Dispersive Estimates for the Dirac–Coulomb Equation

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Abstract We review some recent results on the dispersive estimates for the massless Dirac–Coulomb equation in $3D$.

1 Introduction

The Cauchy problem for the $3D$ massless Dirac–Coulomb equation can be written as follows

$$\begin{cases} i\partial_t u = \mathcal{D}_\nu u, \quad u(t,x) : \mathbb{R}_t \times \mathbb{R}_x^3 \to \mathbb{C}^4 \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where

$$\mathcal{D}_\nu = \mathcal{D} - \frac{\nu}{|x|} I_4 \,.$$

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Here, I_4 is the 4-dimensional identity matrix and $\mathcal{D}$, the (massless) Dirac operator, can be defined as

$$\mathcal{D} = -i\sum_{k=1}^{3}\alpha_k\partial_k = -i(\alpha\cdot\nabla),$$

where the 4×4 Dirac matrices are given by

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3 \tag{2}$$

and σ_j are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

This system can be thought of as a model describing the dynamics of an electron subject to the electric field generated by a charge ν located in the origin. The range of charges ν that make the operator $\mathcal{D}_\nu$ self-adjoint is well understood: $\mathcal{D}_\nu$ is essentially self-adjoint in the range $|\nu| \le \frac{\sqrt{3}}{2}$ and admits a distinguished self-adjoint extension in the range $\frac{\sqrt{3}}{2} < |\nu| \le 1$ (see [19] and the references therein). From a spectral theory point of view, we recall that the continuous spectrum of the operator $\mathcal{D}_\nu$ is the whole real line (as for the case $\nu = 0$); the generalized eigenfunctions are well known and will in fact play a crucial role in our analysis, as we will see. Since the Dirac operator is of first order, the Coulomb potential is a "large" perturbation and, as a consequence, one cannot directly deduce the properties of (1) from the free case $\nu = 0$ using perturbative arguments.

From a dynamical point of view, the Dirac equation falls within the chapter of *dispersive equations* and it is strictly related to the wave equation (and to the Klein-Gordon one in the massive case) due to the fact that the Dirac matrices satisfy the anticommutation relations

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}, \quad j, k = 1, 2, 3,$$

so that by applying the operator $i\partial_t + \mathcal{D}$ to a solution u of the free Dirac equation $i\partial_t u - \mathcal{D}u = 0$ yields

$$\partial_{tt}u - \Delta u = (i\partial_t + \mathcal{D})(i\partial_t - \mathcal{D})u = 0\ .$$

As a consequence, u also satisfies a system of decoupled wave equations. Therefore, most of the results that hold for the free wave flow can be harmlessly translated to the (free) Dirac case by simply applying the identity above. Here, we mean to focus on *dispersive estimates* and, in particular on *Strichartz estimates*: these estimates are a remarkably useful tool in several different contexts (study of local/global well posedness for nonlinear models, scattering,...). Strichartz estimates for the solutions to the $3D$ massless Dirac equation are well known and can be written as follows

$$\|e^{-it\mathcal{D}}u_0\|_{L_t^p L_x^q} \leq \|u_0\|_{\dot{H}^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}} \tag{4}$$

with the notation $L_t^p L_x^q = L^p(\mathbb{R}_t, L^q(\mathbb{R}_x^3))^4$ for the Strichartz spaces, the exponents (p, q) being *wave admissible*, i.e.,

$$\frac{2}{p} + \frac{2}{q} = 1, \qquad 2 \leq p \leq \infty, \quad 2 \leq q < \infty. \tag{5}$$

