Almost Sure Pointwise Convergence of the Cubic Nonlinear Schrödinger Equation on $\mathbb{T}^2$

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Abstract We revisit a result from "Pointwise convergence of the Schrödinger flow, E. Compaan, R. Lucà, G. Staffilani, International Mathematics Research Notices, 2021 (1), 596–647" regarding the pointwise convergence of solutions to the periodic cubic nonlinear Schrödinger equation in dimension $d = 2$.

1 Introduction

We consider the question of pointwise almost everywhere (a.e.) convergence of solutions to the cubic nonlinear Schrödinger equation (NLS) on $\mathbb{T}^2$, namely

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^2 u, \\ u(x,0) \quad\quad = f(x). \end{cases} \tag{1}$$

If $f \in H^s$, for what s do we have that $u(x,t) \to f(x)$ as $t \to 0$ for (Lebesgue) almost every x?

In the linear Euclidean setting, namely when the linear Schrödinger equation equation is posed on $\mathbb{R}^d$, this question was first posed by Carleson [8]. He proved Lebesgue (a.e.) convergence $e^{it\Delta} f(x)$ to $f(x)$ for $f \in H^s(\mathbb{R})$ with $s \leq \frac{1}{4}$. Dahlberg–Kenig [11] showed that this one-dimensional result is sharp, proving the necessity of the regularity condition $s \geq \frac{1}{4}$ in any dimension. The (considerably more difficult) higher dimensional problem has been studied by many authors [1, 4, 10, 12, 16, 20, 22–24, 26, 28, 29, 31, 32, 34]. Recently, Bourgain [5] proved that $s \geq \frac{d}{2(d+1)}$ is necessary (see also [21, 24] for some refinements of this result). This has been proved to be sharp, up to the endpoint, by Du–Guth–Li [15] on $\mathbb{R}^2$ and by

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Du–Zhang [14] in higher dimensions (the endpoint case is still open in dimensions $d \geq 2$).

In the periodic case, much less is known. When $d = 1$, Moyua–Vega [27] proved the sufficiency of $s > \frac{1}{3}$ and necessity of $s \geq \frac{1}{4}$. Their proof, based on Strichartz estimates, has been extended to dimension $d = 2$ in [35] and to higher dimension in [9]. In fact, together with recent improvements in periodic Strichartz estimates [6], one can show that $s > \frac{d}{d+2}$ is a sufficient condition for almost everywhere convergence to initial data. On the other hand, there are several counterexamples showing that we have the same necessary conditions than that on $\mathbb{R}^d$ [9, 17, 27], namely the necessity of $s \geq \frac{d}{2(d+1)}$; in particular, one can "adapt" the counterexamples from $\mathbb{R}^d$ to the periodic setting. At the moment, in the periodic case, almost sure convergence when $s \in \left[\frac{d}{2(d+1)}, \frac{d}{d+2}\right]$ remains an open question.

In the first part of this chapter, we show how to extend the a.e. convergence statement

$$\lim_{t \to 0} e^{it\Delta} f(x) = f(x), \qquad \text{for a.e. } x \in \mathbb{T}^2 \text{ and for all } f \in H^s(\mathbb{T}^2), s > 1/2 \tag{2}$$

to the case of the cubic equation. The following is a special case of Theorem 1.1 in [9].

Theorem 1 *If $f \in H^s(\mathbb{T}^2)$ with $s > 1/2$ and u is the corresponding solution to (1), then*

$$\lim_{t \to 0} u(x,t) = f(x) \quad \textit{for a.e. } x \in \mathbb{T}^2 . \tag{3}$$

Remark 1 By the proof, it will be clear that any improvement of the amount of Sobolev regularity that is sufficient for the convergence of the linear Schrödinger flow on $\mathbb{T}^2$ would imply an analogous improvement in the statement of Theorem 1 as well.

In the second part of this chapter, we consider probabilistic improvements to the convergence problem. More precisely, we will show that a randomization of the Fourier coefficients of the initial data guarantees a better pointwise behavior of the associated linear (and also nonlinear) evolution. To explain why we may expect this, it is worth mentioning that counterexamples to the linear pointwise convergence problem in the periodic setting have been constructed in [17] using as building block for the initial datum the tensor product of Dirichlet kernels

$$\prod_{\ell=1,\dots,d} \sum_{k_\ell \in \mathbb{Z},\, |k_\ell| \leq N} e^{ik_\ell \cdot x_\ell}, \qquad x := (x_1, \dots, x_d), \tag{4}$$

where $N \gg 1$ is a large frequency parameter. It is wort recalling that the pointwise convergence problem is essentially[1] equivalent to establish an $L^2(\mathbb{T}^2)$ estimate for the maximal Schrödinger operator

$$\left\| \sup_{t\in[0,1]} |e^{it\Delta} f| \right\|_{L^2(\mathbb{T}^2)} \lesssim \|f\|_{H^s(\mathbb{T}^2)}. \tag{5}$$

It has been observed in [17, 27] that (5) behaves particularly bad with data of the form (4). It is in fact seen to be false for $s < \frac{n}{2(n+1)}$, taking $N \to \infty$. The moral is that if the bad counterexamples are characterized by having a very rigid structure: the Fourier coefficients in (4) are indeed all equal to 1. This suggest to consider as "good" initial data the following randomized Fourier series

$$f^\omega(x) = \sum_{n\in\mathbb{Z}^d} \frac{g_n^\omega}{\langle n\rangle^{\frac{d}{2}+\alpha}} e^{in\cdot x}, \qquad \alpha > 0, \tag{6}$$

where g_n^ω are independent (complex) standard Gaussian variables. The Japanese brackets are defined as usual as $\langle\cdot\rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$.

It is easy to see that if we fix $t \in \mathbb{R}$, then $e^{it\Delta} f^\omega(x)$ belongs to $\bigcap_{s<\alpha} H^s(\mathbb{T}^d)$ $\mathbb{P}$-almost surely (a.s.), where $\mathbb{P}$ is the probability measure induced by the sequence $\{g_n^\omega\}_{n\in\mathbb{Z}}$. Thus we are working at the $H^{\alpha-}$ level. In fact, more is true, namely that $e^{it\Delta} f^\omega(x)$ belongs to $\bigcap_{s<\alpha} C^s(\mathbb{T}^d)$, $\mathbb{P}$-a.s.; in particular, $e^{it\Delta} f^\omega$ is $\mathbb{P}$-a. s. a continuous function of the x variable. On the other hand, the randomization does not improve the regularity, in the sense that $\|f^\omega\|_{H^\alpha(\mathbb{T}^d)} = \infty$ also holds $\mathbb{P}$-a. s.; see for example Remark 1.2 in [7] and the introduction of [25].

We have the following improved pointwise (a.e.) convergence result for randomized initial data. The following is the first part of Theorem 1.3 in [9].

Proposition 1 *Let $\alpha > 0$, and let f^ω of the form (6). We have $\mathbb{P}$-a. s. the following. For all $t \in \mathbb{R}$, the free solution $e^{it\Delta} f^\omega$ belongs to $\bigcap_{s<\alpha} C^s(\mathbb{T}^d)$ and*

$$e^{it\Delta} f^\omega(x) \to f^\omega(x) \quad \text{as } t \to 0$$

for every $x \in \mathbb{T}^d$ and uniformly.

[1] It is indeed easy to check that the maximal estimate (5) with $s > 1/2$ implies (2) (the argument is the same as used in the proof of Proposition 2). The opposite implication requires the Stein maximal principle. Strictly speaking, there is an equivalence with a weak L^2 estimate. On the other hand, the weak L^2 estimate can be easily promoted to a strong one with an ε-regularity loss. Thus, since we are not interested in endpoint results, we see that (2) and (5) are indeed equivalent.

Finally, we want to prove a similar statement for the cubic NLS (1). In fact, it will be more convenient working with the Wick-ordered version of the equation (WNLS)

$$\begin{cases} i\partial_t u + \Delta u = \mathcal{N}(u), \\ u(x,0) \quad\;\; = f(x), \end{cases} \tag{7}$$

where

$$\mathcal{N}(u) := \pm u\left(|u|^2 - 2\mu\right), \qquad \mu := \fint_{\mathbb{T}^2} |u(x,t)|^2 dx = \fint_{\mathbb{T}^2} |f(x)|^2 dx \tag{8}$$

(recall that μ is a conserved quantity). Since solutions to WNLS are related to that of the cubic NLS by multiplication with a factor $e^{i2\mu t}$, the study of pointwise convergence turns out to be equivalent to that of NLS. The following is the second part of Theorem 1.3 in [9].

Theorem 2 *Let $d = 2$, $\alpha > 0$, and let f^ω of the form (6). Let u be the solution to WNLS (7) with initial data f^ω. We have $\mathbb{P}$-almost surely:*

$$\lim_{t\to 0} u(x,t) = f^\omega(x) \quad \textit{for a.e. } x \in \mathbb{T}^2 . \tag{9}$$

Thus the same is true for solutions to the cubic NLS.

1.1 Notations and Terminology

For a fixed $p \in \mathbb{R}$, we often use the notation $p+ := p + \varepsilon$, $p- := p - \varepsilon$, where ε is any sufficiently small strictly positive real number. When in the same inequality we have two such quantities, we use the following notation to compare them. We write $p + \cdots + := p + \varepsilon \cdot$ (number of $+$), $p - \cdots - := p - \varepsilon \cdot$ (number of $-$). We will use $C > 0$ to denote several constants depending only on fixed parameters, like for instance the dimension d. The value of C may clearly differ from line to line. Let $A, B > 0$. We may write $A \lesssim B$ if $A \leq CB$ when $C > 0$ is such a constant. We write $A \gtrsim B$ if $B \lesssim A$ and $A \sim B$ when $A \lesssim B$ and $A \gtrsim B$. We write $A \ll B$ if $A \leq cB$ for $c > 0$ sufficiently small (and depending only on fixed parameters) and $A \gg B$ if $B \ll A$. We denote $A \wedge B := \min(A, B)$ and $A \vee B := \max(A, B)$. We refer to the following inequality:

$$\|D^s P_N f\|_{L^q} \lesssim N^{s+\frac{d}{p}-\frac{d}{q}} \|P_N f\|_{L^p}, \quad 1 \leq p \leq q \leq \infty ,$$

simply as Bernstein inequality. Here, P_N is the frequency projection on the annulus $\xi \sim N$.

It is useful to recall that the Strichartz estimates are the main tool to study the nonlinear Schrödinger flow. We recall the periodic Strichartz estimates from [2, 6]:

$$\|e^{it\Delta} P_N f\|_{L^p_{x,t}(\Omega^{d+1})} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}+} \|P_N f\|_{L^2_x(\Omega^d)}, \quad p \geq 2\left(\frac{d+2}{d}\right). \tag{10}$$

2 Proof of Theorem 1

Recall that the flow of (1) is locally well defined for initial data in $f \in H^s(\mathbb{T}^2)$ for $s > 0$ [2]. The solutions are constructed via a fixed-point argument in the restriction space $X^{s,b}_\delta$ for $\delta > 0$ sufficiently small (depending polynomially on the $H^s(\mathbb{T}^2)$ norm of f). We recall that

$$\|F\|_{X^{s,b}_\delta} := \inf_{G=F \text{ on } t\in[0,\delta]} \|G\|_{X^{s,b}},$$

where

$$\|F\|^2_{X^{s,b}} := \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}^d} \langle \tau + |n|^2\rangle^{2b} \langle n\rangle^{2s} |\widehat{F}(n,\tau)|^2 d\tau$$

and $\widehat{F}$ is the space–time Fourier transform of F.

