Nonlinear Schrödinger Equation with Singularities

Nevena Dugandžija and Ivana Vojnović

Abstract We describe results on certain types of nonlinear Schrödinger equations, mainly the cubic equation with or without potential. We are interested in singular initial conditions and equations with a delta potential in three dimensions. The existence and uniqueness of solutions are proved in the Colombeau algebra setting and the notion of compatibility of solutions is explored.

1 Introduction

We will analyze the following equations in three dimensions. First, we consider the defocusing cubic Schrödinger equation

$$\begin{aligned} iu_t + \triangle u &= u|u|^2, \\ u(0) &= a, \end{aligned} \tag{1}$$

and then the cubic equation with the delta potential

$$\begin{aligned} iu_t + \triangle u &= u|u|^2 + \delta u, \\ u(0) &= a. \end{aligned} \tag{2}$$

Equation (1) is extensively studied in the classical sense. Applications of (1) are connected with many physical contents such as dynamics of Bose gas, optics, and superfluids.

Well-posedness in Sobolev spaces, and in particular in the energy space $H^1(\mathbb{R}^3)$, is developed in [6] and [8]. Also, it was proved in [11] that global solutions exist in $H^s(\mathbb{R}^3)$ for $s > \frac{4}{5}$. We are interested in initial data which are more singular.

N. Dugandžija (✉) · I. Vojnović
Faculty of Sciences, Novi Sad, Serbia
e-mail: nevena.dugandzija@dmi.uns.ac.rs; ivana.vojnovic@dmi.uns.ac.rs

V. Georgiev et al. (eds.), *Qualitative Properties of Dispersive PDEs*,
Springer INdAM Series 52, https://doi.org/10.1007/978-981-19-6434-3_4

Equation (2) is a model for Bose–Einstein condensates where δ is used to describe a local, short-range potential applied to a condensate. In [17] solutions in weak L^p spaces in one dimension are considered.

We will analyze Eqs. (1) and (2) within the Colombeau algebra setting and for that purpose different spaces of distributions will be embedded in the Colombeau algebra.

We are interested in regularized equations. For instance, the regularized equation for (2) is of the form

$$\begin{aligned} i(u_\varepsilon)_t + \triangle u_\varepsilon &= u_\varepsilon |u_\varepsilon|^2 + \phi_\varepsilon u_\varepsilon, \\ u_\varepsilon(0) &= a_\varepsilon, \end{aligned}$$

for appropriate nets of functions $(u_\varepsilon)_\varepsilon$, $(a_\varepsilon)_\varepsilon$, and $(\phi_\varepsilon)_\varepsilon$ which we will call moderate functions.

Important properties that hold for this equation and that will be used are conservation of charge and energy:

$$\begin{aligned} \|u_\varepsilon(t)\|_2 &= \|a_\varepsilon\|_2, \\ H(u_\varepsilon(t)) &= H(a_\varepsilon), \end{aligned}$$

where $H(u_\varepsilon) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} |u_\varepsilon|^4 dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi_\varepsilon |u_\varepsilon|^2 dx$ is the Hamiltonian. Also, for fixed $\varepsilon > 0$ there is well-posedness in $H^s(\mathbb{R}^3)$ for $s \geq 2$.

The chapter is organized as follows. First, we introduce Colombeau algebras and describe their basic properties and prove theorems which explain how we embed different spaces of distributions into these algebras. The notion of a solution in the sense of Colombeau algebras is also introduced and we define the existence and uniqueness of solutions within this setting. Then we define compatibility between classical solutions and Colombeau solutions and further prove the existence and uniqueness of solutions to Eqs. (1) and (2). We conclude the chapter by analyzing some convergence properties and by giving directions for possible further investigations.

We shortly describe the notation. By $\mathcal{D}(\mathbb{R}^3)$ we denote the space of smooth compactly supported functions $f : \mathbb{R}^3 \to \mathbb{C}$ equipped with the finest locally convex topology for which all the inclusions $\mathcal{D}(K) \hookrightarrow \mathcal{D}(\Omega)$ are continuous (K is an arbitrary compact subset of Ω). Also, $H^s = H^s(\mathbb{R}^3)$, $s \in \mathbb{R}$ is the usual Sobolev space. We say that $f(\varepsilon) \sim g(\varepsilon)$ if $\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = c > 0$. Further, $f(\varepsilon) \lesssim g(\varepsilon)$ if there exists $c > 0$ independent of ε such that $f(\varepsilon) \leq cg(\varepsilon)$. We also use some well-known inequalities, namely Hölder, Young, Gronwall, and Gagliardo–Nirenberg inequalities.

Results presented in this chapter are based on papers [15] and [16].

2 The Colombeau Algebra

In this section we introduce the algebras of Colombeau (see [12, 13]). We construct them as factor algebras of the so-called moderate functions modulo a class of ideals that we call negligible functions, which will be described in the sequel.

In certain examples of partial differential equations with singular coefficients or singular data we need to multiply distributions. For instance, delta waves occur in the analysis of semilinear hyperbolic systems with rough initial data. Many examples of problems (related to elasticity, acoustics, fluid dynamics) where the multiplication of distributions occurs are given in [14] and [25].

However, multiplication of distributions is connected with many difficulties. The product of a smooth function and a distribution is well-defined, but if we try to extend the operation of multiplication to arbitrary distributions we are not able to preserve the associative property:

$$0 = (\delta(x) \cdot x) \cdot vp\frac{1}{x} \neq \delta(x) \cdot (x \cdot vp\frac{1}{x}) = \delta(x),$$

where $vp\frac{1}{x}$ denotes the Cauchy principal value of $\frac{1}{x}$. One possibility to overcome this problem is to embed the space of distributions in some algebra so that we can define a product.

If we denote this algebra by $(\mathcal{A}(\Omega), +, \cdot)$, where $\Omega \subset \mathbb{R}^3$ is an open set, then we would like that the algebra $\mathcal{A}(\Omega)$ satisfies following properties:

1. $\mathcal{D}'(\Omega)$ is linearly embedded into $\mathcal{A}(\Omega)$,
2. there exist differential operators $\partial_i : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$, $i = 1, \dots, n$ that are linear and satisfy the Leibniz rule,
3. $\partial_i|_{\mathcal{D}'}$ is the usual partial derivative, $i = 1, \dots, n$,
4. the restriction $\cdot|_{C^\infty \times C^\infty}$ coincides with the pointwise product of functions.

