On the Cauchy Problem for Quasi-Linear Hamiltonian KdV-Type Equations

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Abstract We prove local in time well-posedness for a class of quasi-linear Hamiltonian KdV-type equations with periodic boundary conditions, more precisely we show existence, uniqueness and continuity of the solution map. We improve the previous result in (Mietka, Ann Math Blaise Pascal 24:83–114, 2017), generalising the considered class of equations and improving the regularity assumption on the initial data.

1 Introduction

In this paper $u(t,x)$ is a function of time $t \in [0,T)$, $T > 0$ and space $x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. $F(x, z_0, z_1)$ is a polynomial function such that $F(x,0,z_1) = F(x,z_0,0) = \partial_{z_0} F(x,0,z_1) = \partial_{z_1} F(x,z_0,0) = 0$. Throughout the paper we shall assume that there exists a constant $\mathfrak{c} > 0$ such that

$$\partial^2_{z_1 z_1} F(x, z_0, z_1) \geq \mathfrak{c}, \tag{1}$$

for any $x \in \mathbb{T}$, z_0, $z_1 \in \mathbb{R}$. We shall denote the partial derivatives of the function u by u_t, u_x, u_{xx} and u_{xxx}, by $\partial_x, \partial_{z_0}, \partial_{z_1}$ the partial derivatives of the function F and by $\frac{d}{dx}$ the total derivative with respect to the variable x. For instance, we have $\frac{d}{dx}F(x,u,u_x) = \partial_x F(x,u,u_x) + \partial_{z_0} F(x,u,u_x)u_x + \partial_{z_1} F(x,u,u_x)u_{xx}$. We consider the equation

$$u_t = \frac{d}{dx}\Big(\nabla_u H(x,u,u_x)\Big), \qquad H(x,u,u_x) := \int_{\mathbb{T}} F(x,u,u_x)dx, \tag{2}$$

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where we denoted by $\nabla_u H$ the L^2-gradient of the Hamiltonian function $H(x, u, u_x)$ on the phase space

$$H_0^s(\mathbb{T}) := \{u(x) \in H^s(\mathbb{T}) : \int_{\mathbb{T}} u(x)dx = 0\}, \tag{3}$$

endowed with the non-degenerate symplectic form $\Omega(u, v) := \int_{\mathbb{T}}(\partial_x^{-1}u)v dx$ (∂_x^{-1} is the periodic primitive of u with zero average) and with the norm $\|u\|_{\dot{H}^s} := \sum_{j\in\mathbb{Z}^*} |u_j|^2|j|^2$ (u_j are the Fourier coefficients of the periodic function u).

The main result is the following.

Theorem 1 *Let $s > 4 + 1/2$ and assume (1). Then for any $u_0 \in H_0^s(\mathbb{T})$ there exists a time $T := T(\|u_0\|_{H^s})$ and a unique solution of (2) with initial condition $u(0, x) = u_0(x)$ satisfying $u(t, x) \in C^0([0, T), H_0^s(\mathbb{T})) \cap C^1([0, T), H_0^{s-3}(\mathbb{T}))$. Moreover the solution map $u_0(x) \mapsto u(t, x)$ is continuous with respect to the H_0^s topology for any t in $[0, T)$.*

This theorem improves the previous one in [15] by Mietka. The result in such a paper holds true if the Hamiltonian function has the form $H(u)$, while here we allow the explicit dependence on the x variable (non-autonomous equation) and the dependence on u_x. We tried to optimise our result in terms of regularity of the initial condition, we do not know if the result is improvable. If we apply our method to the equation considered by Mietka, we find a local well-posedness theorem if the initial condition belongs to the space H_0^s with $s > 3 + 1/2$ (which is natural since the nonlinearity may contain up to three derivatives of u), while in [15] one requires $s \geq 4$. In our statement we need $s > 4 + 1/2$ because our equation is more general and we have the presence of one more derivative in the coefficients with respect to the equation considered in [15].

The proof of Theorem 1 is an application of a method which has been developed in [7, 8] and then improved, in terms of regularity of initial condition, in [1]. Here we follow closely the method in [1] and we use several results proven therein. Both the schemes, the one used in [15] and in [1, 7, 8], rely on solely energy method, the second one is slightly more refined because of the use of paradifferential calculus which allows us to work in fractional Sobolev spaces and to treat more general nonlinear terms. The main idea is to introduce a convenient energy, which is equivalent to the Sobolev norm, which commutes with the principal (quasi-linear) term in the equation (see (40)). In [1, 7, 8] the main difficulty comes from the fact that, after the paralinearization, one needs to prove *a priori* estimates on a system of coupled equations. One needs then to decouple the equations through convenient changes of coordinates which are used to define the modified energy. In the case of KdV equation (2), we have a scalar equation with the sub-principal symbol which is *real* (and so it defines a self-adjoint operator), see (21), therefore it is impossible to obtain energy estimates directly. This term may be completely removed (see Lemma 2) thanks to the Hamiltonian structure. For similar constructions of such kind of energies one can look also at [1, 6, 8–10].

The general equation (2) contains the "classical" KdV equation $u_t+uu_x+u_{xxx}=0$ and the *modified* KdV $u_t+u^pu_x+u_{xxx}=0$, $p\geq 2$. Obviously, for the last two equations better results may be obtained, concerning KdV we quote Bona-Smith [2], Kato [11], Bourgain [3], Kenig-Ponce-Vega [12, 13], Christ-Colliander-Tao [4]. For the general equation, as the one considered in this paper here, several results have been proven by Colliander-Keel-Staffilani-Takaoka-Tao [5], Kenig-Ponce-Vega [14] and the aforementioned Mietka [15].

2 Paradifferential Calculus

In this section we recall some results concerning the paradifferential calculus, we follow [1]. We introduce the Japanese bracket $\langle\xi\rangle=\sqrt{1+\xi^2}$. We denote by $\dot{H}^s$ the homogeneous Sobolev space defined as H^s modulo constant functions.

Definition 1 Given $m,s\in\mathbb{R}$ we denote by Γ^m_s the space of functions $a(x,\xi)$ defined on $\mathbb{T}\times\mathbb{R}$ with values in $\mathbb{C}$, which are C^∞ with respect to the variable $\xi\in\mathbb{R}$ and such that for any $\beta\in\mathbb{N}\cup\{0\}$, there exists a constant $C_\beta>0$ such that

$$\|\partial_\xi^\beta a(\cdot,\xi)\|_{H^s}\leq C_\beta\langle\xi\rangle^{m-\beta},\quad \forall\xi\in\mathbb{R}. \tag{4}$$

We endow the space Γ^m_s with the family of norms

$$|a|_{m,s,n}:=\max_{\beta\leq n}\sup_{\xi\in\mathbb{R}}\|\langle\xi\rangle^{\beta-m}a(\cdot,\xi)\|_{H^s}. \tag{5}$$

Analogously for a given Banach space W we denote by Γ^m_W the space of functions which verify the (4) with the W-norm instead of H^s, we also denote by $|a|_{m,W,n}$ the W based seminorms (5) with $H^s\rightsquigarrow W$.

We say that a symbol $a(x,\xi)$ is spectrally localised if there exists $\delta>0$ such that $\widehat{a}(j,\xi)=0$ for any $|j|\geq\delta\langle\xi\rangle$.

Consider a function $\chi\in C^\infty(\mathbb{R},[0,1])$ such that $\chi(\xi)=1$ if $|\xi|\leq 1.1$ and $\chi(\xi)=0$ if $|\xi|\geq 1.9$. Let $\epsilon\in(0,1)$ and define moreover $\chi_\epsilon(\xi):=\chi(\xi/\epsilon)$. Given $a(x,\xi)$ in Γ^m_s we define the regularised symbol

$$a_\chi(x,\xi):=\sum_{j\in\mathbb{Z}}\widehat{a}(j,\xi)\chi_\epsilon(\tfrac{j}{\langle\xi\rangle})e^{\mathrm{i}jx}.$$

For a symbol $a(x,\xi)$ in Γ^m_s we define its Weyl and Bony-Weyl quantization as

$$Op^{\mathrm{W}}(a(x,\xi))h:=\frac{1}{(2\pi)}\sum_{j\in\mathbb{Z}}e^{\mathrm{i}jx}\sum_{k\in\mathbb{Z}}\widehat{a}\big(j-k,\frac{j+k}{2}\big)\widehat{h}(k), \tag{6}$$

$$Op^{\mathrm{BW}}(a(x,\xi))h := \frac{1}{(2\pi)}\sum_{j\in\mathbb{Z}} e^{\mathrm{i}jx}\sum_{k\in\mathbb{Z}}\chi_\epsilon\Big(\frac{|j-k|}{\langle j+k\rangle}\Big)\widehat{a}\big(j-k,\frac{j+k}{2}\big)\widehat{h}(k). \tag{7}$$

We list below a series of theorems and lemmas that will be used in the paper. All the statements have been taken from [1]. The first one is a result concerning the action of a paradifferential operator on Sobolev spaces. This is Theorem 2.4 in [1].