Let Φ^N_t be the flow associated to the truncated NLS equation

$$i\partial_t \Phi^N_t f + \Delta \Phi^N_t f = \mathrm{P}_{\leq N}\, \mathcal{N}(\Phi^N_t f), \tag{11}$$

with initial datum $\Phi^N_0 f := \mathrm{P}_{\leq N} f$. We denote $\mathrm{P}_{\leq N}$ the frequency projection on the ball of radius N centered in the origin. We write $\Phi_t f := \Phi^\infty_t f$ for the flow of the NLS equation with initial datum $f = \mathrm{P}_\infty f$. We also denote $\mathrm{P}_{>N} := \mathrm{P}_\infty - \mathrm{P}_{\leq N}$ and as already mentioned $\mathrm{P}_N := \mathrm{P}_{\leq N} - \mathrm{P}_{\leq N/2}$.

A similar well-posedness result holds for the truncated flow, uniformly in $N \in \mathbb{N}$. Of course, at fixed N, since Eq. (11) is finite-dimensional, one can construct a global flow in an elementary way using the Cauchy theorem for ODE and the conservation of $\|\Phi^N_t f\|_{L^2(\mathbb{T}^2)}$ (which holds for all $N \in \mathbb{N}$). However, in the following, we will need (as usual in the study of NLS) a control of $\Phi^N_t f$ uniform over N. This is not elementary and will be ensured by the local well-posedness theory in the restriction space.

As already recalled, the main tool in the study of the pointwise convergence properties of the linear Schrödinger equation is the maximal Schrödinger operator

$$t \to \sup_{0\leq t\leq\delta} |e^{it\Delta} f(x)|, \qquad \delta > 0.$$

Assume indeed that for some $\delta \in (0, 1]$, one has

$$\left\| \sup_{0\le t\le\delta} |e^{it\Delta} f(x)| \right\|_{L^2_x(\mathbb{T}^2)} \lesssim \|f\|_{H^s_x(\mathbb{T}^2)}, \tag{12}$$

and then it is not hard to see that $e^{it\Delta} f(x) \to f(x)$ as $t \to 0$ for almost every (with respect to the Lebesgue measure) $x \in \mathbb{T}^2$. The proof is a straightforward modification of the argument that we will use to prove Proposition 2 below.

In the nonlinear setting, we need a (nonlinear) replacement of (12). A convenient replacement is the maximal estimate (13).

Proposition 2 *Let $f \in L^2(\mathbb{T}^2)$ be such that*

$$\lim_{N\to\infty} \left\| \sup_{0\le t\le\delta} |\Phi_t f(x) - \Phi_t^N f(x)| \right\|_{L^2_x(\mathbb{T}^2)} = 0. \tag{13}$$

Then $\Phi_t f(x) \to f(x)$ as $t \to 0$ for almost every $x \in \mathbb{T}^2$.

From the proof, it will be clear that in (13) we can replace the L^2 norm with a weak L^1 norm. However, it is usually convenient to work in the L^2 setting.

Proof To prove Proposition 2, we decompose the difference as follows:

$$|\Phi_t f(x) - f(x)| \le |\Phi_t f(x) - \Phi_t^N f(x)| + |\Phi_t^N f(x) - \mathrm{P}_{\le N} f(x)| + |\mathrm{P}_{>N} f(x)| \tag{14}$$

and pass to the limit $t \to 0$. It is elementary to show that the second term on the right-hand side is zero, namely

$$\lim_{t\to 0} \Phi_t^N f(x) = \mathrm{P}_{\le N} f(x),$$

for all $x \in \mathbb{T}^2$. So we arrive at[2]

$$\limsup_{t\to 0} |\Phi_t f - f| \le \limsup_{t\to 0} |\Phi_t f - \Phi_t^N f| + |\mathrm{P}_{>N} f|.$$

[2] Hereafter, we remove the x variable in the argument of decompositions such as (14) to simplify the notation.

Let $\lambda > 0$. Using the Chebyshev inequality,

$$|\{x \in \mathbb{T}^2 : \limsup_{t\to 0} |\Phi_t f - f| > \lambda\}| \le |\{x \in \mathbb{T}^2 : \sup_{0\le t\le\delta} |\Phi_t f - \Phi_t^N f| > \lambda/2\}| + |\{x \in \mathbb{T}^2 : |\mathrm{P}_{>N} f| > \lambda/2\}| \lesssim \lambda^{-2}\left(\left\|\sup_{0\le t\le\delta} |\Phi_t f - \Phi_t^N f|\right\|^2_{L^2(\mathbb{T}^2)} + \|\mathrm{P}_{>N} f\|^2_{L^2(\mathbb{T}^2)}\right),$$

where $|\cdot|$ is the Lebesgue measure. On the other hand, we have $\|\mathrm{P}_{>N} f\|_{L^2(\mathbb{T}^2)} \to 0$ as $N \to \infty$ (since $f \in L^2(\mathbb{T}^2)$) and

$$\lim_{N\to\infty} \left\|\sup_{0\le t\le\delta} |\Phi_t f - \Phi_t^N f|\right\|_{L^2(\mathbb{T}^2)} = 0$$

by assumption (13). Thus we arrive to

$$|\{x \in \mathbb{T}^2 : \limsup_{t\to 0} |\Phi_t f - f| > \lambda\}| = 0,$$

and the statement follows taking the union over $\lambda > 0$. □

It is not easy to verify the condition (13) directly. However, we can take advantage of a simple lemma that allows to embed a suitable restriction space into the relevant maximal space, namely the space induced by the norm

$$\left\|\sup_{t\in[0,\delta]} |F(x,t)|\right\|_{L^2_x(\mathbb{T}^2)}, \qquad F : (x,t) \in \mathbb{T}^2 \times \mathbb{R} \to F(x,t) \in \mathbb{C}.$$

In other words, we can bound the $L^2_x(\mathbb{T}^2)$ norm of the associated maximal function

$$x \to \sup_{0\le t\le\delta} |F(x,t)|$$

with an appropriate $X^{s,b}_\delta$ norm of F. In fact, this is a rather general property of the restriction spaces $X^{s,b}_\delta$ with $b > \frac{1}{2}$. The proof can be found in [30, Lemma 2.9], in the non-periodic case. The argument adapts to the periodic case as well.

Lemma 1 *Let $b > \frac{1}{2}$, and let Y be a Banach space of functions*

$$F : (x,t) \in \Omega^d \times \mathbb{R} \to F(x,t) \in \mathbb{C}.$$

Let $\alpha \in \mathbb{R}$. Assume

$$\|e^{i\alpha t} e^{it\Delta} f(x)\|_Y \le C\|f\|_{H^s(\Omega^d)}, \tag{15}$$

with a constant $C > 0$ uniform over $\alpha \in \mathbb{R}$. Then

$$\|F\|_Y \le C\|F\|_{X^{s,b}} .$$

Using Lemma 1 with

$$\|F\|_Y = \left\| \sup_{0\le t\le\delta} |F(x,t)| \right\|_{L^2_x(\mathbb{T}^2)}$$

and the fact that the maximal estimate (12) holds for $s > 1/2$, we have the following:

Lemma 2 *Let $b > \frac{1}{2}$ and $s > 1/2$. We have*

$$\left\| \sup_{0\le t\le\delta} |F(x,t)| \right\|_{L^2_x(\mathbb{T}^2)} \lesssim \|F\|_{X^{s,b}_\delta} . \tag{16}$$

We will combine the following lemma with the embedding from Lemma 2 to verify the maximal estimate hypothesis of Proposition 2 for the cubic NLS on $\mathbb{T}^2$.

Lemma 3 *Let $d = 2$ and $s > 0$. Then*

$$\|\mathcal{N}(u) - \mathcal{N}(v)\|_{X^{s,-\frac{1}{2}++}} \lesssim \left(\|u\|^2_{X^{s,\frac{1}{2}+}} + \|v\|^2_{X^{s,\frac{1}{2}+}} \right) \|u-v\|_{X^{s,\frac{1}{2}+}}. \tag{17}$$

In fact, Lemma 3 is a consequence of the following slightly more general statement (that will be useful later) due to Bourgain [3].

Lemma 4 *Let $d = 2$ and $s > 0$. Let $M_1 \ge M_2 \ge M_3$ be dyadic scales. Then*

$$\begin{aligned}\|(\mathrm{P}_{M_1} F)(\mathrm{P}_{M_2} G)(\mathrm{P}_{M_3} H)\|_{X^{s,-\frac{1}{2}++}} \\ \lesssim \|\mathrm{P}_{M_1} F\|_{X^{s,\frac{1}{2}+}} \|\mathrm{P}_{M_2} G\|_{X^{0+,\frac{1}{2}+}} \|\mathrm{P}_{M_3} H\|_{X^{0,\frac{1}{2}+}}.\end{aligned} \tag{18}$$

We denote $R_0 = \|f\|_{H^s(\mathbb{T}^2)}$. Hereafter η will be a smooth cut-off of $[0, 1]$. Taking $\delta = \delta(R_0) < 1$ sufficiently small and combining (25), (26), (27), and Lemma 3, one can show that the map

$$\Gamma(u(x,t)) = \eta(t)e^{it\Delta} \mathrm{P}_{\le N} f(x) - i\eta(t) \int_0^t e^{i(t-t')\Delta} \mathrm{P}_{\le N} \mathcal{N}(u(x,t'))dt' \tag{19}$$

is a contraction on the ball $\{u : \|u\|_{X^{s,\frac{1}{2}+}_\delta} \le 2R_0\}$, for all $N \in 2^{\mathbb{N}} \cup \{\infty\}$. This is a standard argument, so we omit the proof (see for instance [18, Section 3.5.1]).

Moreover, a similar computation is part of the proof of Theorem 1. However, we stress that the value of δ is uniform in $N \in 2^{\mathbb{N}} \cup \{\infty\}$. In particular, we have

$$\|\Phi_t^N f\|_{X_\delta^{s,\frac{1}{2}+}} \leq 2R_0, \qquad \text{for all } N \in 2^{\mathbb{N}} \cup \{\infty\}. \tag{20}$$

We are now ready to prove Theorem 1.

