Schrödinger Flow's Dispersive Estimates in a regime of Re-scaled Potentials

Vladimir Georgiev, Alessandro Michelangeli, and Raffaele Scandone

Abstract The problem of monitoring the (constants in the estimates that quantify the) dispersive behaviour of the flow generated by a Schrödinger operator is posed in terms of the scaling parameter that expresses the small size of the support of the potential, along the scaling limit towards a Hamiltonian of point interaction. At positive size, dispersive estimates are completely classical, but their dependence on the short range of the potential is not explicit, and the understanding of such a dependence would be crucial in connecting the dispersive behaviour of the short-range Schrödinger operator with the zero-range Hamiltonian. The general set-up of the problem is discussed, together with preliminary answers, open questions, and plausible conjectures, in a 'propaganda' spirit for this subject.

1 Introduction and Background

In the context of the dispersive properties of the Schrödinger flow generated by the operator $-\Delta + V(x)$, self-adjointly realised on $L^2(\mathbb{R}^d)$ for a given measurable function $V : \mathbb{R}^d \to \mathbb{R}$, the explicit dependence on V of (the constants in) dispersive

V. Georgiev
Department of Mathematics, University of Pisa, Pisa, Italy

Faculty of Science and Engineering, Waseda University, Tokyo, Japan

Institute of Mathematics and Informatics at Bulgarian Academy of Sciences, Sofia, Bulgaria
e-mail: georgiev@dm.unipi.it

A. Michelangeli
Institute for Applied Mathematics and Hausdorff Centre for Mathematics, University of Bonn, Bonn, Germany

TQT Trieste Institute for Theoretical Quantum Technologies, Trieste, Italy
e-mail: michelangeli@iam.uni-bonn.de

R. Scandone (✉)
Gran Sasso Science Institute, L'Aquila, Italy
e-mail: raffaele.scandone@gssi.it

V. Georgiev et al. (eds.), *Qualitative Properties of Dispersive PDEs*,
Springer INdAM Series 52, https://doi.org/10.1007/978-981-19-6434-3_5

and Strichartz estimates is implicit or tacitly ignored, as V is given and does not represent a relevant parameter, as long as it belongs to a suitable class of potentials satisfying the required working assumptions. The other standard dependence on V in the dispersive estimate is the projection onto the absolutely continuous spectrum of the associated Schrödinger operator: it too is kept at this implicit level.

There are applications, however, where instead an explicit control of the dispersion in terms of V would provide crucial information.

The case that concerns us here is when V approximates in a suitable quantitative sense an actual point-like, 'impurity type' perturbation of $-\Delta$, the well-established construction where, heuristically speaking, one formally adds to $-\Delta$ a potential with delta-like profile supported at some $x_0 \in \mathbb{R}^d$ [3]. In this respect, the problem of comparing the dispersive phenomenon in the limiting case of point-like perturbation with the approximant case of a perturbation of finite size support acquires relevance per se and in application to the study of the solution theory of the associated (linear and) non-linear Schrödinger equations with point-like singular perturbation [1, 9, 16, 18, 27].

In order to place our analysis into context, let us pick for concreteness the *three-dimensional* case and, for $\varepsilon > 0$, let us consider the Schrödinger operator

$$H_\varepsilon \;=\; -\Delta + V_\varepsilon(x), \tag{1}$$

where

$$V_\varepsilon(x) \;:=\; \frac{\eta(\varepsilon)}{\varepsilon^2}\, V\Big(\frac{x}{\varepsilon}\Big), \qquad x \in \mathbb{R}^3\,, \tag{2}$$

for given V, and where the following conditions (or more restrictive versions, as done later) are assumed:

(V1) $\eta : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ is continuous on $\overline{\mathbb{R}^+}$, smooth on $\mathbb{R}^+$, and satisfies $\eta(0) = \eta(1) = 1$ as well as $\sup_{\varepsilon>0} \eta(\varepsilon) < +\infty$;

(V2) V is real-valued, $V \in \mathcal{R}$ (the Rollnik class), $(1 + |\cdot|)V \in L^1(\mathbb{R}^3)$.

Assumption (**V1**) regulates the 'distortion' with respect to the scaling $\varepsilon^{-2}V(x/\varepsilon)$ that has the same behaviour as the scaling of the Laplacian under dilation. Moreover, $H_1 = -\Delta + V$.

Assumption (**V2**), among other consequences, guarantees the self-adjointness of H_ε in $L^2(\mathbb{R}^3)$ with quadratic form domain $H^1(\mathbb{R}^3)$: indeed, under such a condition, V_ε is infinitesimally form bounded with respect to $-\Delta$ [31, Theorems X.17 and X.19]. In fact, for the purposes of the present discussion, it is surely non-restrictive to consider $V \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$, and it is this special choice that we will implicitly have in mind.

The limit $\varepsilon \downarrow 0$ yields distinct constructions depending on whether the additional assumption here below is or is not matched.

(V3) Setting $v(x) := \sqrt{|V(x)|}$ and $u(x) := \sqrt{|V(x)|}\,\mathrm{sign}(V(x))$, the 'Birman-Schwinger' operator $u(-\Delta)^{-1}v$ on $L^2(\mathbb{R}^3)$, which is compact under assumption

(**V2**), admits the simple eigenvalue -1, that is, the equation

$$u(-\Delta)^{-1} v\,\phi \;=\; -\phi \tag{3}$$

has a unique (up to multiples) solution $\phi \in L^2(\mathbb{R}^3) \setminus \{\mathbf{0}\}$, which in fact can be chosen to be real-valued, and for convenience is normalised as

$$\int_{\mathbb{R}^3} \operatorname{sign}(V)|\phi|^2\,\mathrm{d}x \;=\; -1\,, \tag{4}$$

and *in addition* the function

$$\psi \;:=\; (-\Delta)^{-1} v\phi \tag{5}$$

satisfies

$$\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^3) \setminus L^2(\mathbb{R}^3)\,. \tag{6}$$

Assumption (**V3**) is a spectral condition of (simple) *zero-energy resonance* for the Schrödinger operator $-\Delta+V$. In fact, if a non-zero ϕ exists in $L^2(\mathbb{R}^3)$ satisfying (3), then [3, Lemma I.1.2.3] $\psi = (-\Delta)^{-1}v\phi \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$, $\nabla\psi \in L^2(\mathbb{R}^3)$, $(-\Delta + V)\psi = 0$ in the sense of distributions, and moreover

$$\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^3) \setminus L^2(\mathbb{R}^3) \qquad \Leftrightarrow \qquad \int_{\mathbb{R}^3} v\phi\,\mathrm{d}x \;=\; \int_{\mathbb{R}^3} V\psi\,\mathrm{d}x \;\neq\; 0\,. \tag{7}$$

In addition, (**V3**) is a condition of *lack of zero-energy eigenvalue* for $-\Delta+V$: for, if $(-\Delta + V)\psi = 0$ for some $\psi \in H^1(\mathbb{R}^3)$, then $\phi := u\psi \in L^2(\mathbb{R}^3) \setminus \{0\}$ (otherwise, $-\Delta\psi = -v\phi = 0$, which is impossible), and $u(-\Delta)^{-1}v\phi = u(-\Delta)^{-1}V\psi = -u\psi = -\phi$, but by assumption there is only one such ϕ (up to multiples) and the corresponding ψ does not belong to $L^2(\mathbb{R}^3)$. Observe also that the *lack* of eigenvalue -1 for $u(-\Delta)^{-1}v$ is generic; clearly, a suitable scalar dilation $V \mapsto aV$ restores it. (An additional discussion may be found, e.g., in [17].)

