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Qualitative Properties of Dispersive PDEs

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Qualitative Properties of Dispersive PDEs

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Preface

The international workshop *Qualitative Properties of Dispersive PDEs* held in Rome in September 2021 under the auspices of INdAM, the Italian National Institute for Advanced Mathematics, to which the present volume is the natural follow-up initiative, immediately revealed that an actual flurry of research activities was ready to resume after the forced pause imposed by the pandemic, concerning the analysis of linear and non-linear dispersive equations and a whole spectrum of related functional-analytic, operator-theoretic, and probabilistic tools.

In this book, a number of speakers in the above-mentioned meeting, and others selected scholars, provide a valuable collection of contributions presenting the state of the art and some of the most significant latest developments and future challenges for this growing field.

In fact, as emerged already in the process of selecting and assembling the various contributed chapters, the present book eventually displays one even more beneficial feature for that readership represented by graduate students and young researchers, in addition to its role as an updated observatory for the specialists. Indeed, thanks to the quality and effectiveness of the contributors in the first place, the reader is exposed across the various chapters to a true intensive and full-immersion introduction to many crucial mathematical tools whose applicability and versatility go way beyond the specific usage for the model they are discussed for here. This includes updated and comparative literature as well.

It then turns out that fundamental concepts, techniques, and tools can be profitably learnt here, in direct connection with certain applications, on four major lines:

(i) Long-type behaviour of NLS-type equations (the chapters by Cuccagna and by Bellazzini and Forcella),
(ii) Probabilistic and nonstandard methods in the study of the NLS equation (the chapters by Lucà and by Vojnović and Dugandžija),
(iii) Dispersive properties for heat, Schrödinger, and Dirac type flows (the chapters by Georgiev, Michelangeli, and Scandone; by Cacciafesta, Séré, and Junyong Zhang; and by Gallone, Michelangeli, and Pozzoli),

(iv) wave and KdV type equations (the chapters by Iandoli, by Georgiev and Lucente, and by Gallone and Ponno).

We believe that such a joint discussion of theoretical tools and some of their most recent and prolific applications makes this book particularly useful and appealing.

Our thanks and gratitude to the authors cannot be disjointed from our equally warm thanks to the INdAM scientific board, who provided that unique and stimulating opportunity of the INdAM workshop in Rome, to the INdAM administrative staff and the Springer publishing team for their precious support, and to the anonymous reviewers for their careful work and the quality of their reports.

Pisa, Italy — Vladimir Georgiev
Bonn, Germany — Alessandro Michelangeli
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About the Editors

Vladimir Georgiev is a former Alexander von Humboldt fellow. He is Full Professor of Mathematics at the University of Pisa. The main fields of his research interests involve decay estimates for equations of mathematical physics on flat or curved space-time, smoothing and Strichartz estimates for evolution problems, global existence of small and large data solutions to equations of classical quantum mechanics, existence and stability of solitary waves, Maxwell–Dirac and Maxwell–Schrödinger equation, and scattering and long-range effects for relativistic and non-relativistic particles and fields.

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Part I
Long-Time Behavior of NLS-Type Equations

A Note on Small Data Soliton Selection for Nonlinear Schrödinger Equations with Potential

Scipio Cuccagna and Masaya Maeda

Abstract In this note, we give an alternative proof of the theorem on soliton selection for small-energy solutions of nonlinear Schrödinger equations (NLS) studied in (Cuccagna and Maeda, Anal PDE 8(6):1289–1349, 2015; Cuccagna and Maeda, Ann PDE 7:16, 2021). As in (Cuccagna and Maeda, Ann PDE 7:16, 2021), we use the notion of refined profile, but unlike in (Cuccagna and Maeda, Ann PDE 7:16, 2021), we do not modify the modulation coordinates and do not search for Darboux coordinates.

1 Introduction

In this note, we give an alternative and simplified proof of the selection of small-energy standing waves for the nonlinear Schrödinger equation (NLS)

$$\mathrm{i}\partial_t u = Hu + g(|u|^2)u, \quad (t,x) \in \mathbb{R}^{1+3}, \tag{1}$$

where $H := -\Delta + V$ is a Schrödinger operator with $V \in \mathcal{S}(\mathbb{R}^3, \mathbb{R})$ (Schwarz function) and $g \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfies $g(0) = 0$ and the growth condition:

$$\forall n \in \mathbb{N} \cup \{0\},\ \exists C_n > 0,\ |g^{(n)}(s)| \le C_n \langle s \rangle^{2-n} \text{ where } \langle s \rangle := (1+|s|^2)^{1/2}. \tag{2}$$

We consider the Cauchy problem of NLS (1) with the initial condition $u(0) = u_0 \in H^1(\mathbb{R}^3, \mathbb{C})$. It is well known that the NLS (1) is locally well posed (LWP) in $H^1 := H^1(\mathbb{R}^3, \mathbb{C})$, see, e.g., [2, 7]. It is also easy to conclude, by mass and

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energy conservation, that for small initial data $u_0 \in H^1$ the corresponding solution is globally defined.

The aim of this chapter is to revisit the study of asymptotic behavior of small (in H^1) solutions when the Schrödinger operator H has several simple eigenvalues. In such situation, it has been proved that the solutions decouple into a soliton and a dispersive wave [3, 11, 13]. More recently, in [4], we have introduced the notion of refined profile, which simplifies significantly the proof of the result in [3]. In this note, we exploit the notion of refined profile of [4], but we give an alternative proof of the result in [4] that does not exploit directly the Hamiltonian structure of the NLS. In this sense, in this chapter, we are closer in spirit to Soffer and Weinstein [11] and Tsai and Yau [13], but our proof is at the same time simpler and with stronger results.

To state our main result precisely, we introduce some notation and several assumptions. The following two assumptions for the Schrödinger operator H hold for generic V.

Assumption 1 *0 is neither an eigenvalue nor a resonance of H.* □

Assumption 2 *There exists $N \geq 2$ s.t.*

$$\sigma_d(H) = \{\omega_j \mid j = 1, \cdots, N\}, \ \textit{with}\ \omega_1 < \cdots < \omega_N < 0,$$

where $\sigma_d(H)$ is the set of discrete spectra of H. Moreover, we assume all ω_j are simple and

$$\forall \mathbf{m} \in \mathbb{Z}^N \setminus \{0\}, \ \mathbf{m} \cdot \boldsymbol{\omega} \neq 0, \tag{3}$$

where $\boldsymbol{\omega} := (\omega_1, \cdots, \omega_N)$. We set ϕ_j to be the eigenfunction of H associated to the eigenvalue ω_j satisfying $\|\phi_j\|_{L^2} = 1$. We also set $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_N)$. □

Remark 1 The cases $N = 0, 1$ are easier and are not treated it in this chapter. Unfortunately, Assumption (2) excludes radial potentials $V(r)$, for $r = |x|$, where in general we should expect eigenvalues with multiplicity higher than one.

As it is well known, the ϕ_j's are smooth and decay exponentially. For $s \geq 0$, $\gamma \geq 0$, we set

$$H^s_\gamma := \{u \in H^s \mid \|u\|_{H^s_\gamma} := \|\cosh(\gamma x) u\|_{H^s} < \infty\}.$$

The following is well known.

Proposition 1 *There exists $\gamma_0 > 0$ s.t. for all $1 \leq j \leq N$; we have $\phi_j \in \cap_{s \geq 0} H^s_{\gamma_0}$.*

Using $\gamma_0 > 0$, we set

$$\Sigma^s := H^s_{\gamma_0} \text{ if } s \geq 0, \ \Sigma^s := (H^{-s}_{\gamma_0})^* \ \text{ if } s < 0,$$
$$\Sigma^{0-} := (\Sigma^0)^* \text{ and } \Sigma^\infty := \cap_{s \geq 0} \Sigma^s.$$

We will not consider any topology in Σ^∞, and we will only consider it as a set.

In order to introduce the notion of refined profile, we need the following combinatorial setup, exactly that of [4].

We start with the following standard basis of $\mathbb{R}^N$, which we view as "non-resonant" indices,

$$\mathbf{NR}_0 := \{\mathbf{e}_j \mid j = 1, \cdots, N\},\ \mathbf{e}_j := (\delta_{1j}, \cdots, \delta_{Nj}) \in \mathbb{Z}^N, \\ \delta_{ij} \text{ the Kronecker delta.} \tag{4}$$

More generally, the sets of resonant and non-resonant indices **R**, **NR**, are

$$\mathbf{R} := \{\mathbf{m} \in \mathbb{Z}^N \mid \sum \mathbf{m} = 1,\ \boldsymbol{\omega} \cdot \mathbf{m} > 0\}, \\ \mathbf{NR} := \{\mathbf{m} \in \mathbb{Z}^N \mid \sum \mathbf{m} = 1,\ \boldsymbol{\omega} \cdot \mathbf{m} < 0\}, \tag{5}$$

where $\sum \mathbf{m} := \sum_{j=1}^N m_j$ for $\mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{Z}^N$.

From Assumption 2, it is clear that $\{\mathbf{m} \in \mathbb{Z}^N \mid \sum \mathbf{m} = 1\} = \mathbf{R} \cup \mathbf{NR}$ and $\mathbf{NR}_0 \subset \mathbf{NR}$. For $\mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{Z}^N$, we define

$$|\mathbf{m}| := (|m_1|, \cdots, |m_N|) \in \mathbb{Z}^N,\ \|\mathbf{m}\| := \sum |\mathbf{m}| = \sum_{j=1}^N |m_j| \tag{6}$$

and introduce partial orders $\preceq$ and $\prec$ by

$$\mathbf{m} \preceq \mathbf{n} \Leftrightarrow_{\text{def}} \forall j \in \{1, \cdots, N\},\ m_j \le n_j, \\ \mathbf{m} \prec \mathbf{n} \Leftrightarrow_{\text{def}} \mathbf{m} \preceq \mathbf{n} \ \text{ and } \mathbf{m} \neq \mathbf{n}, \tag{7}$$

where $\mathbf{n} = (n_1, \cdots, n_N)$. We define the minimal resonant indices by

$$\mathbf{R}_{\min} := \{\mathbf{m} \in \mathbf{R} \mid \nexists \mathbf{n} \in \mathbf{R} \text{ s.t. } |\mathbf{n}| \prec |\mathbf{m}|\}. \tag{8}$$

We also consider $\mathbf{NR}_1$, formed by the non-resonant indices not larger than resonant indices:

$$\mathbf{NR}_1 := \{\mathbf{m} \in \mathbf{NR} \mid \forall \mathbf{n} \in \mathbf{R}_{\min},\ |\mathbf{n}| \not\prec |\mathbf{m}|\}. \tag{9}$$

Both $\mathbf{R}_{\min}$ and $\mathbf{NR}_1$ are finite sets, see [4] for the elementary proof.

We now introduce the functions $\{G_{\mathbf{m}}\}_{\mathbf{m} \in \mathbf{R}_{\min}} \subset \Sigma^\infty$ that are crucial in our analysis. For $\mathbf{m} \in \mathbf{NR}_1$, we inductively define $\widetilde{\phi}_{\mathbf{m}}(0)$ and $g_{\mathbf{m}}(0)$ by

$$\widetilde{\phi}_{\mathbf{e}_j}(0) := \phi_j,\ g_{\mathbf{e}_j}(0) = 0,\ j = 1, \cdots, N, \tag{10}$$

and, for $\mathbf{m} \in \mathbf{NR}_1 \setminus \mathbf{NR}_0$, by

$$\widetilde{\phi}_{\mathbf{m}}(0) := -(H - \mathbf{m} \cdot \boldsymbol{\omega})^{-1} g_{\mathbf{m}}(0), \tag{11}$$

$$g_{\mathbf{m}}(0) := \sum_{m=1}^{\infty} \frac{1}{m!} g^{(m)}(0) \sum_{(\mathbf{m}_1, \cdots, \mathbf{m}_{2m+1}) \in A(m, \mathbf{m})} \widetilde{\phi}_{\mathbf{m}_1}(0) \cdots \widetilde{\phi}_{\mathbf{m}_{2m+1}}(0), \tag{12}$$

where

$$A(m, \mathbf{m}) := \Big\{ \{\mathbf{m}_j\}_{j=1}^{2m+1} \in (\mathbf{NR}_1)^{2m+1} \mid \sum_{j=0}^{m} \mathbf{m}_{2j+1} - \sum_{j=1}^{m} \mathbf{m}_{2j} = \mathbf{m},\ \sum_{j=0}^{2m+1} |\mathbf{m}_j| = |\mathbf{m}| \Big\}. \tag{13}$$

Remark 2 For each $m \geq 1$ and $\mathbf{m} \in \mathbf{NR}_1$, $A(m, \mathbf{m})$ is a finite set. Furthermore, for sufficiently large m, we have $A(m, \mathbf{m}) = \emptyset$. Thus, even though we are expressing $g_{\mathbf{m}}(0)$ in (12) by a series, the sum is finite.

For $\mathbf{m} \in \mathbf{R}_{\min}$, we define $G_{\mathbf{m}}$ by

$$G_{\mathbf{m}} := \sum_{m=1}^{\infty} \frac{1}{m!} g^{(m)}(0) \sum_{(\mathbf{m}_1, \cdots, \mathbf{m}_{2m+1}) \in A(m, \mathbf{m})} \widetilde{\phi}_{\mathbf{m}_1}(0) \cdots \widetilde{\phi}_{\mathbf{m}_{2m+1}}(0). \tag{14}$$

Remark 3 $g_{\mathbf{m}}(0)$ and $G_{\mathbf{m}}$ are defined similarly. We are using a different notation to emphasize that $g_{\mathbf{m}}(0)$ has $\mathbf{m} \in \mathbf{NR}_1$, while $G_{\mathbf{m}}$ has $\mathbf{m} \in \mathbf{R}_{\min}$.

The following is the nonlinear Fermi Golden Rule (FGR) assumption essential in our analysis.

Assumption 3 *For all* $\mathbf{m} \in \mathbf{R}_{\min}$, *we assume*

$$\int_{|k|^2 = \mathbf{m} \cdot \boldsymbol{\omega}} |\widehat{G}_{\mathbf{m}}(k)|^2 \, dS \neq 0, \tag{15}$$

where $\widehat{G}_{\mathbf{m}}$ *is the distorted Fourier transform associated to* H. □

Remark 4 In the case $N = 2$ and $\omega_1 + 2(\omega_2 - \omega_1) > 0$, we have $G_{\mathbf{m}} = g'(0)\phi_1\phi_2^2$, which corresponds to the condition in Tsai and Yau [14], based on the explicit formulas in Buslaev and Perelman [1] and Soffer and Weinstein [10]. These works are related to Sigal [9]. More general situations are considered in [3], where however the $G_{\mathbf{m}}$ are obtained after a certain number of coordinate changes, so that the relation of the $G_{\mathbf{m}}$ and the ϕ_j's is not discussed in [3] and is not easy to track.

In [4], it is proved that for a generic nonlinear function g, the condition (15) is a consequence of the following simpler one, which is similar to (11.6) in Sigal [9],

$$\int_{|k|^2=\mathbf{m}\cdot\boldsymbol{\omega}} |\widehat{\phi^{\mathbf{m}}}(k)|^2 \, dS \neq 0 \text{ for all } \mathbf{m} \in \mathbf{R}_{\min}, \tag{16}$$

using again the distorted Fourier transform and where $\phi^{\mathbf{m}} := \prod_{j=1,\dots,N} \phi_j^{m_j}$. Specifically, in [4], the following is proved.

Proposition 2 *Let $L = \sup\{\frac{\|\mathbf{m}\|-1}{2} : \mathbf{m} \in \mathbf{R}_{\min}\}$, and suppose that the operator H satisfies condition (16). Then there exists an open dense subset Ω of $\mathbb{R}^L$ s.t. if $(g'(0), \dots, g^{(L)}(0)) \in \Omega$ such that Assumption 3 is true for (1).*

For $\mathbf{z} = (z_1, \cdots, z_N) \in \mathbb{C}^N$, $\mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{Z}^N$, we define

$$\mathbf{z}^{\mathbf{m}} := z_1^{(m_1)} \cdots z_N^{(m_N)} \in \mathbb{C}, \text{ where } z^{(m)} := \begin{cases} z^m & m \geq 0 \\ \bar{z}^{-m} & m < 0, \end{cases} \quad \text{and} \tag{17}$$

$$|\mathbf{z}|^k := (|z_1|^k, \cdots, |z_N|^k) \in \mathbb{R}^N, \ \|\mathbf{z}\| := \sum |\mathbf{z}| = \sum_{j=1}^N |z_j| \in \mathbb{R}. \tag{18}$$

We will use the following notation for a ball in a Banach space B:

$$\mathcal{B}_B(u, r) := \{v \in B \mid \|v - u\|_B < r\}. \tag{19}$$

The refined profile is of the form $\phi(\mathbf{z}) = \mathbf{z} \cdot \boldsymbol{\phi} + o(\|\mathbf{z}\|)$ and is defined by the following proposition, proved in [4].

Proposition 3 (Refined Profile) *For any $s \geq 0$, there exist $\delta_s > 0$ and $C_s > 0$ s.t. δ_s is nonincreasing w.r.t. $s \geq 0$, and there exist*

$$\{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{NR}_1} \in C^\infty(\mathcal{B}_{\mathbb{R}^N}(0, \delta_s^2), (\Sigma^s)^{\sharp\mathbf{NR}_1}), \ \boldsymbol{\varpi}(\cdot) \in C^\infty(\mathcal{B}_{\mathbb{R}^N}(0, \delta_s^2), \mathbb{R}^N)$$
$$\text{and } \mathcal{R} \in C^\infty(\mathcal{B}_{\mathbb{C}^N}(0, \delta_s), \Sigma^s),$$

s.t. $\boldsymbol{\varpi}(0, \cdots, 0) = \boldsymbol{\omega}$, $\psi_{\mathbf{m}}(0) = 0$ for all $\mathbf{m} \in \mathbf{NR}_1$ and

$$\|\mathcal{R}(\mathbf{z})\|_{\Sigma^s} \leq C_s \|\mathbf{z}\|^2 \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|, \tag{20}$$

where $B_X(a, r) := \{u \in X \mid \|u - a\|_X < r\}$, and if we set

$$\phi(\mathbf{z}) := \mathbf{z} \cdot \boldsymbol{\phi} + \sum_{\mathbf{m}\in\mathbf{NR}_1} \mathbf{z}^{\mathbf{m}} \psi_{\mathbf{m}}(|\mathbf{z}|^2) \text{ and } z_j(t) = e^{-\mathrm{i}\varpi_j(|\mathbf{z}|^2)t} z_j, \tag{21}$$

then, setting $\mathbf{z}(t) = (z_1(t), \cdots, z_n(t))$*, the function* $u(t) := \phi(\mathbf{z}(t))$ *satisfies*

$$\mathrm{i}\partial_t u - Hu - g(|u|^2)u = -\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}} - \mathcal{R}(\mathbf{z}), \tag{22}$$

where $\{G_{\mathbf{m}}\}_{\mathbf{R}_{\min}} \subset (\Sigma^\infty)^{\sharp\mathbf{R}_{\min}}$ *is given in* (14)*. Finally, writing* $\psi_{\mathbf{m}} = \psi_{\mathbf{m}}^{(s)}$, $\boldsymbol{\varpi} = \boldsymbol{\varpi}^{(s)}$ *and* $\mathcal{R} = \mathcal{R}^{(s)}$*, for* $s_1 < s_2$*, we have* $\psi_{\mathbf{m}}^{(s_1)}(|\cdot|^2) = \psi_{\mathbf{m}}^{(s_2)}(|\cdot|^2)$, $\boldsymbol{\varpi}^{(s_1)}(|\cdot|^2) = \boldsymbol{\varpi}^{(s_2)}(|\cdot|^2)$*, and* $\mathcal{R}^{(s_1)} = \mathcal{R}^{(s_2)}$ *in* $\mathcal{B}_{\mathbb{R}^N}(0, \delta_{s_2})$.

Remark 5 Notice that solitons, or standing waves, are exact solutions to the NLS generated from the refined profile setting

$$\phi_j(z_j) := \phi(z_j \mathbf{e}_j) \text{ for } z_j \in \mathcal{B}_{\mathbb{C}}(0, \delta_s). \tag{23}$$

So the refined profile fails to be an exact solution precisely when there are at least two nonzero coordinates in $\mathbf{z}$, which, under our hypotheses, make the defect on the right-hand side of (22) nonzero. Notice in particular that (20) states that the error term $\mathcal{R}(\mathbf{z})$ is not just small, but that it has a specific combinatorial structure. A monomial of the form $z_j|z_j|^{2N}$ cannot be a term in $R(\mathbf{z})$, since it does not have the required combinatorial structure. These $z_j|z_j|^{2N}$ terms are in the left-hand side of (22) and cancel out because the refined profile encodes the standing waves, as

$$\phi_j(z_j) = \phi(z_j \mathbf{e}_j) = \left[\mathbf{z}\cdot\boldsymbol{\phi} + \mathbf{z}^{\mathbf{e}_j}\psi_{\mathbf{e}_j}(|\mathbf{z}|^2)\right]\Big|_{\mathbf{z}=z_j\mathbf{e}_j}.$$

We give now several formulae related to the refined profile. Let X be a Banach space and $F \in C^1(\mathcal{B}_{\mathbb{C}^N}(0, \delta), X)$ for some $\delta > 0$. For $\mathbf{z} \in \mathcal{B}_{\mathbb{C}^N}(0, \delta)$ and $\mathbf{w} \in \mathbb{C}^N$, we set

$$D_{\mathbf{z}} F(\mathbf{z})\mathbf{w} := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F(\mathbf{z} + \epsilon\mathbf{w}).$$

For $\mathbf{z}(t)$ given by the 2nd equation of (21), that is $z_j(t) = e^{-\mathrm{i}\varpi_j(|\mathbf{z}|^2)t} z_j$, we have

$$\mathrm{i}\partial_t \mathbf{z} = \boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}, \text{ where } \boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z} := (\varpi_1(|\mathbf{z}|^2)z_1, \cdots, \varpi_N(|\mathbf{z}|^2)z_N).$$

Thus, $\mathrm{i}\partial_t\phi(\mathbf{z}(t)) = \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z}(t))(-\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}(t)|^2)\mathbf{z}(t))$, and we have the following formula, identically satisfied by $\phi(\mathbf{z})$,

$$\mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})(-\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}) = H\phi(\mathbf{z}) + g(|\phi(\mathbf{z})|^2)\phi(\mathbf{z}) - \sum_{\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}} - \mathcal{R}(\mathbf{z}). \tag{24}$$

Furthermore, differentiating (24) w.r.t. $\mathbf{z}$ in any given direction $\widetilde{\mathbf{z}} \in \mathbb{C}^N$, we obtain

$$H[\mathbf{z}]D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}} = \mathrm{i}D^2_{\mathbf{z}}\phi(\mathbf{z})(-\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}, \widetilde{\mathbf{z}}) + \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})\left(D_{\mathbf{z}}(-\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z})\widetilde{\mathbf{z}}\right) + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} D_{\mathbf{z}}(\mathbf{z}^{\mathbf{m}})\widetilde{\mathbf{z}}G_{\mathbf{m}} + D_{\mathbf{z}}\mathcal{R}(\mathbf{z})\widetilde{\mathbf{z}}, \tag{25}$$

where the operator $H[\mathbf{z}]$ is defined by

$$H[\mathbf{z}]f := Hf + g(|\phi(\mathbf{z})|^2)f + 2g'(|\phi(\mathbf{z})|^2)\mathrm{Re}\left(\overline{\phi(\mathbf{z})}\, f\right)\phi(\mathbf{z}) \tag{26}$$

and is self-adjoint for the inner product $\langle u, v\rangle = \mathrm{Re}\int_{\mathbb{R}^3} u\overline{v}dx$.

As mentioned above, the refined profile $\phi(\mathbf{z})$ contains as a special case the small standing waves bifurcating from the eigenvalues, when they are simple.

Corollary 1 *Let $s > 0$ and $j \in \{1, \cdots, N\}$. Then, $\phi\left(z(t)\mathbf{e}_j\right)$ solves (1) if $z \in \mathcal{B}_{\mathbb{C}}(0, \delta_s)$ and $z(t) = e^{-\mathrm{i}\varpi_j(|z\mathbf{e}_j|^2)t}z$.*

Proof Since $(z\mathbf{e}_j)^{\mathbf{m}} = 0$ for $\mathbf{m} \in \mathbf{R}_{\min}$, we see that from (20) and (22), the remainder terms $\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}(t)^{\mathbf{m}}G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}(t))$ are 0 in (22). Therefore, we have the conclusion. □

Remark 6 If the eigenvalues of H are not simple, the above does not hold anymore in general. See Gustafson–Phan [6].

The main result, which we have first proved in [3], is the following.

Theorem 4 *Under Assumptions 1, 2 and 3, there exist $\delta_0 > 0$ and $C > 0$ s.t. for all $u_0 \in H^1$ with $\epsilon_0 := \|u_0\|_{H^1} < \delta_0$, and there exists $j \in \{1, \cdots, N\}$, $z \in C^1(\mathbb{R}, \mathbb{C})$, $\eta_+ \in H^1$, and $\rho_+ \geq 0$ s.t.*

$$\lim_{t\to\infty} \|u(t) - \phi_j(z(t)) - e^{\mathrm{i}t\Delta}\eta_+\|_{H^1} = 0, \tag{27}$$

with $C^{-1}\epsilon_0^2 \leq \rho_+^2 + \|\eta_+\|^2_{H^1} \leq C\epsilon_0^2$ and

$$\lim_{t\to+\infty} |z(t)| = \rho_+ . \tag{28}$$

When written in the modulation parameters, the NLS appears like a complicated system where some discrete modes are coupled to radiation. The discrete modes tend to produce complicated patterns, similar to the ones of a linear system with eigenvalues. However, asymptotically in time, the nonlinear interaction is responsible of spilling of energy into radiation that disperses at space infinity and to the selection of a unique nonlinear standing wave. Theorem 4 is the same of the main theorem in [4] and is very similar to the main theorem in [3]. The proofs here and in [4] are much simpler than in [3] or in earlier papers containing early partial results, such as [11, 13]. In [3], in order to detect the nonlinear redistribution of the energy, it was necessary to make full use of the Hamiltonian structure of our NLS, by first introducing Darboux coordinates and by then considering a normal forms argument. The discovery of the notion of refined profile made in [8] and its further development in [4] allows to forgo the normal forms argument because an almost optimal system of coordinates is provided automatically by the refined profile. In [4], we introduced Darboux coordinates in a way much simpler than in [3]. Undoubtedly, Darboux coordinates are quite natural for a Hamiltonian system, and in [4], they contribute to simplify the system. In the present note however, we provide a different proof that, except for the information that mass and energy are constant, thus guaranteeing the global existence of our small H^1 solutions, does not make explicit use of the Hamiltonian structure of the equations.

2 The Proof

We start from constructing the modulation coordinate. First, we have the following.

Lemma 1 *There exist* $\delta > 0$ *and* $\mathbf{z} \in C^\infty(\mathcal{B}_{\Sigma^{-1}}(0,\delta), \mathbb{C}^N)$ *s.t.*

$$\forall \widetilde{\mathbf{z}} \in \mathbb{C}^N, \ \langle \mathrm{i}\,(u - \phi(\mathbf{z}(u)))\,, D_{\mathbf{z}}\phi(\mathbf{z}(u))\widetilde{\mathbf{z}}\rangle = 0.$$

Proof Standard. □

We set

$$\eta(u) := u - \phi(\mathbf{z}(u)). \tag{29}$$

In the following, we write $\eta = \eta(u)$ and $\mathbf{z} = \mathbf{z}(u)$. Substituting $u = \phi(\mathbf{z}) + \eta$ to (1) and using (24), we have

$$\mathrm{i}\partial_t \eta + \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})\left(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right) = H[\mathbf{z}]\eta + \sum_{\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}) + F(\mathbf{z},\eta), \tag{30}$$

where

$$F(\mathbf{z},\eta) = g(|\phi(\mathbf{z}) + \eta|^2)(\phi(\mathbf{z}) + \eta) - g(|\phi(\mathbf{z})|^2)\phi(\mathbf{z}) - g(|\phi(\mathbf{z})|^2)\eta \\ -2g'(|\phi(\mathbf{z})|^2)\mathrm{Re}\left(\overline{\phi(\mathbf{z})}\eta\right)\phi(\mathbf{z}).$$

Given an interval $I \subseteq \mathbb{R}$, we set

$$\begin{aligned} &\mathrm{Stz}^j(I) := L_t^\infty H^j(I) \cap L_t^2 W^{j,6}(I), \\ &\qquad\qquad \mathrm{Stz}^{*j}(I) := L_t^1 H^j(I) + L_t^2 W^{j,6/5}(I), \;\; j = 0, 1, \end{aligned} \tag{31}$$

where $H^0 = L^2$ and $W^{0,p} = L^p$, and use Yajima's [15] Strichartz inequalities, for $t_0 \in \overline{I}$,

$$\begin{aligned} &\|e^{-\mathrm{i}tH} P_c v\|_{\mathrm{Stz}^j(\mathbb{R})} \lesssim \|v\|_{H^j}, \\ &\qquad \|\int_{t_0}^t e^{-\mathrm{i}(t-s)H} P_c f(s)\, ds\|_{\mathrm{Stz}^j(I)} \lesssim \|f\|_{\mathrm{Stz}^{*j}(I)}, \;\; j = 0, 1. \end{aligned} \tag{32}$$

Under the assumptions of Theorem 4, we have $\|u\|_{L^\infty H^1(\mathbb{R})} \lesssim \epsilon_0$ from energy and mass conservation. Since $\|u\|_{H^1} \sim \|\mathbf{z}\| + \|\eta\|_{H^1}$, we conclude

$$\|\mathbf{z}\|_{L_t^\infty(\mathbb{R})} + \|\eta\|_{L_t^\infty H^1(\mathbb{R})} \lesssim \epsilon_0.$$

Theorem 5 (Main Estimates) *There exist* $\delta_0 > 0$ *and* $C_0 > 0$ *s.t. if* $\epsilon_0 = \|u_0\|_{H^1} < \delta_0$*, we have*

$$\|\eta\|_{\mathrm{Stz}^1(I)} + \sum_{\mathbf{m} \in \mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L_t^2(I)} + \|\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\|_{L_t^2(I)} \le C\epsilon_0, \tag{33}$$

for $I = [0, \infty)$ *and* $C = C_0$.

Notice that (33), Eq. (30) satisfied by η, estimate (20) for $\mathcal{R}(\mathbf{z})$, and Lemma 2 below for $F(\mathbf{z}, \eta)$ allow to prove in a standard and elementary fashion that $\eta(t)$ scatters as $t \to +\infty$, i.e., there exists $\eta_+ \in H^1$ such that $\|\eta(t) - e^{\mathrm{i}t\Delta}\eta_+\|_{H^1} \xrightarrow{t\to+\infty} 0$. From (33), we have $\|\eta_+\|_{H^1} \le C\epsilon_0$.

Using mass conservation, we have

$$\begin{aligned} &\|\phi(\mathbf{z}(t))\|_{L^2}^2 = \\ &\|u_0\|_{L^2}^2 - 2\langle\phi(\mathbf{z}(t)), e^{\mathrm{i}t\Delta}\eta_+\rangle - 2\langle\phi(\mathbf{z}(t)), \eta(t) - e^{\mathrm{i}t\Delta}\eta_+\rangle - \|\eta(t)\|_{L^2}^2 \\ &\xrightarrow{t\to+\infty} \|u_0\|_{L^2}^2 - \|\eta_+\|_{L^2}^2. \end{aligned}$$

So, by $\|\phi(\mathbf{z}(t))\|_{L^2}^2 = \|\mathbf{z}(t)\|^2 + o(\|\mathbf{z}(t)\|^2)$, we get $\lim_{t\to+\infty} \|\mathbf{z}(t)\|^2 = \rho_+^2$ for some $0 \le \rho_+ \le 2C\epsilon_0$.

The fact that $\mathbf{z}^{\mathbf{m}} \in L^2(\mathbb{R}_+)$ and, as it is easy to see, $\partial_t(\mathbf{z}^{\mathbf{m}}) \in L^\infty(\mathbb{R}_+) \cap C^0([0,\infty)$ implies $\mathbf{z}^{\mathbf{m}} \xrightarrow{t\to+\infty} 0$ for any $\mathbf{m} \in \mathbf{R}_{\min}$. This implies $z_k \xrightarrow{t\to+\infty} 0$ for all k except at most for one, yielding the selection of one coordinate j in the statement of Theorem 4. The proof that Theorem 5 implies Theorem 4 is like in [3].

By complete routine arguments discussed in [3], (33) for $I = [0, \infty)$ is a consequence of the following proposition.

Proposition 4 *There exists a constant $c_0 > 0$ s.t. for any $C_0 > c_0$, there is a value $\delta_0 = \delta_0(C_0)$ s.t. if (33) holds for $I = [0, T]$ for some $T > 0$, for $C = C_0$, and for $u_0 \in B_{H^1}(0, \delta_0)$, then in fact for $I = [0, T]$ the inequalities (33) hold for $C = C_0/2$.*

In the remainder of the paper, we prove Proposition 4.

2.1 Estimate of the Continuous Variable η

In the following, we set $\epsilon_0 = \|u_0\|_{H^1}$. Further, when we use $\lesssim$, the implicit constant will not depend on C_0. We start from the estimate of the remainder term F.

Lemma 2 *Under the assumption of Proposition 4, we have*

$$\|F(\mathbf{z}, \eta)\|_{\mathrm{Stz}^{*1}(I)} \lesssim C_0 \epsilon_0^3. \tag{34}$$

Proof By (2), we have the pointwise bound

$$|F(\mathbf{z}, \eta)| + |\nabla_x F(\mathbf{z}, \eta)| \lesssim \left(1 + |\eta|^2\right) (|\phi(\mathbf{z})| + |\nabla_x \phi(\mathbf{z})| + |\eta|) |\eta| (|\eta| + |\nabla_x \eta|) . \tag{35}$$

Using this, we obtain the conclusion by Hölder and Sobolev estimates. □

We set

$$\mathcal{H}_c[\mathbf{z}] := \{v \in L^2 \mid \forall \widetilde{\mathbf{z}} \in \mathbb{C}^N, \ \langle \mathrm{i} v, D_{\mathbf{z}} \phi(\mathbf{z}) \widetilde{\mathbf{z}} \rangle = 0\}. \tag{36}$$

Notice that for $u \in H^1$, $\eta(u) \in \mathcal{H}_c[\mathbf{z}(u)] \cap H^1$. Following Gustafson, Nakanishi, and Tsai [5], we can construct an inverse of P_c on $\mathcal{H}_c[\mathbf{z}]$.

Lemma 3 *There exists $\delta > 0$ s.t. there exists*

$$\{a_{jA}\}_{j=1,\cdots,N, A=\mathrm{R},\mathrm{I}} \in C^\infty(\mathcal{B}_{\mathbb{C}^N(0,\delta)}, \Sigma^1) \ \textit{s.t.}$$

$$\|a_{jA}(\mathbf{z})\|_{\Sigma^1} \lesssim \|\mathbf{z}\|^2, \ j = 1, \cdots, N, \ A = \mathrm{R}, \mathrm{I} \tag{37}$$

and

$$R[\mathbf{z}] := \mathrm{Id} - \sum_{j=1}^{N} \left(\langle \cdot, a_{j\mathrm{R}}(\mathbf{z})\rangle \phi_j + \langle \cdot, a_{j\mathrm{R}}(\mathbf{z})\rangle \mathrm{i}\phi_j\right), \tag{38}$$

satisfies $R[\mathbf{z}] P_c|_{\mathcal{H}_c[\mathbf{z}]} = \mathrm{Id}|_{\mathcal{H}_c[\mathbf{z}]}, \ P_c R[\mathbf{z}]|_{P_c L^2} = \mathrm{Id}|_{P_c L^2}.$

Proof A proof is in [3]. □

We set $\widetilde{\eta} = P_c\eta$. By Lemma 3, we have $\eta = R[z]\widetilde{\eta}$ and $\|\eta\|_{\mathrm{Stz}^1} \sim \|\widetilde{\eta}\|_{\mathrm{Stz}^1}$. Applying P_c to (30), we have

$$\begin{aligned} \mathrm{i}\partial_t\widetilde{\eta} =& H\widetilde{\eta} - \mathrm{i}P_c D_{\mathbf{z}}\phi(\mathbf{z}) \left(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right) + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} P_c G_{\mathbf{m}} \\ &+ P_c\mathcal{R}(\mathbf{z}) + P_c F(\mathbf{z},\eta) + P_c\left(H[\mathbf{z}] - H\right)\eta. \end{aligned} \tag{39}$$

Lemma 4 *Under the assumption of Proposition 4, we have*

$$\|\eta\|_{\mathrm{Stz}^1(I)} \lesssim \epsilon_0 + C(C_0)\epsilon_0^3 + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)}. \tag{40}$$

Proof Obviously, from $\|\eta\|_{\mathrm{Stz}^1} \sim \|\widetilde{\eta}\|_{\mathrm{Stz}^1}$, it is enough to bound the latter. By Strichartz estimates (32) and Lemma 2, we easily obtain

$$\begin{aligned} \|\widetilde{\eta}\|_{\mathrm{Stz}^1(I)} \lesssim \epsilon_0 + C(C_0)\epsilon_0^3 + \|P_c D_{\mathbf{z}}\phi(\mathbf{z}) \left(\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\|_{L^2(I)} \\ + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)}. \end{aligned}$$

Using the fact that $\|P_c D_{\mathbf{z}}\phi(\mathbf{z})\|_{\Sigma^1} = O\left(\|\mathbf{z}\|^2\right)$, we obtain (40). □

We set $Z(\mathbf{z}) := -\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} R_+(\mathbf{m}\cdot\boldsymbol{\omega}) P_c G_{\mathbf{m}}$ and $\xi := \widetilde{\eta} + Z$, where $R_+(\lambda) := (H - \lambda - \mathrm{i}0)^{-1}$. Using the identity

$$\left(D_{\mathbf{z}}\mathbf{z}^{\mathbf{m}}\right)(\mathrm{i}\boldsymbol{\omega}\mathbf{z}) = \mathrm{i}\mathbf{m}\cdot\boldsymbol{\omega}\,\mathbf{z}^{\mathbf{m}} \tag{41}$$

with, in the left-hand side, $\boldsymbol{\omega}\mathbf{z} := (\omega_1 z_1, \cdots, \omega_N z_N)$, we see that Z satisfies

$$-\mathrm{i}\partial_t Z(\mathbf{z}) + HZ(\mathbf{z}) = \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} P_c G_{\mathbf{m}} + \mathcal{R}_Z(\mathbf{z}), \tag{42}$$

where

$$\begin{aligned} \mathcal{R}_Z(\mathbf{z}) = \mathrm{i} \sum_{\mathbf{m}\in\mathbf{R}_{\min}} D_{\mathbf{z}}\left(\mathbf{z}^{\mathbf{m}}\right) \Big[\left(\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right) \\ + \left(\mathrm{i}\boldsymbol{\omega}\mathbf{z} - \mathrm{i}\boldsymbol{\varpi}(|z|^2)\mathbf{z}\right)\Big] R_+(\mathbf{m}\cdot\boldsymbol{\omega}) P_c G_{\mathbf{m}}. \end{aligned}$$

Substituting $\widetilde{\eta} = \xi - Z(\mathbf{z})$ into (39), we obtain

$$\mathrm{i}\partial_t \xi = H\xi - \mathrm{i}P_c D_{\mathbf{z}}\phi(\mathbf{z}) \left(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right) + P_c \mathcal{R}(\mathbf{z}) + P_c F(\mathbf{z}, \eta) + P_c (H[\mathbf{z}] - H)\eta + \mathcal{R}_Z(\mathbf{z}). \tag{43}$$

Lemma 5 *Under the assumption of Proposition 4, we have*

$$\|\xi\|_{L^2\Sigma^{0-}(I)} \lesssim \epsilon_0 + C_0\epsilon_0^3.$$

Proof By $\|\cdot\|_{L^2\Sigma^{0-}} \lesssim \|\cdot\|_{\mathrm{Stz}^0}$ and Strichartz estimates (32), we have

$$\begin{aligned}
&\|\xi\|_{L^2\Sigma^{0-}} \leq \|\widetilde{\eta}(0)\|_{L^2} + \|e^{-\mathrm{i}tH} Z(\mathbf{z}(0))\|_{L^2\Sigma^{0-}} \\
&+ \|\int_0^t e^{-\mathrm{i}(t-s)H}\mathcal{R}_Z(\mathbf{z}(u(s)))\, ds\|_{L^2\Sigma^{0-}} \\
&+ \|\mathrm{i}P_c D_{\mathbf{z}}\phi(\mathbf{z}) \left(\dot{\mathbf{z}} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right) - \mathcal{R}(\mathbf{z}) - F(\mathbf{z}, \eta) - (H[\mathbf{z}] - H)\eta\|_{\mathrm{Stz}^{*0}},
\end{aligned} \tag{44}$$

where $\mathbf{z}(t) = \mathbf{z}(u(t))$. One can bound the contribution of the 2nd line of (44) by $\lesssim C(C_0)\epsilon_0^3$ using, as in Lemma 4, $\|P_c D_{\mathbf{z}}\phi(\mathbf{z})\|_{\Sigma^1} = O\left(\|\mathbf{z}\|^2\right)$ and

$$D_{\mathbf{z}}\left(\mathbf{z}^{\mathbf{m}}\right)\mathrm{i}\left(\boldsymbol{\omega} - \boldsymbol{\varpi}(|\mathbf{z}|^2)\right)\mathbf{z} = \mathrm{i}\mathbf{m}\cdot\left(\boldsymbol{\omega} - \boldsymbol{\varpi}(|\mathbf{z}|^2)\right)\mathbf{z}^{\mathbf{m}} = O\left(\|\mathbf{z}\|^2\right)\mathbf{z}^{\mathbf{m}} \tag{45}$$

by (41) and $\boldsymbol{\varpi}(|\mathbf{z}|^2)|_{\mathbf{z}=0} = \boldsymbol{\omega}$. Similarly, the first term in the r.h.s. of (44) can be bounded by $\lesssim \epsilon_0$. For the 2nd and 3rd terms in the r.h.s. of (44), we will now use the estimate

$$\|e^{-\mathrm{i}tH} R_+(\mathbf{m}\cdot\boldsymbol{\omega})P_c f\|_{\Sigma^{0-}} \lesssim \langle t\rangle^{-3/2}\|f\|_{\Sigma^0}. \tag{46}$$

By (46), we have

$$\|e^{-\mathrm{i}tH} Z(\mathbf{z}(0))\|_{L^2\Sigma^{0-}(I)} \lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}(0)|\, \|\langle t\rangle^{-3/2}\|_{L^2}\|G_{\mathbf{m}}\|_{\Sigma^0} \lesssim \epsilon_0,$$

and

$$\begin{aligned}
&\|\int_0^t e^{-\mathrm{i}(t-s)H}\mathcal{R}_Z(\mathbf{z}(u(s)))\, ds\|_{L^2\Sigma^{0-}(I)} \\
&\leq \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\| \int_0^t \|e^{-\mathrm{i}(t-s)H} R_+(\mathbf{m}\cdot\boldsymbol{\omega})P_c G_{\mathbf{m}}\|_{\Sigma^{0-}}\right.
\end{aligned}$$

$$
\begin{aligned}
&\Big(\Big| D_{\mathbf{z}} \left(\mathbf{z}^{\mathbf{m}}\right)(s) \left(\partial_t \mathbf{z}(s) + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}(s)|^2)\mathbf{z}(s)\right)\Big| \\
&\quad + \Big| D_{\mathbf{z}} \left(\mathbf{z}^{\mathbf{m}}\right)(s)\mathrm{i}\left(\boldsymbol{\omega} - \boldsymbol{\varpi}(|\mathbf{z}(s)|^2)\right)\mathbf{z}(s)\Big| \Big) ds \Big\|_{L^2(I)} \\
&\lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\epsilon_0^2 \int_0^t \left(|\partial_t \mathbf{z}(s) + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2(s))\mathbf{z}(s)| + |\mathbf{z}^{\mathbf{m}}(s)|\right) \langle t-s\rangle^{-3/2} \|_{L^2(I)} \\
&\lesssim C(C_0)\epsilon_0^3,
\end{aligned}
$$

where we have used (45) in the 2nd inequality and Young's convolution inequality in the 3rd inequality. Therefore, we have the conclusion. □

2.2 Estimate of Discrete Variables

We next estimate the quantities $\|\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\|_{L^2}$ and $\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2}$. To do so, we first compute the inner product $\langle(30), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle$ for any given $\widetilde{\mathbf{z}} \in \mathbb{C}^N$. First, notice that by $\eta \in \mathcal{H}_c[\mathbf{z}]$, we obtain the orthogonality relation

$$\langle \mathrm{i}\partial_t \eta, D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle = -\langle \mathrm{i}\eta, D_{\mathbf{z}}^2\phi(\mathbf{z})(\partial_t \mathbf{z}, \widetilde{\mathbf{z}})\rangle.$$

Second, applying the inner product $\langle \eta, \cdot\rangle$ to Eq. (25), we have

$$
\begin{aligned}
\langle H[\mathbf{z}]\eta, D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle &= \langle \mathrm{i}\eta, D_{\mathbf{z}}^2\phi(\mathbf{z})(\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}, \widetilde{\mathbf{z}})\rangle \\
&\quad + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \langle \eta, \left(D_{\mathbf{z}}\left(\mathbf{z}^{\mathbf{m}}\right)\widetilde{\mathbf{z}}\right) G_{\mathbf{m}}\rangle + \langle \eta, D_{\mathbf{z}}\mathcal{R}(\mathbf{z})\widetilde{\mathbf{z}}\rangle,
\end{aligned}
$$

where we exploited the self-adjointness of $H[\mathbf{z}]$ and the orthogonality in Lemma 1. Thus, applying $\langle \cdot, D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle$ to Eq. (30) for η and using the last two equalities, we obtain

$$
\begin{aligned}
&\langle \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle = \\
&\langle \mathrm{i}\eta, D_{\mathbf{z}}^2\phi(\mathbf{z})\left(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}, \widetilde{\mathbf{z}}\right)\rangle \\
&+ \langle \eta, D_{\mathbf{z}}\mathcal{R}(\mathbf{z})\widetilde{\mathbf{z}}\rangle + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \langle \eta, \left(D_{\mathbf{z}}\left(\mathbf{z}^{\mathbf{m}}\right)\widetilde{\mathbf{z}}\right) G_{\mathbf{m}}\rangle \\
&+ \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle + \langle \mathcal{R}(\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle + \langle F(\mathbf{z}, \eta), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle.
\end{aligned} \tag{47}
$$

Using $\widetilde{\mathbf{z}} = \mathbf{e}_j, \mathrm{i}\mathbf{e}_j$, we have the following.

Lemma 6 *Under the assumption of Proposition 4, we have*

$$\partial_t z_j + \mathrm{i}\varpi_j(|\mathbf{z}|^2)z_j = -\mathrm{i}\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}\langle G_{\mathbf{m}}, \phi_j\rangle + r_j(\mathbf{z},\eta), \tag{48}$$

where $r_j(\mathbf{z},\eta)$ *satisfies*

$$\|r_j(\mathbf{z},\eta)\|_{L^2(I)} \lesssim C(C_0)\epsilon_0^3.$$

In particular, we have

$$\|\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\|_{L^2(I)} \lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)} + C(C_0)\epsilon_0^3. \tag{49}$$

Proof First since $D_{\mathbf{z}}\phi(0)\widetilde{\mathbf{z}} = \widetilde{\mathbf{z}}\cdot\boldsymbol{\phi}$, we have

$$\begin{aligned}&\langle \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})(\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle = \\ &\qquad \sum_{j=1}^N \mathrm{Re}(\mathrm{i}(\partial_t z_j + \mathrm{i}\varpi_j(|\mathbf{z}|^2)z_j)\overline{\widetilde{z}_j}) + r(\mathbf{z},\widetilde{\mathbf{z}}),\end{aligned} \tag{50}$$

where

$$\begin{aligned} r(\mathbf{z},\widetilde{\mathbf{z}}) =&\langle \mathrm{i}\left(D_{\mathbf{z}}\phi(\mathbf{z}) - D_{\mathbf{z}}\phi(0)\right)(\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle \\ &+ \langle \mathrm{i}D_{\mathbf{z}}\phi(0)(\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}), \left(D_{\mathbf{z}}\phi(\mathbf{z}) - D_{\mathbf{z}}\phi(0)\right)\widetilde{\mathbf{z}}\rangle. \end{aligned} \tag{51}$$

Since $\|D_{\mathbf{z}}\phi(\mathbf{z}) - D_{\mathbf{z}}\phi(0)\|_{L^2} \lesssim |\mathbf{z}|^2 \lesssim \epsilon_0^2$, by the assumptions of Proposition 4, we have

$$\|r(\mathbf{z},\widetilde{\mathbf{z}})\|_{L^2(I)} \lesssim C(C_0)\epsilon_0^3 \text{ for all } \widetilde{\mathbf{z}} = \mathbf{e}_1, \mathrm{i}\mathbf{e}_1, \cdots, \mathbf{e}_N, \mathrm{i}\mathbf{e}_N. \tag{52}$$

Setting

$$\begin{aligned} \widetilde{r}(\mathbf{z},\widetilde{\mathbf{z}},\eta) :=&\langle \mathrm{i}\eta, D_{\mathbf{z}}^2\phi(\mathbf{z})\left(\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}, \widetilde{\mathbf{z}}\right)\rangle + \langle \eta, D_{\mathbf{z}}\mathcal{R}(\mathbf{z})\widetilde{\mathbf{z}}\rangle \\ &+ \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \langle \eta, \left(D_{\mathbf{z}}\left(\mathbf{z}^{\mathbf{m}}\right)\widetilde{\mathbf{z}}\right) G_{\mathbf{m}}\rangle \\ &+ \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \left(D_{\mathbf{z}}\phi(\mathbf{z}) - D_{\mathbf{z}}\phi(0)\right)\widetilde{\mathbf{z}}\rangle \\ &+ \langle \mathcal{R}(\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle + \langle F(\mathbf{z},\eta), D_{\mathbf{z}}\phi(\mathbf{z})\widetilde{\mathbf{z}}\rangle, \end{aligned} \tag{53}$$

by the assumptions of Proposition 4, we have

$$\|\widetilde{r}(\mathbf{z},\widetilde{\mathbf{z}},\eta)\|_{L^2(I)} \lesssim C(C_0)\epsilon_0^3 \text{ for all } \widetilde{\mathbf{z}} = \mathbf{e}_1, \mathrm{i}\mathbf{e}_1, \cdots, \mathbf{e}_N, \mathrm{i}\mathbf{e}_N. \tag{54}$$

Therefore, since $D\phi(0)\mathrm{i}^k\mathbf{e}_j = \mathrm{i}^k\phi_j$ $(k=0,1)$, we have

$$-\mathrm{Im}\left(\partial_t z_j + \mathrm{i}\varpi_j(|\mathbf{z}|^2)z_j\right) = \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, \phi_j\rangle - r(\mathbf{z},\mathbf{e}_j) + \widetilde{r}(\mathbf{z},\mathbf{e}_j,\eta),$$
$$\mathrm{Re}\left(\partial_t z_j + \mathrm{i}\varpi_j(|\mathbf{z}|^2)z_j\right) = \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, \mathrm{i}\phi_j\rangle - r(\mathbf{z},\mathrm{i}\mathbf{e}_j) + \widetilde{r}(\mathbf{z},\mathrm{i}\mathbf{e}_j,\eta).$$

Since $G_{\mathbf{m}}$ (as can be seen from the proof in [4]) and ϕ_j are $\mathbb{R}$-valued, we have

$$\partial_t z_j + \mathrm{i}\varpi_j(|\mathbf{z}|^2)z_j = -\mathrm{i}\sum_{\mathbf{m}}\langle G_{\mathbf{m}},\phi_j\rangle \mathbf{z}^{\mathbf{m}} - r(\mathbf{z},\mathrm{i}\mathbf{e}_j) + \mathrm{i}r(\mathbf{z},\mathbf{e}_j) + \widetilde{r}(\mathbf{z},\mathrm{i}\mathbf{e}_j,\eta) - \mathrm{i}\widetilde{r}(\mathbf{z},\mathbf{e}_j,\eta).$$

Therefore, from (52) and (54), we have the conclusion with $r_j(\mathbf{z},\eta) = -r(\mathbf{z},\mathrm{i}\mathbf{e}_j) + \mathrm{i}r(\mathbf{z},\mathbf{e}_j) + \widetilde{r}(\mathbf{z},\mathrm{i}\mathbf{e}_j,\eta) - \mathrm{i}\widetilde{r}(\mathbf{z},\mathbf{e}_j,\eta)$. □

Having estimated η and $\partial_t\mathbf{z}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}$ in terms of $\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\|\mathbf{z}^{\mathbf{m}}\|_{L^2(I)}$, we need to estimate the latter quantity. Here we use the Fermi Golden Rule.

Lemma 7 *Under the assumption of Proposition 4, we have*

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2} \lesssim \epsilon_0 + (C_0\epsilon_0)\epsilon_0. \tag{55}$$

Proof We substitute $\widetilde{\mathbf{z}} = \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}$ in (47), and we make various simplifications. First, by $\langle f, \mathrm{i}f\rangle = 0$, the right-hand side of (47) can be rewritten as

$$\begin{aligned}\langle \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})(\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\rangle &= \\ \langle \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})(\partial_t\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\rangle&.\end{aligned} \tag{56}$$

Next, we consider the 3rd line of (47), which we rewrite as

$$\begin{aligned}&\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\rangle = \\ &\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\left(\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\rangle \\ &- \langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\partial_t\mathbf{z}\rangle.\end{aligned} \tag{57}$$

The term in the 1st line of the r.h.s. of (57) can be written as

$$\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, D_{\mathbf{z}}\phi(0)\left(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\rangle + R_1(\mathbf{z}), \tag{58}$$

where

$$\begin{aligned} R_1(\mathbf{z}) =& \langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, (D_{\mathbf{z}}\phi(\mathbf{z}) - D_{\mathbf{z}}\phi(0))\left(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\rangle \\ &+ \langle \mathcal{R}(\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\left(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\rangle, \end{aligned}$$

satisfies

$$\int_0^T |R_1(\mathbf{z}(t))|\, dt \lesssim C_0^2 \epsilon_0^4. \tag{59}$$

Using the stationary refined profile equation (24), the last line of (57) can be written as

$$\begin{aligned} -\langle H\phi(\mathbf{z}) + g(|\phi(\mathbf{z})|^2)\phi(\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\partial_t \mathbf{z}\rangle \\ + \langle D_{\mathbf{z}}\phi(\mathbf{z})(\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2))\mathbf{z}, \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})\partial_t \mathbf{z}\rangle. \end{aligned} \tag{60}$$

Notice that the 2nd term of (60) coincides with the right-hand side of (56), which lies in the left-hand side of (47), so that the two cancel each other. On the other hand, we have

$$\langle H\phi(\mathbf{z}) + g(|\phi(\mathbf{z})|^2)\phi(\mathbf{z}), D_{\mathbf{z}}\phi(\mathbf{z})\partial_t \mathbf{z}\rangle = \frac{d}{dt} E(\phi(\mathbf{z})). \tag{61}$$

Therefore, from (47) with $\widetilde{\mathbf{z}} = \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}$, (56), (57), (58), (60) and (61), we have

$$\begin{aligned} &\frac{d}{dt} E(\phi(\mathbf{z})) - \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{m}\cdot\boldsymbol{\omega}\langle \eta, \mathrm{i}\mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}\rangle \\ &\quad= \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, D_{\mathbf{z}}\phi(0)\left(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\rangle + R_2(\mathbf{z}, \eta), \end{aligned} \tag{62}$$

where

$$\begin{aligned} R_2(\mathbf{z},\eta) = & R_1(\mathbf{z}) + \langle \mathrm{i}\eta, D^2_{\mathbf{z}}\phi(\mathbf{z})\left(\partial_t \mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}, \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\rangle \\ & + \langle \eta, D_{\mathbf{z}}\mathcal{R}(\mathbf{z})\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\rangle \\ & + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} (\boldsymbol{\varpi}(|\mathbf{z}|^2) - \boldsymbol{\omega})\langle \eta, \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}\rangle + \langle F(\mathbf{z},\eta), D_{\mathbf{z}}\phi(\mathbf{z})\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\rangle, \end{aligned} \tag{63}$$

satisfies

$$\int_0^T |R_2(\mathbf{z}(t),\eta(t))|\,dt \lesssim \left(C_0^2\epsilon_0^2 + C_0^5\epsilon_0^5\right)\epsilon_0^2. \tag{64}$$

By Lemma 6 and $D_{\mathbf{z}}\phi(0)\widetilde{\mathbf{z}} = \widetilde{\mathbf{z}}\cdot\boldsymbol{\phi}$, the 1st term of right-hand side of (62) can be written as

$$\begin{aligned} & \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \boldsymbol{\phi}\cdot\left(\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\right)\rangle \\ & = \sum_{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}} \sum_{j=1}^{N} \langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \phi_j\left(-\mathrm{i}\mathbf{z}^{\mathbf{n}} g_{\mathbf{n},j} + r_j(\mathbf{z},\eta)\right)\rangle \\ & = \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}} \sum_{j=1}^{N} \mathrm{Re}\left(\mathrm{i}\mathbf{z}^{\mathbf{m}}\overline{\mathbf{z}^{\mathbf{n}}}\right) g_{\mathbf{m},j} g_{\mathbf{n},j} + \sum_{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}} \sum_{j=1}^{N} \langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, r_j(\mathbf{z},\eta)\phi_j\rangle, \end{aligned}$$

where we have set $g_{\mathbf{m},j} := \langle G_{\mathbf{m}}, \phi_j\rangle$ and used the fact that $\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, -\mathrm{i}\mathbf{z}^{\mathbf{m}}\phi_j\rangle = 0$ due to $G_{\mathbf{m}}$ and ϕ_j being $\mathbb{R}$-valued. Now, for $\mathbf{m}\neq\mathbf{n}$, we have

$$\begin{aligned} \partial_t(\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}}) = & \mathrm{i}(\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega}\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}} + \mathrm{i}(\mathbf{m}-\mathbf{n})\cdot\left(\boldsymbol{\varpi}(|\mathbf{z}|^2) - \boldsymbol{\omega}\right)\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}} \\ & + D_{\mathbf{z}}(\mathbf{z}^{\mathbf{n}})(\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2\mathbf{z}))\overline{\mathbf{z}^{\mathbf{m}}} + \mathbf{z}^{\mathbf{n}}\overline{D_{\mathbf{z}}(\mathbf{z}^{\mathbf{m}})((\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2\mathbf{z})))}. \end{aligned}$$

Thus, since $(\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega}\neq 0$ from Assumption 2, we have

$$\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}} = \frac{1}{\mathrm{i}((\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega})}\partial_t(\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}}) + r_{\mathbf{n},\mathbf{m}}(\mathbf{z}), \tag{65}$$

where

$$\begin{aligned} r_{\mathbf{n},\mathbf{m}}(\mathbf{z}) = & -\frac{(\mathbf{m}-\mathbf{n})\cdot\left(\boldsymbol{\varpi}(|\mathbf{z}|^2) - \boldsymbol{\omega}\right)}{(\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega}}\mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}} + \frac{\mathrm{i}}{(\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega}} \\ & \left(D_{\mathbf{z}}(\mathbf{z}^{\mathbf{n}})(\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2\mathbf{z}))\overline{\mathbf{z}^{\mathbf{m}}} + \mathbf{z}^{\mathbf{n}}\overline{D_{\mathbf{z}}(\mathbf{z}^{\mathbf{m}})((\partial_t\mathbf{z} + \mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2\mathbf{z})))}\right). \end{aligned}$$

Then, by the hypotheses of Proposition 4, we have

$$\int_0^T |r_{\mathbf{m},\mathbf{n}}(\mathbf{z})|\,dt \lesssim C_0^2\epsilon_0^4. \tag{66}$$

Thus, we have

$$\sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}}\sum_{j=1}^N \operatorname{Re}\left(\mathrm{i}\mathbf{z}^{\mathbf{m}}\overline{\mathbf{z}^{\mathbf{n}}}\right) g_{\mathbf{m},j}g_{\mathbf{n},j} = \partial_t A_1(\mathbf{z}) + R_3(\mathbf{z}),$$

where

$$A_1(\mathbf{z}) = \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}}\sum_{j=1}^N \frac{1}{(\mathbf{n}-\mathbf{m})\cdot\boldsymbol{\omega}}\operatorname{Re}(\mathbf{z}^{\mathbf{m}}\overline{\mathbf{z}^{\mathbf{n}}})g_{\mathbf{m},j}g_{\mathbf{n},j}, \text{ and}$$

$$R_3(\mathbf{z}) = \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}}\sum_{j=1}^N \operatorname{Re}\left(\mathrm{i}r_{\mathbf{n},\mathbf{m}}(\mathbf{z})\right) g_{\mathbf{m},j}g_{\mathbf{n},j}.$$

Thus,

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, \boldsymbol{\phi}\cdot\Big(\partial_t\mathbf{z}+\mathrm{i}\boldsymbol{\varpi}(|\mathbf{z}|^2)\mathbf{z}\Big)\rangle = \partial_t A_1(\mathbf{z}) + R_4(\mathbf{z},\eta),$$

where

$$R_4(\mathbf{z},\eta) = R_3(\mathbf{z}) + \sum_{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}}\sum_{j=1}^N\langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, r_j(\mathbf{z},\eta)\phi_j\rangle.$$

By (66) and Lemma 6, we have

$$\int_0^T |R_4(\mathbf{z}(t),\eta(t))|\,dt \lesssim C_0^2\epsilon_0^4.$$

Substituting $\eta = R[\mathbf{z}]\xi - (R[\mathbf{z}] - 1)Z(\mathbf{z}) - Z(\mathbf{z})$ into the 2nd term of the l.h.s. of (62), we have

$$\begin{aligned}\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{m}\cdot\boldsymbol{\omega}\langle\eta, \mathrm{i}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}\rangle = \\ &- \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{m}\cdot\boldsymbol{\omega}|\mathbf{z}^{\mathbf{m}}|^2\langle R_+(\mathbf{m}\cdot\boldsymbol{\omega})P_c G_{\mathbf{m}}, \mathrm{i}G_{\mathbf{m}}\rangle \\ &- \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}} \mathbf{m}\cdot\boldsymbol{\omega}\langle\mathbf{z}^{\mathbf{n}}R_+(\mathbf{n}\cdot\boldsymbol{\omega})P_c G_{\mathbf{n}}, \mathrm{i}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}\rangle \\ &+ \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{m}\cdot\boldsymbol{\omega}\langle R[\mathbf{z}]\xi - (R[\mathbf{z}]-1)Z(\mathbf{z}), \mathrm{i}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}\rangle.\end{aligned} \tag{67}$$

By (65), the 2nd term of the r.h.s. of (67) can be written as

$$- \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}} \mathbf{m}\cdot\boldsymbol{\omega}\langle\mathbf{z}^{\mathbf{n}}R_+(\mathbf{n}\cdot\boldsymbol{\omega})P_c G_{\mathbf{n}}, \mathrm{i}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}\rangle = \partial_t A_2(\mathbf{z}) + R_5(\mathbf{z}),$$

where

$$A_2(\mathbf{z}) = -\mathrm{Re} \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}} \frac{\mathbf{m}\cdot\boldsymbol{\omega}}{\mathrm{i}\,(\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega}} \mathbf{z}^{\mathbf{n}}\overline{\mathbf{z}^{\mathbf{m}}}\langle R_+(\mathbf{n}\cdot\boldsymbol{\omega})P_c G_{\mathbf{n}}, \mathrm{i}G_{\mathbf{m}}\rangle,$$

$$R_5(\mathbf{z}) = - \sum_{\substack{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min}\\ \mathbf{m}\neq\mathbf{n}}} \mathbf{m}\cdot\boldsymbol{\omega}\langle r_{\mathbf{n},\mathbf{m}}(\mathbf{z})R_+(\mathbf{n}\cdot\boldsymbol{\omega})P_c G_{\mathbf{n}}, \mathrm{i}G_{\mathbf{m}}\rangle,$$

with

$$\int_0^T |R_5(\mathbf{z}(t))|\,dt \lesssim C_0^2\epsilon_0^4.$$

The last term of r.h.s. of (67) can be written as

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{m}\cdot\boldsymbol{\omega}\langle R[z]\xi, \mathrm{i}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}\rangle + R_6(\mathbf{z}),$$

with $R_6(\mathbf{z})$ satisfying

$$\int_0^T |R_6(\mathbf{z}(t))|\,dt \lesssim C_0^2\epsilon_0^4.$$

Therefore, we have

$$\begin{aligned}\frac{d}{dt}\left(E(\phi(\mathbf{z}))-A_1(\mathbf{z})-A_2(\mathbf{z})\right)=\\ -\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{m}\cdot\boldsymbol{\omega}|\mathbf{z}^{\mathbf{m}}|^2\langle R_+(\mathbf{m}\cdot\boldsymbol{\omega})P_cG_{\mathbf{m}},\mathrm{i}P_cG_{\mathbf{m}}\rangle\\ +\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{m}\cdot\boldsymbol{\omega}\langle R[\mathbf{z}]\xi,\mathrm{i}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}\rangle+R_7(\mathbf{z},\eta),\end{aligned}\tag{68}$$

where $R_7(\mathbf{z},\eta)=R_2(\mathbf{z})+R_4(\mathbf{z})+R_5+R_6$.