One example of $(\mathcal{A}(\Omega), +, \cdot)$ is the following special Colombeau algebra which we define in the sequel (for details see [19]). We introduce spaces :

$$\begin{aligned}
\mathcal{E}^s(\Omega) &:= (C^\infty(\Omega))^{(0,1]}, \\
\mathcal{E}^s_M(\Omega) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{E}^s(\Omega) \mid \forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^n\ \exists N \in \mathbb{N} \text{ with} \\
&\qquad \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{-N}),\ \varepsilon \to 0\}, \\
\mathcal{N}^s(\Omega) &:= \{(u_\varepsilon)_\varepsilon \in \mathcal{E}^s(\Omega) \mid \forall K \subset\subset \Omega\ \forall \alpha \in \mathbb{N}_0^n\ \forall m \in \mathbb{N} \text{ with} \\
&\qquad \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{m}),\ \varepsilon \to 0\}.
\end{aligned}$$

Here $K \subset\subset \Omega$ means that K is a compact subset of Ω. Elements of $\mathcal{E}_M^s(\Omega)$ are called moderate functions and elements of $\mathcal{N}^s(\Omega)$ are called negligible functions. The special Colombeau algebra is defined as the quotient space

$$\mathcal{G}^s(\Omega) := \mathcal{E}_M^s(\Omega)/\mathcal{N}^s(\Omega).$$

In the sequel we assume that $n = 3$, unless otherwise stated. The embedding of the space of distributions $\mathcal{D}'(\Omega) \hookrightarrow \mathcal{G}^s(\Omega)$ is given by

$$u \mapsto [(u * \rho_\varepsilon)_\varepsilon],$$

where $\rho \in S(\mathbb{R}^3)$ is a mollifier which satisfies conditions

$$\int \rho(x)dx = 1, \tag{3}$$

$$\int x^\alpha \rho(x)dx = 0, \quad \forall|\alpha| \geq 1, \tag{4}$$

and $\rho_\varepsilon(x) = \varepsilon^{-3}\rho\Big(\dfrac{x}{\varepsilon}\Big)$. One can prove that there is no mollifier in $\mathcal{D}(\mathbb{R}^3)$ which satisfies both (3) and (4). However, $\rho \in S(\mathbb{R}^3)$ can be constructed by taking the inverse Fourier transform of a function from $S(\mathbb{R}^3)$ which equals one in a neighborhood of zero.

Next we define the H^2-based Colombeau algebra as in [24] (for a similar construction see [23]). This type of algebra is appropriate for the equations that we consider.

We denote by $\mathcal{E}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ (respectively, $\mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$), $T > 0$ the vector space of nets $(u_\varepsilon)_\varepsilon$ of functions

$$u_\varepsilon \in C([0,T), H^2(\mathbb{R}^3)) \cap C^1([0,T), L^2(\mathbb{R}^3)),\ \varepsilon \in (0,1),$$

such that there exists $N \in \mathbb{N}$ (respectively, for every $M \in \mathbb{N}$):

$$\max\{\sup_{t\in[0,T)} \|u_\varepsilon(t)\|_{H^2}, \sup_{t\in[0,T)} \|\partial_t u_\varepsilon(t)\|_2\} = \mathcal{O}(\varepsilon^{-N}),\ \varepsilon \to 0$$

$$\Bigg(\text{respectively,}$$

$$\max\{\sup_{t\in[0,T)} \|u_\varepsilon(t)\|_{H^2}, \sup_{t\in[0,T)} \|\partial_t u_\varepsilon(t)\|_2\} = \mathcal{O}(\varepsilon^{M}),\ \varepsilon \to 0\Bigg).$$

Then we define the quotient space

$$\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3) = \mathcal{E}_{C^1,H^2}([0,T) \times \mathbb{R}^3)/\mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$$

which is a Colombeau type vector space.

We also define the space $\mathcal{G}_{H^2}(\mathbb{R}^3)$ in a similar manner:

$$\mathcal{E}^2(\mathbb{R}^3) := (H^2(\mathbb{R}^3))^{(0,1]},$$
$$\mathcal{E}_{H^2}(\mathbb{R}^3) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}^2(\mathbb{R}^3) \mid \exists N \in \mathbb{N}\ \|u_\varepsilon\|_{H^2} = O(\varepsilon^{-N}),\ \varepsilon \to 0\},$$
$$\mathcal{N}_{H^2}(\mathbb{R}^3) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}^2(\mathbb{R}^3) \mid \forall m \in \mathbb{N}\ \|u_\varepsilon\|_{H^2} = O(\varepsilon^{m}),\ \varepsilon \to 0\},$$
$$\mathcal{G}_{H^2}(\mathbb{R}^3) := \mathcal{E}_{H^2}(\mathbb{R}^3)/\mathcal{N}_{H^2}(\mathbb{R}^3).$$

Operations of addition, multiplication, and differentiation are defined component-wise, that is

$$u + v = [(u_\varepsilon + v_\varepsilon)_\varepsilon], \quad u \cdot v = [(u_\varepsilon \cdot v_\varepsilon)_\varepsilon], \quad \partial^\alpha u = [(\partial^\alpha u_\varepsilon)_\varepsilon].$$

Differentiation on H^2-based algebra is not a closed operation. If $u \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$, then $\partial^\alpha u$ for $|\alpha| \leq 2$ is represented by $(\partial^\alpha u_\varepsilon)_\varepsilon$ which has moderate growth in $L^2(\mathbb{R}^3)$ and is an element of a quotient vector space $\mathcal{G}_{C,L^2}([0,T) \times \mathbb{R}^3)$. The vector space $\mathcal{G}_{C,L^2}([0,T) \times \mathbb{R}^3)$ is defined analogously as $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$. Difference is that representatives have bounded growth only in L^2-norm, for any $t \in [0,T)$. It is clear that $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3) \subset \mathcal{G}_{C,L^2}([0,T) \times \mathbb{R}^3)$.

Notice that spaces $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ and $\mathcal{G}_{H^2}(\mathbb{R}^3)$ are multiplicative algebras because $H^2(\mathbb{R}^3)$ is an algebra (the same holds for $\mathbb{R}^n$ when $n \leq 3$).

Since $\delta * \rho_\varepsilon = \rho_\varepsilon$ it is clear that $(\rho_\varepsilon)_\varepsilon$ itself is a representative of the delta distribution. Here ρ_ε is given by (3) and (4).

Next we define a *strict delta net* because another representative of the delta distribution is given by this type of net (cf. [19]).

Definition 1 A strict delta net is a net $(\phi_\varepsilon)_{0<\varepsilon\leq 1}$, $\phi_\varepsilon \in \mathcal{D}(\mathbb{R}^3)$ which satisfies

(i) $\operatorname{supp}(\phi_\varepsilon) \to \{0\},\ \varepsilon \to 0,$
(ii) $\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} \phi_\varepsilon(x)dx = 1,$
(iii) $\int |\phi_\varepsilon(x)|dx$ is bounded uniformly in ε.