Based on the above-mentioned consequences of (**V3**), we may further assume:

(**V4**) For given $\alpha \in \mathbb{R} \cup \{\infty\}$, η and V satisfy

$$\alpha \;=\; -\frac{\eta'(0)}{\big|\int_{\mathbb{R}^3} V\psi\,\mathrm{d}x\,\big|^2}\,. \tag{8}$$

As anticipated, the above assumptions regulate the limit $\varepsilon \downarrow 0$. More precisely (see, e.g., [3, Theorem I.1.2.5]),

- if all (**V1**)–(**V4**) hold true, then $H_\varepsilon \xrightarrow{\varepsilon\downarrow 0} -\Delta_\alpha$,
- if, under (**V1**)–(**V2**), (3) has no non-trivial solution in $L^2(\mathbb{R}^3)$, then $H_\varepsilon \xrightarrow{\varepsilon\downarrow 0} -\Delta$

in the *norm resolvent sense* [30, Section VIII.7], where $-\Delta_\alpha$, for α given by (8), is the point-like perturbation of the (negative) Laplacian at the origin, namely the self-adjoint extension in $L^2(\mathbb{R}^3)$ of $-\Delta|_{C_c^\infty(\mathbb{R}^3\setminus\{0\})}$ with s-wave scattering length $-(4\pi\alpha)^{-1}$ and zero effective range.

The latter is by now a standard construction in various equivalent self-adjoint extension schemes (see, e.g., [3, Section I.1.1] and [28, Section 3]). Explicitly, for arbitrary $\lambda > 0$ (and $\lambda \neq (4\pi\alpha)^2$ if $\alpha < 0$),

$$\begin{aligned} \mathrm{dom}(-\Delta_\alpha) &= \left\{ u \in L^2(\mathbb{R}^3) \,\middle|\, \begin{array}{c} \exists\, \varphi_\lambda \in H^2(\mathbb{R}^3) \text{ such that} \\ u = \varphi_\lambda + \dfrac{\varphi_\lambda(0)}{4\pi\alpha + \sqrt{\lambda}}\, \dfrac{e^{-|x|\sqrt{\lambda}}}{|x|} \end{array} \right\}, \\ (-\Delta_\alpha + \lambda)u &= (-\Delta + \lambda)\varphi_\lambda\,. \end{aligned} \tag{9}$$

In particular, $\alpha = \infty$ selects $-\Delta$, with self-adjointness domain $H^2(\mathbb{R}^3)$. One also has the explicit resolvent difference

$$(-\Delta_\alpha + \lambda\mathbb{1})^{-1} - (-\Delta + \lambda\mathbb{1})^{-1} = (4\pi(4\pi\alpha+1))^{-1} \left| \frac{e^{-|x|\sqrt{\lambda}}}{|x|} \right\rangle\!\left\langle \frac{e^{-|x|\sqrt{\lambda}}}{|x|} \right| \tag{10}$$

(with the customary notation $|\psi\rangle\langle\psi|$ for the orthogonal projection in $L^2(\mathbb{R}^3)$ onto the linear span of ψ. Concerning the spectrum of $-\Delta_\alpha$,

$$\begin{aligned} \sigma_{\mathrm{ess}}(-\Delta_\alpha) &= \sigma_{\mathrm{ac}}(-\Delta_\alpha) = [0, +\infty)\,, \\ \sigma_{\mathrm{sc}}(-\Delta_\alpha) &= \emptyset\,, \\ \sigma_{\mathrm{p}}(-\Delta_\alpha) &= \begin{cases} \emptyset\,, & \text{if } \alpha \geqslant 0\,, \\ \{-(4\pi\alpha)^2\} & \text{if } \alpha < 0\,. \end{cases} \end{aligned} \tag{11}$$

The negative eigenvalue, when existing, is non-degenerate.

As a consequence of the above norm resolvent convergence (strong resolvent convergence would have sufficed), Trotter's theorem (see, e.g., [30, Theorem VIII.21] implies

$$\begin{aligned} &\| e^{-\mathrm{i}t(-\Delta+V_\varepsilon)} f - e^{\mathrm{i}t\Delta_\alpha} f \|_{L^2} \xrightarrow{\varepsilon\downarrow 0} 0 \qquad \text{(resonant case)}\,, \\ &\| e^{-\mathrm{i}t(-\Delta+V_\varepsilon)} f - e^{\mathrm{i}t\Delta} f \|_{L^2} \xrightarrow{\varepsilon\downarrow 0} 0 \qquad \text{(non-resonant case)}\,, \\ &\forall t \in \mathbb{R}\,,\ \forall f \in L^2(\mathbb{R}^3)\,, \end{aligned} \tag{12}$$

that is, *strong* convergence of the unitary groups. Observe that instead *norm operator* convergence cannot hold in general (as emerges, e.g., from the proof of [30, Theorem VIII.20]).

Thus, next to the classical and comprehensive knowledge of dispersive, smoothing, and Strichartz estimates for the Schrödinger unitary propagator $e^{-\mathrm{i}tH_\varepsilon}$ (we refer, among others, to the monographs[10, 26, 36, 37] and the multiple references therein), it is relevant in the present context to monitor the dispersive features of $e^{-\mathrm{i}tH_\varepsilon}$ *in terms of the scaling parameter* ε.

As mentioned, this has at least a two-fold motivation. For one thing, there is an abstract interest per se in comparing the dispersive estimates of $e^{-\mathrm{i}tH_\varepsilon}$ and of $e^{\mathrm{i}t\Delta_\alpha}$: notably, for the latter, the explicit knowledge [2, 34] of the integral kernel (see (31) below) actually allows for an explicit derivation of dispersive and Strichartz estimates [12, 13, 21] (see Remark 2 and (35)–(39)). Furthermore, there is a crucial relevance in applications to semi-linear Schrödinger equations induced by $-\Delta_\alpha$: for such equations, whose study, albeit at an early stage, has already produced important well-posedness results [9, 18, 19, 27], and in particular for their physical relevance as effective dynamical equations for large Bose gases with impurities, one natural and open problem is the approximation of the solution u by means of the solution u_ε of the corresponding semi-linear equation induced by H_ε, a question that would require Strichartz estimates for $e^{-\mathrm{i}tH_\varepsilon}$ quantitatively expressed in terms of ε, so as to monitor the $\varepsilon \downarrow 0$ limit.

