

Dynamics of Solutions to the Gross–Pitaevskii Equation Describing Dipolar Bose–Einstein Condensates



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Abstract We review some recent results on the long-time dynamics of solutions to the Gross–Pitaevskii equation (GPE) governing non-trapped dipolar quantum gases. We describe the asymptotic behaviors of solutions for different initial configurations of the initial datum in the energy space, specifically for data below, above, and at the mass–energy threshold. We revisit some properties of powers of the Riesz transforms by means of the decay properties of the integral kernel associated to the parabolic biharmonic equation. These decay properties play a fundamental role in establishing the dynamical features of the solutions to the studied GPE.

1 Introduction

In this chapter, we review some recent progresses concerning the dynamics of solutions to the following Gross–Pitaevskii equation (GPE) that models a so-called dipolar Bose–Einstein condensate (BEC) at low temperatures, see [4, 30, 33, 35–37]:

$$ih \frac{\partial u}{\partial t} = -\frac{\hbar^2}{2m} \Delta u + W(x)u + U_0 |u|^2 u + (V_{dip} * |u|^2)u. \quad (1)$$

In the equation above, t is the time variable, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the space variable, $*$ denotes the convolution, and $u = u(t, x)$ is a complex function. The physical parameters appearing in (1) are: the Planck constant \hbar , the mass m of a dipolar particle, and $U_0 = 4\pi\hbar^2 a_s/m$ describes the strength of the local interaction

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between dipoles in the condensate, where a_s is the s -wave scattering length, which may have positive or negative sign according to the repulsive/attractive nature of the interaction. The non-local, long-range dipolar interaction potential between two dipoles is given instead by the convolution through the potential

$$V_{dip}(x) = \frac{\mu_0 \mu_{dip}^2}{4\pi} \frac{1 - 3 \cos^2(\theta)}{|x|^3}, \quad x \in \mathbb{R}^3,$$

where μ_0 is the vacuum magnetic permeability, μ_{dip} is the permanent magnetic dipole moment, and θ is the angle between the dipole axis and the vector x . Without loss of generality, we can assume the dipole axis to be the vector $(0, 0, 1)$. The potential $W(x)$ is an external trapping potential that will be not considered in the sequel; namely, we study the case $W(x) = 0$.

In the next subsections, we describe the mathematical background on a rescaled version of the model (1), and we state the main results.

1.1 Background

For a mathematical treatment of the equation above, we consider (1) in its dimensionless form, and in particular we study the associated Cauchy problem in the energy space (i.e., $H^1(\mathbb{R}^3)$) as follows:

$$\begin{cases} i \partial_t u + \frac{1}{2} \Delta u = \lambda_1 |u|^2 u + \lambda_2 (K * |u|^2) u, & (t, x) \in \mathbb{R} \times \mathbb{R}^3 \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}^3) \end{cases}, \quad (2)$$

where the dipolar kernel K is now given by

$$K(x) = \frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^5}. \quad (3)$$

Provided we normalize the wave function according to $\int_{\mathbb{R}^3} |u(x, t)|^2 dx = N$, whereas N is the total number of dipolar particles in the dipolar BEC, then the two real coefficients λ_1 and λ_2 are defined by $\lambda_1 = 4\pi a_s N \sqrt{\frac{m}{\hbar}}$, and $\lambda_2 = \frac{N \mu_0 \mu_{dip}^2}{4\pi} \sqrt{\frac{m^3}{\hbar^3}}$, and they are two physical parameters describing the strength of the non-linearities involved in the equation, specifically the local one given by $|u|^2 u$, and the non-local one given by $(K * |u|^2) u$, respectively.

At least formally, the solution $u(t)$ to (2) preserves the mass and the energy of the initial datum $u(0) = u_0$, specifically

$$M(u(t)) := \|u(t)\|_{L^2(\mathbb{R}^3)}^2 = M(u(0)) \quad (4)$$

and

$$\begin{aligned} E(u(t)) &:= \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u(t)|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} \left(\lambda_1 |u(t)|^4 + \lambda_2 (K * |u(t)|^2) |u(t)|^2 \right) dx \\ &= E(u(0)), \end{aligned} \tag{5}$$

where $M(u(t))$ and $E(u(t))$ define the mass and the energy, respectively. For later purpose, we introduce the notation

$$H(f) := \|\nabla f\|_{L^2(\mathbb{R}^3)}^2$$

for the kinetic energy, and

$$P(f) := \int_{\mathbb{R}^3} \left(\lambda_1 |f(x)|^4 + \lambda_2 (K * |f(x)|^2) |f(x)|^2 \right) dx$$

for the potential energy; hence, we rewrite

$$E(u(t)) = \frac{1}{2} (H(u(t)) + P(u(t))).$$

Assuming a local-in-time existence theory for (2) (which is guaranteed by the work of Carles, Markowich, and Sparber, see [9]), and assuming enough regularity of the solutions, the conservation laws (4) and (5) can be proved by a simple integration by parts; a rigorous justification in the energy space $H^1(\mathbb{R}^3)$ (note that in this Sobolev space, the energy functional is well defined) can be done by an approximation argument. Besides the functionals E , H , and P above, we introduce the Pohozaev functional

$$G(f) := H(f) + \frac{3}{2} P(f). \tag{6}$$

It is worth observing that the functional G is (up to a 1/4 factor) the second derivative in time of the virial functional associated to (2), i.e.,

$$G(u(t)) = \frac{1}{4} \frac{d^2}{dt^2} V(t),$$

where $V(t) := V(u(t))$ stands for the variance at time t of the mass density, namely

$$V(t) := \int_{\mathbb{R}^3} |x|^2 |u(t, x)|^2 dx. \tag{7}$$

Motivated by the definition of the functional V , we introduce the space of functions $\Sigma \subset H^1(\mathbb{R}^3)$ as $\Sigma := H^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3; |x|^2 dx)$.

Following the work by Carles, Markowich, and Sparber [9], we introduce the partition of the coordinate plane (λ_1, λ_2) given by the two sets below:

$$UR := \begin{cases} \lambda_1 - \frac{4\pi}{3}\lambda_2 < 0 & \text{if } \lambda_2 > 0 \\ \lambda_1 + \frac{8\pi}{3}\lambda_2 < 0 & \text{if } \lambda_2 < 0 \end{cases}, \quad (8)$$

and its complementary set in \mathbb{R}^2 , namely

$$SR := \begin{cases} \lambda_1 - \frac{4\pi}{3}\lambda_2 \geq 0 & \text{if } \lambda_2 > 0 \\ \lambda_1 + \frac{8\pi}{3}\lambda_2 \geq 0 & \text{if } \lambda_2 < 0 \end{cases}. \quad (9)$$

The two sets above are called *unstable regime* (see (8)) and *stable regime* (see (9)), respectively.

The separation of the parameters λ_1 and λ_2 as in the regions (8) and (9) is crucial in establishing the dynamics of solutions to (2). Indeed, there are two main differences when working in the unstable regime instead of the stable regime. First, in (8), the conservation of the energy does not imply a boundedness in the kinetic term; second, the solutions to the stationary equation (see (10) below) associated to (2) do exist. Hence, at least in a naive way, we can think to the unstable/stable regimes as the analogous for the Gross–Pitaevskii equation (2) of the focusing/defocusing characters for the usual cubic NLS equation. However, note that here it is improper to speak about defocusing/focusing character for (2) since even for two positive coefficients of the nonlinear terms $0 < \lambda_1 < \frac{4\pi}{3}\lambda_2$ finite-time blow-up solutions may come up. See [9, Lemma 5.1], where negative energy solutions are constructed. We also mention here that in the stable regime, we proved in [5] that for any initial datum in $H^1(\mathbb{R}^3)$ the corresponding solution to (2) is global in time and scatters.

Similarly to the classical NLS equation (and more in general to other dispersive PDEs), a fundamental tool toward a classification of Cauchy data $u_0 \in H^1(\mathbb{R}^3)$ as in (2) leading to global (and scattering) solutions versus blowing-up solutions is given by means of quantities related to the solutions of the stationary equation associated to (2):

$$-\frac{1}{2}\Delta Q_\mu + \mu Q_\mu + \lambda_1 |Q_\mu|^2 Q_\mu + \lambda_2 (K * |Q_\mu|^2) Q_\mu = 0, \quad \mu > 0. \quad (10)$$

Notice that if Q_μ solves (10), then $u(t, x) := e^{-i\mu t} Q_\mu(x)$ solves (2). Moreover, by an elementary scaling argument, $E(Q_\mu)M(Q_\mu) = E(Q_1)M(Q_1)$ for all $\mu > 0$. For sake of simplicity in the notation, we will call Q the standing wave solutions with $\mu = 1$. In particular, some bounds for the product of the mass and the energy of an initial datum in terms of the mass and energy of solutions Q to (10) allow to

determine whether a solution $u(t)$ to (2) exists for all time and scatters, or formation of singularities in finite (or infinite) time may arise. Indeed, sufficient conditions on $u_0 \in H^1(\mathbb{R}^3)$ for the scattering/blow-up scenario are given by the relations below:

$$(SC) := \begin{cases} E(u_0)M(u_0) < E(Q)M(Q) \\ H(u_0)M(u_0) < H(Q)M(Q) \end{cases}, \quad (11)$$

and

$$(BC) := \begin{cases} E(u_0)M(u_0) < E(Q)M(Q) \\ H(u_0)M(u_0) > H(Q)M(Q) \end{cases}, \quad (12)$$

respectively. The above conditions on initial data are referred to as the mass–energy (of the initial datum) below the threshold, the latter given by the quantity $E(Q)M(Q)$.

As mentioned above, in the unstable regime (8), the existences of solutions to (10) do exist, and it was proved in two different papers by Antonelli and Sparber, see [2], and later by the first author and Jeanjean, see [8], by employing two different methods. In the former work, the existences of ground states (i.e., standing wave solutions that minimize the energy functional $E(u)$ among all the standing solutions with prescribed mass) are proved by means of minimizing a Weinstein-type functional, while in the latter a geometrical approach is used, specifically by proving that the energy functional satisfied a mountain pass geometry. As for the usual cubic NLS, it turns out that a ground state Q related to the elliptic equation gives an optimizer for the Gagliardo–Nirenberg-type inequality

$$-P(f) \leq C_{GN}(H(f))^{\frac{3}{2}}(M(f))^{\frac{1}{2}}, \quad (13)$$

for $f \in H^1(\mathbb{R}^3)$, meaning that $C_{GN} = -P(Q)/(H(Q))^{\frac{3}{2}}(M(Q))^{\frac{1}{2}}$. Furthermore, the Pohozaev identities tell us that $H(Q) = 6M(Q) = -\frac{3}{2}P(Q)$, and by the latter relations, we have that $E(Q) = \frac{1}{6}H(Q) = -\frac{1}{4}P(Q)$ and that

$$E(Q)M(Q) = \frac{1}{6}H(Q)M(Q) = -\frac{1}{4}P(Q)M(Q) = \frac{2}{27}(C_{GN})^{-2}. \quad (14)$$

It is important to remark that uniqueness of ground states—even up to the action of some symmetry—is unknown; nonetheless, by (14), we can see that the quantities $E(Q)M(Q)$, $H(Q)M(Q)$, and $P(Q)M(Q)$ are independent of the choice of the ground state.