Now, by $R_+(\boldsymbol{\omega}\cdot\mathbf{m})=\mathrm{P.V.}\frac{1}{H-\boldsymbol{\omega}\cdot\mathbf{m}}+\mathrm{i}\pi\delta(H-\boldsymbol{\omega}\cdot\mathbf{m})$ and formula (2.5) p. 156 [12] and Assumption 3, we have

$$\langle \mathrm{i}G_{\mathbf{m}},(H-\boldsymbol{\omega}\cdot\mathbf{m}-\mathrm{i}0)^{-1}G_{\mathbf{m}}\rangle=\frac{1}{16\pi\sqrt{\boldsymbol{\omega}\cdot\mathbf{m}}}\int_{|k|^2=\boldsymbol{\omega}\cdot\mathbf{m}}|\widehat{G_{\mathbf{m}}}(k)|\,dS(k)\gtrsim 1,$$

with $\widehat{G_{\mathbf{m}}}(k)$ like in Assumption 3. Thus, we have

$$\|\mathbf{z}^{\mathbf{m}}\|^2_{L^2(I)}\lesssim\epsilon_0^2+\delta^{-1}\|\xi\|^2_{L^2\Sigma^{0-}(I)}+\delta\|\mathbf{z}^{\mathbf{m}}\|^2_{L^2(I)}+C_0^2\epsilon_0^4,$$

where we have used Schwarz inequality. Taking δ so that the $\|\mathbf{z}^{\mathbf{m}}\|^2_{L^2(I)}\lesssim\epsilon_0^2+\delta^{-1}\|\xi\|^2_{L^2\Sigma^{0-}(I)}+C_0^2\epsilon_0^4$ and using $\|\xi\|_{L^2\Sigma^{0-}(I)}\lesssim\epsilon_0$ by Lemma 5, we obtain (55). □

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Dynamics of Solutions to the Gross–Pitaevskii Equation Describing Dipolar Bose–Einstein Condensates

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Abstract We review some recent results on the long-time dynamics of solutions to the Gross–Pitaevskii equation (GPE) governing non-trapped dipolar quantum gases. We describe the asymptotic behaviors of solutions for different initial configurations of the initial datum in the energy space, specifically for data below, above, and at the mass–energy threshold. We revisit some properties of powers of the Riesz transforms by means of the decay properties of the integral kernel associated to the parabolic biharmonic equation. These decay properties play a fundamental role in establishing the dynamical features of the solutions to the studied GPE.

1 Introduction

In this chapter, we review some recent progresses concerning the dynamics of solutions to the following Gross–Pitaevskii equation (GPE) that models a so-called dipolar Bose–Einstein condensate (BEC) at low temperatures, see [4, 30, 33, 35–37]:

$$ih\frac{\partial u}{\partial t} = -\frac{h^2}{2m}\Delta u + W(x)u + U_0|u|^2u + (V_{dip} * |u|^2)u. \tag{1}$$

In the equation above, t is the time variable, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the space variable, $*$ denotes the convolution, and $u = u(t, x)$ is a complex function. The physical parameters appearing in (1) are: the Planck constant h, the mass m of a dipolar particle, and $U_0 = 4\pi h^2 a_s/m$ describes the strength of the local interaction

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between dipoles in the condensate, where a_s is the $s-$wave scattering length, which may have positive or negative sign according to the repulsive/attractive nature of the interaction. The non-local, long-range dipolar interaction potential between two dipoles is given instead by the convolution through the potential

$$V_{dip}(x) = \frac{\mu_0 \mu_{dip}^2}{4\pi} \frac{1 - 3\cos^2(\theta)}{|x|^3}, \quad x \in \mathbb{R}^3,$$

where μ_0 is the vacuum magnetic permeability, μ_{dip} is the permanent magnetic dipole moment, and θ is the angle between the dipole axis and the vector x. Without loss of generality, we can assume the dipole axis to be the vector $(0, 0, 1)$. The potential $W(x)$ is an external trapping potential that will be not considered in the sequel; namely, we study the case $W(x) = 0$.

In the next subsections, we describe the mathematical background on a rescaled version of the model (1), and we state the main results.

1.1 Background

For a mathematical treatment of the equation above, we consider (1) in its dimensionless form, and in particular we study the associated Cauchy problem in the energy space (i.e., $H^1(\mathbb{R}^3)$) as follows:

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda_1 |u|^2 u + \lambda_2 (K * |u|^2) u, & (t, x) \in \mathbb{R} \times \mathbb{R}^3 \\ u(0, x) = u_0(x) \in H^1(\mathbb{R}^3) \end{cases}, \tag{2}$$

where the dipolar kernel K is now given by

$$K(x) = \frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^5}. \tag{3}$$

Provided we normalize the wave function according to $\int_{\mathbb{R}^3} |u(x,t)|^2 dx = N$, whereas N is the total number of dipolar particles in the dipolar BEC, then the two real coefficients λ_1 and λ_2 are defined by $\lambda_1 = 4\pi a_s N \sqrt{\frac{m}{h}}$, and $\lambda_2 = \frac{N\mu_0 \mu_{dip}^2}{4\pi} \sqrt{\frac{m^3}{h^5}}$, and they are two physical parameters describing the strength of the non-linearities involved in the equation, specifically the local one given by $|u|^2 u$, and the non-local one given by $(K * |u|^2)u$, respectively.

At least formally, the solution $u(t)$ to (2) preserves the mass and the energy of the initial datum $u(0) = u_0$, specifically

$$M(u(t)) := \|u(t)\|_{L^2(\mathbb{R}^3)}^2 = M(u(0)) \tag{4}$$

and

$$E(u(t)) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u(t)|^2 dx + \frac{1}{2}\int_{\mathbb{R}^3} \left(\lambda_1 |u(t)|^4 + \lambda_2 (K * |u(t)|^2)|u(t)|^2\right) dx$$
$$= E(u(0)), \tag{5}$$

where $M(u(t))$ and $E(u(t))$ define the mass and the energy, respectively. For later purpose, we introduce the notation

$$H(f) := \|\nabla f\|^2_{L^2(\mathbb{R}^3)}$$

for the kinetic energy, and

$$P(f) := \int_{\mathbb{R}^3} \left(\lambda_1 |f(x)|^4 + \lambda_2 (K * |f(x)|^2)|f(x)|^2\right) dx$$

for the potential energy; hence, we rewrite

$$E(u(t)) = \frac{1}{2}\left(H(u(t)) + P(u(t))\right).$$

Assuming a local-in-time existence theory for (2) (which is guaranteed by the work of Carles, Markowich, and Sparber, see [9]), and assuming enough regularity of the solutions, the conservation laws (4) and (5) can be proved by a simple integration by parts; a rigorous justification in the energy space $H^1(\mathbb{R}^3)$ (note that in this Sobolev space, the energy functional is well defined) can be done by an approximation argument. Besides the functionals E, H, and P above, we introduce the Pohozaev functional

$$G(f) := H(f) + \frac{3}{2}P(f). \tag{6}$$

It is worth observing that the functional G is (up to a $1/4$ factor) the second derivative in time of the virial functional associated to (2), i.e.,

$$G(u(t)) = \frac{1}{4}\frac{d^2}{dt^2}V(t),$$

where $V(t) := V(u(t))$ stands for the variance at time t of the mass density, namely

$$V(t) := \int_{\mathbb{R}^3} |x|^2 |u(t,x)|^2 dx. \tag{7}$$

Motivated by the definition of the functional V, we introduce the space of functions $\Sigma \subset H^1(\mathbb{R}^3)$ as $\Sigma := H^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3; |x|^2 dx)$.

Following the work by Carles, Markowich, and Sparber [9], we introduce the partition of the coordinate plane (λ_1, λ_2) given by the two sets below:

$$UR := \begin{cases} \lambda_1 - \dfrac{4\pi}{3}\lambda_2 < 0 & \text{if} \quad \lambda_2 > 0 \\ \lambda_1 + \dfrac{8\pi}{3}\lambda_2 < 0 & \text{if} \quad \lambda_2 < 0 \end{cases}, \tag{8}$$

and its complementary set in $\mathbb{R}^2$, namely

$$SR := \begin{cases} \lambda_1 - \dfrac{4\pi}{3}\lambda_2 \geq 0 & \text{if} \quad \lambda_2 > 0 \\ \lambda_1 + \dfrac{8\pi}{3}\lambda_2 \geq 0 & \text{if} \quad \lambda_2 < 0 \end{cases}. \tag{9}$$

The two sets above are called *unstable regime* (see (8)) and *stable regime* (see (9)), respectively.

The separation of the parameters λ_1 and λ_2 as in the regions (8) and (9) is crucial in establishing the dynamics of solutions to (2). Indeed, there are two main differences when working in the unstable regime instead of the stable regime. First, in (8), the conservation of the energy does not imply a boundedness in the kinetic term; second, the solutions to the stationary equation (see (10) below) associated to (2) do exist. Hence, at least in a naive way, we can think to the unstable/stable regimes as the analogous for the Gross–Pitaevskii equation (2) of the focusing/defocusing characters for the usual cubic NLS equation. However, note that here it is improper to speak about defocusing/focusing character for (2) since even for two positive coefficients of the nonlinear terms $0 < \lambda_1 < \frac{4\pi}{3}\lambda_2$ finite-time blow-up solutions may come up. See [9, Lemma 5.1], where negative energy solutions are constructed. We also mention here that in the stable regime, we proved in [5] that for any initial datum in $H^1(\mathbb{R}^3)$ the corresponding solution to (2) is global in time and scatters.

Similarly to the classical NLS equation (and more in general to other dispersive PDEs), a fundamental tool toward a classification of Cauchy data $u_0 \in H^1(\mathbb{R}^3)$ as in (2) leading to global (and scattering) solutions versus blowing-up solutions is given by means of quantities related to the solutions of the stationary equation associated to (2):

$$-\frac{1}{2}\Delta Q_\mu + \mu Q_\mu + \lambda_1 |Q_\mu|^2 Q_\mu + \lambda_2 (K * |Q_\mu|^2) Q_\mu = 0, \quad \mu > 0. \tag{10}$$

Notice that if Q_μ solves (10), then $u(t, x) := e^{-i\mu t} Q_\mu(x)$ solves (2). Moreover, by an elementary scaling argument, $E(Q_\mu)M(Q_\mu) = E(Q_1)M(Q_1)$ for all $\mu > 0$. For sake of simplicity in the notation, we will call Q the standing wave solutions with $\mu = 1$. In particular, some bounds for the product of the mass and the energy of an initial datum in terms of the mass and energy of solutions Q to (10) allow to determine whether a solution $u(t)$ to (2) exists for all time and scatters, or formation of singularities in finite (or infinite) time may arise. Indeed, sufficient conditions on $u_0 \in H^1(\mathbb{R}^3)$ for the scattering/blow-up scenario are given by the relations below:

$$(SC) := \begin{cases} E(u_0)M(u_0) < E(Q)M(Q) \\ H(u_0)M(u_0) < H(Q)M(Q) \end{cases}, \tag{11}$$

and

$$(BC) := \begin{cases} E(u_0)M(u_0) < E(Q)M(Q) \\ H(u_0)M(u_0) > H(Q)M(Q) \end{cases}, \tag{12}$$

respectively. The above conditions on initial data are referred to as the mass–energy (of the initial datum) below the threshold, the latter given by the quantity $E(Q)M(Q)$.

As mentioned above, in the unstable regime (8), the existences of solutions to (10) do exist, and it was proved in two different papers by Antonelli and Sparber, see [2], and later by the first author and Jeanjean, see [8], by employing two different methods. In the former work, the existences of ground states (i.e., standing wave solutions that minimize the energy functional $E(u)$ among all the standing solutions with prescribed mass) are proved by means of minimizing a Weinstein-type functional, while in the latter a geometrical approach is used, specifically by proving that the energy functional satisfied a mountain pass geometry. As for the usual cubic NLS, it turns out that a ground state Q related to the elliptic equation gives an optimizer for the Gagliardo–Nirenberg-type inequality

$$-P(f) \leq C_{GN}(H(f))^{\frac{3}{2}}(M(f))^{\frac{1}{2}}, \tag{13}$$

for $f \in H^1(\mathbb{R}^3)$, meaning that $C_{GN} = -P(Q)/(H(Q))^{\frac{3}{2}}(M(Q))^{\frac{1}{2}}$. Furthermore, the Pohozaev identities tell us that $H(Q) = 6M(Q) = -\frac{3}{2}P(Q)$, and by the latter relations, we have that $E(Q) = \frac{1}{6}H(Q) = -\frac{1}{4}P(Q)$ and that

$$E(Q)M(Q) = \frac{1}{6}H(Q)M(Q) = -\frac{1}{4}P(Q)M(Q) = \frac{2}{27}(C_{GN})^{-2}. \tag{14}$$

It is important to remark that uniqueness of ground states—even up to the action of some symmetry—is unknown; nonetheless, by (14), we can see that the quantities $E(Q)M(Q)$, $H(Q)M(Q)$, and $P(Q)M(Q)$ are independent of the choice of the ground state.

In the paper, we will also give dynamics results for solutions with arbitrarily large initial data (although by imposing some other hypothesis on u_0 and/or by further restricting the conditions on the parameters λ_1 and λ_2 to a subset of the unstable regime), hence by considering data such that $E(u_0)M(u_0) > E(Q)M(Q)$, and for data exactly at the threshold, i.e., for data satisfying $E(u_0)M(u_0) = E(Q)M(Q)$.

See the next subsection, where we enunciate the main results on the dynamics of solutions to (2).

1.2 Main Results

We conclude the Introduction by stating the main results contained in the paper. We separate them according to the fact that the initial data are below, above, or at the threshold determined by $E(Q)M(Q)$.

1.2.1 Dynamics Below the Threshold

We start by giving the scattering theorem and the blow-up in finite-time theorem, for solutions to (2) arising from initial data below the mass–energy threshold, described in terms of a solution Q of the elliptic equation (10). In what follows, $e^{it\frac{1}{2}\Delta}$ denotes the unitary Schrödinger propagator, namely $v(t,x) = e^{it\frac{1}{2}\Delta}v_0$ solves $i\partial_t v + \frac{1}{2}\Delta v = 0$, with $v(0,x) = v_0$. As already mentioned above, local well-posedness for (2) was established in [9], by a usual fixed-point argument based on Strichartz spaces, and upon having established some basic properties on the convolution kernel K, see Proposition 1 and Lemma 1 below. In what follows, we denote by $T_{min} > 0$ and $T_{max} > 0$ the minimal and maximal times of existence of a solution to (2), respectively.

The asymptotic dynamics for data below the threshold has been proved by the authors in [5] and [6]. In [5], we proved the following.

Theorem 1 *Let λ_1 and λ_2 satisfy (8), namely they belong to the unstable regime. Let $u_0 \in H^1(\mathbb{R}^3)$ satisfy (11), where Q is a ground state related to (10). Then the corresponding solution $u(t)$ to (2) exists globally in time and scatters in $H^1(\mathbb{R}^3)$ in both directions, that is, there exist $u_0^\pm \in H^1(\mathbb{R}^3)$ such that*

$$\lim_{t\to\pm\infty} \|u(t) - e^{it\frac{1}{2}\Delta}u_0^\pm\|_{H^1(\mathbb{R}^3)} = 0.$$

The theorem above is obtained by implementing a concentration/compactness and rigidity scheme, as we will explain in the next subsections.

In order to state the blow-up results that we proved in [6], let us define $\bar{x} = (x_1, x_2)$, and let us introduce the functional space

$$\Sigma_3 = \left\{ u \in H^1(\mathbb{R}^3) \quad s.t. \quad u(x) = u(|\bar{x}|, x_3) \text{ and } u \in L^2(\mathbb{R}^3; x_3^2\,dx) \right\},$$

namely the space of cylindrical symmetric functions (note that with an abuse of notation, we indicate with u both the function in the three variables (x_1, x_2, x_3) and the function in the two variables $(\bar{x}, x_3)$) with finite variance in the x_3 direction.

We have the following.

Theorem 2 *Assume that* λ_1 *and* λ_2 *satisfy* (8)*, namely they belong to the unstable regime. Let* $u(t) \in \Sigma_3$ *be a solution to* (2) *defined on* $(-T_{min}, T_{max})$*, with initial datum* u_0 *satisfying* (11)*, where* Q *is a ground state related to* (10)*. Then* T_{min} *and* T_{max} *are finite, namely* $u(t)$ *blows up in finite time.*

It is worth mentioning that for both the scattering and the blow-up result, the main difficulty with respect to other NLS non-local models is the precise structure of the dipolar kernel. Moreover, no radial symmetry for the solutions can be assumed in our context, as the convolution with radial function would make disappear the contribution of the non-local term, hence reducing the equation to a standard cubic NLS. Thus, the blow-up result above for cylindrical symmetric solution is somehow the best one may obtain; let us recall that finite-time blow-up without assuming any structure on the solutions is still unknown even for the usual focusing cubic NLS equation. Moreover, we point out that the dipolar kernel K enjoys a cylindrical symmetry, so our assumption is also physically consistent.

As said above, similarly to the classical cubic focusing NLS, if we do not assume any additional hypothesis on the initial datum, as in Theorem 2 for example, we cannot prove that the solutions blow up in finite time. Nonetheless, in [10], Dinh, Hajaiej, and the second author proved the following.

Theorem 3 *Let* λ_1 *and* λ_2 *satisfy* (8)*. Let* $u(t)$ *be a* $H^1(\mathbb{R}^3)$ *solution to* (2)*, defined on the maximal forward time interval* $[0, T_{max})$*. Assume that there exists a positive constant* $\delta > 0$ *such that*

$$\sup_{t\in[0,T_{max})} G(u(t)) \leq -\delta. \tag{15}$$

Then either the maximal forward time $T_{max} < \infty$ *or* $T_{max} = \infty$*, and there exists a diverging sequence of times, say* $t_n \to \infty$ *as* $n \to \infty$*, such that* $\lim_{n\to\infty} \|u(t_n)\|_{\dot{H}^1(\mathbb{R}^3)} = \infty$*. In the latter case, we say that the solution grows up.*

The next corollary actually shows that the condition given in Theorem 3 is non-empty, as an initial datum belonging to the region (BC), i.e., (12) is satisfied, leads to a solution satisfying (15) (see our paper [6, Section 3]).

Corollary 1 *Let* λ_1 *and* λ_2 *satisfy* (8)*, and* Q *be a ground state related to* (10)*. Assume that* $u_0 \in H^1(\mathbb{R}^3)$ *satisfies* (12)*, and let* $u(t)$ *the corresponding solution to* (2)*. Then* (15) *holds, and therefore either* $T_{max} < \infty$ *or* $T_{max} = \infty$*, and* $u(t)$ *grows up.*

1.2.2 Dynamics Above the Threshold

For the dynamical properties of solutions to (2) above the threshold, we need to further restrict the unstable regime, and we introduce the restricted unstable regime as follows:

$$RUR := \begin{cases} \lambda_1 + \frac{8\pi}{3}\lambda_2 < 0 & \text{if} \quad \lambda_2 > 0 \\ \lambda_1 - \frac{4\pi}{3}\lambda_2 < 0 & \text{if} \quad \lambda_2 < 0 \end{cases}. \tag{16}$$

For a ground state Q related to (10), we also give the scattering or blow-up conditions above the threshold:

$$(SC') := \begin{cases} E(u_0)M(u_0) \geq E(Q)M(Q) \\ \frac{E(u_0)M(u_0)}{E(Q)M(Q)}\left(1 - \frac{(V'(0))^2}{8E(u_0)V(0)}\right) \leq 1 \\ -P(u_0)M(u_0) < -P(Q)M(Q) \\ V'(0) \geq 0 \end{cases}, \tag{17}$$

and

$$(BC') := \begin{cases} E(u_0)M(u_0) \geq E(Q)M(Q) \\ \frac{E(u_0)M(u_0)}{E(Q)M(Q)}\left(1 - \frac{(V'(0))^2}{8E(u_0)V(0)}\right) \leq 1 \\ -P(u_0)M(u_0) > -P(Q)M(Q) \\ V'(0) \leq 0 \end{cases}, \tag{18}$$

respectively. Initial data satisfying (17) or (18) can be constructed by a simple scaling argument, see [10] and [20].

The following blow-up result above the threshold has been given by Gao and Wang in [20].

Theorem 4 *Let λ_1 and λ_2 satisfy* (16)*. Let Q be a ground state related to* (10)*, and $u_0 \in \Sigma$ satisfy* (18)*. Then the corresponding solution $u(t)$ to* (2) *blows up forward in finite time.*

The counterpart of Theorem 4 is the following scattering result, given for initial data satisfying (17). It is one of the main theorems contained in the paper by Dinh, Hajaiej, and the second author [10].

Theorem 5 *Let λ_1 and λ_2 satisfy* (16)*. Let Q be a ground state related to* (10)*, and $u_0 \in \Sigma$ be such that* (17) *holds true. Then the corresponding solution $u(t)$ to* (2) *exists globally and scatters in $H^1(\mathbb{R}^3)$ forward in time.*

Concerning the theorems above, it is worth mentioning the reason why we have to consider the subset RUR of the unstable regime, see (16), instead of the whole configurations of the parameters λ_1 and λ_2 as in (8). Condition (16) implies a control on the potential energy sign; specifically, it is negative for any time along the evolution of the solution. This will play a crucial role in the proof of the scattering criterion Theorem 8 below.

1.2.3 Dynamics at the Threshold

The next theorem deals with the long-time dynamics for solutions to (2) at the mass–energy threshold, i.e., when the initial datum satisfies

$$E(u_0)M(u_0) = E(Q)M(Q). \tag{19}$$

In [10], Dinh, Hajaiej, and the second present author gave a complete picture of the dynamics under the hypothesis (19), analyzing several different scenarios described in terms of the quantity $H(u_0)M(u_0)$. To the best of our knowledge, early results for the focusing cubic NLS at the threshold are given in the work of Duyckaerts and Roudenko [14]. The theorem is as follows.

Theorem 6 *Let* λ_1 *and* λ_2 *satisfy* (8)*. Let* Q *be a ground state related to* (10)*. Suppose that* $u_0 \in H^1(\mathbb{R}^3)$ *satisfies the mass–energy threshold condition* (19)*. We have the following three scenarios.*

(i) *In addition to* (19)*, suppose that*

$$H(u_0)M(u_0) < H(Q)M(Q) \tag{20}$$

and that the corresponding solution $u(t)$ *to* (2) *is defined on the maximal interval of existence* $(-T_{min}, T_{max})$*. Then for every* $t \in (-T_{min}, T_{max})$

$$H(u(t))M(u(t)) < H(Q)M(Q)$$

and in particular $T_{min} = T_{max} = \infty$*. Moreover, provided* λ_1 *and* λ_2 *satisfy* (16)*, the solution:*

- *Either scatters in* $H^1(\mathbb{R}^3)$ *forward in time*
- *Or there exist a diverging sequence of times* $t_n \to \infty$ *as* $n \to \infty$*, a ground state* $\tilde{Q}$ *related to* (10)*, and a sequence* $\{y_n\}_{n\geq 1} \subset \mathbb{R}^3$ *such that for some* $\theta \in \mathbb{R}$ *and* $\mu > 0$

$$u(t_n, \cdot - y_n) \to e^{i\theta}\mu\tilde{Q}(\mu\cdot) \tag{21}$$

strongly in $H^1(\mathbb{R}^3)$ *as* $n \to \infty$*.*

(ii) *In addition to* (19)*, suppose that*

$$H(u_0)M(u_0) = H(Q)M(Q), \tag{22}$$

then there exists a ground state $\tilde{Q}$ *related to* (10) *such that the solution* $u(t)$ *to* (2) *satisfies* $u(t,x) = e^{i\mu^2 t}e^{i\theta}\mu\tilde{Q}(\mu x)$ *for some* $\theta \in \mathbb{R}$ *and* $\mu > 0$*, and hence the solution is global.*

(iii) *In addition to* (19)*, suppose that*

$$H(u_0)M(u_0) > H(Q)M(Q) \tag{23}$$

and that the corresponding solution $u(t)$ *to* (2) *is defined on the maximal interval of existence* $(-T_{min}, T_{max})$*. Then for every* $t \in (-T_{min}, T_{max})$*,*

$$H(u(t))M(u(t)) > H(Q)M(Q).$$

Furthermore, the solution:

- *Either blows up forward in finite time*
- *Or it grows up along some diverging sequence of times* $t_n \to \infty$ *as* $n \to \infty$
- *Or there exists a diverging sequence of times* $t_n \to \infty$ *as* $n \to \infty$ *such that* (21) *holds for some sequence* $\{y_n\}_{n\geq 1} \subset \mathbb{R}^3$*, and some parameters* $\theta \in \mathbb{R}$*, and* $\mu > 0$.

Provided $u_0 \in \Sigma$*, the grow-up scenario as in the second point is ruled out.*

2 Decay for Powers of Riesz Transforms and Virial Arguments

This section provides the first technical tools we need in order to prove our main results. Moreover, we present the strategy we adopt to prove the main theorems, which strongly rely on virial arguments based on the decay for powers of Riesz transforms that we are going to prove.

First of all, we recall the fact that the dipolar kernel defines a Calderón–Zygmund operator; hence, it is a well-known fact that it yields to a map continuous from L^p into itself, for non-end-point Lebesgue exponents, namely for $p \neq 1$ and $p \neq \infty$. For a proof, see [9, Lemma 2.1].

Proposition 1 *The convolution operator* $f \mapsto K * f$ *can be extended as a continuous operator from* L^p *into itself, for any* $p \in (1, \infty)$.

Moreover, in [9], an explicit computation of the Fourier transform of the dipolar kernel K defined in (3) is given. Precisely, we have the following.

Lemma 1 *The Fourier transform of the dipolar kernel K is given by*

$$\hat{K}(\xi) = \frac{4\pi}{3}\frac{2\xi_3^2 - \xi_2^2 - \xi_1^2}{|\xi|^2}, \qquad \xi \in \mathbb{R}^3. \tag{24}$$

Straightforwardly, it follows that $\hat{K} \in \left[-\frac{4}{3}\pi, \frac{8}{3}\pi\right]$.

For a proof of (24), we refer to [9, Lemma 2.3]. The explicit calculation of $\hat{K}$ is done by means of the decomposition in spherical harmonics of the Fourier character $e^{-ix\cdot\xi}$.

2.1 Integral Estimates for $\mathcal{R}_j^4$

In the next propositions, we prove some decay estimates—point-wise and integral ones—regarding the square and the fourth power of the Riesz transforms when acting on suitably localized functions. First, we disclose a link between the fourth power of the Riesz transform $\mathcal{R}_j^4$ and the linear propagator associated to the parabolic biharmonic equation, defined in terms of the Bessel functions. With this correspondence and some decay estimates for the parabolic biharmonic integral kernel, we are able to show the decay estimates for $\langle \mathcal{R}_j^4 f, g\rangle$. Here $\langle\cdot,\cdot\rangle$ stands for the usual $L^2(\mathbb{R}^3)$ inner product. We start with the integral estimates for the fourth power of the Riesz transforms, and, as anticipated above, we do it by means of some decay properties of the kernel associated to the parabolic biharmonic equation

$$\partial_t w + \Delta^2 w = 0, \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^3. \tag{25}$$

We denote by P_t the linear propagator associated to (25), namely $w(t,x) := P_t w_0(x)$ denotes the solution to Eq. (25) with initial datum w_0. We begin with the following proposition that provides a representation of $\mathcal{R}_j^4$ by using the functional calculus. Since now on, we will omit—unless necessary—the notation $\mathbb{R}^3$, as we are concerned only with the three-dimensional model.

Lemma 2 *For any two functions in L^2, we have the following identity:*

$$\langle \mathcal{R}_i^4 f, g\rangle = -\int_0^\infty \langle \partial_{x_i}^4 \frac{d}{dt} P_t f, g\rangle t\, dt. \tag{26}$$

Proof By passing in the frequencies space, it is easy to see that $\widehat{P_t f}(\xi) := e^{-t|\xi|^4}\hat{f}(\xi)$, and we observe, by integration by parts, that

$$\xi_i^4|\xi|^4\int_0^\infty e^{-t|\xi|^4}t\,dt = \frac{\xi_i^4}{|\xi|^4}; \tag{27}$$

hence,

$$\begin{aligned}\int_0^\infty \langle \partial_{x_i}^4 \frac{d}{dt}P_t f, g\rangle t\,dt &= \langle \int_0^\infty \partial_{x_i}^4 \frac{d}{dt}(P_t f)t\,dt, g\rangle \\ &= (2\pi)^{-3}\langle \int_0^\infty \xi_i^4 \frac{d}{dt}(e^{-t|\xi|^4}\hat{f})t\,dt, \hat{g}\rangle \\ &= -(2\pi)^{-3}\langle \xi_i^4|\xi|^4\hat{f}\int_0^\infty e^{-t|\xi|^4}t\,dt, \hat{g}\rangle \\ &= -(2\pi)^{-3}\langle \frac{\xi_i^4}{|\xi|^4}\hat{f}, \hat{g}\rangle \\ &= -\langle \mathcal{R}_i^4 f, g\rangle,\end{aligned}$$

where the change of order of integration (in time and in space) is justified by means of the Fubini–Tonelli's theorem, and we used the Plancherel identity when passing from the frequencies space to the physical space, and vice versa. □

We are now in position to prove a decay estimate for functions supported outside a cylinder of radius $\sim R$. In order to do that, we explicitly write the integral kernel of the propagator P_t. We introduce, for $t > 0$ and $x \in \mathbb{R}^3$,

$$p_t(x) = \alpha\frac{k(\mu)}{t^{3/4}}, \quad \mu = \frac{|x|}{t^{1/4}},$$

and

$$k(\mu) = \mu^{-2}\int_0^\infty e^{-s^4}(\mu s)^{3/2}J_{1/2}(\mu s)\,ds,$$

where $J_{1/2}$ is the $\frac{1}{2}$-th Bessel function, and $\alpha^{-1} := \frac{4\pi}{3}\int_0^\infty s^2 k(s)\,ds$ is a positive normalization constant. We refer to [17] for these definitions and further discussions about the integral kernel of the parabolic biharmonic equation. We recall that the $\frac{1}{2}$-th Bessel function is given by

$$J_{1/2}(s) = (\pi/2)^{-1/2}s^{-1/2}\sin(s),$$

then

$$P_t f(x) = (p_t * f)(x) = c \int f(x-y) \int_0^\infty \frac{1}{|y|^3} e^{-ts^4/|y|^4} s \sin(s)\, ds\, dy,$$

and therefore,

$$\frac{d}{dt} P_t f(x) = -c \int f(x-y) \int_0^\infty \frac{1}{|y|^3} e^{-ts^4/|y|^4} \frac{s^5}{|y|^4} \sin(s)\, ds\, dy.$$

We are ready to prove the following result.

Proposition 2 *Assume that $f, g \in L^1 \cap L^2$ and that f is supported in $\{|\bar{x}| \geq \gamma_2 R\}$, while g is supported in $\{|\bar{x}| \leq \gamma_1 R\}$, for some positive parameters $\gamma_{1,2}$ satisfying $d := \gamma_2 - \gamma_1 > 0$. Then*

$$|\langle \mathcal{R}_i^4 f, g \rangle| \lesssim R^{-1} \|g\|_{L^1} \|f\|_{L^1}. \tag{28}$$

Proof With the change of variable $s^4|y|^{-4} = \tau$, we get

$$\frac{d}{dt} P_t f = -\frac{c}{4} \int \int_0^\infty \frac{1}{|y|} e^{-t\tau} \tau^{1/2} \sin(\tau^{1/4}|y|) f(x-y)\, d\tau\, dy,$$

and hence, by a change of variable in space,

$$\frac{d}{dt} P_t f = -\frac{c}{4} \int \int_0^\infty \frac{1}{|x-y|} e^{-t\tau} \tau^{1/2} \sin(\tau^{1/4}|x-y|) f(y)\, d\tau\, dy.$$

We will use the following, by adopting the notation deg for the degree of a polynomial.

Claim There exist $M \geq 1$ and M pairs of polynomials $(\tilde{q}_k, q_k)_{k \in \{1,\dots,M\}}$ with nonnegative coefficients, such that

$$\min_{k \in \{1,\dots,M\}} \{deg(q_k)\} \geq 1,$$

and satisfying

$$\left| \partial_{x_i}^4 \left(\frac{1}{|x-y|} \sin(\tau^{1/4}|x-y|) \right) \right| \lesssim \sum_{k=1}^M \frac{\tilde{q}_k(\tau^{1/4})}{q_k(|x-y|)}.$$

At this point, by using the identity (26), we infer the following:

$$
\begin{aligned}
|\langle \mathcal{R}_i^4 f, g\rangle| &= \left| \int_0^\infty \langle \partial_{x_i}^4 \frac{d}{dt} P_t f, g\rangle t \, dt \right| \\
&\lesssim \Bigg| \int_0^\infty t \int g(x) \times \\
&\qquad \left(\int \int_0^\infty \partial_{x_i}^4 \left(\frac{1}{|x-y|} \sin(\tau^{1/4}|x-y|) \right) e^{-t\tau} \tau^{1/2} f(y) \, d\tau \, dy \right) dx \, dt \Bigg| \\
&\lesssim \int_0^\infty t \int |g(x)| \times \\
&\qquad \left(\int \int_0^\infty \sum_{k=1}^M \frac{\tilde{q}_k(\tau^{1/4})}{q_k(|x-y|)} e^{-t\tau} \tau^{1/2} |f(y)| \, d\tau \, dy \right) dx \, dt \\
&= \int_0^\infty t \int |g(x)| \times \\
&\qquad \left(\int \int_0^\infty \sum_{k=1}^M \frac{\tilde{q}_k(\tau^{1/4})}{q_k(|y|)} e^{-t\tau} \tau^{1/2} |f(x-y)| \, d\tau \, dy \right) dx \, dt \\
&\le \int_0^\infty t \int_{\{|\bar{x}| \le \gamma_1 R\}} |g(x)| \times \\
&\qquad \left(\int_{\{|\bar{x}-\bar{y}| \ge \gamma_2 R\}} \int_0^\infty \sum_{k=1}^M \frac{\tilde{q}_k(\tau^{1/4})}{q_k(|\bar{y}|)} e^{-t\tau} \tau^{1/2} |f(x-y)| \, d\tau \, dy \right) dx \, dt.
\end{aligned}
$$

Therefore, as the support of $f(x-y)$ is contained in $|\bar{x}-\bar{y}| \ge \gamma_2 R$ and the one of g is contained in $|\bar{x}| \le \gamma_1 R$, we get that $|\bar{y}| \ge dR$. Hence, by defining $\beta = \frac{1}{4} \max_{k \in \{1,\dots,M\}} \{deg(\tilde{q}_k))\}$, we can bound

$$
\begin{aligned}
|\langle \mathcal{R}_i^4 f, g\rangle| &\lesssim R^{-1} \|f\|_{L^1} \|g\|_{L^1} \sum_{k=1}^M \int_0^\infty \int_0^\infty e^{-t\tau} \tau^{1/2} \tilde{q}_k(\tau^{1/4}) \, t \, d\tau \, dt \\
&\lesssim R^{-1} \|f\|_{L^1} \|g\|_{L^1} \int_1^\infty \int_1^\infty e^{-t\tau} \tau^{1/2} \tau^{(\max_{k \in \{1,\dots,M\}} deg(\tilde{q}_k))/4} \, t \, d\tau \, dt \\
&= R^{-1} \|f\|_{L^1} \|g\|_{L^1} \int_1^\infty \int_1^\infty e^{-t\tau} \tau^{\beta+1/2} \, t \, d\tau \, dt \\
&\lesssim R^{-1} \|f\|_{L^1} \|g\|_{L^1},
\end{aligned}
$$

where we used the Fubini–Tonelli's theorem. The proof of (28) is concluded. □

We now give the proof of the claim above.

Proof of the Claim As the derivative is invariant under translations and by defining $c = \tau^{1/4}$, we can reduce everything to the estimate of $\partial^4_{x_i}\left(|x|^{-1}\sin(c|x|)\right)$. By setting $f(r) = r^{-1}\sin(cr)$ and $g(x) = |x|$, we can see

$$|x|^{-1}\sin(c|x|) = (f \circ g)(x),$$

and without loss of generality, we assume $i = 3$. Then we see g as a function of x_3 alone, i.e., $g(x_3) = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}$. We first collect some identities.

$$\begin{aligned}
f'(r) &= cr^{-1}\cos(cr) - r^{-2}\sin(cr),\\
f''(r) &= -c^2r^{-1}\sin(cr) - 2cr^{-2}\cos(cr) + 2r^{-3}\sin(cr),\\
f'''(r) &= -c^3r^{-1}\cos(cr) + c^2r^{-2}\sin(cr) + 8cr^{-3}\cos(cr) - 6r^{-4}\sin(cr),\\
f''''(r) &= c^4r^{-1}\sin(cr) + 2c^3r^{-2}\cos(cr) - 10c^2r^{-3}\sin(cr)\\
&\qquad - 30cs^{-4}\cos(cr) + 24r^{-5}\sin(cr),
\end{aligned}$$

and

$$\begin{aligned}
g' &= \partial_{x_3} g(x_3) = \frac{x_3}{(x_1^2+x_2^2+x_3^2)^{1/2}},\\
g'' &= \partial^2_{x_3} g(x_3) = \frac{1}{(x_1^2+x_2^2+x_3^2)^{1/2}} - \frac{x_3^2}{(x_1^2+x_2^2+x_3^2)^{3/2}},\\
g''' &= \partial^3_{x_3} g(x_3) = -\frac{3}{(x_1^2+x_2^2+x_3^2)^{3/2}} + \frac{3x_3^3}{(x_1^2+x_2^2+x_3^2)^{5/2}},\\
g'''' &= \partial^4_{x_3} g(x_3) = -\frac{3}{(x_1^2+x_2^2+x_3^2)^{3/2}} + \frac{18x_3^2}{(x_1^2+x_2^2+x_3^2)^{5/2}} - \frac{15x_3^4}{(x_1^2+x_2^2+x_3^2)^{7/2}}.
\end{aligned}$$

At this point, we recall that by the Faà di Bruno's formula

$$\begin{aligned}
\partial^4_{x_3}(f\circ g)(x) &= f''''(|x|)[g'(x)]^4 + 6f'''(|x|)g''(x)[g'(x)]^2 + 3f''(|x|)[g''(x)]^2\\
&\quad + 4f''(|x|)g'''(x)g'(x) + f'(|x|)g''''(x),
\end{aligned}$$

and the claim easily follows by replacing $c = \tau^{1/4}$ and translating back to $x \mapsto x - y$. □

Remark 1 It is straightforward to observe that in (27) we can replace the symbol ξ_j^4 with $\xi_k^2\xi_h^2$, for $k \neq h$, to get

$$\frac{\xi_k^2\xi_h^2}{|\xi|^4} = \xi_k^2\xi_h^2|\xi|^4\int_0^\infty e^{-t|\xi|^4}t\,dt, \tag{29}$$

and consequently,

$$\langle \mathcal{R}_k^2 \mathcal{R}_h^2 f, g \rangle = -\int_0^\infty \langle \partial_{x_k}^2 \partial_{x_h}^2 \frac{d}{dt} P_t f, g \rangle t \, dt. \tag{30}$$

The identities (29) and (30) of the remark above easily imply an analogous of Proposition 2 (by repeating its proof with the obvious modifications) for the operator $\mathcal{R}_k^2 \mathcal{R}_h^2$ replacing $\mathcal{R}_j^4$. More precisely:

Proposition 3 *Assume that $f, g \in L^1 \cap L^2$ and that f is supported in $\{|\bar{x}| \geq \gamma_2 R\}$, while g is supported in $\{|\bar{x}| \leq \gamma_1 R\}$, for some $\gamma_{1,2} > 0$ satisfying $d := \gamma_2 - \gamma_1 > 0$. Then*

$$|\langle \mathcal{R}_k^2 \mathcal{R}_h^2 f, g \rangle| \lesssim R^{-1} \|g\|_{L^1} \|f\|_{L^1}.$$

2.2 Point-Wise Estimates for $\mathcal{R}_j^2$

We turn now the attention to the square of the Riesz transforms. In the subsequent results, we will use a cut-off function χ satisfying the following: $\chi(x)$ is a localization function supported in the cylinder $\{|\bar{x}| \leq 1\}$ that is nonnegative and bounded, with $\|\chi\|_{L^\infty} \leq 1$. For a positive parameter γ, we define by $\chi_{\{|\bar{x}| \leq \gamma R\}}$ the rescaled function $\chi(x/\gamma R)$ (hence $\chi_{\{|\bar{x}| \leq \gamma R\}}$ is bounded, positive, and supported in the cylinder of radius γR). The proof of the next propositions is inspired by [31].

Proposition 4 *For any (regular) function f, the following point-wise estimate is satisfied: provided $d := \gamma_2 - \gamma_1 > 0$, there exists a universal constant $C = C(d) > 0$ such that*

$$|\chi_{\{|\bar{x}| \leq \gamma_1 R\}}(x) \mathcal{R}_j^2 [(1 - \chi_{\{|\bar{x}| \leq \gamma_2 R\}}) f](x)| \leq C R^{-3} \chi_{\{|\bar{x}| \leq \gamma_1 R\}}(x) \|f\|_{L^1(|\bar{x}| \geq \gamma_2 R)}. \tag{31}$$

We have an estimate similar to (31) if we localize inside a cylinder the function on which R_j^2 acts, and we then truncate everything with a function supported in the exterior of another cylinder.

Proposition 5 *For any (regular) function f, the following point-wise estimate is satisfied: provided $d := \gamma_1 - \gamma_2 > 0$, there exists a universal constant $C = C(d) > 0$ such that*

$$|(1 - \chi_{|\bar{x}| \leq \gamma_1 R})(x) \mathcal{R}_j^2 [(\chi_{\{|\bar{x}| \leq \gamma_2 R\}}) f](x)| \leq C R^{-3} |(1 - \chi_{\{|\bar{x}| \leq \gamma_1 R\}})(x)| \|f\|_{L^1(|\bar{x}| \leq \gamma_2 R)}.$$

Proof The proofs of the propositions above are analogous, and they can be given by observing that in the principal value sense, the square of the Riesz transform acts on a function g as

$$\mathcal{R}_j^2 g(x) = \iint \frac{x_j - y_j}{|x-y|^{3+1}} \frac{y_j - z_j}{|y-z|^{3+1}} g(z)\, dz\, dy.$$

Without loss of generality, we consider the case depicted in Proposition 4. Let $g(x) = \chi_{\{|\bar{x}| \geq \gamma_2 R\}}(x) f(x)$. Then

$$\chi_{\{|\bar{x}| \leq \gamma_1 R\}}(x) \mathcal{R}_j^2 g(x) = \chi_{\{|\bar{x}| \leq \gamma_1 R\}}(x) \iint \left(\frac{y_j}{|y|^4} \frac{z_j - y_j}{|z-y|^4}\, dy \right) g(x-z)\, dz.$$

Since g is supported in the exterior of a cylinder of radius $\gamma_2 R$, we can assume $|\bar{x} - \bar{z}| \geq \gamma_2 R$, and as the function $\chi_{\{|\bar{x}| \leq \gamma_1 R\}}$ is supported by definition in the cylinder of radius $\gamma_1 R$, we can assume $|\bar{x}| \leq \gamma_1 R$: therefore, we have that $|\bar{z}| \geq dR$. This implies that $\{|\bar{y}| \leq \frac{d}{4} R\} \cap \{|\bar{z} - \bar{y}| \leq \frac{1}{2}|\bar{z}|\} = \emptyset$. Indeed,

$$\frac{1}{2}|\bar{z}| \geq |\bar{z} - \bar{y}| \geq |\bar{z}| - |\bar{y}| \implies |\bar{y}| \geq \frac{1}{2}|\bar{z}| \geq \frac{d}{2} R,$$

hence, we have the following splitting for the inner integral:

$$\begin{aligned} \int \frac{y_j}{|y|^4} \frac{z_1 - y_1}{|z-y|^4}\, dy &= \int_{|\bar{y}| \leq \frac{d}{4} R} \frac{y_j}{|y|^4} \frac{z_j - y_j}{|z-y|^4}\, dy \\ &+ \int_{|\bar{z} - \bar{y}| \leq \frac{1}{2}|\bar{z}|} \frac{y_j}{|y|^4} \frac{z_j - y_j}{|z-y|^4}\, dy \\ &+ \int_{\{|\bar{y}| \geq \frac{d}{4} R\} \cap \{|\bar{z} - \bar{y}| \geq \frac{1}{2}|\bar{z}|\}} \frac{y_j}{|y|^4} \frac{z_j - y_j}{|z-y|^4}\, dy \\ &= \mathcal{A} + \mathcal{B} + \mathcal{C}. \end{aligned}$$

By using the properties of the domains in this splitting, the proof of the proposition can be done by straightforward computations, ending up with

$$\int \frac{y_j}{|y|^4} \frac{z_1 - y_1}{|z-y|^4}\, dy = \mathcal{A} + \mathcal{B} + \mathcal{C} \lesssim R^{-3}.$$

Hence,

$$\begin{aligned}|\chi_{\{|\bar{x}|\leq\gamma_1 R\}}(x)\mathcal{R}_j^2 g(x)| &= \chi_{\{|\bar{x}|\leq\gamma_1 R\}}(x)\left|\iint\left(\frac{y_j}{|y|^4}\frac{z_j-y_j}{|z-y|^4}\,dy\right)g(x-z)\,dz\right|\\ &\lesssim R^{-3}\chi_{\{|\bar{x}|\leq\gamma_1 R\}}(x)\int|g(x-z)|\,dz\\ &\lesssim R^{-3}\chi_{\{|\bar{x}|\leq\gamma_1 R\}}(x)\|f\|_{L^1(|\bar{x}|\geq\gamma_2 R)},\end{aligned}$$

which is the estimate stated in (31). See [6] for the details. □

The proofs of Proposition 2, Proposition 3, Proposition 4, and Proposition 5 can be done by using an alternative approach, by means of a general characterization of homogeneous distribution on $\mathbb{R}^n$ of degree $-n$, coinciding with a regular function in $\mathbb{R}^n \setminus \{0\}$. Indeed, we have the following (we specialize to the three-dimensional case). For a proof, see [6]. In what follows, “dist” denotes the distance function.

Proposition 6 *Let T be an operator defined by means of a Fourier symbol $m(\xi)$, which is smooth in $\mathbb{R}^3 \setminus \{0\}$ and is a homogenous function of degree zero, i.e., $m(\lambda\xi) = m(\xi)$ for any $\lambda > 0$. For any couple of functions $f, g \in L^1$ having disjoint supports, we have the following estimate:*

$$|\langle Tf, g\rangle| \lesssim (dist(supp(f), supp(g)))^{-3}\,\|g\|_{L^1}\|f\|_{L^1}.$$

Remark 2 Keeping in mind the general statement of Proposition 6, it is easy for the reader to see that similar results as in Proposition 2, Proposition 3, Proposition 4, and Proposition 5 can be stated for functions localized outside and inside disjoint balls, instead of disjoint cylinders. Such localizations for functions supported outside and inside balls will be used for the scattering results using a concentration/compactness and rigidity scheme.

2.3 Virial Identities

The main difference between the Gross–Pitaevskii equation (2) and the classical cubic NLS equation is the non-local character of the non-linearity, in conjunction with the fact that the kernel K requires a more careful treatment with respect to the usual Coulomb- or Hartree-type kernels. Hence, we spend a few words here to give an overview on how the results concerning the decay of powers of the Riesz transforms as in the previous subsections will play a central role in the proofs of the main theorems. We show below how the tools above will be used for both the scattering and blow-up/grow-up results:

(i) Standard arguments show that provided (11) is satisfied, then the Pohozaev functional G is bounded from below uniformly in time, in particular there exists a positive α such that $G(u(t)) \geq \alpha > 0$ for all times in the maximal interval of existence of the solution. Similarly, provided (12) holds true, then $G(u(t)) \leq -\delta < 0$ for all times in the maximal interval of existence of the solution, for some positive δ. As a byproduct, $G(u(t)) \lesssim -\delta\|u(t)\|^2_{\dot{H}^1}$.

(ii) Let χ a (regular) nonnegative function, which will be well chosen below. Let us denote by χ_R the rescaled version of χ, defined by $\chi_R = R^2\chi(x/R)$, and let us introduce the quantity

$$V_{\chi_R}(t) := V_{\chi_R}(u(t)) = 2\int \chi_R(x)|u(t,x)|^2\,dx. \tag{32}$$

By formal computations, which can be justified by a classical regularization argument, it is easy to show that

$$\frac{d^2}{dt^2}V_{\chi_R}(t) = 4\int |\nabla u(t)|^2 dx + 6\lambda_1\int |u(t)|^4 dx + H_R(u(t)), \tag{33}$$

where H_R is a term that must be controlled. Our aim is to show that

$$H_R(u(t)) = 6\lambda_2\int (K * |u(t)|^2)|u(t)|^2\,dx + \varepsilon_R, \tag{34}$$

where $\varepsilon_R = o_R(1)$ as $R \to \infty$, uniformly in time in the lifespan of the solution. Let us observe that by gluing together (33) and (34), we get, by recalling the definition of G, see (6),

$$\frac{d^2}{dt^2}V_{\chi_R}(t) = 4G(u(t)) + \varepsilon_R.$$

By using the controls on the Pohozaev functional as described in the first point, and provided that ε_R is made sufficiently small for R large enough, then we are able to conclude the concentration/compactness and rigidity scheme for the scattering results, or we can provide suitable estimates to perform a convexity argument for the blow-up results. See the next points and the discussions in the next sections.

(iii-a) As for the scattering part, let us mention for sake of clarity that the aim of the concentration/compactness and rigidity scheme is to prove that all solutions arising from initial data satisfying (11) are global and scatter. Let us recall that small initial data lead to global and scattering solutions, by a standard perturbative argument. The Kenig and Merle's road map (see [27, 28]) then proceeds as follows: suppose that the threshold for global and scattering solutions is strictly smaller than the claimed one (i.e., $E(Q)M(Q)$); then, by means of a profile decomposition theorem, it is possible to construct a minimal (in terms of the energy) global non-scattering solution at the threshold energy. Moreover, such a solution, called soliton-like solution and denoted by u_{crit}, is precompact in the energy space up to a continuous-in-time translation path $x(t)$, i.e., $\{u(t, x + x(t))\}_{t\in\mathbb{R}^+}$ is precompact in H^1. The crucial fact is that such a path $x(t)$ grows sub-linearly at infinity, and this will rule out the existence of such a soliton-like solution. This latter fact is proved by using the precompactness of the (translated) flow in conjunction with a virial argument, along with the already mentioned growth property of $x(t)$.

For the virial argument in this context, we choose χ to be a cut-off function such that $\chi(x) = |x|^2$ on $|x| \leq 1$ and $supp\,(\chi) \subset B(0, 2)$ (namely, we consider a localized version of (7)). We get

$$\begin{aligned}\frac{d^2}{dt^2}V_{\chi_R}(t) &= 4\int |\nabla u(t)|^2\,dx + 6\lambda_1\int |u(t)|^4\,dx + \varepsilon_{1,R}\\ &\quad - 2\lambda_2 R\int \nabla\chi\left(\frac{x}{R}\right)\cdot\nabla\left(K * |u(t)|^2\right)|u(t)|^2\,dx,\end{aligned} \tag{35}$$

where

$$\varepsilon_{1,R} = C\left(\int_{|x|\geq R} |\nabla u(t)|^2 + R^{-2}|u(t)|^2 + |u(t)|^4\,dx\right). \tag{36}$$

The quantity $\varepsilon_{1,R}$ can be made small, uniformly in time, for R sufficiently large, by using the precompactness of the soliton-like solution constructed with the concentration/compactness scheme. To handle the expression

$$\Lambda := -2\lambda_2 R\int \nabla\chi\left(\frac{x}{R}\right)\cdot\nabla\left(K * |u(t)|^2\right)|u(t)|^2\,dx$$

in (35), we perform a splitting in space of the solution $u(t, x)$ by considering its cut-off inside and outside a ball of radius $\sim R$, eventually obtaining the identity $\Lambda = 6\lambda_2\int(K*|u(t)|^2)|u(t)|^2\,dx + \varepsilon_{2,R}$. It is after the splitting above that we can reduce to a term $\varepsilon_{2,R}$ that fulfills the hypothesis of the point-wise decay of the Riesz transforms as in the previous section; indeed, with such localized functions, we can lead back our term $\varepsilon_{2,R}$ in the framework of Proposition 4 and Proposition 5. By letting $\varepsilon_R := \varepsilon_{1,R} + \varepsilon_{2,R}$, for R sufficiently large, we get

$$\frac{d^2}{dt^2}V_{\chi_R}(t) = 4G(u(t)) + \varepsilon_R \geq 2\alpha,$$

where we used the strictly positive lower bound for G as described in point (i). This latter estimate, in conjunction with the sub-linear growth of $x(t)$, will give a contradiction; hence, the soliton-like solution cannot exist, and therefore, the threshold for the scattering is given precisely by the quantity as in (11).

(iii-b) As for the blow-up in finite time, the last part of strategy can be considered similar, as it is given by a Glassey argument based on virial identities. Nonetheless, the analysis is different and more complicated with respect to two points of the rigidity part for the scattering theorems. Specifically, in the formation of singularities scenario, we cannot rely on some compactness property on the non-linear flow; hence, the control on the remainder $H_R(u(t))$ cannot be given in a full generality. This is why we have to assume some symmetry hypothesis on the solution. It is here that we need to introduce the framework Σ_3, the space of cylindrical symmetric solutions, with finite variance only along the third axis direction. Let us recall that even for the classical cubic NLS (i.e., $\lambda_1 = -1$ and $\lambda_2 = 0$), it is an open problem to show blow-up without assuming any additional symmetry hypothesis or finiteness of the variance, see [1, 12, 21–26, 29, 34]. Here we give the minimal assumptions to obtain formation of singularities in finite time, i.e., the solution is in Σ_3. See also [32] for an early work on NLS in anisotropic spaces and [3, 7, 11, 18] for these techniques applied to other dispersive models.

For the virial argument, we chose here a (rescaled) function χ_R as the sum of a rescaled localization function ρ_R, plus the function x_3^2. Here, ρ_R is a well-constructed function depending only on the two variables $\bar{x} = (x_1, x_2)$ that provides a localization in the exterior of a cylinder, parallel to the x_3 axis and with a radius of size $|\bar{x}| \sim R$. The notation $|\bar{x}|$ clearly stands for $|\bar{x}| := (x_1^2 + x_2^2)^{1/2}$. Moreover, we added the non-localized function x_3^2 in order to obtain a virial-like estimate of the form

$$\frac{d^2}{dt^2} V_{\rho_R + x_3^2}(t) \leq 4 \int |\nabla u(t)|^2 dx + 6\lambda_1 \int |u(t)|^4 dx + H_R(u(t)),$$

where the term H_R is defined by

$$\begin{aligned} H_R(u(t)) = 4\lambda_1 \int a_R(\bar{x}) |u(t)|^4 \, dx + cR^{-2} \\ + 2\lambda_2 \int \nabla \rho_R \cdot \nabla \left(K * |u(t)|^2 \right) |u(t)|^2 \, dx \\ - 4\lambda_2 \int x_3 \partial_{x_3} \left(K * |u(t)|^2 \right) |u(t)|^2 \, dx, \end{aligned}$$

and $a_R(\bar{x})$ is a bounded, nonnegative function supported in the exterior of a cylinder of radius of order R. We estimate $\int a_R(\bar{x}) |u|^4 \, dx = o_R(1) \|u(t)\|_{\dot{H}^1}^2$ by means of a suitable Strauss embedding. Hence, it remains to estimate the non-local terms in $H_R(u(t))$. Similarly to the scattering part, the strategy is to split $u(t, x)$ by separating it into the interior and the exterior of a cylinder, instead of a ball, and computing the interaction given by the dipolar term. The further difficulty (with respect to the virial argument for the scattering theorem) is that $K * \cdot$ is not supported inside any cylinder, even if we localize the function where K is acting on (through the convolution). Therefore, by performing further suitable splittings, we are able to give the identity

$$H_R(u(t)) = 6\lambda_2 \int (K * |u(t)|^2)|u(t)|^2\, dx + \epsilon_R,$$

where the contributes defining ϵ_R consist of terms of the form $\langle \mathcal{R}_3^4 f, g\rangle$ when f is supported in $\{|\bar{x}| \geq \gamma_2 R\}$, while g is supported in $\{|\bar{x}| \leq \gamma_1 R\}$, for some positive parameters γ_1 and γ_2 satisfying $d := \gamma_2 - \gamma_1 > 0$. Clearly, the localizations of $u(t,x)$ play the role of f, g above. Hence, by means of Proposition 2, we can conclude, provided R is large enough, with

$$\frac{d^2}{dt^2} V_{\rho_R + x_3^2}(t) \leq 4G(u(t)) + \epsilon_R \leq -2\delta,$$

which in turn implies the finite-time blow-up via a Glassey convexity argument [21]. Note that we used the strictly negative upper bound for G as described in point (i).

3 Sketch of the Proofs Below the Threshold

3.1 Scattering

As already mentioned in point (iii-a), the scattering result given in Theorem 1 is given by running a concentration/compactness and rigidity scheme, as pioneered by Kenig and Merle in their celebrated works [27, 28]. Nowadays there is a huge literature on this method, applied to several dispersive models, and since the scope of this review paper is not to go over the details of these techniques, we refer the reader to [1, 13, 16, 19, 22, 24] for mass–energy intracritical NLS equations. Let us only mention that the method can be viewed as an induction of the energy method, and it proceeds by contradiction by assuming the threshold for global and scattering solutions is strictly smaller than the claimed one. Hence, we define the threshold for scattering as follows:

$$\mathcal{ME} = \sup \{\delta : M(u_0)E(u_0) < \delta \text{ and } \|u_0\|_{L^2}\|\nabla u_0\| < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$$
$$\text{then the solution to (2) with initial data } u_0 \textit{ is in } L^8L^4 \}.$$

A classical small data theory gives that if the initial datum is small enough in the energy norm, then the corresponding solution scatters, or equivalently, it belongs to $L^8L^4 := L_t^8(\mathbb{R}; L_x^4(\mathbb{R}^3))$. Therefore, the threshold is certainly strictly positive. The goal is therefore to prove that $\mathcal{ME} = M(Q)E(Q)$.

At this point, we assume by contradiction that the threshold is strictly smaller than the given one (i.e., we assume $\mathcal{ME} < M(Q)E(Q)$, and we eventually prove that the latter leads to a contradiction).

Indeed, a linear profile decomposition theorem tailored for Eq. (2), see [5, Theorem 4.1, Proposition 4.3, and Corollary 4.4], and the existence of the wave operator enable us to establish the following.