The purpose of this note is to make propaganda for this and related problems, and to present a first answer in the prototypical three-dimensional set-up. The same issue naturally arises and deserves investigation in two dimensions. The one-dimensional case too is of relevance: that case is somewhat simpler and under more direct control, as in one dimension the singular point-perturbed $-\Delta_\alpha$ is an actual quadratic form sum of $-\Delta$ and (a multiple of) the Dirac δ distribution [3, Chapter I.3].

It is worth observing that in the context of dispersive estimates for Schrödinger operators one is well aware (see, e.g., [35, Section 12.1]) of the very important difference between the one-dimensional dispersive bounds, whose constants do exhibit an explicit dependence on the potential via the Jost solutions, as opposed to the higher dimensional bounds: this general lack of information results, in the present context, in the quest of the ε-dependence.

2 A Preliminary Overview of Relevant Spectral Properties

It is standard that, under the assumptions (**V1**)–(**V2**), H_ε has essential spectrum that is entirely absolutely continuous and amounts to

$$\sigma_{\mathrm{ess}}(H_\varepsilon) = \sigma_{\mathrm{ac}}(H_\varepsilon) = [0, +\infty) \qquad \forall \varepsilon > 0\,. \tag{13}$$

Concerning the (necessarily negative) discrete spectrum, an explicit and detailed discussion is possible, e.g., upon strengthening (**V2**) as:

(**V2′**) V is real-valued and $e^{a|\cdot|}V \in \mathcal{R}$ for some $a > 0$.

In fact, it is known that

- [3, Theorem I.1.3.1(a)] assuming (**V1**)–(**V2′**), any negative eigenvalue E_1 of $H_1 = -\Delta + V$ of multiplicity m gives rise to m (not necessarily distinct) eigenvalues $E_\varepsilon^{(\ell)}$ of H_ε , $\ell \in \{1, \dots, m\}$ running to $-\infty$ as $\varepsilon \downarrow 0$ as

$$E_\varepsilon^{(\ell)} = \varepsilon^{-2} E_1 + O(\varepsilon^{-1}) ; \tag{14}$$

- [3, Theorem I.1.3.1(b)], assuming (**V1**),(**V2′**),(**V3**),(**V4**), and when $\alpha < 0$, H_ε has, for any $\varepsilon > 0$ small enough, the non-degenerate negative eigenvalue $E_\varepsilon^{(\alpha)}$

$$E_\varepsilon^{(\alpha)} = -(4\pi\alpha)^2 + O(\varepsilon) . \tag{15}$$

Last, concerning the nature of the spectral point zero for H_ε, two scenarios are possible under the basic assumptions (**V1**)–(**V2**):

- if, eventually in ε as $\varepsilon \downarrow 0$, one has $\eta(\varepsilon) \equiv 1$, then H_ε and $\varepsilon^{-2} H_1$ are unitarily equivalent, as operators on $L^2(\mathbb{R}^3)$, via the $L^2 \to L^2$ dilation isomorphism U_ε, that is,

$$U_\varepsilon^* H_\varepsilon U_\varepsilon = \frac{1}{\varepsilon^2} H_1 , \qquad (U_\varepsilon f)(x) := \frac{1}{\varepsilon^{3/2}} f\Big(\frac{x}{\varepsilon}\Big) ; \tag{16}$$

  as a consequence, if the spectral point zero is an eigenvalue or a resonance for $-\Delta + V$, so too is it for H_ε;
- on the other hand, in general a re-scaling with $\eta(\varepsilon) \neq 1$ distortion washes out possible eigenvalues or resonance initially present at zero energy for $-\Delta + V$; therefore, if (eventually in ε) $\eta(\varepsilon) = 1 + \kappa\varepsilon$ for some $\kappa \neq 0$, which in fact covers the remaining generality of the present setup (only the quantity $\kappa = \eta'(0)$ enters (8) above), then eventually in ε zero-energy eigenvalues or resonance are absent for H_ε.

We shall refer to the occurrence where all of (**V1**)–(**V4**) hold true as the *resonant regime* (at the given parameter α), and to the occurrence where (**V1**)–(**V2**) are matched, and (3) has no solutions in $L^2(\mathbb{R}^3) \setminus \{0\}$, as the *non-resonant regime*. For what has been just observed, such a terminology refers to the spectral property of $H_1 = -\Delta + V$, and not to the spectrum of H_ε at zero energy. At each ε, H_ε may be well non-resonant even though H_1 is.

3 Dispersive Estimates with ε-Uniform Bound

The $L^q \to L^p$ mapping properties of $e^{-\mathrm{i}tH_\varepsilon}$ depend, as the vast and well-established literature on Schrödinger flow's dispersive estimates shows, on the presence or absence of zero-energy resonance or zero-energy eigenvalues for H_ε, provided that V_ε belongs to certain standard classes of controllable potentials.

In particular [14, 20, 22, 25, 29, 33, 41], $|t|^{-3/2}$ is the typical decay for the norm $\|e^{-\mathrm{i}tH_\varepsilon}P_\varepsilon^{(\mathrm{ac})}\|_{L^1\to L^\infty}$ in the absence of both resonance and eigenvalues at zero energy for H_ε, being in fact the exact decay for the corresponding norm relative to the free Schrödinger propagator $e^{\mathrm{i}t\Delta}$, whereas the *slower* $|t|^{-1/2}$ is typical for the same norm in the presence of resonance at zero. Here $P_\varepsilon^{(\mathrm{ac})}$ is the orthogonal projection onto the absolutely continuous spectral subspace of $L^2(\mathbb{R}^3)$ associated with H_ε (see, e.g., [4, Chapter 4]).

A priori the above norm depends also on ε—an information that, as commented in Sect. 1, would not be of concern if the scaling limit $\varepsilon \downarrow 0$ was not considered.

We show now that the $L^q \to L^p$ bound is actually *uniform* in ε in two meaningful classes of cases.

To this aim, it is convenient to require additional constraints on the size or on the decay of V, and precisely:

(**V_{small}**) V is real-valued and, together with η, it satisfies

$$\|V\|_{\mathcal{R}} := \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|V(x)|\,|V(y)|}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} < 4\pi\left(\sup_{\varepsilon>0}\eta(\varepsilon)\right)^{-1}, \tag{17}$$

$$\|V\|_{\mathcal{K}} := \sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V(y)|}{|x-y|}\,\mathrm{d}y < 4\pi\left(\sup_{\varepsilon>0}\eta(\varepsilon)\right)^{-1} \tag{18}$$

(i.e., respectively, smallness of the *Rollnik norm* and the generalised Kato norm);
($\mathbf{V_{decay}}$) V is real-valued and satisfies $|V(x)| \lesssim \langle x\rangle^{-(7+\delta)}$ for some $\delta > 0$.