Following the paper [29], in the last 20 years a lot of effort has been devoted to investigating the validity of Strichartz estimates for dispersive equations perturbed by various potentials, and many strategies and techniques have been developed and sharpened. It is now well understood that the degree of homogeneity of the differential operator works somehow as a "threshold" for the validity of Strichartz estimates, meaning that for *subcritical* potentials with a faster decay than the critical one, Strichartz estimates can be recovered with more or less standard perturbative arguments, while for *supercritical* potentials with slower decay than the critical one, some non-dispersive solutions can be explicitly built in some cases. Concerning the Dirac equation, we refer to [4, 15–18] for dispersive estimates for subcritical potentials, and [1] for some counterexamples in the supercritical case. Potentials that exhibit the same homogeneity as the free operator correspond thus the *critical* case and typically turn out to represent a delicate and nontrivial problem, as indeed perturbative arguments are ruled out, and a much deeper understanding of the structure of the operator is often needed. Let us try to review some literature on the topic: in [5], [6] the authors proved Strichartz estimates (via Kato-smoothing) for the Schrödinger and wave equations perturbed by an inverse square potential, and more generally zero-order perturbations with critical decay (see also [28]). In [20] (and in subsequent [21, 22]) the authors proved the stronger time-decay estimates for the Schrödinger equation perturbed by critical electromagnetic potentials, exploiting a pseudoconformal transform that allows for an explicit representation of the propagator kernel. Time-decay estimates for the wave equation with critical magnetic potentials in 2d were later obtained in [23] and later on for various flows in [24]. Some results are available for the Dirac equation in Aharonov–Bohm potential that can be somehow thought of as the "magnetic equivalent" to (1); we postpone to Sect. 2.3 a brief overview of the topic. The massless Dirac–Coulomb equation (1) falls within this chapter, as it is indeed invariant under the natural scaling

$$u_\lambda(t, x) = u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right), \quad \lambda > 0$$

(it is thus *scaling critical*). The aim of this note is to present the dispersive estimates available for (1): in particular we will see how to prove a family of local smoothing estimates, and Strichartz estimates with loss of angular derivatives. We stress the fact that most of our results, with suitable differences, can be stated and proved in $2D$ as well. We limit ourselves to present here the $3D$ case that contains the main difficulties.

1.1 The Setup: Partial Wave Decomposition, Spectral Theory, and the Hankel Transform Method

In [5] the authors proved Strichartz estimates for solutions to the Schrödinger and wave equations perturbed with inverse square potentials. The strategy developed in that paper can be roughly summarized in the following steps:

1. Use *spherical harmonics decomposition* to reduce the equation to a radial problem;
2. Use *Hankel transform* to "diagonalize" the reduced problem and to define fractional powers of the operator $-\Delta + \frac{a}{|x|^2}$;
3. Prove a *local smoothing estimate* on a fixed spherical space using Hankel transform properties and the explicit integral representation of the fractional powers;
4. Sum back: use triangle inequality and L^2-orthogonality of spherical harmonics to obtain the desired estimate for the original dynamics;
5. Deduce Strichartz estimates.

In later years, this strategy proved to be quite flexible and was indeed exploited in several other papers in various contexts (see, e.g., [6, 8, 9]). The application of this strategy to system (1) comes with some substantial complications that are the following:

- The Dirac operator does not commute with the representation $\{\psi \to \psi(R^{-1}\cdot), R \in SO_3\}$ of the rotation group SO_3. Instead, it commutes with a spin $\frac{1}{2}$ representation of SU_2. This fact prevents from using the standard spherical harmonics decomposition and forces to rely on the so-called partial wave decomposition (see [32] Sec. 4.6.5), that we briefly review. First of all, we use spherical coordinates to write

$$L^2(\mathbb{R}^3, \mathbb{C}^4) \cong L^2((0,\infty), r^2 dr) \otimes L^2(S^2, \mathbb{C}^4)$$

with S^2 being the unit sphere. Then, we have the orthogonal decomposition on S^2:

$$L^2(S^2, \mathbb{C}^4) \cong \bigoplus_{k\in\mathbb{Z}^*} \bigoplus_{m\in\mathcal{I}_k} \mathfrak{h}_{k,m} \,.$$

