(0, x) = -\frac{\mu}{2} f(x) + g(x). \end{cases}$$

Since $A(t, x, z, z_t, z_{tt}) = |z_t|^p$ satisfies (10) and (11), setting $h(x) = \Delta f + |g(x)|^p$, the result is a direct application of Theorem 2. □

Remark 2 In [4] the case $\alpha = 0$, $\mu > 0$ $f = 0$ is considered. A blow up result for radial solution is established, provided

$$p < \min \left\{ 1 + \frac{2}{\mu}, 1 + \frac{2}{2k + \mu} \right\}, \tag{34}$$

where $k > 0$ is such that $g(|x|) \gtrsim (1 + |x|)^{-k}$. A similar result for the semilinear case is contained in [2]. Let us compare our result with the one in [4]. Though we are considering smooth solution with compact support, our result improves [4], since we do not assume radial solution and we can also treat some $\mu < 0$, for example, for $p = 2$. Moreover our admissible exponents satisfy

$$\frac{\mu}{2} p^2 + \left(1 - \frac{\mu}{2} \right) p - 2 < 0$$

and $1 < p < 2$. At least for $0 < \mu < 3/2$ this range is larger than (34).

Remark 3 The expression (33) shows also the interaction between the potential, the linear operator (29) and nonlinear term. More precisely, following [1], if we describe as Strauss-type exponent a positive solution of an equation like

$$\beta p^2 + (\delta - \beta)p - 2 = 0\,, \quad \beta > 0, \delta > \beta$$

then for $\beta = \frac{\mu}{2}$ and $\delta = 1 - \alpha$ our result provides a subcritical blow up behavior. The word *subcritical* refers to a critical Strauss-type exponent.

The analogous result for (30) with $a = 0$ is the following

Corollary 2 *Let us consider the Cauchy problem*

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t}u_t + \frac{\mu(\mu-2)}{4(1+t)^2}u = (1+t)^\alpha \left|\left(\partial_t + \frac{\mu}{2(1+t)}\right)^2 u\right|^q\,, \quad x \in \mathbb{R}^3, t \geq 0, \\ u(0,x) = f(x)\,, \\ u_t(0,x) = g(x)\,. \end{cases}$$

Let $u(t,x) : [0,T) \to \mathbb{R}$ *be the corresponding maximal solution with* $f, g \in C^2(\mathbb{R}^3)$ *compactly supported. Let* $1 < q < 2$ *and*

$$\alpha > 1 - \frac{2}{q} + \frac{\mu}{2}(q-1)\,, \tag{35}$$

or $q = 2$ *and* $\alpha \geq \frac{\mu}{2}$. *Then* $T < +\infty$.

In a similar way we can treat combined nonlinearity involving time-covariant derivatives. We get a blow up result for classical solution of (30) with smooth and compactly supported initial data for $\mu \leq 0$, $p \leq 2$ or $q \leq 2$ or $\mu > 0$ and

$$\alpha > \frac{\mu}{2}(\max\{p,q\} - 1) + 1 - \frac{2}{\max\{p,q\}}\,.$$

Since (30) is equivalent to (31) we expected such interaction between p and q.

Theorem 2 is very general. Firstly, it gives the possibility to consider second order time derivatives in the nonlinear term. Moreover we are requiring the positivity of the entire nonlinear term, not of any terms which appears in A. For example, take

$$\Box z = N(z)$$

with $N = A_1$ written as

$$A_1 = \alpha_1(1+t)^{\gamma_1}|z_t|^{p_1} + \alpha_2(1+t)^{\gamma_2}|z_t|^{p_2}\,,$$

with $\alpha_i > 0$, $p_i \in (1, 2)$ and $1 - \frac{2}{p_i} < \gamma_i$. Then any classical solution z blows up. But also we have blow up if $N = A_1 + A_0$ or $N = A_1 + A_2$ or $N = A_1 + A_0 + A_2$ being

$$A_0 = \alpha_0(1+t)^{\gamma_0}|z|^{p_0} \quad \alpha_0 \geq 0,\, p_0 > 1,\, \gamma_0 \in \mathbb{R}$$

and

$$A_2 = |z_t|^\ell + |z_{tt}|^m - z_t z_{tt} \quad \frac{1}{\ell} + \frac{1}{m} = 1$$

which is positive due to Young inequality.

Finally this idea can be applied for other scale invariant operators. For example, we can consider

$$u_{tt} - \Delta u + 2b(t)u_t + (b' + b^2)u = (\partial_t + b(t))(\partial_t + b(t))u$$

hence one can put $\partial_{(b),t} = (\partial_t + b(t))$ and study

$$\partial_{(b),t}\partial_{(b),t}u - \Delta u = |\partial_{(b),t}u|^p + |\partial_{(b),t}\partial_{(b),t}u|^q\,.$$

Let

$$B(t) = \int_0^t b(s)ds\,,$$

since $\partial_{(b),t)}(\exp(-B(t))u) = \exp(-B(t))\partial_t u$, setting $u = \exp(-B(t))z$ previous equation becomes

$$z_{tt} - \Delta z = \exp((1-p)B(t))|z_t|^p + \exp((1-q)B(t))|z_{tt}|^q\,.$$

Suitable assumptions on b gives the possibility to apply Theorem 2. For example, negative $b(t)$ leads to the case without potential, while

$$b(t) \leq \frac{C}{1+t}$$

leads to Corollary 1.

4 An Existence Result

First of all, we assert that one can generalize our result, when the nonlinear term in (1) depends also on space-derivatives of the solutions. We leave detailed discussion for a future work, but we shall give a simple example of a suitable variant of (32) so that small data global existence result holds. The example can be considered as a complementary case to our blow up results in Corollary 1 with $p = 2$ and $\alpha = \frac{\mu}{2}$. More precisely, we consider the Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t} u_t + \frac{\mu(\mu-2)}{4(1+t)^2} u = (1+t)^{\frac{\mu}{2}} \left(\left| \left(\partial_t + \frac{\mu}{2(1+t)} \right) u \right|^2 - |\nabla u|^2 \right) , \\ u(0,x) = f(x) , \\ u_t(0,x) = g(x) , \end{cases}$$

being $x \in \mathbb{R}^3$, $t \geq 0$. As usual it becomes

$$\begin{cases} z_{tt} - \Delta z = \left(|z_t|^2 - |\nabla z|^2 \right) , \quad x \in \mathbb{R}^3, t \geq 0 \\ z(0,x) = f(x) , \\ z_t(0,x) = -\frac{\mu}{2} f(x) + g(x) . \end{cases} .$$

Then we can use the Nirenberg transform, see Klainerman in [6]:

$$w = 1 - e^{-z} .$$

We get $w_{tt} - \Delta w = 0$ that gives global existence. We underline that also in this case $\mu < 0$ is admissible.

Acknowledgments The authors are partially supported by INDAM, GNAMPA. The first author was partially supported by contract “Problemi stazionari e di evoluzione nelle equazioni di campo non-lineari dispersive” of GNAMPA– Gruppo Nazionale per l’Analisi Matematica 2020, by the project PRIN 2020XB3EFL by the Italian Ministry of Universities and Research, by Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, by Top Global University Project, Waseda University and the Project PRA 2018 49 of University of Pisa. The second author is supported by the PRIN 2017 project Qualitative and quantitative aspects of nonlinear PDEs.

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