2.1 Proof of Theorem 1

By Lemma 2, we have

$$\left\| \sup_{0\leq t\leq\delta} |\Phi_t f(x) - \Phi_t^N f(x)| \right\|_{L_x^2(\mathbb{T}^2)} \lesssim \|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s,\frac{1}{2}+}}.$$

Thus using Proposition 2, it suffices to show that the right-hand side goes to zero as $N \to \infty$. For $t \in [0, \delta]$, we have (see (19))

$$\begin{aligned} &\Phi_t f(x) - \Phi_t^N f(x) \\ &= \eta(t)e^{it\Delta} \mathrm{P}_{>N} f(x) - i\eta(t) \int_0^t e^{i(t-t')\Delta} \Big(\mathcal{N}(\Phi_{t'} f(x)) - \mathrm{P}_{\leq N} \mathcal{N}(\Phi_{t'}^N f(x))\Big)\, dt'. \end{aligned}$$

Then using (25) and (26), we have

$$\|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s,\frac{1}{2}+}} \lesssim \|\mathrm{P}_{>N} f\|_{H^s(\mathbb{T}^2)} + \|\mathcal{N}(\Phi_t f) - \mathrm{P}_{\leq N} \mathcal{N}(\Phi_t^N f)\|_{X_\delta^{s,-\frac{1}{2}+}}. \tag{21}$$

To handle the nonlinear contribution, we further decompose

$$\mathcal{N}(\Phi_t f) - \mathrm{P}_{\leq N} \mathcal{N}(\Phi_t^N f) = \mathrm{P}_{\leq N} \Big(\mathcal{N}(\Phi_t f) - \mathcal{N}(\Phi_t^N f)\Big) + \mathrm{P}_{>N} \mathcal{N}(\Phi_t f)$$

so that

$$\begin{aligned} \|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s,\frac{1}{2}+}} \lesssim{} & \|\mathrm{P}_{>N} f\|_{H^s(\mathbb{T}^2)} + \|\mathrm{P}_{>N} \mathcal{N}(\Phi_t f)\|_{X_\delta^{s,-\frac{1}{2}+}} \\ & + \|\mathrm{P}_{\leq N} \Big(\mathcal{N}(\Phi_t f) - \mathcal{N}(\Phi_t^N f)\Big)\|_{X_\delta^{s,-\frac{1}{2}+}}. \end{aligned} \tag{22}$$

Then by (27), Lemma 3, and (20), we get

$$\|\mathrm{P}_{\leq N} \Big(\mathcal{N}(\Phi_t f) - \mathcal{N}(\Phi_t^N f)\Big)\|_{X_\delta^{s,-\frac{1}{2}+}} \lesssim \delta^{0+} R_0^2 \|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s,\frac{1}{2}+}}, \tag{23}$$

where we recall $R_0 = \|f\|_{H^s(\mathbb{T}^2)}$. Plugging (23) into (22), taking $\delta = \delta(R_0)$ small enough, and absorbing

$$\delta^{0+} R_0^2 \|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s,\frac{1}{2}+}} \leq \frac{1}{2}\|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s,\frac{1}{2}+}}$$

into the left-hand side, we arrive to

$$\|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s,\frac{1}{2}+}} \lesssim \|\mathrm{P}_{>N} f\|_{H^s(\mathbb{T}^2)} + \|\mathrm{P}_{>N} \mathcal{N}(\Phi_t f)\|_{X_\delta^{s,-\frac{1}{2}+}}. \tag{24}$$

The right-hand side of (24) goes to zero as $N \to \infty$ since $f \in H^s(\mathbb{T}^2)$ and $\mathcal{N}(\Phi_t f) \in X_\delta^{s,-\frac{1}{2}+}$; in fact, applying Lemma 3 with $v = 0$ and recalling (20), we have

$$\|\mathcal{N}(\Phi_t f)\|_{X_\delta^{s,-\frac{1}{2}+}} \lesssim \|\Phi_t f\|^3_{X_\delta^{s,\frac{1}{2}+}} \lesssim R_0^3 .$$

This concludes the proof of (3).

We conclude this section by recalling some well-known properties of restriction spaces that we have used (and that we will use in the rest of the paper). Recall that η is a smooth cut-off of the unit interval.

Lemma 5 *Let $s \in \mathbb{R}$. Then*

$$\|\eta(t)e^{it\Delta} f(x)\|_{X^{s,\frac{1}{2}+}} \lesssim \|f\|_{H^s(\Omega^d)} , \tag{25}$$

$$\left\|\eta(t)\int_0^t e^{i(t-t')\Delta} F(\cdot, t')dt'\right\|_{X^{s,\frac{1}{2}+}} \lesssim \|F\|_{X^{s,-\frac{1}{2}+}} , \tag{26}$$

$$\|F\|_{X_\delta^{s,-\frac{1}{2}+}} \lesssim \delta^{0+}\|F\|_{X_\delta^{s,-\frac{1}{2}++}} . \tag{27}$$

3 Proof of Proposition 1

Here we prove almost surely uniform convergence of the randomized Schrödinger flow to the initial datum, at the H^{0+} level, namely Proposition 1. Thus our goal is to show that $e^{it\Delta} f^\omega \to f^\omega$ as $t \to 0$ uniformly over $x \in \mathbb{T}^d$ and $\mathbb{P}$-almost surely for data f^ω defined as

$$f^\omega(x) = \sum_{n\in\mathbb{Z}^d} \frac{g_n^\omega}{\langle n\rangle^{\frac{d}{2}+\alpha}} e^{in\cdot x}, \qquad x \in \mathbb{T}^d , \tag{28}$$

where $\alpha > 0$ and each g_n^ω is complex and independently drawn from a standard normal distribution. In fact, the argument we present works for independent g_n^ω drawn from any distribution with sufficient decay of the tails (for instance, sub-Gaussian is enough). This will not be the case in Theorem 2, where we will need to take advantage of the hypercontractivity of (multilinear forms of) normal distributions. However, we only present the standard normal case for definiteness, also in this section.

Fix $t \in \mathbb{R}$. We have that $\mathbb{P}$-almost surely

$$e^{it\Delta} f^\omega \in \bigcap_{s<\alpha} H^s(\mathbb{T}^d).$$

This is an immediate consequence of (44) below, taking the union over $\varepsilon > 0$. In fact, for all $t \in \mathbb{R}$, we have $\mathbb{P}$-almost surely

$$e^{it\Delta} f^\omega \in \bigcap_{s<\alpha} C^s(\mathbb{T}^d);$$

thus in particular, $e^{it\Delta} f^\omega$ are $\mathbb{P}$-almost surely continuous functions of the x variable. This is a consequence of the higher integrability property (34) below, from which one can easily deduce uniform convergence as $N \to \infty$ of the sequence $P_{\leq N} f^\omega$, with probability larger than $1 - \varepsilon$. So the limit f^ω is continuous with the same probability, and the almost sure continuity follows taking the union over $\varepsilon > 0$.

Before completing the proof of Proposition 1, we recall few lemmata. We start recalling the following well-known concentration bound:

Lemma 6 ([7, Lemma 3.1]) *There exists a constant C such that*

$$\Big\| \sum_{n\in\mathbb{Z}^d} g_n^\omega\, a_n \Big\|_{L^r_\omega} \leq C r^{\frac{1}{2}} \|a_n\|_{\ell^2_n(\mathbb{Z}^d)} \tag{29}$$

for all $r \geq 2$ and $\{a_n\} \in \ell^2(\mathbb{Z}^d)$.

Using (29) with $a_n = e^{in\cdot x - i|n|^2 t} \langle n \rangle^{-\frac{d}{2}-\alpha}$, we obtain for $r \geq 2$ that for f^ω an in (28)

$$\| \mathrm{P}_N e^{it\Delta} f^\omega \|_{L^r_\omega} \leq C r^{\frac{1}{2}} N^{-\alpha}\,, \tag{30}$$

with a constant uniform in $t \in \mathbb{R}$. From this, we also have improved L^p_x estimates for randomized data.

Lemma 7 *Let $p \in [2, \infty)$. Assume f^ω is as in (28). There exist constants C and c, independent of t $\in \mathbb{R}$, such that*

$$\mathbb{P}(\| \mathrm{P}_N e^{it\Delta} f^\omega \|_{L^p_x(\mathbb{T}^d)} > \lambda) \leq C e^{-cN^{2\alpha}\lambda^2}\,. \tag{31}$$

Thus

$$\mathbb{P}(\| \mathrm{P}_N e^{it\Delta} f^\omega \|_{L^\infty_x(\mathbb{T}^d)} > \lambda) \le C e^{-cN^{2\alpha-}\lambda^2}. \tag{32}$$

In particular, for any $\varepsilon > 0$ *sufficiently small, we have*

$$\| \mathrm{P}_N e^{it\Delta} f^\omega \|_{L^p_x(\mathbb{T}^d)} \lesssim N^{-\alpha} (-\ln \varepsilon)^{1/2}, \quad N \in 2^{\mathbb{Z}} \tag{33}$$

and

$$\| \mathrm{P}_N e^{it\Delta} f^\omega \|_{L^\infty_x(\mathbb{T}^d)} \lesssim N^{-\alpha+} (-\ln \varepsilon)^{1/2}, \quad N \in 2^{\mathbb{Z}}, \tag{34}$$

with probability at least $1 - \varepsilon$.

Proof We prove (31), and then (32) follows by Bernstein inequality. By Minkowski's inequality and Lemma 6 above, we have for any $r \ge p \ge 2$

$$\left(\int \| \mathrm{P}_N e^{it\Delta} f^\omega \|^r_{L^p_x(\mathbb{T}^d)} d\mathbb{P}(\omega) \right)^{\frac{1}{r}} \le \Big\| \| \mathrm{P}_N e^{it\Delta} f^\omega \|_{L^r_\omega} \Big\|_{L^p_x(\mathbb{T}^d)} \le C N^{-\alpha} r^{\frac{1}{2}},$$

which is enough to conclude that $\| \mathrm{P}_N e^{it\Delta} f^\omega \|_{L^p_x(\mathbb{T}^d)}$ is a sub-Gaussian random variable satisfying the tail bound (31).

□

Note that using (31)–(32), the triangle inequality

$$\| P_{>N}(\cdot) \| \le \sum_{M \in 2^{\mathbb{N}} : M > N} \| P_M(\cdot) \|,$$

the union bound, and the fact that

$$\sum_{M \in 2^{\mathbb{N}} : M > N} e^{-cM^{2\alpha}k^{-2}} \lesssim e^{-cN^{2\alpha}k^2},$$

we see that, for all $t \in \mathbb{R}$ and $\alpha > 0$, we have ($p < \infty$)

$$\mathbb{P}\left(\| e^{it\Delta} \mathrm{P}_{>N} f^\omega \|_{L^p_x(\mathbb{T}^d)} > \lambda \right) \lesssim e^{-cN^{2\alpha}\lambda^2} \tag{35}$$

$$\mathbb{P}\left(\| e^{it\Delta} \mathrm{P}_{>N} f^\omega \|_{L^\infty_x(\mathbb{T}^d)} > \lambda \right) \lesssim e^{-cN^{2\alpha-}\lambda^2}. \tag{36}$$

Remark 2 Proceeding as we did to prove (35)–(36), we also easily see that the exceptional set where (33)–(34) are not valid can be chosen to be the same for all $N \in \mathbb{N}$, paying an N^{0+} loss on the right-hand side of the estimates.

Proceeding as in the proof of Lemma 7, we also obtain improved Strichartz estimates for randomized data.