Theorem 2 *Let $a \in \Gamma^m_{s_0}$, $s_0 > 1/2$ and $m \in \mathbb{R}$. Then $Op^{\mathrm{BW}}(a)$ extends as a bounded operator from $\dot{H}^{s-m}(\mathbb{T})$ to $\dot{H}^s(\mathbb{T})$ for any $s \in \mathbb{R}$ with estimate*

$$\|Op^{\mathrm{BW}}(a)u\|_{\dot{H}^{s-m}} \lesssim |a|_{m,s_0,4}\|u\|_{\dot{H}^s}, \tag{8}$$

for any u in $\dot{H}^s(\mathbb{T})$. Moreover for any $\rho \geq 0$ we have for any $u \in \dot{H}^s(\mathbb{T})$

$$\|Op^{\mathrm{BW}}(a)u\|_{\dot{H}^{s-m-\rho}} \lesssim |a|_{m,s_0-\rho,4}\|u\|_{\dot{H}^s}. \tag{9}$$

We now state a result regarding symbolic calculus for the composition of Bony-Weyl paradifferential operators. In the rest of the section, since there is no possibility of confusion, we shall denote the total derivative $\frac{d}{dx}$ as ∂_x with the aim of improving the readability of the formulæ. Given two symbols a and b belonging to $\Gamma^m_{s_0+\rho}$ and $\Gamma^{m'}_{s_0+\rho}$, respectively, we define for $\rho \in (0,3]$

$$a\#_\rho b = \begin{cases} ab & \rho \in (0,1] \\ ab + \frac{1}{2\mathrm{i}}\{a,b\} & \rho \in (1,2], \\ ab + \frac{1}{2\mathrm{i}}\{a,b\} - \frac{1}{8}\mathfrak{s}(a,b) & \rho \in (2,3], \end{cases} \tag{10}$$

where we denoted by $\{a,b\} := \partial_\xi a\partial_x b - \partial_x a\partial_\xi b$ the Poisson's bracket between symbols and $\mathfrak{s}(a,b) := \partial^2_{xx}a\partial^2_{\xi\xi}b - 2\partial^2_{x\xi}a\partial^2_{x\xi}b + \partial^2_{\xi\xi}a\partial^2_{xx}b$.

Remark 1 According to the notation above we have $ab \in \Gamma^{m+m'}_{s_0+\rho}$, $\{a,b\} \in \Gamma^{m+m'-1}_{s_0+\rho-1}$ and $\mathfrak{s}(a,b) \in \Gamma^{m+m'-2}_{s_0+\rho-2}$. Moreover $\{a,b\} = -\{b,a\}$ and $\mathfrak{s}(a,b) = \mathfrak{s}(b,a)$.

The following is essentially Theorem 2.5 of [1], we just need some more precise symbolic calculus since we shall deal with nonlinearities containing three derivatives, while in [1] they have nonlinearities with two derivatives.

Theorem 3 *Let $a \in \Gamma^m_{s_0+\rho}$ and $b \in \Gamma^{m'}_{s_0+\rho}$ with $m, m' \in \mathbb{R}$ and $\rho \in (0,3]$. We have $Op^{\mathrm{BW}}(a) \circ Op^{\mathrm{BW}}(b) = Op^{\mathrm{BW}}(a\#_\rho b) + R^{-\rho}(a,b)$, where the linear operator $R^{-\rho}$ is defined on $\dot{H}^s(\mathbb{T})$ with values in $\dot{H}^{s+\rho-m-m'}$, for any $s \in \mathbb{R}$ and it satisfies*

$$\begin{aligned}&\|R^{-\rho}(a,b)u\|_{\dot{H}^{s-(m+m')+\rho}} \\ &\qquad\lesssim (|a|_{m,s_0+\rho,N}|b|_{m',s_0,N} + |a|_{m,s_0,N}|b|_{m',s_0+\rho,N})\|u\|_{\dot{H}^s},\end{aligned} \tag{11}$$

where $N \geq 8$.

Proof We prove the statement for $\rho \in (2, 3]$, for smaller ρ the reasoning is similar. Recalling formulæ(7) and (6) we have

$$Op^{\mathrm{BW}}(a)Op^{\mathrm{BW}}(b)u = Op^{\mathrm{W}}(a_\chi)Op^{\mathrm{W}}(b_\chi)u$$
$$= \sum_{j,k,\ell} \widehat{a}_\chi(j-k, \frac{j+k}{2})\widehat{b}_\chi(k-\ell, \frac{k+\ell}{2})u_\ell e^{\mathrm{i}jx}.$$

We Taylor expand $\widehat{a}_\chi(j-k, \frac{j+k}{2})$ with respect to the second variable in the point $\frac{j+\ell}{2}$, we have

$$\widehat{a}_\chi(j-k, \tfrac{j+k}{2}) =$$
$$\widehat{a}_\chi(j-k, \tfrac{j+\ell}{2}) + \tfrac{k-\ell}{2}\partial_\xi\widehat{a}_\chi(j-k, \tfrac{j+\ell}{2}) + \tfrac{(k-\ell)^2}{8}\widehat{a}_\chi(j-k, \tfrac{j+\ell}{2})$$
$$+ \tfrac{(k-\ell)^3}{8}\int_0^1 (1-t)^2\partial_\xi^3\widehat{a}_\chi(j-k, \tfrac{j+\ell+t(k-\ell)}{2})dt.$$

Analogously we obtain

$$\widehat{b}_\chi(k-\ell, \tfrac{k+\ell}{2}) =$$
$$+ \tfrac{k-j}{2}\partial_\xi\widehat{b}_\chi(k-\ell, \tfrac{j+\ell}{2}) + \tfrac{(k-j)^2}{8}\partial_\xi^2\widehat{b}_\chi(k-\ell, \tfrac{j+\ell}{2})$$
$$+ \tfrac{(k-j)^3}{8}\int_0^1 (1-t)^2\partial_\xi^3\widehat{b}_\chi(k-\ell, \tfrac{j+\ell+t(k-j)}{2})dt.$$

An explicit computation proves that

$$Op^{\mathrm{BW}}(a)Op^{\mathrm{BW}}(b) - Op^{\mathrm{BW}}(ab + \tfrac{1}{2\mathrm{i}} - \tfrac{1}{8}\mathfrak{s}(a,b))u = \sum_{j=1}^{4} R_i(a,b)u,$$

where

$$R_1 := Op^{\mathrm{W}}\big(a_\chi b_\chi - (ab)_\chi + \tfrac{1}{2\mathrm{i}}(\{a_\chi, b_\chi\} - \{a,b\}_\chi) - \tfrac{1}{8}(\mathfrak{s}(a_\chi, b_\chi) - \mathfrak{s}(a,b)_\chi)\big)u,$$
$$R_2 := \sum Q_3^b\big(\widehat{a}_\chi(j-k, \tfrac{j+\ell}{2}) + \tfrac{k-\ell}{2}\partial_\xi\widehat{a}_\chi(j-k, \tfrac{j+\ell}{2})$$
$$+ \tfrac{(k-\ell)^2}{8}\widehat{a}_\chi(j-k, \tfrac{j+\ell}{2})\big)u_\ell e^{\mathrm{i}jx},$$
$$R_3 := \sum Q_3^a\widehat{b}_\chi(k-\ell, \tfrac{k+\ell}{2})u_\ell e^{\mathrm{i}jx},$$
$$R_4 := -\tfrac{1}{16\mathrm{i}}Op^{\mathrm{W}}(\partial_x^2\partial_\xi a\partial_x\partial_\xi^2 b + \partial_x^2\partial_\xi b\partial_x\partial_\xi^2 a)u + \tfrac{1}{64}Op^{\mathrm{W}}(\partial_x^2\partial_\xi^2 a\partial_x^2\partial_\xi^2 b)u,$$

where we have defined $Q_3^a := \frac{(k-\ell)^3}{8}\int_0^1 (1-t)^2 \partial_\xi^3 \widehat{a}_\chi(j-k, \frac{j+\ell+t(k-\ell)}{2})dt$ and analogously Q_3^b. We prove that each R_i fulfils the estimate (11). The remainders R_1, R_2 and R_3 have to be treated as done in the proof of Theorem 2.5 in [1], we just underline the differences. Concerning R_1 it is enough to prove that for any $\alpha \leq 2$ the symbol $\partial_\xi^\alpha a_\chi \partial_x^\alpha b_\chi - \partial_\xi^\alpha b_\chi \partial_x^\alpha a_\chi$ is a spectrally localised symbol belonging to $\Gamma_{L^\infty}^{m+m'-\rho}$. Following word by word the proof in [1], with $d = 1$ and $\alpha = 2$ (instead of $\alpha = 1$ therein) one may bound $|\partial_\xi^\alpha a_\chi \partial_x^\alpha b_\chi - \partial_\xi^\alpha b_\chi \partial_x^\alpha a_\chi|_{m,W^{1,\infty},n} \lesssim |a|_{m,W^{1,\infty},n+2}|b|_{m',L^\infty,n+2} + |a|_{m,L^\infty,n+2}|b|_{m',W^{1,\infty},n+2}$. The estimate (11) on the remainder R_1 follows from Theorem A.7 in [1]. In order to prove that R_3 and R_2 satisfy (11), one has to follow the proof of Theorem A.5 in [1] with $d = 1, \alpha = 3$ and $\beta \leq 2$ corresponding to the remainder $R_2(a, b)$ therein. Concerning the remainder R_4 we have the following: the symbol of the first summand is in the class $\Gamma_{s_0}^{m+m'-3}$ and the second in $\Gamma_{s_0}^{m+m'-4}$, the estimates follow then by Theorem 2. □