In the paper, we will also give dynamics results for solutions with arbitrarily large initial data (although by imposing some other hypothesis on u_0 and/or by further restricting the conditions on the parameters λ_1 and λ_2 to a subset of the unstable regime), hence by considering data such that $E(u_0)M(u_0) > E(Q)M(Q)$, and for data exactly at the threshold, i.e., for data satisfying $E(u_0)M(u_0) = E(Q)M(Q)$.

See the next subsection, where we enunciate the main results on the dynamics of solutions to (2).

1.2 Main Results

We conclude the Introduction by stating the main results contained in the paper. We separate them according to the fact that the initial data are below, above, or at the threshold determined by $E(Q)M(Q)$.

1.2.1 Dynamics Below the Threshold

We start by giving the scattering theorem and the blow-up in finite-time theorem, for solutions to (2) arising from initial data below the mass–energy threshold, described in terms of a solution Q of the elliptic equation (10). In what follows, $e^{it\frac{1}{2}\Delta}$ denotes the unitary Schrödinger propagator, namely $v(t, x) = e^{it\frac{1}{2}\Delta}v_0$ solves $i\partial_t v + \frac{1}{2}\Delta v = 0$, with $v(0, x) = v_0$. As already mentioned above, local well-posedness for (2) was established in [9], by a usual fixed-point argument based on Strichartz spaces, and upon having established some basic properties on the convolution kernel K , see Proposition 1 and Lemma 1 below. In what follows, we denote by $T_{min} > 0$ and $T_{max} > 0$ the minimal and maximal times of existence of a solution to (2), respectively.

The asymptotic dynamics for data below the threshold has been proved by the authors in [5] and [6]. In [5], we proved the following.

Theorem 1 *Let λ_1 and λ_2 satisfy (8), namely they belong to the unstable regime. Let $u_0 \in H^1(\mathbb{R}^3)$ satisfy (11), where Q is a ground state related to (10). Then the corresponding solution $u(t)$ to (2) exists globally in time and scatters in $H^1(\mathbb{R}^3)$ in both directions, that is, there exist $u_0^\pm \in H^1(\mathbb{R}^3)$ such that*

$$\lim_{t \rightarrow \pm\infty} \|u(t) - e^{it\frac{1}{2}\Delta}u_0^\pm\|_{H^1(\mathbb{R}^3)} = 0.$$

The theorem above is obtained by implementing a concentration/compactness and rigidity scheme, as we will explain in the next subsections.

In order to state the blow-up results that we proved in [6], let us define $\bar{x} = (x_1, x_2)$, and let us introduce the functional space

$$\Sigma_3 = \left\{ u \in H^1(\mathbb{R}^3) \quad s.t. \quad u(x) = u(|\bar{x}|, x_3) \quad \text{and} \quad u \in L^2(\mathbb{R}^3; x_3^2 dx) \right\},$$

namely the space of cylindrical symmetric functions (note that with an abuse of notation, we indicate with u both the function in the three variables (x_1, x_2, x_3) and the function in the two variables (\bar{x}, x_3)) with finite variance in the x_3 direction.

We have the following.

Theorem 2 *Assume that λ_1 and λ_2 satisfy (8), namely they belong to the unstable regime. Let $u(t) \in \Sigma_3$ be a solution to (2) defined on $(-T_{\min}, T_{\max})$, with initial datum u_0 satisfying (11), where Q is a ground state related to (10). Then T_{\min} and T_{\max} are finite, namely $u(t)$ blows up in finite time.*

It is worth mentioning that for both the scattering and the blow-up result, the main difficulty with respect to other NLS non-local models is the precise structure of the dipolar kernel. Moreover, no radial symmetry for the solutions can be assumed in our context, as the convolution with radial function would make disappear the contribution of the non-local term, hence reducing the equation to a standard cubic NLS. Thus, the blow-up result above for cylindrical symmetric solution is somehow the best one may obtain; let us recall that finite-time blow-up without assuming any structure on the solutions is still unknown even for the usual focusing cubic NLS equation. Moreover, we point out that the dipolar kernel K enjoys a cylindrical symmetry, so our assumption is also physically consistent.

As said above, similarly to the classical cubic focusing NLS, if we do not assume any additional hypothesis on the initial datum, as in Theorem 2 for example, we cannot prove that the solutions blow up in finite time. Nonetheless, in [10], Dinh, Hajaiej, and the second author proved the following.

Theorem 3 *Let λ_1 and λ_2 satisfy (8). Let $u(t)$ be a $H^1(\mathbb{R}^3)$ solution to (2), defined on the maximal forward time interval $[0, T_{\max})$. Assume that there exists a positive constant $\delta > 0$ such that*

$$\sup_{t \in [0, T_{\max})} G(u(t)) \leq -\delta. \quad (15)$$

Then either the maximal forward time $T_{\max} < \infty$ or $T_{\max} = \infty$, and there exists a diverging sequence of times, say $t_n \rightarrow \infty$ as $n \rightarrow \infty$, such that $\lim_{n \rightarrow \infty} \|u(t_n)\|_{\dot{H}^1(\mathbb{R}^3)} = \infty$. In the latter case, we say that the solution grows up.

The next corollary actually shows that the condition given in Theorem 3 is non-empty, as an initial datum belonging to the region (BC) , i.e., (12) is satisfied, leads to a solution satisfying (15) (see our paper [6, Section 3]).

Corollary 1 *Let λ_1 and λ_2 satisfy (8), and Q be a ground state related to (10). Assume that $u_0 \in H^1(\mathbb{R}^3)$ satisfies (12), and let $u(t)$ the corresponding solution to (2). Then (15) holds, and therefore either $T_{\max} < \infty$ or $T_{\max} = \infty$, and $u(t)$ grows up.*

1.2.2 Dynamics Above the Threshold

For the dynamical properties of solutions to (2) above the threshold, we need to further restrict the unstable regime, and we introduce the restricted unstable regime

as follows:

$$RUR := \begin{cases} \lambda_1 + \frac{8\pi}{3}\lambda_2 < 0 & \text{if } \lambda_2 > 0 \\ \lambda_1 - \frac{4\pi}{3}\lambda_2 < 0 & \text{if } \lambda_2 < 0 \end{cases}. \quad (16)$$

For a ground state Q related to (10), we also give the scattering or blow-up conditions above the threshold:

$$(SC') := \begin{cases} E(u_0)M(u_0) \geq E(Q)M(Q) \\ \frac{E(u_0)M(u_0)}{E(Q)M(Q)} \left(1 - \frac{(V'(0))^2}{8E(u_0)V(0)}\right) \leq 1, \\ -P(u_0)M(u_0) < -P(Q)M(Q) \\ V'(0) \geq 0 \end{cases}, \quad (17)$$

and

$$(BC') := \begin{cases} E(u_0)M(u_0) \geq E(Q)M(Q) \\ \frac{E(u_0)M(u_0)}{E(Q)M(Q)} \left(1 - \frac{(V'(0))^2}{8E(u_0)V(0)}\right) \leq 1, \\ -P(u_0)M(u_0) > -P(Q)M(Q) \\ V'(0) \leq 0 \end{cases}, \quad (18)$$

respectively. Initial data satisfying (17) or (18) can be constructed by a simple scaling argument, see [10] and [20].

The following blow-up result above the threshold has been given by Gao and Wang in [20].

Theorem 4 *Let λ_1 and λ_2 satisfy (16). Let Q be a ground state related to (10), and $u_0 \in \Sigma$ satisfy (18). Then the corresponding solution $u(t)$ to (2) blows up forward in finite time.*

The counterpart of Theorem 4 is the following scattering result, given for initial data satisfying (17). It is one of the main theorems contained in the paper by Dinh, Hajaiej, and the second author [10].

Theorem 5 *Let λ_1 and λ_2 satisfy (16). Let Q be a ground state related to (10), and $u_0 \in \Sigma$ be such that (17) holds true. Then the corresponding solution $u(t)$ to (2) exists globally and scatters in $H^1(\mathbb{R}^3)$ forward in time.*

Concerning the theorems above, it is worth mentioning the reason why we have to consider the subset RUR of the unstable regime, see (16), instead of the whole configurations of the parameters λ_1 and λ_2 as in (8). Condition (16) implies a

control on the potential energy sign; specifically, it is negative for any time along the evolution of the solution. This will play a crucial role in the proof of the scattering criterion Theorem 8 below.

1.2.3 Dynamics at the Threshold

The next theorem deals with the long-time dynamics for solutions to (2) at the mass–energy threshold, i.e., when the initial datum satisfies

$$E(u_0)M(u_0) = E(Q)M(Q). \quad (19)$$

In [10], Dinh, Hajaiej, and the second present author gave a complete picture of the dynamics under the hypothesis (19), analyzing several different scenarios described in terms of the quantity $H(u_0)M(u_0)$. To the best of our knowledge, early results for the focusing cubic NLS at the threshold are given in the work of Duyckaerts and Roudenko [14]. The theorem is as follows.

Theorem 6 *Let λ_1 and λ_2 satisfy (8). Let Q be a ground state related to (10). Suppose that $u_0 \in H^1(\mathbb{R}^3)$ satisfies the mass–energy threshold condition (19). We have the following three scenarios.*

(i) *In addition to (19), suppose that*

$$H(u_0)M(u_0) < H(Q)M(Q) \quad (20)$$

and that the corresponding solution $u(t)$ to (2) is defined on the maximal interval of existence $(-T_{min}, T_{max})$. Then for every $t \in (-T_{min}, T_{max})$

$$H(u(t))M(u(t)) < H(Q)M(Q)$$

and in particular $T_{min} = T_{max} = \infty$. Moreover, provided λ_1 and λ_2 satisfy (16), the solution:

- *Either scatters in $H^1(\mathbb{R}^3)$ forward in time*
- *Or there exist a diverging sequence of times $t_n \rightarrow \infty$ as $n \rightarrow \infty$, a ground state \tilde{Q} related to (10), and a sequence $\{y_n\}_{n \geq 1} \subset \mathbb{R}^3$ such that for some $\theta \in \mathbb{R}$ and $\mu > 0$*

$$u(t_n, \cdot - y_n) \rightarrow e^{i\theta} \mu \tilde{Q}(\mu \cdot) \quad (21)$$

strongly in $H^1(\mathbb{R}^3)$ as $n \rightarrow \infty$.