Lemma 8 *Let* $p \in [2, \infty)$. *Assume* f^ω *is as in* (28)*. Then we have*

$$\mathbb{P}\left(\|e^{it\Delta} \mathrm{P}_N f^\omega\|_{L^p_{x,t}(\mathbb{T}^{d+1})} > \lambda\right) \leq Ce^{-cN^{2\alpha}\lambda^2}. \tag{37}$$

Thus

$$\mathbb{P}\left(\|e^{it\Delta} \mathrm{P}_N f^\omega\|_{L^\infty_{x,t}(\mathbb{T}^{d+1})} > \lambda\right) \leq Ce^{-cN^{2\alpha-}\lambda^2}. \tag{38}$$

In particular, for any $\varepsilon > 0$ *sufficiently small, we have*

$$\|e^{it\Delta} \mathrm{P}_N f^\omega\|_{L^p_{x,t}(\mathbb{T}^{d+1})} \lesssim N^{-\alpha} (-\ln \varepsilon)^{1/2}, \quad N \in 2^{\mathbb{Z}} \tag{39}$$

and

$$\|e^{it\Delta} \mathrm{P}_N f^\omega\|_{L^\infty_{x,t}(\mathbb{T}^{d+1})} \lesssim N^{-\alpha+} (-\ln \varepsilon)^{1/2}, \quad N \in 2^{\mathbb{Z}}, \tag{40}$$

with probability at least $1 - \varepsilon$.

The bounds (37)–(38) imply

$$\mathbb{P}\left(\|e^{it\Delta} \mathrm{P}_{>N} f^\omega\|_{L^p_{x,t}(\mathbb{T}^{d+1})} > \lambda\right) \lesssim e^{-cN^{2\alpha}\lambda^2} \tag{41}$$

$$\mathbb{P}\left(\|e^{it\Delta} \mathrm{P}_{>N} f^\omega\|_{L^\infty_{x,t}(\mathbb{T}^{d+1})} > \lambda\right) \lesssim e^{-cN^{2\alpha-}\lambda^2} \tag{42}$$

exactly as (31)–(32) imply (35)–(36). Also we have an analogous of Remark (2):

Remark 3 The exceptional set where (39)–(40) are not valid can be chosen to be the same for all $N \in \mathbb{N}$, paying an N^{0+} loss on the right-hand side of the estimates.

Fix $t \in \mathbb{R}$. Later we will also need the following bound for the H^s norm of $e^{it\Delta} f^\omega$ with $s < \alpha$. This is a well-known fact that we recall applying again (29) with $a_n = e^{in\cdot x - |n|^2 t} \langle n \rangle^{-\frac{d}{2} - \alpha + s}$, so that we get for $r \geq 2$

$$\| \mathrm{P}_N \langle D \rangle^s e^{it\Delta} f^\omega \|_{L^r_\omega} \leq C r^{\frac{1}{2}} N^{s-\alpha}, \qquad s < \alpha\,.$$

Here $\langle D \rangle$ denotes the Fourier multiplier operator $\langle n \rangle$. Proceeding as in the proof of Lemma 7, we also obtain

$$\mathbb{P}\left(\|\langle D \rangle^s \mathrm{P}_N e^{it\Delta} f^\omega\|_{L^2_x(\mathbb{T}^d)} > \lambda\right) \leq Ce^{-cN^{2(\alpha-s)}\lambda^2}, \qquad s < \alpha, \tag{43}$$

and in particular, for any $\varepsilon > 0$ sufficiently small

$$\|e^{it\Delta} f^{\omega}\|_{H^s_x(\mathbb{T}^d)} \lesssim (-\ln \varepsilon)^{1/2} \qquad s < \alpha, \quad t \in \mathbb{R}, \tag{44}$$

with probability at least $1 - \varepsilon$. Again the constant is uniform on $t \in \mathbb{R}$.

We are now ready to complete the proof of Proposition 1.

3.1 Proof of Proposition 1

Invoking the Borel–Cantelli lemma, it is enough to show that

$$\mathbb{P}\left(\limsup_{t\to 0} \|e^{it\Delta} f^{\omega} - f^{\omega}\|_{L^{\infty}_x(\mathbb{T}^d)} > 1/k\right) \lesssim \gamma_k, \tag{45}$$

for a summable sequence $\{\gamma_k\}_{k\in\mathbb{N}}$. Let us decompose

$$|e^{it\Delta} f^{\omega} - f^{\omega}| \le |e^{it\Delta} \mathrm{P}_{>N} f^{\omega}| + |e^{it\Delta} \mathrm{P}_{\le N} f^{\omega} - \mathrm{P}_{\le N} f^{\omega}| + |\mathrm{P}_{>N} f^{\omega}|. \tag{46}$$

Using (36) (with $t = 0$) and (42), we see that

$$\|e^{it\Delta} \mathrm{P}_{>N} f^{\omega}\|_{L^{\infty}_{x,t}(\mathbb{T}^{d+1})} + \|\mathrm{P}_{>N} f^{\omega}\|_{L^{\infty}_x(\mathbb{T}^d)} \le \frac{1}{2k} \tag{47}$$

holds for all ω outside an exceptional set of measure $\lesssim e^{-cN^{2\alpha}k^{-2}}$. We choose $N = N_k$ via the identity $N_k^{2\alpha} = k^3$, in such a way that $e^{-cN_k^{2\alpha}k^{-2}} = e^{-ck}$ is summable (over $k \in \mathbb{N}$). Let $s^* > d/2$. Since

$$e^{it\Delta} \mathrm{P}_{\le N_k} f^{\omega} - \mathrm{P}_{\le N_k} f^{\omega} = \sum_{|n|\le N_k} (e^{-it|n|^2} - 1) e^{in\cdot x} \hat{f}^{\omega}(n),$$

using Cauchy–Schwarz, the summability of $\langle n\rangle^{-2s^*}$ (over $n \in \mathbb{Z}^d$) and (44) with $s = 0$, $t = 0$ (in the last inequality), we get

$$\begin{aligned} \|e^{it\Delta} \mathrm{P}_{\le N_k} f^{\omega} - \mathrm{P}_{\le N_k} f^{\omega}\|_{L^{\infty}_x(\mathbb{T}^d)} &\lesssim \sup_{|n|\le N_k} |e^{-it|n|^2} - 1| \left(\sum_{|n|\le N_k} \langle n\rangle^{2s^*} |\hat{f}^{\omega}(n)|^2\right)^{1/2} \\ &\lesssim |t|(N_k)^{s^*+2} \|f^{\omega}\|_{L^2} \le |t|(N_k)^{s^*+2} \frac{1}{k}, \end{aligned} \tag{48}$$

for ω outside an exceptional set of probability $\lesssim e^{-cN_k^{2\alpha}k^{-2}} = e^{-ck}$. From the previous inequality, looking at t so small that $|t|(N_k)^{s^*+2} \leq 1/2$, we have

$$\mathbb{P}\left(\limsup_{t\to 0} \|e^{it\Delta} \mathrm{P}_{\leq N^*} f^\omega - \mathrm{P}_{\leq N^*} f^\omega\|_{L_x^\infty(\mathbb{T}^d)} > 1/k\right) \lesssim e^{-ck}. \tag{49}$$

Combining (36)–(36) and recalling the decomposition (46), the proof is concluded. □

4 Proof of Theorem 2

In this section, we consider the cubic Wick-ordered NLS (8) on $\mathbb{T}^d$ ($d = 1, 2$) as in the work of Bourgain in [3]. Namely, we look at the nonlinearity

$$\mathcal{N}(u) := \pm u\left(|u|^2 - 2\mu\right), \qquad \mu := \fint_{\mathbb{T}^d} |u(x,t)|^2 dx\,.$$

We are interested again in randomized initial data, i.e., f^ω is taken to be of the form (28). Recall (see (44)) that such data is $\mathbb{P}$-almost surely in H^s for all $s < \alpha$ and

$$\|f^\omega\|_{H^s} \lesssim (-\ln\varepsilon)^{1/2}, \quad s < \alpha\,, \tag{50}$$

with probability at least $1-\varepsilon$, for all $\varepsilon \in (0,1)$ sufficiently small. Since we work with any $\alpha > 0$, we are considering initial data in H^{0+}. We approximate Eq. (8) as in (11), for all $N \in 2^{\mathbb{N}} \cup \{\infty\}$. Recall that $\Phi_t^N f^\omega$ denotes the associated flow, with initial datum

$$\Phi_0^N f^\omega := \mathrm{P}_{\leq N} f^\omega = \sum_{|n|\leq N} \frac{g_n^\omega}{\langle n\rangle^{\frac{d}{2}+\alpha}} e^{in\cdot x}\,.$$

We write $\Phi_t f^\omega = \Phi_t^\infty f^\omega$ for the flow of (8) with datum $f^\omega = \mathrm{P}_\infty f^\omega$.

The relevant choice of σ in the following statement is $\sigma = \frac{1}{2}-$ (we will use this to prove Theorem 2).

Proposition 3 *Let $d = 1, 2$ and $\alpha > 0$. Let $N \in 2^{\mathbb{N}} \cup \{\infty\}$. For all $\sigma \in [0, \frac{1}{2})$, the following holds. Assume*

$$u = u(I) + u(II), \quad u(I) = e^{it\Delta} \mathrm{P}_{\leq N} f^\omega, \quad \|u(II)\|_{X^{\alpha+\sigma,\frac{1}{2}+}} < 1 \tag{51}$$

and the same for v. Then

$$\|\mathcal{N}(u)\|_{X^{\alpha+\sigma,-\frac{1}{2}+}} \lesssim (-\ln \varepsilon)^{3/2} \tag{52}$$

$$\|\mathcal{N}(u) - \mathcal{N}(v)\|_{X^{\alpha+\sigma,-\frac{1}{2}++}} \lesssim (-\ln \varepsilon)\,\|u - v\|_{X^{\alpha+\sigma,\frac{1}{2}+}} \tag{53}$$

for initial data of the form (28)*, with probability at least* $1-\varepsilon$*, for all* $\varepsilon \in (0,1)$ *sufficiently small. If we take u as in* (51) *and we instead assume*

$$v = v(I) + u(II), \quad v(I) = e^{it\Delta} f^{\omega}, \quad \|u(II)\|_{X^{\alpha+\sigma,\frac{1}{2}+}} < 1\,,$$

we have

$$\|\mathcal{N}(u) - \mathcal{N}(v)\|_{X^{\alpha+\sigma,-\frac{1}{2}++}} \lesssim N^{-\alpha}\,. \tag{54}$$

Remark 4 Recall that α indicates the regularity of the initial datum. We are denoting by σ the amount of smoothing one can prove for the Wick-ordered cubic nonlinearity $\mathcal{N}$. More precisely, since the initial data (28) belongs to $H^{\alpha-}$, one can interpret this statement as saying that, with arbitrarily large probability, $\mathcal{N}$ is $\sigma+$ smoother than f^{ω}. Since $\sigma < \frac{1}{2}$ is permissible, we reach $\frac{1}{2}-$ smoothing for $\mathcal{N}$ and, combining with (26), also for the Duhamel contribution $\Phi_t^N f^{\omega} - e^{it\Delta} P_{\leq N} f^{\omega}$.