Lemma 1 (Paraproduct) *Fix $s_0 > 1/2$ and let $f, g \in H^s(\mathbb{T}; \mathbb{C})$ for $s \geq s_0$. Then*

$$fg = Op^{\mathrm{BW}}(f)g + Op^{\mathrm{BW}}(g)f + \mathcal{R}(f, g)\,, \tag{12}$$

where

$$\begin{aligned}\widehat{\mathcal{R}(f,g)}(\xi) &= \frac{1}{(2\pi)}\sum_{\eta\in\mathbb{Z}} a(\xi-\eta,\xi)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,,\\ |a(v,w)| &\lesssim \frac{(1+\min(|v|,|w|))^\rho}{(1+\max(|v|,|w|))^\rho}\,,\end{aligned} \tag{13}$$

for any $\rho \geq 0$. For $0 \leq \rho \leq s - s_0$ one has

$$\|\mathcal{R}(f, g)\|_{H^{s+\rho}} \lesssim \|f\|_{H^s}\|g\|_{H^s}\,. \tag{14}$$

Proof Notice that

$$\widehat{(fg)}(\xi) = \sum_{\eta\in\mathbb{Z}} \widehat{f}(\xi-\eta)\widehat{g}(\eta)\,. \tag{15}$$

Consider the cut-off function χ_ϵ and define a new cut-off function $\Theta : \mathbb{R} \to [0, 1]$ as

$$1 = \chi_\epsilon\left(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\right) + \chi_\epsilon\left(\frac{|\eta|}{\langle 2\xi-\eta\rangle}\right) + \Theta(\xi,\eta)\,. \tag{16}$$

Recalling (15) and (7) we note that

$$\begin{aligned}\widehat{(T_f g)}(\xi) &= \sum_{\eta\in\mathbb{Z}} \chi_\epsilon\left(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\right)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,,\\ \widehat{(T_g f)}(\xi) &= \sum_{\eta\in\mathbb{Z}} \chi_\epsilon\left(\frac{|\eta|}{\langle 2\xi-\eta\rangle}\right)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,,\end{aligned} \tag{17}$$

and

$$\mathcal{R} := \mathcal{R}(f,g)\,, \qquad \widehat{\mathcal{R}}(\xi) := \sum_{\eta\in\mathbb{Z}} \Theta(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,. \tag{18}$$

To obtain the second in (17) one has to use the (7) and perform the change of variable $\xi-\eta \rightsquigarrow \eta$. By the definition of the cut-off function $\Theta(\xi,\eta)$ we deduce that, if $\Theta(\xi,\eta)\neq 0$ we must have

$$|\xi-\eta| \geq \frac{5\epsilon}{4}\langle\xi+\eta\rangle \quad \text{and} \quad |\eta| \geq \frac{5\epsilon}{4}\langle 2\xi-\eta\rangle \qquad \Rightarrow \quad \langle\eta\rangle \sim \langle\xi-\eta\rangle\,. \tag{19}$$

This implies that, setting $a(\xi-\eta,\eta) := \Theta(\xi,\eta)$, we get the (13). The (19) also implies that $\langle\xi\rangle \lesssim \max\{\langle\xi-\eta\rangle,\langle\eta\rangle\}$. Then we have

$$\begin{aligned}\|\mathcal{R}h\|^2_{H^{s+\rho}} &\lesssim \sum_{\xi\in\mathbb{Z}}\Big(\sum_{\eta\in\mathbb{Z}}|a(\xi-\eta,\eta)||\widehat{f}(\xi-\eta)||\widehat{g}(\eta)|\langle\xi\rangle^{s+\rho}\Big)^2\\ &\overset{(13)}{\lesssim} \sum_{\xi\in\mathbb{Z}}\Big(\sum_{|\xi-\eta|\geq|\eta|}\langle\xi-\eta\rangle^{s}|\widehat{f}(\xi-\eta)|\langle\eta\rangle^{\rho}|\widehat{g}(\eta)|\Big)^2\\ &+\sum_{\xi\in\mathbb{Z}}\Big(\sum_{|\xi-\eta|\leq|\eta|}\langle\xi-\eta\rangle^{\rho}|\widehat{f}(\xi-\eta)||\widehat{g}(\eta)||\eta|^{s}\Big)^2\\ &\lesssim \sum_{\xi,\eta\in\mathbb{Z}}\langle\eta\rangle^{2(s_0+\rho)}|\widehat{g}(\eta)|^2\langle\xi-\eta\rangle^{2s}|\widehat{f}(\xi-\eta)|^2\\ &+\sum_{\xi,\eta\in\mathbb{Z}}\langle\eta\rangle^{2s}|\widehat{g}(\eta)|^2\langle\xi-\eta\rangle^{2(s_0+\rho)}|\widehat{f}(\xi-\eta)|^2\\ &\lesssim \|f\|^2_{H^s}\|g\|^2_{H^{s_0+\rho}} + \|f\|^2_{H^{s_0+\rho}}\|g\|^2_{H^s}\,,\end{aligned}$$

which implies the (14) for $s_0+\rho\leq s$. □

3 Paralinearization

Equation (2) is equivalent to

$$\begin{aligned} & u_t + u_{xxx}\partial^2_{z_1 z_1} F + 2u_{xx}\partial^3_{z_1 x z_1} F + u^2_{xx}\partial^3_{z_1 z_1 z_1} F + 2u_x u_{xx}\partial^3_{z_1 z_1 z_0} F \\ & \quad + u^2_x \partial^3_{z_1 z_0 z_0} F + u_x(-\partial^2_{z_0 z_0} F + 2\partial^3_{z_1 x z_0} F) - \partial^2_{z_0 x} F + \partial^3_{z_1 xx} F = 0. \end{aligned} \tag{20}$$

We have the following.

Theorem 4 *Equation (20) is equivalent to*

$$u_t + Op^{\mathrm{BW}}(A(u))u + R_0 = 0, \tag{21}$$

where

$$A(u) := \partial^2_{z_1 z_1} F(\mathrm{i}\xi)^3 + \tfrac{1}{2}\tfrac{d}{dx}\Big(\partial^2_{z_1 z_1} F\Big)(\mathrm{i}\xi)^2 + a_1(u, u_x, u_{xx}, u_{xxx})(\mathrm{i}\xi),$$

with a_1 real function and R_0 semi-linear remainder. Moreover we have the following estimates. Let $\sigma \geq s_0 > 1 + 1/2$ and consider $U, V \in \dot{H}^{\sigma+3}$

$$\|R_0(U)\|_{\dot{H}^\sigma} \leq C(\|U\|_{\dot{H}^{s_0+3}})\|U\|_{\dot{H}^\sigma}, \ \ \|R_0(U)\|_{\dot{H}^\sigma} \leq C(\|U\|_{\dot{H}^{s_0}})\|U\|_{\dot{H}^{\sigma+3}}, \tag{22}$$

$$\begin{aligned} \|R_0(U) - R_0(V)\|_{\dot{H}^\sigma} &\leq C(\|U\|_{\dot{H}^{s_0+3}} + \|V\|_{\dot{H}^{s_0+3}})\|U - V\|_{\dot{H}^\sigma} \\ &\quad + C(\|U\|_{\dot{H}^\sigma} + \|V\|_{\dot{H}^\sigma})\|U - V\|_{\dot{H}^{s_0+3}}, \end{aligned} \tag{23}$$

$$\|R_0(U) - R_0(V)\|_{\dot{H}^{s_0}} \leq C(\|U\|_{\dot{H}^{s_0+3}} + \|V\|_{\dot{H}^{s_0+3}})\|U - V\|_{\dot{H}^{s_0}}, \tag{24}$$

where C is a non-decreasing and positive function. Concerning the paradifferential operator we have for any $\sigma \geq 0$

$$\|Op^{\mathrm{BW}}(A(u) - A(w))v\|_{\dot{H}^\sigma} \leq C(\|u\|_{\dot{H}^{s_0}} + \|w\|_{\dot{H}^{s_0}})\|u - w\|_{\dot{H}^{s_0}}\|v\|_{\dot{H}^{s_0+3}}. \tag{25}$$