We can define a strict delta net using ρ_ε as $\phi_\varepsilon(x) = \chi(\frac{x}{\sqrt{\varepsilon}})\rho_\varepsilon(x)$, where χ is a cut-off function and $\rho_\varepsilon \in \mathcal{S}(\mathbb{R}^3)$ is a mollifier defined by (3) and (4). More precisely,

$$\chi \in \mathcal{D}(\mathbb{R}^3), \quad \chi(x) = 1,\ |x| \leq 1 \text{ and } \chi(x) = 0, \quad |x| \geq 2. \tag{5}$$

Since $\mathcal{S}(\mathbb{R}^3) \subset L^p(\mathbb{R}^3)$ the following estimates for ρ_ε and ϕ_ε hold:

$$\begin{aligned} \|\partial^\alpha \rho_\varepsilon\|_p^p = \int_{\mathbb{R}^3} \varepsilon^{-3p}|\partial^\alpha(\rho(\frac{x}{\varepsilon}))|^p dx = \int_{\mathbb{R}^3} \varepsilon^{-3p}|\frac{1}{\varepsilon^{|\alpha|}}(\partial^\alpha\rho)(\frac{x}{\varepsilon})|^p dx = \\ \int_{\mathbb{R}^3} \varepsilon^{-3p+3-|\alpha|p}|\partial^\alpha\rho(t)|^p dt = c\varepsilon^{3(1-p)-|\alpha|p} \lesssim \varepsilon^{-N}, \end{aligned} \tag{6}$$

for some $N \in \mathbb{N}$, $1 \leq p < \infty$ and for any multi-index α. Moreover, $\|\rho_\varepsilon\|_\infty = \varepsilon^{-n} \max |\rho(\frac{x}{\varepsilon})| = c\varepsilon^{-n}$, for any $\varepsilon > 0$.
We also use mollifiers of the type $\rho_{h_\varepsilon} = h_\varepsilon^3 \rho(x h_\varepsilon)$, where $h_\varepsilon \to \infty$, $\varepsilon \to 0$, for example, $h_\varepsilon = \ln \varepsilon^{-1}$, and these mollifiers satisfy analogous estimates.

Furthermore, we derive estimates for $\partial^\alpha(\chi(\frac{x}{\sqrt{\varepsilon}}))$, that is

$$\sup_{x\in\mathbb{R}^3} |\varepsilon^{-|\alpha|/2}(\partial^\alpha \chi)(\frac{x}{\sqrt{\varepsilon}})| \lesssim \varepsilon^{-|\alpha|/2}.$$

Therefore $\phi_\varepsilon(x) = \chi(\frac{x}{\sqrt{\varepsilon}})\rho_\varepsilon(x)$ admits analogous estimates as ρ_ε in the L^p-norm.

Now we prove that we can use a strict delta net to embed delta distribution in $\mathcal{G}_{H^2}(\mathbb{R}^3)$.

Theorem 1 *There exists a strict delta net $(\phi_\varepsilon)_{0<\varepsilon\leq 1}$ such that the difference $(\rho_\varepsilon - \phi_\varepsilon)_\varepsilon$ is an element of $\mathcal{N}_{H^2}(\mathbb{R}^3)$. Both $(\rho_\varepsilon)_\varepsilon$ and $(\phi_\varepsilon)_\varepsilon$ are representatives for the embedded delta distribution $[(\rho_\varepsilon)_\varepsilon] \in \mathcal{G}_{H^2}(\mathbb{R}^3)$.*

Proof Let $\phi_\varepsilon(x) = \chi_\varepsilon(x)\rho_\varepsilon(x) = \chi(\frac{x}{\sqrt{\varepsilon}})\rho_\varepsilon(x)$, where χ is given by (5). Since $\rho_\varepsilon \in \mathcal{S}(\mathbb{R}^3)$ for any $q > 2$ it holds that

$$\begin{aligned}
\|\rho_\varepsilon - \rho_\varepsilon\chi_\varepsilon\|_2^2 &= \int_{\mathbb{R}^3} \rho_\varepsilon^2(x)(1-\chi(\frac{x}{\sqrt{\varepsilon}}))^2 dx \leq \int_{|x|>\sqrt{\varepsilon}} \rho_\varepsilon^2(x)dx \\
&\leq \int_{|x|>\sqrt{\varepsilon}} \varepsilon^{-6}(1+|\frac{x}{\varepsilon}|)^{-2q} dx = \int_{|x|>\sqrt{\varepsilon}} \varepsilon^{-6}(1+\frac{|x|}{\varepsilon})^{-2q+3+1-(3+1)} dx \\
&\leq \varepsilon^{-6} \sup_{x>\sqrt{\varepsilon}} (1+\frac{|x|}{\varepsilon})^{-2q+3+1} \int_{|x|>\sqrt{\varepsilon}} (1+\frac{|x|}{\varepsilon})^{-(3+1)} dx \\
&\leq \varepsilon^{-6}\varepsilon^{q-(3+1)/2}\varepsilon^3 \int_{|y|>1/\sqrt{\varepsilon}} \frac{1}{(1+|y|)^{3+1}} dy \\
&\leq \varepsilon^{q-(3+1)/2-3} \int_{y\in\mathbb{R}^3} \frac{1}{(1+|y|)^{3+1}} dy.
\end{aligned}$$

The above integral is finite and independent of ε. Hence for arbitrary $m \in \mathbb{N}$ we choose $q = m + \frac{10}{2}$ (then $q > 2$) and

$$\|\rho_\varepsilon - \rho_\varepsilon\chi_\varepsilon\|_2^2 \lesssim \varepsilon^m, \quad 0 < \varepsilon \leq 1.$$

Next we need to bound derivatives $\partial^\alpha(\rho_\varepsilon - \rho_\varepsilon\chi_\varepsilon)$ in the L^2-norm, for $|\alpha| = 1$ and $|\alpha| = 2$. This can be done similarly as in the first part of the proof using that the function $1-\chi$ is equal to zero for $|x| \leq \sqrt{\varepsilon}$ and derivatives of the function $1-\chi$ are supported in the set $\sqrt{\varepsilon} \leq |x| \leq 2\sqrt{\varepsilon}$. □

Next we use mollifier ρ_ε given by (3) and (4) to represent functions from $H^2(\mathbb{R}^3)$ as an elements of $\mathcal{G}_{H^2}(\mathbb{R}^3)$.

Theorem 2 *Let $f \in H^2(\mathbb{R}^3)$. Then we can embed $H^2(\mathbb{R}^3)$ into $\mathcal{G}_{H^2}(\mathbb{R}^3)$ such that $f \mapsto [(f * \rho_\varepsilon)_\varepsilon]$.*

Proof For any $|\alpha| \leq 2$ using Young's inequality we have that

$$\|\partial^\alpha (f * \rho_\varepsilon)\|_2 = \|f * \partial^\alpha \rho_\varepsilon\|_2 \leq \|f\|_2 \|\partial^\alpha \rho_\varepsilon\|_1 \lesssim \varepsilon^{-N}$$

for some $N \in \mathbb{N}$, where we use estimates as in (6).