(ii) *In addition to (19), suppose that*

$$H(u_0)M(u_0) = H(Q)M(Q), \quad (22)$$

then there exists a ground state \tilde{Q} related to (10) such that the solution $u(t)$ to (2) satisfies $u(t, x) = e^{i\mu^2 t} e^{i\theta} \mu \tilde{Q}(\mu x)$ for some $\theta \in \mathbb{R}$ and $\mu > 0$, and hence the solution is global.

(iii) In addition to (19), suppose that

$$H(u_0)M(u_0) > H(Q)M(Q) \quad (23)$$

and that the corresponding solution $u(t)$ to (2) is defined on the maximal interval of existence $(-T_{min}, T_{max})$. Then for every $t \in (-T_{min}, T_{max})$,

$$H(u(t))M(u(t)) > H(Q)M(Q).$$

Furthermore, the solution:

- Either blows up forward in finite time
- Or it grows up along some diverging sequence of times $t_n \rightarrow \infty$ as $n \rightarrow \infty$
- Or there exists a diverging sequence of times $t_n \rightarrow \infty$ as $n \rightarrow \infty$ such that (21) holds for some sequence $\{y_n\}_{n \geq 1} \subset \mathbb{R}^3$, and some parameters $\theta \in \mathbb{R}$, and $\mu > 0$.

Provided $u_0 \in \Sigma$, the grow-up scenario as in the second point is ruled out.

2 Decay for Powers of Riesz Transforms and Virial Arguments

This section provides the first technical tools we need in order to prove our main results. Moreover, we present the strategy we adopt to prove the main theorems, which strongly rely on virial arguments based on the decay for powers of Riesz transforms that we are going to prove.

First of all, we recall the fact that the dipolar kernel defines a Calderón–Zygmund operator; hence, it is a well-known fact that it yields to a map continuous from L^p into itself, for non-end-point Lebesgue exponents, namely for $p \neq 1$ and $p \neq \infty$. For a proof, see [9, Lemma 2.1].

Proposition 1 *The convolution operator $f \mapsto K * f$ can be extended as a continuous operator from L^p into itself, for any $p \in (1, \infty)$.*

Moreover, in [9], an explicit computation of the Fourier transform of the dipolar kernel K defined in (3) is given. Precisely, we have the following.

Lemma 1 *The Fourier transform of the dipolar kernel K is given by*

$$\hat{K}(\xi) = \frac{4\pi}{3} \frac{2\xi_3^2 - \xi_2^2 - \xi_1^2}{|\xi|^2}, \quad \xi \in \mathbb{R}^3. \quad (24)$$

Straightforwardly, it follows that $\hat{K} \in \left[-\frac{4}{3}\pi, \frac{8}{3}\pi\right]$.

For a proof of (24), we refer to [9, Lemma 2.3]. The explicit calculation of \hat{K} is done by means of the decomposition in spherical harmonics of the Fourier character $e^{-ix \cdot \xi}$.

2.1 Integral Estimates for \mathcal{R}_j^4

In the next propositions, we prove some decay estimates—point-wise and integral ones—regarding the square and the fourth power of the Riesz transforms when acting on suitably localized functions. First, we disclose a link between the fourth power of the Riesz transform \mathcal{R}_j^4 and the linear propagator associated to the parabolic biharmonic equation, defined in terms of the Bessel functions. With this correspondence and some decay estimates for the parabolic biharmonic integral kernel, we are able to show the decay estimates for $\langle \mathcal{R}_j^4 f, g \rangle$. Here $\langle \cdot, \cdot \rangle$ stands for the usual $L^2(\mathbb{R}^3)$ inner product. We start with the integral estimates for the fourth power of the Riesz transforms, and, as anticipated above, we do it by means of some decay properties of the kernel associated to the parabolic biharmonic equation

$$\partial_t w + \Delta^2 w = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3. \quad (25)$$

We denote by P_t the linear propagator associated to (25), namely $w(t, x) := P_t w_0(x)$ denotes the solution to Eq.(25) with initial datum w_0 . We begin with the following proposition that provides a representation of \mathcal{R}_j^4 by using the functional calculus. Since now on, we will omit—unless necessary—the notation \mathbb{R}^3 , as we are concerned only with the three-dimensional model.

Lemma 2 *For any two functions in L^2 , we have the following identity:*

$$\langle \mathcal{R}_i^4 f, g \rangle = - \int_0^\infty \langle \partial_{x_i}^4 \frac{d}{dt} P_t f, g \rangle t dt. \quad (26)$$

Proof By passing in the frequencies space, it is easy to see that $\widehat{P_t f}(\xi) := e^{-t|\xi|^4} \widehat{f}(\xi)$, and we observe, by integration by parts, that

$$\xi_i^4 |\xi|^4 \int_0^\infty e^{-t|\xi|^4} t dt = \frac{\xi_i^4}{|\xi|^4}; \quad (27)$$

hence,

$$\begin{aligned} \int_0^\infty \langle \partial_{x_i}^4 \frac{d}{dt} P_t f, g \rangle t dt &= \langle \int_0^\infty \partial_{x_i}^4 \frac{d}{dt} (P_t f) t dt, g \rangle \\ &= (2\pi)^{-3} \langle \int_0^\infty \xi_i^4 \frac{d}{dt} (e^{-t|\xi|^4} \widehat{f}) t dt, \widehat{g} \rangle \\ &= -(2\pi)^{-3} \langle \xi_i^4 |\xi|^4 \widehat{f} \int_0^\infty e^{-t|\xi|^4} t dt, \widehat{g} \rangle \\ &= -(2\pi)^{-3} \langle \frac{\xi_i^4}{|\xi|^4} \widehat{f}, \widehat{g} \rangle \\ &= -\langle \mathcal{R}_i^4 f, g \rangle, \end{aligned}$$

where the change of order of integration (in time and in space) is justified by means of the Fubini–Tonelli’s theorem, and we used the Plancherel identity when passing from the frequencies space to the physical space, and vice versa. \square

We are now in position to prove a decay estimate for functions supported outside a cylinder of radius $\sim R$. In order to do that, we explicitly write the integral kernel of the propagator P_t . We introduce, for $t > 0$ and $x \in \mathbb{R}^3$,

$$p_t(x) = \alpha \frac{k(\mu)}{t^{3/4}}, \quad \mu = \frac{|x|}{t^{1/4}},$$

and

$$k(\mu) = \mu^{-2} \int_0^\infty e^{-s^4} (\mu s)^{3/2} J_{1/2}(\mu s) ds,$$

where $J_{1/2}$ is the $\frac{1}{2}$ -th Bessel function, and $\alpha^{-1} := \frac{4\pi}{3} \int_0^\infty s^2 k(s) ds$ is a positive normalization constant. We refer to [17] for these definitions and further discussions about the integral kernel of the parabolic biharmonic equation. We recall that the $\frac{1}{2}$ -th Bessel function is given by

$$J_{1/2}(s) = (\pi/2)^{-1/2} s^{-1/2} \sin(s),$$

then

$$P_t f(x) = (p_t * f)(x) = c \int f(x-y) \int_0^\infty \frac{1}{|y|^3} e^{-ts^4/|y|^4} s \sin(s) ds dy,$$

and therefore,

$$\frac{d}{dt} P_t f(x) = -c \int f(x-y) \int_0^\infty \frac{1}{|y|^3} e^{-ts^4/|y|^4} \frac{s^5}{|y|^4} \sin(s) ds dy.$$

We are ready to prove the following result.

Proposition 2 *Assume that $f, g \in L^1 \cap L^2$ and that f is supported in $\{|\bar{x}| \geq \gamma_2 R\}$, while g is supported in $\{|\bar{x}| \leq \gamma_1 R\}$, for some positive parameters $\gamma_{1,2}$ satisfying $d := \gamma_2 - \gamma_1 > 0$. Then*

$$|\langle \mathcal{R}_t^d f, g \rangle| \lesssim R^{-1} \|g\|_{L^1} \|f\|_{L^1}. \quad (28)$$

Proof With the change of variable $s^4|y|^{-4} = \tau$, we get

$$\frac{d}{dt} P_t f = -\frac{c}{4} \int \int_0^\infty \frac{1}{|y|} e^{-t\tau} \tau^{1/2} \sin(\tau^{1/4}|y|) f(x-y) d\tau dy,$$

and hence, by a change of variable in space,

$$\frac{d}{dt} P_t f = -\frac{c}{4} \int \int_0^\infty \frac{1}{|x-y|} e^{-t\tau} \tau^{1/2} \sin(\tau^{1/4}|x-y|) f(y) d\tau dy.$$

We will use the following, by adopting the notation deg for the degree of a polynomial.

Claim There exist $M \geq 1$ and M pairs of polynomials $(\tilde{q}_k, q_k)_{k \in \{1, \dots, M\}}$ with nonnegative coefficients, such that

$$\min_{k \in \{1, \dots, M\}} \{deg(q_k)\} \geq 1,$$

and satisfying

$$\left| \partial_{x_i}^4 \left(\frac{1}{|x-y|} \sin(\tau^{1/4}|x-y|) \right) \right| \lesssim \sum_{k=1}^M \frac{\tilde{q}_k(\tau^{1/4})}{q_k(|x-y|)}.$$

At this point, by using the identity (26), we infer the following:

$$\begin{aligned}
|\langle \mathcal{R}_i^4 f, g \rangle| &= \left| \int_0^\infty \langle \partial_{x_i}^4 \frac{d}{dt} P_t f, g \rangle t dt \right| \\
&\lesssim \left| \int_0^\infty t \int g(x) \times \right. \\
&\quad \left. \left(\int \int_0^\infty \partial_{x_i}^4 \left(\frac{1}{|x-y|} \sin(\tau^{1/4}|x-y|) \right) e^{-t\tau} \tau^{1/2} f(y) d\tau dy \right) dx dt \right| \\
&\lesssim \int_0^\infty t \int |g(x)| \times \\
&\quad \left(\int \int_0^\infty \sum_{k=1}^M \frac{\tilde{q}_k(\tau^{1/4})}{q_k(|x-y|)} e^{-t\tau} \tau^{1/2} |f(y)| d\tau dy \right) dx dt \\
&= \int_0^\infty t \int |g(x)| \times \\
&\quad \left(\int \int_0^\infty \sum_{k=1}^M \frac{\tilde{q}_k(\tau^{1/4})}{q_k(|y|)} e^{-t\tau} \tau^{1/2} |f(x-y)| d\tau dy \right) dx dt \\
&\leq \int_0^\infty t \int_{\{|\bar{x}| \leq \gamma_1 R\}} |g(x)| \times \\
&\quad \left(\int_{\{|\bar{x}-\bar{y}| \geq \gamma_2 R\}} \int_0^\infty \sum_{k=1}^M \frac{\tilde{q}_k(\tau^{1/4})}{q_k(|\bar{y}|)} e^{-t\tau} \tau^{1/2} |f(x-y)| d\tau dy \right) dx dt.
\end{aligned}$$