Proof In the following we use the Bony paraproduct (Lemma 1) and Proposition 3 and we obtain ($\tilde{R}_0$ is a smoothing remainder satisfying (22), (23) and it possibly changes from line to line)

$$\begin{aligned} u_{xxx}\partial^2_{z_1 z_1} F &= Op^{\mathrm{BW}}(u_{xxx})\partial^2_{z_1 z_1} F + Op^{\mathrm{BW}}(\partial^2_{z_1 z_1} F) \circ Op^{\mathrm{BW}}((\mathrm{i}\xi)^3)u + \tilde{R}_0 \\ &= Op^{\mathrm{BW}}(u_{xxx}) \circ Op^{\mathrm{BW}}(\partial^3_{z_1 z_1 z_1} F) \circ Op^{\mathrm{BW}}(\mathrm{i}\xi)u \\ &\quad + Op^{\mathrm{BW}}(\partial^2_{z_1 z_1} F) \circ Op^{\mathrm{BW}}((\mathrm{i}\xi)^3)u + \tilde{R}_0 \\ &= Op^{\mathrm{BW}}(\partial^2_{z_1 z_1} F(\mathrm{i}\xi)^3)u + \tfrac{3}{2} Op^{\mathrm{BW}}(\tfrac{d}{dx}(\partial^2_{z_1 z_1} F)\xi^2)u \\ &\quad + Op^{\mathrm{BW}}(\tilde{a}_1(\mathrm{i}\xi)) + \tilde{R}_0, \end{aligned} \tag{26}$$

where we have denoted by $\tilde{a}_1$ a real function depending on $x, u, u_x, u_{xx}, u_{xxx}$. Analogously we obtain

$$2u_{xx}\partial^3_{z_1 x z_1} F = 2Op^{\mathrm{BW}}(\partial^3_{z_1 z_1 x} F(\mathrm{i}\xi)^2)u + Op^{\mathrm{BW}}(\tilde{a}_1(\mathrm{i}\xi))u + \tilde{R}_0, \tag{27}$$

$$u^2_{xx}\partial^3_{z_1 z_1 z_1} F = 2Op^{\mathrm{BW}}(u_{xx}\partial^3_{z_1 z_1 z_1} F(\mathrm{i}\xi^2))u + Op^{\mathrm{BW}}(\tilde{a}_1(\mathrm{i}\xi))u + \tilde{R}_0, \tag{28}$$

$$2u_x u_{xx}\partial^3_{z_1 z_1 z_0} F = 2Op^{\mathrm{BW}}\big(u_x\partial^3_{z_1 z_1 z_0} F(\mathrm{i}\xi)^2)u + 2Op^{\mathrm{BW}}(\tilde{a}_1(\mathrm{i}\xi))u + \tilde{R}_0. \tag{29}$$

Summing up the previous equations we get

$$\begin{aligned} &u_t + Op^{\mathrm{BW}}(\partial^2_{z_1 z_1} F(\mathrm{i}\xi)^3)u + \tfrac{1}{2}Op^{\mathrm{BW}}(\tfrac{d}{dx}(\partial^2_{z_1 z_1} F)(\mathrm{i}\xi)^2)u \\ &\quad + Op^{\mathrm{BW}}(a_1(x, u, u_x, u_{xx}, u_{xxx})\mathrm{i}\xi)u + \tilde{R}_0(u) = 0, \end{aligned} \tag{30}$$

where a_1 is real and R_0 is a semi-linear remainder satisfying (22) and (23). □

4 Linear Local Well-Posedness

Proposition 1 *Let* $s_0 > 1 + 1/2$, $\Theta \geq r > 0$, $u \in C^0([0, T]; H_0^{s_0+3}) \cap C^1([0, T]; H_0^{s_0})$ *such that*

$$\|u\|_{L^\infty \dot{H}^{s_0+3}} + \|\partial_t u\|_{\dot{H}^{s_0}} \leq \Theta, \quad \|u\|_{L^\infty \dot{H}^{s_0}} \leq r. \tag{31}$$

Let $\sigma \geq 0$ *and* $t \mapsto R(t) \in C^0([0, T], \dot{H}^\sigma)$. *Then there exists a unique solution* $v \in C^0([0, T]; \dot{H}^\sigma) \cap C^1([0, T]; \dot{H}^{\sigma-3})$ *of the linear inhomogeneous problem*

$$\begin{aligned} &v_t + Op^{\mathrm{BW}}(\partial^2_{z_1 z_1} F(u, u_x)(\mathrm{i}\xi)^3)v + \tfrac{1}{2}Op^{\mathrm{BW}}(\tfrac{d}{dx}(\partial^2_{z_1 z_1} F(u, u_x))(\mathrm{i}\xi)^2)v \\ &\quad + Op^{\mathrm{BW}}(\tilde{a}_1(x, u, u_x, u_{xx}, u_{xxx})(\mathrm{i}\xi))v + R(t) = 0, \\ &v(0, x) = v_0(x). \end{aligned} \tag{32}$$

Moreover the solution satisfies the estimate

$$\|v\|_{L^\infty \dot{H}^\sigma} \leq e^{C_\Theta T}(C_r\|v_0\|_{\dot{H}^\sigma} + C_\Theta T\|R\|_{L^\infty \dot{H}^\sigma}). \tag{33}$$

Consider Eq. (32). We have for any $N \in \mathbb{N}$, $\sigma > 1/2$ and $s \geq 0$

$$
\begin{aligned}
&\|\tilde{a}_1(x, u, u_x, u_{xx}, u_{xxx})\|_{\dot{H}^\sigma} \leq C(\|u\|_{\dot{H}^{\sigma+3}}) \\
&\|\tfrac{d}{dx}(\partial^2_{z_1 z_1} F(u, u_x))\|_{\dot{H}^{\sigma-1}} \leq C(\|u\|_{\dot{H}^{\sigma+2}}) \\
&\|\partial^2_{z_1 z_1} F(u, u_x)\|_{\dot{H}^\sigma} \leq C(\|u\|_{\dot{H}^{\sigma+1}}), \\
&|\partial^2_{z_1 z_1} F(u, u_x)|\xi|^{2s}|_{2s,\sigma,N} \leq C_N(\|u\|_{\dot{H}^{\sigma+1}}), \\
&|\tfrac{d}{dx}(\partial^2_{z_1 z_1} F(u, u_x))(\mathrm{i}\xi)^2|_{2,\sigma,N} \leq C_N(\|u\|_{\dot{H}^{\sigma+2}}), \\
&|\tilde{a}_1(x, u_x, u_{xx}, u_{xxx})|_{1,\sigma,N} \leq C_N(\|u\|_{\dot{H}^{\sigma+2}}).
\end{aligned}
\tag{34}
$$

In the following lemma we prove that, thanks to the Hamiltonian structure, we may eliminate the symbol of order two by means of a paradifferential change of variable. This term is the only one which has positive order and that is not skew-self-adjoint.

Lemma 2 *Define* $\mathtt{d}(x, u, u_x) := \sqrt[6]{\partial^2_{z_1 z_1} F(x, u, u_x)}$. *Then we have*

$$
\begin{aligned}
Op^{\mathrm{BW}}(\mathtt{d}) \circ Op^{\mathrm{BW}}\Big(\partial^2_{z_1 z_1} F(\mathrm{i}\xi)^3 + \tfrac{1}{2}\tfrac{d}{dx}(\partial^2_{z_1 z_1} F)(\mathrm{i}\xi)^2\Big) \circ Op^{\mathrm{BW}}(\mathtt{d}^{-1})v = \\
Op^{\mathrm{BW}}\big(\big[\partial^2_{z_1 z_1} F(\mathrm{i}\xi)^3 + \tilde{a}_1(x, u, u_x, u_{xx}, u_{xxx})(\mathrm{i}\xi)\big]\big)v + R_0,
\end{aligned}
\tag{35}
$$

where $\tilde{a}_1$ is a real function and R_0 is a semi-linear remainder verifying (22)*,* (23)*,* (24)*.*

Proof First of all the function $\mathtt{d}(x, u, u_x)$ is well defined because of hypothesis (1). We recall formula (10) (and the definition of the Poisson's bracket after (10)). By using Theorem 3 with $\rho \in (1, 2]$ we obtain that the L.H.S. of Eq. (35) equals

$$
\begin{aligned}
&-Op^{\mathrm{BW}}(\mathrm{i}\partial_{z_1 z_1} F\xi^3)v - \frac{1}{2} Op^{\mathrm{BW}}(\tfrac{d}{dx}(\partial^2_{z_1 z_1} F)\xi^2)v \\
&\qquad + 3 Op^{\mathrm{BW}}\Big(\mathtt{d}^{-1} \cdot \tfrac{d}{dx}\mathtt{d} \cdot \partial^2_{z_1 z_1} F \cdot \xi^2\Big)v + Op^{\mathrm{BW}}(\tilde{a}_1) + R_0,
\end{aligned}
$$

where $\tilde{a}_1$ is a purely imaginary function and R_0 a semi-linear remainder. One can verify that the symbol of order two equals to zero by direct inspection. □

We consider symbol

$$
\begin{aligned}
&\mathfrak{S}(x, u, \xi) := \\
&\quad \partial^2_{z_1 z_1} F(u, u_x)(\mathrm{i}\xi)^3 + \tfrac{1}{2}\tfrac{d}{dx}(\partial^2_{z_1 z_1} F(u, u_x))(\mathrm{i}\xi)^2 + \tilde{a}_1(u, u_x, u_{xx}, u_{xxx})\mathrm{i}\xi,
\end{aligned}
\tag{36}
$$

and we introduce the smoothed version of the homogeneous part of (32), more precisely

$$\partial_t v^\epsilon = Op^{\mathrm{BW}}(\mathfrak{S}(x, u, u_x, u_{xx}, u_{xxx}; \xi))v^\epsilon - \epsilon\partial_{xx}^4 v_\epsilon. \tag{37}$$

Thanks to the parabolic term $\epsilon\partial_{xx}^4 v_\epsilon$ for any $\epsilon > 0$ there exists a unique solution of the equation, with initial condition in H^σ, (37) which is $C^0([0, T], \dot{H}^\sigma)$ for any $\sigma \geq 0$, where T depends on ϵ. This is the content on the following lemma.