Theorem 7 *There exists a non-trivial initial profile $u_{crit}(0) \in H^1$ such that the following holds true: $M(u_{crit}(0))E(u_{crit}(0)) = \mathcal{ME}$ and $\|u_{crit}(0)\|_{L^2}\|\nabla u_{crit}(0)\| < \|Q\|_{L^2}\|\nabla Q\|_{L^2}$, and the corresponding solution $u_{crit}(t)$ to (2) is globally defined and does not scatter. Moreover, there exists a continuous function $x(t) : \mathbb{R}^+ \mapsto \mathbb{R}^3$ such that $\{u_{crit}(t, x + x(t)),\ t \in \mathbb{R}^+\}$ is precompact as a subset of H^1. Such a function $x(t)$ satisfies $|x(t)| = o(t)$ as $t \to +\infty$, namely it grows sub-linearly at infinity.*

The theorem above says that by assuming $\mathcal{ME} < M(Q)E(Q)$, we are able to construct an initial datum whose nonlinear evolution is global and non-scattering. The precompactness tells us that $u_{crit}(t)$ remains spatially localized (uniformly in time) along the continuous path $x(t) \in \mathbb{R}^3$. Specifically, for any $\varepsilon > 0$, there exists $R_\varepsilon \gg 1$ such that

$$\int_{|x-x(t)|\geq R_\varepsilon} |\nabla u_{crit}(t)|^2 + |u_{crit}(t)|^2 + |u_{crit}(t)|^4 \leq \varepsilon \qquad \text{for any} \quad t \in \mathbb{R}^+. \tag{37}$$

The proof of the growth property of $x(t)$ is inspired by [13], and it is based on Galilean transformations of the solution.

The Kenig–Merle scheme is closed provided we can show that the solution given in Theorem 7 cannot exist. Indeed, as introduced in Sect. 2.3 (iii-a), a virial argument will give, see (35),

$$\begin{aligned}\frac{d^2}{dt^2}V_{\chi_R}(t) &= 4\int |\nabla u(t)|^2\,dx + 6\lambda_1\int |u(t)|^4\,dx + \varepsilon_{1,R} \\ &\quad - 2\lambda_2 R\int \nabla\chi\left(\frac{x}{R}\right)\cdot\nabla\left(K * |u(t)|^2\right)|u(t)|^2\,dx.\end{aligned}$$

The main goal is therefore to estimate the non-local contribution $\Lambda := -2\lambda_2 R\int \nabla\chi\left(\frac{x}{R}\right)\cdot\nabla\left(K * |u(t)|^2\right)|u(t)|^2\,dx$. We introduce the space localization inside and outside a ball of radius $10R$, namely we write (we ignore the time dependence)

$$u = \mathbf{1}_{\{|x|\leq 10R\}}u + \mathbf{1}_{\{|x|>10R\}}u := u_i + u_o.$$

By using the disjointness of the supports, we can rewrite $\Lambda = \Lambda_{i,i} + \Lambda_{o,i}$. In the latter notation, the subscript $\Lambda_{\diamond,\star}$, for $\diamond, \star \in \{i, o\}$, means the following: after some manipulations, the terms we are considering are of the form

$$\Lambda_{\diamond,\star}(u) = \int g\left(K * |u_\diamond|^2\right) h(|u_\star|^2)\, dx,$$

i.e., the dipolar kernel K acts (via the convolution) on the localization (of u) given by the first symbol $\diamond$, while the other term in the integral contains the term localized according to the symbol $\star$. With a careful handling of the expression above, we reduce everything to fulfill the hypothesis of Proposition 4 and Proposition 5, leading to the final estimate

$$\Lambda \geq 6\lambda_2 \int (K * |u(t)|^2)|u(t)|^2\, dx + \varepsilon_{2,R}$$

with

$$\begin{aligned}\varepsilon_{2,R} &\lesssim R^{-1} + R^{-1}\|u(t)\|_{H^1}^2 \|u(t)\|_{L^4(|x|\geq 10R)}^2 + R^{-1}\|u(t)\|_{H^1}^2 \\ &\quad + \|u(t)\|_{L^4(|x|\geq 10R)}^2 + \|u(t)\|_{L^4(|x|\geq 10R)}^4.\end{aligned} \tag{38}$$

Let us observe that the remainder as in (38) has a similar form as the one in (36) describing $\varepsilon_{1,R}$. Hence, they can be controlled in the same fashion. Specifically, we fix a time interval $[T_0, T_1]$ for $0 < T_0 < T_1$, and we take $R \geq \sup_{[T_0,T_1]} |x(t)| + R_\varepsilon$ as in (37) such that $\frac{d^2}{dt^2} z_R(t) \geq \frac{\alpha}{2} > 0$. An integration on $[T_0, T_1]$ yields to

$$R \gtrsim R\|u\|_{L^2}\|\nabla u\|_{L^2} \gtrsim \left|\frac{d}{dt} z_R(T_1) - \frac{d}{dt} z_R(T_0)\right| \geq \frac{\alpha}{2}(T_1 - T_0),$$

i.e., for some $c > 0$, we have $c(T_1 - T_0) \leq R$. Note that by the sub-linearity growth of $x(t)$, once fixed $\delta > 0$, we can guarantee that there exists a time t_δ such that $|x(t)| \leq \delta t$ for any $t \geq t_\delta$. Hence, by picking $\delta = c/2$, and $R = R_\varepsilon + \frac{cT_1}{2}$, we have $\frac{cT_1}{2} \leq R_\varepsilon + cT_0$, and the latter leads a contradiction, as we can let T_1 be as large as we want, while the right-hand side remains bounded.

3.2 Blow-up

As introduced in Sect. 2.3, our goal is to give the following estimate:

$$\frac{d^2}{dt^2} V_{\rho_R + x_3^2}(t) \leq 4\int |\nabla u(t)|^2 dx + 6\lambda_1 \int |u(t)|^4 dx + H_R(u(t)), \tag{39}$$

where H_R is defined by

$$
\begin{aligned}
H_R(u(t)) = 4\lambda_1 \int a_R(\bar{x})|u(t)|^4\,dx + cR^{-2} \\
+ 2\lambda_2 \int \nabla\rho_R \cdot \nabla\left(K * |u(t)|^2\right)|u(t)|^2\,dx \\
- 4\lambda_2 \int x_3\partial_{x_3}\left(K * |u(t)|^2\right)|u(t)|^2\,dx
\end{aligned}
\tag{40}
$$

and $a_R = 0$ in $\{|\bar{x}| \leq R\}$. We recall that the subscript R stands for the rescaling $f_R = R^2 f(x/R)$. To this aim, we consider a regular, nonnegative, radial function $\rho = \rho(|\bar{x}|) = \rho(r)$ such that

$$
\rho(r) = \begin{cases} r^2 & \text{if } r \leq 1 \\ 0 & \text{if } r \geq 2 \end{cases}, \quad \text{such that} \quad \rho'' \leq 2 \quad \text{for any} \quad r \geq 0.
$$

A similar function can be explicitly constructed, see [6, 32, 34], and satisfies (39), (40), with a_R localized in the exterior of a cylinder of radius R. By means of Strauss estimates, it is quite easy to obtain

$$
\begin{aligned}
H_R(u(t)) &= 2\lambda_2\left(\int \nabla\rho_R \cdot \nabla\left(K * |u(t)|^2\right)|u(t)|^2\,dx \right.\\
&\qquad \left. -2\int x_3\partial_{x_3}\left(K * |u(t)|^2\right)|u(t)|^2\,dx\right) \\
&\quad + o_R(1)\|u(t)\|^2_{\dot{H}^1} \\
&:= 2\lambda_2(\Xi + \Upsilon) + o_R(1)\|u(t)\|^2_{\dot{H}^1}.
\end{aligned}
$$

So we are reduced to the estimate of $\Xi + \Upsilon$. In order to use the decays as in Sect. 2, we proceed with several localizations, in order to reduce the problems given by the non-local terms $\Xi + \Upsilon$ to fulfill the hypothesis of the decay properties for powers of the Riesz transforms. The scheme is as follows. We introduce the first localization inside and outside a cylinder of radius $10R$, namely we write (we ignore the time dependence)

$$
u = \mathbf{1}_{\{|\bar{x}| \leq 10R\}}u + \mathbf{1}_{\{|\bar{x}| > 10R\}}u := u_i + u_o.
$$

By using the disjointness of the supports, we can rewrite $\Xi + \Upsilon = \Xi_{o,i} + \Xi_{i,i} + \Upsilon$. The proof of the decay for $\Xi_{o,i}$ can be given, after careful manipulations, by means of the point-wise decay as in Proposition 4 and Proposition 5. The main problem is given by the term $\Xi_{i,i} + \Upsilon$ where we do not have any localization at the exterior of a cylinder, preventing us to obtain some decay straightforwardly. Hence, a further splitting is introduced. We separate u_i as $u_i = w_{i,i} + w_{i,o}$, where

$$w_{i,i} = \mathbf{1}_{\{|\bar{x}|\le R/10\}} u_i \quad \text{and} \quad w_{i,o} = \mathbf{1}_{\{|\bar{x}|> R/10\}} u_i = \mathbf{1}_{\{R/10<|\bar{x}|\le 4R\}} u.$$

Therefore, we generate terms localized outside a cylinder of radius $\sim R$, specifically of the form

$$\mathcal{A}_{i,o}(u_i) = \int g\left(K * |w_{i,i}|^2\right) h(|w_{i,o}|^2)\, dx,$$

$$\mathcal{B}_{o,o}(u_i) = \int \tilde{g}\left(K * |w_{i,o}|^2\right) \tilde{h}(|w_{i,o}|^2)\, dx,$$

plus a quantity

$$C_{i,i}(u) + \Upsilon := 2\int \bar{x}\cdot\nabla_{\bar{x}}\left(K * |u_i|^2\right)|u_i|^2\, dx + 2\int x_3\partial_{x_3}\left(K * |u|^2\right)|u|^2\, dx.$$

By continuing the computations, we end up with a reduction of $\mathcal{A}_{i,o}(u_i)$ and $\mathcal{B}_{o,o}(u_i)$ to a framework as in Proposition 4 and Proposition 5, and we get

$$\mathcal{A}_{i,o}(u_i) = o_R(1)\|u\|^2_{\dot{H}^1} \quad \text{and} \quad \mathcal{B}_{o,o}(u_i) = o_R(1)\|u\|^2_{\dot{H}^1}.$$

In order to control the remainder term $C_{i,i}(u) + \Upsilon$ and to make $6\lambda_2 \int (K * |u|^2)|u|^2\, dx$ appear in (39), which will yield the whole quantity $4G(u(t))$, we need to use the identity

$$2\int x\cdot\nabla(K * f)\, f\, dx = -3\int (K * f)\, f\, dx.$$

The latter follows from the relation $\xi\cdot\nabla_\xi \hat{K} = 0$. By observing that

$$\begin{aligned} C_{i,i}(u) + \Upsilon = {} & 3\int\left(K * |u_i|^2\right)|u_i|^2\, dx - 2\int x_3\partial_{x_3}\left(K * |u_i|^2\right)|u_o|^2\, dx \\ & - 2\int x_3\partial_{x_3}\left(K * |u_o|^2\right)|u_i|^2\, dx - 2\int x_3\partial_{x_3}\left(K * |u_o|^2\right)|u_o|^2\, dx \end{aligned}$$

and that

$$\xi_3\partial_{\xi_3}\hat{K} = 8\pi\frac{\xi_3^2(\xi_1^2+\xi_2^2)}{|\xi|^4} = 8\pi\left(\frac{\xi_3^2}{|\xi|^2} - \frac{\xi_3^4}{|\xi|^4}\right) = 8\pi\widehat{\mathcal{R}_3^2} - 8\pi\widehat{\mathcal{R}_3^4},$$

we reduce the problem to the estimate of $\langle \mathcal{R}_3^4 f, g\rangle_{L^2}$ when f is supported in $\{|\bar{x}| \ge \gamma_2 R\}$, while g is supported in $\{|\bar{x}| \le \gamma_1 R\}$, for some positive parameters γ_1 and γ_2 satisfying $d := \gamma_2 - \gamma_1 > 0$. Note that in the latter identity we used the fact that

$\frac{\xi_3^2}{|\xi|^2}$ and $\frac{\xi_3^4}{|\xi|^4}$ are (up to constants) the symbols, in Fourier space, of the operators $\mathcal{R}_3^2$ and $\mathcal{R}_3^4$, respectively. $\mathcal{R}_j^4$ denotes the fourth power of the Riesz transform, and $\widehat{\mathcal{R}_j^4}$ its symbol in Fourier space. At this point, we use the estimate of Proposition 2 (for the contribution involving $\mathcal{R}_j^4$) and again Proposition 4 and Proposition 5 (for the contribution involving $\mathcal{R}_j^2$). Thus, by summing up together the estimates, we have

$$\frac{d^2}{dt^2}V_{\rho_R+x_3^2}(t) \leq 4G(u(t)) + o_R(1)\|u(t)\|_{\dot{H}^1}^2 \lesssim -1,$$

which allows to close a Glassey-type convexity argument.

3.3 Grow-up

We now give the proof of the grow-up result, by sketching the proof of Theorem 3. The proof follows the approach by Du, Wu, Zhang, see [12] (see also the results by Holmer and Roudenko [25]). It is done by contradiction, and it makes use of the so-called almost finite propagation speed, which enables us to control the quantity $\|u(t)\|_{L^2(|x|\gtrsim R)}$ for sufficiently large times. It is well known that, contrary to the wave equation, the Schrödinger equation does not enjoy a finite propagation speed; nonetheless, we can claim the following: provided $\sup_{t\in[0,\infty)}\|u(t)\|_{\dot{H}^1} < \infty$, then for any $\eta > 0$, there exists a constant $C > 0$ independent of R such that for any $t \in [0, T]$ with $T := \frac{\eta R}{C}$,

$$\int_{|x|\gtrsim R} |u(t,x)|^2 dx \leq \eta + o_R(1). \tag{41}$$

Indeed, let ϑ be a smooth radial function satisfying

$$\vartheta(x) = \vartheta(r) = \begin{cases} 0 \text{ if } r \leq \frac{c}{2}, \\ 1 \text{ if } r \geq c, \end{cases} \qquad \vartheta'(r) \leq 1 \text{ for any } r \geq 0,$$

where $c > 0$ is a given constant. For $R > 1$, we denote the radial function $\psi_R(x) = \psi_R(r) := \vartheta(r/R)$. We plug this function in the virial quantity V_χ defined in (32), and by the fundamental theorem of calculus, and by assuming that $\sup_{t\in[0,\infty)}\|u(t)\|_{\dot{H}^1} < \infty$, we have

$$\begin{aligned} V_{\psi_R}(t) &= V_{\psi_R}(0) + \int_0^t V'_{\psi_R}(s)ds \\ &\leq V_{\psi_R}(0) + t \sup_{s\in[0,t]} |V'_{\psi_R}(s)| \\ &\leq V_{\psi_R}(0) + CR^{-1}t. \end{aligned}$$

By the choice of ϑ, we have $V_{\psi_R}(0) = o_R(1)$ as $R \to \infty$. Since $V_{\psi_R}(t) \geq \int_{|x|\geq cR} |u(t,x)|^2 dx$, we obtain the control on L^2-norm of the solution outside a large ball as in (41). By repeating the estimates as in Sect. 3.1, and by means of the Gagliardo–Nirenberg interpolation inequality applied to (38), we see that

$$V''_{\varphi_R}(t) \lesssim G(u(t)) + \left(R^{-1} + \|u(t)\|^{1/2}_{L^2(|x|\gtrsim R)} + \|u(t)\|_{L^2(|x|\gtrsim R)}\right). \tag{42}$$

Combining (41) and (42), we obtain that for any $\eta \in (0,1)$, there exists a constant $C > 0$ independent of R such that for any $t \in [0, T]$ with $T := \frac{\eta R}{C}$ such that

$$V''_{\varphi_R}(t) \lesssim G(u(t)) + \left((\eta + o_R(1))^{1/4} + (\eta + o_R(1))^{1/2}\right).$$

By the assumption (15), we choose $\eta > 0$ sufficiently small and $R > 1$ sufficiently large to have $V''_{\varphi_R}(t) \lesssim -\delta < 0$ for all $t \in [0, T]$. If we integrate in time twice from 0 to T, we get $V''_{\varphi_R}(T) \leq o_R(1)R^2 - \frac{\delta\eta^2}{2C^2}R^2$, and by choosing R large enough, we obtain $z_{\varphi_R}(T) \leq -\frac{\delta\eta^2}{4C}R^2 < 0$, a contradiction with respect to the fact that $z_{\varphi_R}(T)$ is a nonnegative quantity.

4 Sketch of the Proofs Above the Threshold

The dynamics above the threshold is a consequence of the following general theorem, where a sufficient condition to have global existence and scattering is given. It will be used to establish the asymptotic dynamics when the initial datum lies at the threshold as well (see later on, specifically see the proofs in Sect. 5).

Theorem 8 *Let λ_1 and λ_2 satisfy* (16). *Let Q be a ground state related to* (10). *Let $u(t)$ be a H^1-solution to* (2) *defined on the maximal forward time interval* $[0, T_{max})$. *Assume that*

$$\sup_{t\in[0,T_{max})} -P(u(t))M(u(t)) < -P(Q)M(Q). \tag{43}$$

Then $T_{max} = \infty$ and the solution $u(t)$ scatters in H^1 forward in time.

The proof of the theorem above is done by employing a concentration/compactness and rigidity road map, as for the case below the threshold, see Theorem 1. As mentioned in the paragraph before Theorem 7, the main tool to prove existence of global and non-scattering solution is given by a profile decomposition theorem, which is a linear statement; so, in order to construct nonlinear profiles, the existence of wave operator is used. Moreover, when we are in the case below the threshold, such nonlinear profiles can be proved to be global and scattering. When we do not assume initial data below the threshold, such a control on the nonlinear profiles cannot be given. Nonetheless, we are able to prove a nonlinear profile decomposition theorem along bounded nonlinear flows, which overcomes the lack of finiteness of the scattering norm of the nonlinear profiles. See [10, Lemma 3.1]. The latter result in [10] was inspired to [22], where the NLS case was treated. We also recall here (as we remarked in the Introduction) that the restriction to the region (16) is imposed to guarantee the negative sign of the potential energy, which is fundamental to get the right bounds on the nonlinear profiles constructed when running a Kenig–Merle scheme.

Proof of Theorem 5 Let $u_0 \in \Sigma$ satisfying all the conditions in (17). We will show that (43) holds true, which in turn implies the result, by means of Theorem 8. The strategy is in the spirit of Duyckaerts and Roudenko [15], it is done in three steps, and it is based on an ODE argument. We summarize the main steps by just explaining how the method works and by defining the basic objects. For a comprehensive proof, we refer the reader to [10], where all the details are given.

By easy computations, we have

$$-P(u(t)) = 4E(u) - V''(t), \quad H(u(t)) = 6E(u) - V''(t), \tag{44}$$

and by using that $P(u(t))$ is negative (recall that we are working in RUR), then $V''(t) \leq 4E(u)$. At this point, we recall, see [20], that for any $f \in \Sigma$

$$\left(\operatorname{Im}\int x \cdot \nabla f(x)\overline{f}(x)dx\right)^2 \leq \|xf\|_{L^2}^2 \left(H(f) - \frac{(-P(f))^{\frac{2}{3}}}{(C_{GN})^{\frac{2}{3}}(M(f))^{\frac{1}{3}}}\right). \tag{45}$$

By plugging (44) into (45), we have

$$\left(\frac{V'(t)}{2}\right)^2 \leq V(t)\left[6E(u) - V''(t) - \frac{(4E(u) - V''(t))^{\frac{2}{3}}}{(C_{GN})^{\frac{2}{3}}(M(u))^{\frac{1}{3}}}\right].$$

We introduce the function $z(t) := \sqrt{V(t)}$, and we define $h(\zeta) := 6E(u) - \zeta - \frac{(4E(u)-\zeta)^{\frac{2}{3}}}{(C_{GN})^{\frac{2}{3}}(M(u))^{\frac{1}{3}}}$ for $\zeta \leq 4E(u)$. We can now rewrite the estimate above as $(z'(t))^2 \leq h(V''(t))$. The function $h(\zeta)$ on the unbounded interval $(-\infty, 4E(u))$ has a minimum in ζ_0 defined through

$$1 = \frac{2(4E(u) - \zeta_0)^{-\frac{1}{3}}}{3(C_{GN})^{\frac{2}{3}}(M(u))^{\frac{1}{3}}},$$

and in particular $h(\zeta_0) = \zeta_0/2$. The precise expression for C_{GN} given in (14) yields to

$$\frac{E(u)M(u)}{E(Q)M(Q)}\left(1 - \frac{\zeta_0}{4E(u)}\right) = 1. \tag{46}$$

(i) By using the previous relations, the first point of the ODE argument consists of re-writing the scattering conditions in (17) in an alternative way, by using the functions $z(t)$, $V(t)$, $h(\zeta)$, and the value ζ_0. From the hypothesis that $M(u_0)E(u_0) \geq M(Q)E(Q)$, we get that (46) is equivalent to $\zeta_0 \geq 0$. The second condition in (17) is equivalent to

$$(z'(0))^2 \geq \frac{\zeta_0}{2} = h(\zeta_0),$$

while the third condition in (17) is equivalent to $V''(0) > \zeta_0$. The last condition in (17) is instead equivalent to $z'(0) \geq 0$.

(ii) The previous conditions replacing the ones in (17), jointly with a continuity argument, yield a lower bound

$$V''(t) \geq \zeta_0 + \delta_0, \tag{47}$$

for some $\delta_0 > 0$ and for any $t \in [0, T_{max})$.

(iii) Eventually, we are able to prove (43). It follows from (47) and by using that $\zeta_0 \geq 0$, (46), and (14), that

$$\begin{aligned} -P(u(t))M(u(t)) &= (4E(u) - V''(t))M(u) \leq (4E(u) - \zeta_0 - \delta_0)M(u) \\ &\leq 4E(Q)M(Q) - \delta_0 M(u) = -(1-\eta)P(Q)M(Q) \end{aligned}$$

for all $t \in [0, T_{max})$, where $\eta := \frac{\delta_0 M(u)}{4E(Q)M(Q)} > 0$. This shows (43), and we can conclude the proof of Theorem 5.

□

5 Sketch of the Proofs at the Threshold

We now consider the threshold case, i.e., when the initial data satisfy (19), and we give an overview on the proof of Theorem 6. First, let us observe that in (19), we can assume, by scaling invariance, that $M(u_0) = M(Q)$ and $E(u_0) = E(Q)$. We continue with the proof of the three points in order:

(i) As we are considering $M(u_0) = M(Q)$ and $E(u_0) = E(Q)$, we see that (20) becomes $H(u_0) < H(Q)$. Then in order to prove that (19) and (20) imply that the solution is global, it is enough to prove that the kinetic energy remains bounded by $H(Q)$ (by the blow-up alternative). By contradiction, if we assume that there exists a time τ in the lifespan of the solution such that $H(u(\tau)) = H(Q)$, then we obtain by definition of the energy that $-P(u(\tau)) = H(u(\tau)) - 2E(u(\tau)) = H(Q) - 2E(Q) = -P(Q)$. Namely $u(\tau)$ is an optimizer of (13). A Lions' concentration–compactness-type lemma, see [10, Lemma 5.1], implies that $u(t)$ is a (rescaling of a) ground state related to (10) multiplied by a (time-dependent) phase shift. This yields to a contradiction with respect to the hypothesis, as we would have $H(u_0)M(u_0) = H(Q)M(Q)$; therefore, by the blow-up alternative, $u(t)$ is globally defined.

Under the hypothesis that the coefficients λ_1 and λ_2 satisfy (16), then we are able to prove that we have the result in the second part of Theorem 6 (i), by distinguishing two cases.

We first suppose that $\sup_{t\in[0,\infty)} H(u(t)) < H(Q)$. This means that there exists $\varepsilon > 0$ such that for all $t \in [0,\infty)$ (the solution is global), $H(u(t)) \leq (1-\varepsilon)H(Q)$. By plugging the best constant (given in terms of the ground state to (10)) of the Gagliardo–Sobolev-type estimate (13), it is straightforward to see that

$$-P(u(t))M(u(t)) \leq C_{GN}\,(H(u(t))M(u(t)))^{\frac{3}{2}} \leq -(1-\varepsilon)^{\frac{3}{2}}P(Q)M(Q);$$

hence, the condition (43) of Theorem 8 holds true, and the solution scatters forward in time.

If instead $\sup_{t\in[0,\infty)} H(u(t)) = H(Q)$, then there exists a time sequence $\{t_n\}_{n\geq 1} \subset [0,\infty)$ such that

$$M(u(t_n)) = M(Q), \quad E(u(t_n)) = E(Q), \quad \lim_{n\to\infty} H(u(t_n)) = H(Q).$$

Moreover, $t_n \to \infty$. Indeed, if (up to subsequences) $t_n \to \tau$, as $u(t_n) \to u(\tau)$ strongly in H^1, then it can be shown that $u(\tau)$ is an optimizer for (13). Arguing as above, we have a contradiction. A Lions-type lemma [10, Lemma 5.1] gives the desired result.

(ii) We continue with the proof of the second point. Suppose the initial datum satisfies (19) and (22). By scaling, we reduce to the case $M(u_0) = M(Q)$, $E(u_0) = E(Q)$, and hence, $H(u_0) = H(Q)$. Hence, u_0 is an optimizer for (13). This shows that $u_0(x) = e^{i\theta}\mu\tilde{Q}(\mu x)$ for some $\theta \in \mathbb{R}$, $\mu > 0$ and $\tilde{Q}$ a ground state related to (10). By the uniqueness of solutions, we end up with $u(t,x) = e^{i\mu^2 t}e^{i\tilde{\theta}}\mu\tilde{Q}(\mu x)$ for some $\tilde{\theta} \in \mathbb{R}$.

(iii) Finally, suppose that $u_0 \in H^1$ satisfies (19) and (23). By scaling, we have reduced (23) to $H(u_0) > H(Q)$. By the same argument in the proof of the first point, we claim that $H(u(t)) > H(Q)$, for every time in the lifespan of the solution. If the maximal time of existence is finite, there is nothing to prove. Otherwise, if the solution exists for all times, we separate the analysis in two cases.

Suppose $\sup_{t\in[0,\infty)} H(u(t)) > H(Q)$. Hence, there exists $\varepsilon > 0$ such that for all $t \in [0,\infty)$, $H(u(t)) \geq (1+\eta)H(Q)$. By using the definition (6) of G and the previous property, we have

$$\begin{aligned} G(u(t))M(u(t)) &\leq 3E(Q)M(Q) - \frac{1}{2}(1+\eta)H(Q)M(Q) \\ &= -\frac{\eta}{2}H(Q)M(Q) < 0, \end{aligned}$$

for all $t \in [0, \infty)$, where in the last equality we used (14). By applying Theorem 3, we finish the proof.

If instead $\sup_{t\in[0,\infty)} H(u(t)) = H(Q)$, similarly to above, we have that there exist a diverging sequence of times $\{t_n\}_{n\geq 1}$, a ground state $\tilde{Q}$ related to (10), and a sequence $\{y_n\}_{n\geq 1} \subset \mathbb{R}^3$ such that $u(t_n, \cdot + y_n) \to e^{i\theta}\mu\tilde{Q}(\mu\cdot)$ in H^1, for some $\theta \in \mathbb{R}$ and $\mu > 0$ as $n \to \infty$. This concludes the proof of Theorem 6.

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Part II
Probabilistic and Nonstandard Methods in the Study of NLS Equations

Almost Sure Pointwise Convergence of the Cubic Nonlinear Schrödinger Equation on $\mathbb{T}^2$

Renato Lucà

Abstract We revisit a result from "Pointwise convergence of the Schrödinger flow, E. Compaan, R. Lucà, G. Staffilani, International Mathematics Research Notices, 2021 (1), 596–647" regarding the pointwise convergence of solutions to the periodic cubic nonlinear Schrödinger equation in dimension $d = 2$.

1 Introduction

We consider the question of pointwise almost everywhere (a.e.) convergence of solutions to the cubic nonlinear Schrödinger equation (NLS) on $\mathbb{T}^2$, namely

$$\begin{cases} i\partial_t u + \Delta u = \pm|u|^2 u, \\ u(x,0) \quad\; = f(x). \end{cases} \tag{1}$$

If $f \in H^s$, for what s do we have that $u(x,t) \to f(x)$ as $t \to 0$ for (Lebesgue) almost every x?

In the linear Euclidean setting, namely when the linear Schrödinger equation equation is posed on $\mathbb{R}^d$, this question was first posed by Carleson [8]. He proved Lebesgue (a.e.) convergence $e^{it\Delta}f(x)$ to $f(x)$ for $f \in H^s(\mathbb{R})$ with $s \leq \frac{1}{4}$. Dahlberg–Kenig [11] showed that this one-dimensional result is sharp, proving the necessity of the regularity condition $s \geq \frac{1}{4}$ in any dimension. The (considerably more difficult) higher dimensional problem has been studied by many authors [1, 4, 10, 12, 16, 20, 22–24, 26, 28, 29, 31, 32, 34]. Recently, Bourgain [5] proved that $s \geq \frac{d}{2(d+1)}$ is necessary (see also [21, 24] for some refinements of this result). This has been proved to be sharp, up to the endpoint, by Du–Guth–Li [15] on $\mathbb{R}^2$ and by

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Du–Zhang [14] in higher dimensions (the endpoint case is still open in dimensions $d \geq 2$).

In the periodic case, much less is known. When $d = 1$, Moyua–Vega [27] proved the sufficiency of $s > \frac{1}{3}$ and necessity of $s \geq \frac{1}{4}$. Their proof, based on Strichartz estimates, has been extended to dimension $d = 2$ in [35] and to higher dimension in [9]. In fact, together with recent improvements in periodic Strichartz estimates [6], one can show that $s > \frac{d}{d+2}$ is a sufficient condition for almost everywhere convergence to initial data. On the other hand, there are several counterexamples showing that we have the same necessary conditions than that on $\mathbb{R}^d$ [9, 17, 27], namely the necessity of $s \geq \frac{d}{2(d+1)}$; in particular, one can "adapt" the counterexamples from $\mathbb{R}^d$ to the periodic setting. At the moment, in the periodic case, almost sure convergence when $s \in \left[\frac{d}{2(d+1)}, \frac{d}{d+2}\right]$ remains an open question.

In the first part of this chapter, we show how to extend the a.e. convergence statement

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x), \qquad \text{for a.e. } x \in \mathbb{T}^2 \text{ and for all } f \in H^s(\mathbb{T}^2), s > 1/2 \tag{2}$$

to the case of the cubic equation. The following is a special case of Theorem 1.1 in [9].

Theorem 1 *If $f \in H^s(\mathbb{T}^2)$ with $s > 1/2$ and u is the corresponding solution to* (1), *then*

$$\lim_{t\to 0} u(x,t) = f(x) \quad \textit{for a.e. } x \in \mathbb{T}^2 . \tag{3}$$

Remark 1 By the proof, it will be clear that any improvement of the amount of Sobolev regularity that is sufficient for the convergence of the linear Schrödinger flow on $\mathbb{T}^2$ would imply an analogous improvement in the statement of Theorem 1 as well.

In the second part of this chapter, we consider probabilistic improvements to the convergence problem. More precisely, we will show that a randomization of the Fourier coefficients of the initial data guarantees a better pointwise behavior of the associated linear (and also nonlinear) evolution. To explain why we may expect this, it is worth mentioning that counterexamples to the linear pointwise convergence problem in the periodic setting have been constructed in [17] using as building block for the initial datum the tensor product of Dirichlet kernels

$$\prod_{\ell=1,\dots,d} \sum_{k_\ell \in \mathbb{Z},\, |k_\ell| \leq N} e^{ik_\ell \cdot x_\ell}, \qquad x := (x_1, \dots, x_d), \tag{4}$$

where $N \gg 1$ is a large frequency parameter. It is wort recalling that the pointwise convergence problem is essentially[1] equivalent to establish an $L^2(\mathbb{T}^2)$ estimate for the maximal Schrödinger operator

$$\left\| \sup_{t \in [0,1]} |e^{it\Delta} f| \right\|_{L^2(\mathbb{T}^2)} \lesssim \|f\|_{H^s(\mathbb{T}^2)}. \tag{5}$$

It has been observed in [17, 27] that (5) behaves particularly bad with data of the form (4). It is in fact seen to be false for $s < \frac{n}{2(n+1)}$, taking $N \to \infty$. The moral is that if the bad counterexamples are characterized by having a very rigid structure: the Fourier coefficients in (4) are indeed all equal to 1. This suggest to consider as "good" initial data the following randomized Fourier series

$$f^\omega(x) = \sum_{n \in \mathbb{Z}^d} \frac{g_n^\omega}{\langle n \rangle^{\frac{d}{2}+\alpha}} e^{in\cdot x}, \qquad \alpha > 0, \tag{6}$$

where g_n^ω are independent (complex) standard Gaussian variables. The Japanese brackets are defined as usual as $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$.

It is easy to see that if we fix $t \in \mathbb{R}$, then $e^{it\Delta} f^\omega(x)$ belongs to $\bigcap_{s<\alpha} H^s(\mathbb{T}^d)$ $\mathbb{P}$-almost surely (a.s.), where $\mathbb{P}$ is the probability measure induced by the sequence $\{g_n^\omega\}_{n\in\mathbb{Z}}$. Thus we are working at the $H^{\alpha-}$ level. In fact, more is true, namely that $e^{it\Delta} f^\omega(x)$ belongs to $\bigcap_{s<\alpha} C^s(\mathbb{T}^d)$, $\mathbb{P}$-a.s.; in particular, $e^{it\Delta} f^\omega$ is $\mathbb{P}$-a. s. a continuous function of the x variable. On the other hand, the randomization does not improve the regularity, in the sense that $\|f^\omega\|_{H^\alpha(\mathbb{T}^d)} = \infty$ also holds $\mathbb{P}$-a. s.; see for example Remark 1.2 in [7] and the introduction of [25].

We have the following improved pointwise (a.e.) convergence result for randomized initial data. The following is the first part of Theorem 1.3 in [9].

Proposition 1 *Let $\alpha > 0$, and let f^ω of the form (6). We have $\mathbb{P}$-a. s. the following. For all $t \in \mathbb{R}$, the free solution $e^{it\Delta} f^\omega$ belongs to $\bigcap_{s<\alpha} C^s(\mathbb{T}^d)$ and*

$$e^{it\Delta} f^\omega(x) \to f^\omega(x) \quad \text{as } t \to 0$$

for every $x \in \mathbb{T}^d$ and uniformly.

[1] It is indeed easy to check that the maximal estimate (5) with $s > 1/2$ implies (2) (the argument is the same as used in the proof of Proposition 2). The opposite implication requires the Stein maximal principle. Strictly speaking, there is an equivalence with a weak L^2 estimate. On the other hand, the weak L^2 estimate can be easily promoted to a strong one with an ε-regularity loss. Thus, since we are not interested in endpoint results, we see that (2) and (5) are indeed equivalent.

Finally, we want to prove a similar statement for the cubic NLS (1). In fact, it will be more convenient working with the Wick-ordered version of the equation (WNLS)

$$\begin{cases} i\partial_t u + \Delta u = \mathcal{N}(u), \\ u(x,0) \quad\;\; = f(x), \end{cases} \tag{7}$$

where

$$\mathcal{N}(u) := \pm u\left(|u|^2 - 2\mu\right), \qquad \mu := \fint_{\mathbb{T}^2} |u(x,t)|^2 dx = \fint_{\mathbb{T}^2} |f(x)|^2 dx \tag{8}$$

(recall that μ is a conserved quantity). Since solutions to WNLS are related to that of the cubic NLS by multiplication with a factor $e^{i2\mu t}$, the study of pointwise convergence turns out to be equivalent to that of NLS. The following is the second part of Theorem 1.3 in [9].

Theorem 2 *Let $d = 2$, $\alpha > 0$, and let f^ω of the form* (6)*. Let u be the solution to WNLS* (7) *with initial data f^ω. We have $\mathbb{P}$-almost surely:*

$$\lim_{t\to 0} u(x,t) = f^\omega(x) \quad \textit{for a.e. } x \in \mathbb{T}^2 . \tag{9}$$

Thus the same is true for solutions to the cubic NLS.

1.1 Notations and Terminology

For a fixed $p \in \mathbb{R}$, we often use the notation $p+ := p + \varepsilon$, $p- := p - \varepsilon$, where ε is any sufficiently small strictly positive real number. When in the same inequality we have two such quantities, we use the following notation to compare them. We write $p + \cdots + := p + \varepsilon \cdot (\text{number of } +)$, $p - \cdots - := p - \varepsilon \cdot (\text{number of } -)$. We will use $C > 0$ to denote several constants depending only on fixed parameters, like for instance the dimension d. The value of C may clearly differ from line to line. Let $A, B > 0$. We may write $A \lesssim B$ if $A \leq CB$ when $C > 0$ is such a constant. We write $A \gtrsim B$ if $B \lesssim A$ and $A \sim B$ when $A \lesssim B$ and $A \gtrsim B$. We write $A \ll B$ if $A \leq cB$ for $c > 0$ sufficiently small (and depending only on fixed parameters) and $A \gg B$ if $B \ll A$. We denote $A \wedge B := \min(A, B)$ and $A \vee B := \max(A, B)$. We refer to the following inequality:

$$\|D^s P_N f\|_{L^q} \lesssim N^{s+\frac{d}{p}-\frac{d}{q}} \|P_N f\|_{L^p}, \quad 1 \leq p \leq q \leq \infty ,$$

simply as Bernstein inequality. Here, P_N is the frequency projection on the annulus $\xi \sim N$.

It is useful to recall that the Strichartz estimates are the main tool to study the nonlinear Schrödinger flow. We recall the periodic Strichartz estimates from [2, 6]:

$$\|e^{it\Delta} P_N f\|_{L^p_{x,t}(\Omega^{d+1})} \lesssim N^{\frac{d}{2}-\frac{d+2}{p}+} \|P_N f\|_{L^2_x(\Omega^d)}, \quad p \geq 2\left(\frac{d+2}{d}\right). \tag{10}$$

2 Proof of Theorem 1

Recall that the flow of (1) is locally well defined for initial data in $f \in H^s(\mathbb{T}^2)$ for $s > 0$ [2]. The solutions are constructed via a fixed-point argument in the restriction space $X^{s,b}_\delta$ for $\delta > 0$ sufficiently small (depending polynomially on the $H^s(\mathbb{T}^2)$ norm of f). We recall that

$$\|F\|_{X^{s,b}_\delta} := \inf_{G=F \text{ on } t\in[0,\delta]} \|G\|_{X^{s,b}},$$

where

$$\|F\|^2_{X^{s,b}} := \int_{\mathbb{R}} \sum_{n\in\mathbb{Z}^d} \langle \tau + |n|^2\rangle^{2b} \langle n\rangle^{2s} |\widehat{F}(n,\tau)|^2 d\tau$$

and $\widehat{F}$ is the space–time Fourier transform of F.

Let Φ^N_t be the flow associated to the truncated NLS equation

$$i\partial_t \Phi^N_t f + \Delta \Phi^N_t f = \mathrm{P}_{\leq N} \mathcal{N}(\Phi^N_t f), \tag{11}$$

with initial datum $\Phi^N_0 f := \mathrm{P}_{\leq N} f$. We denote $\mathrm{P}_{\leq N}$ the frequency projection on the ball of radius N centered in the origin. We write $\Phi_t f := \Phi^\infty_t f$ for the flow of the NLS equation with initial datum $f = \mathrm{P}_\infty f$. We also denote $\mathrm{P}_{>N} := \mathrm{P}_\infty - \mathrm{P}_{\leq N}$ and as already mentioned $\mathrm{P}_N := \mathrm{P}_{\leq N} - \mathrm{P}_{\leq N/2}$.

A similar well-posedness result holds for the truncated flow, uniformly in $N \in \mathbb{N}$. Of course, at fixed N, since Eq. (11) is finite-dimensional, one can construct a global flow in an elementary way using the Cauchy theorem for ODE and the conservation of $\|\Phi^N_t f\|_{L^2(\mathbb{T}^2)}$ (which holds for all $N \in \mathbb{N}$). However, in the following, we will need (as usual in the study of NLS) a control of $\Phi^N_t f$ uniform over N. This is not elementary and will be ensured by the local well-posedness theory in the restriction space.

As already recalled, the main tool in the study of the pointwise convergence properties of the linear Schrödinger equation is the maximal Schrödinger operator

$$t \to \sup_{0\leq t\leq\delta} |e^{it\Delta} f(x)|, \qquad \delta > 0.$$

Assume indeed that for some $\delta \in (0, 1]$, one has

$$\left\| \sup_{0\le t\le\delta} |e^{it\Delta} f(x)| \right\|_{L^2_x(\mathbb{T}^2)} \lesssim \|f\|_{H^s_x(\mathbb{T}^2)}, \tag{12}$$

and then it is not hard to see that $e^{it\Delta} f(x) \to f(x)$ as $t \to 0$ for almost every (with respect to the Lebesgue measure) $x \in \mathbb{T}^2$. The proof is a straightforward modification of the argument that we will use to prove Proposition 2 below.

In the nonlinear setting, we need a (nonlinear) replacement of (12). A convenient replacement is the maximal estimate (13).

Proposition 2 *Let $f \in L^2(\mathbb{T}^2)$ be such that*

$$\lim_{N\to\infty} \left\| \sup_{0\le t\le\delta} |\Phi_t f(x) - \Phi_t^N f(x)| \right\|_{L^2_x(\mathbb{T}^2)} = 0. \tag{13}$$

Then $\Phi_t f(x) \to f(x)$ as $t \to 0$ for almost every $x \in \mathbb{T}^2$.

From the proof, it will be clear that in (13) we can replace the L^2 norm with a weak L^1 norm. However, it is usually convenient to work in the L^2 setting.

Proof To prove Proposition 2, we decompose the difference as follows:

$$|\Phi_t f(x) - f(x)| \le |\Phi_t f(x) - \Phi_t^N f(x)| + |\Phi_t^N f(x) - \mathrm{P}_{\le N} f(x)| + |\mathrm{P}_{>N} f(x)| \tag{14}$$

and pass to the limit $t \to 0$. It is elementary to show that the second term on the right-hand side is zero, namely

$$\lim_{t\to 0} \Phi_t^N f(x) = \mathrm{P}_{\le N} f(x),$$

for all $x \in \mathbb{T}^2$. So we arrive at[2]

$$\limsup_{t\to 0} |\Phi_t f - f| \le \limsup_{t\to 0} |\Phi_t f - \Phi_t^N f| + |\mathrm{P}_{>N} f|.$$

[2] Hereafter, we remove the x variable in the argument of decompositions such as (14) to simplify the notation.

Let $\lambda > 0$. Using the Chebyshev inequality,

$$|\{x \in \mathbb{T}^2 : \limsup_{t\to 0} |\Phi_t f - f| > \lambda\}| \leq |\{x \in \mathbb{T}^2 : \sup_{0\leq t\leq \delta} |\Phi_t f - \Phi_t^N f| > \lambda/2\}|$$
$$+ |\{x \in \mathbb{T}^2 : |\mathrm{P}_{>N} f| > \lambda/2\}|$$
$$\lesssim \lambda^{-2} \left(\left\| \sup_{0\leq t\leq \delta} |\Phi_t f - \Phi_t^N f| \right\|_{L^2(\mathbb{T}^2)}^2 + \| \mathrm{P}_{>N} f\|_{L^2(\mathbb{T}^2)}^2 \right),$$

where $|\cdot|$ is the Lebesgue measure. On the other hand, we have $\| \mathrm{P}_{>N} f\|_{L^2(\mathbb{T}^2)} \to 0$ as $N \to \infty$ (since $f \in L^2(\mathbb{T}^2)$) and

$$\lim_{N\to\infty} \left\| \sup_{0\leq t\leq \delta} |\Phi_t f - \Phi_t^N f| \right\|_{L^2(\mathbb{T}^2)} = 0$$

by assumption (13). Thus we arrive to

$$|\{x \in \mathbb{T}^2 : \limsup_{t\to 0} |\Phi_t f - f| > \lambda\}| = 0,$$

and the statement follows taking the union over $\lambda > 0$. □

It is not easy to verify the condition (13) directly. However, we can take advantage of a simple lemma that allows to embed a suitable restriction space into the relevant maximal space, namely the space induced by the norm

$$\left\| \sup_{t\in[0,\delta]} |F(x,t)| \right\|_{L_x^2(\mathbb{T}^2)}, \qquad F : (x,t) \in \mathbb{T}^2 \times \mathbb{R} \to F(x,t) \in \mathbb{C}.$$

In other words, we can bound the $L_x^2(\mathbb{T}^2)$ norm of the associated maximal function

$$x \to \sup_{0\leq t\leq \delta} |F(x,t)|$$

with an appropriate $X_\delta^{s,b}$ norm of F. In fact, this is a rather general property of the restriction spaces $X_\delta^{s,b}$ with $b > \frac{1}{2}$. The proof can be found in [30, Lemma 2.9], in the non-periodic case. The argument adapts to the periodic case as well.

Lemma 1 *Let $b > \frac{1}{2}$, and let Y be a Banach space of functions*

$$F : (x,t) \in \Omega^d \times \mathbb{R} \to F(x,t) \in \mathbb{C}\,.$$

Let $\alpha \in \mathbb{R}$. Assume

$$\|e^{i\alpha t} e^{it\Delta} f(x)\|_Y \leq C \|f\|_{H^s(\Omega^d)}\,, \tag{15}$$

with a constant $C > 0$ uniform over $\alpha \in \mathbb{R}$. Then

$$\|F\|_Y \le C\|F\|_{X^{s,b}} .$$

Using Lemma 1 with

$$\|F\|_Y = \left\| \sup_{0\le t\le\delta} |F(x,t)| \right\|_{L^2_x(\mathbb{T}^2)}$$

and the fact that the maximal estimate (12) holds for $s > 1/2$, we have the following:

Lemma 2 *Let $b > \frac{1}{2}$ and $s > 1/2$. We have*

$$\left\| \sup_{0\le t\le\delta} |F(x,t)| \right\|_{L^2_x(\mathbb{T}^2)} \lesssim \|F\|_{X^{s,b}_\delta} . \tag{16}$$

We will combine the following lemma with the embedding from Lemma 2 to verify the maximal estimate hypothesis of Proposition 2 for the cubic NLS on $\mathbb{T}^2$.

Lemma 3 *Let $d = 2$ and $s > 0$. Then*

$$\|\mathcal{N}(u) - \mathcal{N}(v)\|_{X^{s,-\frac{1}{2}++}} \lesssim \left(\|u\|^2_{X^{s,\frac{1}{2}+}} + \|v\|^2_{X^{s,\frac{1}{2}+}} \right) \|u - v\|_{X^{s,\frac{1}{2}+}}. \tag{17}$$

In fact, Lemma 3 is a consequence of the following slightly more general statement (that will be useful later) due to Bourgain [3].

Lemma 4 *Let $d = 2$ and $s > 0$. Let $M_1 \ge M_2 \ge M_3$ be dyadic scales. Then*

$$\begin{aligned} &\|(\mathrm{P}_{M_1} F)(\mathrm{P}_{M_2} G)(\mathrm{P}_{M_3} H)\|_{X^{s,-\frac{1}{2}++}} \\ &\qquad \lesssim \|\mathrm{P}_{M_1} F\|_{X^{s,\frac{1}{2}+}} \|\mathrm{P}_{M_2} G\|_{X^{0+,\frac{1}{2}+}} \|\mathrm{P}_{M_3} H\|_{X^{0,\frac{1}{2}+}}. \end{aligned} \tag{18}$$

We denote $R_0 = \|f\|_{H^s(\mathbb{T}^2)}$. Hereafter η will be a smooth cut-off of $[0, 1]$. Taking $\delta = \delta(R_0) < 1$ sufficiently small and combining (25), (26), (27), and Lemma 3, one can show that the map

$$\Gamma(u(x,t)) = \eta(t)e^{it\Delta} \mathrm{P}_{\le N} f(x) - i\eta(t) \int_0^t e^{i(t-t')\Delta} \mathrm{P}_{\le N} \mathcal{N}(u(x,t'))dt' \tag{19}$$

is a contraction on the ball $\{u : \|u\|_{X^{s,\frac{1}{2}+}_\delta} \le 2R_0\}$, for all $N \in 2^{\mathbb{N}} \cup \{\infty\}$. This is a standard argument, so we omit the proof (see for instance [18, Section 3.5.1]).

Moreover, a similar computation is part of the proof of Theorem 1. However, we stress that the value of δ is uniform in $N \in 2^{\mathbb{N}} \cup \{\infty\}$. In particular, we have

$$\|\Phi_t^N f\|_{X_\delta^{s,\frac{1}{2}+}} \leq 2R_0, \qquad \text{for all } N \in 2^{\mathbb{N}} \cup \{\infty\}. \tag{20}$$

We are now ready to prove Theorem 1.

2.1 *Proof of Theorem 1*

By Lemma 2, we have

$$\left\| \sup_{0\leq t\leq\delta} |\Phi_t f(x) - \Phi_t^N f(x)| \right\|_{L_x^2(\mathbb{T}^2)} \lesssim \|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s,\frac{1}{2}+}}.$$

Thus using Proposition 2, it suffices to show that the right-hand side goes to zero as $N \to \infty$. For $t \in [0, \delta]$, we have (see (19))

$$\begin{aligned}
&\Phi_t f(x) - \Phi_t^N f(x) \\
&= \eta(t)e^{it\Delta}\,\mathrm{P}_{>N}\, f(x) - i\eta(t)\int_0^t e^{i(t-t')\Delta}\Big(\mathcal{N}(\Phi_{t'} f(x)) - \mathrm{P}_{\leq N}\,\mathcal{N}(\Phi_{t'}^N f(x))\Big)\,dt'.
\end{aligned}$$

Then using (25) and (26), we have

$$\|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s,\frac{1}{2}+}} \lesssim \|\,\mathrm{P}_{>N}\, f\|_{H^s(\mathbb{T}^2)} + \|\mathcal{N}(\Phi_t f) - \mathrm{P}_{\leq N}\,\mathcal{N}(\Phi_t^N f)\|_{X_\delta^{s,-\frac{1}{2}+}}. \tag{21}$$

To handle the nonlinear contribution, we further decompose

$$\mathcal{N}(\Phi_t f) - \mathrm{P}_{\leq N}\,\mathcal{N}(\Phi_t^N f) = \mathrm{P}_{\leq N}\Big(\mathcal{N}(\Phi_t f) - \mathcal{N}(\Phi_t^N f)\Big) + \mathrm{P}_{>N}\,\mathcal{N}(\Phi_t f)$$

so that

$$\begin{aligned}
\|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s,\frac{1}{2}+}} &\lesssim \|\,\mathrm{P}_{>N}\, f\|_{H^s(\mathbb{T}^2)} + \|\,\mathrm{P}_{>N}\,\mathcal{N}(\Phi_t f)\|_{X_\delta^{s,-\frac{1}{2}+}} \\
&\quad + \|\,\mathrm{P}_{\leq N}\Big(\mathcal{N}(\Phi_t f) - \mathcal{N}(\Phi_t^N f)\Big)\|_{X_\delta^{s,-\frac{1}{2}+}}.
\end{aligned} \tag{22}$$

Then by (27), Lemma 3, and (20), we get

$$\|\,\mathrm{P}_{\leq N}\Big(\mathcal{N}(\Phi_t f) - \mathcal{N}(\Phi_t^N f)\Big)\|_{X_\delta^{s,-\frac{1}{2}+}} \lesssim \delta^{0+} R_0^2 \|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s,\frac{1}{2}+}}, \tag{23}$$

where we recall $R_0 = \|f\|_{H^s(\mathbb{T}^2)}$. Plugging (23) into (22), taking $\delta = \delta(R_0)$ small enough, and absorbing

$$\delta^{0+} R_0^2 \|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s,\frac{1}{2}+}} \le \frac{1}{2}\|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s,\frac{1}{2}+}}$$

into the left-hand side, we arrive to

$$\|\Phi_t f - \Phi_t^N f\|_{X_\delta^{s,\frac{1}{2}+}} \lesssim \|\mathrm{P}_{>N} f\|_{H^s(\mathbb{T}^2)} + \|\mathrm{P}_{>N} \mathcal{N}(\Phi_t f)\|_{X_\delta^{s,-\frac{1}{2}+}}. \tag{24}$$

The right-hand side of (24) goes to zero as $N \to \infty$ since $f \in H^s(\mathbb{T}^2)$ and $\mathcal{N}(\Phi_t f) \in X_\delta^{s,-\frac{1}{2}+}$; in fact, applying Lemma 3 with $v = 0$ and recalling (20), we have

$$\|\mathcal{N}(\Phi_t f)\|_{X_\delta^{s,-\frac{1}{2}+}} \lesssim \|\Phi_t f\|^3_{X_\delta^{s,\frac{1}{2}+}} \lesssim R_0^3 .$$

This concludes the proof of (3).

We conclude this section by recalling some well-known properties of restriction spaces that we have used (and that we will use in the rest of the paper). Recall that η is a smooth cut-off of the unit interval.

Lemma 5 *Let $s \in \mathbb{R}$. Then*

$$\|\eta(t)e^{it\Delta} f(x)\|_{X^{s,\frac{1}{2}+}} \lesssim \|f\|_{H^s(\Omega^d)} , \tag{25}$$

$$\left\|\eta(t)\int_0^t e^{i(t-t')\Delta} F(\cdot, t')dt'\right\|_{X^{s,\frac{1}{2}+}} \lesssim \|F\|_{X^{s,-\frac{1}{2}+}} , \tag{26}$$

$$\|F\|_{X_\delta^{s,-\frac{1}{2}+}} \lesssim \delta^{0+}\|F\|_{X_\delta^{s,-\frac{1}{2}++}} . \tag{27}$$

3 Proof of Proposition 1

Here we prove almost surely uniform convergence of the randomized Schrödinger flow to the initial datum, at the H^{0+} level, namely Proposition 1. Thus our goal is to show that $e^{it\Delta} f^\omega \to f^\omega$ as $t \to 0$ uniformly over $x \in \mathbb{T}^d$ and $\mathbb{P}$-almost surely for data f^ω defined as

$$f^\omega(x) = \sum_{n\in\mathbb{Z}^d} \frac{g_n^\omega}{\langle n\rangle^{\frac{d}{2}+\alpha}} e^{in\cdot x}, \qquad x \in \mathbb{T}^d , \tag{28}$$

where $\alpha > 0$ and each g_n^ω is complex and independently drawn from a standard normal distribution. In fact, the argument we present works for independent g_n^ω drawn from any distribution with sufficient decay of the tails (for instance, sub-Gaussian is enough). This will not be the case in Theorem 2, where we will need to take advantage of the hypercontractivity of (multilinear forms of) normal distributions. However, we only present the standard normal case for definiteness, also in this section.

Fix $t \in \mathbb{R}$. We have that $\mathbb{P}$-almost surely

$$e^{it\Delta} f^\omega \in \bigcap_{s<\alpha} H^s(\mathbb{T}^d).$$

This is an immediate consequence of (44) below, taking the union over $\varepsilon > 0$. In fact, for all $t \in \mathbb{R}$, we have $\mathbb{P}$-almost surely

$$e^{it\Delta} f^\omega \in \bigcap_{s<\alpha} C^s(\mathbb{T}^d);$$

thus in particular, $e^{it\Delta} f^\omega$ are $\mathbb{P}$-almost surely continuous functions of the x variable. This is a consequence of the higher integrability property (34) below, from which one can easily deduce uniform convergence as $N \to \infty$ of the sequence $P_{\leq N} f^\omega$, with probability larger than $1 - \varepsilon$. So the limit f^ω is continuous with the same probability, and the almost sure continuity follows taking the union over $\varepsilon > 0$.

Before completing the proof of Proposition 1, we recall few lemmata. We start recalling the following well-known concentration bound:

Lemma 6 ([7, Lemma 3.1]) *There exists a constant C such that*

$$\Big\| \sum_{n\in\mathbb{Z}^d} g_n^\omega \, a_n \Big\|_{L^r_\omega} \leq C r^{\frac{1}{2}} \|a_n\|_{\ell^2_n(\mathbb{Z}^d)} \tag{29}$$

for all $r \geq 2$ *and* $\{a_n\} \in \ell^2(\mathbb{Z}^d)$.

Using (29) with $a_n = e^{in\cdot x - i|n|^2 t} \langle n \rangle^{-\frac{d}{2}-\alpha}$, we obtain for $r \geq 2$ that for f^ω an in (28)

$$\| \mathrm{P}_N e^{it\Delta} f^\omega \|_{L^r_\omega} \leq C r^{\frac{1}{2}} N^{-\alpha}\, , \tag{30}$$

with a constant uniform in $t \in \mathbb{R}$. From this, we also have improved L^p_x estimates for randomized data.

Lemma 7 *Let* $p \in [2, \infty)$. *Assume* f^ω *is as in* (28). *There exist constants C and c, independent of* $t \in \mathbb{R}$, *such that*

$$\mathbb{P}(\| \mathrm{P}_N e^{it\Delta} f^\omega \|_{L^p_x(\mathbb{T}^d)} > \lambda) \leq C e^{-cN^{2\alpha}\lambda^2}\, . \tag{31}$$

Thus

$$\mathbb{P}(\| \mathrm{P}_N e^{it\Delta} f^\omega \|_{L_x^\infty(\mathbb{T}^d)} > \lambda) \le C e^{-cN^{2\alpha-}\lambda^2}. \tag{32}$$

In particular, for any $\varepsilon > 0$ *sufficiently small, we have*

$$\| \mathrm{P}_N e^{it\Delta} f^\omega \|_{L_x^p(\mathbb{T}^d)} \lesssim N^{-\alpha} (-\ln \varepsilon)^{1/2}, \quad N \in 2^{\mathbb{Z}} \tag{33}$$

and

$$\| \mathrm{P}_N e^{it\Delta} f^\omega \|_{L_x^\infty(\mathbb{T}^d)} \lesssim N^{-\alpha+} (-\ln \varepsilon)^{1/2}, \quad N \in 2^{\mathbb{Z}}, \tag{34}$$

with probability at least $1 - \varepsilon$.

Proof We prove (31), and then (32) follows by Bernstein inequality. By Minkowski's inequality and Lemma 6 above, we have for any $r \ge p \ge 2$

$$\left(\int \| \mathrm{P}_N e^{it\Delta} f^\omega \|^r_{L_x^p(\mathbb{T}^d)} d\mathbb{P}(\omega) \right)^{\frac{1}{r}} \le \Big\| \| \mathrm{P}_N e^{it\Delta} f^\omega \|_{L^r_\omega} \Big\|_{L_x^p(\mathbb{T}^d)} \le C N^{-\alpha} r^{\frac{1}{2}},$$

which is enough to conclude that $\| \mathrm{P}_N e^{it\Delta} f^\omega \|_{L_x^p(\mathbb{T}^d)}$ is a sub-Gaussian random variable satisfying the tail bound (31).

□

Note that using (31)–(32), the triangle inequality

$$\| P_{>N}(\cdot) \| \le \sum_{M \in 2^{\mathbb{N}} : M > N} \| P_M(\cdot) \|,$$

the union bound, and the fact that

$$\sum_{M \in 2^{\mathbb{N}} : M > N} e^{-cM^{2\alpha} k^{-2}} \lesssim e^{-cN^{2\alpha} k^2},$$

we see that, for all $t \in \mathbb{R}$ and $\alpha > 0$, we have ($p < \infty$)

$$\mathbb{P}\left(\| e^{it\Delta} \mathrm{P}_{>N} f^\omega \|_{L_x^p(\mathbb{T}^d)} > \lambda \right) \lesssim e^{-cN^{2\alpha}\lambda^2} \tag{35}$$

$$\mathbb{P}\left(\| e^{it\Delta} \mathrm{P}_{>N} f^\omega \|_{L_x^\infty(\mathbb{T}^d)} > \lambda \right) \lesssim e^{-cN^{2\alpha-}\lambda^2}. \tag{36}$$

Remark 2 Proceeding as we did to prove (35)–(36), we also easily see that the exceptional set where (33)–(34) are not valid can be chosen to be the same for all $N \in \mathbb{N}$, paying an N^{0+} loss on the right-hand side of the estimates.

Proceeding as in the proof of Lemma 7, we also obtain improved Strichartz estimates for randomized data.

Lemma 8 *Let* $p \in [2, \infty)$. *Assume* f^ω *is as in* (28)*. Then we have*

$$\mathbb{P}\left(\|e^{it\Delta} \mathrm{P}_N f^\omega\|_{L^p_{x,t}(\mathbb{T}^{d+1})} > \lambda \right) \le Ce^{-cN^{2\alpha}\lambda^2}. \tag{37}$$

Thus

$$\mathbb{P}\left(\|e^{it\Delta} \mathrm{P}_N f^\omega\|_{L^\infty_{x,t}(\mathbb{T}^{d+1})} > \lambda \right) \le Ce^{-cN^{2\alpha-}\lambda^2}. \tag{38}$$

In particular, for any $\varepsilon > 0$ *sufficiently small, we have*

$$\|e^{it\Delta} \mathrm{P}_N f^\omega\|_{L^p_{x,t}(\mathbb{T}^{d+1})} \lesssim N^{-\alpha} (-\ln \varepsilon)^{1/2}, \quad N \in 2^{\mathbb{Z}} \tag{39}$$

and

$$\|e^{it\Delta} \mathrm{P}_N f^\omega\|_{L^\infty_{x,t}(\mathbb{T}^{d+1})} \lesssim N^{-\alpha+} (-\ln \varepsilon)^{1/2}, \quad N \in 2^{\mathbb{Z}}, \tag{40}$$

with probability at least $1 - \varepsilon$.