Observe that ($\mathbf{V_{small}}$) automatically excludes zero-energy eigenvalues or resonance for $-\Delta + V$ (in particular, it excludes (**V3**)), and ($\mathbf{V_{decay}}$) implies (**V2**).

With the extra decay imposed by ($\mathbf{V_{decay}}$) we are surely far from optimality, but in the present context this is not of concern: recall that already the choice $V \in C_c^\infty(\mathbb{R}^3)$ would be completely meaningful and non-restrictive, as it gives rise to both mechanisms $H_\varepsilon \xrightarrow{\varepsilon\downarrow 0} -\Delta_\alpha$ and $H_\varepsilon \xrightarrow{\varepsilon\downarrow 0} -\Delta$ described in Sect. 1.

Theorem 1 *Assume (V1) and (*$\mathbf{V_{small}}$*). Then there exists a constant C, independent of* $\varepsilon > 0$*, such that*

$$\left\| e^{-\mathrm{i}tH_\varepsilon}P_\varepsilon^{(\mathrm{ac})}f\right\|_{L^p} \leqslant C|t|^{-3(\frac{1}{2}-\frac{1}{p})}\|f\|_{L^{p'}} \qquad \begin{array}{l}\forall p\in[2,+\infty]\,,\ p'=\frac{p}{p-1}\,,\\ \forall f\in L^{p'}(\mathbb{R}^3)\,,\\ \forall t\in\mathbb{R}\setminus\{0\}\,.\end{array} \tag{19}$$

Proof It is standard to see that the smallness condition ($\mathbf{V_{small}}$) prevents $-\Delta + V$ to have zero-energy eigenvalues or resonance. The same therefore holds for H_ε, eventually in ε, apart from possible exceptional, isolated values of ε.

In this regime, and at every *fixed* ε at which H_ε is not zero-resonant, the $L^{p'} \to L^p$ boundedness of $e^{-\mathrm{i}tH_\varepsilon}$, with bound $C_\varepsilon |t|^{-3(\frac{1}{2}-\frac{1}{p})}$, is a classical result (we refer to [33]) obtained under the condition $\| V_\varepsilon \|_{\mathcal{R}} < 4\pi$ by means of a Born series expansion for the resolvent with a subsequent estimate of an arising oscillatory integral: this results in a geometric series whose convergence is guaranteed by $\| V_\varepsilon \|_{\mathcal{K}} < 4\pi$.

In fact, owing to (**V1**) and ($\mathbf{V_{small}}$),

$$\begin{aligned} \| V_\varepsilon \|_{\mathcal{K}} \; &\leqslant \; \Big(\sup_{\varepsilon>0} \eta(\varepsilon) \Big) \| V \|_{\mathcal{K}} \; < \; 4\pi \,, \\ \| V_\varepsilon \|_{\mathcal{R}} \; &\leqslant \; \Big(\sup_{\varepsilon>0} \eta(\varepsilon) \Big) \| V \|_{\mathcal{R}} \; < \; 4\pi \,, \end{aligned} \tag{20}$$

thus matching the needed smallness conditions for V_ε.

Moreover, the constant C_ε in the $L^{p'} \to L^p$ bound depends on $\| V_\varepsilon \|_{\mathcal{K}}$ and $\| V_\varepsilon \|_{\mathcal{R}}$, and is therefore uniformly bounded in ε. Estimate (19) is thus established. □

Theorem 2 *Assume (**V1**) with $\eta \equiv 1$, ($\mathbf{V_{decay}}$), and (**V3**), (thereby implying (**V4**) with $\alpha = 0$). In other words, it is assumed that for every $\varepsilon > 0$ H_ε acts self-adjointly on $L^2(\mathbb{R}^3)$ as*

$$H_\varepsilon \; = \; -\Delta + \frac{1}{\varepsilon^2} V\Big(\frac{x}{\varepsilon}\Big) \tag{21}$$

with V satisfying ($\mathbf{V_{decay}}$), and it is assumed furthermore that the spectral value zero is a resonance, but not an eigenvalue for H_1—hence, on account of (16), zero is a resonance but not eigenvalue for H_ε for any $\varepsilon > 0$. Then there exists a constant C, independent of ε, such that

$$\big\| \, e^{-\mathrm{i}tH_\varepsilon} P_\varepsilon^{(\mathrm{ac})} f \, \big\|_{L^p} \; \leqslant \; C |t|^{-3(\frac{1}{2}-\frac{1}{p})} \| f \|_{L^{p'}} \qquad \begin{array}{l} \forall p \in [2,3) \,, \;\; p' = \frac{p}{p-1} \,, \\ \forall f \in L^{p'}(\mathbb{R}^3) \,, \\ \forall t \in \mathbb{R} \setminus \{0\} \,. \end{array} \tag{22}$$

Remark 1 As commented already, H_ε in (21) is zero-energy resonant, without zero-energy eigenvalues, for every $\varepsilon > 0$. For such a Schrödinger operator, the dispersive estimate (22), precisely in the regime $p \in [2,3)$, was established in [41, Theorem 1.3(2)] under the milder decay $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > \frac{11}{2}$, but with an *implicit* dependence of the constant on V_ε, that is, on ε. Theorem 2 adds to this classical picture the novel information that such a bound is *uniform* in ε. It is also worth remarking that [41, Theorem 1.3(2)] prescribes, in addition, that a counterpart to (22) is valid when $p = 3$ provided that the L^3- and $L^{\frac{3}{2}}$-norms are replaced, respectively, by norms of the Lorenz spaces $L^{3,\infty}(\mathbb{R}^3)$ and $L^{\frac{3}{2},1}(\mathbb{R}^3)$.

Remark 2 The dispersive estimate (22), with the uniformity of the bound in terms of ε, is compatible with its known counterpart for the limiting propagator $e^{\mathrm{i}t\Delta_{\alpha=0}}$—recall from Sect. 1 that under the assumptions of Theorem 2 one has $e^{-\mathrm{i}tH_\varepsilon} \xrightarrow{\varepsilon\downarrow 0} e^{\mathrm{i}t\Delta_{\alpha=0}}$ strongly in $L^2(\mathbb{R}^3)$ for every fixed $t \in \mathbb{R}$. Indeed, it was found in [13, 21] that

$$\| e^{\mathrm{i}t\Delta_\alpha} P^{(\mathrm{ac})}_{(\alpha)} f \|_{L^p} \leqslant C|t|^{-3(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^{p'}} \qquad \begin{array}{l} \forall p \in [2,3)\,,\;\; p' = \frac{p}{p-1}\,, \\ \forall f \in L^{p'}(\mathbb{R}^3)\,, \\ \forall t \in \mathbb{R}\setminus\{0\} \end{array} \tag{23}$$

for every $\alpha \in \mathbb{R}$, where now $P^{(\mathrm{ac})}_{(\alpha)}$ is the L^2-orthogonal projection onto the absolutely continuous spectrum $[0, +\infty)$ of $-\Delta_\alpha$.