Hence $f_\varepsilon = f * \rho_\varepsilon$ defines an element $[(f_\varepsilon)_\varepsilon] \in \mathcal{G}_{H^2}(\mathbb{R}^3)$. We also know that $\|f * \phi_\varepsilon - f\|_2 \to 0$. Hence the mapping $f \mapsto [(f_\varepsilon)_\varepsilon]$ is injective. More concretely, if $v \in H^2(\mathbb{R}^3)$ is another function embedded in $\mathcal{G}_{H^2}(\mathbb{R}^3)$ using convolution with ρ_ε, then $(v_\varepsilon)_\varepsilon \in [(f_\varepsilon)_\varepsilon]$ (here $v_\varepsilon = v * \rho_\varepsilon$) and

$$v = \lim_{\varepsilon \to 0} v_\varepsilon = \lim_{\varepsilon \to 0} (f_\varepsilon + n_\varepsilon) = f$$

in $L^2(\mathbb{R}^3)$, where $v_\varepsilon = f_\varepsilon + n_\varepsilon$ and $(n_\varepsilon)_\varepsilon \in \mathcal{N}_{H^2}(\mathbb{R}^3)$. Therefore

$$H^2(\mathbb{R}^3) \hookrightarrow \mathcal{G}_{H^2}(\mathbb{R}^3),$$

what we wanted to prove. □

Another representative of elements from $H^2(\mathbb{R}^3)$ is obtained using a strict delta net.

Theorem 3 *Let $f \in H^2(\mathbb{R}^3)$. Then $f * \rho_\varepsilon - f * \phi_\varepsilon \in \mathcal{N}_{H^2}(\mathbb{R}^3)$, where ϕ_ε is a strict delta net defined by $\phi_\varepsilon = \chi_\varepsilon \rho_\varepsilon$, $\chi_\varepsilon(x) = \chi(\frac{x}{\sqrt{\varepsilon}})$ and χ is a cut-off function as in (5).*

Proof From Young's inequality we have that

$$\|f * (\rho_\varepsilon - \phi_\varepsilon)\|_2 \lesssim \|f\|_2 \|(1 - \chi_\varepsilon)\rho_\varepsilon\|_1.$$

We can estimate $\|(1 - \chi_\varepsilon)\rho_\varepsilon\|_1 \lesssim \varepsilon^m$ for any $m \in \mathbb{N}$, $\varepsilon \to 0$ similarly as in the proof of Theorem 1. Also, $\partial^\alpha (f * (\rho_\varepsilon - \phi_\varepsilon)) = (\partial^\alpha f) * (\rho_\varepsilon - \phi_\varepsilon)$ and therefore the proof follows. □

Further we prove that the product of the delta distribution and an element from $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ remains in $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$.

Theorem 4 *Let $u \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ and ρ_ε is the representative of δ in $\mathcal{G}_{H^2}(\mathbb{R}^3)$. Then $u \cdot [(\rho_\varepsilon)_\varepsilon] \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$.*

Proof Let $u_\varepsilon \in \mathcal{E}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$. We have

$$\|u_\varepsilon \rho_\varepsilon\|_2 \lesssim \|\rho_\varepsilon\|_\infty \|u_\varepsilon(t)\|_2 \lesssim \varepsilon^{-N}, \quad \varepsilon \to 0,$$

for any $t \in [0, T)$. Similar estimates can be derived for $\partial^\alpha(u_\varepsilon \rho_\varepsilon)$, $|\alpha| \leq 2$. In this case we have expressions of form $\partial^\beta u_\varepsilon \partial^\delta \rho_\varepsilon$, $|\beta|, |\delta| \leq 2$, which can be bounded by ε^{-N}, $\varepsilon \to 0$, for some N.

Now let $(v_\varepsilon)_\varepsilon$ be another representative of u and $(\rho_\varepsilon^1)_\varepsilon$ be another representative of δ. Then $\rho_\varepsilon^1 = \rho_\varepsilon + n_\varepsilon^1$ for $n_\varepsilon^1 \in \mathcal{N}_{H^2}(\mathbb{R}^3)$ and $v_\varepsilon = u_\varepsilon + n_\varepsilon^2$ for $n_\varepsilon^2 \in \mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$. Then $u_\varepsilon \rho_\varepsilon - v_\varepsilon \rho_\varepsilon^1 \in \mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$.

Indeed, product of n_ε^1 and n_ε^2 is negligible in $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ and also $u_\varepsilon \cdot n_\varepsilon^2 \in \mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$, $\rho_\varepsilon \cdot n_\varepsilon^1 \in \mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$, where we use that $H^2(\mathbb{R}^3)$ is an algebra. Hence the product is well-defined. □

We also need to define a restriction of an element $u \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ since the initial condition is a function that depends only on x.

Definition 2 Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$. We define the restriction of u to $\{0\} \times \mathbb{R}^3$ as the class $[(u_\varepsilon(0,\cdot))_\varepsilon] \in \mathcal{G}_{H^2}(\mathbb{R}^3)$.

Definition 2 makes sense. Indeed, since $u_\varepsilon \in C([0,T), H^2(\mathbb{R}^3))$, the function $u_\varepsilon(0,\cdot)$ is in $\mathcal{E}_{H^2}(\mathbb{R}^3)$. Also, if $u_\varepsilon \in \mathcal{N}_{C^1,H^2}([0,T), H^2(\mathbb{R}^3))$, then $u_\varepsilon(0,\cdot)$ is in $\mathcal{N}_{H^2}(\mathbb{R}^3)$.

We will also need the definition of an initial condition which is of $(\ln)^j$-type.

Definition 3 We say that $a \in \mathcal{G}_{H^2}(\mathbb{R}^3)$ is of $(\ln)^j$-type, $j \in (0,1]$ if it has a representative $a_\varepsilon \in \mathcal{E}_{H^2}(\mathbb{R}^3)$ such that

$$\|a_\varepsilon\|_2 = O(\ln^j \varepsilon^{-1}), \quad \varepsilon \to 0.$$

Note that a function $a \in H^\infty(\mathbb{R}^3)$ is itself a representative in $\mathcal{G}_{H^2}(\mathbb{R}^3)$ (which will be proved in Theorem 5 in the sequel). This is an example of a function that is of $(\ln)^j$-type for any $j \in (0,1]$ since its L^2–norm is a constant independent of ε. Similarly holds for $a \in L^2(\mathbb{R}^3)$.