Here, $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$, $\mathcal{I}_k := \{-|k| + 1/2, -|k| + 3/2, \cdots, |k| - 1/2\} \subset \mathbb{Z} + 1/2$ and each subspace $\mathfrak{h}_{k,m}$ is two-dimensional, with orthonormal basis

$$\Xi^+_{k,m} = \begin{pmatrix} i\,\Omega_{k,m} \\ 0 \end{pmatrix}, \qquad \Xi^-_{k,m} = \begin{pmatrix} 0 \\ \Omega_{-k,m} \end{pmatrix}.$$

The functions $\Omega_{k,m}$ can be explicitly written in terms of standard spherical harmonics as

$$\Omega_{k,m} = \frac{1}{\sqrt{|2k+1|}} \begin{pmatrix} \sqrt{|k-m+1|}\, Y^{m-1/2}_{|k+1/2|-1/2} \\ \operatorname{sgn}(-k)\sqrt{|k+m+1|}\, Y^{m+1/2}_{|k+1/2|-1/2} \end{pmatrix}.$$

We thus have the unitary isomorphism

$$L^2(\mathbb{R}^3, \mathbb{C}^4) \cong \bigoplus_{\substack{k\in\mathbb{Z}^* \\ m\in\mathcal{I}_k}} L^2((0,\infty), r^2 dr) \otimes \mathfrak{h}_{k,m}$$

given by the decomposition

$$\Phi(x) = \sum_{k\in\mathbb{Z}^*} \sum_{m\in\mathcal{I}_k} f^+_{k,m}(r)\,\Xi^+_{k,m}(\theta,\phi) + f^-_{k,m}(r)\,\Xi^-_{k,m}(\theta,\phi) \tag{6}$$

which holds for any $\Phi \in L^2(\mathbb{R}^3, \mathbb{C}^4)$. The Dirac–Coulomb operator leaves invariant the partial wave subspaces $C^\infty_c(0,\infty) \otimes \mathfrak{h}_{k,m}$ and its action on each column vector of radial functions $f_{k,m} = (f^+_{k,m}, f^-_{k,m})^{\mathrm{T}}$ is given by the radial matrix

$$\mathcal{D}_{\nu,k} = \begin{pmatrix} -\frac{\nu}{r} & -\frac{d}{dr} + \frac{1+k}{r} \\ \frac{d}{dr} - \frac{1-k}{r} & -\frac{\nu}{r} \end{pmatrix}. \tag{7}$$

This isomorphism allows for the following decomposition of the dynamics of the Dirac flow: for any $k \in \mathbb{Z}^*$ the choice of an initial condition as

$$u_{0,k,m}(x) = f^+_{0,k,m}(r)\,\Xi^+_{0,k,m}(\theta,\phi) + f^-_{0,k,m}(r)\,\Xi^-_{0,k,m}(\theta,\phi)$$

implies, by Stone Theorem, that the propagator is given by

$$e^{-it\mathcal{D}_\nu} u_{0,k,m} = f^+_{k,m}(r,t)\,\Xi^+_{k,m}(\theta,\phi) + f^-_{k,m}(r,t)\,\Xi^-_{k,m}(\theta,\phi),$$

where

$$\begin{pmatrix} f^+_{k,m}(r,t) \\ f^-_{k,m}(r,t) \end{pmatrix} = e^{-it\mathcal{D}_{\nu,k}} \begin{pmatrix} f^+_{0,k,m}(r) \\ f^-_{0,k,m}(r) \end{pmatrix}.$$

In what follows, we will in fact use the shortened notation

$$f \cdot \Xi_{k,m} = f^+(r)\Xi^+_{k,m}(\theta,\phi) + f^-(r)\Xi^-_{k,m}(\theta,\phi)\,, \quad f(r) = (f^+(r), f^-(r))^{\mathrm{T}}. \tag{8}$$

- One cannot use the standard Hankel transform: the generalized eigenstates are not Bessel functions, moreover positive and negative energy eigenstates are present and should be dealt with simultaneously. We thus define, for a fixed $k \in \mathbb{Z}^*$, a "relativistic Hankel transform" of the form

$$\mathcal{P}_k f(E) = \int_0^{+\infty} H_k(Er) f(r) r^2 dr \tag{9}$$

where $E \in (0,\infty)$ and, for any $\rho > 0$, $H_k(\rho) = \begin{pmatrix} F_k(\rho) & G_k(\rho) \\ F_k(-\rho) & G_k(-\rho) \end{pmatrix}$.