In fact, a stronger statement than 3 has been proved in [13]. Namely that the reminder can be further decomposed into a sum of two terms. The first one, to which one we refer as paracontrolled, lies in $X^{\frac{1}{2}-,\frac{1}{2}+}$ but has a precise random structure. The second one is a smoother deterministic reminder that lies in $X^{1-,\frac{1}{2}+}$.

Here we only explain how to get Proposition 3 for the first Picard iteration, namely when 4. Recall that η is a smooth cut-off of the unit interval. Let us fix $\alpha > 0$. Using (26), (27), and Proposition 3, one can show that for all $\delta > 0$ sufficiently small, the following holds. For all $N \in 2^{\mathbb{N}} \cup \{\infty\}$, the map

$$\Gamma^N(u) := \eta(t) e^{it\Delta}\,\mathrm{P}_{\leq N}\, f^{\omega} - i\eta(t) \int_0^t e^{i(t-s)\Delta}\,\mathrm{P}_{\leq N}\,\mathcal{N}(u(\cdot,s))\,ds \tag{55}$$

is a contraction on the set

$$\left\{ e^{it\Delta}\,\mathrm{P}_{\leq N}\, f^{\omega} + g, \quad \|g\|_{X_{\delta}^{\alpha+\sigma,\frac{1}{2}+}} < 1 \right\} \tag{56}$$

equipped with the $X_{\delta}^{\alpha+\sigma,\frac{1}{2}+}$ norm, outside an exceptional set (we call it a δ-exceptional set) of initial data of probability smaller than $e^{-\delta^{-\gamma}}$, with $\gamma > 0$ a given small constant. Notice that this holds uniformly over $N \in 2^{\mathbb{N}} \cup \{\infty\}$. Again, this is a standard routine calculation that we omit (see for instance [18, Section 3.5.1]). We only explain how to find the relation between the local existence time δ and the size of the exceptional set. Given any $\varepsilon \in (0, 1)$ sufficiently small, using (26), (27), and Proposition 3, we have

$$\|\Gamma^N(u) - \eta(t)e^{it\Delta}\,\mathrm{P}_{\leq N}\, f^{\omega}\|_{X_\delta^{\alpha+\sigma,\frac{1}{2}+}} \lesssim \delta^{0+}\,(-\ln\varepsilon)^{3/2}\ ,$$

for all f^{ω} outside an exceptional set of probability smaller than ε. Letting δ such that $\varepsilon = e^{-\delta^{-\gamma}}$ with $\gamma > 0$ a fixed small constant, we have $C\delta^{0+}\,(-\ln\varepsilon)^{3/2} < 1$ for all $\delta > 0$ sufficiently small. Note that the measure $e^{-\delta^{-\gamma}}$ of the δ-exceptional set converges to zero as $\delta \to 0$. In particular, for ω outside the δ-exceptional set, the fixed point $\Phi_t^N f^{\omega}$ of the map (55) belongs to the set (56), namely

$$\|\Phi_t^N f^{\omega} - e^{it\Delta}\,\mathrm{P}_{\leq N}\, f^{\omega}\|_{X_\delta^{\alpha+\sigma,\frac{1}{2}+}} < 1, \qquad N \in 2^{\mathbb{N}} \cup \{\infty\}\,. \tag{57}$$

We are now ready to prove Theorem 2.

4.1 Proof of Theorem 2

It suffices to show that

$$\lim_{N\to\infty} \left\| \sup_{0\leq t\leq\delta} |\Phi_t f^{\omega}(x) - \Phi_t^N f^{\omega}(x)| \right\|_{L_x^2(\mathbb{T}^2)} = 0 \tag{58}$$

for all f^{ω} outside a δ-exceptional set A_δ. Note indeed that (58) implies that given f^{ω}, we can find $\mathbb{P}$-almost surely, a δ_ω (which depends on ω) such that (58) is satisfied. Indeed, if we could not do so, this would mean that $f^{\omega} \in \bigcap_{\delta>0} A_\delta$, and the probability of this event is zero since $\mathbb{P}(A_\delta) \to 0$ as $\delta \to 0$.

Once we have (58) with $\delta = \delta_\omega$, we have $\mathbb{P}$-almost surely

$$\lim_{t\to 0} \Phi_t^{\omega} f^{\omega}(x) - f^{\omega}(x) = 0, \qquad \text{for a.e. } x \in \mathbb{T}^2\,,$$

as claimed, simply invoking Proposition 2.

In order to prove (58), we decompose

$$|\Phi_t f^{\omega} - \Phi_t^N f^{\omega}| \leq |e^{it\Delta}\,\mathrm{P}_{>N}\, f^{\omega}| + |\Phi_t f^{\omega} - e^{it\Delta} f^{\omega} - (\Phi_t^N f^{\omega} - e^{it\Delta}\,\mathrm{P}_{\leq N}\, f^{\omega})|\,.$$

Thus, recalling the decay of the high-frequency linear term given by (36), it remains to show that

$$\lim_{N\to\infty}\left\|\sup_{0\le t\le\delta}|\Phi_t f^\omega - e^{it\Delta}f^\omega - (\Phi_t^N f^\omega - e^{it\Delta}\mathrm{P}_{\le N} f^\omega)|\right\|_{L^2(\mathbb{T}^2)} = 0\,, \tag{59}$$

for all f^ω outside a δ-exceptional set.

For any $\alpha > 0$, we can choose σ sufficiently close to $\frac{1}{2}$ that

$$\frac{1}{2} < \alpha + \sigma\,. \tag{60}$$

Thus, using the $X^{s,b}$ space embedding from Lemma 2, it suffices to prove

$$\lim_{N\to\infty}\left\|w - w^N\right\|_{X_\delta^{\alpha+\sigma,\frac{1}{2}+}} = 0\,, \tag{61}$$

where

$$w^N := \Phi_t^N f - e^{it\Delta}\mathrm{P}_{\le N} f^\omega, \qquad w := w^\infty\,.$$

Notice that by (57), we have

$$\|w^N\|_{X_\delta^{\alpha+\sigma,\frac{1}{2}+}} < 1, \qquad N \in 2^{\mathbb{N}} \cup \{\infty\}\,.$$

Since for $t \in [0,\delta]$, we have

$$w - w^N = -i\eta(t)\int_0^{t'} e^{i(t-t')\Delta}\Big(\mathcal{N}(\Phi_{t'} f^\omega) - \mathrm{P}_{\le N}\mathcal{N}(\Phi_{t'}^N f^\omega)\Big)\,dt'\,, \tag{62}$$

using (26), (27), we get

$$\|w - w^N\|_{X_\delta^{\alpha+\sigma,\frac{1}{2}+}} \lesssim \delta^{0+}\|\mathcal{N}(\Phi_t f) - \mathrm{P}_{\le N}\mathcal{N}(\Phi_t^N f)\|_{X_\delta^{\alpha+\sigma,-\frac{1}{2}++}}\,. \tag{63}$$

We decompose

$$\begin{aligned}&\mathcal{N}(\Phi_t f) - \mathrm{P}_{\le N}\mathcal{N}(\Phi_t^N f) = \\ &\qquad \mathrm{P}_{\le N}\Big(\mathcal{N}(e^{it\Delta}\mathrm{P}_{\le N} f^\omega + w) - \mathcal{N}(e^{it\Delta}\mathrm{P}_{\le N} f^\omega + w^N)\Big) + \text{Remainders}\,,\end{aligned} \tag{64}$$

where

$$\text{Remainders} := \mathrm{P}_{\le N}\Big(\mathcal{N}(e^{it\Delta} f^\omega + w) - \mathcal{N}(e^{it\Delta}\mathrm{P}_{\le N} f^\omega + w)\Big) + \mathrm{P}_{>N}\mathcal{N}(\Phi_t f)\,.$$

Notice that by (52), (54), we have

$$\|\text{Remainders}\|_{X_\delta^{\alpha+\sigma,-\frac{1}{2}++}} \to 0 \quad \text{as } N \to \infty\,, \tag{65}$$

with probability at least $1-\varepsilon$. Using (53), we can estimate

$$\begin{aligned}\| \mathrm{P}_{\leq N} \Big(\mathcal{N}(e^{it\Delta}\,\mathrm{P}_{\leq N}\, f^{\omega} + w) - \mathcal{N}(e^{it\Delta}\,\mathrm{P}_{\leq N}\, f^{\omega} + w_N)) \|_{X_\delta^{\alpha+\sigma,-\frac{1}{2}++}} \\ \lesssim (-\ln\varepsilon) \left\| w - w^N \right\|_{X_\delta^{\alpha+\sigma,\frac{1}{2}+}}\,,\end{aligned} \tag{66}$$

and (63), (64), (66) give

$$\left\| w - w^N \right\|_{X_\delta^{\alpha+\sigma,\frac{1}{2}+}} \lesssim \delta^{0+}\,(-\ln\varepsilon) \left\| w - w^N \right\|_{X_\delta^{\alpha+\sigma,\frac{1}{2}+}} + \|\text{Remainders}\|_{X_\delta^{\alpha+\sigma,-\frac{1}{2}++}} \tag{67}$$

with probability at least $1-\varepsilon$. Since with our choice of $\varepsilon = e^{-\delta^{-\gamma}}$, we have $C\delta^{0+}(-\ln\varepsilon)^{3/2} < 1$, we can absorb the first term on the right-hand side into the left-hand side, and we still have that (65) holds outside a δ-exceptional set. Thus letting $N \to \infty$, the proof of (9) is complete.

□

Remark 5 It is worthy to remark that, comparing with for instance [3], the procedure that allows to promote a statement valid on a δ-exceptional set A_δ for arbitrarily small $\delta > 0$ to a statement that is valid with probability $= 1$ is far easier. In particular, it does not involve any control on the evolution of the (Gaussian) measure induced by the random Fourier series. This is because we are considering a property that has to be verified only at time $t = 0$ a.s., instead that in a time interval containing $t = 0$, as in [3].

We now give some hints on the proof of the smoothing estimates given in Proposition 3.

4.2 Proof of Proposition 3

Again it is worthy to recall that an even stronger statement than 3 has been proved in [13]. Here we show how to handle the first Picard iterate. Notice that the Wick-ordered nonlinearity can be written as

$$\mathcal{N}(u(x,\cdot)) = \sum_{n_2\neq n_1,n_3} \widehat{u}(n_1)\widehat{\bar{u}}(n_2)\widehat{u}(n_3)e^{i(n_1-n_2+n_3)\cdot x} - \sum_{n} \widehat{u}(n)|\widehat{u}(n)|^2 e^{in\cdot x}, \tag{68}$$

where we are looking at the nonlinear term for fixed time and $\widehat{u}(\cdot)$ denotes the space Fourier coefficients. Looking at a similar expansion for the difference $\mathcal{N}(u) - \mathcal{N}(v)$, it is easy to see that we can deduce (3) from a slightly more general Lemma 9 given below. It implies the desired statement

$$u_j(n_j) = u(n_j), \quad v(n_j), \quad \text{or } \; u(n_j) - v(n_j)\,.$$

□

We will give a proof of the following Lemma in the fully random case $J_j = I$ for $j = 1, 2, 3,$, which corresponds to the study of the first Picard iterate. Comparing with 9 (and [13]), there is a simplification coming from the fact that our f^ω is slightly more regular, namely we consider $\alpha > 0$ instead of $\alpha = 0$.