Theorem 5 *If $a \in H^\infty(\mathbb{R}^3)$, then $[(a)_\varepsilon] \in \mathcal{G}_{H^2}(\mathbb{R}^3)$.*

Proof We need to show that $(a_\varepsilon - a)_\varepsilon \in \mathcal{N}_{H^2}(\mathbb{R}^3)$, where $a_\varepsilon = a * \rho_\varepsilon$ and ρ_ε is given by (3) and (4).

We follow ideas given in [3]. It holds that

$$\|a_\varepsilon - a\|_2^2 = \|a * \rho_\varepsilon - a\|_2^2 = \int \left| \int (a(x - \varepsilon y) - a(x))\rho(y)dy \right|^2 dx.$$

We can apply Taylor's formula to a up to order m. Since $\int y^\alpha \rho(y)dy = 0$ for $|\alpha| \leq m$ (by (4)) it follows that

$$\|a_\varepsilon - a\|_2^2 = \int | \sum_{|\alpha|=m+1} \int \frac{(-\varepsilon y)^{\alpha})}{m!} \int_0^1 (1-\sigma)^m \partial^\alpha a(x - \sigma\varepsilon y) d\sigma \rho(y) dy|^2 dx$$

$$\le C(m,q) \max_{|\alpha|=m+1} \int \left| \int \frac{(-\varepsilon y)^{\alpha})}{m!} \rho(y) \int_0^1 (1-\sigma)^m \partial^\alpha a(x-\sigma\varepsilon y) d\sigma dy \right|^2 dx$$

$$\le C(m,q) \max_{|\alpha|=m+1} \int \int \left| \frac{(\varepsilon y)^{\alpha})}{m!} \rho(y) \int_0^1 (1-\sigma)^m \partial^\alpha a(x-\sigma\varepsilon y) d\sigma \right|^2 dx dy$$

$$\le \frac{\varepsilon^{m+1}}{m!} C(m,q) \max_{|\alpha|=m+1} \int |y^\alpha \rho(y)| \int \int_0^1 |\partial^\alpha a(y-\sigma\varepsilon y)|^2 d\sigma dx dy$$

$$\le c\varepsilon^{m+1} \int |y|^{m+1} |\rho(y)| dy \max_{|\alpha|=m+1} \|\partial^\alpha a\|_2.$$

Hence for any $m \in \mathbb{N}$ and sufficiently small ε we have

$$\|u_\varepsilon - u\|_2 \le c\varepsilon^m.$$

The same estimates hold for $\partial^\alpha(a_\varepsilon - a)$, $|\alpha| \le 2$. □

2.1 Notion of Colombeau Solution

Let us consider the following Schrödinger equation:

$$\begin{aligned} iu_t + \triangle u + g(u) &= 0, \\ u(0) &= a, \end{aligned} \tag{7}$$

where $g(u)$ is given nonlinearity. Next we define the existence of a solution in the Colombeau sense.

Definition 4 We say that $u \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ is a solution of (7) if for an initial condition a and its representative $a_\varepsilon = a * \rho_\varepsilon$, there exists a representative $(u_\varepsilon)_\varepsilon \in \mathcal{E}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ such that

$$\begin{aligned} i(u_\varepsilon)_t + \triangle u_\varepsilon + g(u_\varepsilon) &= M_\varepsilon, \\ u_\varepsilon(0) &= a_\varepsilon + n_\varepsilon, \end{aligned} \tag{8}$$

for some $n_\varepsilon \in \mathcal{N}_{H^2}(\mathbb{R}^3)$ and we assume that $\sup_{t\in[0,T)} \|M_\varepsilon\|_2 = \mathcal{O}(\varepsilon^M)$, $\varepsilon \to 0$, for any $M \in \mathbb{N}$.

If the above statement holds for some u_ε, then it holds for all representatives of the class $u = [(u_\varepsilon)_\varepsilon]$. We will show that for $g(u_\varepsilon) = -(u_\varepsilon|u_\varepsilon|^2 + \phi_\varepsilon u_\varepsilon)$.

Let $v_\varepsilon = u_\varepsilon + N_\varepsilon$, for some $N_\varepsilon \in \mathcal{N}_{C^1,H^2}(\mathbb{R}^3)$. Then

$$\begin{aligned} i(v_\varepsilon)_t + \triangle v_\varepsilon - (v_\varepsilon|v_\varepsilon|^2 + \phi_\varepsilon v_\varepsilon) &= i(u_\varepsilon)_t + \triangle u_\varepsilon - u_\varepsilon|u_\varepsilon|^2 - \phi_\varepsilon u_\varepsilon \\ &\quad + i(N_\varepsilon)_t + \triangle N_\varepsilon - f(u_\varepsilon, N_\varepsilon, \phi_\varepsilon) \\ &= M_\varepsilon + i(N_\varepsilon)_t + \triangle N_\varepsilon - f(u_\varepsilon, N_\varepsilon, \phi_\varepsilon), \end{aligned}$$

where $\sup_{0\le t<T} \|M_\varepsilon\|_2 = \mathcal{O}(\varepsilon^M)$, $\varepsilon \to 0$, for any $M \in \mathbb{N}$ and

$$f(u_\varepsilon, N_\varepsilon, \phi_\varepsilon) = \overline{N_\varepsilon} u_\varepsilon^2 + 2|u_\varepsilon|^2 N_\varepsilon + 2u_\varepsilon|N_\varepsilon|^2 + \overline{u}_\varepsilon N_\varepsilon^2 + N_\varepsilon|N_\varepsilon|^2 + \phi_\varepsilon N_\varepsilon.$$

Since $N_\varepsilon \in \mathcal{N}_{C^1,H^2}(\mathbb{R}^3)$, it holds that $\|i(N_\varepsilon)_t + \Delta N_\varepsilon\|_2 = \mathcal{O}(\varepsilon^M)$, $\varepsilon \to 0$, for any $M \in \mathbb{N}$. Also, using the Sobolev embedding $\|N_\varepsilon\|_\infty \le c\|N_\varepsilon\|_{H^2}$ we see that $\sup_{0\le t<T} \|f(u_\varepsilon, N_\varepsilon, \phi_\varepsilon)\|_2 = \mathcal{O}(\varepsilon^M)$, $\varepsilon \to 0$. Furthermore,

$$v_\varepsilon(0) = u_\varepsilon(0) + N_\varepsilon(0) = a_\varepsilon + n_\varepsilon + N_\varepsilon(0) = a_\varepsilon + N_\varepsilon^1,$$

where $N_\varepsilon^1 \in \mathcal{N}_{H^2}(\mathbb{R}^3)$. Therefore v_ε satisfies all the conditions from Definition 4.