Proof of Theorem 2 Let us consider on $L^2(\mathbb{R}^3)$ the wave operators

$$W^\pm_\varepsilon \equiv W^\pm(H_\varepsilon, -\Delta) := \lim_{t\to\pm\infty} e^{\mathrm{i}tH_\varepsilon} e^{\mathrm{i}t\Delta} \tag{24}$$

(as strong limits in $L^2(\mathbb{R}^3)$) associated with the pair of self-adjoint operators H_ε and $-\Delta$. Standard arguments from scattering theory (see, e.g., [32, Theorem XI.30]) guarantee that such wave operators exist in $L^2(\mathbb{R}^3)$ and are complete, meaning that

$$\operatorname{ran} W^\pm_\varepsilon = L^2_{\mathrm{ac}}(H_\varepsilon) := P^{(\mathrm{ac})}_\varepsilon L^2(\mathbb{R}^3)\,. \tag{25}$$

Owing to their completeness, W^+_ε and W^-_ε are unitaries from $L^2(\mathbb{R}^3)$ onto $L^2_{\mathrm{ac}}(H_\varepsilon)$ and they intertwine $H_\varepsilon P^{(\mathrm{ac})}_\varepsilon$ and $-\Delta$, in particular,

$$e^{-\mathrm{i}tH_\varepsilon} P^{(\mathrm{ac})}_\varepsilon = W^\pm_\varepsilon e^{\mathrm{i}t\Delta} (W^\pm_\varepsilon)^* \qquad \forall t \in \mathbb{R}\,. \tag{26}$$

In analogy to $W^\pm_\varepsilon$ let us also consider on $L^2(\mathbb{R}^3)$ the wave operators

$$W^\pm_{(\alpha)} \equiv W^\pm(-\Delta_\alpha, -\Delta) := \lim_{t\to\pm\infty} e^{-\mathrm{i}t\Delta_\alpha} e^{\mathrm{i}t\Delta} \tag{27}$$

(as strong limits in $L^2(\mathbb{R}^3)$) associated with $-\Delta_\alpha$ and $-\Delta$. Since the difference of the corresponding resolvents is a rank-one operator (see (10) above), $W^\pm_{(\alpha)}$ too exist and are complete, on account of the Kuroda-Birman theorem (see, e.g., [31, Theorem XI.9].

The intertwining relation (26) allows to deduce the $L^{p'} \to L^p$ boundedness of $e^{-\mathrm{i}tH_\varepsilon} P^{(\mathrm{ac})}_\varepsilon$ directly from the known $L^{p'} \to L^p$ boundedness of $e^{\mathrm{i}t\Delta}$, once one also knows that $W^\pm_\varepsilon$ is bounded on $L^p(\mathbb{R}^3)$: the latter information is classical, and there is in fact a vast literature on the L^p-boundedness of $W^\pm_\varepsilon$ for sufficiently regular V_ε vanishing at spatial infinity [5–8, 11, 15, 23, 24, 38–40, 42, 43]. This yields

$$\| e^{-\mathrm{i}tH_\varepsilon} P^{(\mathrm{ac})}_\varepsilon f \|_{L^p} \leqslant C \| W^+_\varepsilon \|^2_{L^p\to L^p} |t|^{-3(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^{p'}} \tag{28}$$

for any $t \in \mathbb{R} \setminus \{0\}$, any $p \in [2, +\infty]$, and any $f \in L^{p'}(\mathbb{R}^3)$.

On the other hand, it was recently proved in [13] that $W^{\pm}_{(\alpha)}$ are L^p-bounded only for $p \in (1, 3)$ [13, Theorem 1.1] and that

$$\forall u \in L^p(\mathbb{R}^3) \qquad \lim_{\varepsilon \downarrow 0} W^{\pm}_{\varepsilon} u \;=\; W^{\pm}_{(\alpha=0)} u \qquad \text{weakly in } L^p(\mathbb{R}^3)\,, \tag{29}$$

[13, Proposition 7.1]. (Strictly speaking for the latter result both ($\mathbf{V_{decay}}$) and the lack of zero-energy eigenvalue, as well as the special form (21) of H_ε, were all required in [13, Proposition 7.1].) The Banach-Steinhaus theorem then allows to deduce from (29) that

$$\| W^{\pm}_{\varepsilon} \|_{L^p \to L^p} \;\leqslant\; \kappa \;<\; +\infty \tag{30}$$

uniformly in ε. Plugging (30) into (28) finally yields (22). □

4 Outlook on Further Scaling Regimes

The preceding discussion shows that there are relevant scaling regimes that remain uncharted, as far as the ε-dependence of the norm $\| e^{-\mathrm{i}tH_\varepsilon} P^{(\mathrm{ac})}_{\varepsilon} \|_{L^{p'} \to L^p}$ is concerned:

(A) the special resonant case with H_ε given by (21), that is, under assumptions (**V2**) (or stronger spatial decay) and (**V3**), (zero-energy resonance and absence of zero-energy eigenvalue for $-\Delta + V$), and in the dispersive regime $p \in [3, +\infty]$;

(B) the general resonant regime with H_ε given by (1)–(2) under (**V1**)–(**V4**), in the dispersive regime $p \in [2, +3)$;

(C) the general resonant regime with H_ε given by (1)–(2) under (**V1**)–(**V4**), in the dispersive regime $p \in [3, +\infty]$.

Apart from the dependence on ε, the norm $\| e^{-\mathrm{i}tH_\varepsilon} P^{(\mathrm{ac})}_{\varepsilon} \|_{L^{p'} \to L^p}$ is already well controlled in time in all the above cases (A), (B), and (C).

Each one among (A), (B), (C) presents specific difficulties, which justifies listing them separately.

Case (B) is conceptually similar to Theorem 2: when $p \in [2, 3)$ the wave operators $W^{\pm}_{(\alpha)} \equiv W^{\pm}(-\Delta_\alpha, -\Delta)$ are still L^p-bounded, as established in [13, Theorem 1.1], which in turns implies the dispersive estimate (23) for $-\Delta_\alpha$, precisely for $p \in [2, 3)$. This, and the L^2-strong convergence $e^{-\mathrm{i}tH_\varepsilon} \xrightarrow{\varepsilon \downarrow 0} e^{\mathrm{i}t\Delta_\alpha}$ for each $t \in \mathbb{R}$ suggest that in case (B) the propagator $e^{-\mathrm{i}tH_\varepsilon}$ should satisfy the same $L^{p'} \to L^p$ bound as in (22). In order to mimic the scattering scheme of Theorem 2's proof, one would require a version of the key ingredient [13, Proposition 7.1], that is, the same L^p-weak convergence $W^{\pm}_{\varepsilon} \xrightarrow{\varepsilon \downarrow 0} W^{\pm}_{(\alpha)}$ of (29), so as to cover the generic scaling (1)–(2) for H_ε.