Lemma 3 *For any initial condition v_0 in $\dot{H}^\sigma$ with $\sigma \geq 0$, there exists a time $T_\epsilon > 0$ and a unique solution v^ϵ (37) belonging to $C^0([0, T_\epsilon); \dot{H}^\sigma)$.*

Proof We consider the operator

$$\Gamma v := e^{-\epsilon t\partial_x^4} v_0 + \int_0^t e^{-\epsilon(t-t')\partial_x^4} Op^{\mathrm{BW}}(\mathfrak{S}(x, u, u_x, u_{xx}, u_{xxx}; \xi))v^\epsilon(t')dt'.$$

We have $\|e^{-\epsilon t\partial_x^4} v_0\|_{\dot{H}^\sigma} \leq \|v_0\|_{\dot{H}^\sigma}$ and $\|\int_0^t e^{-\epsilon(t-t')\partial_x^4} f(t', \cdot)dt'\|_{\dot{H}^\sigma} \leq t^{\frac{1}{4}}\epsilon^{-\frac{3}{4}}\|f\|_{\dot{H}^{\sigma-3}}$, with these estimates, (34), (31) and Theorem 2 one may apply a fixed point argument in a suitable subspace of $C^0([0, T_\epsilon); \dot{H}^\sigma)$ for a suitable time T_ϵ (going to zero when ϵ goes to zero). Let us prove the second one of the above inequalities. We use the Minkowski inequality and the boundedness of the function $\alpha^{3/2}e^{-\alpha}$ for $\alpha \geq 0$, we get

$$\begin{aligned}
\| \int_0^t e^{-\epsilon(t-t')\partial_x^4} f(t', \cdot)dt'\|_{\dot{H}^\sigma} &\leq \int_0^t \|e^{-\epsilon(t-t')\partial_x^4} f(t', \cdot)\|_{\dot{H}^\sigma} dt' \\
&= \int_0^t \left(\sum_{\xi\in\mathbb{Z}^*} e^{-2\epsilon(t-t')\xi^4} \xi^{2\sigma} |\widehat{f}(t', \xi)|^2 \right)^{1/2} dt' \\
&\lesssim \int_0^t \epsilon^{-\frac{3}{4}}(t-t')^{-\frac{3}{4}} \|f(t', \cdot)\|_{\dot{H}^{\sigma-3}} dt' \\
&\lesssim t^{\frac{1}{4}}\epsilon^{-\frac{3}{4}}\|f\|_{L^\infty \dot{H}^{\sigma-3}}.
\end{aligned}$$

□

We show that (37) equation verifies *a priori* estimates with constants independent of ϵ. We have the following.

Proposition 2 *Let u be a function as in (31). For any $\sigma \geq 0$ there exist constants C_Θ and C_r, such that for any $\epsilon > 0$ the unique solution of (37) verifies*

$$\|v^\epsilon\|_{\dot{H}^\sigma}^2 \leq C_r\|v_0\|_{\dot{H}^\sigma}^2 + C_\Theta \int_0^t \|v^\epsilon(\tau)\|_{\dot{H}^\sigma}^2 d\tau, \forall t \in [0, T]. \tag{38}$$

As a consequence we have

$$\|v^{\epsilon}\|_{\dot{H}^{\sigma}} \leq C_r e^{TC_{\theta}} \|v_0\|_{\dot{H}^{\sigma}}, \forall t \in [0, T]. \tag{39}$$

We define the modified energy

$$\|v\|^2_{\sigma,u} := \left\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1 z_1} F)^{\frac{2}{3}\sigma} \xi^{2\sigma}\right) Op^{\mathrm{BW}}\left(\mathtt{d}(x, u, u_x)\right) v, Op^{\mathrm{BW}}\left(\mathtt{d}(x, u, u_x)\right) v\right\rangle_{L^2}, \tag{40}$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product on $L^2(\mathbb{R})$ and $\mathtt{d}$ is defined in Lemma 2, note that the function $(\partial^2_{z_1 z_1} F(x, u, u_x))^{\frac{2}{3}\sigma}$ is well defined for any $\sigma \in \mathbb{R}$ thanks to (1).

In the following we prove that $\|\cdot\|_{\sigma,u}$ is equivalent to $\|\cdot\|_{\dot{H}^{\sigma}}$.

Lemma 4 *Let $s_0 > 1/2, \sigma \geq 0, r \geq 0$. Then there exists a constant (depending on r and σ) such that for any u such that $\|u\|_{\dot{H}^{s_0}} \leq r$ we have*

$$C_r^{-1} \|v\|^2_{\dot{H}^{\sigma}} - \|v\|^2_{\dot{H}^{-3}} \leq \|v\|^2_{\sigma,u} \leq C_r \|v\|^2_{\dot{H}^{\sigma}} \tag{41}$$

for any v in $\dot{H}^{\sigma}$.

Proof Concerning the second inequality in (41), we reason as follows. We have

$$\begin{aligned} \|v\|^2_{\sigma,u} \leq & \|Op^{\mathrm{BW}}((\partial^2_{z_1 z_1} F(x, u, u_x))^{\frac{2}{3}\sigma} \xi^{2\sigma}) Op^{\mathrm{BW}}(\mathtt{d}(x, u, u_x)) v\|_{\dot{H}^{-\sigma}} \\ & \times \|Op^{\mathrm{BW}}(\mathtt{d}(x, u, u_x)) v\|_{\dot{H}^{\sigma}} \\ \leq & C_r \|v\|_{\dot{H}^{\sigma}}, \end{aligned}$$

where in the last inequality we used Theorem 2 and the fact that $\mathtt{d}$ is a symbol of order zero. We focus on the first inequality in (41). Let $\delta > 0$ be such that $s_0 - \delta = 1/2$, then applying Theorem 3 with $s_0 = \delta$ instead of s_0 and $\rho = \delta$, we have

$$\begin{aligned} & Op^{\mathrm{BW}}((\partial^2_{z_1 z_1} F(x, u, u_x))^{\frac{1}{3}\sigma}) \circ Op^{\mathrm{BW}}(|\xi|^{2\sigma}) \circ Op^{\mathrm{BW}}((\partial^2_{z_1 z_1} F(x, u, u_x))^{\frac{1}{3}\sigma}) \\ & = Op^{\mathrm{BW}}(Op^{\mathrm{BW}}((\partial^2_{z_1 z_1} F(x, u, u_x))^{\frac{2}{3}\sigma} |\xi|^{2\sigma}) + \mathcal{R}^{2\sigma-\delta}(u), \end{aligned} \tag{42}$$

where

$$\|\mathcal{R}^{2\sigma-\delta}(u) f\|_{\dot{H}^{\overline{\sigma}-2\sigma+\delta}} \leq C(r, \overline{\sigma}) \|f\|_{\dot{H}^{\overline{\sigma}}}.$$

Analogously we obtain

$$Op^{\text{BW}}((\partial^2_{z_1z_1}F(x,u,u_x))^{-\frac{1}{3}\sigma})\circ Op^{\text{BW}}(\mathtt{d}^{-1}(x,u,u_x,u_xx))\circ \\ Op^{\text{BW}}((\partial^2_{z_1z_1}F(x,u,u_x))^{\frac{1}{3}\sigma})\circ Op^{\text{BW}}(\mathtt{d}(x,u,u_x,u_xx)) \\ = 1+R^{-\delta}(u), \tag{43}$$

where

$$\|\mathcal{R}^{-\delta}(u)f\|_{\dot H^{\overline{\sigma}}}\leq C(r,\overline{\sigma})\|f\|_{\dot H^{\overline{\sigma}-\delta}},$$

for any f in $\dot H^{\overline{\sigma}-\delta}$. Therefore we have

$$\begin{aligned}\|v\|^2_{\dot H^\sigma} &\overset{(43)}{\lesssim} \\ &\|Op^{\text{BW}}((\partial^2_{z_1z_1}F)^{-\frac{1}{3}\sigma})Op^{\text{BW}}(\mathtt{d})v\|^2_{\dot H^\sigma}+\|v\|^2_{\dot H^{\sigma-\delta}}\\ &\leq C_r(\|Op^{\text{BW}}(\partial^2_{z_1z_1}F)^{\frac{1}{3}\sigma})Op^{\text{BW}}(\mathtt{d})v\|^2_{\dot H^\sigma}+\|v\|^2_{\dot H^{\sigma-\delta}})\\ &\overset{(42)}{=}C_r(\|v\|^2_{u,\sigma}+\|v\|^2_{\dot H^{\sigma-\delta/2}}+\|v\|^2_{\dot H^{\sigma-\delta}}).\end{aligned}$$

Then by using the interpolation inequality $\|f\|_{\dot H^{\theta s_1+(1-\theta)s_2}}\leq\|f\|^\theta_{\dot H^{s_1}}\|f\|^{1-\theta}_{\dot H^{s_2}}$ which is valid for any $s_1<s_2$, $\theta\in[0,1]$ and $f\in\dot H^{s_2}$, we get (by means of the Young inequality $ab\leq p^{-1}a^p+q^{-1}b^q$, with $1/p+1/q=1$ and $p=2(\sigma+3)/\delta$, $q=2(\sigma+3)/[2(\sigma+3)-\delta]$)

$$\begin{aligned}\|v\|^2_{\dot H^{\sigma-\delta/2}}&\leq(\|v\|^2_{\dot H^{-3}})^{\frac{\delta}{2}\frac{1}{\sigma+3}}(\|v\|^2_{\dot H^\sigma})^{\frac{2(\sigma+3)-\delta}{2(\sigma+3)}}\\ &\leq\frac{\delta}{2(\sigma+3)}\|v\|^2_{\dot H^{-3}}\tau^{-\frac{2(\sigma+3)}{\delta}}+\frac{2(\sigma+3)-\delta}{2(\sigma+3)}\tau^{\frac{2(\sigma+3)-\delta}{2(\sigma+3)}}\|v\|^2_{\dot H^\sigma},\end{aligned}$$

for any $\tau>0$. Choosing τ small enough we conclude. □

We shall need the following (weak) Garding type inequality.