When we want to prove the existence of a solution in the Colombeau sense, usually we first solve

$$\begin{aligned} i(u_\varepsilon)_t + \triangle u_\varepsilon + g(u_\varepsilon) &= 0, \\ u_\varepsilon(0) &= a_\varepsilon, \end{aligned}$$

where $a_\varepsilon = a * \rho_\varepsilon$ and then the previous analysis implies that $[(u_\varepsilon)_\varepsilon]$ is indeed a solution.

Definition 5 We say that the solution of (7) is unique if for any two solutions $u, v \in \mathcal{G}_{C^1,H^2}$ it holds $\sup_{t\in[0,T)} \|u_\varepsilon - v_\varepsilon\|_2 = O(\varepsilon^M)$, $\varepsilon \to 0$, for any $M \in \mathbb{N}$. Here $u = [(u_\varepsilon)_\varepsilon]$ and $v = [(v_\varepsilon)_\varepsilon]$.

2.2 Compatibility

If $a \in H^2(\mathbb{R}^3)$, then there exists a unique solution $u \in C([0,T), H^2(\mathbb{R}^3))$ of the cubic equation (1). We proved that the space $H^2(\mathbb{R}^3)$ is embedded in the Colombeau algebra $\mathcal{G}_{H^2}(\mathbb{R}^3)$ (Theorem 2). If there is a unique solution of (1) in $\mathcal{G}_{C^1,H^2}([0,T)\times \mathbb{R}^3)$, then there is its representative $(u_\varepsilon)_\varepsilon$ which solves

$$\begin{aligned} i(u_\varepsilon)_t + \Delta u_\varepsilon &= u_\varepsilon|u_\varepsilon|^2, \\ u_\varepsilon(0) &= a * \rho_\varepsilon, \end{aligned}$$

for $a \in H^2(\mathbb{R}^3)$ (we show that there is a solution to the equation without negligible functions, so the above claim is justified). Classes $[(u_\varepsilon)_\varepsilon]$ and $[(u * \rho_\varepsilon)_\varepsilon]$ may coincide but in general we can prove a weaker version of this equality of classes, which we give in the next definition (see [19], p. 47).

Definition 6 We say that $u \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ is associated with a distribution $v(t) \in \mathcal{D}'(\mathbb{R}^3)$ for any $t \in [0,T)$ if there is a representative $(u_\varepsilon)_\varepsilon$ of u such that $u_\varepsilon \to v$ in $\mathcal{D}'(\mathbb{R}^3)$ for any $t \in [0,T)$ as $\varepsilon \to 0$. We denote association by $u \approx v$.

However, we are sometimes able to prove $\|u - u_\varepsilon\|_2 \to 0,\ \varepsilon \to 0$, for every $t \in [0,T)$ and this implies $[(u_\varepsilon)_\varepsilon] \approx u$. Therefore we introduce the following definition.

Definition 7 We say that there is a compatibility between a classical (Sobolev) solution and the Colombeau solution of the equation

$$
\begin{aligned}
iu_t + \triangle u + g(u) &= 0,\\
u(0) &= a,
\end{aligned}
$$

if $\sup_{[0,T)} \|u_\varepsilon - u\|_2 \to 0$ as $\varepsilon \to 0$, where $(u_\varepsilon)_\varepsilon \in \mathcal{E}_{C^1,H^2}$ is a solution of

$$
\begin{aligned}
i(u_\varepsilon)_t + \triangle u_\varepsilon + g(u_\varepsilon) &= 0\\
u_\varepsilon(0) &= a * \rho_\varepsilon.
\end{aligned}
$$

This definition does not depend on representatives. If $u_\varepsilon \to u$ in L^2 and v_ε is another representative, then

$$
\|v_\varepsilon - u\|_2 \le \|v_\varepsilon - u_\varepsilon\|_2 + \|u_\varepsilon - u\|_2 \to 0,\ \ \varepsilon \to 0.
$$

Note that if $a \in C^1([0,T), H^\infty)$, then a represents itself and the same holds for the corresponding solution $u \in C^1([0,T), H^\infty)$. Hence in this case we automatically have compatibility between two solutions.

Looking outside the context of equivalence classes, estimates that we derive can be useful for discussing different types of convergences. For instance, there is no classical well-posedness theory for (2), but we can analyze the net of solutions and get some insights in that direction. Uniqueness in our setting also differs from the usual notion of uniqueness. Because we define an L^2-type of uniqueness, it can happen that the solution is unique but there are different classes $u,\ v \in \mathcal{G}_{C^1,H^2}$ that solve the equation. Again, if there is convergence in L^2 of representatives of u, then representatives of v also converge to the same limit. Hence notions of compatibility and uniqueness complement each other.

We state a few examples in which notions of compatibility and association were used. In [21], Hörmann showed that there is a unique generalized solution to the linear Schrödinger equation with generalized coefficients and also that this solution is associated with the corresponding distributional solution.

In [3], the generalized solution of the Korteweg-de Vries equation is considered and an interesting result is observed. Namely, for classical initial data, the distribution associated with the generalized solution is not a weak solution of the equation.

Burger's equation is studied in [4] and association of the generalized solution with a classical entropy solution is shown. In [26], hyperbolic conservation laws are considered in the Colombeau framework and the authors prove that the generalized solution is associated with the weak entropic solution.

In some cases (such as ours), it is possible to prove more than association. For example, in [23], the H^2 convergence of regularized solutions is shown. In [24], weak L^2-convergence of the net of solutions is proven.

Non-uniqueness and instability are potential problems in analysis of distributional solutions (cf. [9, 20]). This is another reason to emphasize the importance of compatibility.

3 Existence and Uniqueness of a Singular Solution

We consider a regularized equation of type

$$\begin{aligned} i(u_\varepsilon)_t + \triangle u_\varepsilon &= N(u_\varepsilon), \\ u_\varepsilon(0) &= a_\varepsilon, \end{aligned}$$

where $(a_\varepsilon)_\varepsilon$ is a representative of $a \in \mathcal{G}_{H^2}(\mathbb{R}^3)$ and N is the given nonlinearity. Existence of a unique solution for fixed ε follows from the classical theory of Sobolev solutions. The main ingredient for existence in Colombeau algebra is deriving the estimates of the type

$$\|u_\varepsilon\|_{H^2} \lesssim f(\|a_\varepsilon\|_{H^2}),$$

for any $t \in [0, T)$, since then $(u_\varepsilon)_\varepsilon$ defines an element of $\mathcal{G}_{C^1,H^2}([0, T) \times \mathbb{R}^3)$ and the Definition 4 is satisfied. Note that appropriate bounds for $\|(u_\varepsilon)_t\|_2$ are easily obtained from the equation itself.

Estimates for $\|u_\varepsilon\|_{H^1}$ follow from the conservation of energy and the main difficulty is to bound second order derivatives in the L^2 norm. Besides moderate growth, a usually needs to satisfy additional logarithmic bounds, as we will see in the sequel.