Therefore, as the support of $f(x-y)$ is contained in $|\bar{x}-\bar{y}| \geq \gamma_2 R$ and the one of g is contained in $|\bar{x}| \leq \gamma_1 R$, we get that $|\bar{y}| \geq dR$. Hence, by defining $\beta = \frac{1}{4} \max_{k \in \{1, \dots, M\}} \{deg(\tilde{q}_k)\}$, we can bound

$$\begin{aligned}
|\langle \mathcal{R}_i^4 f, g \rangle| &\lesssim R^{-1} \|f\|_{L^1} \|g\|_{L^1} \sum_{k=1}^M \int_0^\infty \int_0^\infty e^{-t\tau} \tau^{1/2} \tilde{q}_k(\tau^{1/4}) t d\tau dt \\
&\lesssim R^{-1} \|f\|_{L^1} \|g\|_{L^1} \int_1^\infty \int_1^\infty e^{-t\tau} \tau^{1/2} \tau^{(\max_{k \in \{1, \dots, M\}} deg(\tilde{q}_k))/4} t d\tau dt \\
&= R^{-1} \|f\|_{L^1} \|g\|_{L^1} \int_1^\infty \int_1^\infty e^{-t\tau} \tau^{\beta+1/2} t d\tau dt \\
&\lesssim R^{-1} \|f\|_{L^1} \|g\|_{L^1},
\end{aligned}$$

where we used the Fubini–Tonelli’s theorem. The proof of (28) is concluded. \square

We now give the proof of the claim above.

Proof of the Claim As the derivative is invariant under translations and by defining $c = \tau^{1/4}$, we can reduce everything to the estimate of $\partial_{x_i}^4 (|x|^{-1} \sin(c|x|))$. By setting $f(r) = r^{-1} \sin(cr)$ and $g(x) = |x|$, we can see

$$|x|^{-1} \sin(c|x|) = (f \circ g)(x),$$

and without loss of generality, we assume $i = 3$. Then we see g as a function of x_3 alone, i.e., $g(x_3) = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. We first collect some identities.

$$\begin{aligned} f'(r) &= cr^{-1} \cos(cr) - r^{-2} \sin(cr), \\ f''(r) &= -c^2 r^{-1} \sin(cr) - 2cr^{-2} \cos(cr) + 2r^{-3} \sin(cr), \\ f'''(r) &= -c^3 r^{-1} \cos(cr) + c^2 r^{-2} \sin(cr) + 8cr^{-3} \cos(cr) - 6r^{-4} \sin(cr), \\ f''''(r) &= c^4 r^{-1} \sin(cr) + 2c^3 r^{-2} \cos(cr) - 10c^2 r^{-3} \sin(cr) \\ &\quad - 30cs^{-4} \cos(cr) + 24r^{-5} \sin(cr), \end{aligned}$$

and

$$\begin{aligned} g' &= \partial_{x_3} g(x_3) = \frac{x_3}{(x_1^2 + x_2^2 + x_3^2)^{1/2}}, \\ g'' &= \partial_{x_3}^2 g(x_3) = \frac{1}{(x_1^2 + x_2^2 + x_3^2)^{1/2}} - \frac{x_3^2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}, \\ g''' &= \partial_{x_3}^3 g(x_3) = -\frac{3}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} + \frac{3x_3^3}{(x_1^2 + x_2^2 + x_3^2)^{5/2}}, \\ g'''' &= \partial_{x_3}^4 g(x_3) = -\frac{3}{(x_1^2 + x_2^2 + x_3^2)^{3/2}} + \frac{18x_3^2}{(x_1^2 + x_2^2 + x_3^2)^{5/2}} - \frac{15x_3^4}{(x_1^2 + x_2^2 + x_3^2)^{7/2}}. \end{aligned}$$

At this point, we recall that by the Faà di Bruno's formula

$$\begin{aligned} \partial_{x_3}^4 (f \circ g)(x) &= f''''(|x|)[g'(x)]^4 + 6f'''(|x|)g''(x)[g'(x)]^2 + 3f''(|x|)[g''(x)]^2 \\ &\quad + 4f''(|x|)g'''(x)g'(x) + f'(|x|)g''''(x), \end{aligned}$$

and the claim easily follows by replacing $c = \tau^{1/4}$ and translating back to $x \mapsto x - y$. \square

Remark 1 It is straightforward to observe that in (27) we can replace the symbol ξ_j^4 with $\xi_k^2 \xi_h^2$, for $k \neq h$, to get

$$\frac{\xi_k^2 \xi_h^2}{|\xi|^4} = \xi_k^2 \xi_h^2 |\xi|^{-4} \int_0^\infty e^{-t|\xi|^4} t \, dt, \quad (29)$$

and consequently,

$$\langle \mathcal{R}_k^2 \mathcal{R}_h^2 f, g \rangle = - \int_0^\infty \langle \partial_{x_k}^2 \partial_{x_h}^2 \frac{d}{dt} P_t f, g \rangle_t dt. \quad (30)$$

The identities (29) and (30) of the remark above easily imply an analogous of Proposition 2 (by repeating its proof with the obvious modifications) for the operator $\mathcal{R}_k^2 \mathcal{R}_h^2$ replacing \mathcal{R}_j^4 . More precisely:

Proposition 3 *Assume that $f, g \in L^1 \cap L^2$ and that f is supported in $\{|\bar{x}| \geq \gamma_2 R\}$, while g is supported in $\{|\bar{x}| \leq \gamma_1 R\}$, for some $\gamma_{1,2} > 0$ satisfying $d := \gamma_2 - \gamma_1 > 0$. Then*

$$|\langle \mathcal{R}_k^2 \mathcal{R}_h^2 f, g \rangle| \lesssim R^{-1} \|g\|_{L^1} \|f\|_{L^1}.$$

2.2 Point-Wise Estimates for \mathcal{R}_j^2

We turn now the attention to the square of the Riesz transforms. In the subsequent results, we will use a cut-off function χ satisfying the following: $\chi(x)$ is a localization function supported in the cylinder $\{|\bar{x}| \leq 1\}$ that is nonnegative and bounded, with $\|\chi\|_{L^\infty} \leq 1$. For a positive parameter γ , we define by $\chi_{\{|\bar{x}| \leq \gamma R\}}$ the rescaled function $\chi(x/\gamma R)$ (hence $\chi_{\{|\bar{x}| \leq \gamma R\}}$ is bounded, positive, and supported in the cylinder of radius γR). The proof of the next propositions is inspired by [31].

Proposition 4 *For any (regular) function f , the following point-wise estimate is satisfied: provided $d := \gamma_2 - \gamma_1 > 0$, there exists a universal constant $C = C(d) > 0$ such that*

$$|\chi_{\{|\bar{x}| \leq \gamma_1 R\}}(x) \mathcal{R}_j^2 [(1 - \chi_{\{|\bar{x}| \leq \gamma_2 R\}}) f](x)| \leq C R^{-3} \chi_{\{|\bar{x}| \leq \gamma_1 R\}}(x) \|f\|_{L^1(\{|\bar{x}| \geq \gamma_2 R\})}. \quad (31)$$

We have an estimate similar to (31) if we localize inside a cylinder the function on which \mathcal{R}_j^2 acts, and we then truncate everything with a function supported in the exterior of another cylinder.

Proposition 5 *For any (regular) function f , the following point-wise estimate is satisfied: provided $d := \gamma_1 - \gamma_2 > 0$, there exists a universal constant $C = C(d) > 0$ such that*

$$|(1 - \chi_{\{|\bar{x}| \leq \gamma_1 R\}}(x)) \mathcal{R}_j^2 [(\chi_{\{|\bar{x}| \leq \gamma_2 R\}}) f](x)| \leq C R^{-3} (1 - \chi_{\{|\bar{x}| \leq \gamma_1 R\}}(x)) \|f\|_{L^1(\{|\bar{x}| \leq \gamma_2 R\})}.$$

Proof The proofs of the propositions above are analogous, and they can be given by observing that in the principal value sense, the square of the Riesz transform acts on a function g as

$$\mathcal{R}_j^2 g(x) = \iint \frac{x_j - y_j}{|x - y|^{3+1}} \frac{y_j - z_j}{|y - z|^{3+1}} g(z) dz dy.$$

Without loss of generality, we consider the case depicted in Proposition 4. Let $g(x) = \chi_{\{|\bar{x}| \geq \gamma_2 R\}}(x) f(x)$. Then

$$\chi_{\{|\bar{x}| \leq \gamma_1 R\}}(x) \mathcal{R}_j^2 g(x) = \chi_{\{|\bar{x}| \leq \gamma_1 R\}}(x) \iint \left(\frac{y_j}{|y|^4} \frac{z_j - y_j}{|z - y|^4} dy \right) g(x - z) dz.$$

Since g is supported in the exterior of a cylinder of radius $\gamma_2 R$, we can assume $|\bar{x} - \bar{z}| \geq \gamma_2 R$, and as the function $\chi_{\{|\bar{x}| \leq \gamma_1 R\}}$ is supported by definition in the cylinder of radius $\gamma_1 R$, we can assume $|\bar{x}| \leq \gamma_1 R$: therefore, we have that $|\bar{z}| \geq dR$. This implies that $\{|\bar{y}| \leq \frac{d}{4} R\} \cap \{|\bar{z} - \bar{y}| \leq \frac{1}{2} |\bar{z}|\} = \emptyset$. Indeed,

$$\frac{1}{2} |\bar{z}| \geq |\bar{z} - \bar{y}| \geq |\bar{z}| - |\bar{y}| \implies |\bar{y}| \geq \frac{1}{2} |\bar{z}| \geq \frac{d}{2} R,$$

hence, we have the following splitting for the inner integral:

$$\begin{aligned} \int \frac{y_j}{|y|^4} \frac{z_1 - y_1}{|z - y|^4} dy &= \int_{|\bar{y}| \leq \frac{d}{4} R} \frac{y_j}{|y|^4} \frac{z_j - y_j}{|z - y|^4} dy \\ &+ \int_{|\bar{z} - \bar{y}| \leq \frac{1}{2} |\bar{z}|} \frac{y_j}{|y|^4} \frac{z_j - y_j}{|z - y|^4} dy \\ &+ \int_{\{|\bar{y}| \geq \frac{d}{4} R\} \cap \{|\bar{z} - \bar{y}| \geq \frac{1}{2} |\bar{z}|\}} \frac{y_j}{|y|^4} \frac{z_j - y_j}{|z - y|^4} dy \\ &= \mathcal{A} + \mathcal{B} + \mathcal{C}. \end{aligned}$$