The functions

$$\psi_k(\pm Er) = \begin{pmatrix} F_k(\pm Er) \\ G_k(\pm Er) \end{pmatrix} \tag{10}$$

are the generalized eigenstates of the self-adjoint operator $\mathcal{D}_{\nu,k}$ with energies $\pm E$, so that

$$\mathcal{P}_k \mathcal{D}_{\nu,k} = \mathrm{Diag}(E, -E)\mathcal{P}_k\,. \tag{11}$$

In other words, the transform $\mathcal{P}_k$ "diagonalizes" the operator $\mathcal{D}_{\nu,k}$.

Remark 1 The operator $\mathcal{D}_{\nu,k}$, its generalized eigenstates $\psi_k(\pm Er)$, and the transform $\mathcal{P}_k$ are independent of m.

This construction suggests that the functions $\psi_k = \begin{pmatrix} F_k \\ G_k \end{pmatrix}$ play a crucial role, and most of the technical issues in our dispersive estimates will consist in proving suitable estimates for them (or, more precisely, for integrals of products of these functions, as for formula (15)). We therefore recall their precise definition, as given in, e.g., [27], formulas (36.1)-(36-20): for fixed values of $k \in \mathbb{Z}^*$ and $\rho \in \mathbb{R}^*$, with

$$F_k(\rho) = \frac{\sqrt{2}|\Gamma(\gamma+1+i\nu)|}{\Gamma(2\gamma+1)} e^{\pi\nu/2}|2\rho|^{\gamma-1} \tag{12}$$

$$\times \mathrm{Im}\left\{e^{i(\rho+\xi)}{}_1F_1(\gamma - i\nu, 2\gamma+1, -2i\rho)\right\}$$

and

$$G_k(\rho) = \frac{\sqrt{2}|\Gamma(\gamma + 1 + i\nu)|}{\Gamma(2\gamma + 1)} e^{\pi\nu/2} |2\rho|^{\gamma - 1} \tag{13}$$

$$\times \mathrm{Re}\left\{ e^{i(\rho+\xi)} {}_1F_1(\gamma - i\nu, 2\gamma + 1, -2i\rho) \right\},$$

where ${}_1F_1(a, b, z)$ are *confluent hypergeometric functions*, $\gamma = \sqrt{k^2 - \nu^2}$ and $e^{-2i\xi} = \frac{\gamma - i\nu}{k}$ is a phase shift.

One of the key tools of our strategy is represented by the following result, that has been proved in [10]:

Proposition 1 *For any $k \in \mathbb{Z}^*$ the following properties hold:*

1. *$\mathcal{P}_k$ is an L^2-isometry.*
2. *$\mathcal{P}_k \mathcal{D}_{\nu,k} = \sigma_3 \Omega \mathcal{P}_k$, where $\Omega f(x) := |x| f(x)$.*
3. *The inverse transform of $\mathcal{P}_k$ is given by*

$$\mathcal{P}_k^{-1} f(r) = \int_0^{+\infty} H_k^*(Er) f(E) E^{n-1} dE \tag{14}$$

where $H_k^* = \begin{pmatrix} F_k(Er) & F_k(-Er) \\ G_k(Er) & G_k(-Er) \end{pmatrix}$ (notice the misprint in formula (2.18) in [10]).