The bounds (37)–(38) imply

$$\mathbb{P}\left(\|e^{it\Delta} \mathrm{P}_{>N} f^\omega\|_{L^p_{x,t}(\mathbb{T}^{d+1})} > \lambda \right) \lesssim e^{-cN^{2\alpha}\lambda^2} \tag{41}$$

$$\mathbb{P}\left(\|e^{it\Delta} \mathrm{P}_{>N} f^\omega\|_{L^\infty_{x,t}(\mathbb{T}^{d+1})} > \lambda \right) \lesssim e^{-cN^{2\alpha-}\lambda^2} \tag{42}$$

exactly as (31)–(32) imply (35)–(36). Also we have an analogous of Remark (2):

Remark 3 The exceptional set where (39)–(40) are not valid can be chosen to be the same for all $N \in \mathbb{N}$, paying an N^{0+} loss on the right-hand side of the estimates.

Fix $t \in \mathbb{R}$. Later we will also need the following bound for the H^s norm of $e^{it\Delta} f^\omega$ with $s < \alpha$. This is a well-known fact that we recall applying again (29) with $a_n = e^{in\cdot x - |n|^2 t} \langle n \rangle^{-\frac{d}{2}-\alpha+s}$, so that we get for $r \ge 2$

$$\| \mathrm{P}_N \langle D \rangle^s e^{it\Delta} f^\omega \|_{L^r_\omega} \le Cr^{\frac{1}{2}} N^{s-\alpha}, \qquad s < \alpha .$$

Here $\langle D \rangle$ denotes the Fourier multiplier operator $\langle n \rangle$. Proceeding as in the proof of Lemma 7, we also obtain

$$\mathbb{P}\left(\|\langle D \rangle^s \mathrm{P}_N e^{it\Delta} f^\omega\|_{L^2_x(\mathbb{T}^d)} > \lambda \right) \le Ce^{-cN^{2(\alpha-s)}\lambda^2}, \qquad s < \alpha, \tag{43}$$

and in particular, for any $\varepsilon > 0$ sufficiently small

$$\|e^{it\Delta} f^{\omega}\|_{H_x^s(\mathbb{T}^d)} \lesssim (-\ln \varepsilon)^{1/2} \qquad s < \alpha, \quad t \in \mathbb{R}, \tag{44}$$

with probability at least $1-\varepsilon$. Again the constant is uniform on $t \in \mathbb{R}$.

We are now ready to complete the proof of Proposition 1.

3.1 Proof of Proposition 1

Invoking the Borel–Cantelli lemma, it is enough to show that

$$\mathbb{P}\left(\limsup_{t\to 0} \|e^{it\Delta} f^{\omega} - f^{\omega}\|_{L_x^{\infty}(\mathbb{T}^d)} > 1/k\right) \lesssim \gamma_k, \tag{45}$$

for a summable sequence $\{\gamma_k\}_{k\in\mathbb{N}}$. Let us decompose

$$|e^{it\Delta} f^{\omega} - f^{\omega}| \le |e^{it\Delta} \mathrm{P}_{>N} f^{\omega}| + |e^{it\Delta} \mathrm{P}_{\le N} f^{\omega} - \mathrm{P}_{\le N} f^{\omega}| + |\mathrm{P}_{>N} f^{\omega}|. \tag{46}$$

Using (36) (with $t=0$) and (42), we see that

$$\|e^{it\Delta} \mathrm{P}_{>N} f^{\omega}\|_{L_{x,t}^{\infty}(\mathbb{T}^{d+1})} + \|\mathrm{P}_{>N} f^{\omega}\|_{L_x^{\infty}(\mathbb{T}^d)} \le \frac{1}{2k} \tag{47}$$

holds for all ω outside an exceptional set of measure $\lesssim e^{-cN^{2\alpha}k^{-2}}$. We choose $N = N_k$ via the identity $N_k^{2\alpha} = k^3$, in such a way that $e^{-cN_k^{2\alpha}k^{-2}} = e^{-ck}$ is summable (over $k \in \mathbb{N}$). Let $s^* > d/2$. Since

$$e^{it\Delta} \mathrm{P}_{\le N_k} f^{\omega} - \mathrm{P}_{\le N_k} f^{\omega} = \sum_{|n|\le N_k} (e^{-it|n|^2} - 1) e^{in\cdot x} \hat{f}^{\omega}(n),$$

using Cauchy–Schwarz, the summability of $\langle n\rangle^{-2s^*}$ (over $n \in \mathbb{Z}^d$) and (44) with $s=0$, $t=0$ (in the last inequality), we get

$$\begin{aligned}\|e^{it\Delta} \mathrm{P}_{\le N_k} f^{\omega} - \mathrm{P}_{\le N_k} f^{\omega}\|_{L_x^{\infty}(\mathbb{T}^d)} &\lesssim \sup_{|n|\le N_k} |e^{-it|n|^2} - 1| \left(\sum_{|n|\le N_k} \langle n\rangle^{2s^*} |\hat{f}^{\omega}(n)|^2\right)^{1/2} \\ &\lesssim |t|(N_k)^{s^*+2} \|f^{\omega}\|_{L^2} \le |t|(N_k)^{s^*+2}\frac{1}{k}, \end{aligned} \tag{48}$$

for ω outside an exceptional set of probability $\lesssim e^{-cN_k^{2\alpha}k^{-2}} = e^{-ck}$. From the previous inequality, looking at t so small that $|t|(N_k)^{s^*+2} \le 1/2$, we have

$$\mathbb{P}\left(\limsup_{t\to 0} \|e^{it\Delta}\,\mathrm{P}_{\le N^*} f^\omega - \mathrm{P}_{\le N^*} f^\omega\|_{L^\infty_x(\mathbb{T}^d)} > 1/k\right) \lesssim e^{-ck}. \tag{49}$$

Combining (36)–(36) and recalling the decomposition (46), the proof is concluded. □

4 Proof of Theorem 2

In this section, we consider the cubic Wick-ordered NLS (8) on $\mathbb{T}^d$ ($d = 1, 2$) as in the work of Bourgain in [3]. Namely, we look at the nonlinearity

$$\mathcal{N}(u) := \pm u\left(|u|^2 - 2\mu\right), \qquad \mu := \fint_{\mathbb{T}^d} |u(x,t)|^2 dx\,.$$

We are interested again in randomized initial data, i.e., f^ω is taken to be of the form (28). Recall (see (44)) that such data is $\mathbb{P}$-almost surely in H^s for all $s < \alpha$ and

$$\|f^\omega\|_{H^s} \lesssim (-\ln\varepsilon)^{1/2}, \quad s < \alpha\,, \tag{50}$$

with probability at least $1-\varepsilon$, for all $\varepsilon \in (0,1)$ sufficiently small. Since we work with any $\alpha > 0$, we are considering initial data in H^{0+}. We approximate Eq. (8) as in (11), for all $N \in 2^{\mathbb{N}} \cup \{\infty\}$. Recall that $\Phi_t^N f^\omega$ denotes the associated flow, with initial datum

$$\Phi_0^N f^\omega := \mathrm{P}_{\le N}\, f^\omega = \sum_{|n|\le N} \frac{g_n^\omega}{\langle n\rangle^{\frac{d}{2}+\alpha}} e^{in\cdot x}\,.$$

We write $\Phi_t f^\omega = \Phi_t^\infty f^\omega$ for the flow of (8) with datum $f^\omega = \mathrm{P}_\infty\, f^\omega$.

The relevant choice of σ in the following statement is $\sigma = \frac{1}{2}-$ (we will use this to prove Theorem 2).

Proposition 3 *Let $d = 1, 2$ and $\alpha > 0$. Let $N \in 2^{\mathbb{N}} \cup \{\infty\}$. For all $\sigma \in [0, \frac{1}{2})$, the following holds. Assume*

$$u = u(I) + u(II), \quad u(I) = e^{it\Delta}\,\mathrm{P}_{\le N}\, f^\omega, \quad \|u(II)\|_{X^{\alpha+\sigma,\frac{1}{2}+}} < 1 \tag{51}$$

and the same for v. Then

$$\|\mathcal{N}(u)\|_{X^{\alpha+\sigma,-\frac{1}{2}+}} \lesssim (-\ln\varepsilon)^{3/2} \tag{52}$$

$$\|\mathcal{N}(u)-\mathcal{N}(v)\|_{X^{\alpha+\sigma,-\frac{1}{2}++}} \lesssim (-\ln\varepsilon)\,\|u-v\|_{X^{\alpha+\sigma,\frac{1}{2}+}} \tag{53}$$

for initial data of the form (28)*, with probability at least* $1-\varepsilon$*, for all* $\varepsilon \in (0,1)$ *sufficiently small. If we take u as in* (51) *and we instead assume*

$$v = v(I) + u(II), \quad v(I) = e^{it\Delta} f^{\omega}, \quad \|u(II)\|_{X^{\alpha+\sigma,\frac{1}{2}+}} < 1\,,$$

we have

$$\|\mathcal{N}(u)-\mathcal{N}(v)\|_{X^{\alpha+\sigma,-\frac{1}{2}++}} \lesssim N^{-\alpha}\,. \tag{54}$$

Remark 4 Recall that α indicates the regularity of the initial datum. We are denoting by σ the amount of smoothing one can prove for the Wick-ordered cubic nonlinearity $\mathcal{N}$. More precisely, since the initial data (28) belongs to $H^{\alpha-}$, one can interpret this statement as saying that, with arbitrarily large probability, $\mathcal{N}$ is $\sigma+$ smoother than f^{ω}. Since $\sigma < \frac{1}{2}$ is permissible, we reach $\frac{1}{2}-$ smoothing for $\mathcal{N}$ and, combining with (26), also for the Duhamel contribution $\Phi_t^N f^{\omega} - e^{it\Delta} P_{\leq N} f^{\omega}$.

In fact, a stronger statement than 3 has been proved in [13]. Namely that the reminder can be further decomposed into a sum of two terms. The first one, to which one we refer as paracontrolled, lies in $X^{\frac{1}{2}-,\frac{1}{2}+}$ but has a precise random structure. The second one is a smoother deterministic reminder that lies in $X^{1-,\frac{1}{2}+}$.

Here we only explain how to get Proposition 3 for the first Picard iteration, namely when 4. Recall that η is a smooth cut-off of the unit interval. Let us fix $\alpha > 0$. Using (26), (27), and Proposition 3, one can show that for all $\delta > 0$ sufficiently small, the following holds. For all $N \in 2^{\mathbb{N}} \cup \{\infty\}$, the map

$$\Gamma^N(u) := \eta(t) e^{it\Delta} \mathrm{P}_{\leq N} f^{\omega} - i\eta(t) \int_0^t e^{i(t-s)\Delta} \mathrm{P}_{\leq N} \mathcal{N}(u(\cdot,s))\,ds \tag{55}$$

is a contraction on the set

$$\left\{ e^{it\Delta} \mathrm{P}_{\leq N} f^{\omega} + g, \quad \|g\|_{X_{\delta}^{\alpha+\sigma,\frac{1}{2}+}} < 1 \right\} \tag{56}$$

equipped with the $X_{\delta}^{\alpha+\sigma,\frac{1}{2}+}$ norm, outside an exceptional set (we call it a δ-exceptional set) of initial data of probability smaller than $e^{-\delta^{-\gamma}}$, with $\gamma > 0$ a given small constant. Notice that this holds uniformly over $N \in 2^{\mathbb{N}} \cup \{\infty\}$. Again, this is a standard routine calculation that we omit (see for instance [18, Section 3.5.1]). We only explain how to find the relation between the local existence time δ and the size of the exceptional set. Given any $\varepsilon \in (0, 1)$ sufficiently small, using (26), (27), and Proposition 3, we have

$$\|\Gamma^N(u) - \eta(t)e^{it\Delta}\,\mathrm{P}_{\leq N}\, f^\omega\|_{X_\delta^{\alpha+\sigma,\frac{1}{2}+}} \lesssim \delta^{0+}\,(-\ln\varepsilon)^{3/2}\ ,$$

for all f^ω outside an exceptional set of probability smaller than ε. Letting δ such that $\varepsilon = e^{-\delta^{-\gamma}}$ with $\gamma > 0$ a fixed small constant, we have $C\delta^{0+}\,(-\ln\varepsilon)^{3/2} < 1$ for all $\delta > 0$ sufficiently small. Note that the measure $e^{-\delta^{-\gamma}}$ of the δ-exceptional set converges to zero as $\delta \to 0$. In particular, for ω outside the δ-exceptional set, the fixed point $\Phi_t^N f^\omega$ of the map (55) belongs to the set (56), namely

$$\|\Phi_t^N f^\omega - e^{it\Delta}\,\mathrm{P}_{\leq N}\, f^\omega\|_{X_\delta^{\alpha+\sigma,\frac{1}{2}+}} < 1, \qquad N \in 2^{\mathbb{N}} \cup \{\infty\}\,. \tag{57}$$

We are now ready to prove Theorem 2.

4.1 Proof of Theorem 2

It suffices to show that

$$\lim_{N\to\infty} \left\| \sup_{0\leq t\leq\delta} |\Phi_t f^\omega(x) - \Phi_t^N f^\omega(x)| \right\|_{L_x^2(\mathbb{T}^2)} = 0 \tag{58}$$

for all f^ω outside a δ-exceptional set A_δ. Note indeed that (58) implies that given f^ω, we can find $\mathbb{P}$-almost surely, a δ_ω (which depends on ω) such that (58) is satisfied. Indeed, if we could not do so, this would mean that $f^\omega \in \bigcap_{\delta>0} A_\delta$, and the probability of this event is zero since $\mathbb{P}(A_\delta) \to 0$ as $\delta \to 0$.

Once we have (58) with $\delta = \delta_\omega$, we have $\mathbb{P}$-almost surely

$$\lim_{t\to 0} \Phi_t^\omega f^\omega(x) - f^\omega(x) = 0, \qquad \text{for a.e. } x \in \mathbb{T}^2\,,$$

as claimed, simply invoking Proposition 2.

In order to prove (58), we decompose

$$|\Phi_t f^\omega - \Phi_t^N f^\omega| \leq |e^{it\Delta}\,\mathrm{P}_{>N}\, f^\omega| + |\Phi_t f^\omega - e^{it\Delta} f^\omega - (\Phi_t^N f^\omega - e^{it\Delta}\,\mathrm{P}_{\leq N}\, f^\omega)|\,.$$

Thus, recalling the decay of the high-frequency linear term given by (36), it remains to show that

$$\lim_{N\to\infty}\left\|\sup_{0\le t\le\delta}|\Phi_t f^\omega - e^{it\Delta}f^\omega - (\Phi_t^N f^\omega - e^{it\Delta}\,\mathrm{P}_{\le N}\, f^\omega)|\right\|_{L^2(\mathbb{T}^2)} = 0\,, \tag{59}$$

for all f^ω outside a δ-exceptional set.

For any $\alpha > 0$, we can choose σ sufficiently close to $\frac{1}{2}$ that

$$\frac{1}{2} < \alpha + \sigma\,. \tag{60}$$

Thus, using the $X^{s,b}$ space embedding from Lemma 2, it suffices to prove

$$\lim_{N\to\infty}\left\|w - w^N\right\|_{X_\delta^{\alpha+\sigma,\frac{1}{2}+}} = 0\,, \tag{61}$$

where

$$w^N := \Phi_t^N f - e^{it\Delta}\,\mathrm{P}_{\le N}\, f^\omega, \qquad w := w^\infty\,.$$

Notice that by (57), we have

$$\|w^N\|_{X_\delta^{\alpha+\sigma,\frac{1}{2}+}} < 1, \qquad N \in 2^{\mathbb{N}} \cup \{\infty\}\,.$$

Since for $t \in [0,\delta]$, we have

$$w - w^N = -i\eta(t)\int_0^{t'} e^{i(t-t')\Delta}\left(\mathcal{N}(\Phi_{t'} f^\omega) - \mathrm{P}_{\le N}\,\mathcal{N}(\Phi_{t'}^N f^\omega)\right)dt'\,, \tag{62}$$

using (26), (27), we get

$$\|w - w^N\|_{X_\delta^{\alpha+\sigma,\frac{1}{2}+}} \lesssim \delta^{0+}\|\mathcal{N}(\Phi_t f) - \mathrm{P}_{\le N}\,\mathcal{N}(\Phi_t^N f)\|_{X_\delta^{\alpha+\sigma,-\frac{1}{2}++}}\,. \tag{63}$$

We decompose

$$\begin{aligned}\mathcal{N}(\Phi_t f) - \mathrm{P}_{\le N}\,\mathcal{N}(\Phi_t^N f) = \\ \mathrm{P}_{\le N}\left(\mathcal{N}(e^{it\Delta}\,\mathrm{P}_{\le N}\, f^\omega + w) - \mathcal{N}(e^{it\Delta}\,\mathrm{P}_{\le N}\, f^\omega + w^N)\right) + \text{Remainders}\,,\end{aligned} \tag{64}$$

where

$$\text{Remainders} := \mathrm{P}_{\le N}\left(\mathcal{N}(e^{it\Delta} f^\omega + w) - \mathcal{N}(e^{it\Delta}\,\mathrm{P}_{\le N}\, f^\omega + w)\right) + \mathrm{P}_{>N}\,\mathcal{N}(\Phi_t f)\,.$$

Notice that by (52), (54), we have

$$\|\text{Remainders}\|_{X_\delta^{\alpha+\sigma,-\frac{1}{2}++}} \to 0 \quad \text{as } N \to \infty\,, \tag{65}$$

with probability at least $1-\varepsilon$. Using (53), we can estimate

$$\begin{aligned} \| \mathrm{P}_{\leq N} \Big(\mathcal{N}(e^{it\Delta}\mathrm{P}_{\leq N} f^\omega + w) - \mathcal{N}(e^{it\Delta}\mathrm{P}_{\leq N} f^\omega + w_N)) \|_{X_\delta^{\alpha+\sigma,-\frac{1}{2}++}} \\ \lesssim (-\ln\varepsilon) \left\| w - w^N \right\|_{X_\delta^{\alpha+\sigma,\frac{1}{2}+}}, \end{aligned} \tag{66}$$

and (63), (64), (66) give

$$\left\| w - w^N \right\|_{X_\delta^{\alpha+\sigma,\frac{1}{2}+}} \lesssim \delta^{0+}(-\ln\varepsilon) \left\| w - w^N \right\|_{X_\delta^{\alpha+\sigma,\frac{1}{2}+}} + \|\text{Remainders}\|_{X_\delta^{\alpha+\sigma,-\frac{1}{2}++}} \tag{67}$$

with probability at least $1-\varepsilon$. Since with our choice of $\varepsilon = e^{-\delta^{-\gamma}}$, we have $C\delta^{0+}(-\ln\varepsilon)^{3/2} < 1$, we can absorb the first term on the right-hand side into the left-hand side, and we still have that (65) holds outside a δ-exceptional set. Thus letting $N \to \infty$, the proof of (9) is complete.

□

Remark 5 It is worthy to remark that, comparing with for instance [3], the procedure that allows to promote a statement valid on a δ-exceptional set A_δ for arbitrarily small $\delta > 0$ to a statement that is valid with probability $= 1$ is far easier. In particular, it does not involve any control on the evolution of the (Gaussian) measure induced by the random Fourier series. This is because we are considering a property that has to be verified only at time $t = 0$ a.s., instead that in a time interval containing $t = 0$, as in [3].

We now give some hints on the proof of the smoothing estimates given in Proposition 3.

4.2 *Proof of Proposition 3*

Again it is worthy to recall that an even stronger statement than 3 has been proved in [13]. Here we show how to handle the first Picard iterate. Notice that the Wick-ordered nonlinearity can be written as

$$\mathcal{N}(u(x,\cdot)) = \sum_{n_2 \neq n_1, n_3} \widehat{u}(n_1)\widehat{\overline{u}}(n_2)\widehat{u}(n_3)e^{i(n_1-n_2+n_3)\cdot x} - \sum_n \widehat{u}(n)|\widehat{u}(n)|^2 e^{in\cdot x}, \tag{68}$$

where we are looking at the nonlinear term for fixed time and $\widehat{u}(\cdot)$ denotes the space Fourier coefficients. Looking at a similar expansion for the difference $\mathcal{N}(u) - \mathcal{N}(v)$, it is easy to see that we can deduce (3) from a slightly more general Lemma 9 given below. It implies the desired statement

$$u_j(n_j) = u(n_j), \quad v(n_j), \quad \text{or } \; u(n_j) - v(n_j)\,.$$

□

We will give a proof of the following Lemma in the fully random case $J_j = I$ for $j = 1, 2, 3,$, which corresponds to the study of the first Picard iterate. Comparing with 9 (and [13]), there is a simplification coming from the fact that our f^ω is slightly more regular, namely we consider $\alpha > 0$ instead of $\alpha = 0$.

Lemma 9 *Let $d = 1, 2$ and $\alpha > 0$. Let $N \in 2^{\mathbb{N}} \cup \{\infty\}$. For all $\sigma \in [0, \frac{1}{2})$, the following holds. Assume for $j = 1, 2, 3$*

$$u_j(I) = e^{it\Delta}\,\mathrm{P}_{\leq N}\, f^\omega, \qquad \|u_j(II)\|_{X^{\alpha+\sigma, \frac{1}{2}+}} < 1. \tag{69}$$

Let $J_j \in \{I, II\}$, $j = 1, 2, 3$. Then, for all $\varepsilon \in (0, 1)$ sufficiently small, we have the following:

$$\|\mathcal{N}(u_1(J_1), \overline{u_2}(J_2), u_3(J_3))\|_{X^{\alpha+\sigma, -\frac{1}{2}+}} \lesssim (-\ln \varepsilon)^{3/2}\ , \tag{70}$$

and more precisely,

$$\|\mathcal{N}(u_1(II), \overline{u_2}(J_2), u_3(J_3))\|_{X^{\alpha+\sigma, -\frac{1}{2}++}} \lesssim (-\ln \varepsilon)\, \|u_1(II)\|_{X^{\alpha+\sigma, \frac{1}{2}+}}\,, \tag{71}$$

$$\|\mathcal{N}(u_1(J_1), \overline{u_2}(II), u_3(J_3))\|_{X^{\alpha+\sigma, -\frac{1}{2}++}} \lesssim (-\ln \varepsilon)\, \|u_2(II)\|_{X^{\alpha+\sigma, \frac{1}{2}+}}\,, \tag{72}$$

with probability at least $1 - \varepsilon$. Moreover, if in (69) we replace for some $j = j^*$ the projection operator $\mathrm{P}_{\leq N}$ by $\mathrm{P}_{>N}$, then the estimate (70) with $J_{j^*} = I$ holds with an extra factor $N^{-\alpha}$ on the right-hand side.

Remark 6 Saying that these estimates hold with probability at least $1 - \varepsilon$ means, more precisely, that they hold for all ω outside an exceptional set of probability $\leq \varepsilon$. Moreover, this set can be chosen to be independent on $N \in 2^{\mathbb{N}} \cup \{\infty\}$.

Remark 7 Notice that by the symmetry $n_1 \leftrightarrow n_3$ the estimate (71) implies an analogous estimate for $u_3(II)$.

Here we only consider the case $J_j = I$ for $j = 1, 2, 3$, namely the case in which all the contributions are a linear random evolution. We prove the bound (70) relative to this case and to $N = \infty$. Moreover, we split the nonlinearity as a difference of two terms (see (68))

$$\mathcal{N}_1(u_1(J_1), \overline{u_2}(J_2), u_3(J_3)) = \sum_{n_2 \neq n_1, n_3} \widehat{u_1(J_1)}(n_1)\widehat{\overline{u_2}(J_2)}(n_2)\widehat{u_3(J_3)}(n_3)e^{i(n_1-n_2+n_3)\cdot x},$$

$$\mathcal{N}_2(u_1(J_1), \overline{u_2}(J_2), u_3(J_3)) = \sum_{n} \widehat{u_1(J_1)}(n)\widehat{\overline{u_2}(J_2)}(n)\widehat{u_3(J_2)}(n)e^{in\cdot x},$$

and we prove (70) only for $\mathcal{N}_1$, which is the most challenging contribution. The proof for $\mathcal{N}_2$ is indeed elementary.

To prove, (70) will be useful to recall that the space–time Fourier transform of $e^{it\Delta} f^\omega$ is

$$\widehat{e^{it\Delta} f^\omega}(n, \tau) = \frac{g^\omega}{\langle n \rangle^{\frac{d}{2}+\alpha}} \delta(\tau + |n|^2),$$

where δ is the delta function. So a direct computation gives

$$\|e^{it\Delta} f^\omega\|^2_{X^{0+,\frac{1}{2}+}} = \sum_n \frac{|g_n^\omega|^2}{\langle n \rangle^{d+2\alpha-}},$$

which, recalling $\int |g_n^\omega|^2 d\omega = 1$, immediately implies

$$\| \|e^{it\Delta} f^\omega\|_{X^{0+,\frac{1}{2}+}} \|^2_{L^2_\omega} = \sum_n \frac{1}{\langle n \rangle^{d+2\alpha-}} < \infty.$$

Since we can expand the LHS as a bilinear form in the Gaussian variables g_n^ω, we get by Gaussian hypercontractivity

$$\| \|e^{it\Delta} f^\omega\|_{X^{0+,\frac{1}{2}+}} \|^2_{L^q_\omega} = \sum_n \frac{1}{\langle n \rangle^{d+2\alpha-}} < C_q < \infty.$$

Proceeding essentially as in the Proof of Lemmas 7–8 (recall also Remarks 2–RemarkUniform1Bis), this allows to prove a pointwise bound

$$\|e^{it\Delta} f^\omega\|_{X^{0+,\frac{1}{2}+}} \lesssim \sqrt{\ln\left(\frac{1}{\varepsilon}\right)}, \tag{73}$$

with probability larger than $1 - C\varepsilon$ for all sufficiently small $\varepsilon > 0$.

Let N, N_1, N_2, N_3 be dyadic scales. We denote with $\tilde{N}$ the maximum between N_1, N_2, N_3. First we perform a reduction to remove frequencies that are far from the paraboloid. More precisely, we denote with P_A the space–time Fourier projection into the set A, and our goal is to reduce

$$\sum_{N_1,N_2,N_3} \|\mathcal{N}_1\left(\mathrm{P}_{N_1} u_1(I), \mathrm{P}_{N_2} \overline{u_2}(I), \mathrm{P}_{N_3} u_3(I)\right)\|^2_{X^{\alpha+\sigma,-\frac{1}{2}++}} \tag{74}$$

$$= \sum_{N,N_1,N_2,N_3} N^{2\alpha+2\sigma} \| \mathrm{P}_N \, \mathcal{N}_1\left(\mathrm{P}_{N_1} u_1(I), \mathrm{P}_{N_2} \overline{u_2}(I), \mathrm{P}_{N_3} u_3(I)\right) \|^2_{X^{0,-\frac{1}{2}++}}$$

to

$$\sum_{N,N_1,N_2,N_3} N^{2\alpha+2\sigma} \| \mathrm{P}_N \, \mathrm{P}_{\left\{\langle\tau+|n|^2\rangle\le\tilde{N}^{1+\frac{1}{10}}\right\}} \mathcal{N}_1\left(\mathrm{P}_{N_1} u_1(I) \, \mathrm{P}_{N_2} \overline{u_2}(I) \, \mathrm{P}_{N_3} u_3(I)\right) \|^2_{X^{0,-\frac{1}{2}++}}. \tag{75}$$

To obtain this reduction, it is sufficient to show that projection of the nonlinearity onto the complementary set is appropriately bounded, i.e., that

$$\sum_{N,N_1,N_2,N_3} N^{2\alpha+2\sigma} \| \mathrm{P}_N \, \mathrm{P}_{\left\{\langle\tau+|n|^2\rangle>\tilde{N}^{\frac{11}{10}}\right\}} \mathcal{N}_1\left(\mathrm{P}_{N_1} u_1(I), \mathrm{P}_{N_2} \overline{u_2}(I), \mathrm{P}_{N_3} u_3(I)\right) \|^2_{X^{0,-\frac{1}{2}++}} \tag{76}$$

$$\lesssim (-\ln \varepsilon)^3$$

with probability at least $1-\varepsilon$. To do so, we abbreviate

$$\mathcal{N}_1^{N_1,N_2,N_3}(\cdot) := \mathcal{N}_1\left(\mathrm{P}_{N_1} u_1(I), \mathrm{P}_{N_2} \overline{u_2}(I), \mathrm{P}_{N_3} u_3(I)\right),$$

and we bound

$$\sum_{N_1,N_2,N_3} N^{2\alpha+2\sigma} \| \mathrm{P}_N \, \mathrm{P}_{\left\{\langle\tau+|n|^2\rangle>\tilde{N}^{\frac{11}{10}}\right\}} \mathcal{N}_1^{N_1,N_2,N_3} \|^2_{X^{0,-\frac{1}{2}++}} \tag{77}$$

$$\sim N^{2\alpha+2\sigma} \sum_{\substack{N_1,N_2,N_3\\ n\sim N}} \int \frac{\chi_{\{\langle\tau+|n|^2\rangle>\tilde{N}^{\frac{11}{10}}\}}}{\langle\tau+|n|^2\rangle^{1--}} \left|\widehat{\mathcal{N}_1^{N_1,N_2,N_3}}(\cdot)(n,\tau)\right|^2 d\tau$$

$$\lesssim N^{2\alpha+2\sigma-1-\frac{1}{10}+3(0+)} \sum_{\substack{N_1,N_2,N_3\\ n\sim N}} \int \left|\widehat{\mathcal{N}_1^{N_1,N_2,N_3}}(\cdot)(n,\tau)\right|^2 d\tau$$

$$\sim N^{2\alpha-\frac{1}{20}} \sum_{N_1,N_2,N_3} \| \mathrm{P}_N \, \mathcal{N}_1^{N_1,N_2,N_3} \|^2_{L^2_{x,t}},$$

recalling that $\sigma < 1/2$ (here in fact we may have more smoothing than $\frac{1}{2}-$). We have used the fact that at least one of the frequency scales N_j has to be comparable to N; otherwise, the contribution is zero by orthogonality, and so particular, we have

$N \lesssim \tilde{N}$ (recall that $\tilde{N} = \max(N_1, N_2, N_3)$). In order to continue the estimate, we assume for definiteness that $N_1 \sim N$. The other possible case is $N_2 \sim N$ (since everything is symmetric under $n_1 \leftrightarrow n_3$), and one can indeed immediately check that the estimate (78) below is still valid in this case, with obvious changes. Thus we have using Hölder's inequality, the improved Strichartz inequality (40) for randomized functions (for the L^∞ norm of $u_1(I)$), and the Strichartz inequality (10) (for the L^4 norms of $u_2(I)$ and $u_3(I)$), we obtain

$$
\begin{aligned}
&\| \mathrm{P}_N \mathcal{N}_1^{N_1,N_2,N_3} \|^2_{L^2_{x,t}} \qquad (78)\\
&\le \| \mathrm{P}_{N_1} u_1(I)\|^2_{L^\infty_{x,t}} \| \mathrm{P}_{N_2} \overline{u_2}(I)\|^2_{L^4_{x,t}} \| \mathrm{P}_{N_3} u_3(I)\|^2_{L^4_{x,t}} .\\
&\lesssim (-\ln \varepsilon)\, N_1^{-2\alpha} \| \mathrm{P}_{N_2} \overline{u_2}(I)\|^2_{L^4_{x,t}} \| \mathrm{P}_{N_3} w_3(I)\|^2_{L^4_{x,t}} ,\\
&\lesssim (-\ln \varepsilon)\, N^{-2\alpha} \| \mathrm{P}_{N_2} \overline{u_2}(I)\|^2_{X^{0+,\frac{1}{2}+}} \| \mathrm{P}_{N_3} u_3(I)\|^2_{X^{0+,\frac{1}{2}+}} ,
\end{aligned}
$$

this holds on a set of probability larger than $1-\varepsilon$, and this set may be chosen to be independent on $N_1 \in \mathbb{N} \cup \{\infty\}$ (see Remark 3) and thus on $N \in \mathbb{N} \cup \{\infty\}$. Plugging (78) into (77), summing over the N_j, and using (73), we arrive to the needed bound

$$
\begin{aligned}
\text{LHS of (76)} &\lesssim (-\ln \varepsilon) \sum_{N,N_1} N^{-\frac{1}{20}} \|u_2(I)\|^2_{X^{0+,\frac{1}{2}+}} \|u_3(I)\|^2_{X^{0+,\frac{1}{2}+}}\\
&\lesssim (-\ln \varepsilon)^3 \sum_{N,N_1} N^{-\frac{1}{40}} N_1^{-\frac{1}{40}} \lesssim (-\ln \varepsilon)^3 .
\end{aligned}
$$

Note that in (78), we could also use a weaker bound replacing the L^4 norm with the L^∞ and that in the fully random case $J_j = I$ for all j is controlled invoking (40) for all $j = 1, 2, 3$. However, the L^4 bound is more robust since it works also in the other cases, where the contributions are not all random (namely if some J_j is of the form II).

So we have reduced to (75). We have

$$
\begin{aligned}
&\mathrm{P}_N \mathrm{P}_{\left\{\langle \tau+|n|^2\rangle \le \tilde{N}^{\frac{11}{10}}\right\}} \mathcal{N}_1^{N_1,N_2,N_3}(\cdot) \qquad (79)\\
&= \mathrm{P}_N \mathrm{P}_{\left\{\langle \tau+|n|^2\rangle \le \tilde{N}^{\frac{11}{10}}\right\}} \left(\sum_{|n_j|\sim N_j} e^{ix\cdot(n_1-n_2+n_3)} e^{-it(|n_1|^2-|n_2|^2+|n_3|^2)} \right.\\
&\qquad \left. \times \frac{g^\omega_{n_1}}{\langle n_1\rangle^{1+\alpha}} \frac{\overline{g^\omega_{n_2}}}{\langle n_2\rangle^{1+\alpha}} \frac{g^\omega_{n_3}}{\langle n_3\rangle^{1+\alpha}} \right) .
\end{aligned}
$$

Thus we see that (75) satisfies the desired inequalities (70) as long as we can bound

$$N^{2\alpha+2\sigma}\Bigg\| \sum_{N_1,N_2,N_3} \mathrm{P}_N\,\mathrm{P}_{\left\{\langle\tau+|n|^2\rangle\le\tilde{N}^{\frac{11}{10}}\right\}}\left(\sum_{|n_j|\sim N_j} e^{ix\cdot(n_1-n_2+n_3)}e^{-it(|n_1|^2-|n_2|^2+|n_3|^2)}\right.$$
$$\left.\times\,\frac{g^{\omega}_{n_1}}{\langle n_1\rangle^{1+\alpha}}\frac{\overline{g^{\omega}_{n_2}}}{\langle n_2\rangle^{1+\alpha}}\frac{g^{\omega}_{n_3}}{\langle n_3\rangle^{1+\alpha}}\right)\Bigg\|^2_{X^{0,-\frac{1}{2}++}} \lesssim (-\ln\varepsilon)^3\,N^{0-}\,, \tag{80}$$

on a set of probability larger than $1-\varepsilon$.

Since

$$\mathcal{F}\Big(e^{ix\cdot(n_1-n_2+n_3)}e^{-it(|n_1|^2-|n_2|^2+|n_3|^2)}\Big)(n,\tau) = \sum_{n_1-n_2+n_3=n}\delta(\tau+|n_1|^2-|n_2|^2+|n_3|^2)\,, \tag{81}$$

where $\mathcal{F}$ is the space–time Fourier transform and δ is the delta function, we reduce (80) to showing that

$$N^{2\alpha+2\sigma}\sum_{N_1,N_2,N_3}\sum_{|n|\sim N}\int \frac{\chi_{\{\langle\tau+|n|^2\rangle\le\tilde{N}^{\frac{11}{10}}\}}}{\langle\tau+|n|^2\rangle^{1--}}$$
$$\times\left|\sum_{\substack{|n_j|\sim N_j,\, n_2\neq n_1,n_3\\ n=n_1-n_2+n_3\\ \tau+|n_1|^2-|n_2|^2+|n_3|^2=0}} \frac{g^{\omega}_{n_1}}{\langle n_1\rangle^{1+\alpha}}\frac{\overline{g^{\omega}_{n_2}}}{\langle n_2\rangle^{1+\alpha}}\frac{g^{\omega}_{n_3}}{\langle n_3\rangle^{1+\alpha}}\right|^2 d\tau \lesssim (-\ln\varepsilon)^3\,N^{0-}, \tag{82}$$

with probability at least $1-\varepsilon$. Letting

$$\mu := |n|^2+\tau = |n|^2-|n_1|^2+|n_2|^2-|n_3|^2$$

(the second identity holds over the integration set, since we have a factor

$$\delta(\tau+|n_1|^2-|n_2|^2+|n_3|^2)$$

in the integrand) and recalling that $N \lesssim \tilde{N}$, this follows by

$$N^{2\alpha+2\sigma} \sum_{N_1,N_2,N_3} \sum_{|n|\sim N} \int \frac{\chi_{\{\langle\mu\rangle \leq \tilde{N}^{\frac{11}{10}}\}}}{\langle\mu\rangle^{1--}} \times \left| \sum_{\substack{|n_j|\sim N_j,\, n_2\neq n_1,n_3 \\ n=n_1-n_2+n_3 \\ -|n|^2+|n_1|^2-|n_2|^2+|n_3|^2=\mu}} \frac{g_{n_1}^{\omega}}{\langle n_1\rangle^{1+\alpha}} \frac{\overline{g_{n_2}^{\omega}}}{\langle n_2\rangle^{1+\alpha}} \frac{g_{n_3}^{\omega}}{\langle n_3\rangle^{1+\alpha}} \right|^2 d\tau \lesssim (-\ln\varepsilon)^3 N^{0-}, \tag{83}$$

with probability at least $1-\varepsilon$. Using Hölder inequality in $d\mu$, we reduce to prove (here we use the symmetry $\mu \leftrightarrow -\mu$)

$$N^{2\alpha+2\sigma} \tilde{N}^{0+} \sum_{N_1,N_2,N_3} \sup_{|\mu|\lesssim \tilde{N}^{\frac{11}{10}}} \sum_{|n|\sim N} \times \left| \sum_{\substack{|n_j|\sim N_j,\, n_2\neq n_1,n_3 \\ n=n_1-n_2+n_3 \\ \mu=|n|^2-|n_1|^2+|n_2|^2-|n_3|^2}} \frac{g_{n_1}^{\omega}}{\langle n_1\rangle^{1+\alpha}} \frac{\overline{g_{n_2}^{\omega}}}{\langle n_2\rangle^{1+\alpha}} \frac{g_{n_3}^{\omega}}{\langle n_3\rangle^{1+\alpha}} \right|^2 \lesssim (-\ln\varepsilon)^3 N^{0-}, \tag{84}$$

with probability at least $1-\varepsilon$. We rewrite (84) as

$$N^{2\alpha+2\sigma} \tilde{N}^{0+} \sum_{N_1,N_2,N_3} \sup_{|\mu|\lesssim \tilde{N}^{\frac{11}{10}}} \sum_{|n|\sim N} \left| \sum_{R_n(n_1,n_2,n_3)} \frac{g_{n_1}^{\omega}}{\langle n_1\rangle^{1+\alpha}} \frac{\overline{g_{n_2}^{\omega}}}{\langle n_2\rangle^{1+\alpha}} \frac{g_{n_3}^{\omega}}{\langle n_3\rangle^{1+\alpha}} \right|^2 \lesssim (-\ln\varepsilon)^3 N^{0-}, \tag{85}$$

where for fixed n, μ we have denoted

$$R_n(n_1,n_2,n_3) := \Big\{ (n_1,n_2,n_3) \in (\mathbb{Z}^2)^3 \ : \ |n_j| \sim N_j,\ j=1,2,3, \tag{86}$$
$$n_2 \neq n_1, n_3,\ n_1-n_2+n_3 = n,\ \mu = |n|^2-|n_1|^2+|n_2|^2-|n_3|^2 \Big\}.$$

The set $R_n(\cdot)$ depends on μ also (like all the sets we will define below). However, we omit this dependence to simplify the notation. Notice that in the definition of $R_n(\cdot)$ the condition

$$|n|^2 - |n_1|^2 + |n_2|^2 - |n_3|^2 = \mu$$

can be equivalently replaced by

$$2(n_1 - n_2) \cdot (n_3 - n_2) = \mu\,.$$

We also note that we have reduced to a case in which at least one of the frequencies N_1, N_3 is comparable to $\tilde{N}$. Indeed, if both $N_1 \ll \tilde{N}$ and $N_3 \ll \tilde{N}$, we must have $N_2 = \tilde{N}$ and $\mu \sim N^2$, which contradicts the fact that $\mu \lesssim N^{\frac{11}{10}}$. Since the roles of N_1 and N_3 are symmetric (they are always the size of the indices of the Fourier coefficients of u_1, u_3), hereafter we assume that

$$N_1 \sim \tilde{N} \gtrsim N.$$

To estimate, (85) will be also useful to introduce the set

$$S(n_1, n_2, n_3) := \Big\{ (n_1, n_2, n_3) \in (\mathbb{Z}^2)^3 \;:\; |n_j| \sim N_j,\, j = 1, 2, 3, \qquad (87)$$
$$n_2 \neq n_1, n_3,\; \mu = 2(n_1 - n_2) \cdot (n_3 - n_2) \Big\}\,.$$

We recall that the Gaussian variables contract in the following way:

$$\int g_n^\omega g_{n'}^\omega d\mathbb{P}(\omega) = 0, \qquad \int g_n^\omega \overline{g_{n'}^\omega} d\mathbb{P}(\omega) = \begin{cases} 0 \text{ if } n \neq n' \\ 1 \text{ if } n = n' \end{cases}. \qquad (88)$$

Along with the fact that the sum is restricted over $n_1, n_3 \neq n_2$ and symmetric under $n_1 \leftrightarrow n_3$, we get

$$\int \Bigg| \sum_{R_n(n_1,n_2,n_3)} \frac{g_{n_1}^\omega}{\langle n_1 \rangle^{1+\alpha}} \frac{\overline{g_{n_2}^\omega}}{\langle n_2 \rangle^{1+\alpha}} \frac{g_{n_3}^\omega}{\langle n_3 \rangle^{1+\alpha}} \Bigg|^2 d\mathbb{P}(\omega) \qquad (89)$$
$$= 2 \sum_{R_n(n_1,n_2,n_3)} \frac{1}{\langle n_1 \rangle^{2\alpha+2}} \frac{1}{\langle n_2 \rangle^{2\alpha+2}} \frac{1}{\langle n_3 \rangle^{2\alpha+2}} = 2 \sum_{R_n(n_1,n_2,n_3)} N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2}\,.$$

In other words, the $L^2(d\omega)$ norm of the Gaussian trilinear form is controlled by square root of the right-hand side of (89). Using the hypercontractivity of the Gaussians (see [19, 33]), we can promote this to an $L^q(d\omega)$ bound, with a multiplicative factor that is factor $q^{3/2}$. Then using Minkowski integral inequality and Bernstein inequality (as we did in Sect. 3), this also gives to us a (uniform) pointwise bound

$$\left| \sum_{R_n(n_1,n_2,n_3)} \frac{g^{\omega}_{n_1}}{\langle n_1\rangle^{1+\alpha}} \frac{\overline{g^{\omega}_{n_2}}}{\langle n_2\rangle^{1+\alpha}} \frac{g^{\omega}_{n_3}}{\langle n_3\rangle^{1+\alpha}} \right|^2 \lesssim (-\ln\varepsilon)^3 N_1^{0+} \sum_{R_n(n_1,n_2,n_3)} N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2}\,, \tag{90}$$

with an extra N_1^{0+} loss, that is valid for ω outside an exceptional set of probability $\leq \varepsilon$ (again, proceeding as in Sect. 3, we see that this exceptional set can be chosen to be independent on N, as required).

We finally distinguish two last possibilities. First restrict the summation over $(n_1, n_2, n_3) \in R_n(n_1, n_2, n_3)$ such that $n_1 \neq n_3$ (with a small abuse of notation, we do not introduce additional notation for this restriction). In this case, we get, with probability $> 1 - \varepsilon$, the following estimate

$$\begin{aligned} &\sum_{|n|\sim N} \left| \sum_{R_n(n_1,n_2,n_3)} \frac{g^{\omega}_{n_1}}{\langle n_1\rangle^{1+\alpha}} \frac{\overline{g^{\omega}_{n_2}}}{\langle n_2\rangle^{1+\alpha}} \frac{g^{\omega}_{n_3}}{\langle n_3\rangle^{1+\alpha}} \right|^2 \\ &\lesssim (-\ln\varepsilon)^3 \sum_{|n|\sim N} \sum_{R_n(n_1,n_2,n_3)} N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2} \\ &\lesssim (-\ln\varepsilon)^3 \sum_{S(n_1,n_2,n_3)} N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2} \\ &\sim (-\ln\varepsilon)^3 \sum_{S(n_1,n_2,n_3)} N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2} \\ &\lesssim (-\ln\varepsilon)^3 N_1^{-2\alpha-2} N_2^{-2\alpha-2} N_3^{-2\alpha-2} \# S(n_1,n_2,n_3) \\ &\lesssim (-\ln\varepsilon)^3 N_1^{-2\alpha-1} N_2^{-2\alpha} N_3^{-2\alpha}\,, \end{aligned} \tag{91}$$

where we used that if $n_1 \neq n_3$, then

$$\# S(n_1,n_2,n_3) \lesssim N_1 N_2^2 N_3^2\,;$$

this is because once we have fixed n_2, n_3 in $N_2^2 N_3^2$ possible ways, we remain with at most N_1 choices for n_1 by the relation $\mu = 2(n_1 - n_2)\cdot(n_3 - n_2)$. This fact has a clear geometric interpretation, namely that this relation forces the (two-dimensional) lattice point n_1 to belong to the portion of a line that lies inside a ball of radius $\lesssim N_1$ (and there are $\lesssim N_1$ such lattice points n_1).

The second possibility is that we sum over $(n_1, n_2, n_3) \in R_n(n_1, n_2, n_3)$ such that $n_1 = n_3$. In this case restriction, $\mu = 2|n_1 - n_2|^2$ implies that once we have chosen n_2 in N_2^2 possible ways, we remain with $\lesssim \mu^{0+} \lesssim N_1^{0++}$ choices for $n_1 = n_3$ (since a circle of radius μ contains $\lesssim \mu^{0+}$ lattice points). This gives an even better bound than the one above.

Thus, summing the (91) over N_2, N_3 and recalling that $N_1 \sim \tilde{N} \gtrsim N$, we have bounded, with probability $> 1 - \varepsilon$, the expression (85) by

$$N^{2\alpha+2\sigma} N_1^{0+} \sum_{N_1} N_1^{-2\alpha-1} \lesssim (-\ln \varepsilon)^3 N_1^{2\sigma-1+0+} \lesssim (-\ln \varepsilon)^3 N^{0-} ,$$

where we used $\sigma < \frac{1}{2}$.

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Nonlinear Schrödinger Equation with Singularities

Nevena Dugandžija and Ivana Vojnović

Abstract We describe results on certain types of nonlinear Schrödinger equations, mainly the cubic equation with or without potential. We are interested in singular initial conditions and equations with a delta potential in three dimensions. The existence and uniqueness of solutions are proved in the Colombeau algebra setting and the notion of compatibility of solutions is explored.

1 Introduction

We will analyze the following equations in three dimensions. First, we consider the defocusing cubic Schrödinger equation

$$\begin{aligned} iu_t + \triangle u &= u|u|^2, \\ u(0) &= a, \end{aligned} \tag{1}$$

and then the cubic equation with the delta potential

$$\begin{aligned} iu_t + \triangle u &= u|u|^2 + \delta u, \\ u(0) &= a. \end{aligned} \tag{2}$$

Equation (1) is extensively studied in the classical sense. Applications of (1) are connected with many physical contents such as dynamics of Bose gas, optics, and superfluids.

Well-posedness in Sobolev spaces, and in particular in the energy space $H^1(\mathbb{R}^3)$, is developed in [6] and [8]. Also, it was proved in [11] that global solutions exist in $H^s(\mathbb{R}^3)$ for $s > \frac{4}{5}$. We are interested in initial data which are more singular.

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Equation (2) is a model for Bose–Einstein condensates where δ is used to describe a local, short-range potential applied to a condensate. In [17] solutions in weak L^p spaces in one dimension are considered.

We will analyze Eqs. (1) and (2) within the Colombeau algebra setting and for that purpose different spaces of distributions will be embedded in the Colombeau algebra.

We are interested in regularized equations. For instance, the regularized equation for (2) is of the form

$$i(u_\varepsilon)_t + \triangle u_\varepsilon = u_\varepsilon |u_\varepsilon|^2 + \phi_\varepsilon u_\varepsilon,$$
$$u_\varepsilon(0) = a_\varepsilon,$$

for appropriate nets of functions $(u_\varepsilon)_\varepsilon$, $(a_\varepsilon)_\varepsilon$, and $(\phi_\varepsilon)_\varepsilon$ which we will call moderate functions.

Important properties that hold for this equation and that will be used are conservation of charge and energy:

$$\|u_\varepsilon(t)\|_2 = \|a_\varepsilon\|_2,$$
$$H(u_\varepsilon(t)) = H(a_\varepsilon),$$

where $H(u_\varepsilon) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} |u_\varepsilon|^4 dx + \frac{1}{2}\int_{\mathbb{R}^3} \phi_\varepsilon |u_\varepsilon|^2 dx$ is the Hamiltonian. Also, for fixed $\varepsilon > 0$ there is well-posedness in $H^s(\mathbb{R}^3)$ for $s \geq 2$.

The chapter is organized as follows. First, we introduce Colombeau algebras and describe their basic properties and prove theorems which explain how we embed different spaces of distributions into these algebras. The notion of a solution in the sense of Colombeau algebras is also introduced and we define the existence and uniqueness of solutions within this setting. Then we define compatibility between classical solutions and Colombeau solutions and further prove the existence and uniqueness of solutions to Eqs. (1) and (2). We conclude the chapter by analyzing some convergence properties and by giving directions for possible further investigations.

We shortly describe the notation. By $\mathcal{D}(\mathbb{R}^3)$ we denote the space of smooth compactly supported functions $f : \mathbb{R}^3 \to \mathbb{C}$ equipped with the finest locally convex topology for which all the inclusions $\mathcal{D}(K) \hookrightarrow \mathcal{D}(\Omega)$ are continuous (K is an arbitrary compact subset of Ω). Also, $H^s = H^s(\mathbb{R}^3)$, $s \in \mathbb{R}$ is the usual Sobolev space. We say that $f(\varepsilon) \sim g(\varepsilon)$ if $\lim_{\varepsilon\to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = c > 0$. Further, $f(\varepsilon) \lesssim g(\varepsilon)$ if there exists $c > 0$ independent of ε such that $f(\varepsilon) \leq cg(\varepsilon)$. We also use some well-known inequalities, namely Hölder, Young, Gronwall, and Gagliardo–Nirenberg inequalities.

Results presented in this chapter are based on papers [15] and [16].

2 The Colombeau Algebra

In this section we introduce the algebras of Colombeau (see [12, 13]). We construct them as factor algebras of the so-called moderate functions modulo a class of ideals that we call negligible functions, which will be described in the sequel.

In certain examples of partial differential equations with singular coefficients or singular data we need to multiply distributions. For instance, delta waves occur in the analysis of semilinear hyperbolic systems with rough initial data. Many examples of problems (related to elasticity, acoustics, fluid dynamics) where the multiplication of distributions occurs are given in [14] and [25].

However, multiplication of distributions is connected with many difficulties. The product of a smooth function and a distribution is well-defined, but if we try to extend the operation of multiplication to arbitrary distributions we are not able to preserve the associative property:

$$0 = (\delta(x) \cdot x) \cdot vp\frac{1}{x} \neq \delta(x) \cdot (x \cdot vp\frac{1}{x}) = \delta(x),$$

where $vp\frac{1}{x}$ denotes the Cauchy principal value of $\frac{1}{x}$. One possibility to overcome this problem is to embed the space of distributions in some algebra so that we can define a product.

If we denote this algebra by $(\mathcal{A}(\Omega), +, \cdot)$, where $\Omega \subset \mathbb{R}^3$ is an open set, then we would like that the algebra $\mathcal{A}(\Omega)$ satisfies following properties:

1. $\mathcal{D}'(\Omega)$ is linearly embedded into $\mathcal{A}(\Omega)$,
2. there exist differential operators $\partial_i : \mathcal{A}(\Omega) \to \mathcal{A}(\Omega)$, $i = 1, \dots, n$ that are linear and satisfy the Leibniz rule,
3. $\partial_i|_{\mathcal{D}'}$ is the usual partial derivative, $i = 1, \dots, n$,
4. the restriction $\cdot|_{C^\infty \times C^\infty}$ coincides with the pointwise product of functions.

One example of $(\mathcal{A}(\Omega), +, \cdot)$ is the following special Colombeau algebra which we define in the sequel (for details see [19]). We introduce spaces :

$$\mathcal{E}^s(\Omega) := (C^\infty(\Omega))^{(0,1]},$$

$$\mathcal{E}^s_M(\Omega) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}^s(\Omega) \mid \forall K \subset\subset \Omega \; \forall \alpha \in \mathbb{N}_0^n \; \exists N \in \mathbb{N} \text{ with } \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{-N}), \; \varepsilon \to 0\},$$

$$\mathcal{N}^s(\Omega) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}^s(\Omega) \mid \forall K \subset\subset \Omega \; \forall \alpha \in \mathbb{N}_0^n \; \forall m \in \mathbb{N} \text{ with } \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}(\varepsilon^{m}), \; \varepsilon \to 0\}.$$

Here $K \subset\subset \Omega$ means that K is a compact subset of Ω. Elements of $\mathcal{E}^s_M(\Omega)$ are called moderate functions and elements of $\mathcal{N}^s(\Omega)$ are called negligible functions. The special Colombeau algebra is defined as the quotient space

$$\mathcal{G}^s(\Omega) := \mathcal{E}^s_M(\Omega)/\mathcal{N}^s(\Omega).$$

In the sequel we assume that $n = 3$, unless otherwise stated. The embedding of the space of distributions $\mathcal{D}'(\Omega) \hookrightarrow \mathcal{G}^s(\Omega)$ is given by

$$u \mapsto [(u * \rho_\varepsilon)_\varepsilon],$$

where $\rho \in S(\mathbb{R}^3)$ is a mollifier which satisfies conditions

$$\int \rho(x)dx = 1, \tag{3}$$

$$\int x^\alpha \rho(x)dx = 0, \quad \forall|\alpha| \geq 1, \tag{4}$$

and $\rho_\varepsilon(x) = \varepsilon^{-3}\rho\left(\frac{x}{\varepsilon}\right)$. One can prove that there is no mollifier in $\mathcal{D}(\mathbb{R}^3)$ which satisfies both (3) and (4). However, $\rho \in S(\mathbb{R}^3)$ can be constructed by taking the inverse Fourier transform of a function from $S(\mathbb{R}^3)$ which equals one in a neighborhood of zero.

Next we define the H^2-based Colombeau algebra as in [24] (for a similar construction see [23]). This type of algebra is appropriate for the equations that we consider.

We denote by $\mathcal{E}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ (respectively, $\mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$), $T > 0$ the vector space of nets $(u_\varepsilon)_\varepsilon$ of functions

$$u_\varepsilon \in C([0,T), H^2(\mathbb{R}^3)) \cap C^1([0,T), L^2(\mathbb{R}^3)),\ \varepsilon \in (0,1),$$

such that there exists $N \in \mathbb{N}$ (respectively, for every $M \in \mathbb{N}$):

$$\max\{\sup_{t\in[0,T)} \|u_\varepsilon(t)\|_{H^2}, \sup_{t\in[0,T)} \|\partial_t u_\varepsilon(t)\|_2\} = \mathcal{O}(\varepsilon^{-N}),\ \varepsilon \to 0$$

$$\Bigg(\text{respectively,}$$

$$\max\{\sup_{t\in[0,T)} \|u_\varepsilon(t)\|_{H^2}, \sup_{t\in[0,T)} \|\partial_t u_\varepsilon(t)\|_2\} = \mathcal{O}(\varepsilon^{M}),\ \varepsilon \to 0\Bigg).$$

Then we define the quotient space

$$\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3) = \mathcal{E}_{C^1,H^2}([0,T) \times \mathbb{R}^3)/\mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$$

which is a Colombeau type vector space.

We also define the space $\mathcal{G}_{H^2}(\mathbb{R}^3)$ in a similar manner:

$$\mathcal{E}^2(\mathbb{R}^3) := (H^2(\mathbb{R}^3))^{(0,1]},$$
$$\mathcal{E}_{H^2}(\mathbb{R}^3) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}^2(\mathbb{R}^3) \mid \exists N \in \mathbb{N}\ \|u_\varepsilon\|_{H^2} = O(\varepsilon^{-N}),\ \varepsilon \to 0\},$$
$$\mathcal{N}_{H^2}(\mathbb{R}^3) := \{(u_\varepsilon)_\varepsilon \in \mathcal{E}^2(\mathbb{R}^3) \mid \forall m \in \mathbb{N}\ \|u_\varepsilon\|_{H^2} = O(\varepsilon^{m}),\ \varepsilon \to 0\},$$
$$\mathcal{G}_{H^2}(\mathbb{R}^3) := \mathcal{E}_{H^2}(\mathbb{R}^3)/\mathcal{N}_{H^2}(\mathbb{R}^3).$$

Operations of addition, multiplication, and differentiation are defined component-wise, that is

$$u + v = [(u_\varepsilon + v_\varepsilon)_\varepsilon], \quad u \cdot v = [(u_\varepsilon \cdot v_\varepsilon)_\varepsilon], \quad \partial^\alpha u = [(\partial^\alpha u_\varepsilon)_\varepsilon].$$

Differentiation on H^2-based algebra is not a closed operation. If $u \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$, then $\partial^\alpha u$ for $|\alpha| \le 2$ is represented by $(\partial^\alpha u_\varepsilon)_\varepsilon$ which has moderate growth in $L^2(\mathbb{R}^3)$ and is an element of a quotient vector space $\mathcal{G}_{C,L^2}([0,T) \times \mathbb{R}^3)$. The vector space $\mathcal{G}_{C,L^2}([0,T) \times \mathbb{R}^3)$ is defined analogously as $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$. Difference is that representatives have bounded growth only in L^2-norm, for any $t \in [0,T)$. It is clear that $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3) \subset \mathcal{G}_{C,L^2}([0,T) \times \mathbb{R}^3)$.

Notice that spaces $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ and $\mathcal{G}_{H^2}(\mathbb{R}^3)$ are multiplicative algebras because $H^2(\mathbb{R}^3)$ is an algebra (the same holds for $\mathbb{R}^n$ when $n \le 3$).

Since $\delta * \rho_\varepsilon = \rho_\varepsilon$ it is clear that $(\rho_\varepsilon)_\varepsilon$ itself is a representative of the delta distribution. Here ρ_ε is given by (3) and (4).

Next we define a *strict delta net* because another representative of the delta distribution is given by this type of net (cf. [19]).

Definition 1 A strict delta net is a net $(\phi_\varepsilon)_{0<\varepsilon\le 1}$, $\phi_\varepsilon \in \mathcal{D}(\mathbb{R}^3)$ which satisfies

(i) $\operatorname{supp}(\phi_\varepsilon) \to \{0\},\ \varepsilon \to 0$,
(ii) $\lim_{\varepsilon\to 0} \int_{\mathbb{R}^3} \phi_\varepsilon(x)dx = 1$,
(iii) $\int |\phi_\varepsilon(x)|dx$ is bounded uniformly in ε.

We can define a strict delta net using ρ_ε as $\phi_\varepsilon(x) = \chi(\frac{x}{\sqrt{\varepsilon}})\rho_\varepsilon(x)$, where χ is a cut-off function and $\rho_\varepsilon \in \mathcal{S}(\mathbb{R}^3)$ is a mollifier defined by (3) and (4). More precisely,

$$\chi \in \mathcal{D}(\mathbb{R}^3), \quad \chi(x) = 1,\ |x| \le 1 \text{ and } \chi(x) = 0, \quad |x| \ge 2. \tag{5}$$

Since $\mathcal{S}(\mathbb{R}^3) \subset L^p(\mathbb{R}^3)$ the following estimates for ρ_ε and ϕ_ε hold:

$$\begin{aligned}\|\partial^\alpha \rho_\varepsilon\|_p^p = \int_{\mathbb{R}^3} \varepsilon^{-3p} |\partial^\alpha(\rho(\frac{x}{\varepsilon}))|^p dx = \int_{\mathbb{R}^3} \varepsilon^{-3p} |\frac{1}{\varepsilon^{|\alpha|}}(\partial^\alpha \rho)(\frac{x}{\varepsilon})|^p dx = \\ \int_{\mathbb{R}^3} \varepsilon^{-3p+3-|\alpha|p} |\partial^\alpha \rho(t)|^p dt = c\varepsilon^{3(1-p)-|\alpha|p} \lesssim \varepsilon^{-N},\end{aligned} \tag{6}$$

for some $N \in \mathbb{N}$, $1 \leq p < \infty$ and for any multi-index α. Moreover, $\|\rho_\varepsilon\|_\infty = \varepsilon^{-n} \max |\rho(\frac{x}{\varepsilon})| = c\varepsilon^{-n}$, for any $\varepsilon > 0$.
We also use mollifiers of the type $\rho_{h_\varepsilon} = h_\varepsilon^3 \rho(x h_\varepsilon)$, where $h_\varepsilon \to \infty$, $\varepsilon \to 0$, for example, $h_\varepsilon = \ln \varepsilon^{-1}$, and these mollifiers satisfy analogous estimates.

Furthermore, we derive estimates for $\partial^\alpha(\chi(\frac{x}{\sqrt{\varepsilon}}))$, that is

$$\sup_{x \in \mathbb{R}^3} |\varepsilon^{-|\alpha|/2} (\partial^\alpha \chi)(\frac{x}{\sqrt{\varepsilon}})| \lesssim \varepsilon^{-|\alpha|/2}.$$

Therefore $\phi_\varepsilon(x) = \chi(\frac{x}{\sqrt{\varepsilon}})\rho_\varepsilon(x)$ admits analogous estimates as ρ_ε in the L^p-norm.

Now we prove that we can use a strict delta net to embed delta distribution in $\mathcal{G}_{H^2}(\mathbb{R}^3)$.