By using the properties of the domains in this splitting, the proof of the proposition can be done by straightforward computations, ending up with

$$\int \frac{y_j}{|y|^4} \frac{z_1 - y_1}{|z - y|^4} dy = \mathcal{A} + \mathcal{B} + \mathcal{C} \lesssim R^{-3}.$$

Hence,

$$\begin{aligned}
|\chi_{\{|\bar{x}|\leq\gamma_1 R\}}(x)\mathcal{R}_j^2 g(x)| &= \chi_{\{|\bar{x}|\leq\gamma_1 R\}}(x) \left| \iint \left(\frac{y_j}{|y|^4} \frac{z_j - y_j}{|z - y|^4} dy \right) g(x - z) dz \right| \\
&\lesssim R^{-3} \chi_{\{|\bar{x}|\leq\gamma_1 R\}}(x) \int |g(x - z)| dz \\
&\lesssim R^{-3} \chi_{\{|\bar{x}|\leq\gamma_1 R\}}(x) \|f\|_{L^1(|\bar{x}|\geq\gamma_2 R)},
\end{aligned}$$

which is the estimate stated in (31). See [6] for the details. \square

The proofs of Proposition 2, Proposition 3, Proposition 4, and Proposition 5 can be done by using an alternative approach, by means of a general characterization of homogeneous distribution on \mathbb{R}^n of degree $-n$, coinciding with a regular function in $\mathbb{R}^n \setminus \{0\}$. Indeed, we have the following (we specialize to the three-dimensional case). For a proof, see [6]. In what follows, “dist” denotes the distance function.

Proposition 6 *Let T be an operator defined by means of a Fourier symbol $m(\xi)$, which is smooth in $\mathbb{R}^3 \setminus \{0\}$ and is a homogenous function of degree zero, i.e., $m(\lambda\xi) = m(\xi)$ for any $\lambda > 0$. For any couple of functions $f, g \in L^1$ having disjoint supports, we have the following estimate:*

$$|\langle Tf, g \rangle| \lesssim (\text{dist}(\text{supp}(f), \text{supp}(g)))^{-3} \|g\|_{L^1} \|f\|_{L^1}.$$

Remark 2 Keeping in mind the general statement of Proposition 6, it is easy for the reader to see that similar results as in Proposition 2, Proposition 3, Proposition 4, and Proposition 5 can be stated for functions localized outside and inside disjoint balls, instead of disjoint cylinders. Such localizations for functions supported outside and inside balls will be used for the scattering results using a concentration/compactness and rigidity scheme.

2.3 Virial Identities

The main difference between the Gross–Pitaevskii equation (2) and the classical cubic NLS equation is the non-local character of the non-linearity, in conjunction with the fact that the kernel K requires a more careful treatment with respect to the usual Coulomb- or Hartree-type kernels. Hence, we spend a few words here to give an overview on how the results concerning the decay of powers of the Riesz transforms as in the previous subsections will play a central role in the proofs of the main theorems. We show below how the tools above will be used for both the scattering and blow-up/grow-up results:

- (i) Standard arguments show that provided (11) is satisfied, then the Pohozaev functional G is bounded from below uniformly in time, in particular there

exists a positive α such that $G(u(t)) \geq \alpha > 0$ for all times in the maximal interval of existence of the solution. Similarly, provided (12) holds true, then $G(u(t)) \leq -\delta < 0$ for all times in the maximal interval of existence of the solution, for some positive δ . As a byproduct, $G(u(t)) \lesssim -\delta \|u(t)\|_{\dot{H}^1}^2$.

- (ii) Let χ a (regular) nonnegative function, which will be well chosen below. Let us denote by χ_R the rescaled version of χ , defined by $\chi_R = R^2 \chi(x/R)$, and let us introduce the quantity

$$V_{\chi_R}(t) := V_{\chi_R}(u(t)) = 2 \int \chi_R(x) |u(t, x)|^2 dx. \quad (32)$$

By formal computations, which can be justified by a classical regularization argument, it is easy to show that

$$\frac{d^2}{dt^2} V_{\chi_R}(t) = 4 \int |\nabla u(t)|^2 dx + 6\lambda_1 \int |u(t)|^4 dx + H_R(u(t)), \quad (33)$$

where H_R is a term that must be controlled. Our aim is to show that

$$H_R(u(t)) = 6\lambda_2 \int (K * |u(t)|^2) |u(t)|^2 dx + \varepsilon_R, \quad (34)$$

where $\varepsilon_R = o_R(1)$ as $R \rightarrow \infty$, uniformly in time in the lifespan of the solution. Let us observe that by gluing together (33) and (34), we get, by recalling the definition of G , see (6),

$$\frac{d^2}{dt^2} V_{\chi_R}(t) = 4G(u(t)) + \varepsilon_R.$$

By using the controls on the Pohozaev functional as described in the first point, and provided that ε_R is made sufficiently small for R large enough, then we are able to conclude the concentration/compactness and rigidity scheme for the scattering results, or we can provide suitable estimates to perform a convexity argument for the blow-up results. See the next points and the discussions in the next sections.

- (iii-a) As for the scattering part, let us mention for sake of clarity that the aim of the concentration/compactness and rigidity scheme is to prove that all solutions arising from initial data satisfying (11) are global and scatter. Let us recall that small initial data lead to global and scattering solutions, by a standard perturbative argument. The Kenig and Merle's road map (see [27, 28]) then proceeds as follows: suppose that the threshold for global and scattering solutions is strictly smaller than the claimed one (i.e., $E(Q)M(Q)$); then, by means of a profile decomposition theorem, it is possible to construct a minimal (in terms of the energy) global non-scattering solution at the threshold energy. Moreover, such a solution, called soliton-like solution and

denoted by u_{crit} , is precompact in the energy space up to a continuous-in-time translation path $x(t)$, i.e., $\{u(t, x + x(t))\}_{t \in \mathbb{R}^+}$ is precompact in H^1 . The crucial fact is that such a path $x(t)$ grows sub-linearly at infinity, and this will rule out the existence of such a soliton-like solution. This latter fact is proved by using the precompactness of the (translated) flow in conjunction with a virial argument, along with the already mentioned growth property of $x(t)$.

For the virial argument in this context, we choose χ to be a cut-off function such that $\chi(x) = |x|^2$ on $|x| \leq 1$ and $supp(\chi) \subset B(0, 2)$ (namely, we consider a localized version of (7)). We get

$$\begin{aligned} \frac{d^2}{dt^2} V_{\chi R}(t) &= 4 \int |\nabla u(t)|^2 dx + 6\lambda_1 \int |u(t)|^4 dx + \varepsilon_{1,R} \\ &\quad - 2\lambda_2 R \int \nabla \chi \left(\frac{x}{R} \right) \cdot \nabla \left(K * |u(t)|^2 \right) |u(t)|^2 dx, \end{aligned} \quad (35)$$

where

$$\varepsilon_{1,R} = C \left(\int_{|x| \geq R} |\nabla u(t)|^2 + R^{-2} |u(t)|^2 + |u(t)|^4 dx \right). \quad (36)$$

The quantity $\varepsilon_{1,R}$ can be made small, uniformly in time, for R sufficiently large, by using the precompactness of the soliton-like solution constructed with the concentration/compactness scheme. To handle the expression

$$\Lambda := -2\lambda_2 R \int \nabla \chi \left(\frac{x}{R} \right) \cdot \nabla \left(K * |u(t)|^2 \right) |u(t)|^2 dx$$

in (35), we perform a splitting in space of the solution $u(t, x)$ by considering its cut-off inside and outside a ball of radius $\sim R$, eventually obtaining the identity $\Lambda = 6\lambda_2 \int (K * |u(t)|^2) |u(t)|^2 dx + \varepsilon_{2,R}$. It is after the splitting above that we can reduce to a term $\varepsilon_{2,R}$ that fulfills the hypothesis of the pointwise decay of the Riesz transforms as in the previous section; indeed, with such localized functions, we can lead back our term $\varepsilon_{2,R}$ in the framework of Proposition 4 and Proposition 5. By letting $\varepsilon_R := \varepsilon_{1,R} + \varepsilon_{2,R}$, for R sufficiently large, we get

$$\frac{d^2}{dt^2} V_{\chi R}(t) = 4G(u(t)) + \varepsilon_R \geq 2\alpha,$$

where we used the strictly positive lower bound for G as described in point (i). This latter estimate, in conjunction with the sub-linear growth of $x(t)$, will give a contradiction; hence, the soliton-like solution cannot exist, and therefore, the threshold for the scattering is given precisely by the quantity as in (11).

(iii-b) As for the blow-up in finite time, the last part of strategy can be considered similar, as it is given by a Glassey argument based on virial identities. Nonetheless, the analysis is different and more complicated with respect to two points of the rigidity part for the scattering theorems. Specifically, in the formation of singularities scenario, we cannot rely on some compactness property on the non-linear flow; hence, the control on the remainder $H_R(u(t))$ cannot be given in a full generality. This is why we have to assume some symmetry hypothesis on the solution. It is here that we need to introduce the framework Σ_3 , the space of cylindrical symmetric solutions, with finite variance only along the third axis direction. Let us recall that even for the classical cubic NLS (i.e., $\lambda_1 = -1$ and $\lambda_2 = 0$), it is an open problem to show blow-up without assuming any additional symmetry hypothesis or finiteness of the variance, see [1, 12, 21–26, 29, 34]. Here we give the minimal assumptions to obtain formation of singularities in finite time, i.e., the solution is in Σ_3 . See also [32] for an early work on NLS in anisotropic spaces and [3, 7, 11, 18] for these techniques applied to other dispersive models.