4. *For every $\sigma \in \mathbb{R}$ we can define the fractional operators*

$$A_k^\sigma f(r) = \mathcal{P}_k \sigma_3 \Omega^\sigma \mathcal{P}_k^{-1} f(r) = \int_0^{+\infty} S_k^\sigma(r, s) \cdot f(s) s^2 ds, \tag{15}$$

where the integral kernel $S_k^\sigma(r, s)$ is the 2×2 matrix given by

$$S_k^\sigma(r, s) = \int_0^{+\infty} H_k(Er) \cdot H_k^*(Es) E^{2+\sigma} dE. \tag{16}$$

Remark 2 When summing on k, property (15) allows to define in a standard way the fractional powers of the operator $|\mathcal{D}_\nu|$, which will be used in forthcoming Theorem 1.

As a consequence of this Proposition, given a function $u_0 = \sum_{\substack{k \in \mathbb{Z}^* \\ m \in \mathcal{I}_k}} f_{0,k,m} \cdot \Xi_{k,m}$ we can decompose the solution to Eq. (1) as follows:

$$e^{-it\mathcal{D}_\nu} u_0 = \sum_{\substack{k \in \mathbb{Z}^* \\ m \in \mathcal{I}_k}} (e^{-it\mathcal{D}_{\nu,k}} f_{0,k,m}) \cdot \Xi_{k,m} = \sum_{\substack{k \in \mathbb{Z}^* \\ m \in \mathcal{I}_k}} \mathcal{P}_k^{-1} \left[e^{-itE\sigma_3} \big(\mathcal{P}_k f_{0,k,m}\big)(E) \right] \cdot \Xi_{k,m}. \tag{17}$$

This decomposition represents the essential starting point of our analysis.

2 Dispersive Estimates

2.1 Local Smoothing

The main result of [10] is the following local smoothing (or Morawetz-type) estimate:

Theorem 1 ([10]) *Let K be a positive integer, and set*

$$\mathfrak{h}_{\geq K} = \bigoplus_{|k|\geq K} \bigoplus_{m\in\mathcal{I}_k} \mathfrak{h}_{k,m}.$$

Let u be a solution to (1). Then for any

$$1/2 < \varepsilon < \sqrt{K^2 - \nu^2} + 1/2$$

and any $f \in L^2((0,\infty), r^2 dr) \otimes \mathfrak{h}_{\geq K}$ there exists a constant $C = C(\nu, \varepsilon, K)$ such that the following estimate holds

$$\| |x|^{-\varepsilon} |\mathcal{D}_\nu|^{1/2-\varepsilon} u \|_{L^2_t L^2_x} \leq C \|u_0\|_{L^2_x}. \tag{18}$$

Remark 3 Notice that the range of ε gets wider if we require the initial condition to be orthogonal to some of the first partial wave subspaces: this also happens for the Schrödinger and wave equations with inverse square potentials (see [5]).

Remark 4 In order to deduce Strichartz estimates in a "standard" way (by the use of Duhamel formula and the application of the local smoothing estimate above twice), it would be necessary to prove (18) for $\varepsilon = 1/2$: this estimate, even if we do not have a concrete counterexample, is most likely false. The requirement of additional regularity on the initial condition seems not to help either. Therefore, at this stage, it does not seem to be possible to obtain (any kind of) Strichartz estimates with this strategy.

We mention the fact that the proof of this result turns out to be quite delicate, as it forces to provide uniform-in-k estimates for integrals in the form of (16), that involves products of confluent hypergeometric functions.

2.2 Strichartz Estimates with Loss of Angular Derivatives

Subsequently, we tried to tackle the problem of proving Strichartz estimates with loss of angular derivatives *without* using local smoothing, working directly on decomposition (17). The steps of the strategy that was inspired by [28] are, roughly speaking, the following:

1. Use partial wave decomposition and relativistic Hankel transform to decompose the flow as in (17);
2. Prove Strichartz estimates for fixed k and with unit frequency, that is assuming that $\operatorname{supp}\mathcal{P}_k(f_k)(\rho) \subset [1, 2]$;
3. Deduce Strichartz estimates for the complete dynamics using scaling argument and a dyadic decomposition.