Lemma 5 (Weak Garding) *Let* $\mathtt{d}$ *as in Lemma 2 and* $\mathfrak{c}>0$ *as in (1) and define* $g:=\partial^2_{z_2z_2}F$*, we have the following inequalities*

$$\langle Op^{\text{BW}}(\mathtt{d})Op^{\text{BW}}(g^{\frac{2}{3}\sigma}\xi^{2\sigma})Op^{\text{BW}}(\mathtt{d})Op^{\text{BW}}(\xi^4)w,w\rangle_{L^2}\geq\frac{\mathfrak{c}_\sigma}{2}\|w\|_{\dot H^{\sigma+2}}-\mathfrak{K}\|w\|_{\dot H^\sigma},$$

$$\langle Op^{\text{BW}}(\xi^4)Op^{\text{BW}}(\mathtt{d})Op^{\text{BW}}(g^{\frac{2}{3}\sigma}\xi^{2\sigma})Op^{\text{BW}}(\mathtt{d})w,w\rangle_{L^2}\geq\frac{\mathfrak{c}_\sigma}{2}\|w\|_{\dot H^{\sigma+2}}-\mathfrak{K}\|w\|_{\dot H^\sigma},$$

for any w *in* $\dot H^{\sigma+2}$ *and where* $\mathfrak{K}>0$ *depends on* Θ *in (31) and* $\mathfrak{c}_\sigma:=\mathfrak{c}^{\frac{1}{3}+\frac{2}{3}\sigma}$.

Proof We prove the first inequality, the second one is similar. By using Theorem 3 with $\rho = 1$ we get

$$\begin{aligned} & Op^{\mathrm{BW}}(\mathtt{d}) Op^{\mathrm{BW}}(g^{\frac{2}{3}\sigma}\xi^{2\sigma}) Op^{\mathrm{BW}}(\mathtt{d}) Op^{\mathrm{BW}}(\xi^4) w \\ &= Op^{\mathrm{BW}}(\mathtt{d}^2 g^{\frac{2}{3}\sigma}\xi^{2\sigma}\xi^4) w + R_{2\sigma+3} w \\ &= Op^{\mathrm{BW}}(g^{\frac{1}{3}+\frac{2}{3}\sigma}\xi^{2\sigma}\xi^4) w + R_{2\sigma+3} w, \end{aligned}$$

where $\|R_{2\sigma+3}w\|_{\dot{H}^{-\sigma-2}} \leq C_\Theta \|w\|_{\dot{H}^{\sigma+1}}$. Now we set

$$p(x,\xi) = \sqrt{g^{\frac{1}{3}+\frac{2}{3}\sigma}\xi^{2\sigma+4} - \frac{\mathfrak{c}_\sigma}{2}\xi^{2\sigma+4}}, \quad |\xi| \geq 1, \quad \mathfrak{c}_\sigma = \mathfrak{c}^{\frac{1}{3}+\frac{2}{3}\sigma}. \tag{44}$$

We have

$$\begin{aligned} 0 \leq \|Op^{\mathrm{BW}}(p) w\|_{L^2} &= \langle Op^{\mathrm{BW}}(p) Op^{\mathrm{BW}}(p) w, w\rangle_{L^2} \\ &= \langle Op^{\mathrm{BW}}(g^{\frac{1}{3}+\frac{2}{3}\sigma}\xi^{2\sigma+4}) w, w\rangle_{L^2} \\ &\quad - \frac{\mathfrak{c}_\sigma}{2}\|w\|^2_{\dot{H}^{\sigma+2}} + \langle \tilde{R}_{2\sigma+3} w, w\rangle, \end{aligned}$$

where $\tilde{R}_{2\sigma+3}$ verifies the same estimate as $R_{2\sigma+3}$ and where we used Theorem 3 with $\rho = 1$. Summing up we obtain

$$\begin{aligned} & \langle Op^{\mathrm{BW}}(\mathtt{d}) Op^{\mathrm{BW}}(g^{\frac{2}{3}\sigma}\xi^{2\sigma}) Op^{\mathrm{BW}}(\mathtt{d}) Op^{\mathrm{BW}}(\xi^4) w, w\rangle_{L^2} \geq \\ & \frac{\mathfrak{c}_\sigma}{2}\|w\|^2_{\dot{H}^{\sigma+2}} - 2C_\Theta \|w\|_{\dot{H}^{\sigma+1}} \|w\|_{\dot{H}^{\sigma+2}}. \end{aligned}$$

We need to estimate from above the last summand, for any $\varepsilon, \eta > 0$ we have

$$\begin{aligned} \|w\|_{\dot{H}^{\sigma+1}} \|w\|_{\dot{H}^{\sigma+2}} &\leq \varepsilon \|w\|^2_{\dot{H}^{\sigma+2}} + C_\varepsilon \|w\|_{\dot{H}^{\sigma}} \|w\|_{\dot{H}^{\sigma+2}} \\ &\leq \varepsilon \|w\|^2_{\dot{H}^{\sigma+2}} + C_\varepsilon(\eta \|w\|^2_{\dot{H}^{\sigma+2}} + \eta^{-1}\|w\|^2_{\dot{H}^{\sigma}}), \end{aligned}$$

we conclude by choosing ε and η in such a way that $2C_\Theta(\varepsilon + C_\varepsilon \eta) \leq \mathfrak{c}_\sigma/4$. □

We are in position to prove Proposition 2.

Proof of Proposition 2 We take the derivative with respect to t of the modified energy (40) along the solution v^ϵ of Eq. (37). We have

$$\frac{d}{dt}\|v^\epsilon\|_{\sigma,u} = \langle Op^{\mathrm{BW}}\left(\frac{d}{dt}(\partial^2_{z_1 z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right) Op^{\mathrm{BW}}(\mathtt{d})\, v^\epsilon, Op^{\mathrm{BW}}(\mathtt{d})\, v^\epsilon\rangle_{L^2} \tag{45}$$

$$+\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1 z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right) Op^{\mathrm{BW}}\left(\frac{d}{dt}\mathtt{d}\right) v^\epsilon, Op^{\mathrm{BW}}(\mathtt{d})\, v^\epsilon\rangle_{L^2} \tag{46}$$

$$+\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1 z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right) Op^{\mathrm{BW}}(\mathtt{d})\, \frac{d}{dt}v^\epsilon, Op^{\mathrm{BW}}(\mathtt{d})\, v^\epsilon\rangle_{L^2} \tag{47}$$

$$+\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1 z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,v^{\epsilon}, Op^{\mathrm{BW}}\left(\tfrac{d}{dt}\mathtt{d}\right)v^{\epsilon}\rangle_{L^2} \tag{48}$$

$$+\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1 z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,v^{\epsilon}, Op^{\mathrm{BW}}(\mathtt{d})\,\tfrac{d}{dt}v^{\epsilon}\rangle_{L^2}. \tag{49}$$

The most important term, where we have to see a cancellation, is the one given by (47)+(49). Using Eq. (37) we deduce that (47)+(49) equals to

$$\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1 z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,v^{\epsilon}, Op^{\mathrm{BW}}(\mathtt{d})\,Op^{\mathrm{BW}}(\mathfrak{S})v^{\epsilon}\rangle_{L^2} \tag{50}$$

$$+\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1 z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,Op^{\mathrm{BW}}(\mathfrak{S})v^{\epsilon}, Op^{\mathrm{BW}}(\mathtt{d})\,v^{\epsilon}\rangle_{L^2} \tag{51}$$

$$-\epsilon\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1 z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,Op^{\mathrm{BW}}(\xi^4)v^{\epsilon}, Op^{\mathrm{BW}}(\mathtt{d})\,v^{\epsilon}\rangle_{L^2} \tag{52}$$

$$-\epsilon\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1 z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,v^{\epsilon}, Op^{\mathrm{BW}}(\mathtt{d})\,Op^{\mathrm{BW}}(\xi^4)v^{\epsilon}\rangle_{L^2}, \tag{53}$$

where $\mathfrak{S}$ has been defined in (36). For the moment we consider just the first two summands (50)+(51) in the above equation. We note that by using Theorem 3 with $\rho = 3$ we obtain

$$Op^{\mathrm{BW}}(\mathtt{d}^{-1})Op^{\mathrm{BW}}(\mathtt{d})v^{\epsilon} = v^{\epsilon} + \mathcal{R}^{-3}(u)v^{\epsilon},$$

where $\mathcal{R}^{-3}$ verifies (11) with $\rho = 3$. We plug this identity in (50)+(51) and we note that the contribution coming from $\mathcal{R}^{-3}$ is bounded by $C_r\|v^{\epsilon}\|^2_{\dot{H}^{\sigma}}$ thanks to Theorems 3, 2, to the Cauchy Schwartz inequality and to the assumption (31). We are left with

$$\begin{aligned}&\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1 z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,v^{\epsilon}, Op^{\mathrm{BW}}(\mathtt{d})\\ &\quad\times Op^{\mathrm{BW}}(\mathfrak{S})Op^{\mathrm{BW}}(\mathtt{d}^{-1})Op^{\mathrm{BW}}(\mathtt{d})v^{\epsilon}\rangle_{L^2}+\\ &\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1 z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,Op^{\mathrm{BW}}(\mathfrak{S})\\ &\quad\times Op^{\mathrm{BW}}(\mathtt{d}^{-1})Op^{\mathrm{BW}}(\mathtt{d})v^{\epsilon}, Op^{\mathrm{BW}}(\mathtt{d})\,v^{\epsilon}\rangle_{L^2}.\end{aligned}$$