For the virial argument, we chose here a (rescaled) function χ_R as the sum of a rescaled localization function ρ_R , plus the function x_3^2 . Here, ρ_R is a well-constructed function depending only on the two variables $\bar{x} = (x_1, x_2)$ that provides a localization in the exterior of a cylinder, parallel to the x_3 axis and with a radius of size $|\bar{x}| \sim R$. The notation $|\bar{x}|$ clearly stands for $|\bar{x}| := (x_1^2 + x_2^2)^{1/2}$. Moreover, we added the non-localized function x_3^2 in order to obtain a virial-like estimate of the form

$$\frac{d^2}{dt^2} V_{\rho_R + x_3^2}(t) \leq 4 \int |\nabla u(t)|^2 dx + 6\lambda_1 \int |u(t)|^4 dx + H_R(u(t)),$$

where the term H_R is defined by

$$\begin{aligned} H_R(u(t)) &= 4\lambda_1 \int a_R(\bar{x}) |u(t)|^4 dx + cR^{-2} \\ &\quad + 2\lambda_2 \int \nabla \rho_R \cdot \nabla (K * |u(t)|^2) |u(t)|^2 dx \\ &\quad - 4\lambda_2 \int x_3 \partial_{x_3} (K * |u(t)|^2) |u(t)|^2 dx, \end{aligned}$$

and $a_R(\bar{x})$ is a bounded, nonnegative function supported in the exterior of a cylinder of radius of order R . We estimate $\int a_R(\bar{x}) |u|^4 dx = o_R(1) \|u(t)\|_{\dot{H}^1}^2$ by means of a suitable Strauss embedding. Hence, it remains to estimate the non-local terms in $H_R(u(t))$. Similarly to the scattering part, the strategy is to split $u(t, x)$ by separating it into the interior and the exterior of a cylinder, instead of a ball, and computing the interaction given by the dipolar term. The further difficulty (with respect to the virial argument for the scattering theorem) is that $K * \cdot$ is not supported

inside any cylinder, even if we localize the function where K is acting on (through the convolution). Therefore, by performing further suitable splittings, we are able to give the identity

$$H_R(u(t)) = 6\lambda_2 \int (K * |u(t)|^2) |u(t)|^2 dx + \epsilon_R,$$

where the contributes defining ϵ_R consist of terms of the form $\langle \mathcal{R}_3^4 f, g \rangle$ when f is supported in $\{|\bar{x}| \geq \gamma_2 R\}$, while g is supported in $\{|\bar{x}| \leq \gamma_1 R\}$, for some positive parameters γ_1 and γ_2 satisfying $d := \gamma_2 - \gamma_1 > 0$. Clearly, the localizations of $u(t, x)$ play the role of f, g above. Hence, by means of Proposition 2, we can conclude, provided R is large enough, with

$$\frac{d^2}{dt^2} V_{\rho_R + x_3^2}(t) \leq 4G(u(t)) + \epsilon_R \leq -2\delta,$$

which in turn implies the finite-time blow-up via a Glassey convexity argument [21]. Note that we used the strictly negative upper bound for G as described in point (i).

3 Sketch of the Proofs Below the Threshold

3.1 Scattering

As already mentioned in point (iii-a), the scattering result given in Theorem 1 is given by running a concentration/compactness and rigidity scheme, as pioneered by Kenig and Merle in their celebrated works [27, 28]. Nowadays there is a huge literature on this method, applied to several dispersive models, and since the scope of this review paper is not to go over the details of these techniques, we refer the reader to [1, 13, 16, 19, 22, 24] for mass–energy intracritical NLS equations. Let us only mention that the method can be viewed as an induction of the energy method, and it proceeds by contradiction by assuming the threshold for global and scattering solutions is strictly smaller than the claimed one. Hence, we define the threshold for scattering as follows:

$$\mathcal{ME} = \sup \{ \delta : M(u_0)E(u_0) < \delta \text{ and } \|u_0\|_{L^2} \|\nabla u_0\| < \|Q\|_{L^2} \|\nabla Q\|_{L^2} \}$$

then the solution to (2) with initial data u_0 is in $L^8 L^4$ }.

A classical small data theory gives that if the initial datum is small enough in the energy norm, then the corresponding solution scatters, or equivalently, it belongs to $L^8 L^4 := L_t^8(\mathbb{R}; L_x^4(\mathbb{R}^3))$. Therefore, the threshold is certainly strictly positive. The goal is therefore to prove that $\mathcal{ME} = M(Q)E(Q)$.

At this point, we assume by contradiction that the threshold is strictly smaller than the given one (i.e., we assume $\mathcal{ME} < M(Q)E(Q)$, and we eventually prove that the latter leads to a contradiction).

Indeed, a linear profile decomposition theorem tailored for Eq. (2), see [5, Theorem 4.1, Proposition 4.3, and Corollary 4.4], and the existence of the wave operator enable us to establish the following.

Theorem 7 *There exists a non-trivial initial profile $u_{crit}(0) \in H^1$ such that the following holds true: $M(u_{crit}(0))E(u_{crit}(0)) = \mathcal{ME}$ and $\|u_{crit}(0)\|_{L^2}\|\nabla u_{crit}(0)\| < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$, and the corresponding solution $u_{crit}(t)$ to (2) is globally defined and does not scatter. Moreover, there exists a continuous function $x(t) : \mathbb{R}^+ \mapsto \mathbb{R}^3$ such that $\{u_{crit}(t, x + x(t)), t \in \mathbb{R}^+\}$ is precompact as a subset of H^1 . Such a function $x(t)$ satisfies $|x(t)| = o(t)$ as $t \rightarrow +\infty$, namely it grows sub-linearly at infinity.*

The theorem above says that by assuming $\mathcal{ME} < M(Q)E(Q)$, we are able to construct an initial datum whose nonlinear evolution is global and non-scattering. The precompactness tells us that $u_{crit}(t)$ remains spatially localized (uniformly in time) along the continuous path $x(t) \in \mathbb{R}^3$. Specifically, for any $\varepsilon > 0$, there exists $R_\varepsilon \gg 1$ such that

$$\int_{|x-x(t)| \geq R_\varepsilon} |\nabla u_{crit}(t)|^2 + |u_{crit}(t)|^2 + |u_{crit}(t)|^4 \leq \varepsilon \quad \text{for any } t \in \mathbb{R}^+. \quad (37)$$

The proof of the growth property of $x(t)$ is inspired by [13], and it is based on Galilean transformations of the solution.

The Kenig–Merle scheme is closed provided we can show that the solution given in Theorem 7 cannot exist. Indeed, as introduced in Sect. 2.3 (iii-a), a virial argument will give, see (35),

$$\begin{aligned} \frac{d^2}{dt^2} V_{\chi R}(t) &= 4 \int |\nabla u(t)|^2 dx + 6\lambda_1 \int |u(t)|^4 dx + \varepsilon_{1,R} \\ &\quad - 2\lambda_2 R \int \nabla \chi \left(\frac{x}{R} \right) \cdot \nabla \left(K * |u(t)|^2 \right) |u(t)|^2 dx. \end{aligned}$$

The main goal is therefore to estimate the non-local contribution $\Lambda := -2\lambda_2 R \int \nabla \chi \left(\frac{x}{R} \right) \cdot \nabla \left(K * |u(t)|^2 \right) |u(t)|^2 dx$. We introduce the space localization inside and outside a ball of radius $10R$, namely we write (we ignore the time dependence)

$$u = \mathbf{1}_{\{|x| \leq 10R\}} u + \mathbf{1}_{\{|x| > 10R\}} u := u_i + u_o.$$

By using the disjointness of the supports, we can rewrite $\Lambda = \Lambda_{i,i} + \Lambda_{o,i}$. In the latter notation, the subscript $\Lambda_{\diamond, \star}$, for $\diamond, \star \in \{i, o\}$, means the following: after some manipulations, the terms we are considering are of the form

$$\Lambda_{\diamond, \star}(u) = \int g \left(K * |u_{\diamond}|^2 \right) h(|u_{\star}|^2) dx,$$

i.e., the dipolar kernel K acts (via the convolution) on the localization (of u) given by the first symbol \diamond , while the other term in the integral contains the term localized according to the symbol \star . With a careful handling of the expression above, we reduce everything to fulfill the hypothesis of Proposition 4 and Proposition 5, leading to the final estimate

$$\Lambda \geq 6\lambda_2 \int (K * |u(t)|^2) |u(t)|^2 dx + \varepsilon_{2,R}$$

with

$$\begin{aligned} \varepsilon_{2,R} &\lesssim R^{-1} + R^{-1} \|u(t)\|_{H^1}^2 \|u(t)\|_{L^4(|x| \geq 10R)}^2 + R^{-1} \|u(t)\|_{H^1}^2 \\ &\quad + \|u(t)\|_{L^4(|x| \geq 10R)}^2 + \|u(t)\|_{L^4(|x| \geq 10R)}^4. \end{aligned} \quad (38)$$

Let us observe that the remainder as in (38) has a similar form as the one in (36) describing $\varepsilon_{1,R}$. Hence, they can be controlled in the same fashion. Specifically, we fix a time interval $[T_0, T_1]$ for $0 < T_0 < T_1$, and we take $R \geq \sup_{[T_0, T_1]} |x(t)| + R_\varepsilon$ as in (37) such that $\frac{d^2}{dt^2} z_R(t) \geq \frac{\alpha}{2} > 0$. An integration on $[T_0, T_1]$ yields to

$$R \gtrsim R \|u\|_{L^2} \|\nabla u\|_{L^2} \gtrsim \left| \frac{d}{dt} z_R(T_1) - \frac{d}{dt} z_R(T_0) \right| \geq \frac{\alpha}{2} (T_1 - T_0),$$

i.e., for some $c > 0$, we have $c(T_1 - T_0) \leq R$. Note that by the sub-linearity growth of $x(t)$, once fixed $\delta > 0$, we can guarantee that there exists a time t_δ such that $|x(t)| \leq \delta t$ for any $t \geq t_\delta$. Hence, by picking $\delta = c/2$, and $R = R_\varepsilon + \frac{cT_1}{2}$, we have $\frac{cT_1}{2} \leq R_\varepsilon + cT_0$, and the latter leads a contradiction, as we can let T_1 be as large as we want, while the right-hand side remains bounded.