In the dispersive regime $p \in [3, +\infty]$ of cases (A) and (C), instead, no $L^{p'} \to L^p$ boundedness of $e^{\mathrm{i}t\Delta_\alpha}$ is possible: this is ultimately a consequence of the fact that the linear Schrödinger dynamics develops, at almost every instant $t > 0$, a $|x|^{-1}$-singularity in $(e^{\mathrm{i}t\Delta_\alpha} f)(x)$, clearly not locally L^p-integrable for $p \geqslant 3$. This can be argued from the explicit form [2, 34] of the integral kernel $K_\alpha(x, y; t)$ of the propagator $e^{\mathrm{i}t\Delta_\alpha}$:

$$K_\alpha(x,y;t) = \begin{cases} K(x,y;t) + \dfrac{1}{|x|\,|y|}\displaystyle\int_0^{+\infty} e^{-4\pi\alpha u}(u+|x|+|y|)\times \\ \qquad \times K(u+|x|+|y|,0;t)\,\mathrm{d}u\,, & \text{if } \alpha > 0\,, \\ K(x,y;t) + \dfrac{2\,\mathrm{i}\,t}{|x|\,|y|}\,K(|x|+|y|,0;t)\,, & \text{if } \alpha = 0\,, \\ K(x,y;t) + e^{\mathrm{i}t(4\pi\alpha)^2}\Psi_\alpha(x)\Psi_\alpha(y) \\ +\dfrac{1}{|x|\,|y|}\displaystyle\int_0^{+\infty} e^{-4\pi|\alpha|u}(u-|x|-|y|)\times & \text{if } \alpha < 0\,, \\ \qquad \times K(u-|x|-|y,0.t)\,\mathrm{d}u\,, \end{cases} \tag{31}$$

where

$$K(x,y;t) := \frac{e^{-\frac{|x-y|^2}{4\mathrm{i}t}}}{(4\pi\mathrm{i}t)^{\frac{3}{2}}}\,, \qquad t > 0\,, \tag{32}$$

and

$$\Psi_\alpha(x) := \sqrt{-2|\alpha|}\,\frac{e^{-4\pi|\alpha||x|}}{|x|}\,. \tag{33}$$

In fact, the $L^{p'} \to L^p$ *unboundedness* of $e^{\mathrm{i}t\Delta_\alpha}$ when $p \geqslant 3$, and the L^2-strong convergence $e^{-\mathrm{i}tH_\varepsilon} \xrightarrow{\varepsilon\downarrow 0} e^{\mathrm{i}t\Delta_\alpha}$, prevent the norm $\|\, e^{-\mathrm{i}tH_\varepsilon} P_\varepsilon^{(\mathrm{ac})} \|_{L^{p'}\to L^p}$ to be uniformly bounded in ε when $p \geqslant 3$ (cases (A) and (C) above). For, if at an instant t when the evolution $e^{\mathrm{i}t\Delta_\alpha} f$ of a generic $f \in \big(\bigcap_\varepsilon P_\varepsilon^{(\mathrm{ac})} L^2(\mathbb{R}^3)\big) \cap L^{p'}(\mathbb{R}^3)$ is $|x|^{-1}$-singular around the origin one had

$$\|\, e^{-\mathrm{i}tH_\varepsilon} f \|_{L^p} \leqslant C_\varepsilon(t)\|f\|_{L^{p'}} \tag{34}$$

with $C_\varepsilon(t) \leqslant C(t)$ for some ε-*independent* $C(t) \geqslant 0$ (eventually as $\varepsilon \downarrow 0$), then from the sequence $(f_n)_{n\in\mathbb{N}}$ defined by

$$f_n := e^{-\mathrm{i}tH_{\varepsilon_n}} f\,, \qquad \varepsilon_n := n^{-1}\,,$$

which would then be uniformly bounded in $L^p(\mathbb{R}^3)$, one would have $f_n \to f_*$ L^p-weakly as $n \to \infty$, up to extracting a subsequence, for some $f_* \in L^p(\mathbb{R}^3)$. Since, on the other hand, $f_n \xrightarrow{n\to\infty} e^{\mathrm{i}t\Delta_\alpha} f$ in $L^2(\mathbb{R}^3)$, one should necessarily conclude $e^{\mathrm{i}t\Delta_\alpha} f = f_* \in L^p(\mathbb{R}^3)$. This is, however, incompatible with the $|x|^{-1}$-singularity of $e^{\mathrm{i}t\Delta_\alpha} f$, since $p \geqslant 3$. Necessarily $C_\varepsilon(t)$ in (34) blows up in ε, that is, $\| e^{-\mathrm{i}tH_\varepsilon} P_\varepsilon^{(\mathrm{ac})} \|_{L^{p'} \to L^p}$ becomes singular in ε as $\varepsilon \downarrow 0$ and $p \geqslant 3$. Observe that this argument sheds no light on the blow-up rate of $C_\varepsilon(t)$ as $\varepsilon \downarrow 0$ or on the short-time and long-time behaviour of $C_\varepsilon(t)$: actually, such a behaviour depends, at every fixed ε, on the presence or absence or zero-energy resonance and eigenvalue(s) for H_ε.

The above reasoning naturally suggests that the dispersive regime $p \geqslant 3$ for $e^{-\mathrm{i}tH_\varepsilon}$ (cases (A) and (C) above) could be meaningfully monitored, as far as the ε dependence is concerned, in suitably *weighted* $L^{p'} \to L^p$ norms—so as to absorb, informally speaking, the 'emergent' $|x|^{-1}$-singularity.