At this point we are ready to use Lemma 2 and we obtain that the previous quantity equals

$$\begin{aligned}&\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1 z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,v^{\epsilon}, Op^{\mathrm{BW}}\left(\partial^2_{z_1 z_1}F(\mathrm{i}\xi)^3+\tilde{a}_1(\mathrm{i}\xi)\right)Op^{\mathrm{BW}}(\mathtt{d})v^{\epsilon}\rangle_{L^2}+\\ &\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1 z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}\left(\partial^2_{z_1 z_1}F(\mathrm{i}\xi)^3+\tilde{a}_1(\mathrm{i}\xi)\right)Op^{\mathrm{BW}}(\mathtt{d})v^{\epsilon}, Op^{\mathrm{BW}}(\mathtt{d})\,v^{\epsilon}\rangle_{L^2}.\end{aligned}$$

By using the skew self-adjoint character of the operators, we deduce that the main term to estimate is the commutator

$$\Big[Op^{\mathrm{BW}}\Big((\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\Big), Op^{\mathrm{BW}}\Big(\partial^2_{z_1z_1}F(\mathrm{i}\xi)^3+\tilde{a}_1(\mathrm{i}\xi)\Big)\Big]Op^{\mathrm{BW}}(\mathtt{d})v^{\epsilon}. \tag{54}$$

We start from the first summand. By using Theorem 3 and Remark 1 with $\rho = 3$ we obtain that

$$C := \Big[Op^{\mathrm{BW}}\Big((\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\Big), Op^{\mathrm{BW}}\Big(\partial^2_{z_1z_1}F(\mathrm{i}\xi)^3\Big)\Big]Op^{\mathrm{BW}}(\mathtt{d})v^{\epsilon} =$$
$$\frac{1}{\mathrm{i}}Op^{\mathrm{BW}}\Big(\Big\{(\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}, \partial^2_{z_1z_1}F(\mathrm{i}\xi)^3\Big\}\Big)Op^{\mathrm{BW}}(\mathtt{d})v^{\epsilon} + \mathcal{R}^0(u)Op^{\mathrm{BW}}(\mathtt{d})v^{\epsilon}.$$

By direct inspection we see that the Poisson bracket above equals to 0. Recalling that $\mathtt{d}$ is a symbol of order 0, by using also Theorem 2 and the assumption (31), we may obtain the bound $\langle C, Op^{\mathrm{BW}}(\mathtt{d})v^{\epsilon}\rangle \leq C_r\|v^{\epsilon}\|^2_{\dot{H}^{\sigma}}$. The second summand, i.e. the one coming from $\tilde{a}_1(\mathrm{i}\xi)$ in (54), may be treated in a similar way: one uses Theorem 3 with $\rho = 1$, at the first order the contribution is equal to zero, then the remainder is a bounded operator from $\dot{H}^{2\sigma}$ to $\dot{H}^0$ and one concludes as before, by using also the duality inequality $\langle f, g\rangle_{L^2} \leq \|f\|_{\dot{H}^{-\sigma}}\|g\|_{\dot{H}^{\sigma}}$, bounding everything by $C_r\|v^{\epsilon}\|^2_{\dot{H}^{\sigma}}$. This concludes the analysis of (50)+(51).

Concerning (52)+(53) we use Lemma 5 and the fact that

$$(52) + (53) \leq -\epsilon\mathfrak{c}_{\sigma}\|v^{\epsilon}\|_{\dot{H}^{\sigma+2}} + 2\mathfrak{K}\|v^{\varepsilon}\|_{\dot{H}^{\sigma}} \leq 2\mathfrak{K}\|v^{\varepsilon}\|_{\dot{H}^{\sigma}},$$

with $\mathfrak{K}$ depending on Θ and $\mathfrak{c}_{\sigma} = \mathfrak{c}^{\frac{1}{3}+\frac{2}{3}\sigma}$, recall (1).

We are left with (45), (46) and (48). These terms are simpler, one just has to use the duality inequality recalled above, then Theorem 2 and the fact that

$$|\tfrac{d}{dt}\mathtt{d}(x, u, u_x)|_{0,\sigma,4}, \quad |\tfrac{d}{dt}(\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}|_{0,0,4} \leq C_{\Theta}\|u\|_{\dot{H}^{\sigma}},$$

where we have used the first one of the assumptions (31).

We eventually obtained $\frac{d}{dt}\|v^{\epsilon}\|^2_{\sigma,u} \leq C_{\Theta}\|v^{\epsilon}\|^2_{\dot{H}^{\sigma}}$, integrating over the time interval $[0, t)$ we obtain

$$\|v^{\epsilon}\|^2_{\sigma,u(t)} \leq \|v^{\epsilon}(0)\|^2_{\sigma,u(0)} + C_{\Theta}\int_0^t \|v^{\epsilon}(\tau)\|^2_{\dot{H}^{\sigma}}d\tau$$
$$\leq C_r\|v^{\epsilon}(0)\|^2_{\dot{H}^{\sigma}} + C_{\Theta}\int_0^t \|v^{\epsilon}(\tau)\|^2_{\dot{H}^{\sigma}}d\tau.$$

We now use (41) and the fact that $\|\partial_t v^{\epsilon}\|_{\dot{H}^{-3}} \leq C_{\Theta}\|v^{\epsilon}\|_{\dot{H}^0} \leq C_{\Theta}\|v^{\epsilon}\|_{\dot{H}^{\sigma}}$ since $\sigma \geq 0$. □

We may now prove Proposition 1.

Proof of Proposition 1 Let v_0 be in $\dot{H}^\sigma$, we consider the smoothed initial condition

$$v_0^\epsilon = \chi(|D|\epsilon^{\frac{1}{8}})v_0 = \mathcal{F}^{-1}(\chi(|\xi|\epsilon^{\frac{1}{8}})\widehat{v_0}(\xi)),$$

for a C_0^∞ cut-off function supported on $(-2, 2)$ and equal to one on $[-1, 1]$. Let v^ϵ the solution of (37) with initial condition v_0^ϵ. By Lemma 3 v^ϵ is a continuous function with values in $\dot{H}^\sigma$ for a short time T_ϵ. By Proposition 2 the $\dot{H}^\sigma$ norm of the solution v^ϵ is bounded from above by a constant depending only on $\|v_0\|_{\dot{H}^\sigma}$, r and Θ in (31). Therefore if we proved that Γ in the proof of Lemma 3 was a contraction on the ball of radius M in $C^0([0, T); \dot{H}^\sigma)$ with M big enough with respect to $\|v_0\|_{\dot{H}^\sigma}$, r and Θ, then we have that there exists a time $T > 0$ depending only on $\|v_0\|_{\dot{H}^\sigma}$, r and Θ such that the solution verifies $\sup_{[0,T_\epsilon)} \|v^\epsilon\|_{\dot{H}^\sigma} \leq M/2$ for any $T_\epsilon \leq T$. For this reason we may iterate the proof of Lemma 3 on the interval $[T_\epsilon, 2T_\epsilon]$ etc… We conclude that there exists a common time of existence $T > 0$ for each solution v^ϵ such that $\sup_{[0,T_\epsilon)} \|v^\epsilon\|_{\dot{H}^\sigma} \leq \mathtt{M}$ with $\mathtt{M}$ depending on $\|v_0\|_{\dot{H}^\sigma}$, Θ and r in (31).