3.2 Blow-up

As introduced in Sect. 2.3, our goal is to give the following estimate:

$$\frac{d^2}{dt^2} V_{\rho_R + x_3^2}(t) \leq 4 \int |\nabla u(t)|^2 dx + 6\lambda_1 \int |u(t)|^4 dx + H_R(u(t)), \quad (39)$$

where H_R is defined by

$$\begin{aligned}
H_R(u(t)) &= 4\lambda_1 \int a_R(\bar{x}) |u(t)|^4 dx + cR^{-2} \\
&\quad + 2\lambda_2 \int \nabla \rho_R \cdot \nabla \left(K * |u(t)|^2 \right) |u(t)|^2 dx \\
&\quad - 4\lambda_2 \int x_3 \partial_{x_3} \left(K * |u(t)|^2 \right) |u(t)|^2 dx
\end{aligned} \tag{40}$$

and $a_R = 0$ in $\{|\bar{x}| \leq R\}$. We recall that the subscript R stands for the rescaling $f_R = R^2 f(x/R)$. To this aim, we consider a regular, nonnegative, radial function $\rho = \rho(|\bar{x}|) = \rho(r)$ such that

$$\rho(r) = \begin{cases} r^2 & \text{if } r \leq 1 \\ 0 & \text{if } r \geq 2 \end{cases}, \quad \text{such that } \rho'' \leq 2 \quad \text{for any } r \geq 0.$$

A similar function can be explicitly constructed, see [6, 32, 34], and satisfies (39), (40), with a_R localized in the exterior of a cylinder of radius R . By means of Strauss estimates, it is quite easy to obtain

$$\begin{aligned}
H_R(u(t)) &= 2\lambda_2 \left(\int \nabla \rho_R \cdot \nabla \left(K * |u(t)|^2 \right) |u(t)|^2 dx \right. \\
&\quad \left. - 2 \int x_3 \partial_{x_3} \left(K * |u(t)|^2 \right) |u(t)|^2 dx \right) \\
&\quad + o_R(1) \|u(t)\|_{\dot{H}^1}^2 \\
&:= 2\lambda_2 (\Xi + \Upsilon) + o_R(1) \|u(t)\|_{\dot{H}^1}^2.
\end{aligned}$$

So we are reduced to the estimate of $\Xi + \Upsilon$. In order to use the decays as in Sect. 2, we proceed with several localizations, in order to reduce the problems given by the non-local terms $\Xi + \Upsilon$ to fulfill the hypothesis of the decay properties for powers of the Riesz transforms. The scheme is as follows. We introduce the first localization inside and outside a cylinder of radius $10R$, namely we write (we ignore the time dependence)

$$u = \mathbf{1}_{\{|\bar{x}| \leq 10R\}} u + \mathbf{1}_{\{|\bar{x}| > 10R\}} u := u_i + u_o.$$

By using the disjointness of the supports, we can rewrite $\Xi + \Upsilon = \Xi_{o,i} + \Xi_{i,i} + \Upsilon$. The proof of the decay for $\Xi_{o,i}$ can be given, after careful manipulations, by means of the point-wise decay as in Proposition 4 and Proposition 5. The main problem is given by the term $\Xi_{i,i} + \Upsilon$ where we do not have any localization at the exterior of

a cylinder, preventing us to obtain some decay straightforwardly. Hence, a further splitting is introduced. We separate u_i as $u_i = w_{i,i} + w_{i,o}$, where

$$w_{i,i} = \mathbf{1}_{\{|\bar{x}| \leq R/10\}} u_i \quad \text{and} \quad w_{i,o} = \mathbf{1}_{\{|\bar{x}| > R/10\}} u_i = \mathbf{1}_{\{R/10 < |\bar{x}| \leq 4R\}} u_i.$$

Therefore, we generate terms localized outside a cylinder of radius $\sim R$, specifically of the form

$$\begin{aligned} \mathcal{A}_{i,o}(u_i) &= \int g \left(K * |w_{i,i}|^2 \right) h(|w_{i,o}|^2) dx, \\ \mathcal{B}_{o,o}(u_i) &= \int \tilde{g} \left(K * |w_{i,o}|^2 \right) \tilde{h}(|w_{i,o}|^2) dx, \end{aligned}$$

plus a quantity

$$C_{i,i}(u) + \Upsilon := 2 \int \bar{x} \cdot \nabla_{\bar{x}} \left(K * |u_i|^2 \right) |u_i|^2 dx + 2 \int x_3 \partial_{x_3} \left(K * |u|^2 \right) |u|^2 dx.$$

By continuing the computations, we end up with a reduction of $\mathcal{A}_{i,o}(u_i)$ and $\mathcal{B}_{o,o}(u_i)$ to a framework as in Proposition 4 and Proposition 5, and we get

$$\mathcal{A}_{i,o}(u_i) = o_R(1) \|u\|_{\dot{H}^1}^2 \quad \text{and} \quad \mathcal{B}_{o,o}(u_i) = o_R(1) \|u\|_{\dot{H}^1}^2.$$

In order to control the remainder term $C_{i,i}(u) + \Upsilon$ and to make $6\lambda_2 \int (K * |u|^2) |u|^2 dx$ appear in (39), which will yield the whole quantity $4G(u(t))$, we need to use the identity

$$2 \int x \cdot \nabla (K * f) f dx = -3 \int (K * f) f dx.$$

The latter follows from the relation $\xi \cdot \nabla_{\xi} \hat{K} = 0$. By observing that

$$\begin{aligned} C_{i,i}(u) + \Upsilon &= 3 \int \left(K * |u_i|^2 \right) |u_i|^2 dx - 2 \int x_3 \partial_{x_3} \left(K * |u_i|^2 \right) |u_o|^2 dx \\ &\quad - 2 \int x_3 \partial_{x_3} \left(K * |u_o|^2 \right) |u_i|^2 dx - 2 \int x_3 \partial_{x_3} \left(K * |u_o|^2 \right) |u_o|^2 dx \end{aligned}$$

and that

$$\xi_3 \partial_{\xi_3} \hat{K} = 8\pi \frac{\xi_3^2 (\xi_1^2 + \xi_2^2)}{|\xi|^4} = 8\pi \left(\frac{\xi_3^2}{|\xi|^2} - \frac{\xi_3^4}{|\xi|^4} \right) = 8\pi \widehat{\mathcal{R}}_3^2 - 8\pi \widehat{\mathcal{R}}_3^4,$$

we reduce the problem to the estimate of $\langle \widehat{\mathcal{R}}_3^4 f, g \rangle_{L^2}$ when f is supported in $\{|\bar{x}| \geq \gamma_2 R\}$, while g is supported in $\{|\bar{x}| \leq \gamma_1 R\}$, for some positive parameters γ_1 and γ_2 satisfying $d := \gamma_2 - \gamma_1 > 0$. Note that in the latter identity we used the fact that

$\frac{\xi_3^2}{|\xi|^2}$ and $\frac{\xi_3^4}{|\xi|^4}$ are (up to constants) the symbols, in Fourier space, of the operators \mathcal{R}_3^2 and \mathcal{R}_3^4 , respectively. \mathcal{R}_j^4 denotes the fourth power of the Riesz transform, and $\widehat{\mathcal{R}_j^4}$ its symbol in Fourier space. At this point, we use the estimate of Proposition 2 (for the contribution involving \mathcal{R}_j^4) and again Proposition 4 and Proposition 5 (for the contribution involving \mathcal{R}_j^2). Thus, by summing up together the estimates, we have

$$\frac{d^2}{dt^2} V_{\rho_R+x_3^2}(t) \leq 4G(u(t)) + o_R(1)\|u(t)\|_{\dot{H}^1}^2 \lesssim -1,$$

which allows to close a Glassey-type convexity argument.

3.3 Grow-up

We now give the proof of the grow-up result, by sketching the proof of Theorem 3. The proof follows the approach by Du, Wu, Zhang, see [12] (see also the results by Holmer and Roudenko [25]). It is done by contradiction, and it makes use of the so-called almost finite propagation speed, which enables us to control the quantity $\|u(t)\|_{L^2(|x| \gtrsim R)}$ for sufficiently large times. It is well known that, contrary to the wave equation, the Schrödinger equation does not enjoy a finite propagation speed; nonetheless, we can claim the following: provided $\sup_{t \in [0, \infty)} \|u(t)\|_{\dot{H}^1} < \infty$, then for any $\eta > 0$, there exists a constant $C > 0$ independent of R such that for any $t \in [0, T]$ with $T := \frac{\eta R}{C}$,

$$\int_{|x| \gtrsim R} |u(t, x)|^2 dx \leq \eta + o_R(1). \quad (41)$$

Indeed, let ϑ be a smooth radial function satisfying

$$\vartheta(x) = \vartheta(r) = \begin{cases} 0 & \text{if } r \leq \frac{c}{2}, \\ 1 & \text{if } r \geq c, \end{cases} \quad \vartheta'(r) \leq 1 \text{ for any } r \geq 0,$$

where $c > 0$ is a given constant. For $R > 1$, we denote the radial function $\psi_R(x) = \psi_R(r) := \vartheta(r/R)$. We plug this function in the virial quantity V_χ defined in (32), and by the fundamental theorem of calculus, and by assuming that $\sup_{t \in [0, \infty)} \|u(t)\|_{\dot{H}^1} < \infty$, we have

$$\begin{aligned} V_{\psi_R}(t) &= V_{\psi_R}(0) + \int_0^t V'_{\psi_R}(s) ds \\ &\leq V_{\psi_R}(0) + t \sup_{s \in [0, t]} |V'_{\psi_R}(s)| \\ &\leq V_{\psi_R}(0) + CR^{-1}t. \end{aligned}$$

By the choice of ϑ , we have $V_{\psi_R}(0) = o_R(1)$ as $R \rightarrow \infty$. Since $V_{\psi_R}(t) \geq \int_{|x| \geq cR} |u(t, x)|^2 dx$, we obtain the control on L^2 -norm of the solution outside a large ball as in (41). By repeating the estimates as in Sect. 3.1, and by means of the Gagliardo–Nirenberg interpolation inequality applied to (38), we see that

$$V_{\varphi_R}''(t) \lesssim G(u(t)) + \left(R^{-1} + \|u(t)\|_{L^2(|x| \gtrsim R)}^{1/2} + \|u(t)\|_{L^2(|x| \gtrsim R)} \right). \quad (42)$$

Combining (41) and (42), we obtain that for any $\eta \in (0, 1)$, there exists a constant $C > 0$ independent of R such that for any $t \in [0, T]$ with $T := \frac{\eta R}{C}$ such that

$$V_{\varphi_R}''(t) \lesssim G(u(t)) + \left((\eta + o_R(1))^{1/4} + (\eta + o_R(1))^{1/2} \right).$$

By the assumption (15), we choose $\eta > 0$ sufficiently small and $R > 1$ sufficiently large to have $V_{\varphi_R}''(t) \lesssim -\delta < 0$ for all $t \in [0, T]$. If we integrate in time twice from 0 to T , we get $V_{\varphi_R}''(T) \leq o_R(1)R^2 - \frac{\delta \eta^2}{2C^2}R^2$, and by choosing R large enough, we obtain $z_{\varphi_R}(T) \leq -\frac{\delta \eta^2}{4C}R^2 < 0$, a contradiction with respect to the fact that $z_{\varphi_R}(T)$ is a nonnegative quantity.

4 Sketch of the Proofs Above the Threshold

The dynamics above the threshold is a consequence of the following general theorem, where a sufficient condition to have global existence and scattering is given. It will be used to establish the asymptotic dynamics when the initial datum lies at the threshold as well (see later on, specifically see the proofs in Sect. 5).

Theorem 8 *Let λ_1 and λ_2 satisfy (16). Let Q be a ground state related to (10). Let $u(t)$ be a H^1 -solution to (2) defined on the maximal forward time interval $[0, T_{max})$. Assume that*

$$\sup_{t \in [0, T_{max})} -P(u(t))M(u(t)) < -P(Q)M(Q). \quad (43)$$

Then $T_{max} = \infty$ and the solution $u(t)$ scatters in H^1 forward in time.