Weighted $L^1 \to L^\infty$ dispersive estimates for $-\Delta_\alpha$ were originally established in [12, Theorem 1], directly from (31), in a form that, interpolated with the trivial L^2-bound, reads (see [21, Proposition 4])

$$\big\| w^{-(1-\frac{2}{p})}\, e^{\mathrm{i}t\Delta_\alpha} P_{(\alpha)}^{(\mathrm{ac})} f \big\|_{L^p} \;\leqslant\; C|t|^{-3(\frac{1}{2}-\frac{1}{p})} \big\| w^{\frac{2}{p'}-1} f \big\|_{L^{p'}}\,, \qquad p \in [2,+\infty] \tag{35}$$

when $\alpha \neq 0$, and

$$\big\| w^{-(1-\frac{2}{p})}\, e^{\mathrm{i}t\Delta_{\alpha=0}} f \big\|_{L^p} \;\leqslant\; C|t|^{-(\frac{1}{2}-\frac{1}{p})} \big\| w^{\frac{2}{p'}-1} f \big\|_{L^{p'}}\,, \qquad p \in [2,+\infty] \tag{36}$$

in the case $\alpha = 0$, with weight

$$w(x) \;:=\; 1 + \frac{1}{|x|}\,. \tag{37}$$

In fact $-\Delta_\alpha$ has a zero-energy resonance when $\alpha = 0$, and the slower time-decay (36) totally resembles what happens for actual Schrödinger operators with threshold resonances. From a more refined manipulation of (31) the weight-less version (23) in the range $p \in [2,3)$ was later obtained in [21, Proposition 5] (and subsequently in [13, Corollary 1.3]), which, by interpolation with the weighted $L^1 \to L^\infty$ estimate above, allows to improve the powers of the weights in (35)–(36) in the regime $p \in [3,+\infty]$ to almost optimal ones, respectively ([21, Corollary 1]),

$$\big\| w^{-(1-\frac{3-\delta}{p})}\, e^{\mathrm{i}t\Delta_\alpha} P_{(\alpha)}^{(\mathrm{ac})} f \big\|_{L^p} \;\leqslant\; C|t|^{-3(\frac{1}{2}-\frac{1}{p})} \big\| w^{1-\frac{3-\delta}{p}} f \big\|_{L^{p'}}\,, \qquad \alpha \neq 0\,,\ p \in [3,+\infty] \tag{38}$$

and

$$\big\| w^{-(1-\frac{3-\delta}{p})} e^{\mathrm{i}t\Delta_0} f \big\|_{L^p} \leqslant C|t|^{-\frac{1}{2}+\frac{\delta}{p}} \big\| w^{1-\frac{3-\delta}{p}} f \big\|_{L^{p'}}, \qquad p \in [3, +\infty] \tag{39}$$

for arbitrarily small $\delta > 0$.

It is natural to expect that the wave operators $W^{\pm}_{(\alpha)} \equiv W^{\pm}(-\Delta_\alpha, -\Delta)$, $\alpha \in \mathbb{R}\backslash\{0\}$, can be extended as continuous maps from $L^{p'}(\mathbb{R}^3, w_p^{-1}\mathrm{d}x)$ to $L^p(\mathbb{R}^3, w_p\mathrm{d}x)$ for $p \in (3, +\infty)$ (the 'endpoint' case $p = +\infty$ is typically more subtle), where

$$w_p(x) := w(x)^{-p+3+\delta} = \Big(1 + \frac{1}{|x|}\Big)^{-p+3+\delta} \tag{40}$$

for some delta $\delta > 0$ (that can be chosen arbitrarily small). Observe that $|x|^{-1} \in L^p(\mathbb{R}^3, w_p\mathrm{d}x)$, i.e., the weight w_p cancels out the local singularity generated by the point interaction. We also point out that we do not expect the boundedness of the wave operators in the zero-energy resonant case $\alpha = 0$, as this would lead to weighted $L^{p'} - L^p$ estimates with a time-decay $|t|^{-3(\frac{1}{2}-\frac{1}{p})}$ instead of the resonant time-decay $|t|^{-\frac{1}{2}+\frac{\delta}{p}}$.

It is also conceivable, under assumptions (**V1**), (**V**$_{\mathbf{decay}}$), (**V3**), and (**V4**) with $\alpha \neq 0$, that the wave operators $W^{\pm}_{\varepsilon} \equiv W^{\pm}(H_\varepsilon, -\Delta)$ can be extended as bounded maps from $L^{p'}(\mathbb{R}^3, w_p^{-1}\mathrm{d}x)$ to $L^p(\mathbb{R}^3, w_p\mathrm{d}x)$, and that $W^{\pm}_{\varepsilon}$ converges to $W^{\pm}_{(\alpha)}$, as $\varepsilon \downarrow 0$, in the weak topology of $\mathcal{B}(L^{p'}(\mathbb{R}^3, w_p^{-1}\mathrm{d}x; L^p(\mathbb{R}^3, w_p\mathrm{d}x))$.

All the ingredients above would allow to prove, by adapting the proof of Theorem 2, that under assumptions (**V1**), (**V**$_{\mathbf{decay}}$), (**V3**), and (**V4**) with $\alpha \neq 0$, weighted dispersive estimates analogous to (38) (with $p \in [3, \infty)$) hold true also for H_ε with an ε-independent constant.

In addition, by combining the above ε-uniform weighted dispersive estimates, a space-time re-scaling argument and suitable weighted resolvent bounds, it should be possible to provide (almost) optimal bounds for the blow-up rate as $\varepsilon \downarrow 0$ of the weight-less $L^{p'} - L^p$ estimates for H_ε, in the regime $p \geq 3$.

As already mentioned, the explicit dependence on the potential V in the dispersive estimates for $H = -\Delta + V$ cannot be in general directly deduced from the standard proofs, for these rely on the spectral behaviour of H at zero energy, which is unstable even with respect small perturbation of V in the Rollnik and (generalised) Kato norms.

Understanding the technical mechanisms at the basis of such an explicit dependence deserves further investigation, and the prototypical case of re-scaled potentials may serve as a starting point in this direction.

Acknowledgments This work is partially supported by the Italian National Institute for Higher Mathematics—INdAM (V.G., A.M., R.S.), the project 'Problemi stazionari e di evoluzione nelle equazioni di campo non-lineari dispersive' of GNAMPA—Gruppo Nazionale per l'Analisi Matematica (V.G.), the PRIN project no. 2020XB3EFL of the MIUR—Italian Ministry of University and Research (V.G.), the Institute of Mathematics and Informatics at the Bulgarian Academy of

Sciences (V.G.), the Top Global University Project at Waseda University (V.G.), and the Alexander von Humboldt Foundation, Bonn (A.M.).