Theorem 1 *There exists a strict delta net $(\phi_\varepsilon)_{0<\varepsilon\leq 1}$ such that the difference $(\rho_\varepsilon - \phi_\varepsilon)_\varepsilon$ is an element of $\mathcal{N}_{H^2}(\mathbb{R}^3)$. Both $(\rho_\varepsilon)_\varepsilon$ and $(\phi_\varepsilon)_\varepsilon$ are representatives for the embedded delta distribution $[(\rho_\varepsilon)_\varepsilon] \in \mathcal{G}_{H^2}(\mathbb{R}^3)$.*

Proof Let $\phi_\varepsilon(x) = \chi_\varepsilon(x)\rho_\varepsilon(x) = \chi(\frac{x}{\sqrt{\varepsilon}})\rho_\varepsilon(x)$, where χ is given by (5). Since $\rho_\varepsilon \in \mathcal{S}(\mathbb{R}^3)$ for any $q > 2$ it holds that

$$\begin{aligned}
\|\rho_\varepsilon - \rho_\varepsilon\chi_\varepsilon\|_2^2 &= \int_{\mathbb{R}^3} \rho_\varepsilon^2(x)(1-\chi(\frac{x}{\sqrt{\varepsilon}}))^2 dx \leq \int_{|x|>\sqrt{\varepsilon}} \rho_\varepsilon^2(x) dx \\
&\leq \int_{|x|>\sqrt{\varepsilon}} \varepsilon^{-6}(1+|\frac{x}{\varepsilon}|)^{-2q} dx = \int_{|x|>\sqrt{\varepsilon}} \varepsilon^{-6}(1+\frac{|x|}{\varepsilon})^{-2q+3+1-(3+1)} dx \\
&\leq \varepsilon^{-6} \sup_{x>\sqrt{\varepsilon}} (1+\frac{|x|}{\varepsilon})^{-2q+3+1} \int_{|x|>\sqrt{\varepsilon}} (1+\frac{|x|}{\varepsilon})^{-(3+1)} dx \\
&\leq \varepsilon^{-6}\varepsilon^{q-(3+1)/2}\varepsilon^3 \int_{|y|>1/\sqrt{\varepsilon}} \frac{1}{(1+|y|)^{3+1}} dy \\
&\leq \varepsilon^{q-(3+1)/2-3} \int_{y\in\mathbb{R}^3} \frac{1}{(1+|y|)^{3+1}} dy.
\end{aligned}$$

The above integral is finite and independent of ε. Hence for arbitrary $m \in \mathbb{N}$ we choose $q = m + \frac{10}{2}$ (then $q > 2$) and

$$\|\rho_\varepsilon - \rho_\varepsilon\chi_\varepsilon\|_2^2 \lesssim \varepsilon^m, \quad 0 < \varepsilon \leq 1.$$

Next we need to bound derivatives $\partial^\alpha(\rho_\varepsilon - \rho_\varepsilon\chi_\varepsilon)$ in the L^2-norm, for $|\alpha| = 1$ and $|\alpha| = 2$. This can be done similarly as in the first part of the proof using that the function $1-\chi$ is equal to zero for $|x| \leq \sqrt{\varepsilon}$ and derivatives of the function $1-\chi$ are supported in the set $\sqrt{\varepsilon} \leq |x| \leq 2\sqrt{\varepsilon}$. □

Next we use mollifier ρ_ε given by (3) and (4) to represent functions from $H^2(\mathbb{R}^3)$ as an elements of $\mathcal{G}_{H^2}(\mathbb{R}^3)$.

Theorem 2 *Let $f \in H^2(\mathbb{R}^3)$. Then we can embed $H^2(\mathbb{R}^3)$ into $\mathcal{G}_{H^2}(\mathbb{R}^3)$ such that $f \mapsto [(f * \rho_\varepsilon)_\varepsilon]$.*

Proof For any $|\alpha| \leq 2$ using Young's inequality we have that

$$\|\partial^\alpha (f * \rho_\varepsilon)\|_2 = \|f * \partial^\alpha \rho_\varepsilon\|_2 \leq \|f\|_2 \|\partial^\alpha \rho_\varepsilon\|_1 \lesssim \varepsilon^{-N}$$

for some $N \in \mathbb{N}$, where we use estimates as in (6).

Hence $f_\varepsilon = f * \rho_\varepsilon$ defines an element $[(f_\varepsilon)_\varepsilon] \in \mathcal{G}_{H^2}(\mathbb{R}^3)$. We also know that $\|f * \phi_\varepsilon - f\|_2 \to 0$. Hence the mapping $f \mapsto [(f_\varepsilon)_\varepsilon]$ is injective. More concretely, if $v \in H^2(\mathbb{R}^3)$ is another function embedded in $\mathcal{G}_{H^2}(\mathbb{R}^3)$ using convolution with ρ_ε, then $(v_\varepsilon)_\varepsilon \in [(f_\varepsilon)_\varepsilon]$ (here $v_\varepsilon = v * \rho_\varepsilon$) and

$$v = \lim_{\varepsilon \to 0} v_\varepsilon = \lim_{\varepsilon \to 0} (f_\varepsilon + n_\varepsilon) = f$$

in $L^2(\mathbb{R}^3)$, where $v_\varepsilon = f_\varepsilon + n_\varepsilon$ and $(n_\varepsilon)_\varepsilon \in \mathcal{N}_{H^2}(\mathbb{R}^3)$. Therefore

$$H^2(\mathbb{R}^3) \hookrightarrow \mathcal{G}_{H^2}(\mathbb{R}^3),$$

what we wanted to prove. □

Another representative of elements from $H^2(\mathbb{R}^3)$ is obtained using a strict delta net.

Theorem 3 *Let $f \in H^2(\mathbb{R}^3)$. Then $f * \rho_\varepsilon - f * \phi_\varepsilon \in \mathcal{N}_{H^2}(\mathbb{R}^3)$, where ϕ_ε is a strict delta net defined by $\phi_\varepsilon = \chi_\varepsilon \rho_\varepsilon$, $\chi_\varepsilon(x) = \chi(\frac{x}{\sqrt{\varepsilon}})$ and χ is a cut-off function as in (5).*

Proof From Young's inequality we have that

$$\|f * (\rho_\varepsilon - \phi_\varepsilon)\|_2 \lesssim \|f\|_2 \|(1 - \chi_\varepsilon)\rho_\varepsilon\|_1.$$

We can estimate $\|(1 - \chi_\varepsilon)\rho_\varepsilon\|_1 \lesssim \varepsilon^m$ for any $m \in \mathbb{N}$, $\varepsilon \to 0$ similarly as in the proof of Theorem 1. Also, $\partial^\alpha(f * (\rho_\varepsilon - \phi_\varepsilon)) = (\partial^\alpha f) * (\rho_\varepsilon - \phi_\varepsilon)$ and therefore the proof follows. □

Further we prove that the product of the delta distribution and an element from $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ remains in $\mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$.

Theorem 4 *Let $u \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ and ρ_ε is the representative of δ in $\mathcal{G}_{H^2}(\mathbb{R}^3)$. Then $u \cdot [(\rho_\varepsilon)_\varepsilon] \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$.*

Proof Let $u_\varepsilon \in \mathcal{E}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$. We have

$$\|u_\varepsilon \rho_\varepsilon\|_2 \lesssim \|\rho_\varepsilon\|_\infty \|u_\varepsilon(t)\|_2 \lesssim \varepsilon^{-N}, \quad \varepsilon \to 0,$$

for any $t \in [0, T)$. Similar estimates can be derived for $\partial^\alpha(u_\varepsilon \rho_\varepsilon)$, $|\alpha| \leq 2$. In this case we have expressions of form $\partial^\beta u_\varepsilon \partial^\delta \rho_\varepsilon$, $|\beta|, |\delta| \leq 2$, which can be bounded by ε^{-N}, $\varepsilon \to 0$, for some N.

Now let $(v_\varepsilon)_\varepsilon$ be another representative of u and $(\rho_\varepsilon^1)_\varepsilon$ be another representative of δ. Then $\rho_\varepsilon^1 = \rho_\varepsilon + n_\varepsilon^1$ for $n_\varepsilon^1 \in \mathcal{N}_{H^2}(\mathbb{R}^3)$ and $v_\varepsilon = u_\varepsilon + n_\varepsilon^2$ for $n_\varepsilon^2 \in \mathcal{N}_{C^1,H^2}([0, T) \times \mathbb{R}^3)$. Then $u_\varepsilon \rho_\varepsilon - v_\varepsilon \rho_\varepsilon^1 \in \mathcal{N}_{C^1,H^2}([0, T) \times \mathbb{R}^3)$.

Indeed, product of n_ε^1 and n_ε^2 is negligible in $\mathcal{G}_{C^1,H^2}([0, T) \times \mathbb{R}^3)$ and also $u_\varepsilon \cdot n_\varepsilon^2 \in \mathcal{N}_{C^1,H^2}([0, T) \times \mathbb{R}^3)$, $\rho_\varepsilon \cdot n_\varepsilon^1 \in \mathcal{N}_{C^1,H^2}([0, T) \times \mathbb{R}^3)$, where we use that $H^2(\mathbb{R}^3)$ is an algebra. Hence the product is well-defined. □

We also need to define a restriction of an element $u \in \mathcal{G}_{C^1,H^2}([0, T) \times \mathbb{R}^3)$ since the initial condition is a function that depends only on x.

Definition 2 Let $u = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{C^1,H^2}([0, T) \times \mathbb{R}^3)$. We define the restriction of u to $\{0\} \times \mathbb{R}^3$ as the class $[(u_\varepsilon(0, \cdot))_\varepsilon] \in \mathcal{G}_{H^2}(\mathbb{R}^3)$.

Definition 2 makes sense. Indeed, since $u_\varepsilon \in C([0, T), H^2(\mathbb{R}^3))$, the function $u_\varepsilon(0, \cdot)$ is in $\mathcal{E}_{H^2}(\mathbb{R}^3)$. Also, if $u_\varepsilon \in \mathcal{N}_{C^1,H^2}([0, T), H^2(\mathbb{R}^3))$, then $u_\varepsilon(0, \cdot)$ is in $\mathcal{N}_{H^2}(\mathbb{R}^3)$.

We will also need the definition of an initial condition which is of $(\ln)^j$-type.

Definition 3 We say that $a \in \mathcal{G}_{H^2}(\mathbb{R}^3)$ is of $(\ln)^j$-type, $j \in (0, 1]$ if it has a representative $a_\varepsilon \in \mathcal{E}_{H^2}(\mathbb{R}^3)$ such that

$$\|a_\varepsilon\|_2 = \mathcal{O}(\ln^j \varepsilon^{-1}), \quad \varepsilon \to 0.$$

Note that a function $a \in H^\infty(\mathbb{R}^3)$ is itself a representative in $\mathcal{G}_{H^2}(\mathbb{R}^3)$ (which will be proved in Theorem 5 in the sequel). This is an example of a function that is of $(\ln)^j$-type for any $j \in (0, 1]$ since its L^2–norm is a constant independent of ε. Similarly holds for $a \in L^2(\mathbb{R}^3)$.

Theorem 5 *If $a \in H^\infty(\mathbb{R}^3)$, then $[(a)_\varepsilon] \in \mathcal{G}_{H^2}(\mathbb{R}^3)$.*

Proof We need to show that $(a_\varepsilon - a)_\varepsilon \in \mathcal{N}_{H^2}(\mathbb{R}^3)$, where $a_\varepsilon = a * \rho_\varepsilon$ and ρ_ε is given by (3) and (4).

We follow ideas given in [3]. It holds that

$$\|a_\varepsilon - a\|_2^2 = \|a * \rho_\varepsilon - a\|_2^2 = \int \left| \int (a(x - \varepsilon y) - a(x))\rho(y)dy \right|^2 dx.$$

We can apply Taylor's formula to a up to order m. Since $\int y^\alpha \rho(y)dy = 0$ for $|\alpha| \leq m$ (by (4)) it follows that

$$\|a_\varepsilon - a\|_2^2 = \int | \sum_{|\alpha|=m+1} \int \frac{(-\varepsilon y)^{\alpha})}{m!} \int_0^1 (1-\sigma)^m \partial^\alpha a(x - \sigma\varepsilon y) d\sigma \rho(y)dy|^2 dx$$

$$\leq C(m,q)\max_{|\alpha|=m+1}\int\left|\int\frac{(-\varepsilon y)^{\alpha})}{m!}\rho(y)\int_0^1(1-\sigma)^m\partial^\alpha a(x-\sigma\varepsilon y)d\sigma dy\right|^2 dx$$
$$\leq C(m,q)\max_{|\alpha|=m+1}\int\int\left|\frac{(\varepsilon y)^{\alpha})}{m!}\rho(y)\int_0^1(1-\sigma)^m\partial^\alpha a(x-\sigma\varepsilon y)d\sigma\right|^2 dxdy$$
$$\leq\frac{\varepsilon^{m+1}}{m!}C(m,q)\max_{|\alpha|=m+1}\int|y^\alpha\rho(y)|\int\int_0^1|\partial^\alpha a(y-\sigma\varepsilon y)|^2 d\sigma dxdy$$
$$\leq c\varepsilon^{m+1}\int|y|^{m+1}|\rho(y)|dy\max_{|\alpha|=m+1}\|\partial^\alpha a\|_2.$$

Hence for any $m\in\mathbb{N}$ and sufficiently small ε we have

$$\|u_\varepsilon-u\|_2\leq c\varepsilon^m.$$

The same estimates hold for $\partial^\alpha(a_\varepsilon-a)$, $|\alpha|\leq 2$. □

2.1 Notion of Colombeau Solution

Let us consider the following Schrödinger equation:

$$\begin{aligned} iu_t+\triangle u+g(u)&=0,\\ u(0)&=a,\end{aligned}\tag{7}$$

where $g(u)$ is given nonlinearity. Next we define the existence of a solution in the Colombeau sense.

Definition 4 We say that $u\in\mathcal{G}_{C^1,H^2}([0,T)\times\mathbb{R}^3)$ is a solution of (7) if for an initial condition a and its representative $a_\varepsilon=a*\rho_\varepsilon$, there exists a representative $(u_\varepsilon)_\varepsilon\in\mathcal{E}_{C^1,H^2}([0,T)\times\mathbb{R}^3)$ such that

$$\begin{aligned} i(u_\varepsilon)_t+\triangle u_\varepsilon+g(u_\varepsilon)&=M_\varepsilon,\\ u_\varepsilon(0)&=a_\varepsilon+n_\varepsilon,\end{aligned}\tag{8}$$

for some $n_\varepsilon\in\mathcal{N}_{H^2}(\mathbb{R}^3)$ and we assume that $\sup_{t\in[0,T)}\|M_\varepsilon\|_2=\mathcal{O}(\varepsilon^M)$, $\varepsilon\to 0$, for any $M\in\mathbb{N}$.

If the above statement holds for some u_ε, then it holds for all representatives of the class $u=[(u_\varepsilon)_\varepsilon]$. We will show that for $g(u_\varepsilon)=-(u_\varepsilon|u_\varepsilon|^2+\phi_\varepsilon u_\varepsilon)$.

Let $v_\varepsilon = u_\varepsilon + N_\varepsilon$, for some $N_\varepsilon \in \mathcal{N}_{C^1,H^2}(\mathbb{R}^3)$. Then

$$\begin{aligned} i(v_\varepsilon)_t + \triangle v_\varepsilon - (v_\varepsilon|v_\varepsilon|^2 + \phi_\varepsilon v_\varepsilon) &= i(u_\varepsilon)_t + \triangle u_\varepsilon - u_\varepsilon|u_\varepsilon|^2 - \phi_\varepsilon u_\varepsilon \\ &\quad + i(N_\varepsilon)_t + \triangle N_\varepsilon - f(u_\varepsilon, N_\varepsilon, \phi_\varepsilon) \\ &= M_\varepsilon + i(N_\varepsilon)_t + \triangle N_\varepsilon - f(u_\varepsilon, N_\varepsilon, \phi_\varepsilon), \end{aligned}$$

where $\sup_{0\le t<T} \|M_\varepsilon\|_2 = \mathcal{O}(\varepsilon^M)$, $\varepsilon \to 0$, for any $M \in \mathbb{N}$ and

$$f(u_\varepsilon, N_\varepsilon, \phi_\varepsilon) = \overline{N_\varepsilon} u_\varepsilon^2 + 2|u_\varepsilon|^2 N_\varepsilon + 2u_\varepsilon|N_\varepsilon|^2 + \overline{u}_\varepsilon N_\varepsilon^2 + N_\varepsilon|N_\varepsilon|^2 + \phi_\varepsilon N_\varepsilon.$$

Since $N_\varepsilon \in \mathcal{N}_{C^1,H^2}(\mathbb{R}^3)$, it holds that $\|i(N_\varepsilon)_t + \Delta N_\varepsilon\|_2 = \mathcal{O}(\varepsilon^M)$, $\varepsilon \to 0$, for any $M \in \mathbb{N}$. Also, using the Sobolev embedding $\|N_\varepsilon\|_\infty \le c\|N_\varepsilon\|_{H^2}$ we see that $\sup_{0\le t<T} \|f(u_\varepsilon, N_\varepsilon, \phi_\varepsilon)\|_2 = \mathcal{O}(\varepsilon^M)$, $\varepsilon \to 0$. Furthermore,

$$v_\varepsilon(0) = u_\varepsilon(0) + N_\varepsilon(0) = a_\varepsilon + n_\varepsilon + N_\varepsilon(0) = a_\varepsilon + N_\varepsilon^1,$$

where $N_\varepsilon^1 \in \mathcal{N}_{H^2}(\mathbb{R}^3)$. Therefore v_ε satisfies all the conditions from Definition 4.

When we want to prove the existence of a solution in the Colombeau sense, usually we first solve

$$\begin{aligned} i(u_\varepsilon)_t + \triangle u_\varepsilon + g(u_\varepsilon) &= 0, \\ u_\varepsilon(0) &= a_\varepsilon, \end{aligned}$$

where $a_\varepsilon = a * \rho_\varepsilon$ and then the previous analysis implies that $[(u_\varepsilon)_\varepsilon]$ is indeed a solution.

Definition 5 We say that the solution of (7) is unique if for any two solutions $u, v \in \mathcal{G}_{C^1,H^2}$ it holds $\sup_{t\in[0,T)} \|u_\varepsilon - v_\varepsilon\|_2 = O(\varepsilon^M)$, $\varepsilon \to 0$, for any $M \in \mathbb{N}$. Here $u = [(u_\varepsilon)_\varepsilon]$ and $v = [(v_\varepsilon)_\varepsilon]$.

2.2 Compatibility

If $a \in H^2(\mathbb{R}^3)$, then there exists a unique solution $u \in C([0,T), H^2(\mathbb{R}^3))$ of the cubic equation (1). We proved that the space $H^2(\mathbb{R}^3)$ is embedded in the Colombeau algebra $\mathcal{G}_{H^2}(\mathbb{R}^3)$ (Theorem 2). If there is a unique solution of (1) in $\mathcal{G}_{C^1,H^2}([0,T)\times \mathbb{R}^3)$, then there is its representative $(u_\varepsilon)_\varepsilon$ which solves

$$\begin{aligned} i(u_\varepsilon)_t + \Delta u_\varepsilon &= u_\varepsilon|u_\varepsilon|^2, \\ u_\varepsilon(0) &= a * \rho_\varepsilon, \end{aligned}$$

for $a \in H^2(\mathbb{R}^3)$ (we show that there is a solution to the equation without negligible functions, so the above claim is justified). Classes $[(u_\varepsilon)_\varepsilon]$ and $[(u * \rho_\varepsilon)_\varepsilon]$ may coincide but in general we can prove a weaker version of this equality of classes, which we give in the next definition (see [19], p. 47).

Definition 6 We say that $u \in \mathcal{G}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ is associated with a distribution $v(t) \in \mathcal{D}'(\mathbb{R}^3)$ for any $t \in [0,T)$ if there is a representative $(u_\varepsilon)_\varepsilon$ of u such that $u_\varepsilon \to v$ in $\mathcal{D}'(\mathbb{R}^3)$ for any $t \in [0,T)$ as $\varepsilon \to 0$. We denote association by $u \approx v$.

However, we are sometimes able to prove $\|u - u_\varepsilon\|_2 \to 0,\ \varepsilon \to 0$, for every $t \in [0,T)$ and this implies $[(u_\varepsilon)_\varepsilon] \approx u$. Therefore we introduce the following definition.

Definition 7 We say that there is a compatibility between a classical (Sobolev) solution and the Colombeau solution of the equation

$$iu_t + \triangle u + g(u) = 0,$$
$$u(0) = a,$$

if $\sup_{[0,T)} \|u_\varepsilon - u\|_2 \to 0$ as $\varepsilon \to 0$, where $(u_\varepsilon)_\varepsilon \in \mathcal{E}_{C^1,H^2}$ is a solution of

$$i(u_\varepsilon)_t + \triangle u_\varepsilon + g(u_\varepsilon) = 0$$
$$u_\varepsilon(0) = a * \rho_\varepsilon.$$

This definition does not depend on representatives. If $u_\varepsilon \to u$ in L^2 and v_ε is another representative, then

$$\|v_\varepsilon - u\|_2 \le \|v_\varepsilon - u_\varepsilon\|_2 + \|u_\varepsilon - u\|_2 \to 0,\ \ \varepsilon \to 0.$$

Note that if $a \in C^1([0,T), H^\infty)$, then a represents itself and the same holds for the corresponding solution $u \in C^1([0,T), H^\infty)$. Hence in this case we automatically have compatibility between two solutions.

Looking outside the context of equivalence classes, estimates that we derive can be useful for discussing different types of convergences. For instance, there is no classical well-posedness theory for (2), but we can analyze the net of solutions and get some insights in that direction. Uniqueness in our setting also differs from the usual notion of uniqueness. Because we define an L^2-type of uniqueness, it can happen that the solution is unique but there are different classes $u,\ v \in \mathcal{G}_{C^1,H^2}$ that solve the equation. Again, if there is convergence in L^2 of representatives of u, then representatives of v also converge to the same limit. Hence notions of compatibility and uniqueness complement each other.

We state a few examples in which notions of compatibility and association were used. In [21], Hörmann showed that there is a unique generalized solution to the linear Schrödinger equation with generalized coefficients and also that this solution is associated with the corresponding distributional solution.

In [3], the generalized solution of the Korteweg-de Vries equation is considered and an interesting result is observed. Namely, for classical initial data, the distribution associated with the generalized solution is not a weak solution of the equation.

Burger's equation is studied in [4] and association of the generalized solution with a classical entropy solution is shown. In [26], hyperbolic conservation laws are considered in the Colombeau framework and the authors prove that the generalized solution is associated with the weak entropic solution.

In some cases (such as ours), it is possible to prove more than association. For example, in [23], the H^2 convergence of regularized solutions is shown. In [24], weak L^2-convergence of the net of solutions is proven.

Non-uniqueness and instability are potential problems in analysis of distributional solutions (cf. [9, 20]). This is another reason to emphasize the importance of compatibility.

3 Existence and Uniqueness of a Singular Solution

We consider a regularized equation of type

$$\begin{aligned} i(u_\varepsilon)_t + \triangle u_\varepsilon &= N(u_\varepsilon), \\ u_\varepsilon(0) &= a_\varepsilon, \end{aligned}$$

where $(a_\varepsilon)_\varepsilon$ is a representative of $a \in \mathcal{G}_{H^2}(\mathbb{R}^3)$ and N is the given nonlinearity. Existence of a unique solution for fixed ε follows from the classical theory of Sobolev solutions. The main ingredient for existence in Colombeau algebra is deriving the estimates of the type

$$\|u_\varepsilon\|_{H^2} \lesssim f(\|a_\varepsilon\|_{H^2}),$$

for any $t \in [0, T)$, since then $(u_\varepsilon)_\varepsilon$ defines an element of $\mathcal{G}_{C^1,H^2}([0, T) \times \mathbb{R}^3)$ and the Definition 4 is satisfied. Note that appropriate bounds for $\|(u_\varepsilon)_t\|_2$ are easily obtained from the equation itself.

Estimates for $\|u_\varepsilon\|_{H^1}$ follow from the conservation of energy and the main difficulty is to bound second order derivatives in the L^2 norm. Besides moderate growth, a usually needs to satisfy additional logarithmic bounds, as we will see in the sequel.

For the simpler cubic equation without potential we can claim existence in the Colombeau sense in dimensions $n = 2$ and $n = 3$.

Theorem 6 *Let $n \in \{2, 3\}$, $T > 0$, $a \in \mathcal{G}_{H^2}(\mathbb{R}^n)$ such that there exists a representative $(a_\varepsilon)_\varepsilon$ which satisfies condition*

$$\|a_\varepsilon\|_{H^2} \le h_\varepsilon \tag{9}$$

with $h_\varepsilon \sim \varepsilon^{-N}$ for $n = 2$ and $h_\varepsilon \sim N \ln \varepsilon^{-1}$ for $n = 3$, for some $N \in \mathbb{N}$. Then there exists a solution $u \in \mathcal{G}_{C^1,H^2}([0, T) \times \mathbb{R}^n)$ of (1).

The cubic equation in 3D satisfies a growth estimate proved in [5]:

$$\|u_\varepsilon(t)\|_{H^2} \le c \exp(\|a_\varepsilon\|_{H^2}) \quad \forall t \ge 0. \tag{10}$$

In [10] it was shown that there is also an estimate of the type:

$$\|u_\varepsilon(t)\|_{H^2} \le c\|a_\varepsilon\|_{H^2} \quad \forall t \ge 0. \tag{11}$$

Therefore proof of Theorem 6 follows from these bounds.

Proving an analogous theorem for Eq. (2) required deriving new estimates. This leads us to different conditions for initial data, presented in the following theorem.

Theorem 7 *Let $a \in \mathcal{G}_{H^2}(\mathbb{R}^3)$ such that there exists a representative $(a_\varepsilon)_\varepsilon$ which satisfies the following:*

$$\|a_\varepsilon\|_{H^3} = \mathcal{O}(\varepsilon^{-N}), \quad \text{and} \quad \|a_\varepsilon\|_{H^1} = \mathcal{O}(h_\varepsilon) \quad for\ some\ N \in \mathbb{N},\ \varepsilon \to 0, \tag{12}$$

where $h_\varepsilon \sim (\ln \varepsilon^{-1})^{\frac{5}{11}}$. Then for any $T > 0$ there exists a generalized solution $u \in \mathcal{G}_{C^1,H^2}([0, T) \times \mathbb{R}^3)$ of (2).

We will describe the main ingredients of the proof. In this case, we need estimates for the following regularized equation:

$$\begin{aligned} i(u_\varepsilon)_t + \triangle u_\varepsilon &= \phi_{h_\varepsilon} u_\varepsilon + u_\varepsilon |u_\varepsilon|^2, \\ u_\varepsilon(0) &= a_\varepsilon. \end{aligned} \tag{13}$$

For simplicity, we regularize the delta function with the same h_ε used to bound the initial condition. Denote by $\mathcal{T}(t)$ the usual Schrödinger evolution operator which satisfies an estimate:

$$\|\mathcal{T}(t)\phi\|_{L^p} \le (4\pi|t|)^{-n(\frac{1}{2}-\frac{1}{p})} \|\phi\|_{L^{p'}}, \quad \forall \phi \in L^{p'}. \tag{14}$$

The solution of (13) is given by Duhamel's formula:

$$u_\varepsilon(t) = \mathcal{T}(t)a_\varepsilon - i \int_0^t \mathcal{T}(t-s) \left(\phi_{h_\varepsilon} u_\varepsilon + u_\varepsilon |u_\varepsilon|^2\right) ds. \tag{15}$$

Estimates that we need can be described with the following steps.

- Differentiate (3)—take the second order derivative in x and apply the L^2 norm; the main expression to bound after this is the following:

$$\|u_\varepsilon^2\partial^\alpha u_\varepsilon + u_\varepsilon\partial^\beta u_\varepsilon\partial^\gamma u_\varepsilon\|_2 + \|\phi_{h_\varepsilon}\partial^\alpha u_\varepsilon + \partial^\beta\phi_{h_\varepsilon}\partial^\gamma u_\varepsilon + \partial^\alpha\phi_{h_\varepsilon}u_\varepsilon\|_2,$$

  where $|\alpha| = 2$ and $|\beta| = |\gamma| = 1$.
- We are able to bound each term by a product of a known quantity ($\|\phi_{h_\varepsilon}\|_p$, $\|u_\varepsilon\|_2$ or $\|u_\varepsilon\|_{H^1}$) and $\|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}}$; for this we used Hölder and Gagliardo–Nirenberg inequality.
- Moreover, we bound $\|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}}$ by $\|\partial^\alpha u_\varepsilon\|_{\frac{10}{7}}$ using the estimate (14) and further bound the $L^{\frac{10}{7}}$ norm with the $L^{\frac{10}{3}}$ norm by Hölder and Gagliardo–Nirenberg inequality.
- In this way, we are able to use Gronwall's inequality and bound $\|\partial^\alpha u_\varepsilon\|_{\frac{10}{3}}$ and by that $\|\partial^\alpha u_\varepsilon\|_2$ also.

The resulting estimate is exponential in $\|a_\varepsilon\|_{H^1}$ but not in higher norms of a_ε. Specifically,

$$\begin{aligned}\sup_{[0,T)}\|\partial^\alpha u_\varepsilon\|_2 \le{}& \|a_\varepsilon\|_{H^2} + g_\varepsilon f_\varepsilon^{\frac{3}{2}}\|a_\varepsilon\|_2^{\frac{1}{2}} + H(a_\varepsilon)^{\frac{1}{2}} g_\varepsilon^{\frac{20}{13}}\|a_\varepsilon\|_2^{\frac{6}{13}} \\ &+ \|\partial^\alpha\phi_{h_\varepsilon}\|_\infty\|a_\varepsilon\|_2 + H(a_\varepsilon)^{\frac{1}{2}}\|\partial^\beta\phi_{h_\varepsilon}\|_\infty + g_\varepsilon\|\phi_{h_\varepsilon}\|_5, \end{aligned} \tag{16}$$

where

$$\begin{aligned} f_\varepsilon &= c_1(a_\varepsilon, \phi_{h_\varepsilon})\cdot\exp(c_2(a_\varepsilon, \phi_{h_\varepsilon})), \\ c_1(a_\varepsilon, \phi_{h_\varepsilon}) &= \|a_\varepsilon\|_{H^2} + T^{\frac{2}{5}}(\|a_\varepsilon\|_2\|\partial^\gamma\phi_{h_\varepsilon}\|_5 + H(a_\varepsilon)^{\frac{1}{2}}\|\phi_{h_\varepsilon}\|_5), \\ c_2(a_\varepsilon, \phi_{h_\varepsilon}) &= T^{\frac{2}{5}}H(a_\varepsilon)^{\frac{1}{10}}\|a_\varepsilon\|_2^{\frac{9}{5}}; \\ g_\varepsilon &= (\|a_\varepsilon\|_{H^3} + c_4(a_\varepsilon, \phi_{h_\varepsilon}))\exp(c_3(a_\varepsilon, \phi_{h_\varepsilon})\cdot T^{\frac{2}{5}}), \\ c_3(a_\varepsilon, \phi_{h_\varepsilon}) &= H(a_\varepsilon)^{\frac{1}{10}}\|a_\varepsilon\|_2^{\frac{9}{5}} + \|\phi_{h_\varepsilon}\|_{\frac{5}{2}}, \\ c_4(a_\varepsilon, \phi_{h_\varepsilon}) &= H(a_\varepsilon)^{\frac{1}{2}}\|\partial^\beta\phi_{h_\varepsilon}\|_5 + \|a_\varepsilon\|_2\|\partial^\alpha\phi_{h_\varepsilon}\|_5 + \|a_\varepsilon\|_2^{\frac{1}{2}} f_\varepsilon^{\frac{7}{2}} \end{aligned}$$

and $H(\cdot)$ is the Hamiltonian.

Let us now turn to uniqueness of a solution in the sense of Definition 5. We assume that there is another solution $v \in \mathcal{G}_{C^1,H^2}([0,T)\times\mathbb{R}^3)$. Then, a representative v_ε of v solves

$$\begin{aligned} i(v_\varepsilon)_t + \triangle v_\varepsilon &= N(v_\varepsilon) + n_\varepsilon, \\ v_\varepsilon(0) &= a_\varepsilon + m_\varepsilon, \end{aligned} \tag{17}$$

where N is again the given nonlinearity, $n_\varepsilon \in \mathcal{N}_{C,L^2}$ and $m_\varepsilon \in \mathcal{N}_{H^2}$. Then the difference $w_\varepsilon = u_\varepsilon - v_\varepsilon$ satisfies an appropriate equation from which we derive the following estimate

$$\|w_\varepsilon\|_2 \lesssim \varepsilon^M \exp(\|u_\varepsilon\|_\infty^2 + \|u_\varepsilon\|_\infty \|v_\varepsilon\|_\infty),$$

for any $M \in \mathbb{N}$ and any $t \in [0, T)$. This estimate is obtained by energy methods and it holds for both (1) and (2).

To complete the proof, we need to control the infinity norm of a solution by an appropriate H^2 norm of the initial condition. If we use the Sobolev embedding, we see that we have already achieved this for the solution of (13), but v_ε solves a slightly more complicated (inhomogeneous) Eq. (17). For this reason, we have to derive analogous estimates for $\|v_\varepsilon\|_{H^2}$ and also to ask for a more strict condition on n_ε. This leads us to a modified version of uniqueness.

Definition 8 Let $u, v \in \mathcal{G}_{C^1,H^2}(\mathbb{R}^3)$ be two classes such that for each class there exists a representative that solves

$$\begin{aligned} i\partial_t u_\varepsilon + \triangle u_\varepsilon &= N(u_\varepsilon) + n_\varepsilon, \\ u_\varepsilon(0) &= a_\varepsilon + m_\varepsilon, \end{aligned} \tag{18}$$

where $n_\varepsilon \in \mathcal{N}_{C^1,H^2}([0,T) \times \mathbb{R}^3)$ and $m_\varepsilon \in \mathcal{N}_{H^2}(\mathbb{R}^3)$ (similarly for v). If $\sup_{[0,T)} \|u_\varepsilon - v_\varepsilon\|_2 = \mathcal{O}(\varepsilon^M)$, $\varepsilon \to 0$ for any $M \in \mathbb{N}$, then we say that the solution is unique.

Now we can formulate the following theorem.

Theorem 8 *If $\|a_\varepsilon\|_{H^3} \sim \ln^s \ln^q \varepsilon^{-1}$, where $s = \frac{5}{7}$, $q = \frac{1}{24}$, the solution of (1) is unique in the sense of Definition 8. If $\|a_\varepsilon\|_{H^3} \sim \ln^s \ln^q \varepsilon^{-1}$, where $s = \frac{7}{25}$, $q = \frac{1}{500}$ the solution of (2) is unique in the sense of Definition 8.*

As mentioned, the proofs are now essentially the same for both equations, but with estimates for $\|v_\varepsilon\|_{H^2}$ being slightly different depending on the equation.

4 Convergence Properties

We prove compatibility for the cubic equation (1) (notion presented in Sect. 2). If $a \in H^2(\mathbb{R}^3)$ there is a unique solution $u \in H^2(\mathbb{R}^3)$. Such function a can be embedded in $\mathcal{G}_{H^2}(\mathbb{R}^3)$ by convolution with a mollifier. For an appropriate mollifier ρ_ε, the norm $\|a * \rho_\varepsilon\|_{H^3}$ satisfies all the necessary estimates of Theorems 6 and 8. Hence for $a \in H^2(\mathbb{R}^3)$ there is a unique solution $[(u_\varepsilon)_\varepsilon] \in \mathcal{G}_{C^1,H^2}(\mathbb{R}^3)$. We already used that the cubic equation satisfies an estimate proved by Bourgain which applied to the regularized equation is in the form

$$\|u_\varepsilon(t)\|_{H^2} \leq c\exp(\|a * \rho_\varepsilon\|_{H^2}) \quad \forall t \geq 0. \tag{19}$$

The expression $\|a * \rho_\varepsilon\|_{H^2}$ is bounded uniformly in ε due to Young's inequality. Using this fact, energy methods and Gronwall inequality, we prove that

$$\|u_\varepsilon - u\|_2 \to 0$$

and hence the Sobolev and the Colombeau solution are compatible.

Regarding Eq. (2), some possible future directions are to compare our approach with other settings, like the one given in [17] where the authors consider solutions in weak L^p spaces. Also, we would like to consider the Hartree equation

$$\begin{aligned} iu_t + \triangle u &= (w * |u|^2)u + \delta u, \\ u(0) &= a, \end{aligned} \tag{20}$$

in the Colombeau setting. More precisely it would be interesting to study NLS (nonlinear Schrödinger equations) in which the linear part is characterized by a Schrödinger operator with point-interaction. These operators provide an alternative way to model a zero-range potential. They are well-studied by means of classical techniques (see, e.g., [2]), and also within the framework of generalized functions (cf. [27]). The associated nonlinear problem has recently attracted attention—see, e.g., [1, 7, 22] and [18]. In these papers NLS with point interactions have been analyzed by classical techniques, and it would be interesting to exploit also the Colombeau approach based on generalized functions.

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Part III
Dispersive Properties

Schrödinger Flow's Dispersive Estimates in a regime of Re-scaled Potentials

Vladimir Georgiev, Alessandro Michelangeli, and Raffaele Scandone

Abstract The problem of monitoring the (constants in the estimates that quantify the) dispersive behaviour of the flow generated by a Schrödinger operator is posed in terms of the scaling parameter that expresses the small size of the support of the potential, along the scaling limit towards a Hamiltonian of point interaction. At positive size, dispersive estimates are completely classical, but their dependence on the short range of the potential is not explicit, and the understanding of such a dependence would be crucial in connecting the dispersive behaviour of the short-range Schrödinger operator with the zero-range Hamiltonian. The general set-up of the problem is discussed, together with preliminary answers, open questions, and plausible conjectures, in a 'propaganda' spirit for this subject.

1 Introduction and Background

In the context of the dispersive properties of the Schrödinger flow generated by the operator $-\Delta + V(x)$, self-adjointly realised on $L^2(\mathbb{R}^d)$ for a given measurable function $V : \mathbb{R}^d \to \mathbb{R}$, the explicit dependence on V of (the constants in) dispersive

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and Strichartz estimates is implicit or tacitly ignored, as V is given and does not represent a relevant parameter, as long as it belongs to a suitable class of potentials satisfying the required working assumptions. The other standard dependence on V in the dispersive estimate is the projection onto the absolutely continuous spectrum of the associated Schrödinger operator: it too is kept at this implicit level.

There are applications, however, where instead an explicit control of the dispersion in terms of V would provide crucial information.

The case that concerns us here is when V approximates in a suitable quantitative sense an actual point-like, 'impurity type' perturbation of $-\Delta$, the well-established construction where, heuristically speaking, one formally adds to $-\Delta$ a potential with delta-like profile supported at some $x_0 \in \mathbb{R}^d$ [3]. In this respect, the problem of comparing the dispersive phenomenon in the limiting case of point-like perturbation with the approximant case of a perturbation of finite size support acquires relevance per se and in application to the study of the solution theory of the associated (linear and) non-linear Schrödinger equations with point-like singular perturbation [1, 9, 16, 18, 27].

In order to place our analysis into context, let us pick for concreteness the *three-dimensional* case and, for $\varepsilon > 0$, let us consider the Schrödinger operator

$$H_\varepsilon \;=\; -\Delta + V_\varepsilon(x), \tag{1}$$

where

$$V_\varepsilon(x) \;:=\; \frac{\eta(\varepsilon)}{\varepsilon^2}\, V\Big(\frac{x}{\varepsilon}\Big), \qquad x \in \mathbb{R}^3\,, \tag{2}$$

for given V, and where the following conditions (or more restrictive versions, as done later) are assumed:

(V1) $\eta : \overline{\mathbb{R}^+} \to \overline{\mathbb{R}^+}$ is continuous on $\overline{\mathbb{R}^+}$, smooth on $\mathbb{R}^+$, and satisfies $\eta(0) = \eta(1) = 1$ as well as $\sup_{\varepsilon>0} \eta(\varepsilon) < +\infty$;

(V2) V is real-valued, $V \in \mathcal{R}$ (the Rollnik class), $(1 + |\cdot|)V \in L^1(\mathbb{R}^3)$.

Assumption (**V1**) regulates the 'distortion' with respect to the scaling $\varepsilon^{-2}V(x/\varepsilon)$ that has the same behaviour as the scaling of the Laplacian under dilation. Moreover, $H_1 = -\Delta + V$.

Assumption (**V2**), among other consequences, guarantees the self-adjointness of H_ε in $L^2(\mathbb{R}^3)$ with quadratic form domain $H^1(\mathbb{R}^3)$: indeed, under such a condition, V_ε is infinitesimally form bounded with respect to $-\Delta$ [31, Theorems X.17 and X.19]. In fact, for the purposes of the present discussion, it is surely non-restrictive to consider $V \in C_c^\infty(\mathbb{R}^3, \mathbb{R})$, and it is this special choice that we will implicitly have in mind.

The limit $\varepsilon \downarrow 0$ yields distinct constructions depending on whether the additional assumption here below is or is not matched.

(V3) Setting $v(x) := \sqrt{|V(x)|}$ and $u(x) := \sqrt{|V(x)|}\,\mathrm{sign}(V(x))$, the 'Birman-Schwinger' operator $u(-\Delta)^{-1}v$ on $L^2(\mathbb{R}^3)$, which is compact under assumption

(**V2**), admits the simple eigenvalue -1, that is, the equation

$$u(-\Delta)^{-1}v\,\phi \;=\; -\phi \tag{3}$$

has a unique (up to multiples) solution $\phi \in L^2(\mathbb{R}^3)\setminus\{\mathbf{0}\}$, which in fact can be chosen to be real-valued, and for convenience is normalised as

$$\int_{\mathbb{R}^3} \mathrm{sign}(V)|\phi|^2\,\mathrm{d}x \;=\; -1\,, \tag{4}$$

and *in addition* the function

$$\psi \;:=\; (-\Delta)^{-1}v\phi \tag{5}$$

satisfies

$$\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^3)\setminus L^2(\mathbb{R}^3)\,. \tag{6}$$

Assumption (**V3**) is a spectral condition of (simple) *zero-energy resonance* for the Schrödinger operator $-\Delta+V$. In fact, if a non-zero ϕ exists in $L^2(\mathbb{R}^3)$ satisfying (3), then [3, Lemma I.1.2.3] $\psi = (-\Delta)^{-1}v\phi \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$, $\nabla\psi \in L^2(\mathbb{R}^3)$, $(-\Delta + V)\psi = 0$ in the sense of distributions, and moreover

$$\psi \in L^2_{\mathrm{loc}}(\mathbb{R}^3)\setminus L^2(\mathbb{R}^3) \qquad \Leftrightarrow \qquad \int_{\mathbb{R}^3} v\phi\,\mathrm{d}x \;=\; \int_{\mathbb{R}^3} V\psi\,\mathrm{d}x \;\neq\; 0\,. \tag{7}$$

In addition, (**V3**) is a condition of *lack of zero-energy eigenvalue* for $-\Delta+V$: for, if $(-\Delta+V)\psi = 0$ for some $\psi \in H^1(\mathbb{R}^3)$, then $\phi := u\psi \in L^2(\mathbb{R}^3)\setminus\{0\}$ (otherwise, $-\Delta\psi = -v\phi = 0$, which is impossible), and $u(-\Delta)^{-1}v\phi = u(-\Delta)^{-1}V\psi = -u\psi = -\phi$, but by assumption there is only one such ϕ (up to multiples) and the corresponding ψ does not belong to $L^2(\mathbb{R}^3)$. Observe also that the *lack* of eigenvalue -1 for $u(-\Delta)^{-1}v$ is generic; clearly, a suitable scalar dilation $V \mapsto aV$ restores it. (An additional discussion may be found, e.g., in [17].)

Based on the above-mentioned consequences of (**V3**), we may further assume:

(**V4**) For given $\alpha \in \mathbb{R}\cup\{\infty\}$, η and V satisfy

$$\alpha \;=\; -\frac{\eta'(0)}{\left|\int_{\mathbb{R}^3} V\psi\,\mathrm{d}x\right|^2}\,. \tag{8}$$

As anticipated, the above assumptions regulate the limit $\varepsilon \downarrow 0$. More precisely (see, e.g., [3, Theorem I.1.2.5]),

- if all (**V1**)–(**V4**) hold true, then $H_\varepsilon \xrightarrow{\varepsilon\downarrow 0} -\Delta_\alpha$,
- if, under (**V1**)–(**V2**), (3) has no non-trivial solution in $L^2(\mathbb{R}^3)$, then $H_\varepsilon \xrightarrow{\varepsilon\downarrow 0} -\Delta$

in the *norm resolvent sense* [30, Section VIII.7], where $-\Delta_\alpha$, for α given by (8), is the point-like perturbation of the (negative) Laplacian at the origin, namely the self-adjoint extension in $L^2(\mathbb{R}^3)$ of $-\Delta|_{C^\infty_c(\mathbb{R}^3\setminus\{0\})}$ with s-wave scattering length $-(4\pi\alpha)^{-1}$ and zero effective range.

The latter is by now a standard construction in various equivalent self-adjoint extension schemes (see, e.g., [3, Section I.1.1] and [28, Section 3]). Explicitly, for arbitrary $\lambda > 0$ (and $\lambda \neq (4\pi\alpha)^2$ if $\alpha < 0$),

$$\begin{aligned} \mathrm{dom}(-\Delta_\alpha) &= \left\{ u \in L^2(\mathbb{R}^3) \;\middle|\; \begin{array}{c} \exists\, \varphi_\lambda \in H^2(\mathbb{R}^3) \text{ such that} \\ u = \varphi_\lambda + \dfrac{\varphi_\lambda(0)}{4\pi\alpha+\sqrt{\lambda}}\, \dfrac{e^{-|x|\sqrt{\lambda}}}{|x|} \end{array} \right\}, \\ (-\Delta_\alpha+\lambda)u &= (-\Delta+\lambda)\varphi_\lambda\,. \end{aligned} \tag{9}$$

In particular, $\alpha = \infty$ selects $-\Delta$, with self-adjointness domain $H^2(\mathbb{R}^3)$. One also has the explicit resolvent difference

$$(-\Delta_\alpha+\lambda\mathbb{1})^{-1}-(-\Delta+\lambda\mathbb{1})^{-1} = (4\pi(4\pi\alpha+1))^{-1}\left|\frac{e^{-|x|\sqrt{\lambda}}}{|x|}\right\rangle\left\langle\frac{e^{-|x|\sqrt{\lambda}}}{|x|}\right| \tag{10}$$

(with the customary notation $|\psi\rangle\langle\psi|$ for the orthogonal projection in $L^2(\mathbb{R}^3)$ onto the linear span of ψ. Concerning the spectrum of $-\Delta_\alpha$,

$$\begin{aligned} \sigma_{\mathrm{ess}}(-\Delta_\alpha) = \sigma_{\mathrm{ac}}(-\Delta_\alpha) &= [0,+\infty)\,, \\ \sigma_{\mathrm{sc}}(-\Delta_\alpha) &= \emptyset\,, \\ \sigma_{\mathrm{p}}(-\Delta_\alpha) &= \begin{cases} \emptyset\,, & \text{if } \alpha \geqslant 0\,, \\ \{-(4\pi\alpha)^2\} & \text{if } \alpha < 0\,. \end{cases} \end{aligned} \tag{11}$$

The negative eigenvalue, when existing, is non-degenerate.

As a consequence of the above norm resolvent convergence (strong resolvent convergence would have sufficed), Trotter's theorem (see, e.g., [30, Theorem VIII.21] implies

$$\begin{aligned} \|\, e^{-\mathrm{i}t(-\Delta+V_\varepsilon)}f - e^{\mathrm{i}t\Delta_\alpha}f\,\|_{L^2} &\xrightarrow{\varepsilon\downarrow 0} 0 \qquad \text{(resonant case)}\,, \\ \|\, e^{-\mathrm{i}t(-\Delta+V_\varepsilon)}f - e^{\mathrm{i}t\Delta}f\,\|_{L^2} &\xrightarrow{\varepsilon\downarrow 0} 0 \qquad \text{(non-resonant case)}\,, \\ \forall t \in \mathbb{R}\,,\ \forall f &\in L^2(\mathbb{R}^3)\,, \end{aligned} \tag{12}$$

that is, *strong* convergence of the unitary groups. Observe that instead *norm operator* convergence cannot hold in general (as emerges, e.g., from the proof of [30, Theorem VIII.20]).

Thus, next to the classical and comprehensive knowledge of dispersive, smoothing, and Strichartz estimates for the Schrödinger unitary propagator $e^{-\mathrm{i}tH_\varepsilon}$ (we refer, among others, to the monographs[10, 26, 36, 37] and the multiple references therein), it is relevant in the present context to monitor the dispersive features of $e^{-\mathrm{i}tH_\varepsilon}$ *in terms of the scaling parameter* ε.

As mentioned, this has at least a two-fold motivation. For one thing, there is an abstract interest per se in comparing the dispersive estimates of $e^{-\mathrm{i}tH_\varepsilon}$ and of $e^{\mathrm{i}t\Delta_\alpha}$: notably, for the latter, the explicit knowledge [2, 34] of the integral kernel (see (31) below) actually allows for an explicit derivation of dispersive and Strichartz estimates [12, 13, 21] (see Remark 2 and (35)–(39)). Furthermore, there is a crucial relevance in applications to semi-linear Schrödinger equations induced by $-\Delta_\alpha$: for such equations, whose study, albeit at an early stage, has already produced important well-posedness results [9, 18, 19, 27], and in particular for their physical relevance as effective dynamical equations for large Bose gases with impurities, one natural and open problem is the approximation of the solution u by means of the solution u_ε of the corresponding semi-linear equation induced by H_ε, a question that would require Strichartz estimates for $e^{-\mathrm{i}tH_\varepsilon}$ quantitatively expressed in terms of ε, so as to monitor the $\varepsilon \downarrow 0$ limit.

The purpose of this note is to make propaganda for this and related problems, and to present a first answer in the prototypical three-dimensional set-up. The same issue naturally arises and deserves investigation in two dimensions. The one-dimensional case too is of relevance: that case is somewhat simpler and under more direct control, as in one dimension the singular point-perturbed $-\Delta_\alpha$ is an actual quadratic form sum of $-\Delta$ and (a multiple of) the Dirac δ distribution [3, Chapter I.3].

It is worth observing that in the context of dispersive estimates for Schrödinger operators one is well aware (see, e.g., [35, Section 12.1]) of the very important difference between the one-dimensional dispersive bounds, whose constants do exhibit an explicit dependence on the potential via the Jost solutions, as opposed to the higher dimensional bounds: this general lack of information results, in the present context, in the quest of the ε-dependence.

2 A Preliminary Overview of Relevant Spectral Properties

It is standard that, under the assumptions (**V1**)–(**V2**), H_ε has essential spectrum that is entirely absolutely continuous and amounts to

$$\sigma_{\mathrm{ess}}(H_\varepsilon) = \sigma_{\mathrm{ac}}(H_\varepsilon) = [0, +\infty) \qquad \forall \varepsilon > 0\,. \tag{13}$$

Concerning the (necessarily negative) discrete spectrum, an explicit and detailed discussion is possible, e.g., upon strengthening (**V2**) as:

(**V2′**) $\quad V$ is real-valued and $e^{a|\cdot|}V \in \mathcal{R}$ for some $a > 0$.

In fact, it is known that

- [3, Theorem I.1.3.1(a)] assuming (**V1**)–(**V2′**), any negative eigenvalue E_1 of $H_1 = -\Delta + V$ of multiplicity m gives rise to m (not necessarily distinct) eigenvalues $E_\varepsilon^{(\ell)}$ of H_ε , $\ell \in \{1, \dots, m\}$ running to $-\infty$ as $\varepsilon \downarrow 0$ as

$$E_\varepsilon^{(\ell)} = \varepsilon^{-2} E_1 + O(\varepsilon^{-1}) ; \tag{14}$$

- [3, Theorem I.1.3.1(b)], assuming (**V1**),(**V2′**),(**V3**),(**V4**), and when $\alpha < 0$, H_ε has, for any $\varepsilon > 0$ small enough, the non-degenerate negative eigenvalue $E_\varepsilon^{(\alpha)}$

$$E_\varepsilon^{(\alpha)} = -(4\pi\alpha)^2 + O(\varepsilon) . \tag{15}$$

Last, concerning the nature of the spectral point zero for H_ε, two scenarios are possible under the basic assumptions (**V1**)–(**V2**):

- if, eventually in ε as $\varepsilon \downarrow 0$, one has $\eta(\varepsilon) \equiv 1$, then H_ε and $\varepsilon^{-2} H_1$ are unitarily equivalent, as operators on $L^2(\mathbb{R}^3)$, via the $L^2 \to L^2$ dilation isomorphism U_ε, that is,

$$U_\varepsilon^* H_\varepsilon U_\varepsilon = \frac{1}{\varepsilon^2} H_1 , \qquad (U_\varepsilon f)(x) := \frac{1}{\varepsilon^{3/2}} f\Big(\frac{x}{\varepsilon}\Big) ; \tag{16}$$

  as a consequence, if the spectral point zero is an eigenvalue or a resonance for $-\Delta + V$, so too is it for H_ε;
- on the other hand, in general a re-scaling with $\eta(\varepsilon) \neq 1$ distortion washes out possible eigenvalues or resonance initially present at zero energy for $-\Delta + V$; therefore, if (eventually in ε) $\eta(\varepsilon) = 1 + \kappa\varepsilon$ for some $\kappa \neq 0$, which in fact covers the remaining generality of the present setup (only the quantity $\kappa = \eta'(0)$ enters (8) above), then eventually in ε zero-energy eigenvalues or resonance are absent for H_ε.

We shall refer to the occurrence where all of (**V1**)–(**V4**) hold true as the *resonant regime* (at the given parameter α), and to the occurrence where (**V1**)–(**V2**) are matched, and (3) has no solutions in $L^2(\mathbb{R}^3) \setminus \{0\}$, as the *non-resonant regime*. For what has been just observed, such a terminology refers to the spectral property of $H_1 = -\Delta + V$, and not to the spectrum of H_ε at zero energy. At each ε, H_ε may be well non-resonant even though H_1 is.

3 Dispersive Estimates with ε-Uniform Bound

The $L^q \to L^p$ mapping properties of $e^{-\mathrm{i}tH_\varepsilon}$ depend, as the vast and well-established literature on Schrödinger flow's dispersive estimates shows, on the presence or absence of zero-energy resonance or zero-energy eigenvalues for H_ε, provided that V_ε belongs to certain standard classes of controllable potentials.

In particular [14, 20, 22, 25, 29, 33, 41], $|t|^{-3/2}$ is the typical decay for the norm $\|e^{-\mathrm{i}tH_\varepsilon}P_\varepsilon^{(\mathrm{ac})}\|_{L^1\to L^\infty}$ in the absence of both resonance and eigenvalues at zero energy for H_ε, being in fact the exact decay for the corresponding norm relative to the free Schrödinger propagator $e^{\mathrm{i}t\Delta}$, whereas the *slower* $|t|^{-1/2}$ is typical for the same norm in the presence of resonance at zero. Here $P_\varepsilon^{(\mathrm{ac})}$ is the orthogonal projection onto the absolutely continuous spectral subspace of $L^2(\mathbb{R}^3)$ associated with H_ε (see, e.g., [4, Chapter 4]).

A priori the above norm depends also on ε—an information that, as commented in Sect. 1, would not be of concern if the scaling limit $\varepsilon\downarrow 0$ was not considered.

We show now that the $L^q \to L^p$ bound is actually *uniform* in ε in two meaningful classes of cases.

To this aim, it is convenient to require additional constraints on the size or on the decay of V, and precisely:

($\mathbf{V_{small}}$) V is real-valued and, together with η, it satisfies

$$\|V\|_{\mathcal{R}} := \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|V(x)|\,|V(y)|}{|x-y|^2}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}} < 4\pi\left(\sup_{\varepsilon>0}\eta(\varepsilon)\right)^{-1}, \tag{17}$$

$$\|V\|_{\mathcal{K}} := \sup_{x\in\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V(y)|}{|x-y|}\,\mathrm{d}y < 4\pi\left(\sup_{\varepsilon>0}\eta(\varepsilon)\right)^{-1} \tag{18}$$

(i.e., respectively, smallness of the *Rollnik norm* and the generalised Kato norm);
($\mathbf{V_{decay}}$) V is real-valued and satisfies $|V(x)|\lesssim\langle x\rangle^{-(7+\delta)}$ for some $\delta>0$.

Observe that ($\mathbf{V_{small}}$) automatically excludes zero-energy eigenvalues or resonance for $-\Delta+V$ (in particular, it excludes (**V3**)), and ($\mathbf{V_{decay}}$) implies (**V2**).

With the extra decay imposed by ($\mathbf{V_{decay}}$) we are surely far from optimality, but in the present context this is not of concern: recall that already the choice $V\in C_c^\infty(\mathbb{R}^3)$ would be completely meaningful and non-restrictive, as it gives rise to both mechanisms $H_\varepsilon \xrightarrow{\varepsilon\downarrow 0} -\Delta_\alpha$ and $H_\varepsilon \xrightarrow{\varepsilon\downarrow 0} -\Delta$ described in Sect. 1.

Theorem 1 *Assume (V1) and (*$\mathbf{V_{small}}$*). Then there exists a constant C, independent of $\varepsilon>0$, such that*

$$\big\|\,e^{-\mathrm{i}tH_\varepsilon}P_\varepsilon^{(\mathrm{ac})}f\,\big\|_{L^p} \leqslant C|t|^{-3(\frac{1}{2}-\frac{1}{p})}\|f\|_{L^{p'}} \qquad \begin{array}{l}\forall p\in[2,+\infty]\,,\ p'=\frac{p}{p-1}\,,\\ \forall f\in L^{p'}(\mathbb{R}^3)\,,\\ \forall t\in\mathbb{R}\setminus\{0\}\,.\end{array} \tag{19}$$

Proof It is standard to see that the smallness condition ($\mathbf{V_{small}}$) prevents $-\Delta+V$ to have zero-energy eigenvalues or resonance. The same therefore holds for H_ε, eventually in ε, apart from possible exceptional, isolated values of ε.

In this regime, and at every *fixed* ε at which H_ε is not zero-resonant, the $L^{p'} \to L^p$ boundedness of $e^{-\mathrm{i}tH_\varepsilon}$, with bound $C_\varepsilon|t|^{-3(\frac{1}{2}-\frac{1}{p})}$, is a classical result (we refer to [33]) obtained under the condition $\|V_\varepsilon\|_{\mathcal{R}} < 4\pi$ by means of a Born series expansion for the resolvent with a subsequent estimate of an arising oscillatory integral: this results in a geometric series whose convergence is guaranteed by $\|V_\varepsilon\|_{\mathcal{K}} < 4\pi$.

In fact, owing to (**V1**) and ($\mathbf{V_{small}}$),

$$\begin{aligned} \|V_\varepsilon\|_{\mathcal{K}} &\leqslant \Big(\sup_{\varepsilon>0}\eta(\varepsilon)\Big)\|V\|_{\mathcal{K}} \;<\; 4\pi\,, \\ \|V_\varepsilon\|_{\mathcal{R}} &\leqslant \Big(\sup_{\varepsilon>0}\eta(\varepsilon)\Big)\|V\|_{\mathcal{R}} \;<\; 4\pi\,, \end{aligned} \tag{20}$$

thus matching the needed smallness conditions for V_ε.

Moreover, the constant C_ε in the $L^{p'} \to L^p$ bound depends on $\|V_\varepsilon\|_{\mathcal{K}}$ and $\|V_\varepsilon\|_{\mathcal{R}}$, and is therefore uniformly bounded in ε. Estimate (19) is thus established.

□

Theorem 2 *Assume (**V1**) with $\eta \equiv 1$, ($\mathbf{V_{decay}}$), and (**V3**), (thereby implying (**V4**) with $\alpha = 0$). In other words, it is assumed that for every $\varepsilon > 0$ H_ε acts self-adjointly on $L^2(\mathbb{R}^3)$ as*

$$H_\varepsilon \;=\; -\Delta + \frac{1}{\varepsilon^2}V\Big(\frac{x}{\varepsilon}\Big) \tag{21}$$

with V satisfying ($\mathbf{V_{decay}}$), and it is assumed furthermore that the spectral value zero is a resonance, but not an eigenvalue for H_1—hence, on account of (16), zero is a resonance but not eigenvalue for H_ε for any $\varepsilon > 0$. Then there exists a constant C, independent of ε, such that

$$\big\|\, e^{-\mathrm{i}tH_\varepsilon}P_\varepsilon^{(\mathrm{ac})}f\,\big\|_{L^p} \;\leqslant\; C|t|^{-3(\frac{1}{2}-\frac{1}{p})}\|f\|_{L^{p'}} \qquad \begin{array}{l} \forall p\in[2,3)\,,\;\; p' = \frac{p}{p-1}\,, \\ \forall f\in L^{p'}(\mathbb{R}^3)\,, \\ \forall t\in\mathbb{R}\setminus\{0\}\,. \end{array} \tag{22}$$

Remark 1 As commented already, H_ε in (21) is zero-energy resonant, without zero-energy eigenvalues, for every $\varepsilon > 0$. For such a Schrödinger operator, the dispersive estimate (22), precisely in the regime $p \in [2, 3)$, was established in [41, Theorem 1.3(2)] under the milder decay $|V(x)| \lesssim \langle x\rangle^{-\beta}$ for some $\beta > \frac{11}{2}$, but with an *implicit* dependence of the constant on V_ε, that is, on ε. Theorem 2 adds to this classical picture the novel information that such a bound is *uniform* in ε. It is also worth remarking that [41, Theorem 1.3(2)] prescribes, in addition, that a counterpart to (22) is valid when $p = 3$ provided that the L^3- and $L^{\frac{3}{2}}$-norms are replaced, respectively, by norms of the Lorenz spaces $L^{3,\infty}(\mathbb{R}^3)$ and $L^{\frac{3}{2},1}(\mathbb{R}^3)$.