The proof of the theorem above is done by employing a concentration/compactness and rigidity road map, as for the case below the threshold, see Theorem 1. As mentioned in the paragraph before Theorem 7, the main tool to prove existence of global and non-scattering solution is given by a profile decomposition theorem, which is a linear statement; so, in order to construct nonlinear profiles, the existence of wave operator is used. Moreover, when we are in the case below the threshold, such nonlinear profiles can be proved to be global and scattering. When we do not assume initial data below the threshold, such a control on the nonlinear profiles

cannot be given. Nonetheless, we are able to prove a nonlinear profile decomposition theorem along bounded nonlinear flows, which overcomes the lack of finiteness of the scattering norm of the nonlinear profiles. See [10, Lemma 3.1]. The latter result in [10] was inspired to [22], where the NLS case was treated. We also recall here (as we remarked in the Introduction) that the restriction to the region (16) is imposed to guarantee the negative sign of the potential energy, which is fundamental to get the right bounds on the nonlinear profiles constructed when running a Kenig–Merle scheme.

Proof of Theorem 5 Let $u_0 \in \Sigma$ satisfying all the conditions in (17). We will show that (43) holds true, which in turn implies the result, by means of Theorem 8. The strategy is in the spirit of Duyckaerts and Roudenko [15], it is done in three steps, and it is based on an ODE argument. We summarize the main steps by just explaining how the method works and by defining the basic objects. For a comprehensive proof, we refer the reader to [10], where all the details are given.

By easy computations, we have

$$-P(u(t)) = 4E(u) - V''(t), \quad H(u(t)) = 6E(u) - V''(t), \quad (44)$$

and by using that $P(u(t))$ is negative (recall that we are working in RUR), then $V''(t) \leq 4E(u)$. At this point, we recall, see [20], that for any $f \in \Sigma$

$$\left(\operatorname{Im} \int x \cdot \nabla f(x) \bar{f}(x) dx \right)^2 \leq \|xf\|_{L^2}^2 \left(H(f) - \frac{(-P(f))^{\frac{2}{3}}}{(C_{GN})^{\frac{2}{3}}(M(f))^{\frac{1}{3}}} \right). \quad (45)$$

By plugging (44) into (45), we have

$$\left(\frac{V'(t)}{2} \right)^2 \leq V(t) \left[6E(u) - V''(t) - \frac{(4E(u) - V''(t))^{\frac{2}{3}}}{(C_{GN})^{\frac{2}{3}}(M(u))^{\frac{1}{3}}} \right].$$

We introduce the function $z(t) := \sqrt{V(t)}$, and we define $h(\zeta) := 6E(u) - \zeta - \frac{(4E(u) - \zeta)^{\frac{2}{3}}}{(C_{GN})^{\frac{2}{3}}(M(u))^{\frac{1}{3}}}$ for $\zeta \leq 4E(u)$. We can now rewrite the estimate above as $(z'(t))^2 \leq h(V''(t))$. The function $h(\zeta)$ on the unbounded interval $(-\infty, 4E(u))$ has a minimum in ζ_0 defined through

$$1 = \frac{2(4E(u) - \zeta_0)^{-\frac{1}{3}}}{3(C_{GN})^{\frac{2}{3}}(M(u))^{\frac{1}{3}}},$$

and in particular $h(\zeta_0) = \zeta_0/2$. The precise expression for C_{GN} given in (14) yields to

$$\frac{E(u)M(u)}{E(Q)M(Q)} \left(1 - \frac{\zeta_0}{4E(u)} \right) = 1. \quad (46)$$

- (i) By using the previous relations, the first point of the ODE argument consists of re-writing the scattering conditions in (17) in an alternative way, by using the functions $z(t)$, $V(t)$, $h(\zeta)$, and the value ζ_0 . From the hypothesis that $M(u_0)E(u_0) \geq M(Q)E(Q)$, we get that (46) is equivalent to $\zeta_0 \geq 0$. The second condition in (17) is equivalent to

$$(z'(0))^2 \geq \frac{\zeta_0}{2} = h(\zeta_0),$$

while the third condition in (17) is equivalent to $V''(0) > \zeta_0$. The last condition in (17) is instead equivalent to $z'(0) \geq 0$.

- (ii) The previous conditions replacing the ones in (17), jointly with a continuity argument, yield a lower bound

$$V''(t) \geq \zeta_0 + \delta_0, \quad (47)$$

for some $\delta_0 > 0$ and for any $t \in [0, T_{max})$.

- (iii) Eventually, we are able to prove (43). It follows from (47) and by using that $\zeta_0 \geq 0$, (46), and (14), that

$$\begin{aligned} -P(u(t))M(u(t)) &= (4E(u) - V''(t))M(u) \leq (4E(u) - \zeta_0 - \delta_0)M(u) \\ &\leq 4E(Q)M(Q) - \delta_0M(u) = -(1 - \eta)P(Q)M(Q) \end{aligned}$$

for all $t \in [0, T_{max})$, where $\eta := \frac{\delta_0 M(u)}{4E(Q)M(Q)} > 0$. This shows (43), and we can conclude the proof of Theorem 5. \square

5 Sketch of the Proofs at the Threshold

We now consider the threshold case, i.e., when the initial data satisfy (19), and we give an overview on the proof of Theorem 6. First, let us observe that in (19), we can assume, by scaling invariance, that $M(u_0) = M(Q)$ and $E(u_0) = E(Q)$. We continue with the proof of the three points in order:

- (i) As we are considering $M(u_0) = M(Q)$ and $E(u_0) = E(Q)$, we see that (20) becomes $H(u_0) < H(Q)$. Then in order to prove that (19) and (20) imply that the solution is global, it is enough to prove that the kinetic energy remains bounded by $H(Q)$ (by the blow-up alternative). By contradiction, if we assume that there exists a time τ in the lifespan of the solution such that $H(u(\tau)) = H(Q)$, then we obtain by definition of the energy that $-P(u(\tau)) = H(u(\tau)) - 2E(u(\tau)) = H(Q) - 2E(Q) = -P(Q)$. Namely $u(\tau)$ is an optimizer of (13). A Lions' concentration–compactness-type lemma, see [10, Lemma 5.1], implies that $u(t)$ is a (rescaling of a) ground state related to (10) multiplied by

a (time-dependent) phase shift. This yields to a contradiction with respect to the hypothesis, as we would have $H(u_0)M(u_0) = H(Q)M(Q)$; therefore, by the blow-up alternative, $u(t)$ is globally defined.

Under the hypothesis that the coefficients λ_1 and λ_2 satisfy (16), then we are able to prove that we have the result in the second part of Theorem 6 (i), by distinguishing two cases.

We first suppose that $\sup_{t \in [0, \infty)} H(u(t)) < H(Q)$. This means that there exists $\varepsilon > 0$ such that for all $t \in [0, \infty)$ (the solution is global), $H(u(t)) \leq (1 - \varepsilon)H(Q)$. By plugging the best constant (given in terms of the ground state to (10)) of the Gagliardo–Sobolev-type estimate (13), it is straightforward to see that

$$-P(u(t))M(u(t)) \leq C_{GN} (H(u(t))M(u(t)))^{\frac{3}{2}} \leq -(1 - \varepsilon)^{\frac{3}{2}} P(Q)M(Q);$$

hence, the condition (43) of Theorem 8 holds true, and the solution scatters forward in time.

If instead $\sup_{t \in [0, \infty)} H(u(t)) = H(Q)$, then there exists a time sequence $\{t_n\}_{n \geq 1} \subset [0, \infty)$ such that

$$M(u(t_n)) = M(Q), \quad E(u(t_n)) = E(Q), \quad \lim_{n \rightarrow \infty} H(u(t_n)) = H(Q).$$

Moreover, $t_n \rightarrow \infty$. Indeed, if (up to subsequences) $t_n \rightarrow \tau$, as $u(t_n) \rightarrow u(\tau)$ strongly in H^1 , then it can be shown that $u(\tau)$ is an optimizer for (13). Arguing as above, we have a contradiction. A Lions-type lemma [10, Lemma 5.1] gives the desired result.

- (ii) We continue with the proof of the second point. Suppose the initial datum satisfies (19) and (22). By scaling, we reduce to the case $M(u_0) = M(Q)$, $E(u_0) = E(Q)$, and hence, $H(u_0) = H(Q)$. Hence, u_0 is an optimizer for (13). This shows that $u_0(x) = e^{i\theta} \mu \tilde{Q}(\mu x)$ for some $\theta \in \mathbb{R}$, $\mu > 0$ and \tilde{Q} a ground state related to (10). By the uniqueness of solutions, we end up with $u(t, x) = e^{i\mu^2 t} e^{i\tilde{\theta}} \mu \tilde{Q}(\mu x)$ for some $\tilde{\theta} \in \mathbb{R}$.
- (iii) Finally, suppose that $u_0 \in H^1$ satisfies (19) and (23). By scaling, we have reduced (23) to $H(u_0) > H(Q)$. By the same argument in the proof of the first point, we claim that $H(u(t)) > H(Q)$, for every time in the lifespan of the solution. If the maximal time of existence is finite, there is nothing to prove. Otherwise, if the solution exists for all times, we separate the analysis in two cases.

Suppose $\sup_{t \in [0, \infty)} H(u(t)) > H(Q)$. Hence, there exists $\varepsilon > 0$ such that for all $t \in [0, \infty)$, $H(u(t)) \geq (1 + \eta)H(Q)$. By using the definition (6) of G and the previous property, we have

$$\begin{aligned} G(u(t))M(u(t)) &\leq 3E(Q)M(Q) - \frac{1}{2}(1 + \eta)H(Q)M(Q) \\ &= -\frac{\eta}{2}H(Q)M(Q) < 0, \end{aligned}$$

for all $t \in [0, \infty)$, where in the last equality we used (14). By applying Theorem 3, we finish the proof.

If instead $\sup_{t \in [0, \infty)} H(u(t)) = H(Q)$, similarly to above, we have that there exist a diverging sequence of times $\{t_n\}_{n \geq 1}$, a ground state \tilde{Q} related to (10), and a sequence $\{y_n\}_{n \geq 1} \subset \mathbb{R}^3$ such that $u(t_n, \cdot + y_n) \rightarrow e^{i\theta} \mu \tilde{Q}(\mu \cdot)$ in H^1 , for some $\theta \in \mathbb{R}$ and $\mu > 0$ as $n \rightarrow \infty$. This concludes the proof of Theorem 6.

Acknowledgments J.B. was partially supported by the project “Problemi stazionari e di evoluzione nelle equazioni di campo nonlineari dispersive” of GNAMPA 2020, and by the project PRIN grant 2020XB3EFL. L.F. was supported by the EPSRC New Investigator Award (grant no. EP/S033157/1).

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