References

1. Adami, R., Boni, F., Carlone, R., Tentarelli, L.: Ground states for the planar NLSE with a point defect as minimizers of the constrained energy. Calc. Var. **61**, 195 (2022)
2. Albeverio, S., Brzeźniak, Z., Dabrowski, L.: Fundamental solution of the heat and Schrödinger equations with point interaction. J. Funct. Anal. **130**, 220–254 (1995)
3. Albeverio, S., Gesztesy, F., Høegh-Krohn, R., Holden, H.Solvable Models in Quantum Mechanics, 2nd edn. AMS Chelsea Publishing, Providence (2005). With an appendix by Pavel Exner
4. Amrein, W.O.: Hilbert Space Methods in Quantum Mechanics. Fundamental Sciences. EPFL Press, Lausanne; distributed by CRC Press, Boca Raton, FL (2009)
5. Artbazar, G., Yajima, K.; The L^p-continuity of wave operators for one dimensional Schrödinger operators. J. Math. Sci. Univ. Tokyo **7**, 221–240 (2000)
6. Beceanu, M.: Structure of wave operators for a scaling-critical class of potentials. Am. J. Math. **136**, 255–308 (2014)
7. Beceanu, M., Schlag, W.: Structure formulas for wave operators under a small scaling invariant condition. J. Spectr. Theory **9**, 967–990 (2019)
8. Beceanu, M., Schlag, W.: Structure formulas for wave operators. Am. J. Math. **142**, 751–807 (2020)
9. Cacciapuoti, C., Finco, D., Noja, D.: Well posedness of the nonlinear Schrödinger equation with isolated singularities. J. Differ. Equ. **305**, 288–318 (2021)
10. Cazenave, T.: Semilinear Schrödinger Equations, vol. 10 of Courant Lecture Notes in Mathematics. New York University Courant Institute of Mathematical Sciences, New York (2003)
11. D'Ancona, P., Fanelli, L.: L^p-boundedness of the wave operator for the one dimensional Schrödinger operator. Commun. Math. Phys. **268**, 415–438 (2006)
12. D'Ancona, P., Pierfelice, V., Teta, A.: Dispersive estimate for the Schrödinger equation with point interactions. Math. Methods Appl. Sci. **29**, 309–323 (2006)
13. Dell'Antonio, G., Michelangeli, A., Scandone, R., Yajima, K.: L^p-boundedness of wave operators for the three-dimensional multi-centre point interaction. Ann. Henri Poincaré **19**, 283–322 (2018)
14. Erdoğan, M.B., Schlag, W.: Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three. I. Dyn. Partial Differ. Equ. **1**, 359–379 (2004)
15. Finco, D., Yajima, K.: The L^p boundedness of wave operators for Schrödinger operators with threshold singularities. II. Even dimensional case. J. Math. Sci. Univ. Tokyo **13**, 277–346 (2006)
16. Fukaya, N., Georgiev, V., Ikeda, M.: On stability and instability of standing waves for 2d-nonlinear Schrödinger equations with point interaction (2021). arXiv:2109.04680
17. Georgiev, V., Giammetta, A.R.: Sectorial Hamiltonians without zero resonance in one dimension, in Recent Advances in Partial Differential Equations and Applications, vol. 666 of Contemp. Math., pp. 225–237. Amer. Math. Soc., Providence (2016)
18. Georgiev, V., Michelangeli, A., Scandone, R.: On fractional powers of singular perturbations of the Laplacian. J. Funct. Anal. **275**, 1551–1602 (2018)
19. Georgiev, V., Michelangeli, A., Scandone, R.: Standing waves and global well-posedness for the 2d Hartree equation with a point interaction (2022). arXiv.org:2204.05053
20. Goldberg, M.: Dispersive bounds for the three-dimensional Schrödinger equation with almost critical potentials. Geom. Funct. Anal. **16**, 517–536 (2006)

21. Iandoli, F., Scandone, R.: Dispersive estimates for Schrödinger operators with point interactions in $\mathbb{R}^3$. In: Michelangeli, A., Dell'Antonio, G. (eds.), Advances in Quantum Mechanics: Contemporary Trends and Open Problems. Springer INdAM Series, vol. 18, pp. 187–199. Springer, Berlin
22. Jensen, A., Kato, T.: Spectral properties of Schrödinger operators and time-decay of the wave functions. Duke Math. J. **46**, 583–611 (1979)
23. Jensen, A., Yajima, K.: A remark on L^p-boundedness of wave operators for two-dimensional Schrödinger operators. Commun. Math. Phys. **225**, 633–637 (2002)
24. Jensen, A., Yajima, K.: On L^p boundedness of wave operators for 4-dimensional Schrödinger operators with threshold singularities. Proc. Lond. Math. Soc. (3) **96**, 136–162 (2008)
25. Journé, J.-L., Soffer, A., Sogge, C.D.: Decay estimates for Schrödinger operators. Commun. Pure Appl. Math. **44**, 573–604 (1991)
26. Linares, F., Ponce, G.: Introduction to Nonlinear Dispersive Equations. Universitext, 2nd edn. Springer, New York (2015)
27. Michelangeli, A., Olgiati, A., Scandone, R.: Singular Hartree equation in fractional perturbed Sobolev spaces. J. Nonlinear Math. Phys. **25**, 558–588 (2018)
28. Michelangeli, A., Ottolini, A.: On point interactions realised as Ter-Martirosyan-Skornyakov Hamiltonians. Rep. Math. Phys. **79**, 215–260 (2017)
29. Rauch, J.: Local decay of scattering solutions to Schrödinger's equation. Commun. Math. Phys. **61**, 149–168 (1978)
30. Reed, M., Simon, B.: Methods of Modern Mathematical Physics, vol. 1. Academic Press, New York (1972)
31. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. II. Fourier Analysis, Self-Adjointness. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London (1975)
32. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. III. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London (1979). Scattering theory
33. Rodnianski, I., Schlag, W.: Time decay for solutions of Schrödinger equations with rough and time-dependent potentials. Invent. Math. **155**, 451–513 (2004)
34. Scarlatti, S., Teta, A.: Derivation of the time-dependent propagator for the three-dimensional Schrödinger equation with one-point interaction. J. Phys. A **23**, L1033–L1035 (1990)
35. Schlag, W.: Dispersive estimates for Schrödinger operators: a survey. In: Mathematical Aspects of Nonlinear Dispersive Equations, vol. 163 of Ann. of Math. Stud., pp. 255–285. Princeton Univ. Press, Princeton (2007)
36. Sulem, C., Sulem, P.-L.: The nonlinear Schrödinger equation, vol. 139 of Applied Mathematical Sciences. Springer, New York (1999). Self-focusing and wave collapse
37. Tao, T.: Nonlinear dispersive equations, vol. 106 of CBMS Regional Conference Series in Mathematics, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence (2006). Local and global analysis
38. Weder, R.: L^p-$L^{\dot{p}}$ estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential. J. Funct. Anal. **170**, 37–68 (2000)
39. Yajima, K.: The $W^{k,p}$-continuity of wave operators for Schrödinger operators. J. Math. Soc. Jpn. **47**, 551–581 (1995)
40. Yajima, K.: L^p-boundedness of wave operators for two-dimensional Schrödinger operators. Commun. Math. Phys. **208**, 125–152 (1999)
41. Yajima, K.: Dispersive estimates for Schrödinger equations with threshold resonance and eigenvalue. Commun. Math. Phys. **259**, 475–509 (2005)
42. Yajima, K.: Remarks on L^p-boundedness of wave operators for Schrödinger operators with threshold singularities. Doc. Math. **21**, 391–443 (2016)
43. Yajima, K.: On wave operators for Schrödinger operators with threshold singularities in three dimensions (2016). arXiv:1606.03575