Lemma 9 *Let $d = 1, 2$ and $\alpha > 0$. Let $N \in 2^{\mathbb{N}} \cup \{\infty\}$. For all $\sigma \in [0, \frac{1}{2})$, the following holds. Assume for $j = 1, 2, 3$*

$$u_j(I) = e^{it\Delta}\, \mathrm{P}_{\leq N}\, f^\omega, \qquad \|u_j(II)\|_{X^{\alpha+\sigma, \frac{1}{2}+}} < 1. \tag{69}$$

Let $J_j \in \{I, II\}$, $j = 1, 2, 3$. Then, for all $\varepsilon \in (0, 1)$ sufficiently small, we have the following:

$$\|\mathcal{N}(u_1(J_1), \overline{u_2}(J_2), u_3(J_3))\|_{X^{\alpha+\sigma, -\frac{1}{2}+}} \lesssim (-\ln \varepsilon)^{3/2}\ , \tag{70}$$

and more precisely,

$$\|\mathcal{N}(u_1(II), \overline{u_2}(J_2), u_3(J_3))\|_{X^{\alpha+\sigma, -\frac{1}{2}++}} \lesssim (-\ln \varepsilon)\, \|u_1(II)\|_{X^{\alpha+\sigma, \frac{1}{2}+}}\,, \tag{71}$$

$$\|\mathcal{N}(u_1(J_1), \overline{u_2}(II), u_3(J_3))\|_{X^{\alpha+\sigma, -\frac{1}{2}++}} \lesssim (-\ln \varepsilon)\, \|u_2(II)\|_{X^{\alpha+\sigma, \frac{1}{2}+}}\,, \tag{72}$$

with probability at least $1 - \varepsilon$. Moreover, if in (69) we replace for some $j = j^*$ the projection operator $\mathrm{P}_{\leq N}$ by $\mathrm{P}_{>N}$, then the estimate (70) with $J_{j^*} = I$ holds with an extra factor $N^{-\alpha}$ on the right-hand side.

Remark 6 Saying that these estimates hold with probability at least $1 - \varepsilon$ means, more precisely, that they hold for all ω outside an exceptional set of probability $\leq \varepsilon$. Moreover, this set can be chosen to be independent on $N \in 2^{\mathbb{N}} \cup \{\infty\}$.

Remark 7 Notice that by the symmetry $n_1 \leftrightarrow n_3$ the estimate (71) implies an analogous estimate for $u_3(II)$.

Here we only consider the case $J_j = I$ for $j = 1, 2, 3$, namely the case in which all the contributions are a linear random evolution. We prove the bound (70) relative to this case and to $N = \infty$. Moreover, we split the nonlinearity as a difference of two terms (see (68))

$$\mathcal{N}_1(u_1(J_1), \overline{u_2}(J_2), u_3(J_3)) = \sum_{n_2 \neq n_1, n_3} \widehat{u_1(J_1)}(n_1)\widehat{\overline{u_2}(J_2)}(n_2)\widehat{u_3(J_3)}(n_3)e^{i(n_1-n_2+n_3)\cdot x},$$

$$\mathcal{N}_2(u_1(J_1), \overline{u_2}(J_2), u_3(J_3)) = \sum_{n} \widehat{u_1(J_1)}(n)\widehat{\overline{u_2}(J_2)}(n)\widehat{u_3(J_2)}(n)e^{in\cdot x},$$

and we prove (70) only for $\mathcal{N}_1$, which is the most challenging contribution. The proof for $\mathcal{N}_2$ is indeed elementary.

To prove, (70) will be useful to recall that the space–time Fourier transform of $e^{it\Delta} f^\omega$ is

$$\widehat{e^{it\Delta} f^\omega}(n, \tau) = \frac{g^\omega}{\langle n \rangle^{\frac{d}{2}+\alpha}} \delta(\tau + |n|^2),$$

where δ is the delta function. So a direct computation gives

$$\| e^{it\Delta} f^\omega \|^2_{X^{0+,\frac{1}{2}+}} = \sum_n \frac{|g_n^\omega|^2}{\langle n \rangle^{d+2\alpha-}},$$

which, recalling $\int |g_n^\omega|^2 d\omega = 1$, immediately implies

$$\| \| e^{it\Delta} f^\omega \|_{X^{0+,\frac{1}{2}+}} \|^2_{L^2_\omega} = \sum_n \frac{1}{\langle n \rangle^{d+2\alpha-}} < \infty .$$

Since we can expand the LHS as a bilinear form in the Gaussian variables g_n^ω, we get by Gaussian hypercontractivity

$$\| \| e^{it\Delta} f^\omega \|_{X^{0+,\frac{1}{2}+}} \|^2_{L^q_\omega} = \sum_n \frac{1}{\langle n \rangle^{d+2\alpha-}} < C_q < \infty .$$

Proceeding essentially as in the Proof of Lemmas 7–8 (recall also Remarks 2–RemarkUniform1Bis), this allows to prove a pointwise bound

$$\| e^{it\Delta} f^\omega \|_{X^{0+,\frac{1}{2}+}} \lesssim \sqrt{\ln\left(\frac{1}{\varepsilon}\right)}, \tag{73}$$

with probability larger than $1 - C\varepsilon$ for all sufficiently small $\varepsilon > 0$.

Let N, N_1, N_2, N_3 be dyadic scales. We denote with $\tilde{N}$ the maximum between N_1, N_2, N_3. First we perform a reduction to remove frequencies that are far from the paraboloid. More precisely, we denote with P_A the space–time Fourier projection into the set A, and our goal is to reduce

$$\sum_{N_1,N_2,N_3} \|\mathcal{N}_1\left(\mathrm{P}_{N_1} u_1(I), \mathrm{P}_{N_2} \overline{u_2}(I), \mathrm{P}_{N_3} u_3(I)\right)\|^2_{X^{\alpha+\sigma,-\frac{1}{2}++}} \tag{74}$$

$$= \sum_{N,N_1,N_2,N_3} N^{2\alpha+2\sigma} \| \mathrm{P}_N \mathcal{N}_1\left(\mathrm{P}_{N_1} u_1(I), \mathrm{P}_{N_2} \overline{u_2}(I), \mathrm{P}_{N_3} u_3(I)\right) \|^2_{X^{0,-\frac{1}{2}++}}$$

to

$$\sum_{N,N_1,N_2,N_3} N^{2\alpha+2\sigma} \| \mathrm{P}_N \mathrm{P}_{\left\{\langle\tau+|n|^2\rangle \le \tilde{N}^{1+\frac{1}{10}}\right\}} \mathcal{N}_1\left(\mathrm{P}_{N_1} u_1(I)\, \mathrm{P}_{N_2} \overline{u_2}(I)\, \mathrm{P}_{N_3} u_3(I)\right) \|^2_{X^{0,-\frac{1}{2}++}}. \tag{75}$$

To obtain this reduction, it is sufficient to show that projection of the nonlinearity onto the complementary set is appropriately bounded, i.e., that

$$\sum_{N,N_1,N_2,N_3} N^{2\alpha+2\sigma} \| \mathrm{P}_N \mathrm{P}_{\left\{\langle\tau+|n|^2\rangle > \tilde{N}^{\frac{11}{10}}\right\}} \mathcal{N}_1\left(\mathrm{P}_{N_1} u_1(I), \mathrm{P}_{N_2} \overline{u_2}(I), \mathrm{P}_{N_3} u_3(I)\right) \|^2_{X^{0,-\frac{1}{2}++}} \tag{76}$$

$$\lesssim (-\ln \varepsilon)^3$$

with probability at least $1-\varepsilon$. To do so, we abbreviate

$$\mathcal{N}_1^{N_1,N_2,N_3}(\cdot) := \mathcal{N}_1\left(\mathrm{P}_{N_1} u_1(I), \mathrm{P}_{N_2} \overline{u_2}(I), \mathrm{P}_{N_3} u_3(I)\right),$$

and we bound

$$\sum_{N_1,N_2,N_3} N^{2\alpha+2\sigma} \| \mathrm{P}_N \mathrm{P}_{\left\{\langle\tau+|n|^2\rangle > \tilde{N}^{\frac{11}{10}}\right\}} \mathcal{N}_1^{N_1,N_2,N_3} \|^2_{X^{0,-\frac{1}{2}++}} \tag{77}$$

$$\sim N^{2\alpha+2\sigma} \sum_{\substack{N_1,N_2,N_3\\ n\sim N}} \int \frac{\chi_{\{\langle\tau+|n|^2\rangle > \tilde{N}^{\frac{11}{10}}\}}}{\langle\tau+|n|^2\rangle^{1--}} \left|\widehat{\mathcal{N}_1^{N_1,N_2,N_3}}(\cdot)(n,\tau)\right|^2 d\tau$$

$$\lesssim N^{2\alpha+2\sigma-1-\frac{1}{10}+3(0+)} \sum_{\substack{N_1,N_2,N_3\\ n\sim N}} \int \left|\widehat{\mathcal{N}_1^{N_1,N_2,N_3}}(\cdot)(n,\tau)\right|^2 d\tau$$

$$\sim N^{2\alpha-\frac{1}{20}} \sum_{N_1,N_2,N_3} \| \mathrm{P}_N \mathcal{N}_1^{N_1,N_2,N_3} \|^2_{L^2_{x,t}},$$

recalling that $\sigma < 1/2$ (here in fact we may have more smoothing than $\frac{1}{2}-$). We have used the fact that at least one of the frequency scales N_j has to be comparable to N; otherwise, the contribution is zero by orthogonality, and so particular, we have

$N \lesssim \tilde{N}$ (recall that $\tilde{N} = \max(N_1, N_2, N_3)$). In order to continue the estimate, we assume for definiteness that $N_1 \sim N$. The other possible case is $N_2 \sim N$ (since everything is symmetric under $n_1 \leftrightarrow n_3$), and one can indeed immediately check that the estimate (78) below is still valid in this case, with obvious changes. Thus we have using Hölder's inequality, the improved Strichartz inequality (40) for randomized functions (for the L^∞ norm of $u_1(I)$), and the Strichartz inequality (10) (for the L^4 norms of $u_2(I)$ and $u_3(I)$), we obtain

$$\begin{aligned} &\| \mathrm{P}_N \mathcal{N}_1^{N_1,N_2,N_3} \|^2_{L^2_{x,t}} \qquad (78)\\ &\leq \| \mathrm{P}_{N_1} u_1(I)\|^2_{L^\infty_{x,t}} \| \mathrm{P}_{N_2} \overline{u_2}(I)\|^2_{L^4_{x,t}} \| \mathrm{P}_{N_3} u_3(I)\|^2_{L^4_{x,t}} . \\ &\lesssim (-\ln \varepsilon)\, N_1^{-2\alpha} \| \mathrm{P}_{N_2} \overline{u_2}(I)\|^2_{L^4_{x,t}} \| \mathrm{P}_{N_3} w_3(I)\|^2_{L^4_{x,t}} , \\ &\lesssim (-\ln \varepsilon)\, N^{-2\alpha} \| \mathrm{P}_{N_2} \overline{u_2}(I)\|^2_{X^{0+,\frac{1}{2}+}} \| \mathrm{P}_{N_3} u_3(I)\|^2_{X^{0+,\frac{1}{2}+}} , \end{aligned}$$

this holds on a set of probability larger than $1-\varepsilon$, and this set may be chosen to be independent on $N_1 \in \mathbb{N} \cup \{\infty\}$ (see Remark 3) and thus on $N \in \mathbb{N} \cup \{\infty\}$. Plugging (78) into (77), summing over the N_j, and using (73), we arrive to the needed bound

$$\begin{aligned} \text{LHS of (76)} &\lesssim (-\ln \varepsilon) \sum_{N,N_1} N^{-\frac{1}{20}} \|u_2(I)\|^2_{X^{0+,\frac{1}{2}+}} \|u_3(I)\|^2_{X^{0+,\frac{1}{2}+}} \\ &\lesssim (-\ln \varepsilon)^3 \sum_{N,N_1} N^{-\frac{1}{40}} N_1^{-\frac{1}{40}} \lesssim (-\ln \varepsilon)^3 . \end{aligned}$$

Note that in (78), we could also use a weaker bound replacing the L^4 norm with the L^∞ and that in the fully random case $J_j = I$ for all j is controlled invoking (40) for all $j = 1, 2, 3$. However, the L^4 bound is more robust since it works also in the other cases, where the contributions are not all random (namely if some J_j is of the form II).