Remark 2 The dispersive estimate (22), with the uniformity of the bound in terms of ε, is compatible with its known counterpart for the limiting propagator $e^{\mathrm{i}t\Delta_{\alpha=0}}$—recall from Sect. 1 that under the assumptions of Theorem 2 one has $e^{-\mathrm{i}tH_\varepsilon} \xrightarrow{\varepsilon\downarrow 0} e^{\mathrm{i}t\Delta_{\alpha=0}}$ strongly in $L^2(\mathbb{R}^3)$ for every fixed $t \in \mathbb{R}$. Indeed, it was found in [13, 21] that

$$\| e^{\mathrm{i}t\Delta_\alpha} P^{(\mathrm{ac})}_{(\alpha)} f \|_{L^p} \leqslant C|t|^{-3(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^{p'}} \qquad \begin{array}{l} \forall p \in [2,3)\,,\ p' = \frac{p}{p-1}\,, \\ \forall f \in L^{p'}(\mathbb{R}^3)\,, \\ \forall t \in \mathbb{R}\setminus\{0\} \end{array} \tag{23}$$

for every $\alpha \in \mathbb{R}$, where now $P^{(\mathrm{ac})}_{(\alpha)}$ is the L^2-orthogonal projection onto the absolutely continuous spectrum $[0,+\infty)$ of $-\Delta_\alpha$.

Proof of Theorem 2 Let us consider on $L^2(\mathbb{R}^3)$ the wave operators

$$W^{\pm}_{\varepsilon} \equiv W^{\pm}(H_\varepsilon, -\Delta) := \lim_{t\to\pm\infty} e^{\mathrm{i}tH_\varepsilon} e^{\mathrm{i}t\Delta} \tag{24}$$

(as strong limits in $L^2(\mathbb{R}^3)$) associated with the pair of self-adjoint operators H_ε and $-\Delta$. Standard arguments from scattering theory (see, e.g., [32, Theorem XI.30]) guarantee that such wave operators exist in $L^2(\mathbb{R}^3)$ and are complete, meaning that

$$\mathrm{ran}\, W^{\pm}_{\varepsilon} = L^2_{\mathrm{ac}}(H_\varepsilon) := P^{(\mathrm{ac})}_{\varepsilon} L^2(\mathbb{R}^3)\,. \tag{25}$$

Owing to their completeness, W^{+}_{ε} and W^{-}_{ε} are unitaries from $L^2(\mathbb{R}^3)$ onto $L^2_{\mathrm{ac}}(H_\varepsilon)$ and they intertwine $H_\varepsilon P^{(\mathrm{ac})}_{\varepsilon}$ and $-\Delta$, in particular,

$$e^{-\mathrm{i}tH_\varepsilon} P^{(\mathrm{ac})}_{\varepsilon} = W^{\pm}_{\varepsilon} e^{\mathrm{i}t\Delta} (W^{\pm}_{\varepsilon})^* \qquad \forall t \in \mathbb{R}\,. \tag{26}$$

In analogy to $W^{\pm}_{\varepsilon}$ let us also consider on $L^2(\mathbb{R}^3)$ the wave operators

$$W^{\pm}_{(\alpha)} \equiv W^{\pm}(-\Delta_\alpha, -\Delta) := \lim_{t\to\pm\infty} e^{-\mathrm{i}t\Delta_\alpha} e^{\mathrm{i}t\Delta} \tag{27}$$

(as strong limits in $L^2(\mathbb{R}^3)$) associated with $-\Delta_\alpha$ and $-\Delta$. Since the difference of the corresponding resolvents is a rank-one operator (see (10) above), $W^{\pm}_{(\alpha)}$ too exist and are complete, on account of the Kuroda-Birman theorem (see, e.g., [31, Theorem XI.9].

The intertwining relation (26) allows to deduce the $L^{p'} \to L^p$ boundedness of $e^{-\mathrm{i}tH_\varepsilon} P^{(\mathrm{ac})}_{\varepsilon}$ directly from the known $L^{p'} \to L^p$ boundedness of $e^{\mathrm{i}t\Delta}$, once one also knows that $W^{\pm}_{\varepsilon}$ is bounded on $L^p(\mathbb{R}^3)$: the latter information is classical, and there is in fact a vast literature on the L^p-boundedness of $W^{\pm}_{\varepsilon}$ for sufficiently regular V_ε vanishing at spatial infinity [5–8, 11, 15, 23, 24, 38–40, 42, 43]. This yields

$$\| e^{-\mathrm{i}tH_\varepsilon} P^{(\mathrm{ac})}_{\varepsilon} f \|_{L^p} \leqslant C \|W^{+}_{\varepsilon}\|^2_{L^p\to L^p} |t|^{-3(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^{p'}} \tag{28}$$

for any $t \in \mathbb{R} \setminus \{0\}$, any $p \in [2, +\infty]$, and any $f \in L^{p'}(\mathbb{R}^3)$.

On the other hand, it was recently proved in [13] that $W^{\pm}_{(\alpha)}$ are L^p-bounded only for $p \in (1, 3)$ [13, Theorem 1.1] and that

$$\forall u \in L^p(\mathbb{R}^3) \qquad \lim_{\varepsilon \downarrow 0} W^{\pm}_{\varepsilon} u \;=\; W^{\pm}_{(\alpha=0)} u \qquad \text{weakly in } L^p(\mathbb{R}^3)\,, \tag{29}$$

[13, Proposition 7.1]. (Strictly speaking for the latter result both ($\mathbf{V_{decay}}$) and the lack of zero-energy eigenvalue, as well as the special form (21) of H_ε, were all required in [13, Proposition 7.1].) The Banach-Steinhaus theorem then allows to deduce from (29) that

$$\| W^{\pm}_{\varepsilon} \|_{L^p \to L^p} \;\leqslant\; \kappa \;<\; +\infty \tag{30}$$

uniformly in ε. Plugging (30) into (28) finally yields (22). □

4 Outlook on Further Scaling Regimes

The preceding discussion shows that there are relevant scaling regimes that remain uncharted, as far as the ε-dependence of the norm $\| e^{-\mathrm{i}tH_\varepsilon} P^{(\mathrm{ac})}_{\varepsilon} \|_{L^{p'} \to L^p}$ is concerned:

(A) the special resonant case with H_ε given by (21), that is, under assumptions (**V2**) (or stronger spatial decay) and (**V3**), (zero-energy resonance and absence of zero-energy eigenvalue for $-\Delta + V$), and in the dispersive regime $p \in [3, +\infty]$;
(B) the general resonant regime with H_ε given by (1)–(2) under (**V1**)–(**V4**), in the dispersive regime $p \in [2, +3)$;
(C) the general resonant regime with H_ε given by (1)–(2) under (**V1**)–(**V4**), in the dispersive regime $p \in [3, +\infty]$.

Apart from the dependence on ε, the norm $\| e^{-\mathrm{i}tH_\varepsilon} P^{(\mathrm{ac})}_{\varepsilon} \|_{L^{p'} \to L^p}$ is already well controlled in time in all the above cases (A), (B), and (C).

Each one among (A), (B), (C) presents specific difficulties, which justifies listing them separately.

Case (B) is conceptually similar to Theorem 2: when $p \in [2, 3)$ the wave operators $W^{\pm}_{(\alpha)} \equiv W^{\pm}(-\Delta_\alpha, -\Delta)$ are still L^p-bounded, as established in [13, Theorem 1.1], which in turns implies the dispersive estimate (23) for $-\Delta_\alpha$, precisely for $p \in [2, 3)$. This, and the L^2-strong convergence $e^{-\mathrm{i}tH_\varepsilon} \xrightarrow{\varepsilon \downarrow 0} e^{\mathrm{i}t\Delta_\alpha}$ for each $t \in \mathbb{R}$ suggest that in case (B) the propagator $e^{-\mathrm{i}tH_\varepsilon}$ should satisfy the same $L^{p'} \to L^p$ bound as in (22). In order to mimic the scattering scheme of Theorem 2's proof, one would require a version of the key ingredient [13, Proposition 7.1], that is, the same L^p-weak convergence $W^{\pm}_{\varepsilon} \xrightarrow{\varepsilon \downarrow 0} W^{\pm}_{(\alpha)}$ of (29), so as to cover the generic scaling (1)–(2) for H_ε.

In the dispersive regime $p \in [3, +\infty]$ of cases (A) and (C), instead, no $L^{p'} \to L^p$ boundedness of $e^{\mathrm{i}t\Delta_\alpha}$ is possible: this is ultimately a consequence of the fact that the linear Schrödinger dynamics develops, at almost every instant $t > 0$, a $|x|^{-1}$-singularity in $(e^{\mathrm{i}t\Delta_\alpha} f)(x)$, clearly not locally L^p-integrable for $p \geqslant 3$. This can be argued from the explicit form [2, 34] of the integral kernel $K_\alpha(x, y; t)$ of the propagator $e^{\mathrm{i}t\Delta_\alpha}$:

$$K_\alpha(x,y;t) \;=\; \begin{cases} K(x,y;t) + \dfrac{1}{|x|\,|y|}\displaystyle\int_0^{+\infty} e^{-4\pi\alpha u}(u+|x|+|y|)\times \\ \qquad\qquad \times K(u+|x|+|y|,0;t)\,\mathrm{d}u\,, & \text{if } \alpha > 0\,, \\[2ex] K(x,y;t) + \dfrac{2\,\mathrm{i}\,t}{|x|\,|y|}\,K(|x|+|y|,0;t)\,, & \text{if } \alpha = 0\,, \\[2ex] K(x,y;t) + e^{\mathrm{i}t(4\pi\alpha)^2}\Psi_\alpha(x)\Psi_\alpha(y) \\ +\dfrac{1}{|x|\,|y|}\displaystyle\int_0^{+\infty} e^{-4\pi|\alpha|u}(u-|x|-|y|)\times & \text{if } \alpha < 0\,, \\ \qquad\qquad \times K(u-|x|-|y,0.t)\,\mathrm{d}u\,, \end{cases} \tag{31}$$

where

$$K(x,y;t) \;:=\; \frac{e^{-\frac{|x-y|^2}{4\mathrm{i}t}}}{(4\pi\mathrm{i}t)^{\frac{3}{2}}}\,, \qquad t>0\,, \tag{32}$$

and

$$\Psi_\alpha(x) \;:=\; \sqrt{-2|\alpha|}\,\frac{e^{-4\pi|\alpha||x|}}{|x|}\,. \tag{33}$$

In fact, the $L^{p'} \to L^p$ *unboundedness* of $e^{\mathrm{i}t\Delta_\alpha}$ when $p \geqslant 3$, and the L^2-strong convergence $e^{-\mathrm{i}tH_\varepsilon} \xrightarrow{\varepsilon\downarrow 0} e^{\mathrm{i}t\Delta_\alpha}$, prevent the norm $\|\, e^{-\mathrm{i}tH_\varepsilon} P_\varepsilon^{(\mathrm{ac})} \|_{L^{p'}\to L^p}$ to be uniformly bounded in ε when $p \geqslant 3$ (cases (A) and (C) above). For, if at an instant t when the evolution $e^{\mathrm{i}t\Delta_\alpha} f$ of a generic $f \in \big(\bigcap_\varepsilon P_\varepsilon^{(\mathrm{ac})} L^2(\mathbb{R}^3)\big) \cap L^{p'}(\mathbb{R}^3)$ is $|x|^{-1}$-singular around the origin one had

$$\|\, e^{-\mathrm{i}tH_\varepsilon} f \,\|_{L^p} \;\leqslant\; C_\varepsilon(t)\|f\|_{L^{p'}} \tag{34}$$

with $C_\varepsilon(t) \leqslant C(t)$ for some ε-*independent* $C(t) \geqslant 0$ (eventually as $\varepsilon \downarrow 0$), then from the sequence $(f_n)_{n\in\mathbb{N}}$ defined by

$$f_n \;:=\; e^{-\mathrm{i}tH_{\varepsilon_n}} f\,, \qquad \varepsilon_n \;:=\; n^{-1}\,,$$

which would then be uniformly bounded in $L^p(\mathbb{R}^3)$, one would have $f_n \to f_*$ L^p-weakly as $n \to \infty$, up to extracting a subsequence, for some $f_* \in L^p(\mathbb{R}^3)$. Since, on the other hand, $f_n \xrightarrow{n\to\infty} e^{\mathrm{i}t\Delta_\alpha} f$ in $L^2(\mathbb{R}^3)$, one should necessarily conclude $e^{\mathrm{i}t\Delta_\alpha} f = f_* \in L^p(\mathbb{R}^3)$. This is, however, incompatible with the $|x|^{-1}$-singularity of $e^{\mathrm{i}t\Delta_\alpha} f$, since $p \geqslant 3$. Necessarily $C_\varepsilon(t)$ in (34) blows up in ε, that is, $\| e^{-\mathrm{i}tH_\varepsilon} P_\varepsilon^{(\mathrm{ac})} \|_{L^{p'} \to L^p}$ becomes singular in ε as $\varepsilon \downarrow 0$ and $p \geqslant 3$. Observe that this argument sheds no light on the blow-up rate of $C_\varepsilon(t)$ as $\varepsilon \downarrow 0$ or on the short-time and long-time behaviour of $C_\varepsilon(t)$: actually, such a behaviour depends, at every fixed ε, on the presence or absence or zero-energy resonance and eigenvalue(s) for H_ε.

The above reasoning naturally suggests that the dispersive regime $p \geqslant 3$ for $e^{-\mathrm{i}tH_\varepsilon}$ (cases (A) and (C) above) could be meaningfully monitored, as far as the ε dependence is concerned, in suitably *weighted* $L^{p'} \to L^p$ norms—so as to absorb, informally speaking, the 'emergent' $|x|^{-1}$-singularity.

Weighted $L^1 \to L^\infty$ dispersive estimates for $-\Delta_\alpha$ were originally established in [12, Theorem 1], directly from (31), in a form that, interpolated with the trivial L^2-bound, reads (see [21, Proposition 4])

$$\big\| w^{-(1-\frac{2}{p})} e^{\mathrm{i}t\Delta_\alpha} P_{(\alpha)}^{(\mathrm{ac})} f \big\|_{L^p} \leqslant C|t|^{-3(\frac{1}{2}-\frac{1}{p})} \big\| w^{\frac{2}{p'}-1} f \big\|_{L^{p'}}, \qquad p \in [2,+\infty] \tag{35}$$

when $\alpha \neq 0$, and

$$\big\| w^{-(1-\frac{2}{p})} e^{\mathrm{i}t\Delta_{\alpha=0}} f \big\|_{L^p} \leqslant C|t|^{-(\frac{1}{2}-\frac{1}{p})} \big\| w^{\frac{2}{p'}-1} f \big\|_{L^{p'}}, \qquad p \in [2,+\infty] \tag{36}$$

in the case $\alpha = 0$, with weight

$$w(x) := 1 + \frac{1}{|x|}. \tag{37}$$

In fact $-\Delta_\alpha$ has a zero-energy resonance when $\alpha = 0$, and the slower time-decay (36) totally resembles what happens for actual Schrödinger operators with threshold resonances. From a more refined manipulation of (31) the weight-less version (23) in the range $p \in [2, 3)$ was later obtained in [21, Proposition 5] (and subsequently in [13, Corollary 1.3]), which, by interpolation with the weighted $L^1 \to L^\infty$ estimate above, allows to improve the powers of the weights in (35)–(36) in the regime $p \in [3, +\infty]$ to almost optimal ones, respectively ([21, Corollary 1]),

$$\big\| w^{-(1-\frac{3-\delta}{p})} e^{\mathrm{i}t\Delta_\alpha} P_{(\alpha)}^{(\mathrm{ac})} f \big\|_{L^p} \leqslant C|t|^{-3(\frac{1}{2}-\frac{1}{p})} \big\| w^{1-\frac{3-\delta}{p}} f \big\|_{L^{p'}}, \qquad \begin{array}{l} \alpha \neq 0, \\ p \in [3,+\infty] \end{array} \tag{38}$$

and

$$\big\| w^{-(1-\frac{3-\delta}{p})}\, e^{\mathrm{i}t\Delta_0} f \big\|_{L^p} \;\leqslant\; C|t|^{-\frac{1}{2}+\frac{\delta}{p}} \big\| w^{1-\frac{3-\delta}{p}} f \big\|_{L^{p'}}\,, \qquad p \in [3,+\infty] \tag{39}$$

for arbitrarily small $\delta > 0$.

It is natural to expect that the wave operators $W^{\pm}_{(\alpha)} \equiv W^{\pm}(-\Delta_\alpha, -\Delta)$, $\alpha \in \mathbb{R}\backslash\{0\}$, can be extended as continuous maps from $L^{p'}(\mathbb{R}^3, w_p^{-1}\mathrm{d}x)$ to $L^p(\mathbb{R}^3, w_p\mathrm{d}x)$ for $p \in (3,+\infty)$ (the 'endpoint' case $p = +\infty$ is typically more subtle), where

$$w_p(x) \;:=\; w(x)^{-p+3+\delta} \;=\; \Big(1+\frac{1}{|x|}\Big)^{-p+3+\delta} \tag{40}$$

for some delta $\delta > 0$ (that can be chosen arbitrarily small). Observe that $|x|^{-1} \in L^p(\mathbb{R}^3, w_p\mathrm{d}x)$, i.e., the weight w_p cancels out the local singularity generated by the point interaction. We also point out that we do not expect the boundedness of the wave operators in the zero-energy resonant case $\alpha = 0$, as this would lead to weighted $L^{p'} - L^p$ estimates with a time-decay $|t|^{-3(\frac{1}{2}-\frac{1}{p})}$ instead of the resonant time-decay $|t|^{-\frac{1}{2}+\frac{\delta}{p}}$.

It is also conceivable, under assumptions (**V1**), ($\mathbf{V_{decay}}$), (**V3**), and (**V4**) with $\alpha \neq 0$, that the wave operators $W^{\pm}_{\varepsilon} \equiv W^{\pm}(H_\varepsilon, -\Delta)$ can be extended as bounded maps from $L^{p'}(\mathbb{R}^3, w_p^{-1}\mathrm{d}x)$ to $L^p(\mathbb{R}^3, w_p\mathrm{d}x)$, and that $W^{\pm}_{\varepsilon}$ converges to $W^{\pm}_{(\alpha)}$, as $\varepsilon \downarrow 0$, in the weak topology of $\mathcal{B}(L^{p'}(\mathbb{R}^3, w_p^{-1}\mathrm{d}x; L^p(\mathbb{R}^3, w_p\mathrm{d}x))$.

All the ingredients above would allow to prove, by adapting the proof of Theorem 2, that under assumptions (**V1**), ($\mathbf{V_{decay}}$), (**V3**), and (**V4**) with $\alpha \neq 0$, weighted dispersive estimates analogous to (38) (with $p \in [3,\infty)$) hold true also for H_ε with an ε-independent constant.

In addition, by combining the above ε-uniform weighted dispersive estimates, a space-time re-scaling argument and suitable weighted resolvent bounds, it should be possible to provide (almost) optimal bounds for the blow-up rate as $\varepsilon \downarrow 0$ of the weight-less $L^{p'} - L^p$ estimates for H_ε, in the regime $p \geq 3$.

As already mentioned, the explicit dependence on the potential V in the dispersive estimates for $H = -\Delta + V$ cannot be in general directly deduced from the standard proofs, for these rely on the spectral behaviour of H at zero energy, which is unstable even with respect small perturbation of V in the Rollnik and (generalised) Kato norms.

Understanding the technical mechanisms at the basis of such an explicit dependence deserves further investigation, and the prototypical case of re-scaled potentials may serve as a starting point in this direction.

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Dispersive Estimates for the Dirac–Coulomb Equation

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Abstract We review some recent results on the dispersive estimates for the massless Dirac–Coulomb equation in $3D$.

1 Introduction

The Cauchy problem for the $3D$ massless Dirac–Coulomb equation can be written as follows

$$\begin{cases} i\partial_t u = \mathcal{D}_\nu u, \quad u(t,x) : \mathbb{R}_t \times \mathbb{R}_x^3 \to \mathbb{C}^4 \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

where

$$\mathcal{D}_\nu = \mathcal{D} - \frac{\nu}{|x|} I_4 \,.$$

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Here, I_4 is the 4-dimensional identity matrix and $\mathcal{D}$, the (massless) Dirac operator, can be defined as

$$\mathcal{D} = -i \sum_{k=1}^{3} \alpha_k \partial_k = -i(\alpha \cdot \nabla),$$

where the 4×4 Dirac matrices are given by

$$\alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3 \tag{2}$$

and σ_j are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

This system can be thought of as a model describing the dynamics of an electron subject to the electric field generated by a charge ν located in the origin. The range of charges ν that make the operator $\mathcal{D}_\nu$ self-adjoint is well understood: $\mathcal{D}_\nu$ is essentially self-adjoint in the range $|\nu| \leq \frac{\sqrt{3}}{2}$ and admits a distinguished self-adjoint extension in the range $\frac{\sqrt{3}}{2} < |\nu| \leq 1$ (see [19] and the references therein). From a spectral theory point of view, we recall that the continuous spectrum of the operator $\mathcal{D}_\nu$ is the whole real line (as for the case $\nu = 0$); the generalized eigenfunctions are well known and will in fact play a crucial role in our analysis, as we will see. Since the Dirac operator is of first order, the Coulomb potential is a "large" perturbation and, as a consequence, one cannot directly deduce the properties of (1) from the free case $\nu = 0$ using perturbative arguments.

From a dynamical point of view, the Dirac equation falls within the chapter of *dispersive equations* and it is strictly related to the wave equation (and to the Klein-Gordon one in the massive case) due to the fact that the Dirac matrices satisfy the anticommutation relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk}, \quad j, k = 1, 2, 3,$$

so that by applying the operator $i\partial_t + \mathcal{D}$ to a solution u of the free Dirac equation $i\partial_t u - \mathcal{D}u = 0$ yields

$$\partial_{tt} u - \Delta u = (i\partial_t + \mathcal{D})(i\partial_t - \mathcal{D})u = 0 .$$

As a consequence, u also satisfies a system of decoupled wave equations. Therefore, most of the results that hold for the free wave flow can be harmlessly translated to the (free) Dirac case by simply applying the identity above. Here, we mean to focus on *dispersive estimates* and, in particular on *Strichartz estimates*: these estimates are a remarkably useful tool in several different contexts (study of local/global well posedness for nonlinear models, scattering,...). Strichartz estimates for the solutions to the $3D$ massless Dirac equation are well known and can be written as follows

$$\|e^{-it\mathcal{D}}u_0\|_{L_t^p L_x^q} \leq \|u_0\|_{\dot{H}^{\frac{1}{2}+\frac{1}{p}-\frac{1}{q}}} \tag{4}$$

with the notation $L_t^p L_x^q = L^p(\mathbb{R}_t, L^q(\mathbb{R}_x^3))^4$ for the Strichartz spaces, the exponents (p, q) being *wave admissible*, i.e.,

$$\frac{2}{p} + \frac{2}{q} = 1, \qquad 2 \leq p \leq \infty, \quad 2 \leq q < \infty. \tag{5}$$

Following the paper [29], in the last 20 years a lot of effort has been devoted to investigating the validity of Strichartz estimates for dispersive equations perturbed by various potentials, and many strategies and techniques have been developed and sharpened. It is now well understood that the degree of homogeneity of the differential operator works somehow as a "threshold" for the validity of Strichartz estimates, meaning that for *subcritical* potentials with a faster decay than the critical one, Strichartz estimates can be recovered with more or less standard perturbative arguments, while for *supercritical* potentials with slower decay than the critical one, some non-dispersive solutions can be explicitly built in some cases. Concerning the Dirac equation, we refer to [4, 15–18] for dispersive estimates for subcritical potentials, and [1] for some counterexamples in the supercritical case. Potentials that exhibit the same homogeneity as the free operator correspond thus the *critical* case and typically turn out to represent a delicate and nontrivial problem, as indeed perturbative arguments are ruled out, and a much deeper understanding of the structure of the operator is often needed. Let us try to review some literature on the topic: in [5], [6] the authors proved Strichartz estimates (via Kato-smoothing) for the Schrödinger and wave equations perturbed by an inverse square potential, and more generally zero-order perturbations with critical decay (see also [28]). In [20] (and in subsequent [21, 22]) the authors proved the stronger time-decay estimates for the Schrödinger equation perturbed by critical electromagnetic potentials, exploiting a pseudoconformal transform that allows for an explicit representation of the propagator kernel. Time-decay estimates for the wave equation with critical magnetic potentials in 2d were later obtained in [23] and later on for various flows in [24]. Some results are available for the Dirac equation in Aharonov–Bohm potential that can be somehow thought of as the "magnetic equivalent" to (1); we postpone to Sect. 2.3 a brief overview of the topic. The massless Dirac–Coulomb equation (1) falls within this chapter, as it is indeed invariant under the natural scaling

$$u_\lambda(t, x) = u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right), \quad \lambda > 0$$

(it is thus *scaling critical*). The aim of this note is to present the dispersive estimates available for (1): in particular we will see how to prove a family of local smoothing estimates, and Strichartz estimates with loss of angular derivatives. We stress the fact that most of our results, with suitable differences, can be stated and proved in $2D$ as well. We limit ourselves to present here the $3D$ case that contains the main difficulties.

1.1 The Setup: Partial Wave Decomposition, Spectral Theory, and the Hankel Transform Method

In [5] the authors proved Strichartz estimates for solutions to the Schrödinger and wave equations perturbed with inverse square potentials. The strategy developed in that paper can be roughly summarized in the following steps:

1. Use *spherical harmonics decomposition* to reduce the equation to a radial problem;
2. Use *Hankel transform* to “diagonalize” the reduced problem and to define fractional powers of the operator $-\Delta + \frac{a}{|x|^2}$;
3. Prove a *local smoothing estimate* on a fixed spherical space using Hankel transform properties and the explicit integral representation of the fractional powers;
4. Sum back: use triangle inequality and L^2-orthogonality of spherical harmonics to obtain the desired estimate for the original dynamics;
5. Deduce Strichartz estimates.

In later years, this strategy proved to be quite flexible and was indeed exploited in several other papers in various contexts (see, e.g., [6, 8, 9]). The application of this strategy to system (1) comes with some substantial complications that are the following:

- The Dirac operator does not commute with the representation $\{\psi \to \psi(R^{-1}\cdot), R \in SO_3\}$ of the rotation group SO_3. Instead, it commutes with a spin $\frac{1}{2}$ representation of SU_2. This fact prevents from using the standard spherical harmonics decomposition and forces to rely on the so-called partial wave decomposition (see [32] Sec. 4.6.5), that we briefly review. First of all, we use spherical coordinates to write

$$L^2(\mathbb{R}^3, \mathbb{C}^4) \cong L^2((0,\infty), r^2 dr) \otimes L^2(S^2, \mathbb{C}^4)$$

with S^2 being the unit sphere. Then, we have the orthogonal decomposition on S^2:

$$L^2(S^2, \mathbb{C}^4) \cong \bigoplus_{k\in\mathbb{Z}^*} \bigoplus_{m\in\mathcal{I}_k} \mathfrak{h}_{k,m} \, .$$

Here, $\mathbb{Z}^* = \mathbb{Z}\backslash\{0\}$, $\mathcal{I}_k := \{-|k|+1/2, -|k|+3/2, \cdots, |k|-1/2\} \subset \mathbb{Z}+1/2$ and each subspace $\mathfrak{h}_{k,m}$ is two-dimensional, with orthonormal basis

$$\Xi^+_{k,m} = \begin{pmatrix} i\,\Omega_{k,m} \\ 0 \end{pmatrix}, \qquad \Xi^-_{k,m} = \begin{pmatrix} 0 \\ \Omega_{-k,m} \end{pmatrix}.$$

The functions $\Omega_{k,m}$ can be explicitly written in terms of standard spherical harmonics as

$$\Omega_{k,m} = \frac{1}{\sqrt{|2k+1|}} \begin{pmatrix} \sqrt{|k-m+1|}\, Y^{m-1/2}_{|k+1/2|-1/2} \\ \operatorname{sgn}(-k)\sqrt{|k+m+1|}\, Y^{m+1/2}_{|k+1/2|-1/2} \end{pmatrix}.$$

We thus have the unitary isomorphism

$$L^2(\mathbb{R}^3, \mathbb{C}^4) \cong \bigoplus_{\substack{k\in\mathbb{Z}^* \\ m\in\mathcal{I}_k}} L^2((0,\infty), r^2 dr) \otimes \mathfrak{h}_{k,m}$$

given by the decomposition

$$\Phi(x) = \sum_{k\in\mathbb{Z}^*} \sum_{m\in\mathcal{I}_k} f^+_{k,m}(r)\,\Xi^+_{k,m}(\theta,\phi) + f^-_{k,m}(r)\,\Xi^-_{k,m}(\theta,\phi) \tag{6}$$

which holds for any $\Phi \in L^2(\mathbb{R}^3, \mathbb{C}^4)$. The Dirac–Coulomb operator leaves invariant the partial wave subspaces $C^\infty_c(0,\infty) \otimes \mathfrak{h}_{k,m}$ and its action on each column vector of radial functions $f_{k,m} = (f^+_{k,m}, f^-_{k,m})^{\mathrm{T}}$ is given by the radial matrix

$$\mathcal{D}_{\nu,k} = \begin{pmatrix} -\frac{\nu}{r} & -\frac{d}{dr} + \frac{1+k}{r} \\ \frac{d}{dr} - \frac{1-k}{r} & -\frac{\nu}{r} \end{pmatrix}. \tag{7}$$

This isomorphism allows for the following decomposition of the dynamics of the Dirac flow: for any $k \in \mathbb{Z}^*$ the choice of an initial condition as

$$u_{0,k,m}(x) = f^+_{0,k,m}(r)\,\Xi^+_{0,k,m}(\theta,\phi) + f^-_{0,k,m}(r)\,\Xi^-_{0,k,m}(\theta,\phi)$$

implies, by Stone Theorem, that the propagator is given by

$$e^{-it\mathcal{D}_\nu} u_{0,k,m} = f^+_{k,m}(r,t)\,\Xi^+_{k,m}(\theta,\phi) + f^-_{k,m}(r,t)\,\Xi^-_{k,m}(\theta,\phi),$$

where

$$\begin{pmatrix} f^+_{k,m}(r,t) \\ f^-_{k,m}(r,t) \end{pmatrix} = e^{-it\mathcal{D}_{\nu,k}} \begin{pmatrix} f^+_{0,k,m}(r) \\ f^-_{0,k,m}(r) \end{pmatrix}.$$

In what follows, we will in fact use the shortened notation

$$f \cdot \Xi_{k,m} = f^+(r)\Xi^+_{k,m}(\theta,\phi) + f^-(r)\Xi^-_{k,m}(\theta,\phi)\,, \quad f(r) = (f^+(r), f^-(r))^{\mathrm{T}}. \tag{8}$$

- One cannot use the standard Hankel transform: the generalized eigenstates are not Bessel functions, moreover positive and negative energy eigenstates are present and should be dealt with simultaneously. We thus define, for a fixed $k \in \mathbb{Z}^*$, a "relativistic Hankel transform" of the form

$$\mathcal{P}_k f(E) = \int_0^{+\infty} H_k(Er) f(r) r^2 dr \tag{9}$$

where $E \in (0,\infty)$ and, for any $\rho > 0$, $H_k(\rho) = \begin{pmatrix} F_k(\rho) & G_k(\rho) \\ F_k(-\rho) & G_k(-\rho) \end{pmatrix}$.

The functions

$$\psi_k(\pm Er) = \begin{pmatrix} F_k(\pm Er) \\ G_k(\pm Er) \end{pmatrix} \tag{10}$$

are the generalized eigenstates of the self-adjoint operator $\mathcal{D}_{\nu,k}$ with energies $\pm E$, so that

$$\mathcal{P}_k \mathcal{D}_{\nu,k} = \mathrm{Diag}(E, -E)\mathcal{P}_k\,. \tag{11}$$

In other words, the transform $\mathcal{P}_k$ "diagonalizes" the operator $\mathcal{D}_{\nu,k}$.

Remark 1 The operator $\mathcal{D}_{\nu,k}$, its generalized eigenstates $\psi_k(\pm Er)$, and the transform $\mathcal{P}_k$ are independent of m.

This construction suggests that the functions $\psi_k = \begin{pmatrix} F_k \\ G_k \end{pmatrix}$ play a crucial role, and most of the technical issues in our dispersive estimates will consist in proving suitable estimates for them (or, more precisely, for integrals of products of these functions, as for formula (15)). We therefore recall their precise definition, as given in, e.g., [27], formulas (36.1)-(36-20): for fixed values of $k \in \mathbb{Z}^*$ and $\rho \in \mathbb{R}^*$, with

$$F_k(\rho) = \frac{\sqrt{2}|\Gamma(\gamma+1+i\nu)|}{\Gamma(2\gamma+1)} e^{\pi\nu/2}|2\rho|^{\gamma-1} \tag{12}$$

$$\times \mathrm{Im}\left\{e^{i(\rho+\xi)}{}_1F_1(\gamma - i\nu, 2\gamma+1, -2i\rho)\right\}$$

and

$$G_k(\rho) = \frac{\sqrt{2}|\Gamma(\gamma+1+i\nu)|}{\Gamma(2\gamma+1)} e^{\pi\nu/2}|2\rho|^{\gamma-1} \tag{13}$$

$$\times \mathrm{Re}\left\{e^{i(\rho+\xi)}{}_1F_1(\gamma-i\nu, 2\gamma+1, -2i\rho)\right\},$$

where ${}_1F_1(a,b,z)$ are *confluent hypergeometric functions*, $\gamma = \sqrt{k^2-\nu^2}$ and $e^{-2i\xi} = \frac{\gamma-i\nu}{k}$ is a phase shift.

One of the key tools of our strategy is represented by the following result, that has been proved in [10]:

Proposition 1 *For any $k \in \mathbb{Z}^*$ the following properties hold:*

1. *$\mathcal{P}_k$ is an L^2-isometry.*
2. *$\mathcal{P}_k\mathcal{D}_{\nu,k} = \sigma_3\Omega\mathcal{P}_k$, where $\Omega f(x) := |x|f(x)$.*
3. *The inverse transform of $\mathcal{P}_k$ is given by*

$$\mathcal{P}_k^{-1}f(r) = \int_0^{+\infty} H_k^*(Er)f(E)E^{n-1}dE \tag{14}$$

where $H_k^* = \begin{pmatrix} F_k(Er) & F_k(-Er) \\ G_k(Er) & G_k(-Er) \end{pmatrix}$ (notice the misprint in formula (2.18) in [10]).

4. *For every $\sigma \in \mathbb{R}$ we can define the fractional operators*

$$A_k^\sigma f(r) = \mathcal{P}_k\sigma_3\Omega^\sigma\mathcal{P}_k^{-1}f(r) = \int_0^{+\infty} S_k^\sigma(r,s)\cdot f(s)s^2ds, \tag{15}$$

where the integral kernel $S_k^\sigma(r,s)$ is the 2×2 matrix given by

$$S_k^\sigma(r,s) = \int_0^{+\infty} H_k(Er)\cdot H_k^*(Es)E^{2+\sigma}dE. \tag{16}$$

Remark 2 When summing on k, property (15) allows to define in a standard way the fractional powers of the operator $|\mathcal{D}_\nu|$, which will be used in forthcoming Theorem 1.

As a consequence of this Proposition, given a function $u_0 = \sum_{\substack{k\in\mathbb{Z}^* \\ m\in\mathcal{I}_k}} f_{0,k,m}\cdot\Xi_{k,m}$ we can decompose the solution to Eq. (1) as follows:

$$e^{-it\mathcal{D}_\nu}u_0 = \sum_{\substack{k\in\mathbb{Z}^* \\ m\in\mathcal{I}_k}} (e^{-it\mathcal{D}_{\nu,k}}f_{0,k,m})\cdot\Xi_{k,m} = \sum_{\substack{k\in\mathbb{Z}^* \\ m\in\mathcal{I}_k}} \mathcal{P}_k^{-1}\left[e^{-itE\sigma_3}\big(\mathcal{P}_kf_{0,k,m}\big)(E)\right]\cdot\Xi_{k,m}. \tag{17}$$

This decomposition represents the essential starting point of our analysis.

2 Dispersive Estimates

2.1 Local Smoothing

The main result of [10] is the following local smoothing (or Morawetz-type) estimate:

Theorem 1 ([10]) *Let K be a positive integer, and set*

$$\mathfrak{h}_{\geq K} = \bigoplus_{|k|\geq K} \bigoplus_{m\in\mathcal{I}_k} \mathfrak{h}_{k,m}.$$

Let u be a solution to (1). Then for any

$$1/2 < \varepsilon < \sqrt{K^2 - \nu^2} + 1/2$$

and any $f \in L^2((0,\infty), r^2 dr) \otimes \mathfrak{h}_{\geq K}$ there exists a constant $C = C(\nu, \varepsilon, K)$ such that the following estimate holds

$$\| |x|^{-\varepsilon} |\mathcal{D}_\nu|^{1/2-\varepsilon} u \|_{L^2_t L^2_x} \leq C \|u_0\|_{L^2_x}. \tag{18}$$

Remark 3 Notice that the range of ε gets wider if we require the initial condition to be orthogonal to some of the first partial wave subspaces: this also happens for the Schrödinger and wave equations with inverse square potentials (see [5]).

Remark 4 In order to deduce Strichartz estimates in a "standard" way (by the use of Duhamel formula and the application of the local smoothing estimate above twice), it would be necessary to prove (18) for $\varepsilon = 1/2$: this estimate, even if we do not have a concrete counterexample, is most likely false. The requirement of additional regularity on the initial condition seems not to help either. Therefore, at this stage, it does not seem to be possible to obtain (any kind of) Strichartz estimates with this strategy.

We mention the fact that the proof of this result turns out to be quite delicate, as it forces to provide uniform-in-k estimates for integrals in the form of (16), that involves products of confluent hypergeometric functions.

2.2 Strichartz Estimates with Loss of Angular Derivatives

Subsequently, we tried to tackle the problem of proving Strichartz estimates with loss of angular derivatives *without* using local smoothing, working directly on decomposition (17). The steps of the strategy that was inspired by [28] are, roughly speaking, the following:

1. Use partial wave decomposition and relativistic Hankel transform to decompose the flow as in (17);
2. Prove Strichartz estimates for fixed k and with unit frequency, that is assuming that $\operatorname{supp}\mathcal{P}_k(f_k)(\rho) \subset [1, 2]$;
3. Deduce Strichartz estimates for the complete dynamics using scaling argument and a dyadic decomposition.

The crucial technical step is (2), and some explicit estimates on the generalized eigenfunctions ψ_k are needed. In [28] indeed, the following bound on standard Bessel functions for $\lambda \gg 1$ plays an essential role:

$$|J_\lambda(\rho)| \leq C \times \begin{cases} e^{-D\lambda}, & 0 < \rho \leq \lambda/2, \\ \lambda^{-1/4}(|\rho - \lambda| + \lambda^{1/3})^{-1/4}, & \lambda/2 < \rho \leq 2\lambda, \\ \rho^{-1/2}, & 2\lambda < \rho \end{cases} \tag{19}$$

(notice that in our context λ has to be thought of as, roughly speaking, the "angular parameter"), and for some positive constants C and D independent on ρ and λ (for this estimate see, e.g., [2–31]). Therefore, it is necessary to provide an analog of estimate (19) for confluent hypergeometric functions. The main result obtained in [11] is indeed the following:

Theorem 2 *Let $\psi_k(\rho)$ be a generalized eigenfunction of $\mathcal{D}_\nu$ as given in* (10)*, with $|\nu| < 1$ and $\gamma := \sqrt{k^2 - \nu^2} \gg 1$. Then there exists a constant C independent on γ such that the following estimates for $F_k(\rho)$ and $G_k(\rho)$ in* (10) *hold:*

$$|F_k(\rho)|, |G_k(\rho)| \leq C \begin{cases} e^{-C\gamma}, & 0 < \rho \leq \gamma/2, \\ \gamma^{-\frac{3}{4}}\left(|\gamma - \rho| + \gamma^{\frac{1}{3}}\right)^{-\frac{1}{4}}, & \frac{\gamma}{2} \leq \rho \leq 2\gamma, \\ \rho^{-1}, & \rho > 2\gamma. \end{cases} \tag{20}$$

Remark 5 We stress the fact that while the proof of (19) is based on the Van der Corput method in which the oscillations play a crucial role, the proof of (20) relies on the construction of a steepest descent path which allows to apply Laplace's method. We should also point out that the limits of F_k, G_k as $\nu \to 0$ can be expressed in terms of the Bessel function $J_{k-1/2}$. This is consistent with the similar form of estimates (19) and (20).

With Theorem 2 at our disposal, developing the strategy presented in the previous subsection, we are able to prove the following Strichartz estimates

Theorem 3 *Let* $|\nu| < \frac{\sqrt{15}}{4}$. *For any* $u_0 \in \dot{H}^s$, *the following Strichartz estimates hold*

$$\|e^{-it\mathcal{D}_\nu}u_0\|_{L_t^2 L_r^q L_\omega^2} \leq C\|u_0\|_{\dot{H}^s} \tag{21}$$

provided

$$4 < q < \frac{3}{1-\sqrt{1-\nu^2}}m, \qquad s = 1 - \frac{3}{q}. \tag{22}$$

Remark 6 The upper bound $|\nu| < \frac{\sqrt{15}}{4}$ seems to have no physical meaning and it is a byproduct of our proof; notice anyway that as $\frac{\sqrt{15}}{4} > \frac{\sqrt{3}}{2}$, this range includes the set of charges that make the Dirac–Coulomb operator essentially self-adjoint.

Remark 7 We notice that this strategy could be developed in the 2-dimensional case as well; on the other hand, L_t^2-Strichartz estimates do not hold in 2d even for the free wave equation. Nevertheless, it might be possible to obtain some L_t^p, $p > 2$, estimates as done in [12], but this would require a fair amount of additional work, therefore we prefer to limit the estimates to the 3d case.

2.3 Open Problems and Related Models

As it is seen, the understanding of dispersive dynamics for Eq. (1) is far from being satisfactory, and many questions need to be answered. Also, there is a number of related problems and models that would certainly deserve further investigation: here we list a few of them.

- A first natural step would be trying to understand whether the estimates reviewed above hold in the massive case, that is for the operator $\mathcal{D}_\nu^m := \mathcal{D}_\nu + m\beta$ with $m > 0$: the restriction to $m = 0$ is quite structural, as indeed the massless equation exhibits a scaling that can be exploited, as opposed to the case $m > 0$ (e.g., Proposition 1 does not work properly any more when $m > 0$). Also, when $m > 0$ it is a well-known fact that the Dirac–Coulomb operator has eigenvalues in the gap $(-m, m)$, and eigenvalues represent an obstacle to dispersion; this problem can typically be bypassed by projecting the dynamics onto the absolute spectrum of the operator (see [26]). Anyway, it is not entirely clear how to deduce estimates for the massive case from the massless ones; a good starting point might be trying to adapt the results proved in [14], in which estimates for the Klein-Gordon flow are deduced from the corresponding ones for the wave flow by some kind of "shifting argument" for the estimates on the resolvent. This kind of strategy might work (with some additional care due to the fact that the presence of a mass "opens a gap" in the continuous spectrum of the operator) at least to extend the local smoothing estimate (18) to the massive case.

- The problem of proving Strichartz estimates without angular regularity for solutions to (1) remains open, and at the moment seems to be out of reach. A possible approach might be trying to prove time-decay estimates by providing a suitable representation for the integral kernel of the propagator, essentially writing it as an integral transform of the Green function (which is explicit, see [31]). Again, the complexity of the structure of the eigenfunctions will represent a technical obstruction.
- From a purely mathematical point of view, a model related to the Dirac–Coulomb equation is the Dirac equation perturbed with Aharonov–Bohm potential: the massless Dirac Hamiltonian in the Aharonov–Bohm magnetic field is

$$\mathcal{D}_A = \sigma_1(i\partial_1 + A^1) + \sigma_2(i\partial_2 + A^2), \tag{23}$$

where σ_j are the Pauli matrices and the magnetic potential $A_B(x) = (A^1(x), A^2(x))$ is given by

$$A_B : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2, \quad A_B(x) = \alpha\left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\right), \quad \alpha \in \mathbb{R}, \quad x = (x_1, x_2). \tag{24}$$

The Cauchy problem associated with the Hamiltonian (23) takes the form

$$\begin{cases} i\partial_t u = \mathcal{D}_A u, \quad u(t,x) : \mathbb{R}_t \times \mathbb{R}_x^2 \to \mathbb{C}^2 \\ u(0,x) = u_0(x). \end{cases} \tag{25}$$

We refer to [8] and the references therein for further details on the model. As it is seen, equation $i\partial_t u = \mathcal{D}_A u$ is still scaling-invariant, and in this sense we can consider system (25) similar to (1). On the other hand, the study of dispersive estimates for system (25) turns out to be remarkably simpler, and this is due to the fact that the generalized eigenfunctions of the operator $\mathcal{D}_A$ only involve standard Bessel functions (see, e.g., [25]), which are much simpler to deal with, and for which several very precise estimates are available. Therefore, mainly relying on the crucial estimate (19), generalized Strichartz estimates with loss of angular derivatives were obtained in [12]. In this case, it seems simpler to recover the full set of Strichartz estimates (without any loss): this could be done by following the strategy developed in [24], in which the propagator for the Schrödinger and wave/Klein-Gordon equations with scaling critical magnetic perturbations is explicitly built using the corresponding eigenfunctions. This strategy seems to be adaptable to deal with the Dirac case, with additional care due to the much richer structure of the equation: this is a current work in progress.

- Lastly, we mention the fact that scaling critical perturbations appear in a somehow natural way when studying the dynamical Dirac equation on curved spaces: in [7]-[3], the authors have proved, respectively, local and global in time weighted Strichartz estimates for the Dirac dynamics in some spherically symmetric spaces. The main tool in those papers consists in exploiting the spherical structure of the manifolds and to introduce suitably chosen weighted spinors, in order to translate the free dynamics on the curved space into a dynamics on the Minkowski space with a (scaling critical) potential perturbation, and then to rely on existing theory for the latter. Therefore, a better understanding of dispersive estimates for the Dirac equation with a Coulomb (or, more in general, scaling critical) perturbation would also allow to improve the estimates on non-flat manifolds with spherical symmetry.

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Heat Equation with Inverse-Square Potential of Bridging Type Across Two Half-Lines

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Abstract The heat equation with inverse-square potential on both half-lines of $\mathbb{R}$ is discussed in the presence of *bridging* boundary conditions at the origin. The problem is the lowest energy (zero-momentum) mode of the transmission of the heat flow across a Grushin-type cylinder, a generalisation of an almost-Riemannian structure with compact singularity set. This and related models are reviewed, and the issue is posed of the analysis of the dispersive properties for the heat kernel generated by the underlying positive self-adjoint operator. Numerical integration is shown that provides a first insight and relevant qualitative features of the solution at later times.

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1 Introduction: The Bridging-Heat Equation in 1D

For fixed $\alpha \in [0,1)$ we discuss in this note the following initial value problem in the unknown $u \equiv u(t,x)$, with $t \geqslant 0$ and $x \in \mathbb{R} \setminus \{0\}$:

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\alpha(\alpha+2)}{4x^2} u \;=\; 0\,, \\
u_0^-(t) \;=\; u_0^+(t) \qquad \text{where } u_0^{\pm}(t) := \lim\limits_{x\to 0^{\pm}} |x|^{\frac{\alpha}{2}} u(t,x)\,, \\
u_1^-(t) \;=\; -u_1^+(t) \qquad \text{where } u_1^{\pm}(t) := \lim\limits_{x\to 0^{\pm}} |x|^{-(1+\frac{\alpha}{2})}(u(t,x) - |x|^{-\frac{\alpha}{2}} u_0^{\pm}(t))\,, \\
u(0,x) \;=\; \varphi(x) \qquad \text{where } \varphi \in L^2(\mathbb{R})\,,
\end{cases}
\tag{1}
$$

seeking for solutions u that, for (almost every) t, belong to $L^2(\mathbb{R})$. In the above formulation (1) the existence of the limits $u_0^{\pm}$ and $u_1^{\pm}$ is part of the problem. We shall also consider the special case where the initial datum itself satisfies the very boundary conditions required at later times.

We are in particular concerned with the well-posedness of the problem and the dispersive properties of the solution(s).

At the same time, in this note we review the origin and meaning of the problem (1) in the context of geometric quantum confinement or transmission across the metric's singularity for a particle constrained on a degenerate Riemannian manifold and only subject to the geometry of the constraining manifold, thus "free to evolve" over that manifold in analogy to a classical particle moving along geodesics.

The latter viewpoint is attracting an increasing amount of interest in recent years, making it natural to investigate the time-dependent equations arising in such contexts. Ours here is a 'pilot' analysis of a more systematic study that unfolds ahead of us concerning dispersive and Strichartz estimates, and it has therefore the purpose of some propaganda and overview of the state of the art and on the future perspectives. Moreover, here we only deal with the heat evolution, and not the Schrödinger evolution, as we shall comment in due time.

Prior to outlining the geometric background, let us comment on the structure of the problem (1). The considered PDE is a heat type equation governed by the second order, elliptic (Schrödinger) differential operator

$$
-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{C_\alpha}{x^2}\,, \qquad C_\alpha := \frac{\alpha(\alpha+2)}{4x^2} \tag{2}
$$

(the precise meaning of the parameter α and its presence through the coefficient C_α will be clear after discussing the parent geometric model). As such, the complete description of square-integrable solutions to the associated heat equation is achieved through a standard PDE analysis, once certain features of (2) are known as a linear operator on $L^2(\mathbb{R})$. For concreteness, a limit-circle/limit-point argument [24, Theorems X.11] shows that when $C_\alpha \geqslant \frac{3}{4}$, i.e., $\alpha \in (-\infty,-3] \cup [1,+\infty)$, the linear operator (2) minimally defined on smooth functions compactly supported away from $x = 0$ is actually essentially self-adjoint on $L^2(\mathbb{R})$. Denoting its closure with A, one concludes that A is a self-adjoint operator with strictly positive spectrum and domain $\mathcal{D}(A)$ that explicitly, when $C_\alpha > \frac{3}{4}$, is the Sobolev space $H^2_0(\mathbb{R})$. As a straightforward consequence of the abstract theory of differential equation on Hilbert space [25, Proposition 6.6], one then concludes that the heat equation $\frac{\mathrm{d}}{\mathrm{d}t}u = -Au$ with initial datum $\varphi \in L^2(\mathbb{R})$ admits a unique solution in $C^1(\mathbb{R}^+_t, L^2(\mathbb{R}_x))$, with $u(t,\cdot) \in \mathcal{D}(A)$ at ever later $t > 0$, explicitly given by $u(t,x) = (e^{-tA}\varphi)(x)$.

In fact, it is worth recalling that the inverse-square potential differential operator (2) is greatly studied and deeply understood from many standpoints, in particular, both as far radial space-time (Strichartz) estimates are concerned both in the linear and non-linear Schrödinger evolution (see, e.g., [10, 20] and the references therein), and as a Bessel operator on the L^2-space of the half-line (see, e.g., [12] and its precursors in that prolific research line).

Yet, in addition to the differential side, the problem (1) prescribes the solutions u to satisfy certain boundary conditions at $x = 0$. The first one, $g_0^+(t) = g_0^-(t)$, can be interpreted as the continuity of the function, up to the weight $|x|^{\frac{\alpha}{2}}$ that allows u to have some degree of singularity at the origin; analogously, the condition $g_1^+(t) = -g_1^-(t)$ links the right and left derivative at zero, up to certain weights, and taken directionally from each side. In the regime $C_\alpha > \frac{3}{4}$ such conditions are obviously redundant, but when $C_\alpha \leqslant \frac{3}{4}$ an ad hoc analysis is needed to recognise that the prescribed behaviour at the origin expresses another condition of self-adjointness and positivity. Such an analysis has been carried out in several recent works [8, 16, 18, 19, 21] and is concisely reviewed in Sect. 2. The net result, for the sake of the present discussion, is the following.

Theorem 1 ([19]) *Let* $\alpha \in [0,1)$ *and let* C_α *be given by* (2).

(i) The space

$$\mathcal{D} := \left\{ g \in L^2(\mathbb{R}) \,\middle|\, \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{C_\alpha}{x^2}\right) g \in L^2(\mathbb{R}) \right\}$$

is a dense subspace of $L^2(\mathbb{R})$ *and for every* $g \in \mathcal{D}$ *the following limits exist and are finite:*

$$\begin{aligned} g_0^\pm &= \lim_{x\to 0^\pm} |x|^{\frac{\alpha}{2}} g^\pm(x)\,, \\ g_1^\pm &= \lim_{x\to 0^\pm} |x|^{-(1+\frac{\alpha}{2})}\big(g^\pm(x) - g_0^\pm |x|^{-\frac{\alpha}{2}}\big)\,. \end{aligned} \tag{3}$$

(ii) The operator

$$\begin{aligned} \mathcal{D}\big(A_\alpha^{\mathrm{B}}\big) &= \{g \in \mathcal{D} \,|\, g_0^+ = g_0^-\,,\; g_1^+ = -g_1^-\}\,, \\ A_\alpha^{\mathrm{B}} g &= -g'' + C_\alpha |x|^{-2} g \end{aligned} \tag{4}$$

is self-adjoint on $L^2(\mathbb{R})$ *and non-negative. Its spectrum is* $[0, +\infty)$ *and is all essential and absolutely continuous.*

In Theorem 1 we only consider the regime $\alpha \in [0, 1)$. The remaining regime $\alpha \in [-3, 0)$ is simply less relevant from the viewpoint of the underlying geometric model, as will be argued in Sect. 2. And, as discussed above, when $\alpha \in (-\infty, -3) \cup (1, +\infty)$ one applies standard limit-point/limit-circle considerations.

In view of Theorem 1, the initial value problem (1) is immediately interpreted as the problem for the one-dimensional heat equation governed by the positive and self-adjoint operator A_α^{B}, and therefore it admits unique solution $u(t) = e^{-tA_\alpha^{\mathrm{B}}}\varphi$, again by abstract facts of differential equations on Hilbert space [25, Proposition 6.6].

The *well-posedness* of (1) is therefore fully controlled in $C^1(\mathbb{R}_t^+, L^2(\mathbb{R}_x))$ with $u(t, \cdot) \in \mathcal{D}(A_\alpha^{\mathrm{B}})$ at every $t > 0$.

The superscript 'B' in A_α^{B} is to refer to certain '*bridging*' features of optimal transmission across the origin, allowing in a precise sense complete communication between the right and left half-line, induced by A_α^{B}, as compared to a whole family of similar transmission protocols.

Indeed, in Sect. 2 it will be recapped how initial value problems like (1), and their counterparts with the Schrödinger equation, describe the heat type or Schrödinger-type propagation over a particular almost-Riemannian structure, customarily referred to as a '*Grushin cylinder*', constituted by an infinite two-dimensional cylinder with a non-flat metric that becomes suitably singular on a given orthogonal section. Depending on the magnitude of the metric's singularity, which is quantified by the parameter α, the transmission is either inhibited, so that a function initially supported on one half-cylinder remains confined in that half at later times, or on the opposite it is allowed, through a precise set of boundary conditions between the two halves. This can be qualitatively visualised as in Fig. 1 with cylinders that shrink to one point or get flattened in correspondence of the given singular section. *One-dimensional* problems like (1) emerge for the evolution on the lowest energy mode, which corresponds to functions on the cylinder that are constant along the compact variable. It turns out that in certain regimes of metric's singularity (and $\alpha \in [0, 1)$ is the physically most significant regime) an infinity of transmission protocols emerge, each characterised by suitable boundary conditions of self-adjointness, and each yielding a heat type or Schrödinger-type equation. Among them, the *bridging* protocol described by the operator A_α^{B} given by (4) displays distinguished features of optimal transmission. In practice—see equations (30)–(31) below—the heat (and Schrödinger) equation of bridging type between the two half-lines is the one that describes a crossing at $x = 0$ 'without spatial filter'

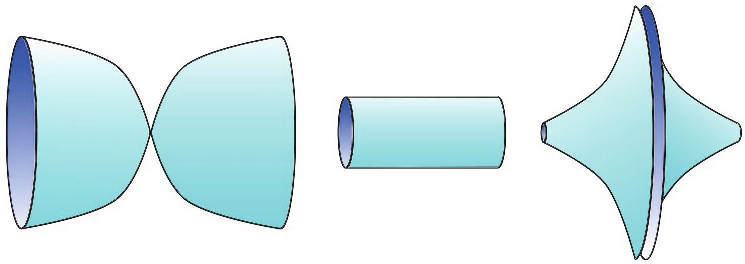

Fig. 1 Manifold M_α for different α: $\alpha < 0$ (left), $\alpha = 0$ (center) and $\alpha > 0$ (right)

(continuity of the function) and 'without energy filter' (the fraction of transmitted flux does not depend on the incident energy).

The bridging protocol A_α^{B} was first identified in the recent work [8]. The comparison analysis of the bridging protocol with respect to the whole family of the other physically meaningful ones was subsequently analysed in [16, 18]. Section 3 reports on the recent literature of closely related models and results. Moreover, in [7, 8] the bridging and some other protocols were analysed from the perspective of the conservation of the total heat, or equivalently, the perspective of infinite lifespan of the stochastic processes generated by such operators (stochastic completeness).

It should be then sufficiently clear at this point that the initial value problem (1) describes the low-energy transmission of bridging type between the two halves of a Grushin cylinder with metric singularity at zero. In view of that, beside the already guaranteed well-posedness, *dispersive properties* of the solution come to have great relevance in connection with the underlying physical transmission protocol, in particular L^p-L^q estimates, smoothing estimates, and space-time (Strichartz) estimates for the heat semi-group associated with A_α^{B}.

These are the analogue of the well-known estimates for the classical heat equation $(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2})u(t,x) = 0$, $u(0,x) = \varphi(x)$, for which one has [26, Section 2.2]

$$\begin{aligned}
\big\| u(t,\cdot) \big\|_{L^p} &\lesssim t^{-\frac{1}{2}(\frac{1}{r}-\frac{1}{p})} \|\varphi\|_{L^r} & & 1 \leqslant r \leqslant p \leqslant \infty\,, \\
\Big\| \frac{\partial}{\partial x} u(t,\cdot) \Big\|_{L^p} &\lesssim t^{-\frac{1}{2}(\frac{1}{r}-\frac{1}{p}+1)} \|\varphi\|_{L^r} & & 1 \leqslant r \leqslant p \leqslant \infty\,, \\
\| u \|_{L^q(\mathbb{R}_t^+, L_x^p)} &\lesssim \|\varphi\|_{L^2} & & 2 \leqslant p < \infty\,,\ q = \frac{4p}{p-2}\,, \\
\Big\| \frac{\partial}{\partial x} u \Big\|_{L^2(\mathbb{R}_t^+, L_x^2)} &\lesssim \|\varphi\|_{L^2}\,.
\end{aligned} \tag{5}$$

Establishing the analogue of (5) for the heat type semi-group $\exp(-tA_\alpha^{\mathrm{B}})$, and in fact eventually also for the Schrödinger unitary group $\exp(-\mathrm{i}tA_\alpha^{\mathrm{B}})$, as well as for the corresponding semi-groups and groups induced by other transmission protocols, and more generally on other geometries and classes of almost-Riemannian structures beyond the Grushin cylinder, appear to be one of most relevant challenges in this field, with abstract interest per se and impact on applications, of quantum control concern in the first place.

In this respect we intend with this note to promote the above questions and advertise them for future investigations, in particular posing them in the rigorous context of geometric confinement and transmission protocols. Such an overview is given, as mentioned, in Sects. 2 and 3.

Last, in Sect. 4 we present a glance at numerical computations of the solution u of (1) when the initial datum is well localised on one half-line. The numerical evidence is strong on the dynamical formation of the bridging boundary conditions at $x = 0$ at later times, and on a general behaviour that is qualitatively comparable to the classical heat propagation.

In fact, at present no analytic computation is available of the heat (and, in the future, the Schrödinger) propagator generated by A_α^{B}, and numerics is a first natural approach to infer meaningful properties to be rigorously proved in forthcoming investigations.

2 A Concise Review of Geometric Confinement and Transmission Protocols in a Grushin Cylinder

We have already anticipated that the problem (1) provides the one-dimensional description for the heat flow across the singularity of a Grushin cylinder, and let us give in this section a concise overview of the problem from that perspective.