For the simpler cubic equation without potential we can claim existence in the Colombeau sense in dimensions $n = 2$ and $n = 3$.

Theorem 6 *Let* $n \in \{2, 3\}$, $T > 0$, $a \in \mathcal{G}_{H^2}(\mathbb{R}^n)$ *such that there exists a representative* $(a_\varepsilon)_\varepsilon$ *which satisfies condition*

$$\|a_\varepsilon\|_{H^2} \leq h_\varepsilon \tag{9}$$

with $h_\varepsilon \sim \varepsilon^{-N}$ *for* $n = 2$ *and* $h_\varepsilon \sim N \ln \varepsilon^{-1}$ *for* $n = 3$, *for some* $N \in \mathbb{N}$. *Then there exists a solution* $u \in \mathcal{G}_{C^1,H^2}([0, T) \times \mathbb{R}^n)$ *of* (1).

The cubic equation in 3D satisfies a growth estimate proved in [5]:

$$\|u_\varepsilon(t)\|_{H^2} \leq c \exp(\|a_\varepsilon\|_{H^2}) \quad \forall t \geq 0. \tag{10}$$

In [10] it was shown that there is also an estimate of the type:

$$\|u_\varepsilon(t)\|_{H^2} \leq c\|a_\varepsilon\|_{H^2} \quad \forall t \geq 0. \tag{11}$$

Therefore proof of Theorem 6 follows from these bounds.

Proving an analogous theorem for Eq. (2) required deriving new estimates. This leads us to different conditions for initial data, presented in the following theorem.

Theorem 7 *Let* $a \in \mathcal{G}_{H^2}(\mathbb{R}^3)$ *such that there exists a representative* $(a_\varepsilon)_\varepsilon$ *which satisfies the following:*

$$\|a_\varepsilon\|_{H^3} = \mathcal{O}(\varepsilon^{-N}), \quad \text{and} \quad \|a_\varepsilon\|_{H^1} = \mathcal{O}(h_\varepsilon) \quad for\ some\ N \in \mathbb{N},\ \varepsilon \to 0, \tag{12}$$

where $h_\varepsilon \sim (\ln \varepsilon^{-1})^{\frac{5}{11}}$. *Then for any* $T > 0$ *there exists a generalized solution* $u \in \mathcal{G}_{C^1,H^2}([0, T) \times \mathbb{R}^3)$ *of* (2).

We will describe the main ingredients of the proof. In this case, we need estimates for the following regularized equation:

$$\begin{aligned} i(u_\varepsilon)_t + \triangle u_\varepsilon &= \phi_{h_\varepsilon} u_\varepsilon + u_\varepsilon |u_\varepsilon|^2, \\ u_\varepsilon(0) &= a_\varepsilon. \end{aligned} \tag{13}$$

For simplicity, we regularize the delta function with the same h_ε used to bound the initial condition. Denote by $\mathcal{T}(t)$ the usual Schrödinger evolution operator which satisfies an estimate:

$$\|\mathcal{T}(t)\phi\|_{L^p} \leq (4\pi|t|)^{-n(\frac{1}{2}-\frac{1}{p})} \|\phi\|_{L^{p'}}, \quad \forall \phi \in L^{p'}. \tag{14}$$

The solution of (13) is given by Duhamel's formula:

$$u_\varepsilon(t) = \mathcal{T}(t)a_\varepsilon - i \int_0^t \mathcal{T}(t-s) \left(\phi_{h_\varepsilon} u_\varepsilon + u_\varepsilon |u_\varepsilon|^2\right) ds. \tag{15}$$

Estimates that we need can be described with the following steps.

- Differentiate (3)—take the second order derivative in x and apply the L^2 norm; the main expression to bound after this is the following:

$$\|u_\varepsilon^2 \partial^\alpha u_\varepsilon + u_\varepsilon \partial^\beta u_\varepsilon \partial^\gamma u_\varepsilon\|_2 + \|\phi_{h_\varepsilon} \partial^\alpha u_\varepsilon + \partial^\beta \phi_{h_\varepsilon} \partial^\gamma u_\varepsilon + \partial^\alpha \phi_{h_\varepsilon} u_\varepsilon\|_2,$$

  where $|\alpha| = 2$ and $|\beta| = |\gamma| = 1$.
- We are able to bound each term by a product of a known quantity ($\|\phi_{h_\varepsilon}\|_p$, $\|u_\varepsilon\|_2$ or $\|u_\varepsilon\|_{H^1}$) and $\|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}}$; for this we used Hölder and Gagliardo–Nirenberg inequality.
- Moreover, we bound $\|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}}$ by $\|\partial^\alpha u_\varepsilon\|_{\frac{10}{7}}$ using the estimate (14) and further bound the $L^{\frac{10}{7}}$ norm with the $L^{\frac{10}{3}}$ norm by Hölder and Gagliardo–Nirenberg inequality.
- In this way, we are able to use Gronwall's inequality and bound $\|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}}$ and by that $\|\partial^\alpha u_\varepsilon\|_2$ also.

The resulting estimate is exponential in $\|a_\varepsilon\|_{H^1}$ but not in higher norms of a_ε. Specifically,

$$\begin{aligned}\sup_{[0,T)} \|\partial^\alpha u_\varepsilon\|_2 \leq & \|a_\varepsilon\|_{H^2} + g_\varepsilon f_\varepsilon^{\frac{3}{2}} \|a_\varepsilon\|_2^{\frac{1}{2}} + H(a_\varepsilon)^{\frac{1}{2}} g_\varepsilon^{\frac{20}{13}} \|a_\varepsilon\|_2^{\frac{6}{13}} \\ & + \|\partial^\alpha \phi_{h_\varepsilon}\|_\infty \|a_\varepsilon\|_2 + H(a_\varepsilon)^{\frac{1}{2}} \|\partial^\beta \phi_{h_\varepsilon}\|_\infty + g_\varepsilon \|\phi_{h_\varepsilon}\|_5, \end{aligned} \tag{16}$$