The crucial technical step is (2), and some explicit estimates on the generalized eigenfunctions ψ_k are needed. In [28] indeed, the following bound on standard Bessel functions for $\lambda \gg 1$ plays an essential role:

$$|J_\lambda(\rho)| \leq C \times \begin{cases} e^{-D\lambda}, & 0 < \rho \leq \lambda/2, \\ \lambda^{-1/4}(|\rho - \lambda| + \lambda^{1/3})^{-1/4}, & \lambda/2 < \rho \leq 2\lambda, \\ \rho^{-1/2}, & 2\lambda < \rho \end{cases} \tag{19}$$

(notice that in our context λ has to be thought of as, roughly speaking, the "angular parameter"), and for some positive constants C and D independent on ρ and λ (for this estimate see, e.g., [2–31]). Therefore, it is necessary to provide an analog of estimate (19) for confluent hypergeometric functions. The main result obtained in [11] is indeed the following:

Theorem 2 *Let $\psi_k(\rho)$ be a generalized eigenfunction of $\mathcal{D}_\nu$ as given in (10), with $|\nu| < 1$ and $\gamma := \sqrt{k^2 - \nu^2} \gg 1$. Then there exists a constant C independent on γ such that the following estimates for $F_k(\rho)$ and $G_k(\rho)$ in (10) hold:*

$$|F_k(\rho)|, |G_k(\rho)| \leq C \begin{cases} e^{-C\gamma}, & 0 < \rho \leq \gamma/2, \\ \gamma^{-\frac{3}{4}}\left(|\gamma - \rho| + \gamma^{\frac{1}{3}}\right)^{-\frac{1}{4}}, & \frac{\gamma}{2} \leq \rho \leq 2\gamma, \\ \rho^{-1}, & \rho > 2\gamma. \end{cases} \tag{20}$$

Remark 5 We stress the fact that while the proof of (19) is based on the Van der Corput method in which the oscillations play a crucial role, the proof of (20) relies on the construction of a steepest descent path which allows to apply Laplace's method. We should also point out that the limits of F_k , G_k as $\nu \to 0$ can be expressed in terms of the Bessel function $J_{k-1/2}$. This is consistent with the similar form of estimates (19) and (20).

With Theorem 2 at our disposal, developing the strategy presented in the previous subsection, we are able to prove the following Strichartz estimates

Theorem 3 *Let* $|\nu| < \frac{\sqrt{15}}{4}$*. For any* $u_0 \in \dot{H}^s$*, the following Strichartz estimates hold*

$$\|e^{-it\mathcal{D}_\nu} u_0\|_{L^2_t L^q_r L^2_\omega} \leq C\|u_0\|_{\dot{H}^s} \tag{21}$$

provided

$$4 < q < \frac{3}{1-\sqrt{1-\nu^2}} m, \qquad s = 1 - \frac{3}{q}. \tag{22}$$

Remark 6 The upper bound $|\nu| < \frac{\sqrt{15}}{4}$ seems to have no physical meaning and it is a byproduct of our proof; notice anyway that as $\frac{\sqrt{15}}{4} > \frac{\sqrt{3}}{2}$, this range includes the set of charges that make the Dirac–Coulomb operator essentially self-adjoint.

Remark 7 We notice that this strategy could be developed in the 2-dimensional case as well; on the other hand, L^2_t-Strichartz estimates do not hold in 2d even for the free wave equation. Nevertheless, it might be possible to obtain some L^p_t, $p > 2$, estimates as done in [12], but this would require a fair amount of additional work, therefore we prefer to limit the estimates to the 3d case.

2.3 Open Problems and Related Models

As it is seen, the understanding of dispersive dynamics for Eq. (1) is far from being satisfactory, and many questions need to be answered. Also, there is a number of related problems and models that would certainly deserve further investigation: here we list a few of them.