Grushin cylinders are Riemannian manifolds $M_\alpha \equiv (M, g_\alpha)$, with parameter $\alpha \in \mathbb{R}$, where

$$M^{\pm} := \mathbb{R}_x^{\pm} \times \mathbb{S}_y^1 \qquad \mathcal{Z} := \{0\} \times \mathbb{S}_y^1 , \qquad M := M^+ \cup M^- \tag{6}$$

and with degenerate Riemannian metric

$$g_\alpha := \mathrm{d}x \otimes \mathrm{d}x + |x|^{-2\alpha} \mathrm{d}y \otimes \mathrm{d}y . \tag{7}$$

Thus, M_α is a two-dimensional manifold built upon the cylinder $\mathbb{R} \times \mathbb{S}^1$, with singularity locus $\mathcal{Z}$ and incomplete Riemannian metric both on the right and the left half-cylinder $\mathbb{R}^{\pm} \times \mathbb{S}^1$ meaning that geodesics cross smoothly the singularity $\mathcal{Z}$ at finite times). The values $\alpha = -1$, $\alpha = 0$, and $\alpha = 1$ select, respectively, the flat cone, the Euclidean cylinder, and the standard '*Grushin cylinder*' [11, Chapter 11]: in the latter case one has an '*almost-Riemannian structure*' on $\mathbb{R} \times \mathbb{S}^1 = M^+ \cup \mathcal{Z} \cup M^-$ in the rigorous sense of [3, Sec. 1] or [23, Sect. 7.1]. Actually,

g_α is defined as the unique metric for which the distribution of vector fields globally defined on $\mathbb{R} \times \mathbb{S}^1$ as

$$X_1(x,y) \;:=\; \frac{\partial}{\partial x}\,, \qquad X_2^{(\alpha)}(x,y) \;:=\; |x|^\alpha \frac{\partial}{\partial y} \tag{8}$$

is an orthonormal frame at every $(x,y) \in \mathbb{R}\times\mathbb{S}^1$: in this regard, the Grushin cylinder ($\alpha = 1$) is a two-dimensional almost-Riemannian manifold of step two, meaning that

$$\mathrm{span}\Big\{X_1, X_2^{(1)}, \Big[X_1, X_2^{(1)}\Big]\Big\}\Big|_{(x,y)} \;=\; \mathbb{R}^2 \qquad \forall (x,y) \in \mathbb{R}\times\mathbb{S}^1\,,$$

where $[X_1, X_2^\alpha]$ denotes the Lie brackets of vector fields. In fact, M_α is a hyperbolic manifold whenever $\alpha > 0$, with Gaussian (sectional) curvature

$$K_\alpha(x,y) \;=\; -\frac{\alpha(\alpha+1)}{x^2}\,. \tag{9}$$

To each M_α one naturally associates the Riemannian volume form

$$\mu_\alpha \;:=\; \mathrm{vol}_{g_\alpha} \;=\; \sqrt{\det g_\alpha}\,\mathrm{d}x\wedge\mathrm{d}y \;=\; |x|^{-\alpha}\,\mathrm{d}x\wedge\mathrm{d}y\,, \tag{10}$$

the Hilbert space

$$\mathcal{H}_\alpha \;:=\; L^2(M, \mathrm{d}\mu_\alpha)\,, \tag{11}$$

understood as the completion of $C_c^\infty(M)$ with respect to the scalar product

$$\langle\psi,\varphi\rangle_\alpha \;:=\; \iint_{\mathbb{R}\times\mathbb{S}^1} \overline{\psi(x,y)}\,\varphi(x,y)\,\frac{1}{|x|^\alpha}\,\mathrm{d}x\,\mathrm{d}y\,, \tag{12}$$

and the (Riemannian) Laplace–Beltrami operator $\Delta_{\mu_\alpha} := \mathrm{div}_{\mu_\alpha} \circ \nabla$ acting on functions over M_α. A standard computation (see, e.g., [18, Sect. 2]) yields explicitly

$$\Delta_{\mu_\alpha} \;=\; \frac{\partial^2}{\partial x^2} + |x|^{2\alpha}\frac{\partial^2}{\partial y^2} - \frac{\alpha}{|x|}\frac{\partial}{\partial x}\,. \tag{13}$$

As a linear operator on $\mathcal{H}_\alpha$ one minimally defines Δ_{μ_α} on the dense subspace of $L^2(M, \mathrm{d}\mu_\alpha)$-functions that are smooth and compactly supported away from the metric's singularity locus $\mathcal{Z}$, thus introducing

$$H_\alpha \;:=\; -\Delta_{\mu_\alpha}\,, \qquad \mathcal{D}(H_\alpha) \;:=\; C_c^\infty(M)\,. \tag{14}$$

The Green identity implies that H_α is symmetric and non-negative. One has the following classification.

Theorem 2 ([6, 8, 18])

(i) If $\alpha \in (-\infty, -3] \cup [1, +\infty)$*, then the operator* H_α *is essentially self-adjoint.*
(ii) If $\alpha \in (-3, -1]$*, then* H_α *is not essentially self-adjoint with deficiency index* 2*.*
(iii) If $\alpha \in (-1, 1)$*, then* H_α *is not essentially self-adjoint and it has infinite deficiency index.*

The same holds, separately, for the symmetric operators $H_\alpha^\pm$ minimally defined on the L^2-space of each half-cylinder.

With respect to the Hilbert space orthogonal decomposition

$$\mathcal{H}_\alpha \;=\; L^2(M, \mathrm{d}\mu_\alpha) \;\cong\; L^2(M^-, \mathrm{d}\mu_\alpha) \oplus L^2(M^+, \mathrm{d}\mu_\alpha) \tag{15}$$

the operator H_α is reduced as $H_\alpha = H_\alpha^- \oplus H_\alpha^+$, and therefore in the regime of essential self-adjointness its closure is the reduced, non-negative, self-adjoint operator $\overline{H_\alpha} = \overline{H_\alpha^-} \oplus \overline{H_\alpha^+}$. This implies that both the Schrödinger equation $\partial_t u = -\mathrm{i}\overline{H_\alpha}u$ and the heat equation $\partial_t u = -\overline{H_\alpha}u$ decompose to uncoupled equations on each half-cylinder, or, better to say, group and semi-group decompose, respectively, as $e^{-\mathrm{i}t\overline{H_\alpha}} = e^{-\mathrm{i}t\overline{H_\alpha^-}} \oplus e^{-\mathrm{i}t\overline{H_\alpha^+}}$ and $e^{-t\overline{H_\alpha}} = e^{-t\overline{H_\alpha^-}} \oplus e^{-t\overline{H_\alpha^+}}$, with the consequence that an initial datum supported, say, only on one half, keeps evolving in that half at each later time. This phenomenon is customarily referred to as '*heat-geometric confinement*' and '*quantum-geometric confinement*', respectively, to emphasise the sole effect of the geometry (meaning that we are not considering any potential energy on the manifold, but only the kinetic one), with no coupling boundary conditions—hence no interaction—declared at $\mathcal{Z}$. Quantum-mechanically, in this regime of the Grushin metric, a quantum particle constrained on M_α and left 'free' to evolve only under the effect of the underlying geometry never happens to cross the singularity locus $\mathcal{Z}$.

The scenario becomes much more diversified when H_α is *not* essentially self-adjoint and therefore admits non-trivial self-adjoint extensions. Our regime of interest includes $\alpha \in (0, 1)$, the sub-case of greatest relevance because it corresponds to an actual local singularity (and not vanishing) of the metric g_α, and for the purposes of this note we shall only consider such α's. Qualitatively analogous results can be established in the remaining non-self-adjoint regime $\alpha \in (-3, 0)$.

There is in fact a giant family of inequivalent self-adjoint realisations of H_α when $\alpha \in [0, 1)$, as the deficiency index is infinite. Each one is characterised by boundary conditions at $\mathcal{Z}$ that prescribe a one-sided or two-sided interaction with the boundary, or more generally a protocol of left↔right transmission. Such a family includes physically unstable realisations (those that are not lower semi-bounded), as well as a huge amount of unphysical realisations, such as those with non-local boundary conditions at $\mathcal{Z}$.

An extensive and fairly explicit classification of physical extensions of H_α was recently completed in [19].

Theorem 3 ([19]) *Let* $\alpha \in [0,1)$. H_α *defined in* (14) *admits, among others, the following families of self-adjoint extensions with respect to* $L^2(M, \mathrm{d}\mu_\alpha)$*:*

- *Friedrichs extension:* $H_{\alpha,\mathrm{F}}$*;*
- *Family* $\mathrm{I_R}$*:* $\{H_{\alpha,\mathrm{R}}^{[\gamma]} \mid \gamma \in \mathbb{R}\}$*;*
- *Family* $\mathrm{I_L}$*:* $\{H_{\alpha,\mathrm{L}}^{[\gamma]} \mid \gamma \in \mathbb{R}\}$*;*
- *Family* II_a *with* $a \in \mathbb{C}$*:* $\{H_{\alpha,a}^{[\gamma]} \mid \gamma \in \mathbb{R}\}$*;*
- *Family* III*:* $\{H_{\alpha}^{[\Gamma]} \mid \Gamma \equiv (\gamma_1, \gamma_2, \gamma_3, \gamma_4) \in \mathbb{R}^4\}$.

Each member of any such family acts precisely as the differential operator $-\Delta_{\mu_\alpha}$ *on a domain of functions* $f \in L^2(M, \mathrm{d}\mu_\alpha)$ *satisfying the following properties.*

(i) Integrability and regularity:

$$\sum_{\pm} \iint_{\mathbb{R}_x^{\pm} \times \mathbb{S}_y^1} \big|(\Delta_{\mu_\alpha} f)(x,y)\big|^2 \, \mathrm{d}\mu_\alpha(x,y) \; < \; +\infty \,. \tag{16}$$

(ii) Boundary condition: The limits

$$f_0^{\pm}(y) = \lim_{x \to 0^{\pm}} f(x,y)\,, \tag{17}$$

$$f_1^{\pm}(y) = \pm(1+\alpha)^{-1} \lim_{x \to 0^{\pm}} \Big(\frac{1}{|x|^\alpha} \frac{\partial f(x,y)}{\partial x} \Big) \tag{18}$$

exist and are finite for almost every $y \in \mathbb{S}^1$*, and depending on the considered type of extension, and for almost every* $y \in \mathbb{R}$*,*

$$f_0^{\pm}(y) \; = \; 0 \qquad \textit{if } \; f \in \mathcal{D}(H_{\alpha,\mathrm{F}})\,, \tag{19}$$

$$\begin{cases} f_0^{-}(y) = 0 \\ f_1^{+}(y) = \gamma f_0^{+}(y) \end{cases} \qquad \textit{if } \; f \in \mathcal{D}(H_{\alpha,\mathrm{R}}^{[\gamma]})\,, \tag{20}$$

$$\begin{cases} f_1^{-}(y) = \gamma f_0^{-}(y) \\ f_0^{+}(y) = 0 \end{cases} \qquad \textit{if } \; f \in \mathcal{D}(H_{\alpha,\mathrm{L}}^{[\gamma]})\,, \tag{21}$$

$$\begin{cases} f_0^{+}(y) = a\, f_0^{-}(y) \\ f_1^{-}(y) + \overline{a}\, f_1^{+}(y) = \gamma f_0^{-}(y) \end{cases} \qquad \textit{if } \; f \in \mathcal{D}(H_{\alpha,a}^{[\gamma]})\,, \tag{22}$$

$$\begin{cases} f_1^{-}(y) = \gamma_1 f_0^{-}(y) + (\gamma_2 + \mathrm{i}\gamma_3) f_0^{+}(y) \\ f_1^{+}(y) = (\gamma_2 - \mathrm{i}\gamma_3) f_0^{-}(y) + \gamma_4 f_0^{+}(y) \end{cases} \qquad \textit{if } \; f \in \mathcal{D}(H_{\alpha}^{[\Gamma]})\,. \tag{23}$$

One can further select those extensions that are non-negative and then induce a heat type flow.

Theorem 4 ([16])

- *The Friedrichs extension* $H_{\alpha,\mathrm{F}}$ *is non-negative.*
- *Extensions in the family* $\mathrm{I_R}$, $\mathrm{I_L}$, *and* II_a, $a \in \mathbb{C}$, *are non-negative if and only if* $\gamma \geqslant 0$.
- *Extensions in the family* III *are non-negative if and only if so is the matrix*

$$\widetilde{\Gamma} := \begin{pmatrix} \gamma_1 & \gamma_2 + \mathrm{i}\gamma_3 \\ \gamma_2 - \mathrm{i}\gamma_3 & \gamma_4 \end{pmatrix},$$

i.e., if and only if $\gamma_1 + \gamma_4 > 0$ *and* $\gamma_1\gamma_4 \geqslant \gamma_2^2 + \gamma_3^2$.

A customary *quantum-mechanical* quantification of the transmission modelled by each extension is the fraction of Schrödinger flux that gets transmitted vs reflected when a beam of particles are shot free from infinity towards $\mathcal{Z}$. This analysis, albeit in a *Schrödinger equation* framework, elucidates the qualitative properties of the crossing at $x = 0$ and was recently done in [16]. Intuitively speaking, far away from $\mathcal{Z}$ the metric tends to become flat and the action $-\Delta_{\mu_\alpha}$ of each self-adjoint 'free Hamiltonian' tends to resemble that of the free Laplacian $-\Delta$, plus the correction due to the $(|x|^{-1}\partial_x)$-term, on wave functions $f(x, y)$ that are constant in y. This suggests that at very large distances a quantum particle evolves free from the effects of the underlying geometry, and one can speak of scattering states of energy $E > 0$. The precise shape of the wave function f_{scatt} of such a scattering state can be easily guessed to be of the form

$$f_{\mathrm{scatt}}(x, y) \sim |x|^{\frac{\alpha}{2}} e^{\pm\mathrm{i}x\sqrt{E}} \qquad \text{as } |x| \to +\infty\,. \tag{24}$$

Indeed, $-\Delta_{\mu_\alpha} f_{\mathrm{scatt}} \sim E f_{\mathrm{scatt}} + \frac{\alpha(2+\alpha)}{4|x|^2} f_{\mathrm{scatt}}$, that is, up to a very small $O(|x|^{-2})$-correction, f_{scatt} is a generalised eigenfunction of $-\Delta_{\mu_\alpha}$ with eigenvalue E. All this can be fully justified on rigorous grounds [16] and leads naturally to the definition of the '*transmission coefficient*' and '*reflection coefficient*' for the scattering, namely the spatial density of the transmitted flux and the reflected flux, normalised with respect to the density of the incident flux. Obviously, no scattering across the singularity occurs for Friedrichs, or type-$\mathrm{I_R}$, or type-$\mathrm{I_L}$ quantum protocols, whereas in type-II_a scattering one obtains the following (analogous conclusions can be made for type-III scattering).

Theorem 5 ([16]) *Let $\alpha \in [0,1)$, $a \in \mathbb{C}$, $\gamma \in \mathbb{R}$. The transmission coefficient $T_{\alpha,a,\gamma}(E)$ and the reflection coefficient $R_{\alpha,a,\gamma}(E)$ at given energy $E > 0$ for the Schrödinger transmission protocol governed by $H_{\alpha,a}^{[\gamma]}$ are given by*

$$
\begin{aligned}
T_{\alpha,a,\gamma}(E) \;&=\; \left| \frac{E^{\frac{1+\alpha}{2}}\,(1+e^{\mathrm{i}\pi\alpha})\,\Gamma(\frac{1-\alpha}{2})\,\overline{a}}{E^{\frac{1+\alpha}{2}}\,\Gamma(\frac{1-\alpha}{2})(1+|a|^2)+\mathrm{i}\,\gamma\,2^{1+\alpha}e^{\mathrm{i}\frac{\pi}{2}\alpha}\Gamma(\frac{3+\alpha}{2})} \right|^2 ,\\
R_{\alpha,a,\gamma}(E) \;&=\; \left| \frac{E^{\frac{1+\alpha}{2}}\,\Gamma(\frac{1-\alpha}{2})\,(1-|a|^2\,e^{\mathrm{i}\pi\alpha})+\mathrm{i}\,\gamma\,2^{1+\alpha}e^{\mathrm{i}\frac{\pi}{2}\alpha}\Gamma(\frac{3+\alpha}{2})}{E^{\frac{1+\alpha}{2}}\,\Gamma(\frac{1-\alpha}{2})(1+|a|^2)+\mathrm{i}\,\gamma\,2^{1+\alpha}e^{\mathrm{i}\frac{\pi}{2}\alpha}\Gamma(\frac{3+\alpha}{2})} \right|^2 .
\end{aligned}
\tag{25}
$$

They satisfy

$$
T_{\alpha,a,\gamma}(E) + R_{\alpha,a,\gamma}(E) \;=\; 1\,, \tag{26}
$$

and when $\gamma = 0$ they are independent of E . The scattering is reflection-less ($R_{\alpha,a,\gamma}(E) = 0$) when

$$
E \;=\; \left(\frac{2^{1+\alpha}\,\gamma\,\Gamma(\frac{3+\alpha}{2})\,\sin\frac{\pi}{2}\alpha}{\Gamma(\frac{1-\alpha}{2})(1-\cos\pi\alpha)} \right)^{\frac{2}{1+\alpha}} , \tag{27}
$$

provided that $\alpha \in (0,1)$, $|a| = 1$, and $\gamma > 0$. In the high energy limit the scattering is independent of the extension parameter γ and one has

$$
\begin{aligned}
\lim_{E\to+\infty} T_{\alpha,a,\gamma}(E) \;&=\; \frac{2\,|a|^2(1+\cos\pi\alpha)}{(1+|a|^2)^2}\,,\\
\lim_{E\to+\infty} R_{\alpha,a,\gamma}(E) \;&=\; \frac{1+|a|^4-2|a|^2\cos\pi\alpha}{(1+|a|^2)^2}\,,
\end{aligned}
\tag{28}
$$

whereas in the low-energy limit, for $\gamma \neq 0$,

$$
\begin{aligned}
\lim_{E\downarrow 0} T_{\alpha,a,\gamma}(E) \;&=\; 0\,,\\
\lim_{E\downarrow 0} R_{\alpha,a,\gamma}(E) \;&=\; 1\,.
\end{aligned}
\tag{29}
$$

Upon inspection of the boundary conditions (19)–(23) one sees that the type-II$_a$ extension $H_{\alpha,a}^{[\gamma]}$ with $a = 1$ and $\gamma = 0$ imposes the local behaviour

$$
\begin{aligned}
\lim_{x\to 0^-} f(x,y) \;&=\; \lim_{x\to 0^+} f(x,y)\\
\lim_{x\to 0^-} \Big(\frac{1}{|x|^\alpha}\,\frac{\partial f(x,y)}{\partial x} \Big) \;&=\; \lim_{x\to 0^+} \Big(\frac{1}{|x|^\alpha}\,\frac{\partial f(x,y)}{\partial x} \Big)\,,
\end{aligned}
\tag{30}
$$

namely the distinguished feature of having a domain of functions that are continuous across the Grushin singularity, together with their weighted derivative. $H_{\alpha,1}^{[0]}$ is called the '*bridging extension*' of H_α, and for it we shall simply write H_α^{B}. In view of the results reviewed so far, the transmission modelled by the bridging extension

- has *no spatial filter* in the sense of (30) (in fact, all type-II$_a$ protocols with $a = 1$ impose local continuity at $\mathcal{Z}$; quantum-mechanically this is interpreted as a lack of jump in the particle's probability density from one side to the other of the singularity),
- and has *no energy filter* in the Schrödinger scattering, indeed, H_α^{B} and all type-II$_a$ protocols with $\gamma = 0$ induce a scattering where the fraction of transmitted and reflected flux does not depend on the incident energy (see (25) above),

$$\begin{aligned} T_\alpha^{\mathrm{B}} &:= T_{\alpha,1,0}(E) = \frac{1}{2}(1+\cos\pi\alpha)\,, \\ R_\alpha^{\mathrm{B}} &:= R_{\alpha,1,0}(E) = \frac{1}{2}(1-\cos\pi\alpha)\,, \end{aligned} \tag{31}$$

meaning that the singularity does not act as a filter in the energy.

The overall picture surveyed so far poses naturally the question of the analysis of the heat type flow generated by the *positive* and *self-adjoint* realisations of the Laplace–Beltrami operator on $\mathcal{H}_\alpha$ (Theorem 4), as well as the Schrödinger-type flow generated by *self-adjoint realisations* (Theorem 3), let alone the study of *non-linear* heat and Schrödinger equations on M_α with linear part given by a self-adjoint Laplace–Beltrami operator. This appears to be a completely uncharted territory of notable relevance in abstract terms and for applications. The gap between such future goals and the current knowledge is a lack of informative characterisation of the heat and Schrödinger propagator's kernel.

To complete the present review, let us make the connection explicit between the two-dimensional heat type equation induced by H_α^{B} and the one-dimensional problem (1).

This is done [8, 19] by means of the canonical Hilbert space unitary isomorphism $\mathcal{H}_\alpha \xrightarrow{\cong} \mathcal{H}$, where

$$\begin{aligned} \mathcal{H} &:= \mathcal{F}_2 U_\alpha L^2(M, \mathrm{d}\mu_\alpha) \cong \ell^2(\mathbb{Z}, L^2(\mathbb{R}, \mathrm{d}x)) \cong \mathcal{H}^- \oplus \mathcal{H}^+ \cong \bigoplus_{k\in\mathbb{Z}} \mathfrak{h}\,, \\ \mathfrak{h} &:= L^2(\mathbb{R}^-, \mathrm{d}x) \oplus L^2(\mathbb{R}^+, \mathrm{d}x) \cong L^2(\mathbb{R}, \mathrm{d}x)\,, \end{aligned} \tag{32}$$

recalling that

$$\mathcal{H}_\alpha \cong L^2(M^-, \mathrm{d}\mu_\alpha) \oplus L^2(M^+, \mathrm{d}\mu_\alpha)\,, \tag{33}$$

and where the unitary transformations $U_\alpha := U_\alpha^- \oplus U_\alpha^+$ and $\mathcal{F}_2 := \mathcal{F}_2^- \oplus \mathcal{F}_2^+$ are defined, respectively, as

$$\begin{aligned} U_\alpha^\pm : L^2(\mathbb{R}^\pm \times \mathbb{S}^1, |x|^{-\alpha}\mathrm{d}x\mathrm{d}y) &\xrightarrow{\cong} L^2(\mathbb{R}^\pm \times \mathbb{S}^1, \mathrm{d}x\mathrm{d}y)\,, \\ f &\mapsto \phi := |x|^{-\frac{\alpha}{2}} f\,, \end{aligned} \tag{34}$$

and

$$\begin{aligned} \mathcal{F}_2^\pm : L^2(\mathbb{R}^\pm \times \mathbb{S}^1, \mathrm{d}x\mathrm{d}y) &\xrightarrow{\cong} L^2(\mathbb{R}^\pm, \mathrm{d}x) \otimes \ell^2(\mathbb{Z})\,, \\ \phi &\mapsto \psi \equiv (\psi_k)_{k\in\mathbb{Z}}\,, \\ e_k(y) := \frac{e^{\mathrm{i}ky}}{\sqrt{2\pi}}\,, \qquad \psi_k(x) &:= \int_0^{2\pi} \overline{e_k(y)}\,\phi(x,y)\,\mathrm{d}y\,, \qquad x \in \mathbb{R}^\pm \end{aligned} \tag{35}$$

(thus, $\phi(x,y) = \sum_{k\in\mathbb{Z}} \psi_k(x) e_k(y)$ in the L^2-convergent sense). This provides, up to isomorphism, the orthogonal sum decomposition of the Hilbert space of interest into identical 'bilateral' *fibres* $\mathfrak{h} = L^2(\mathbb{R}^-, \mathrm{d}x) \oplus L^2(\mathbb{R}^+, \mathrm{d}x) \cong L^2(\mathbb{R}, \mathrm{d}x)$. The decomposition is discrete, as a consequence of having taken the Fourier transform $\mathcal{F}_2$ only in the compact variable y.

Theorem 6 ([19]) *Let $\alpha \in [0,1)$. Through the isomorphism (32) the self-adjoint bridging operator H_α^{B} on $\mathcal{H}_\alpha = L^2(M, \mathrm{d}\mu_\alpha)$ is unitarily equivalent to the self-adjoint operator $\mathscr{H}_\alpha^{\mathrm{B}}$ on $\mathcal{H} \cong \ell^2(\mathbb{Z}, L^2(\mathbb{R}))$, namely*

$$H_\alpha^{\mathrm{B}} = (U_\alpha)^{-1}(\mathcal{F}_2)^{-1} \mathscr{H}_\alpha^{\mathrm{B}}\, \mathcal{F}_2\, U_\alpha\,, \tag{36}$$

where

$$\mathscr{H}_\alpha^{\mathrm{B}} = \bigoplus_{k\in\mathbb{Z}} A_\alpha(k) \tag{37}$$

and each $A_\alpha(k)$ is the self-adjoint operator on $L^2(\mathbb{R})$ given by

$$\begin{aligned} \mathcal{D}(A_\alpha(k)) &= \left\{ \begin{array}{c} g = g^- \oplus g^+\,,\ g^\pm \in L^2(\mathbb{R}^\pm, \mathrm{d}x) \ \textit{such that} \\ \big(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2|x|^{2\alpha} + \frac{\alpha(2+\alpha)}{4x^2}\big) g^\pm \in L^2(\mathbb{R}^\pm, \mathrm{d}x) \\ g_0^- = g_0^+\,, \quad g_1^- = -g_1^+ \end{array} \right\}, \\ A_\alpha(k) g &= \bigoplus_{\pm} \Big(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k^2|x|^{2\alpha} + \frac{\alpha(2+\alpha)}{4x^2} \Big) g^\pm\,, \end{aligned} \tag{38}$$

where $g_0^\pm, g_1^\pm \in \mathbb{C}$ *are the existing and finite limits*

$$\begin{aligned} g_0^\pm &= \lim_{x\to 0^\pm} |x|^{\frac{\alpha}{2}} g(x)\,, \\ g_1^\pm &= \lim_{x\to 0^\pm} |x|^{-(1+\frac{\alpha}{2})}\big(g(x) - g_0^\pm |x|^{-\frac{\alpha}{2}}\big)\,. \end{aligned} \tag{39}$$

In Theorem 6 the existence and finiteness of the limits (39) is guaranteed by the distributional constraint $\big(-\frac{d^2}{dx^2}+k^2|x|^{2\alpha}+\frac{\alpha(2+\alpha)}{4x^2}\big)g^\pm \in L^2(\mathbb{R}^\pm, dx)$. A completely analogous unitary equivalence and fibred decomposition like (36)–(37) holds for all other self-adjoint realisations of the Laplace–Beltrami operator on Grushin cylinder, as classified in Theorem 3 [19].

Each $A_\alpha(k)$ is the k-th transversal momentum mode of the operator H_α^{B} on cylinder, in the sense of the isomorphism (32), namely with respect to the momentum conjugate to the y-variable. By compactness, these are discrete modes and, as seen from (38), the boundary condition at $x = 0$ has the *same* form ($g_0^- = g_0^+, g_1^- = -g_1^+$) in *each* mode, and moreover it *does not couple distinct modes*. Because of this structure, the bridging operator H_α^{B} is said to be '*uniformly fibred*', and in fact all other extensions classified in Theorem 3 are uniformly fibred too [19]. Uniformly fibred extensions generate a heat or Schrödinger flow that is reduced into the discrete modes k.

A careful spectral analysis [16] shows that for each (uniformly fibred) extension from Theorem 3, the transversal momentum modes are energetically increasingly ordered in the sense of increasing $|k|$, meaning in particular that the zero-th mode is the lowest energy one, and that for the bridging operator all modes have only non-negative, essential, absolutely continuous spectrum.

Comparing (38) with (4) one recognises that $A_\alpha(0) = A_\alpha^{\mathrm{B}}$. This and the considerations made in Sect. 1 finally show that the heat flow generated by the bridging operator H_α^{B} starting with a function f_{init} on the cylinder which belongs to the zero-th transversal momentum mode and therefore is constant in y, say, $f_{\mathrm{init}}(x, y) = \varphi(x)$, produces at times $t > 0$ and evoluted function

$$f(t; x, y) = u(t, x) \tag{40}$$

(still belonging to the zero mode) where u solves the one-dimensional initial value problem (1) with initial datum φ.

3 Related Settings: Grushin Planes and Almost-Riemannian Manifolds

The subject of geometric quantum confinement away from the metric's singularity, and transmission across it, for quantum particles or for the heat flow on degenerate

Riemannian manifolds is experiencing a fast growth in the recent years. Such themes are particularly active with reference to Grushin structures on cylinder, cone, and plane [4–9, 18, 21], as well as, more generally, on two-dimensional orientable compact almost-Riemannian manifolds of step two [6], d-dimensional regular almost-Riemannian and sub-Riemannian manifolds [15, 23].

Of significant relevance is the counterpart model to the Grushin-type cylinder, but in the lack of compact variable. This leads to related almost-Riemannian structures called '*Grushin-type planes*'. In complete analogy to Sect. 2, these are Riemannian manifolds $M_\alpha \equiv (M, g_\alpha)$, for some $\alpha \in \mathbb{R}$, where now

$$M^{\pm} := \mathbb{R}^{\pm}_{x} \times \mathbb{R}_y \qquad \mathcal{Z} := \{0\} \times \mathbb{R}_y\,, \qquad M := M^+ \cup M^- \tag{41}$$

and again with degenerate Riemannian metric

$$g_\alpha := \mathrm{d}x \otimes \mathrm{d}x + |x|^{-2\alpha}\mathrm{d}y \otimes \mathrm{d}y\,. \tag{42}$$

The standard '*Grushin plane*' corresponds to $\alpha = 1$. Also for a Grushin-type plane one builds the Hilbert space $\mathcal{H}_\alpha$, defined as in (11), and the Laplace–Beltrami differential operator $\Delta_{\mu_\alpha} := \mathrm{div}_{\mu_\alpha} \circ \nabla$, explicitly given again by the analogue of (13), and minimally realised as the analogue of (14) on smooth functions compactly supported within each open half-plane. This yields the densely defined, non-negative, symmetric operator H_α, and poses the problem of self-adjointness of H_α, in order to analyse the generated heat or Schrödinger flow.

Theorem 7 ([15, 18, 21, 22])

(i) If $\alpha \in [-1, 1)$, then H_α is not essentially self-adjoint in $\mathcal{H}_\alpha$ and has infinite deficiency index.

(ii) If $\alpha \in (-\infty, -1) \cup [1, +\infty)$, then H_α is essentially self-adjoint and therefore the Grushin-type plane M_α induces geometric quantum confinement.

The above regime of essential self-adjointness was implicitly established in [15] as an adaptation of the previous perturbative analysis [23] devised for the compactified version of the manifold; the complete identification of essential self-adjointness and lack thereof was subsequently obtained in [18, 21, 22] within a non-perturbative, novel scheme of *constant-fibre direct integral decomposition* of the Hilbert space $\mathcal{H}_\alpha = L^2(M, \mathrm{d}\mu_\alpha)$ that generalises the direct integral decomposition (32)–(35) one performs in the compact case. This replaces uniformly fibred extensions on cylinder of the form (37) discussed above, namely

$$\bigoplus_{k\in\mathbb{Z}} A_\alpha(k)\,,$$

$A_\alpha(k)$ acting self-adjointly on the fibre Hilbert space $\mathfrak{h} = L^2(\mathbb{R})$, with uniformly fibred direct integral extensions

$$\int_{\mathbb{R}}^{\oplus} A_\alpha(\xi)\,\mathrm{d}\xi\,,$$

where the fibre operator $A_\alpha(\xi)$ on $\mathfrak{h}$ now depends on the continuous Fourier mode ξ, dual to the non-compact variable y.

It is worth observing that the regime of self-adjointness for α-Grushin cylinders and planes differ when $\alpha \in (-3, -1)$ (compare Theorems 2 and 7). This is due to the different nature of the direct sum and direct integral decompositions: indeed, when $\alpha \in (-3, -1)$, the only Fourier mode that is *not* self-adjoint is the zero-th one, which brings a non-trivial contribution to the sum, but not to the integral. As a consequence, when $\alpha \in (-3, -1)$ the zero mode of a generic function $\psi \equiv \psi(x, y)$ hitting $\mathcal{Z}$ in the cylinder, namely the average on $\mathbb{S}^1$

$$\psi_0(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{S}^1} \psi(x, y)\,\mathrm{d}y\,,$$

does cross the singularity, whereas the zero mode of ψ on the plane, namely

$$\psi_0(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \psi(x, y)\,\mathrm{d}y\,,$$

does not cross the singularity. The case $\alpha = -1$ is different as well between cylinder and plane: indeed, the non-self-adjoint Fourier modes are $\xi \in (-1, 1)$ for the plane, and $k = 0$ for the cylinder, thus yielding deficiency index of H_α equal to infinity for the plane, and equal to 2 for the cylinder.

As a matter of fact, the lack of compactness makes the systematic identification of non-trivial self-adjoint extensions of H_α considerably harder and so far no explicit classification is available that mirrors Theorem 3 for the plane.

Beside the above concrete cylindrical and planar settings, the deep connection between geometry and self-adjointness is investigated for the problem of geometric confinement on more general almost-Riemannian structures. This includes '*two-step two-dimensional almost-Riemannian structures*', characterised by an orthonormal frame for the metric in the vicinity of the singularity locus $\mathcal{Z}$ of the form [3]

$$X_1(x, y) = \frac{\partial}{\partial x}\,, \qquad X_2(x, y) = x e^{\phi(x,y)} \frac{\partial}{\partial y} \tag{43}$$

(to be compared to (8) with $\alpha = 1$). The essential self-adjointness of the corresponding minimally defined Laplace–Beltrami in the case of compactified $\mathcal{Z}$ was established in [6].

From a related perspective, the already observed circumstance that Grushin-type cylinders or planes are, classically, geodesically incomplete, but can induce, quantum-mechanically, geometric confinement (a condition that occurs more generally for regular almost-Riemannian manifold with compact singular set), poses an intriguing problem as far as semi-classical analysis is concerned. Indeed, reinstating Planck's constant in the Schrödinger equation

$$\mathrm{i}\partial_t \psi + \varepsilon^2 \Delta_{\mu_\alpha} \psi \;=\; 0\,, \qquad \varepsilon > 0 \tag{44}$$

(in the regime of α in which the minimally defined Δ_{μ_α} is unambiguously realised self-adjointly), semi-classics show, informally speaking, that as $\varepsilon \downarrow 0$ solutions get concentrated and evolve around geodesics. Therefore, the above-mentioned classical/quantum discrepancy makes the semi-classical analysis necessarily break down in the limit.

Such a discordance between classical and quantum picture can be at least partially resolved by appealing to different quantisation procedures on the considered Riemannian manifold, in practice considering corrections of the Laplace–Beltrami operator that have a suitable interpretation of free kinetic energy, much in the original spirit of [13]. Most of coordinate-invariant quantisation procedures (including path integral quantisation, covariant Weyl quantisation, geometric quantisation, and finite-dimensional approximation to Wiener measures) modify Δ_{μ_α} with a term that depends on the scalar curvature R_α (which, in two dimensions, is twice the Gaussian curvature K_α). This produces a replacement in (44), in two dimensions, of $-\Delta_{\mu_\alpha}$ with the '*curvature Laplacian*'

$$-\,\Delta_{\mu_\alpha} + cK_\alpha \tag{45}$$

for suitable $c \geqslant 0$. In the recent work [5] it was indeed shown, for generic two-step two-dimensional almost-Riemannian manifolds with compact singular set, that irrespective of $c \in (0, \frac{1}{2})$ the above correction washes essential self-adjointness out, yielding a quantum picture where the Schrödinger particle does reach the singularity much as the classical particle does. (At the expenses of some further technicalities, the whole regime $c > 0$ can be covered as well.) For concreteness, in the Grushin cylinder the effect of the curvature correction is evidently understood as a compensation between $K = -\frac{2}{x^2}$ (see (9) above) and the singular term $\frac{3}{4x^2}$ of the (unitary equivalent) Laplace–Beltrami operator. Still, the classical/quantum discrepancy discussed so far remains unexplained in more general settings.

Concerning, instead, the heat flow, a satisfactory interpretation of the heat-confinement in the Grushin cylinder is known in terms of Brownian motions [7] and random walks [2]: roughly speaking, random particles are lost in the infinite-area strip around $\mathcal{Z}$: the latter, in practice, acts as a barrier. Clearly, whereas curvature Laplacians are meaningful in the above context of inducing a non-confining (transmitting) Schrödinger flow on two-step two-dimensional almost-Riemannian manifolds (including the Grushin cylinders), thus making quantum and classical picture more alike and well connected by semi-classics, this has no direct meaning instead in application to the heat flow on Riemannian or almost-Riemannian manifolds. Indeed, as long as one regards the heat equation on manifold as a limit of a space-time discretised random walk, the stochastic process' generator *is* the Laplace–Beltrami operator.

Generalisations of [6] have been established in [15, 23] from two-step two-dimensional almost-Riemannian structures to any dimensions, any step, and even to sub-Riemannian geometries, provided that certain geometrical assumptions on the singular set are taken. The main difficulty is the treatment of the '*tangency*' (or '*characteristic*') points: these are points belonging to the singularity of the metric structure where the vector distribution is tangent to the singularity. They are never present in Grushin cylinder or two-step almost-Riemannian structures but may appear, for example, in three-step structures, such as, for instance,

$$X_1(x,y) \;=\; \frac{\partial}{\partial x}\,, \qquad X_2(x,y) \;=\; (y-x^2)\frac{\partial}{\partial y}\,, \quad (x,y)\in\mathbb{R}^2\,, \tag{46}$$

where the singularity is the parabola $y=x^2$ and the origin $(0,0)$ is a tangency point. Virtually nothing is known on the heat or the quantum confinement on such singular structures, including the simplest example (46) (see [14] for further remarks). First preliminary results in this respect were recently obtained in [4], where the interpretation of almost-Riemannian structures as special Lie manifolds permits to study some closure properties of singular perturbations of the Laplace–Beltrami operator even in the presence of tangency points. This opens new perspectives of treating several types of different singularities in sub-Riemannian geometry within the same unifying theory.

4 A Numerical Glance at the Bridging-Heat Evolution

In this final section we present and comment on qualitative features of the solution to the one-dimensional problem (1), *obtained by numerical integration*, also in comparison with the initial value problem for the classical heat equation on $\mathbb{R}$.

As already argued, it is the determination of the (integral kernel of) the heat propagator $\exp(-tA_\alpha^{\mathrm{B}})$, $t>0$, to be hard analytically, and this is due to the presence of boundary conditions for the solution at $x=0$ and any positive time.

Numerics then represent a first, valuable way to access relevant aspects of the transmission of the heat flow between positive and negative half-line with bridging boundary conditions, and one may envisage that a systematic comparison will be launched numerically between analogous heat flows with different transmission protocols among those surveyed in Sect. 2. Ours, here, is only an initial numerical glance at the bridging-heat evolution to provide some insight and anticipate future investigations.

Our numerical approach consists in approximating the solution $u=e^{-tA_\alpha^{\mathrm{B}}}\varphi$ to the problem (1) by means of an approximated version of both the *spatial* convolution integral between propagator's kernel and φ, and the *complex* line integral that turns the resolvent of A_α^{B} into its semi-group.

More precisely, let us write

$$u(t,x) \;=\; (e^{-tA_\alpha^{\mathrm{B}}}\varphi)(x) \;=\; \int_{\mathbb{R}} \mathcal{K}_\alpha^{\mathrm{B}}(t;x,y)\varphi(x)\,\mathrm{d}x\,, \tag{47}$$

where $\mathcal{K}_\alpha^{\mathrm{B}}(\cdot,\cdot)$ is the integral kernel of the bridging-heat propagator. In turn, let us exploit the relation

$$\begin{aligned} e^{-tA_\alpha^{\mathrm{B}}} \;&=\; \mathscr{L}^{-1}\big((A_\alpha^{\mathrm{B}}-(\cdot)\mathbb{1})^{-1}\big)(t) \\ &=\; \frac{1}{2\pi\mathrm{i}}\int_\Gamma e^{-zt}\big((A_\alpha^{\mathrm{B}}-z\mathbb{1})^{-1}\big)\,\mathrm{d}z\,, \qquad t>0\,, \end{aligned} \tag{48}$$

as an identity between bounded operators on $L^2(\mathbb{R})$ and with the integral understood in the Riemann sense in the strong operator topology, Γ being a straight line in $\mathbb{C}$ orthogonal to the real axis in the open left half-plane, and $\mathscr{L}^{-1}$ denoting the inverse Laplace transform (the non-negativity of A_α^{B} has led here to the choice $\mathfrak{Re}\,z<0$). (48) connects the resolvent of A_α^{B} at the complex point z with the semi-group at time $t>0$, and in terms of the integral kernels $(A_\alpha^{\mathrm{B}}-z\mathbb{1})^{-1}(x,y)$ of the resolvent and $\mathcal{K}_\alpha^{\mathrm{B}}(t;x,y)$ of the propagator it reads

$$\mathcal{K}_\alpha^{\mathrm{B}}(t;x,y) \;=\; \frac{1}{2\pi\mathrm{i}}\int_{-1-\mathrm{i}\cdot\infty}^{-1+\mathrm{i}\cdot\infty} e^{-zt}\big((A_\alpha^{\mathrm{B}}-z\mathbb{1})^{-1}\big)(x,y)\,\mathrm{d}z\,, \qquad t>0\,. \tag{49}$$

The combinations of (47) and (49) produce the solution u and the two integrations contained therein may be computed numerically with standard packages.

Of course, for (47) and (49) to be implementable one needs to know the (integral kernel of) the resolvent $(A_\alpha^{\mathrm{B}}-z\mathbb{1})^{-1}$. This is a not so hard knowledge to achieve from the underlying structure (4) of the operator A_α^{B}, once it is interpreted as a self-adjoint extension of the differential operator (2) minimally defined on smooth functions compactly supported on $\mathbb{R}$ away from the origin. For this status of extension operator, one can appeal to the general Kreĭn-Višik-Birman theory of self-adjoint extensions of lower semi-bounded and densely defined symmetric operators on Hilbert space [17], and obtain $(A_\alpha^{\mathrm{B}}-z\mathbb{1})^{-1}$ fairly explicitly.

The net result of this computation gives the following expression for the integral kernel of $(A_\alpha^{\mathrm{B}}-z\mathbb{1})^{-1}$. With respect to the canonical decomposition

$$L^2(\mathbb{R},\mathrm{d}x) \;\xrightarrow{\cong}\; L^2(\mathbb{R}^+,\mathrm{d}x)\oplus L^2(\mathbb{R}^-,\mathrm{d}x)\,, \qquad u\mapsto \begin{pmatrix} u^+ \\ u^- \end{pmatrix} \tag{50}$$

(that is, $u^{\pm}(x) := u(x)$ for $x \gtrless 0$), consider the unitary transformation

$$\begin{aligned} U \;:\; L^2(\mathbb{R}^+, \mathrm{d}x) \oplus L^2(\mathbb{R}^-, \mathrm{d}x) \;\xrightarrow{\cong}\; L^2(\mathbb{R}^+, \mathrm{d}x) \oplus L^2(\mathbb{R}^+, \mathrm{d}x)\,, \\ U \begin{pmatrix} u^+ \\ u^- \end{pmatrix}(x) \;=\; \begin{pmatrix} u^+(x) \\ u^-(-x) \end{pmatrix}, \qquad x > 0\,, \end{aligned} \tag{51}$$

and set $\mathcal{R}_\alpha^{\mathrm{B}}(z) := U(A_\alpha^{\mathrm{B}} - z\mathbb{1})^{-1}U^{-1}$. Then

$$(A_\alpha^{\mathrm{B}} - z\mathbb{1})^{-1} \;=\; U^{-1}\,\mathcal{R}_\alpha^{\mathrm{B}}(z)\,U \tag{52}$$

and the integral kernel of $\mathcal{R}_\alpha^{\mathrm{B}}(z)$ is given by

$$\begin{aligned} \mathcal{R}_\alpha^{\mathrm{B}}(z)(x,y) \;=\; \mathcal{G}_{\alpha,z}(x,y) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{\mathrm{i}\pi}{8}\cos\Big(\frac{\pi\alpha}{2}\Big)\, e^{\mathrm{i}\frac{\pi\alpha}{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} P_{\alpha,z}(x)P_{\alpha,z}(y)\,, \\ x > 0\,,\; y > 0\,, \end{aligned} \tag{53}$$

where, in terms of the Bessel functions of first and second kind J_ν and Y_ν [1, equations (9.1.10)-(9.1.2)],

$$\begin{aligned} P_{\alpha,z}(x) \;&=\; \sqrt{x}\, J_{\frac{1+\alpha}{2}}(x\sqrt{z}) + \mathrm{i}\sqrt{x}\, Y_{\frac{1+\alpha}{2}}(x\sqrt{z}) \\ Q_{\alpha,z}(x) \;&=\; 2\sqrt{x}\, J_{\frac{1+\alpha}{2}}(x\sqrt{z}) \end{aligned} \qquad (\Im\mathfrak{m}\sqrt{z} > 0)\,, \tag{54}$$

and

$$\mathcal{G}_{\alpha,z}(x,y) \;=\; -\frac{\mathrm{i}\pi}{4} \begin{cases} P_{\alpha,z}(x)\,Q_{\alpha,z}(y)\,, & \text{if} \quad 0 < y < x\,, \\ Q_{\alpha,z}(x)\,P_{\alpha,z}(y)\,, & \text{if} \quad 0 < x < y\,. \end{cases} \tag{55}$$

When formulas (47), (49), (52), and (53) are implemented numerically we obtain a scenario exemplified in Figs. 2, 3, 4, and 5 below.

For concreteness, the bridging-heat evolution is considered, namely the solution $u \equiv u(t,x)$ to (1), of an initial datum φ essentially supported on the right half-line. An initial Gaussian is seen to evolve at later times with the typical heat-flow flattening of the solution, with the immediate formation of the characteristic bridging behaviour at $x = 0$ (Fig. 2).

Notably, if φ is additionally shot with an initial non-zero momentum towards the singularity, its evolution displays an oscillation given by the superposition of an in-going wave and a component that bounces backwards (Fig. 3), as compared with the evolution at the same time of the same Gaussian with no initial momentum.

It is also pretty transparent that the bridging-heat flow has a regularising effect at every $t > 0$, as observed with the evolution of an initial step function (Fig. 4).

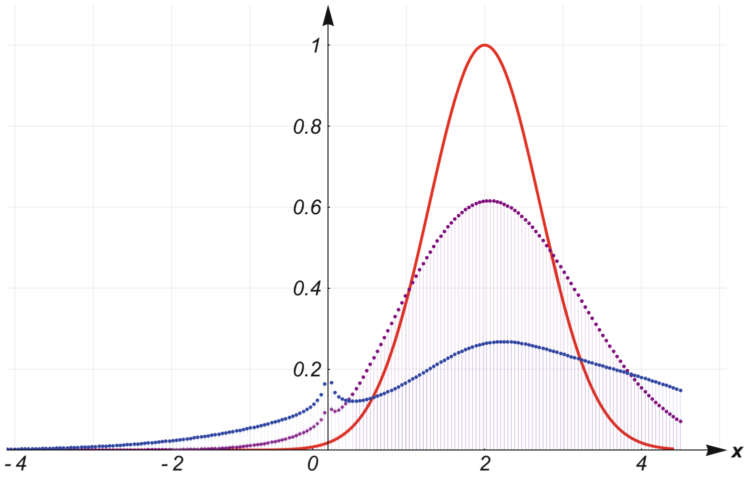

Fig. 2 Solution $u(t,x)$ to the heat-bridging initial value problem (1) with Gaussian initial datum $\varphi(x) = e^{-(x-2)^2}$ (red curve). Plot of $|u(t,\cdot)|$ at $t = 0.5$ (magenta dotted line) and $t = 2$ (blue dotted line)

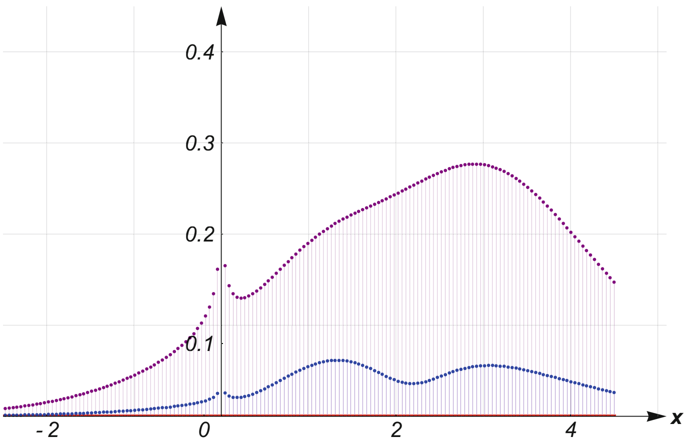

Fig. 3 Comparison at time $t = 1.5$ between the solution to the heat-bridging initial value problem (1) with zero-momentum Gaussian initial datum $\varphi(x) = e^{-(x-2)^2}$ (magenta dotted curve) and with non-zero momentum Gaussian initial datum $\varphi_2(x) = e^{-3ix}e^{-(x-2)^2}$ towards left. Both plots are of $|u(t,\cdot)|$. The evolution of the Gaussian initially shot towards left displays in-going + outgoing oscillation

We have also further evidence of a qualitatively similar behaviour of the free heat flow and the bridging-heat flow, but of course for the characteristic boundary condition of bridging type at the origin (Fig. 5).

Whereas, as said, this provides only a first glance at the qualitative properties of the bridging-heat evolution on two connected half-lines, the evidences collected

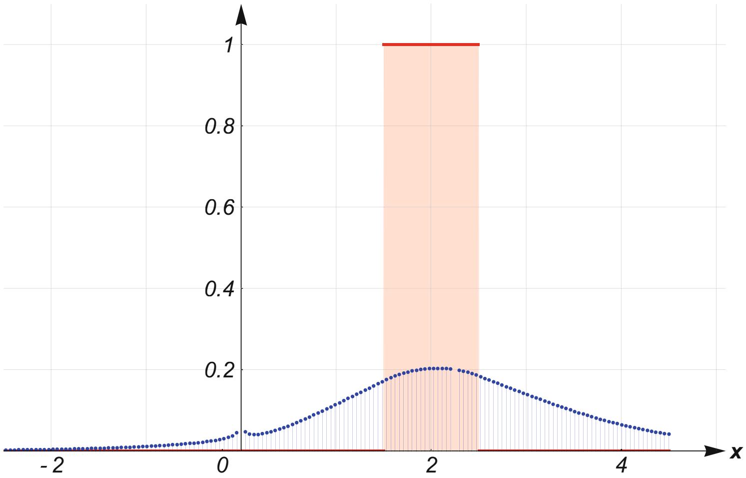

Fig. 4 Solution $u(t, x)$ to the heat-bridging initial value problem (1) with initial datum $\varphi(x)$ given by the characteristic function of the interval $[\frac{1}{2}, \frac{3}{2}]$ (red curve). Plot of $|u(t, \cdot)|$ at $t = 0.5$ (blue dotted line)

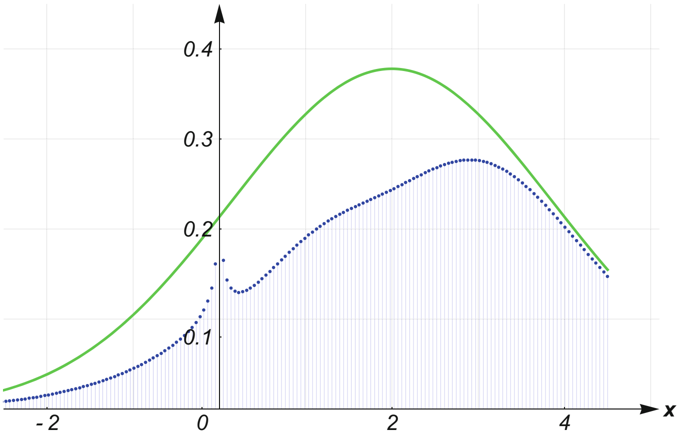

Fig. 5 Comparison at time $t = 1.5$ between the solution to the heat-bridging initial value problem (1) with Gaussian initial datum $\varphi(x) = e^{-(x-2)^2}$ (blue dotted curve) and the solution to the ordinary heat equation on $\mathbb{R}$ (green curve)

here are encouraging and further corroborate the quest for the analytic identification of counterpart L^p-L^q estimates, smoothing estimates, and space-time (Strichartz) estimates for the bridging-heat flow, as compared to (5) for the classical heat flow.

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Part IV
Wave- and KdV-Type Equations

On the Cauchy Problem for Quasi-Linear Hamiltonian KdV-Type Equations

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Abstract We prove local in time well-posedness for a class of quasi-linear Hamiltonian KdV-type equations with periodic boundary conditions, more precisely we show existence, uniqueness and continuity of the solution map. We improve the previous result in (Mietka, Ann Math Blaise Pascal 24:83–114, 2017), generalising the considered class of equations and improving the regularity assumption on the initial data.

1 Introduction

In this paper $u(t, x)$ is a function of time $t \in [0, T)$, $T > 0$ and space $x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$. $F(x, z_0, z_1)$ is a polynomial function such that $F(x, 0, z_1) = F(x, z_0, 0) = \partial_{z_0} F(x, 0, z_1) = \partial_{z_1} F(x, z_0, 0) = 0$. Throughout the paper we shall assume that there exists a constant $\mathfrak{c} > 0$ such that

$$\partial^2_{z_1 z_1} F(x, z_0, z_1) \geq \mathfrak{c}, \tag{1}$$

for any $x \in \mathbb{T}$, z_0, $z_1 \in \mathbb{R}$. We shall denote the partial derivatives of the function u by u_t, u_x, u_{xx} and u_{xxx}, by $\partial_x, \partial_{z_0}, \partial_{z_1}$ the partial derivatives of the function F and by $\frac{d}{dx}$ the total derivative with respect to the variable x. For instance, we have $\frac{d}{dx} F(x, u, u_x) = \partial_x F(x, u, u_x) + \partial_{z_0} F(x, u, u_x) u_x + \partial_{z_1} F(x, u, u_x) u_{xx}$. We consider the equation

$$u_t = \frac{d}{dx}\Big(\nabla_u H(x, u, u_x)\Big), \quad H(x, u, u_x) := \int_{\mathbb{T}} F(x, u, u_x) dx, \tag{2}$$

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where we denoted by $\nabla_u H$ the L^2-gradient of the Hamiltonian function $H(x, u, u_x)$ on the phase space

$$H_0^s(\mathbb{T}) := \{u(x) \in H^s(\mathbb{T}) : \int_{\mathbb{T}} u(x)dx = 0\}, \tag{3}$$

endowed with the non-degenerate symplectic form $\Omega(u, v) := \int_{\mathbb{T}}(\partial_x^{-1}u)v dx$ (∂_x^{-1} is the periodic primitive of u with zero average) and with the norm $\|u\|_{\dot{H}^s} := \sum_{j\in\mathbb{Z}^*} |u_j|^2|j|^2$ (u_j are the Fourier coefficients of the periodic function u).

The main result is the following.

Theorem 1 *Let* $s > 4 + 1/2$ *and assume* (1)*. Then for any* $u_0 \in H_0^s(\mathbb{T})$ *there exists a time* $T := T(\|u_0\|_{H^s})$ *and a unique solution of* (2) *with initial condition* $u(0, x) = u_0(x)$ *satisfying* $u(t, x) \in C^0([0, T), H_0^s(\mathbb{T})) \cap C^1([0, T), H_0^{s-3}(\mathbb{T}))$. *Moreover the solution map* $u_0(x) \mapsto u(t, x)$ *is continuous with respect to the* H_0^s *topology for any* t *in* $[0, T)$.

This theorem improves the previous one in [15] by Mietka. The result in such a paper holds true if the Hamiltonian function has the form $H(u)$, while here we allow the explicit dependence on the x variable (non-autonomous equation) and the dependence on u_x. We tried to optimise our result in terms of regularity of the initial condition, we do not know if the result is improvable. If we apply our method to the equation considered by Mietka, we find a local well-posedness theorem if the initial condition belongs to the space H_0^s with $s > 3 + 1/2$ (which is natural since the nonlinearity may contain up to three derivatives of u), while in [15] one requires $s \geq 4$. In our statement we need $s > 4 + 1/2$ because our equation is more general and we have the presence of one more derivative in the coefficients with respect to the equation considered in [15].

The proof of Theorem 1 is an application of a method which has been developed in [7, 8] and then improved, in terms of regularity of initial condition, in [1]. Here we follow closely the method in [1] and we use several results proven therein. Both the schemes, the one used in [15] and in [1, 7, 8], rely on solely energy method, the second one is slightly more refined because of the use of paradifferential calculus which allows us to work in fractional Sobolev spaces and to treat more general nonlinear terms. The main idea is to introduce a convenient energy, which is equivalent to the Sobolev norm, which commutes with the principal (quasi-linear) term in the equation (see (40)). In [1, 7, 8] the main difficulty comes from the fact that, after the paralinearization, one needs to prove *a priori* estimates on a system of coupled equations. One needs then to decouple the equations through convenient changes of coordinates which are used to define the modified energy. In the case of KdV equation (2), we have a scalar equation with the sub-principal symbol which is *real* (and so it defines a self-adjoint operator), see (21), therefore it is impossible to obtain energy estimates directly. This term may be completely removed (see Lemma 2) thanks to the Hamiltonian structure. For similar constructions of such kind of energies one can look also at [1, 6, 8–10].

The general equation (2) contains the "classical" KdV equation $u_t+uu_x+u_{xxx} = 0$ and the *modified* KdV $u_t + u^p u_x + u_{xxx} = 0$, $p \geq 2$. Obviously, for the last two equations better results may be obtained, concerning KdV we quote Bona-Smith [2], Kato [11], Bourgain [3], Kenig-Ponce-Vega [12, 13], Christ-Colliander-Tao [4]. For the general equation, as the one considered in this paper here, several results have been proven by Colliander-Keel-Staffilani-Takaoka-Tao [5], Kenig-Ponce-Vega [14] and the aforementioned Mietka [15].

2 Paradifferential Calculus

In this section we recall some results concerning the paradifferential calculus, we follow [1]. We introduce the Japanese bracket $\langle\xi\rangle = \sqrt{1+\xi^2}$. We denote by $\dot{H}^s$ the homogeneous Sobolev space defined as H^s modulo constant functions.

Definition 1 Given $m, s \in \mathbb{R}$ we denote by Γ^m_s the space of functions $a(x,\xi)$ defined on $\mathbb{T}\times\mathbb{R}$ with values in $\mathbb{C}$, which are C^∞ with respect to the variable $\xi \in \mathbb{R}$ and such that for any $\beta \in \mathbb{N}\cup\{0\}$, there exists a constant $C_\beta > 0$ such that

$$\|\partial_\xi^\beta a(\cdot,\xi)\|_{H^s} \leq C_\beta \langle\xi\rangle^{m-\beta}, \quad \forall \xi \in \mathbb{R}. \tag{4}$$

We endow the space Γ^m_s with the family of norms

$$|a|_{m,s,n} := \max_{\beta\leq n} \sup_{\xi\in\mathbb{R}} \|\langle\xi\rangle^{\beta-m} a(\cdot,\xi)\|_{H^s}. \tag{5}$$

Analogously for a given Banach space W we denote by Γ^m_W the space of functions which verify the (4) with the W-norm instead of H^s, we also denote by $|a|_{m,W,n}$ the W based seminorms (5) with $H^s \rightsquigarrow W$.

We say that a symbol $a(x,\xi)$ is spectrally localised if there exists $\delta > 0$ such that $\widehat{a}(j,\xi) = 0$ for any $|j| \geq \delta\langle\xi\rangle$.

Consider a function $\chi \in C^\infty(\mathbb{R},[0,1])$ such that $\chi(\xi) = 1$ if $|\xi| \leq 1.1$ and $\chi(\xi) = 0$ if $|\xi| \geq 1.9$. Let $\epsilon \in (0,1)$ and define moreover $\chi_\epsilon(\xi) := \chi(\xi/\epsilon)$. Given $a(x,\xi)$ in Γ^m_s we define the regularised symbol

$$a_\chi(x,\xi) := \sum_{j\in\mathbb{Z}} \widehat{a}(j,\xi)\chi_\epsilon(\tfrac{j}{\langle\xi\rangle})e^{\mathrm{i}jx}.$$

For a symbol $a(x,\xi)$ in Γ^m_s we define its Weyl and Bony-Weyl quantization as

$$Op^{\mathrm{W}}(a(x,\xi))h := \frac{1}{(2\pi)} \sum_{j\in\mathbb{Z}} e^{\mathrm{i}jx} \sum_{k\in\mathbb{Z}} \widehat{a}\big(j-k, \frac{j+k}{2}\big)\widehat{h}(k), \tag{6}$$

$$Op^{\mathrm{BW}}(a(x,\xi))h := \frac{1}{(2\pi)}\sum_{j\in\mathbb{Z}} e^{\mathrm{i}jx}\sum_{k\in\mathbb{Z}} \chi_\epsilon\Big(\frac{|j-k|}{\langle j+k\rangle}\Big)\widehat{a}\big(j-k, \frac{j+k}{2}\big)\widehat{h}(k). \tag{7}$$

We list below a series of theorems and lemmas that will be used in the paper. All the statements have been taken from [1]. The first one is a result concerning the action of a paradifferential operator on Sobolev spaces. This is Theorem 2.4 in [1].

Theorem 2 *Let $a \in \Gamma^m_{s_0}$, $s_0 > 1/2$ and $m \in \mathbb{R}$. Then $Op^{\mathrm{BW}}(a)$ extends as a bounded operator from $\dot{H}^{s-m}(\mathbb{T})$ to $\dot{H}^s(\mathbb{T})$ for any $s \in \mathbb{R}$ with estimate*

$$\|Op^{\mathrm{BW}}(a)u\|_{\dot{H}^{s-m}} \lesssim |a|_{m,s_0,4}\|u\|_{\dot{H}^s}, \tag{8}$$

for any u in $\dot{H}^s(\mathbb{T})$. Moreover for any $\rho \geq 0$ we have for any $u \in \dot{H}^s(\mathbb{T})$

$$\|Op^{\mathrm{BW}}(a)u\|_{\dot{H}^{s-m-\rho}} \lesssim |a|_{m,s_0-\rho,4}\|u\|_{\dot{H}^s}. \tag{9}$$

We now state a result regarding symbolic calculus for the composition of Bony-Weyl paradifferential operators. In the rest of the section, since there is no possibility of confusion, we shall denote the total derivative $\frac{d}{dx}$ as ∂_x with the aim of improving the readability of the formulæ. Given two symbols a and b belonging to $\Gamma^m_{s_0+\rho}$ and $\Gamma^{m'}_{s_0+\rho}$, respectively, we define for $\rho \in (0,3]$

$$a\#_\rho b = \begin{cases} ab \quad \rho \in (0,1] \\ ab + \frac{1}{2\mathrm{i}}\{a,b\} \quad \rho \in (1,2], \\ ab + \frac{1}{2\mathrm{i}}\{a,b\} - \frac{1}{8}\mathfrak{s}(a,b) \quad \rho \in (2,3], \end{cases} \tag{10}$$

where we denoted by $\{a,b\} := \partial_\xi a\partial_x b - \partial_x a\partial_\xi b$ the Poisson's bracket between symbols and $\mathfrak{s}(a,b) := \partial^2_{xx}a\partial^2_{\xi\xi}b - 2\partial^2_{x\xi}a\partial^2_{x\xi}b + \partial^2_{\xi\xi}a\partial^2_{xx}b$.

Remark 1 According to the notation above we have $ab \in \Gamma^{m+m'}_{s_0+\rho}$, $\{a,b\} \in \Gamma^{m+m'-1}_{s_0+\rho-1}$ and $\mathfrak{s}(a,b) \in \Gamma^{m+m'-2}_{s_0+\rho-2}$. Moreover $\{a,b\} = -\{b,a\}$ and $\mathfrak{s}(a,b) = \mathfrak{s}(b,a)$.