So we have reduced to (75). We have

$$\begin{aligned} &\mathrm{P}_N \mathrm{P}_{\left\{\langle \tau+|n|^2\rangle \leq \tilde{N}^{\frac{11}{10}}\right\}} \mathcal{N}_1^{N_1,N_2,N_3}(\cdot) \qquad (79)\\ &= \mathrm{P}_N \mathrm{P}_{\left\{\langle \tau+|n|^2\rangle \leq \tilde{N}^{\frac{11}{10}}\right\}} \left(\sum_{|n_j|\sim N_j} e^{ix\cdot(n_1-n_2+n_3)} e^{-it(|n_1|^2-|n_2|^2+|n_3|^2)} \right. \\ &\qquad \left. \times \frac{g^\omega_{n_1}}{\langle n_1\rangle^{1+\alpha}} \frac{\overline{g^\omega_{n_2}}}{\langle n_2\rangle^{1+\alpha}} \frac{g^\omega_{n_3}}{\langle n_3\rangle^{1+\alpha}} \right). \end{aligned}$$

Thus we see that (75) satisfies the desired inequalities (70) as long as we can bound

$$N^{2\alpha+2\sigma}\Bigg\| \sum_{N_1,N_2,N_3} \mathrm{P}_N\, \mathrm{P}_{\left\{\langle\tau+|n|^2\rangle\le \tilde{N}^{\frac{11}{10}}\right\}} \Bigg(\sum_{|n_j|\sim N_j} e^{ix\cdot(n_1-n_2+n_3)} e^{-it(|n_1|^2-|n_2|^2+|n_3|^2)} \Bigg) \tag{80}$$

$$\times \frac{g^{\omega}_{n_1}}{\langle n_1\rangle^{1+\alpha}} \frac{\overline{g^{\omega}_{n_2}}}{\langle n_2\rangle^{1+\alpha}} \frac{g^{\omega}_{n_3}}{\langle n_3\rangle^{1+\alpha}} \Bigg\|^2_{X^{0,-\frac{1}{2}++}} \lesssim (-\ln\varepsilon)^3\, N^{0-}\,,$$

on a set of probability larger than $1-\varepsilon$.

Since

$$\mathcal{F}\Big(e^{ix\cdot(n_1-n_2+n_3)} e^{-it(|n_1|^2-|n_2|^2+|n_3|^2)}\Big)(n,\tau) \tag{81}$$
$$= \sum_{n_1-n_2+n_3=n} \delta(\tau+|n_1|^2-|n_2|^2+|n_3|^2)\,,$$

where $\mathcal{F}$ is the space–time Fourier transform and δ is the delta function, we reduce (80) to showing that

$$N^{2\alpha+2\sigma} \sum_{N_1,N_2,N_3} \sum_{|n|\sim N} \int \frac{\chi_{\{\langle\tau+|n|^2\rangle\le\tilde{N}^{\frac{11}{10}}\}}}{\langle\tau+|n|^2\rangle^{1--}}$$
$$\times \Bigg| \sum_{\substack{|n_j|\sim N_j,\ n_2\neq n_1,n_3 \\ n=n_1-n_2+n_3 \\ \tau+|n_1|^2-|n_2|^2+|n_3|^2=0}} \frac{g^{\omega}_{n_1}}{\langle n_1\rangle^{1+\alpha}} \frac{\overline{g^{\omega}_{n_2}}}{\langle n_2\rangle^{1+\alpha}} \frac{g^{\omega}_{n_3}}{\langle n_3\rangle^{1+\alpha}} \Bigg|^2 d\tau \lesssim (-\ln\varepsilon)^3\, N^{0-}, \tag{82}$$

with probability at least $1-\varepsilon$. Letting

$$\mu := |n|^2+\tau = |n|^2-|n_1|^2+|n_2|^2-|n_3|^2$$

(the second identity holds over the integration set, since we have a factor

$$\delta(\tau+|n_1|^2-|n_2|^2+|n_3|^2)$$

in the integrand) and recalling that $N \lesssim \tilde{N}$, this follows by

$$N^{2\alpha+2\sigma} \sum_{N_1,N_2,N_3} \sum_{|n|\sim N} \int \frac{\chi_{\{\langle\mu\rangle \le \tilde{N}^{\frac{11}{10}}\}}}{\langle\mu\rangle^{1--}} \times \left| \sum_{\substack{|n_j|\sim N_j,\, n_2\neq n_1,n_3 \\ n=n_1-n_2+n_3 \\ -|n|^2+|n_1|^2-|n_2|^2+|n_3|^2=\mu}} \frac{g_{n_1}^{\omega}}{\langle n_1\rangle^{1+\alpha}} \frac{\overline{g_{n_2}^{\omega}}}{\langle n_2\rangle^{1+\alpha}} \frac{g_{n_3}^{\omega}}{\langle n_3\rangle^{1+\alpha}} \right|^2 d\tau \lesssim (-\ln\varepsilon)^3 N^{0-}, \tag{83}$$

with probability at least $1-\varepsilon$. Using Hölder inequality in $d\mu$, we reduce to prove (here we use the symmetry $\mu \leftrightarrow -\mu$)

$$N^{2\alpha+2\sigma} \tilde{N}^{0+} \sum_{N_1,N_2,N_3} \sup_{|\mu|\lesssim \tilde{N}^{\frac{11}{10}}} \sum_{|n|\sim N} \times \left| \sum_{\substack{|n_j|\sim N_j,\, n_2\neq n_1,n_3 \\ n=n_1-n_2+n_3 \\ \mu=|n|^2-|n_1|^2+|n_2|^2-|n_3|^2}} \frac{g_{n_1}^{\omega}}{\langle n_1\rangle^{1+\alpha}} \frac{\overline{g_{n_2}^{\omega}}}{\langle n_2\rangle^{1+\alpha}} \frac{g_{n_3}^{\omega}}{\langle n_3\rangle^{1+\alpha}} \right|^2 \lesssim (-\ln\varepsilon)^3 N^{0-}, \tag{84}$$

with probability at least $1-\varepsilon$. We rewrite (84) as

$$N^{2\alpha+2\sigma} \tilde{N}^{0+} \sum_{N_1,N_2,N_3} \sup_{|\mu|\lesssim \tilde{N}^{\frac{11}{10}}} \sum_{|n|\sim N} \left| \sum_{R_n(n_1,n_2,n_3)} \frac{g_{n_1}^{\omega}}{\langle n_1\rangle^{1+\alpha}} \frac{\overline{g_{n_2}^{\omega}}}{\langle n_2\rangle^{1+\alpha}} \frac{g_{n_3}^{\omega}}{\langle n_3\rangle^{1+\alpha}} \right|^2 \lesssim (-\ln\varepsilon)^3 N^{0-}, \tag{85}$$

where for fixed n, μ we have denoted

$$R_n(n_1,n_2,n_3) := \Big\{ (n_1,n_2,n_3) \in (\mathbb{Z}^2)^3 \;:\; |n_j| \sim N_j,\ j=1,2,3, \\ n_2 \neq n_1, n_3,\ n_1-n_2+n_3=n,\ \mu = |n|^2-|n_1|^2+|n_2|^2-|n_3|^2 \Big\}. \tag{86}$$

The set $R_n(\cdot)$ depends on μ also (like all the sets we will define below). However, we omit this dependence to simplify the notation. Notice that in the definition of $R_n(\cdot)$ the condition

$$|n|^2 - |n_1|^2 + |n_2|^2 - |n_3|^2 = \mu$$

can be equivalently replaced by

$$2(n_1 - n_2) \cdot (n_3 - n_2) = \mu\,.$$

We also note that we have reduced to a case in which at least one of the frequencies N_1, N_3 is comparable to $\tilde{N}$. Indeed, if both $N_1 \ll \tilde{N}$ and $N_3 \ll \tilde{N}$, we must have $N_2 = \tilde{N}$ and $\mu \sim N^2$, which contradicts the fact that $\mu \lesssim N^{\frac{11}{10}}$. Since the roles of N_1 and N_3 are symmetric (they are always the size of the indices of the Fourier coefficients of u_1, u_3), hereafter we assume that

$$N_1 \sim \tilde{N} \gtrsim N.$$

To estimate, (85) will be also useful to introduce the set

$$S(n_1, n_2, n_3) := \Big\{ (n_1, n_2, n_3) \in (\mathbb{Z}^2)^3 \;:\; |n_j| \sim N_j,\, j = 1, 2, 3, \quad n_2 \neq n_1, n_3,\; \mu = 2(n_1 - n_2) \cdot (n_3 - n_2) \Big\}. \tag{87}$$

We recall that the Gaussian variables contract in the following way:

$$\int g_n^{\omega} g_{n'}^{\omega} d\mathbb{P}(\omega) = 0, \qquad \int g_n^{\omega} \overline{g_{n'}^{\omega}} d\mathbb{P}(\omega) = \begin{cases} 0 \text{ if } n \neq n' \\ 1 \text{ if } n = n' \end{cases}. \tag{88}$$

Along with the fact that the sum is restricted over $n_1, n_3 \neq n_2$ and symmetric under $n_1 \leftrightarrow n_3$, we get

$$\begin{aligned} &\int \Bigg| \sum_{R_n(n_1,n_2,n_3)} \frac{g_{n_1}^{\omega}}{\langle n_1 \rangle^{1+\alpha}} \frac{\overline{g_{n_2}^{\omega}}}{\langle n_2 \rangle^{1+\alpha}} \frac{g_{n_3}^{\omega}}{\langle n_3 \rangle^{1+\alpha}} \Bigg|^2 d\mathbb{P}(\omega) \\ &= 2 \sum_{R_n(n_1,n_2,n_3)} \frac{1}{\langle n_1 \rangle^{2\alpha+2}} \frac{1}{\langle n_2 \rangle^{2\alpha+2}} \frac{1}{\langle n_3 \rangle^{2\alpha+2}} = 2 \sum_{R_n(n_1,n_2,n_3)} N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2}. \end{aligned} \tag{89}$$

In other words, the $L^2(d\omega)$ norm of the Gaussian trilinear form is controlled by square root of the right-hand side of (89). Using the hypercontractivity of the Gaussians (see [19, 33]), we can promote this to an $L^q(d\omega)$ bound, with a multiplicative factor that is factor $q^{3/2}$. Then using Minkowski integral inequality and Bernstein inequality (as we did in Sect. 3), this also gives to us a (uniform) pointwise bound

$$\left| \sum_{R_n(n_1,n_2,n_3)} \frac{g^\omega_{n_1}}{\langle n_1\rangle^{1+\alpha}} \frac{\overline{g^\omega_{n_2}}}{\langle n_2\rangle^{1+\alpha}} \frac{g^\omega_{n_3}}{\langle n_3\rangle^{1+\alpha}} \right|^2 \lesssim (-\ln\varepsilon)^3 N_1^{0+} \sum_{R_n(n_1,n_2,n_3)} N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2}\,, \tag{90}$$

with an extra N_1^{0+} loss, that is valid for ω outside an exceptional set of probability $\leq \varepsilon$ (again, proceeding as in Sect. 3, we see that this exceptional set can be chosen to be independent on N, as required).