- A first natural step would be trying to understand whether the estimates reviewed above hold in the massive case, that is for the operator $\mathcal{D}^m_\nu := \mathcal{D}_\nu + m\beta$ with $m > 0$: the restriction to $m = 0$ is quite structural, as indeed the massless equation exhibits a scaling that can be exploited, as opposed to the case $m > 0$ (e.g., Proposition 1 does not work properly any more when $m > 0$). Also, when $m > 0$ it is a well-known fact that the Dirac–Coulomb operator has eigenvalues in the gap $(-m, m)$, and eigenvalues represent an obstacle to dispersion; this problem can typically be bypassed by projecting the dynamics onto the absolute spectrum of the operator (see [26]). Anyway, it is not entirely clear how to deduce estimates for the massive case from the massless ones; a good starting point might be trying to adapt the results proved in [14], in which estimates for the Klein-Gordon flow are deduced from the corresponding ones for the wave flow by some kind of "shifting argument" for the estimates on the resolvent. This kind of strategy might work (with some additional care due to the fact that the presence of a mass "opens a gap" in the continuous spectrum of the operator) at least to extend the local smoothing estimate (18) to the massive case.

- The problem of proving Strichartz estimates without angular regularity for solutions to (1) remains open, and at the moment seems to be out of reach. A possible approach might be trying to prove time-decay estimates by providing a suitable representation for the integral kernel of the propagator, essentially writing it as an integral transform of the Green function (which is explicit, see [31]). Again, the complexity of the structure of the eigenfunctions will represent a technical obstruction.
- From a purely mathematical point of view, a model related to the Dirac–Coulomb equation is the Dirac equation perturbed with Aharonov–Bohm potential: the massless Dirac Hamiltonian in the Aharonov–Bohm magnetic field is

$$\mathcal{D}_A = \sigma_1(i\partial_1 + A^1) + \sigma_2(i\partial_2 + A^2), \tag{23}$$

where σ_j are the Pauli matrices and the magnetic potential $A_B(x) = (A^1(x), A^2(x))$ is given by

$$A_B : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2, \quad A_B(x) = \alpha\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right), \quad \alpha \in \mathbb{R}, \quad x = (x_1, x_2). \tag{24}$$

The Cauchy problem associated with the Hamiltonian (23) takes the form

$$\begin{cases} i\partial_t u = \mathcal{D}_A u, & u(t,x) : \mathbb{R}_t \times \mathbb{R}^2_x \to \mathbb{C}^2 \\ u(0,x) = u_0(x). \end{cases} \tag{25}$$

We refer to [8] and the references therein for further details on the model. As it is seen, equation $i\partial_t u = \mathcal{D}_A u$ is still scaling-invariant, and in this sense we can consider system (25) similar to (1). On the other hand, the study of dispersive estimates for system (25) turns out to be remarkably simpler, and this is due to the fact that the generalized eigenfunctions of the operator $\mathcal{D}_A$ only involve standard Bessel functions (see, e.g., [25]), which are much simpler to deal with, and for which several very precise estimates are available. Therefore, mainly relying on the crucial estimate (19), generalized Strichartz estimates with loss of angular derivatives were obtained in [12]. In this case, it seems simpler to recover the full set of Strichartz estimates (without any loss): this could be done by following the strategy developed in [24], in which the propagator for the Schrödinger and wave/Klein-Gordon equations with scaling critical magnetic perturbations is explicitly built using the corresponding eigenfunctions. This strategy seems to be adaptable to deal with the Dirac case, with additional care due to the much richer structure of the equation: this is a current work in progress.

- Lastly, we mention the fact that scaling critical perturbations appear in a somehow natural way when studying the dynamical Dirac equation on curved spaces: in [7]-[3], the authors have proved, respectively, local and global in time weighted Strichartz estimates for the Dirac dynamics in some spherically symmetric spaces. The main tool in those papers consists in exploiting the spherical structure of the manifolds and to introduce suitably chosen weighted spinors, in order to translate the free dynamics on the curved space into a dynamics on the Minkowski space with a (scaling critical) potential perturbation, and then to rely on existing theory for the latter. Therefore, a better understanding of dispersive estimates for the Dirac equation with a Coulomb (or, more in general, scaling critical) perturbation would also allow to improve the estimates on non-flat manifolds with spherical symmetry.

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