where

$$\begin{aligned} f_\varepsilon &= c_1(a_\varepsilon, \phi_{h_\varepsilon}) \cdot \exp(c_2(a_\varepsilon, \phi_{h_\varepsilon})), \\ c_1(a_\varepsilon, \phi_{h_\varepsilon}) &= \|a_\varepsilon\|_{H^2} + T^{\frac{2}{5}}(\|a_\varepsilon\|_2 \|\partial^\gamma \phi_{h_\varepsilon}\|_5 + H(a_\varepsilon)^{\frac{1}{2}} \|\phi_{h_\varepsilon}\|_5), \\ c_2(a_\varepsilon, \phi_{h_\varepsilon}) &= T^{\frac{2}{5}} H(a_\varepsilon)^{\frac{1}{10}} \|a_\varepsilon\|_2^{\frac{9}{5}}; \\ g_\varepsilon &= (\|a_\varepsilon\|_{H^3} + c_4(a_\varepsilon, \phi_{h_\varepsilon})) \exp(c_3(a_\varepsilon, \phi_{h_\varepsilon}) \cdot T^{\frac{2}{5}}), \\ c_3(a_\varepsilon, \phi_{h_\varepsilon}) &= H(a_\varepsilon)^{\frac{1}{10}} \|a_\varepsilon\|_2^{\frac{9}{5}} + \|\phi_{h_\varepsilon}\|_{\frac{5}{2}}, \\ c_4(a_\varepsilon, \phi_{h_\varepsilon}) &= H(a_\varepsilon)^{\frac{1}{2}} \|\partial^\beta \phi_{h_\varepsilon}\|_5 + \|a_\varepsilon\|_2 \|\partial^\alpha \phi_{h_\varepsilon}\|_5 + \|a_\varepsilon\|_2^{\frac{1}{2}} f_\varepsilon^{\frac{7}{2}} \end{aligned}$$

and $H(\cdot)$ is the Hamiltonian.

Let us now turn to uniqueness of a solution in the sense of Definition 5. We assume that there is another solution $v \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$. Then, a representative v_ε of v solves

$$\begin{aligned} i(v_\varepsilon)_t + \triangle v_\varepsilon &= N(v_\varepsilon) + n_\varepsilon, \\ v_\varepsilon(0) &= a_\varepsilon + m_\varepsilon, \end{aligned} \tag{17}$$

where N is again the given nonlinearity, $n_\varepsilon \in \mathcal{N}_{C,L^2}$ and $m_\varepsilon \in \mathcal{N}_{H^2}$. Then the difference $w_\varepsilon = u_\varepsilon - v_\varepsilon$ satisfies an appropriate equation from which we derive the following estimate

$$\|w_\varepsilon\|_2 \lesssim \varepsilon^M \exp(\|u_\varepsilon\|_\infty^2 + \|u_\varepsilon\|_\infty \|v_\varepsilon\|_\infty),$$

for any $M \in \mathbb{N}$ and any $t \in [0, T)$. This estimate is obtained by energy methods and it holds for both (1) and (2).

To complete the proof, we need to control the infinity norm of a solution by an appropriate H^2 norm of the initial condition. If we use the Sobolev embedding, we see that we have already achieved this for the solution of (13), but v_ε solves a slightly more complicated (inhomogeneous) Eq. (17). For this reason, we have to derive analogous estimates for $\|v_\varepsilon\|_{H^2}$ and also to ask for a more strict condition on n_ε. This leads us to a modified version of uniqueness.

Definition 8 Let $u, v \in \mathcal{G}_{C^1,H^2}(\mathbb{R}^3)$ be two classes such that for each class there exists a representative that solves

$$\begin{aligned} i\partial_t u_\varepsilon + \triangle u_\varepsilon &= N(u_\varepsilon) + n_\varepsilon, \\ u_\varepsilon(0) &= a_\varepsilon + m_\varepsilon, \end{aligned} \tag{18}$$

where $n_\varepsilon \in \mathcal{N}_{C^1,H^2}([0, T) \times \mathbb{R}^3)$ and $m_\varepsilon \in \mathcal{N}_{H^2}(\mathbb{R}^3)$ (similarly for v). If $\sup_{[0,T)} \|u_\varepsilon - v_\varepsilon\|_2 = \mathcal{O}(\varepsilon^M)$, $\varepsilon \to 0$ for any $M \in \mathbb{N}$, then we say that the solution is unique.

Now we can formulate the following theorem.

Theorem 8 *If $\|a_\varepsilon\|_{H^3} \sim \ln^s \ln^q \varepsilon^{-1}$, where $s = \frac{5}{7}$, $q = \frac{1}{24}$, the solution of (1) is unique in the sense of Definition 8. If $\|a_\varepsilon\|_{H^3} \sim \ln^s \ln^q \varepsilon^{-1}$, where $s = \frac{7}{25}$, $q = \frac{1}{500}$ the solution of (2) is unique in the sense of Definition 8.*

As mentioned, the proofs are now essentially the same for both equations, but with estimates for $\|v_\varepsilon\|_{H^2}$ being slightly different depending on the equation.

4 Convergence Properties

We prove compatibility for the cubic equation (1) (notion presented in Sect. 2). If $a \in H^2(\mathbb{R}^3)$ there is a unique solution $u \in H^2(\mathbb{R}^3)$. Such function a can be embedded in $\mathcal{G}_{H^2}(\mathbb{R}^3)$ by convolution with a mollifier. For an appropriate mollifier ρ_ε, the norm $\|a * \rho_\varepsilon\|_{H^3}$ satisfies all the necessary estimates of Theorems 6 and 8. Hence for $a \in H^2(\mathbb{R}^3)$ there is a unique solution $[(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{C^1,H^2}(\mathbb{R}^3)$. We already used that the cubic equation satisfies an estimate proved by Bourgain which applied to the regularized equation is in the form

$$\|u_\varepsilon(t)\|_{H^2} \leq c\exp(\|a * \rho_\varepsilon\|_{H^2}) \quad \forall t \geq 0. \tag{19}$$

The expression $\|a * \rho_\varepsilon\|_{H^2}$ is bounded uniformly in ε due to Young's inequality. Using this fact, energy methods and Gronwall inequality, we prove that

$$\|u_\varepsilon - u\|_2 \to 0$$

and hence the Sobolev and the Colombeau solution are compatible.

Regarding Eq. (2), some possible future directions are to compare our approach with other settings, like the one given in [17] where the authors consider solutions in weak L^p spaces. Also, we would like to consider the Hartree equation

$$\begin{aligned} iu_t + \triangle u &= (w * |u|^2)u + \delta u, \\ u(0) &= a, \end{aligned} \tag{20}$$

in the Colombeau setting. More precisely it would be interesting to study NLS (nonlinear Schrödinger equations) in which the linear part is characterized by a Schrödinger operator with point-interaction. These operators provide an alternative way to model a zero-range potential. They are well-studied by means of classical techniques (see, e.g., [2]), and also within the framework of generalized functions (cf. [27]). The associated nonlinear problem has recently attracted attention—see, e.g., [1, 7, 22] and [18]. In these papers NLS with point interactions have been analyzed by classical techniques, and it would be interesting to exploit also the Colombeau approach based on generalized functions.

Acknowledgments The authors acknowledge the financial support of the Ministry of Education, Science and Technological Development of the Republic of Serbia (Grant No. 451-03-68/2022-14/200125). The second author acknowledges the financial support of the Croatian Science Foundation under project 2449 MiTPDE.

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