We finally distinguish two last possibilities. First restrict the summation over $(n_1, n_2, n_3) \in R_n(n_1, n_2, n_3)$ such that $n_1 \neq n_3$ (with a small abuse of notation, we do not introduce additional notation for this restriction). In this case, we get, with probability $> 1 - \varepsilon$, the following estimate

$$\begin{aligned} &\sum_{|n|\sim N} \left| \sum_{R_n(n_1,n_2,n_3)} \frac{g^\omega_{n_1}}{\langle n_1\rangle^{1+\alpha}} \frac{\overline{g^\omega_{n_2}}}{\langle n_2\rangle^{1+\alpha}} \frac{g^\omega_{n_3}}{\langle n_3\rangle^{1+\alpha}} \right|^2 \\ &\lesssim (-\ln\varepsilon)^3 \sum_{|n|\sim N} \sum_{R_n(n_1,n_2,n_3)} N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2} \\ &\lesssim (-\ln\varepsilon)^3 \sum_{S(n_1,n_2,n_3)} N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2} \\ &\sim (-\ln\varepsilon)^3 \sum_{S(n_1,n_2,n_3)} N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2} \\ &\lesssim (-\ln\varepsilon)^3 N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2} \# S(n_1,n_2,n_3) \\ &\lesssim (-\ln\varepsilon)^3 N_1^{-2\alpha-1} N_2^{-2\alpha} N_3^{-2\alpha}\,, \end{aligned} \tag{91}$$

where we used that if $n_1 \neq n_3$, then

$$\# S(n_1,n_2,n_3) \lesssim N_1 N_2^2 N_3^2\,;$$

this is because once we have fixed n_2, n_3 in $N_2^2 N_3^2$ possible ways, we remain with at most N_1 choices for n_1 by the relation $\mu = 2(n_1 - n_2)\cdot(n_3 - n_2)$. This fact has a clear geometric interpretation, namely that this relation forces the (two-dimensional) lattice point n_1 to belong to the portion of a line that lies inside a ball of radius $\lesssim N_1$ (and there are $\lesssim N_1$ such lattice points n_1).

The second possibility is that we sum over $(n_1, n_2, n_3) \in R_n(n_1, n_2, n_3)$ such that $n_1 = n_3$. In this case restriction, $\mu = 2|n_1 - n_2|^2$ implies that once we have chosen n_2 in N_2^2 possible ways, we remain with $\lesssim \mu^{0+} \lesssim N_1^{0++}$ choices for $n_1 = n_3$ (since a circle of radius μ contains $\lesssim \mu^{0+}$ lattice points). This gives an even better bound than the one above.

Thus, summing the (91) over N_2, N_3 and recalling that $N_1 \sim \tilde{N} \gtrsim N$, we have bounded, with probability $> 1 - \varepsilon$, the expression (85) by

$$N^{2\alpha+2\sigma} N_1^{0+} \sum_{N_1} N_1^{-2\alpha-1} \lesssim (-\ln \varepsilon)^3 N_1^{2\sigma-1+0+} \lesssim (-\ln \varepsilon)^3 N^{0-} ,$$

where we used $\sigma < \frac{1}{2}$.

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References

1. Bourgain, J.: A remark on Schrödinger operators. Israel J. Math. **77**(1–2), 1–16 (1992)
2. Bourgain, J.: Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations. Geom. Funct. Anal. **3**(2), 107–156 (1993)
3. Bourgain, J.: Invariant measures for the 2d-defocusing nonlinear Schrödinger equation. Commun. Math. Phys. **176**(2), 421–445 (1996)
4. Bourgain, J.: On the Schrödinger maximal function in higher dimension. Tr. Mat. Inst. Steklova, 280 (Ortogonal'nye Ryady, Teoriya Priblizheniĭ i Smezhnye Voprosy), pp. 53–66 (2013)
5. Bourgain, J.: A note on the Schrödinger maximal function. J. Anal. Math. **130**, 393–396 (2016)
6. Bourgain, J., Demeter, C.: The proof of the l^2 decoupling conjecture. Ann. of Math. (2) **182**(1), 351–389 (2015)
7. Burq, N., Tzvetkov, N.: Random data Cauchy theory for supercritical wave equations. I. Local theory. Invent. Math. **173**(3), 449–475 (2008)
8. Carleson, L.: Some analytic problems related to statistical mechanics. In: Euclidean harmonic analysis (Proc. Sem., Univ. Maryland, College Park, Md., 1979), volume 779 of Lecture Notes in Math., pp. 5–45. Springer, Berlin (1980)
9. Compaan, E., Lucà, G. Staffilani, R.: Pointwise convergence of the Schrödinger flow. Int. Math. Res. Not. **1**, 596–647 (2021)
10. Cowling, M.G.: Pointwise behavior of solutions to Schrödinger equations. In: Harmonic analysis (Cortona, 1982), volume 992 of Lecture Notes in Math., pp. 83–90. Springer, Berlin, (1983)
11. Dahlberg, B.E.J., Kenig, C.E.: A note on the almost everywhere behavior of solutions to the Schrödinger equation. In: Harmonic Analysis (Minneapolis, Minn., 1981), volume 908 of Lecture Notes in Math., pp. 205–209. Springer, Berlin-New York (1982)
12. Demeter, C., Guo, S.: Schrödinger maximal function estimates via the pseudoconformal transformation (2016)
13. Deng, Y., Nahmod, A., Yue, H.: Invariant Gibbs measures and global strong solutions for nonlinear Schrödinger equations in dimension two (2019). arXiv:1910.08492

14. Du, X., Zhang, R.: Sharp l^2 estimates of the Schrödinger maximal function in higher dimensions. Ann. of Math. **189**(3), 837 (2019)
15. Du, X., Guth, L., Li, X.: A sharp Schrödinger maximal estimate in $\mathbb{R}^2$. Ann. Math. **186**(2), 607–640 (2017)
16. Du, X., Guth, L., Li, X., Zhang, R.: Pointwise convergence of Schrödinger solutions and multilinear refined Strichartz estimates. Forum Math. Sigma **6**, e14, 18 (2018)
17. Eceizabarrena, D., Lucà, R.: Convergence over fractals for the periodic Schrödinger equation. Anal. PDE (2020). arXiv preprint arXiv:2005.07581
18. Erdoğan, M.B., Tzirakis, N.: Dispersive Partial Differential Equations: Wellposedness and Applications. London Mathematical Society Student Texts. Cambridge University Press, Cambridge (2016)
19. Genovese, G., Lucà, R., Valeri, D.: Gibbs measures associated to the integrals of motion of the periodic derivative nonlinear Schrödinger equation. Selecta Math. (N.S.) **22**(3), 1663–1702 (2016)
20. Lee, S.: On pointwise convergence of the solutions to Schrödinger equations in $\mathbb{R}^2$. Int. Math. Res. Not. Art. ID 32597, 21 (2006)
21. Lucà, R., Ponce-Vanegas, F.: Convergence over fractals for the Schrödinger equation. Indiana Univ. Math. J.. Preprint on arXiv:2101.02495
22. Lucà, R., Rogers, K.: Average decay of the Fourier transform of measures with applications. J. Eur. Math. Soc. **21**(2), 465–506 (2018)
23. Lucà, R., Rogers, K.M.: Coherence on fractals versus pointwise convergence for the Schrödinger equation. Commun. Math. Phys. **351**(1), 341–359 (2017)
24. Lucà, R., Rogers, K.M.: A note on pointwise convergence for the Schrödinger equation. Math. Proc. Camb. Philos. Soc. **166**(2), 209–218 (2019)
25. Lührmann, J., Mendelson, D.: Random data Cauchy theory for nonlinear wave equations of power-type on $\mathbb{R}^3$. Commun. Partial Differ. Equ. **39**(12), 2262–2283 (2014)
26. Moyua, A., Vargas, A., Vega, L.: Restriction theorems and maximal operators related to oscillatory integrals in $\mathbb{R}^3$. Duke Math. J. **96**(3), 547–574 (1999)
27. Moyua, A., Vega, L.: Bounds for the maximal function associated to periodic solutions of one-dimensional dispersive equations. Bull. Lond. Math. Soc. **40**(1), 117–128 (2008)
28. Sjölin, P.: Regularity of solutions to the Schrödinger equation. Duke Math. J. **55**(3), 699–715 (1987)
29. Tao, T.: A sharp bilinear restrictions estimate for paraboloids. Geom. Funct. Anal. **13**(6), 1359–1384 (2003)
30. Tao, T., C.B. of the Mathematical Sciences: Nonlinear Dispersive Equations: Local and Global Analysis. Conference Board of the Mathematical Sciences. Regional Conference Series in Mathematics. American Mathematical Society (2006)
31. Tao, T., Vargas, A.: A bilinear approach to cone multipliers. I. Restriction estimates. Geom. Funct. Anal. **10**(1), 185–215 (2000)
32. Tao, T., Vargas, A.: A bilinear approach to cone multipliers. II. Applications. Geom. Funct. Anal. **10**(1), 216–258 (2000)
33. Thomann, L., Tzvetkov, N.: Gibbs measure for the periodic derivative nonlinear Schrödinger equation. Nonlinearity **23**(11), 2771–2791 (2010)
34. Vega, L.: Schrödinger equations: pointwise convergence to the initial data. Proc. Am. Math. Soc. **102**(4), 874–878 (1988)
35. Wang, X., Zhang, C.: Pointwise convergence of solutions to the Schrödinger equation on manifolds. Can. J. Math. **71**(4), 983—995 (2019)