We show that v^ϵ is a Cauchy sequence in $C([0, T); \dot{H}^\sigma)$. Let $0 < \delta \leq \epsilon$ and set $z = v^\epsilon - v^\delta$, then we have $\partial_t z = Op^{\mathrm{BW}}(\mathfrak{S})z - \epsilon\partial_x^4 z + \partial_x^4 v^\epsilon(\delta - \epsilon)$. By Lemma 3 we have that the flow Φ_ϵ of $\partial_t z_1 = Op^{\mathrm{BW}}(\mathfrak{S})z_1 - \epsilon\partial_x^4 z_1$ exists and by Proposition 2, it has estimates independent of ϵ. By Duhamel formulation we have

$$z(t, x) = \Phi_\epsilon(t)(v_0^\epsilon(x) - v_0^\delta(x)) + (\delta - \epsilon)\Phi_\epsilon(t)\int_0^t \Phi_\epsilon(s)^{-1}\partial_x^4 v^\epsilon(s, x)ds.$$

By the estimate (39) (on the flow Φ_ϵ) and the Minkowski inequality we get $\|z(t, x)\|_{\dot{H}^\sigma} \leq (\epsilon - \delta)C\|\partial_x^4 v^\epsilon\|_{\dot{H}^\sigma}$, for a constant depending on Θ and r in (31). Applying again (39) on the function v^ϵ we get $\|z(t, x)\|_{\dot{H}^\sigma} \leq C(\epsilon - \delta)\|v_0^\epsilon\|_{\dot{H}^{\sigma+4}}$ for another constant C depending on r and Θ. At this point we may use that $\|\chi(|D|\epsilon^{\frac{1}{8}})v_0\|_{\dot{H}^{\sigma+4}} \leq \epsilon^{-\frac{1}{2}}\|v_0\|_{\dot{H}^\sigma}$. Since $0 < \delta < \epsilon$ we have that $\|z(t, x)\|_{\dot{H}^\sigma} \leq \tilde{C}\epsilon^{\frac{1}{2}}\|v_0\|_{\dot{H}^\sigma}$, hence $z(t, x)$ is a Cauchy sequence in $\dot{H}^\sigma$ and converges to a solution of (37) with $\epsilon = 0$ and initial condition $v_0 \in \dot{H}^\sigma$.

The flow $\Phi(t)$ of Eq. (32) with $R(t) = 0$ is well defined as a bounded operator form $\dot{H}^\sigma$ to $\dot{H}^\sigma$ and satisfies the estimate

$$\|\Phi(t)v_0\|_{\dot{H}^\sigma} \leq C_r e^{C_\Theta t}\|v_0\|_{\dot{H}^\sigma}.$$

One concludes by using the Duhamel formulation of (32). □

5 Nonlinear Local Well-Posedness

To build the solutions of the nonlinear problem (32), we shall consider a classical quasi-linear iterative scheme, we follow the approach in [1, 7, 8, 15]. Set

$$A(u) := Op^{\mathrm{BW}}\left(\partial^2_{z_1 z_1} F(\mathrm{i}\xi)^3 + \tfrac{1}{2}\tfrac{d}{dx}(\partial^2_{z_1 z_1} F)(\mathrm{i}\xi)^2 + \tilde{a}_1(x, u, u_x, u_{xx}, u_{xxx})(\mathrm{i}\xi)\right)$$

and define

$$\mathcal{P}_1: \quad \partial_t u_1 = A(u_0)u_1;$$
$$\mathcal{P}_n: \quad \partial_t u_n = A(u_{n-1})u_n + R(u_{n-1}), \quad n \geq 2.$$

The proof of the main Theorem 1 is a consequence of the next lemma. Owing to such a lemma one can follow closely the proof of Lemma 4.8 and Proposition 4.1 in [1] or the proof of Theorem 1.2 in [8](this is the classical Bona-Smith technique [2], but we followed the notation of [1, 8]). We do not reproduce here such a proof.

Lemma 6 *Let $s > \frac{1}{2} + 4$. Set $r := \|u_0\|_{\dot{H}^{s_0}}$ and $s_0 > 1 + 1/2$. There exists a time $T := T(\|u_0\|_{\dot{H}^{s_0+3}})$ such that for any $n \in \mathbb{N}$ the following statements are true.*

$(\mathcal{S}0)_n$: *There exists a unique solution u_n of the problem $\mathcal{P}_n$ belonging to the space $C^0([0,T); \dot{H}^s) \cap C^1([0,T); \dot{H}^{s-3})$.*

$(\mathcal{S}1)_n$: *There exists a constant $\mathcal{C}_r \geq 1$ such that if $\Theta = 4\ \mathcal{C}_r\|u_0\|_{\dot{H}^{s_0+3}}$ and $M = 4\mathcal{C}_r\|u_0\|_{\dot{H}^s}$, for any $1 \leq m \leq n$, for any $1 \leq m \leq n$ we have*

$$\|u_m\|_{L^\infty \dot{H}^{s_0}} \leq \mathcal{C}_r, \tag{55}$$

$$\|u_m\|_{L^\infty \dot{H}^{s_0+3}} \leq \Theta, \quad \|\partial_t u_m\|_{L^\infty \dot{H}^{s_0}} \leq \mathcal{C}_r \Theta, \tag{56}$$

$$\|u_m\|_{L^\infty \dot{H}^s} \leq M, \quad \|\partial_t u_m\|_{L^\infty \dot{H}^s} \leq \mathcal{C}_r M. \tag{57}$$

$(\mathcal{S}2)_n$: *For any $1 \leq m \leq n$ we have*

$$\|u_1\|_{L^\infty \dot{H}^{s_0}} \leq \mathcal{C}_r, \quad \|u_m - u_{m-1}\|_{L^\infty \dot{H}^{s_0}} \leq 2^{-m}\mathcal{C}_r, \quad m \geq 2. \tag{58}$$

Proof We proceed by induction over n. We prove $(S0)_1$, by using Proposition 1 with $R(t) = 0$, $u \rightsquigarrow u_0$ and $v \rightsquigarrow u_1$; we obtain a solution u_1 which is defined on every interval $[0, T)$ and verifies the estimate $\|u_1\|_{L^\infty_T \dot{H}^\sigma} \leq e^{T\|u_0\|_{\dot{H}^\sigma}} C_r \|u_0\|_{\dot{H}^\sigma}$, $\sigma \geq 0$ with $C_r > 0$ given by Proposition 1. $(S1)_1$ is a consequence of the previous estimate applied with $\sigma = s_0$ for (55) and (56), with $\sigma = s$ for (57). In order to obtain the seconds in (56) and (57), one has to fix $T \leq 1/\|u_0\|_{\dot{H}^{s_0}}$ and use the equation for u_1 together with Theorem 2 and one finds M which depends on $\|u_0\|_{\dot{H}^s}$ and Θ which depends on $\|u_0\|_{\dot{H}^{s_0}}$ and on a constant C_r depending only on $\|u_0\|_{\dot{H}^{s_0}}$. $(S2)_1$ is trivial.

We assume that $(SJ)_{n-1}$ holds true for any $J = 0, 1, 2$ and we prove that $(SJ)_n$.

Owing to $(S0)_{n-1}$ and $(S1)_{n-1}$, the $(S0)_n$ is a direct consequence of Proposition 1. Let us prove (55) with $m = n$. By using (33) applied to the problem solved by u_n, the estimate (22) with $\sigma = s_0$, (55) with $m = n-1$ and $(S0)_{n-1}$, we obtain $\|u_n\|_{L^\infty \dot{H}^{s_0}} \le e^{C_\Theta T}(C_r\|u_0\|_{\dot{H}^{s_0}} + \mathtt{C}_r C_\Theta T)$, the thesis follows by choosing $e^{C_\Theta T} C_\Theta T < 1/4$ and $\mathtt{C}_r \ge \|u_0\|_{\dot{H}^{s_0}}/4C_r$.

We prove the first in (56). Applying (33) with $\sigma = s_0 + 3$ and $v \rightsquigarrow u_n$, $u \rightsquigarrow u_{n-1}$, the estimate on the remainder (22) and using $(S1)_{n-1}$ we obtain $\|u_n\|_{\dot{H}^{s_0+3}} \le e^{C_\Theta T}\mathtt{C}_r\|u_0\|_{\dot{H}^{s_0+3}} + \Theta C_\Theta T e^{C_\Theta T}$, fixing T small enough such that $TC_\Theta \le 1$ and $TC_\Theta e^{C_\Theta T} \le 1/4$, the thesis follows from the definition $\Theta := 4\mathtt{C}_r\|u_0\|_{\dot{H}^{s_0}}$. The second in (56) may be proven by using the equation for u_n and the second in (22)

$$\|\partial_t u_n\|_{\dot{H}^{s_0}} \le \|A(u_{n-1})u_n\|_{\dot{H}^{s_0}} + \|R(u_{n-1})\|_{\dot{H}^{s_0}} \le C(\|u_{n-1}\|_{\dot{H}^{s_0}})\|u_n\|_{\dot{H}^{s_0+3}} \le \mathtt{C}_r\Theta.$$

The (57) is similar. We prove $(S2)_n$, we write the equation solved by $v_n = u_n - u_{n-1}$

$$\partial_t v_n = A(u_{n-1})v_n + f_n, \qquad f_n = \big[A(u_{n-1}) - A(u_{n-2})\big]u_{n-1} + R(u_{n-1}) - R(u_{n-2}).$$

By using (23), (25) and the $(S2)_{n-1}$ we may prove that $\|f_n\|_{\dot{H}^{s_0}} \le C_\Theta\|v_{n-1}\|_{\dot{H}^{s_0}}$. We apply again Proposition 1 with $\sigma = s_0$ and we find $\|v_n\|_{\dot{H}^{s_0}} \le C_\Theta T e^{C_\Theta T}\|v_{n-1}\|_{\dot{H}^{s_0}}$, as T has been chosen small enough we conclude the proof. □

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