The following is essentially Theorem 2.5 of [1], we just need some more precise symbolic calculus since we shall deal with nonlinearities containing three derivatives, while in [1] they have nonlinearities with two derivatives.

Theorem 3 *Let $a \in \Gamma^m_{s_0+\rho}$ and $b \in \Gamma^{m'}_{s_0+\rho}$ with $m, m' \in \mathbb{R}$ and $\rho \in (0,3]$. We have $Op^{\mathrm{BW}}(a) \circ Op^{\mathrm{BW}}(b) = Op^{\mathrm{BW}}(a\#_\rho b) + R^{-\rho}(a,b)$, where the linear operator $R^{-\rho}$ is defined on $\dot{H}^s(\mathbb{T})$ with values in $\dot{H}^{s+\rho-m-m'}$, for any $s \in \mathbb{R}$ and it satisfies*

$$\begin{aligned} &\|R^{-\rho}(a,b)u\|_{\dot{H}^{s-(m+m')+\rho}} \\ &\qquad \lesssim (|a|_{m,s_0+\rho,N}|b|_{m',s_0,N} + |a|_{m,s_0,N}|b|_{m',s_0+\rho,N})\|u\|_{\dot{H}^s}, \end{aligned} \tag{11}$$

where $N \geq 8$.

Proof We prove the statement for $\rho \in (2, 3]$, for smaller ρ the reasoning is similar. Recalling formulæ(7) and (6) we have

$$Op^{BW}(a)Op^{BW}(b)u = Op^{W}(a_\chi)Op^{W}(b_\chi)u$$
$$= \sum_{j,k,\ell} \widehat{a}_\chi(j-k, \frac{j+k}{2})\widehat{b}_\chi(k-\ell, \frac{k+\ell}{2})u_\ell e^{\mathrm{i}jx}.$$

We Taylor expand $\widehat{a}_\chi(j-k, \frac{j+k}{2})$ with respect to the second variable in the point $\frac{j+\ell}{2}$, we have

$$\widehat{a}_\chi(j-k, \tfrac{j+k}{2}) =$$
$$\widehat{a}_\chi(j-k, \tfrac{j+\ell}{2}) + \tfrac{k-\ell}{2}\partial_\xi \widehat{a}_\chi(j-k, \tfrac{j+\ell}{2}) + \tfrac{(k-\ell)^2}{8}\widehat{a}_\chi(j-k, \tfrac{j+\ell}{2})$$
$$+ \tfrac{(k-\ell)^3}{8}\int_0^1 (1-t)^2\partial_\xi^3\widehat{a}_\chi(j-k, \tfrac{j+\ell+t(k-\ell)}{2})dt.$$

Analogously we obtain

$$\widehat{b}_\chi(k-\ell, \tfrac{k+\ell}{2}) =$$
$$+ \tfrac{k-j}{2}\partial_\xi \widehat{b}_\chi(k-\ell, \tfrac{j+\ell}{2}) + \tfrac{(k-j)^2}{8}\partial_\xi^2\widehat{b}_\chi(k-\ell, \tfrac{j+\ell}{2})$$
$$+ \tfrac{(k-j)^3}{8}\int_0^1 (1-t)^2\partial_\xi^3\widehat{b}_\chi(k-\ell, \tfrac{j+\ell+t(k-j)}{2})dt.$$

An explicit computation proves that

$$Op^{BW}(a)Op^{BW}(b) - Op^{BW}(ab + \tfrac{1}{2\mathrm{i}} - \tfrac{1}{8}\mathfrak{s}(a,b))u = \sum_{j=1}^{4} R_i(a,b)u,$$

where

$$R_1 := Op^{W}\big(a_\chi b_\chi - (ab)_\chi + \tfrac{1}{2\mathrm{i}}(\{a_\chi, b_\chi\} - \{a,b\}_\chi) - \tfrac{1}{8}(\mathfrak{s}(a_\chi, b_\chi) - \mathfrak{s}(a,b)_\chi)\big)u,$$
$$R_2 := \sum Q_3^b\big(\widehat{a}_\chi(j-k, \tfrac{j+\ell}{2}) + \tfrac{k-\ell}{2}\partial_\xi\widehat{a}_\chi(j-k, \tfrac{j+\ell}{2})$$
$$+ \tfrac{(k-\ell)^2}{8}\widehat{a}_\chi(j-k, \tfrac{j+\ell}{2})\big)u_\ell e^{\mathrm{i}jx},$$
$$R_3 := \sum Q_3^a\widehat{b}_\chi(k-\ell, \tfrac{k+\ell}{2})u_\ell e^{\mathrm{i}jx},$$
$$R_4 := -\tfrac{1}{16\mathrm{i}}Op^{W}(\partial_x^2\partial_\xi a\partial_x\partial_\xi^2 b + \partial_x^2\partial_\xi b\partial_x\partial_\xi^2 a)u + \tfrac{1}{64}Op^{W}(\partial_x^2\partial_\xi^2 a\partial_x^2\partial_\xi^2 b)u,$$

where we have defined $Q_3^a := \frac{(k-\ell)^3}{8}\int_0^1(1-t)^2\partial_\xi^3\widehat{a}_\chi(j-k, \frac{j+\ell+t(k-\ell)}{2})dt$ and analogously Q_3^b. We prove that each R_i fulfils the estimate (11). The remainders R_1, R_2 and R_3 have to be treated as done in the proof of Theorem 2.5 in [1], we just underline the differences. Concerning R_1 it is enough to prove that for any $\alpha \leq 2$ the symbol $\partial_\xi^\alpha a_\chi \partial_x^\alpha b_\chi - \partial_\xi^\alpha b_\chi \partial_x^\alpha a_\chi$ is a spectrally localised symbol belonging to $\Gamma_{L^\infty}^{m+m'-\rho}$. Following word by word the proof in [1], with $d = 1$ and $\alpha = 2$ (instead of $\alpha = 1$ therein) one may bound $|\partial_\xi^\alpha a_\chi \partial_x^\alpha b_\chi - \partial_\xi^\alpha b_\chi \partial_x^\alpha a_\chi|_{m,W^{1,\infty},n} \lesssim |a|_{m,W^{1,\infty},n+2}|b|_{m',L^\infty,n+2} + |a|_{m,L^\infty,n+2}|b|_{m',W^{1,\infty},n+2}$. The estimate (11) on the remainder R_1 follows from Theorem A.7 in [1]. In order to prove that R_3 and R_2 satisfy (11), one has to follow the proof of Theorem A.5 in [1] with $d = 1, \alpha = 3$ and $\beta \leq 2$ corresponding to the remainder $R_2(a, b)$ therein. Concerning the remainder R_4 we have the following: the symbol of the first summand is in the class $\Gamma_{s_0}^{m+m'-3}$ and the second in $\Gamma_{s_0}^{m+m'-4}$, the estimates follow then by Theorem 2. □

Lemma 1 (Paraproduct) *Fix* $s_0 > 1/2$ *and let* $f, g \in H^s(\mathbb{T}; \mathbb{C})$ *for* $s \geq s_0$. *Then*

$$fg = Op^{\mathrm{BW}}(f)g + Op^{\mathrm{BW}}(g)f + \mathcal{R}(f, g)\,, \tag{12}$$

where

$$\begin{aligned} \widehat{\mathcal{R}(f, g)}(\xi) &= \frac{1}{(2\pi)}\sum_{\eta\in\mathbb{Z}} a(\xi-\eta, \xi)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,, \\ |a(v, w)| &\lesssim \frac{(1+\min(|v|, |w|))^\rho}{(1+\max(|v|, |w|))^\rho}\,, \end{aligned} \tag{13}$$

for any $\rho \geq 0$. *For* $0 \leq \rho \leq s - s_0$ *one has*

$$\|\mathcal{R}(f, g)\|_{H^{s+\rho}} \lesssim \|f\|_{H^s}\|g\|_{H^s}\,. \tag{14}$$

Proof Notice that

$$\widehat{(fg)}(\xi) = \sum_{\eta\in\mathbb{Z}} \widehat{f}(\xi-\eta)\widehat{g}(\eta)\,. \tag{15}$$

Consider the cut-off function χ_ϵ and define a new cut-off function $\Theta : \mathbb{R} \to [0, 1]$ as

$$1 = \chi_\epsilon\left(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\right) + \chi_\epsilon\left(\frac{|\eta|}{\langle 2\xi-\eta\rangle}\right) + \Theta(\xi, \eta)\,. \tag{16}$$

Recalling (15) and (7) we note that

$$
\begin{aligned}
\widehat{(T_f g)}(\xi) &= \sum_{\eta\in\mathbb{Z}} \chi_\epsilon\left(\frac{|\xi-\eta|}{\langle\xi+\eta\rangle}\right)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,,\\
\widehat{(T_g f)}(\xi) &= \sum_{\eta\in\mathbb{Z}} \chi_\epsilon\left(\frac{|\eta|}{\langle 2\xi-\eta\rangle}\right)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,,
\end{aligned}
\tag{17}
$$

and

$$
\mathcal{R} := \mathcal{R}(f,g)\,, \qquad \widehat{\mathcal{R}}(\xi) := \sum_{\eta\in\mathbb{Z}} \Theta(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,. \tag{18}
$$

To obtain the second in (17) one has to use the (7) and perform the change of variable $\xi-\eta \rightsquigarrow \eta$. By the definition of the cut-off function $\Theta(\xi,\eta)$ we deduce that, if $\Theta(\xi,\eta)\neq 0$ we must have

$$
|\xi-\eta| \geq \frac{5\epsilon}{4}\langle\xi+\eta\rangle \quad \text{and} \quad |\eta| \geq \frac{5\epsilon}{4}\langle 2\xi-\eta\rangle \qquad \Rightarrow \quad \langle\eta\rangle \sim \langle\xi-\eta\rangle\,. \tag{19}
$$

This implies that, setting $a(\xi-\eta,\eta) := \Theta(\xi,\eta)$, we get the (13). The (19) also implies that $\langle\xi\rangle \lesssim \max\{\langle\xi-\eta\rangle, \langle\eta\rangle\}$. Then we have

$$
\begin{aligned}
\|\mathcal{R}h\|^2_{H^{s+\rho}} &\lesssim \sum_{\xi\in\mathbb{Z}}\Big(\sum_{\eta\in\mathbb{Z}} |a(\xi-\eta,\eta)||\widehat{f}(\xi-\eta)||\widehat{g}(\eta)|\langle\xi\rangle^{s+\rho}\Big)^2\\
&\overset{(13)}{\lesssim} \sum_{\xi\in\mathbb{Z}}\Big(\sum_{|\xi-\eta|\geq|\eta|} \langle\xi-\eta\rangle^s|\widehat{f}(\xi-\eta)|\langle\eta\rangle^\rho|\widehat{g}(\eta)|\Big)^2\\
&+ \sum_{\xi\in\mathbb{Z}}\Big(\sum_{|\xi-\eta|\leq|\eta|} \langle\xi-\eta\rangle^\rho|\widehat{f}(\xi-\eta)||\widehat{g}(\eta)||\eta|^s\Big)^2\\
&\lesssim \sum_{\xi,\eta\in\mathbb{Z}} \langle\eta\rangle^{2(s_0+\rho)}|\widehat{g}(\eta)|^2\langle\xi-\eta\rangle^{2s}|\widehat{f}(\xi-\eta)|^2\\
&+ \sum_{\xi,\eta\in\mathbb{Z}} \langle\eta\rangle^{2s}|\widehat{g}(\eta)|^2\langle\xi-\eta\rangle^{2(s_0+\rho)}|\widehat{f}(\xi-\eta)|^2\\
&\lesssim \|f\|^2_{H^s}\|g\|^2_{H^{s_0+\rho}} + \|f\|^2_{H^{s_0+\rho}}\|g\|^2_{H^s}\,,
\end{aligned}
$$

which implies the (14) for $s_0+\rho\leq s$. □

3 Paralinearization

Equation (2) is equivalent to

$$
\begin{aligned}
&u_t + u_{xxx}\partial^2_{z_1z_1}F + 2u_{xx}\partial^3_{z_1xz_1}F + u_{xx}^2\partial^3_{z_1z_1z_1}F + 2u_xu_{xx}\partial^3_{z_1z_1z_0}F \\
&\quad + u_x^2\partial^3_{z_1z_0z_0}F + u_x(-\partial^2_{z_0z_0}F + 2\partial^3_{z_1xz_0}F) - \partial^2_{z_0x}F + \partial^3_{z_1xx}F = 0.
\end{aligned} \tag{20}
$$

We have the following.

Theorem 4 *Equation (20) is equivalent to*

$$
u_t + Op^{\mathrm{BW}}(A(u))u + R_0 = 0, \tag{21}
$$

where

$$
A(u) := \partial^2_{z_1z_1}F(\mathrm{i}\xi)^3 + \tfrac{1}{2}\tfrac{d}{dx}\Big(\partial^2_{z_1z_1}F\Big)(\mathrm{i}\xi)^2 + a_1(u, u_x, u_{xx}, u_{xxx})(\mathrm{i}\xi),
$$

with a_1 real function and R_0 semi-linear remainder. Moreover we have the following estimates. Let $\sigma \geq s_0 > 1 + 1/2$ and consider $U, V \in \dot{H}^{\sigma+3}$

$$
\|R_0(U)\|_{\dot{H}^\sigma} \leq C(\|U\|_{\dot{H}^{s_0+3}})\|U\|_{\dot{H}^\sigma}, \ \ \|R_0(U)\|_{\dot{H}^\sigma} \leq C(\|U\|_{\dot{H}^{s_0}})\|U\|_{\dot{H}^{\sigma+3}}, \tag{22}
$$

$$
\begin{aligned}
\|R_0(U) - R_0(V)\|_{\dot{H}^\sigma} &\leq C(\|U\|_{\dot{H}^{s_0+3}} + \|V\|_{\dot{H}^{s_0+3}})\|U - V\|_{\dot{H}^\sigma} \\
&\quad + C(\|U\|_{\dot{H}^\sigma} + \|V\|_{\dot{H}^\sigma})\|U - V\|_{\dot{H}^{s_0+3}},
\end{aligned} \tag{23}
$$

$$
\|R_0(U) - R_0(V)\|_{\dot{H}^{s_0}} \leq C(\|U\|_{\dot{H}^{s_0+3}} + \|V\|_{\dot{H}^{s_0+3}})\|U - V\|_{\dot{H}^{s_0}}, \tag{24}
$$

where C is a non-decreasing and positive function. Concerning the paradifferential operator we have for any $\sigma \geq 0$

$$
\|Op^{\mathrm{BW}}(A(u) - A(w))v\|_{\dot{H}^\sigma} \leq C(\|u\|_{\dot{H}^{s_0}} + \|w\|_{\dot{H}^{s_0}})\|u - w\|_{\dot{H}^{s_0}}\|v\|_{\dot{H}^{s_0+3}}. \tag{25}
$$

Proof In the following we use the Bony paraproduct (Lemma 1) and Proposition 3 and we obtain ($\tilde{R}_0$ is a smoothing remainder satisfying (22), (23) and it possibly changes from line to line)

$$
\begin{aligned}
u_{xxx}\partial^2_{z_1z_1}F &= Op^{\mathrm{BW}}(u_{xxx})\partial^2_{z_1z_1}F + Op^{\mathrm{BW}}(\partial^2_{z_1z_1}F) \circ Op^{\mathrm{BW}}((\mathrm{i}\xi)^3)u + \tilde{R}_0 \\
&= Op^{\mathrm{BW}}(u_{xxx}) \circ Op^{\mathrm{BW}}(\partial^3_{z_1z_1z_1}F) \circ Op^{\mathrm{BW}}(\mathrm{i}\xi)u \\
&\quad + Op^{\mathrm{BW}}(\partial^2_{z_1z_1}F) \circ Op^{\mathrm{BW}}((\mathrm{i}\xi)^3)u + \tilde{R}_0 \\
&= Op^{\mathrm{BW}}(\partial^2_{z_1z_1}F(\mathrm{i}\xi)^3)u + \tfrac{3}{2}Op^{\mathrm{BW}}(\tfrac{d}{dx}(\partial^2_{z_1z_1}F)\xi^2)u \\
&\quad + Op^{\mathrm{BW}}(\tilde{a}_1(\mathrm{i}\xi)) + \tilde{R}_0,
\end{aligned} \tag{26}
$$

where we have denoted by $\tilde{a}_1$ a real function depending on $x, u, u_x, u_{xx}, u_{xxx}$. Analogously we obtain

$$2u_{xx}\partial^3_{z_1xz_1}F = 2Op^{\mathrm{BW}}(\partial^3_{z_1z_1x}F(\mathrm{i}\xi)^2)u + Op^{\mathrm{BW}}(\tilde{a}_1(\mathrm{i}\xi))u + \tilde{R}_0, \tag{27}$$

$$u^2_{xx}\partial^3_{z_1z_1z_1}F = 2Op^{\mathrm{BW}}(u_{xx}\partial^3_{z_1z_1z_1}F(\mathrm{i}\xi^2))u + Op^{\mathrm{BW}}(\tilde{a}_1(\mathrm{i}\xi))u + \tilde{R}_0, \tag{28}$$

$$2u_xu_{xx}\partial^3_{z_1z_1z_0}F = 2Op^{\mathrm{BW}}\big(u_x\partial^3_{z_1z_1z_0}F(\mathrm{i}\xi)^2)u + 2Op^{\mathrm{BW}}(\tilde{a}_1(\mathrm{i}\xi))u + \tilde{R}_0. \tag{29}$$

Summing up the previous equations we get

$$\begin{aligned} u_t &+ Op^{\mathrm{BW}}(\partial^2_{z_1z_1}F(\mathrm{i}\xi)^3)u + \tfrac{1}{2}Op^{\mathrm{BW}}(\tfrac{d}{dx}(\partial^2_{z_1z_1}F)(\mathrm{i}\xi)^2)u \\ &+ Op^{\mathrm{BW}}(a_1(x, u, u_x, u_{xx}, u_{xxx})\mathrm{i}\xi)u + \tilde{R}_0(u) = 0, \end{aligned} \tag{30}$$

where a_1 is real and R_0 is a semi-linear remainder satisfying (22) and (23). □

4 Linear Local Well-Posedness

Proposition 1 *Let* $s_0 > 1 + 1/2$, $\Theta \geq r > 0$, $u \in C^0([0, T]; H_0^{s_0+3}) \cap C^1([0, T]; H_0^{s_0})$ *such that*

$$\|u\|_{L^\infty \dot{H}^{s_0+3}} + \|\partial_t u\|_{\dot{H}^{s_0}} \leq \Theta, \quad \|u\|_{L^\infty \dot{H}^{s_0}} \leq r. \tag{31}$$

Let $\sigma \geq 0$ *and* $t \mapsto R(t) \in C^0([0, T], \dot{H}^\sigma)$. *Then there exists a unique solution* $v \in C^0([0, T]; \dot{H}^\sigma) \cap C^1([0, T]; \dot{H}^{\sigma-3})$ *of the linear inhomogeneous problem*

$$\begin{aligned} &v_t + Op^{\mathrm{BW}}(\partial^2_{z_1z_1}F(u, u_x)(\mathrm{i}\xi)^3)v + \tfrac{1}{2}Op^{\mathrm{BW}}(\tfrac{d}{dx}(\partial^2_{z_1z_1}F(u, u_x))(\mathrm{i}\xi)^2)v \\ &\quad + Op^{\mathrm{BW}}(\tilde{a}_1(x, u, u_x, u_{xx}, u_{xxx})(\mathrm{i}\xi))v + R(t) = 0, \\ &v(0, x) = v_0(x). \end{aligned} \tag{32}$$

Moreover the solution satisfies the estimate

$$\|v\|_{L^\infty \dot{H}^\sigma} \leq e^{C_\Theta T}(C_r\|v_0\|_{\dot{H}^\sigma} + C_\Theta T\|R\|_{L^\infty \dot{H}^\sigma}). \tag{33}$$

Consider Eq. (32). We have for any $N \in \mathbb{N}$, $\sigma > 1/2$ and $s \geq 0$

$$\begin{aligned}
&\|\tilde{a}_1(x, u, u_x, u_{xx}, u_{xxx})\|_{\dot{H}^\sigma} \leq C(\|u\|_{\dot{H}^{\sigma+3}})\\
&\|\tfrac{d}{dx}(\partial^2_{z_1 z_1} F(u, u_x))\|_{\dot{H}^{\sigma-1}} \leq C(\|u\|_{\dot{H}^{\sigma+2}})\\
&\|\partial^2_{z_1 z_1} F(u, u_x)\|_{\dot{H}^\sigma} \leq C(\|u\|_{\dot{H}^{\sigma+1}}),\\
&|\partial^2_{z_1 z_1} F(u, u_x)|\xi|^{2s}|_{2s,\sigma,N} \leq C_N(\|u\|_{\dot{H}^{\sigma+1}}),\\
&|\tfrac{d}{dx}(\partial^2_{z_1 z_1} F(u, u_x))(\mathrm{i}\xi)^2|_{2,\sigma,N} \leq C_N(\|u\|_{\dot{H}^{\sigma+2}}),\\
&|\tilde{a}_1(x, u_x, u_{xx}, u_{xxx})|_{1,\sigma,N} \leq C_N(\|u\|_{\dot{H}^{\sigma+2}}).
\end{aligned} \tag{34}$$

In the following lemma we prove that, thanks to the Hamiltonian structure, we may eliminate the symbol of order two by means of a paradifferential change of variable. This term is the only one which has positive order and that is not skew-self-adjoint.

Lemma 2 *Define* $\mathtt{d}(x, u, u_x) := \sqrt[6]{\partial^2_{z_1 z_1} F(x, u, u_x)}$. *Then we have*

$$\begin{aligned}
Op^{\mathrm{BW}}(\mathtt{d}) \circ Op^{\mathrm{BW}}\Big(\partial^2_{z_1 z_1} F(\mathrm{i}\xi)^3 + \tfrac{1}{2}\tfrac{d}{dx}(\partial^2_{z_1 z_1} F)(\mathrm{i}\xi)^2\Big) \circ Op^{\mathrm{BW}}(\mathtt{d}^{-1})v = \\
Op^{\mathrm{BW}}\big(\big[\partial^2_{z_1 z_1} F(\mathrm{i}\xi)^3 + \tilde{a}_1(x, u, u_x, u_{xx}, u_{xxx})(\mathrm{i}\xi)\big]\big)v + R_0,
\end{aligned} \tag{35}$$

where $\tilde{a}_1$ is a real function and R_0 is a semi-linear remainder verifying (22), (23), (24).

Proof First of all the function $\mathtt{d}(x, u, u_x)$ is well defined because of hypothesis (1). We recall formula (10) (and the definition of the Poisson's bracket after (10)). By using Theorem 3 with $\rho \in (1, 2]$ we obtain that the L.H.S. of Eq. (35) equals

$$\begin{aligned}
-Op^{\mathrm{BW}}(\mathrm{i}\partial_{z_1 z_1} F\xi^3)v - \frac{1}{2} Op^{\mathrm{BW}}(\tfrac{d}{dx}(\partial^2_{z_1 z_1} F)\xi^2)v \\
+ 3 Op^{\mathrm{BW}}\Big(\mathtt{d}^{-1} \cdot \tfrac{d}{dx}\mathtt{d} \cdot \partial^2_{z_1 z_1} F \cdot \xi^2\Big)v + Op^{\mathrm{BW}}(\tilde{a}_1) + R_0,
\end{aligned}$$

where $\tilde{a}_1$ is a purely imaginary function and R_0 a semi-linear remainder. One can verify that the symbol of order two equals to zero by direct inspection. □

We consider symbol

$$\begin{aligned}
&\mathfrak{S}(x, u, \xi) := \\
&\partial^2_{z_1 z_1} F(u, u_x)(\mathrm{i}\xi)^3 + \tfrac{1}{2}\tfrac{d}{dx}(\partial^2_{z_1 z_1} F(u, u_x))(\mathrm{i}\xi)^2 + \tilde{a}_1(u, u_x, u_{xx}, u_{xxx})\mathrm{i}\xi,
\end{aligned} \tag{36}$$

and we introduce the smoothed version of the homogeneous part of (32), more precisely

$$\partial_t v^\epsilon = Op^{\mathrm{BW}}(\mathfrak{S}(x, u, u_x, u_{xx}, u_{xxx}; \xi))v^\epsilon - \epsilon\partial_{xx}^4 v_\epsilon. \tag{37}$$

Thanks to the parabolic term $\epsilon\partial_{xx}^4 v_\epsilon$ for any $\epsilon > 0$ there exists a unique solution of the equation, with initial condition in H^σ, (37) which is $C^0([0, T], \dot{H}^\sigma)$ for any $\sigma \geq 0$, where T depends on ϵ. This is the content on the following lemma.

Lemma 3 *For any initial condition v_0 in $\dot{H}^\sigma$ with $\sigma \geq 0$, there exists a time $T_\epsilon > 0$ and a unique solution v^ϵ (37) belonging to $C^0([0, T_\epsilon); \dot{H}^\sigma)$.*

Proof We consider the operator

$$\Gamma v := e^{-\epsilon t\partial_x^4} v_0 + \int_0^t e^{-\epsilon(t-t')\partial_x^4} Op^{\mathrm{BW}}(\mathfrak{S}(x, u, u_x, u_{xx}, u_{xxx}; \xi))v^\epsilon(t')dt'.$$

We have $\|e^{-\epsilon t\partial_x^4} v_0\|_{\dot{H}^\sigma} \leq \|v_0\|_{\dot{H}^\sigma}$ and $\|\int_0^t e^{-\epsilon(t-t')\partial_x^4} f(t', \cdot)dt'\|_{\dot{H}^\sigma} \leq t^{\frac{1}{4}}\epsilon^{-\frac{3}{4}}\|f\|_{\dot{H}^{\sigma-3}}$, with these estimates, (34), (31) and Theorem 2 one may apply a fixed point argument in a suitable subspace of $C^0([0, T_\epsilon); \dot{H}^\sigma)$ for a suitable time T_ϵ (going to zero when ϵ goes to zero). Let us prove the second one of the above inequalities. We use the Minkowski inequality and the boundedness of the function $\alpha^{3/2}e^{-\alpha}$ for $\alpha \geq 0$, we get

$$\begin{aligned}
\|\int_0^t e^{-\epsilon(t-t')\partial_x^4} f(t', \cdot)dt'\|_{\dot{H}^\sigma} &\leq \int_0^t \|e^{-\epsilon(t-t')\partial_x^4} f(t', \cdot)\|_{\dot{H}^\sigma} dt' \\
&= \int_0^t \left(\sum_{\xi\in\mathbb{Z}^*} e^{-2\epsilon(t-t')\xi^4}\xi^{2\sigma}|\widehat{f}(t', \xi)|^2\right)^{1/2} dt' \\
&\lesssim \int_0^t \epsilon^{-\frac{3}{4}}(t-t')^{-\frac{3}{4}}\|f(t', \cdot)\|_{\dot{H}^{\sigma-3}} dt' \\
&\lesssim t^{\frac{1}{4}}\epsilon^{-\frac{3}{4}}\|f\|_{L^\infty\dot{H}^{\sigma-3}}.
\end{aligned}$$

□

We show that (37) equation verifies *a priori* estimates with constants independent of ϵ. We have the following.

Proposition 2 *Let u be a function as in (31). For any $\sigma \geq 0$ there exist constants C_Θ and C_r, such that for any $\epsilon > 0$ the unique solution of (37) verifies*

$$\|v^\epsilon\|_{\dot{H}^\sigma}^2 \leq C_r\|v_0\|_{\dot{H}^\sigma}^2 + C_\Theta\int_0^t \|v^\epsilon(\tau)\|_{\dot{H}^\sigma}^2 d\tau, \forall t \in [0, T]. \tag{38}$$

As a consequence we have

$$\|v^{\epsilon}\|_{\dot{H}^{\sigma}} \le C_r e^{TC_{\theta}} \|v_0\|_{\dot{H}^{\sigma}}, \forall t \in [0, T]. \tag{39}$$

We define the modified energy

$$\|v\|^2_{\sigma,u} := \left\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1 z_1} F)^{\frac{2}{3}\sigma} \xi^{2\sigma}\right) Op^{\mathrm{BW}}\left(\mathtt{d}(x, u, u_x)\right) v, Op^{\mathrm{BW}}\left(\mathtt{d}(x, u, u_x)\right) v\right\rangle_{L^2}, \tag{40}$$

where $\langle\cdot,\cdot\rangle$ is the standard scalar product on $L^2(\mathbb{R})$ and $\mathtt{d}$ is defined in Lemma 2, note that the function $(\partial^2_{z_1 z_1} F(x, u, u_x))^{\frac{2}{3}\sigma}$ is well defined for any $\sigma \in \mathbb{R}$ thanks to (1).

In the following we prove that $\|\cdot\|_{\sigma,u}$ is equivalent to $\|\cdot\|_{\dot{H}^{\sigma}}$.

Lemma 4 *Let $s_0 > 1/2, \sigma \ge 0, r \ge 0$. Then there exists a constant (depending on r and σ) such that for any u such that $\|u\|_{\dot{H}^{s_0}} \le r$ we have*

$$C_r^{-1}\|v\|^2_{\dot{H}^{\sigma}} - \|v\|^2_{\dot{H}^{-3}} \le \|v\|^2_{\sigma,u} \le C_r\|v\|^2_{\dot{H}^{\sigma}} \tag{41}$$

for any v in $\dot{H}^{\sigma}$.

Proof Concerning the second inequality in (41), we reason as follows. We have

$$\begin{aligned}\|v\|^2_{\sigma,u} \le & \|Op^{\mathrm{BW}}((\partial^2_{z_1 z_1} F(x, u, u_x))^{\frac{2}{3}\sigma}\xi^{2\sigma}) Op^{\mathrm{BW}}(\mathtt{d}(x, u, u_x))v\|_{\dot{H}^{-\sigma}} \\ & \times \|Op^{\mathrm{BW}}(\mathtt{d}(x, u, u_x))v\|_{\dot{H}^{\sigma}} \\ \le & C_r\|v\|_{\dot{H}^{\sigma}},\end{aligned}$$

where in the last inequality we used Theorem 2 and the fact that $\mathtt{d}$ is a symbol of order zero. We focus on the first inequality in (41). Let $\delta > 0$ be such that $s_0 - \delta = 1/2$, then applying Theorem 3 with $s_0 = \delta$ instead of s_0 and $\rho = \delta$, we have

$$\begin{aligned}& Op^{\mathrm{BW}}((\partial^2_{z_1 z_1} F(x, u, u_x))^{\frac{1}{3}\sigma}) \circ Op^{\mathrm{BW}}(|\xi|^{2\sigma}) \circ Op^{\mathrm{BW}}((\partial^2_{z_1 z_1} F(x, u, u_x))^{\frac{1}{3}\sigma}) \\ & = Op^{\mathrm{BW}}(Op^{\mathrm{BW}}((\partial^2_{z_1 z_1} F(x, u, u_x))^{\frac{2}{3}\sigma}|\xi|^{2\sigma}) + \mathcal{R}^{2\sigma-\delta}(u),\end{aligned} \tag{42}$$

where

$$\|\mathcal{R}^{2\sigma-\delta}(u) f\|_{\dot{H}^{\overline{\sigma}-2\sigma+\delta}} \le C(r, \overline{\sigma})\|f\|_{\dot{H}^{\overline{\sigma}}}.$$

Analogously we obtain

$$\begin{aligned} & Op^{\mathrm{BW}}((\partial^2_{z_1 z_1} F(x,u,u_x))^{-\frac{1}{3}\sigma}) \circ Op^{\mathrm{BW}}(\mathtt{d}^{-1}(x,u,u_x,u_x x)) \circ \\ & \quad Op^{\mathrm{BW}}((\partial^2_{z_1 z_1} F(x,u,u_x))^{\frac{1}{3}\sigma}) \circ Op^{\mathrm{BW}}(\mathtt{d}(x,u,u_x,u_x x)) \\ & \quad = 1 + R^{-\delta}(u), \end{aligned} \tag{43}$$

where

$$\|\mathcal{R}^{-\delta}(u) f\|_{\dot{H}^{\overline{\sigma}}} \le C(r,\overline{\sigma}) \|f\|_{\dot{H}^{\overline{\sigma}-\delta}},$$

for any f in $\dot{H}^{\overline{\sigma}-\delta}$. Therefore we have

$$\begin{aligned} \|v\|^2_{\dot{H}^\sigma} & \overset{(43)}{\lesssim} \\ & \|Op^{\mathrm{BW}}((\partial^2_{z_1 z_1} F)^{-\frac{1}{3}\sigma}) Op^{\mathrm{BW}}(\mathtt{d}) v\|^2_{\dot{H}^\sigma} + \|v\|^2_{\dot{H}^{\sigma-\delta}} \\ & \le C_r (\|Op^{\mathrm{BW}}(\partial^2_{z_1 z_1} F)^{\frac{1}{3}\sigma}) Op^{\mathrm{BW}}(\mathtt{d}) v\|^2_{\dot{H}^\sigma} + \|v\|^2_{\dot{H}^{\sigma-\delta}}) \\ & \overset{(42)}{=} C_r (\|v\|^2_{u,\sigma} + \|v\|^2_{\dot{H}^{\sigma-\delta/2}} + \|v\|^2_{\dot{H}^{\sigma-\delta}}). \end{aligned}$$

Then by using the interpolation inequality $\|f\|_{\dot{H}^{\theta s_1+(1-\theta)s_2}} \le \|f\|^\theta_{\dot{H}^{s_1}} \|f\|^{1-\theta}_{\dot{H}^{s_2}}$ which is valid for any $s_1 < s_2$, $\theta \in [0,1]$ and $f \in \dot{H}^{s_2}$, we get (by means of the Young inequality $ab \le p^{-1}a^p + q^{-1}b^q$, with $1/p + 1/q = 1$ and $p = 2(\sigma+3)/\delta$, $q = 2(\sigma+3)/[2(\sigma+3)-\delta]$)

$$\begin{aligned} \|v\|^2_{\dot{H}^{\sigma-\delta/2}} & \le (\|v\|^2_{\dot{H}^{-3}})^{\frac{\delta}{2}\frac{1}{\sigma+3}} (\|v\|^2_{\dot{H}^\sigma})^{\frac{2(\sigma+3)-\delta}{2(\sigma+3)}} \\ & \le \frac{\delta}{2(\sigma+3)} \|v\|^2_{\dot{H}^{-3}} \tau^{-\frac{2(\sigma+3)}{\delta}} + \frac{2(\sigma+3)-\delta}{2(\sigma+3)} \tau^{\frac{2(\sigma+3)-\delta}{2(\sigma+3)}} \|v\|^2_{\dot{H}^\sigma}, \end{aligned}$$

for any $\tau > 0$. Choosing τ small enough we conclude. □

We shall need the following (weak) Garding type inequality.

Lemma 5 (Weak Garding) *Let* $\mathtt{d}$ *as in Lemma 2 and* $\mathfrak{c} > 0$ *as in (1) and define* $g := \partial^2_{z_2 z_2} F$*, we have the following inequalities*

$$\langle Op^{\mathrm{BW}}(\mathtt{d}) Op^{\mathrm{BW}}(g^{\frac{2}{3}\sigma} \xi^{2\sigma}) Op^{\mathrm{BW}}(\mathtt{d}) Op^{\mathrm{BW}}(\xi^4) w, w\rangle_{L^2} \ge \frac{\mathfrak{c}_\sigma}{2} \|w\|_{\dot{H}^{\sigma+2}} - \mathfrak{K}\|w\|_{\dot{H}^\sigma},$$
$$\langle Op^{\mathrm{BW}}(\xi^4) Op^{\mathrm{BW}}(\mathtt{d}) Op^{\mathrm{BW}}(g^{\frac{2}{3}\sigma} \xi^{2\sigma}) Op^{\mathrm{BW}}(\mathtt{d}) w, w\rangle_{L^2} \ge \frac{\mathfrak{c}_\sigma}{2} \|w\|_{\dot{H}^{\sigma+2}} - \mathfrak{K}\|w\|_{\dot{H}^\sigma},$$

for any w *in* $\dot{H}^{\sigma+2}$ *and where* $\mathfrak{K} > 0$ *depends on* Θ *in (31) and* $\mathfrak{c}_\sigma := \mathfrak{c}^{\frac{1}{3}+\frac{2}{3}\sigma}$.

Proof We prove the first inequality, the second one is similar. By using Theorem 3 with $\rho = 1$ we get

$$
\begin{aligned}
&Op^{\mathrm{BW}}(\mathtt{d})Op^{\mathrm{BW}}(g^{\frac{2}{3}\sigma}\xi^{2\sigma})Op^{\mathrm{BW}}(\mathtt{d})Op^{\mathrm{BW}}(\xi^4)w\\
&= Op^{\mathrm{BW}}(\mathtt{d}^2 g^{\frac{2}{3}\sigma}\xi^{2\sigma}\xi^4)w + R_{2\sigma+3}w\\
&= Op^{\mathrm{BW}}(g^{\frac{1}{3}+\frac{2}{3}\sigma}\xi^{2\sigma}\xi^4)w + R_{2\sigma+3}w,
\end{aligned}
$$

where $\|R_{2\sigma+3}w\|_{\dot{H}^{-\sigma-2}} \leq C_\Theta \|w\|_{\dot{H}^{\sigma+1}}$. Now we set

$$
p(x,\xi) = \sqrt{g^{\frac{1}{3}+\frac{2}{3}\sigma}\xi^{2\sigma+4} - \tfrac{\mathfrak{c}_\sigma}{2}\xi^{2\sigma+4}}, \quad |\xi| \geq 1, \quad \mathfrak{c}_\sigma = \mathfrak{c}^{\frac{1}{3}+\frac{2}{3}\sigma}. \tag{44}
$$

We have

$$
\begin{aligned}
0 \leq \|Op^{\mathrm{BW}}(p)w\|_{L^2} &= \langle Op^{\mathrm{BW}}(p)Op^{\mathrm{BW}}(p)w, w\rangle_{L^2}\\
&= \langle Op^{\mathrm{BW}}(g^{\frac{1}{3}+\frac{2}{3}\sigma}\xi^{2\sigma+4})w, w\rangle_{L^2}\\
&\quad - \tfrac{\mathfrak{c}_\sigma}{2}\|w\|^2_{\dot{H}^{\sigma+2}} + \langle \tilde{R}_{2\sigma+3}w, w\rangle,
\end{aligned}
$$

where $\tilde{R}_{2\sigma+3}$ verifies the same estimate as $R_{2\sigma+3}$ and where we used Theorem 3 with $\rho = 1$. Summing up we obtain

$$
\begin{aligned}
&\langle Op^{\mathrm{BW}}(\mathtt{d})Op^{\mathrm{BW}}(g^{\frac{2}{3}\sigma}\xi^{2\sigma})Op^{\mathrm{BW}}(\mathtt{d})Op^{\mathrm{BW}}(\xi^4)w, w\rangle_{L^2} \geq\\
&\tfrac{\mathfrak{c}_\sigma}{2}\|w\|^2_{\dot{H}^{\sigma+2}} - 2C_\Theta\|w\|_{\dot{H}^{\sigma+1}}\|w\|_{\dot{H}^{\sigma+2}}.
\end{aligned}
$$

We need to estimate from above the last summand, for any $\varepsilon, \eta > 0$ we have

$$
\begin{aligned}
\|w\|_{\dot{H}^{\sigma+1}}\|w\|_{\dot{H}^{\sigma+2}} &\leq \varepsilon\|w\|^2_{\dot{H}^{\sigma+2}} + C_\varepsilon\|w\|_{\dot{H}^{\sigma}}\|w\|_{\dot{H}^{\sigma+2}}\\
&\leq \varepsilon\|w\|^2_{\dot{H}^{\sigma+2}} + C_\varepsilon(\eta\|w\|^2_{\dot{H}^{\sigma+2}} + \eta^{-1}\|w\|^2_{\dot{H}^{\sigma}}),
\end{aligned}
$$

we conclude by choosing ε and η in such a way that $2C_\Theta(\varepsilon + C_\varepsilon\eta) \leq \mathfrak{c}_\sigma/4$. □

We are in position to prove Proposition 2.

Proof of Proposition 2 We take the derivative with respect to t of the modified energy (40) along the solution v^ϵ of Eq. (37). We have

$$
\frac{d}{dt}\|v^\epsilon\|_{\sigma,u} = \langle Op^{\mathrm{BW}}\left(\tfrac{d}{dt}(\partial^2_{z_1 z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,v^\epsilon, Op^{\mathrm{BW}}(\mathtt{d})\,v^\epsilon\rangle_{L^2} \tag{45}
$$

$$
+\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1 z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}\left(\tfrac{d}{dt}\mathtt{d}\right)v^\epsilon, Op^{\mathrm{BW}}(\mathtt{d})\,v^\epsilon\rangle_{L^2} \tag{46}
$$

$$
+\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1 z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,\tfrac{d}{dt}v^\epsilon, Op^{\mathrm{BW}}(\mathtt{d})\,v^\epsilon\rangle_{L^2} \tag{47}
$$

$$+\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,v^\epsilon, Op^{\mathrm{BW}}\left(\tfrac{d}{dt}\mathtt{d}\right)v^\epsilon\rangle_{L^2} \tag{48}$$

$$+\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,v^\epsilon, Op^{\mathrm{BW}}(\mathtt{d})\,\tfrac{d}{dt}v^\epsilon\rangle_{L^2}. \tag{49}$$

The most important term, where we have to see a cancellation, is the one given by (47)+(49). Using Eq. (37) we deduce that (47)+(49) equals to

$$\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,v^\epsilon, Op^{\mathrm{BW}}(\mathtt{d})\,Op^{\mathrm{BW}}(\mathfrak{S})v^\epsilon\rangle_{L^2} \tag{50}$$

$$+\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,Op^{\mathrm{BW}}(\mathfrak{S})v^\epsilon, Op^{\mathrm{BW}}(\mathtt{d})\,v^\epsilon\rangle_{L^2} \tag{51}$$

$$-\epsilon\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,Op^{\mathrm{BW}}(\xi^4)v^\epsilon, Op^{\mathrm{BW}}(\mathtt{d})\,v^\epsilon\rangle_{L^2} \tag{52}$$

$$-\epsilon\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,v^\epsilon, Op^{\mathrm{BW}}(\mathtt{d})\,Op^{\mathrm{BW}}(\xi^4)v^\epsilon\rangle_{L^2}, \tag{53}$$

where $\mathfrak{S}$ has been defined in (36). For the moment we consider just the first two summands (50)+(51) in the above equation. We note that by using Theorem 3 with $\rho = 3$ we obtain

$$Op^{\mathrm{BW}}(\mathtt{d}^{-1})Op^{\mathrm{BW}}(\mathtt{d})v^\epsilon = v^\epsilon + \mathcal{R}^{-3}(u)v^\epsilon,$$

where $\mathcal{R}^{-3}$ verifies (11) with $\rho = 3$. We plug this identity in (50)+(51) and we note that the contribution coming from $\mathcal{R}^{-3}$ is bounded by $C_r\|v^\epsilon\|^2_{\dot{H}^\sigma}$ thanks to Theorems 3, 2, to the Cauchy Schwartz inequality and to the assumption (31). We are left with

$$\begin{aligned}
&\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,v^\epsilon, Op^{\mathrm{BW}}(\mathtt{d})\\
&\quad\times Op^{\mathrm{BW}}(\mathfrak{S})Op^{\mathrm{BW}}(\mathtt{d}^{-1})Op^{\mathrm{BW}}(\mathtt{d})v^\epsilon\rangle_{L^2}+\\
&\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,Op^{\mathrm{BW}}(\mathfrak{S})\\
&\quad\times Op^{\mathrm{BW}}(\mathtt{d}^{-1})Op^{\mathrm{BW}}(\mathtt{d})v^\epsilon, Op^{\mathrm{BW}}(\mathtt{d})\,v^\epsilon\rangle_{L^2}.
\end{aligned}$$

At this point we are ready to use Lemma 2 and we obtain that the previous quantity equals

$$\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}(\mathtt{d})\,v^\epsilon, Op^{\mathrm{BW}}\left(\partial^2_{z_1z_1}F(\mathrm{i}\xi)^3+\tilde{a}_1(\mathrm{i}\xi)\right)Op^{\mathrm{BW}}(\mathtt{d})v^\epsilon\rangle_{L^2}+$$
$$\langle Op^{\mathrm{BW}}\left((\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\right)Op^{\mathrm{BW}}\left(\partial^2_{z_1z_1}F(\mathrm{i}\xi)^3+\tilde{a}_1(\mathrm{i}\xi)\right)Op^{\mathrm{BW}}(\mathtt{d})v^\epsilon, Op^{\mathrm{BW}}(\mathtt{d})\,v^\epsilon\rangle_{L^2}.$$

By using the skew self-adjoint character of the operators, we deduce that the main term to estimate is the commutator

$$\Big[Op^{\mathrm{BW}}\Big((\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\Big), Op^{\mathrm{BW}}\Big(\partial^2_{z_1z_1}F(\mathrm{i}\xi)^3+\tilde{a}_1(\mathrm{i}\xi)\Big)\Big]Op^{\mathrm{BW}}(\mathtt{d})v^{\epsilon}. \tag{54}$$

We start from the first summand. By using Theorem 3 and Remark 1 with $\rho = 3$ we obtain that

$$C := \Big[Op^{\mathrm{BW}}\Big((\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}\Big), Op^{\mathrm{BW}}\Big(\partial^2_{z_1z_1}F(\mathrm{i}\xi)^3\Big)\Big]Op^{\mathrm{BW}}(\mathtt{d})v^{\epsilon} =$$
$$\frac{1}{\mathrm{i}}Op^{\mathrm{BW}}\Big(\Big\{(\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}\xi^{2\sigma}, \partial^2_{z_1z_1}F(\mathrm{i}\xi)^3\Big\}\Big)Op^{\mathrm{BW}}(\mathtt{d})v^{\epsilon} + \mathcal{R}^0(u)Op^{\mathrm{BW}}(\mathtt{d})v^{\epsilon}.$$

By direct inspection we see that the Poisson bracket above equals to 0. Recalling that $\mathtt{d}$ is a symbol of order 0, by using also Theorem 2 and the assumption (31), we may obtain the bound $\langle C, Op^{\mathrm{BW}}(\mathtt{d})v^{\epsilon}\rangle \le C_r\|v^{\epsilon}\|^2_{\dot{H}^{\sigma}}$. The second summand, i.e. the one coming from $\tilde{a}_1(\mathrm{i}\xi)$ in (54), may be treated in a similar way: one uses Theorem 3 with $\rho = 1$, at the first order the contribution is equal to zero, then the remainder is a bounded operator from $\dot{H}^{2\sigma}$ to $\dot{H}^0$ and one concludes as before, by using also the duality inequality $\langle f, g\rangle_{L^2} \le \|f\|_{\dot{H}^{-\sigma}}\|g\|_{\dot{H}^{\sigma}}$, bounding everything by $C_r\|v^{\epsilon}\|^2_{\dot{H}^{\sigma}}$. This concludes the analysis of (50)+(51).

Concerning (52)+(53) we use Lemma 5 and the fact that

$$(52) + (53) \le -\epsilon\mathfrak{c}_{\sigma}\|v^{\epsilon}\|_{\dot{H}^{\sigma+2}} + 2\mathfrak{K}\|v^{\varepsilon}\|_{\dot{H}^{\sigma}} \le 2\mathfrak{K}\|v^{\varepsilon}\|_{\dot{H}^{\sigma}},$$

with $\mathfrak{K}$ depending on Θ and $\mathfrak{c}_{\sigma} = \mathfrak{c}^{\frac{1}{3}+\frac{2}{3}\sigma}$, recall (1).

We are left with (45), (46) and (48). These terms are simpler, one just has to use the duality inequality recalled above, then Theorem 2 and the fact that

$$|\tfrac{d}{dt}\mathtt{d}(x, u, u_x)|_{0,\sigma,4}, \quad |\tfrac{d}{dt}(\partial^2_{z_1z_1}F)^{\frac{2}{3}\sigma}|_{0,0,4} \le C_{\Theta}\|u\|_{\dot{H}^{\sigma}},$$

where we have used the first one of the assumptions (31).

We eventually obtained $\frac{d}{dt}\|v^{\epsilon}\|^2_{\sigma,u} \le C_{\Theta}\|v^{\epsilon}\|^2_{\dot{H}^{\sigma}}$, integrating over the time interval $[0, t)$ we obtain

$$\begin{aligned}\|v^{\epsilon}\|^2_{\sigma,u(t)} &\le \|v^{\epsilon}(0)\|^2_{\sigma,u(0)} + C_{\Theta}\int_0^t \|v^{\epsilon}(\tau)\|^2_{\dot{H}^{\sigma}}d\tau \\ &\le C_r\|v^{\epsilon}(0)\|^2_{\dot{H}^{\sigma}} + C_{\Theta}\int_0^t \|v^{\epsilon}(\tau)\|^2_{\dot{H}^{\sigma}}d\tau.\end{aligned}$$

We now use (41) and the fact that $\|\partial_t v^{\epsilon}\|_{\dot{H}^{-3}} \le C_{\Theta}\|v^{\epsilon}\|_{\dot{H}^0} \le C_{\Theta}\|v^{\epsilon}\|_{\dot{H}^{\sigma}}$ since $\sigma \ge 0$. □

We may now prove Proposition 1.

Proof of Proposition 1 Let v_0 be in $\dot{H}^\sigma$, we consider the smoothed initial condition

$$v_0^\epsilon = \chi(|D|\epsilon^{\frac{1}{8}})v_0 = \mathcal{F}^{-1}(\chi(|\xi|\epsilon^{\frac{1}{8}})\widehat{v_0}(\xi)),$$

for a C_0^∞ cut-off function supported on $(-2, 2)$ and equal to one on $[-1, 1]$. Let v^ϵ the solution of (37) with initial condition v_0^ϵ. By Lemma 3 v^ϵ is a continuous function with values in $\dot{H}^\sigma$ for a short time T_ϵ. By Proposition 2 the $\dot{H}^\sigma$ norm of the solution v^ϵ is bounded from above by a constant depending only on $\|v_0\|_{\dot{H}^\sigma}$, r and Θ in (31). Therefore if we proved that Γ in the proof of Lemma 3 was a contraction on the ball of radius M in $C^0([0, T); \dot{H}^\sigma)$ with M big enough with respect to $\|v_0\|_{\dot{H}^\sigma}$, r and Θ, then we have that there exists a time $T > 0$ depending only on $\|v_0\|_{\dot{H}^\sigma}$, r and Θ such that the solution verifies $\sup_{[0,T_\epsilon)} \|v^\epsilon\|_{\dot{H}^\sigma} \leq M/2$ for any $T_\epsilon \leq T$. For this reason we may iterate the proof of Lemma 3 on the interval $[T_\epsilon, 2T_\epsilon]$ etc... We conclude that there exists a common time of existence $T > 0$ for each solution v^ϵ such that $\sup_{[0,T_\epsilon)} \|v^\epsilon\|_{\dot{H}^\sigma} \leq \mathtt{M}$ with $\mathtt{M}$ depending on $\|v_0\|_{\dot{H}^\sigma}$, Θ and r in (31).

We show that v^ϵ is a Cauchy sequence in $C([0, T); \dot{H}^\sigma)$. Let $0 < \delta \leq \epsilon$ and set $z = v^\epsilon - v^\delta$, then we have $\partial_t z = Op^{\mathrm{BW}}(\mathfrak{S})z - \epsilon\partial_x^4 z + \partial_x^4 v^\epsilon(\delta - \epsilon)$. By Lemma 3 we have that the flow Φ_ϵ of $\partial_t z_1 = Op^{\mathrm{BW}}(\mathfrak{S})z_1 - \epsilon\partial_x^4 z_1$ exists and by Proposition 2, it has estimates independent of ϵ. By Duhamel formulation we have

$$z(t, x) = \Phi_\epsilon(t)(v_0^\epsilon(x) - v_0^\delta(x)) + (\delta - \epsilon)\Phi_\epsilon(t)\int_0^t \Phi_\epsilon(s)^{-1}\partial_x^4 v^\epsilon(s, x)ds.$$

By the estimate (39) (on the flow Φ_ϵ) and the Minkowski inequality we get $\|z(t, x)\|_{\dot{H}^\sigma} \leq (\epsilon - \delta)C\|\partial_x^4 v^\epsilon\|_{\dot{H}^\sigma}$, for a constant depending on Θ and r in (31). Applying again (39) on the function v^ϵ we get $\|z(t, x)\|_{\dot{H}^\sigma} \leq C(\epsilon - \delta)\|v_0^\epsilon\|_{\dot{H}^{\sigma+4}}$ for another constant C depending on r and Θ. At this point we may use that $\|\chi(|D|\epsilon^{\frac{1}{8}})v_0\|_{\dot{H}^{\sigma+4}} \leq \epsilon^{-\frac{1}{2}}\|v_0\|_{\dot{H}^\sigma}$. Since $0 < \delta < \epsilon$ we have that $\|z(t, x)\|_{\dot{H}^\sigma} \leq \tilde{C}\epsilon^{\frac{1}{2}}\|v_0\|_{\dot{H}^\sigma}$, hence $z(t, x)$ is a Cauchy sequence in $\dot{H}^\sigma$ and converges to a solution of (37) with $\epsilon = 0$ and initial condition $v_0 \in \dot{H}^\sigma$.

The flow $\Phi(t)$ of Eq. (32) with $R(t) = 0$ is well defined as a bounded operator form $\dot{H}^\sigma$ to $\dot{H}^\sigma$ and satisfies the estimate

$$\|\Phi(t)v_0\|_{\dot{H}^\sigma} \leq C_r e^{C_\Theta t}\|v_0\|_{\dot{H}^\sigma}.$$

One concludes by using the Duhamel formulation of (32). □

5 Nonlinear Local Well-Posedness

To build the solutions of the nonlinear problem (32), we shall consider a classical quasi-linear iterative scheme, we follow the approach in [1, 7, 8, 15]. Set

$$A(u) := Op^{\mathrm{BW}}\left(\partial^2_{z_1 z_1} F(\mathrm{i}\xi)^3 + \frac{1}{2}\frac{d}{dx}(\partial^2_{z_1 z_1} F)(\mathrm{i}\xi)^2 + \tilde{a}_1(x, u, u_x, u_{xx}, u_{xxx})(\mathrm{i}\xi)\right)$$

and define

$$\mathcal{P}_1 : \quad \partial_t u_1 = A(u_0)u_1;$$
$$\mathcal{P}_n : \quad \partial_t u_n = A(u_{n-1})u_n + R(u_{n-1}), \quad n \geq 2.$$

The proof of the main Theorem 1 is a consequence of the next lemma. Owing to such a lemma one can follow closely the proof of Lemma 4.8 and Proposition 4.1 in [1] or the proof of Theorem 1.2 in [8](this is the classical Bona-Smith technique [2], but we followed the notation of [1, 8]). We do not reproduce here such a proof.

Lemma 6 *Let $s > \frac{1}{2} + 4$. Set $r := \|u_0\|_{\dot{H}^{s_0}}$ and $s_0 > 1 + 1/2$. There exists a time $T := T(\|u_0\|_{\dot{H}^{s_0+3}})$ such that for any $n \in \mathbb{N}$ the following statements are true.*

$(\mathcal{S}0)_n$: *There exists a unique solution u_n of the problem $\mathcal{P}_n$ belonging to the space $C^0([0, T); \dot{H}^s) \cap C^1([0, T); \dot{H}^{s-3})$.*

$(\mathcal{S}1)_n$: *There exists a constant $\mathtt{C}_r \geq 1$ such that if $\Theta = 4\ \mathtt{C}_r\|u_0\|_{\dot{H}^{s_0+3}}$ and $M = 4\mathtt{C}_r\|u_0\|_{\dot{H}^s}$, for any $1 \leq m \leq n$, for any $1 \leq m \leq n$ we have*

$$\|u_m\|_{L^\infty \dot{H}^{s_0}} \leq \mathtt{C}_r, \tag{55}$$

$$\|u_m\|_{L^\infty \dot{H}^{s_0+3}} \leq \Theta, \quad \|\partial_t u_m\|_{L^\infty \dot{H}^{s_0}} \leq \mathtt{C}_r\Theta, \tag{56}$$

$$\|u_m\|_{L^\infty \dot{H}^s} \leq M, \quad \|\partial_t u_m\|_{L^\infty \dot{H}^s} \leq \mathtt{C}_r M. \tag{57}$$

$(\mathcal{S}2)_n$: *For any $1 \leq m \leq n$ we have*

$$\|u_1\|_{L^\infty \dot{H}^{s_0}} \leq \mathtt{C}_r, \quad \|u_m - u_{m-1}\|_{L^\infty \dot{H}^{s_0}} \leq 2^{-m}\mathtt{C}_r, \quad m \geq 2. \tag{58}$$

Proof We proceed by induction over n. We prove $(S0)_1$, by using Proposition 1 with $R(t) = 0$, $u \rightsquigarrow u_0$ and $v \rightsquigarrow u_1$; we obtain a solution u_1 which is defined on every interval $[0, T)$ and verifies the estimate $\|u_1\|_{L^\infty_T \dot{H}^\sigma} \leq e^{T\|u_0\|_{\dot{H}^\sigma}} C_r\|u_0\|_{\dot{H}^\sigma}$, $\sigma \geq 0$ with $C_r > 0$ given by Proposition 1. $(S1)_1$ is a consequence of the previous estimate applied with $\sigma = s_0$ for (55) and (56), with $\sigma = s$ for (57). In order to obtain the seconds in (56) and (57), one has to fix $T \leq 1/\|u_0\|_{\dot{H}^{s_0}}$ and use the equation for u_1 together with Theorem 2 and one finds M which depends on $\|u_0\|_{\dot{H}^s}$ and Θ which depends on $\|u_0\|_{\dot{H}^{s_0}}$ and on a constant C_r depending only on $\|u_0\|_{\dot{H}^{s_0}}$. $(S2)_1$ is trivial.

We assume that $(SJ)_{n-1}$ holds true for any $J = 0, 1, 2$ and we prove that $(SJ)_n$.

Owing to $(S0)_{n-1}$ and $(S1)_{n-1}$, the $(S0)_n$ is a direct consequence of Proposition 1. Let us prove (55) with $m = n$. By using (33) applied to the problem solved by u_n, the estimate (22) with $\sigma = s_0$, (55) with $m = n - 1$ and $(S0)_{n-1}$, we obtain $\|u_n\|_{L^\infty \dot{H}^{s_0}} \leq e^{C_\Theta T}(C_r\|u_0\|_{\dot{H}^{s_0}} + \mathtt{C}_r C_\Theta T)$, the thesis follows by choosing $e^{C_\Theta T} C_\Theta T < 1/4$ and $\mathtt{C}_r \geq \|u_0\|_{\dot{H}^{s_0}}/4C_r$.

We prove the first in (56). Applying (33) with $\sigma = s_0 + 3$ and $v \rightsquigarrow u_n$, $u \rightsquigarrow u_{n-1}$, the estimate on the remainder (22) and using $(S1)_{n-1}$ we obtain $\|u_n\|_{\dot{H}^{s_0+3}} \leq e^{C_\Theta T}\mathtt{C}_r\|u_0\|_{\dot{H}^{s_0+3}} + \Theta C_\Theta T e^{C_\Theta T}$, fixing T small enough such that $TC_\Theta \leq 1$ and $TC_\Theta e^{C_\Theta T} \leq 1/4$, the thesis follows from the definition $\Theta := 4\mathtt{C}_r\|u_0\|_{\dot{H}^{s_0}}$. The second in (56) may be proven by using the equation for u_n and the second in (22)

$$\|\partial_t u_n\|_{\dot{H}^{s_0}} \leq \|A(u_{n-1})u_n\|_{\dot{H}^{s_0}} + \|R(u_{n-1})\|_{\dot{H}^{s_0}} \leq C(\|u_{n-1}\|_{\dot{H}^{s_0}})\|u_n\|_{\dot{H}^{s_0+3}} \leq \mathtt{C}_r\Theta.$$

The (57) is similar. We prove $(S2)_n$, we write the equation solved by $v_n = u_n - u_{n-1}$

$$\partial_t v_n = A(u_{n-1})v_n + f_n, \qquad f_n = \big[A(u_{n-1}) - A(u_{n-2})\big]u_{n-1} + R(u_{n-1}) - R(u_{n-2}).$$

By using (23), (25) and the $(S2)_{n-1}$ we may prove that $\|f_n\|_{\dot{H}^{s_0}} \leq C_\Theta\|v_{n-1}\|_{\dot{H}^{s_0}}$. We apply again Proposition 1 with $\sigma = s_0$ and we find $\|v_n\|_{\dot{H}^{s_0}} \leq C_\Theta T e^{C_\Theta T}\|v_{n-1}\|_{\dot{H}^{s_0}}$, as T has been chosen small enough we conclude the proof. □

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Quasilinear Wave Equations with Decaying Time-Potential

Vladimir Georgiev and Sandra Lucente

Abstract An active area of recent research is the study of global existence and blow up for nonlinear wave equations where time depending mass or damping are involved. The interaction between linear and nonlinear terms is a crucial point in determination of global evolution dynamics. When the nonlinear term depends on the derivatives of the solution, the situation is even more delicate. Indeed, even in the constant coefficients case, the null conditions strongly relate the symbol of the linear operator with the form of admissible nonlinear terms which leads to global existence. Some peculiar operators with time-dependent coefficients lead to a wave operator in which the time derivative becomes a covariant time derivative. In this paper we give a blow up result for a class of quasilinear wave equations in which the nonlinear term is a combination of powers of first and second order time derivatives and a time-dependent factor. Then we apply this result to scale invariant damped wave equations with nonlinearity involving the covariant time derivatives.

1 Introduction

We study the following Cauchy Problem:

$$\begin{cases} z_{tt} - \Delta z = (1+t)^{\gamma} A(t, x, z, z_t, z_{tt})\,, \\ z(0, x) = f(x)\,, \\ z_t(0, x) = g(x)\,, \end{cases} \tag{1}$$

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with $x \in \mathbb{R}^3, t \geq 0$ and $\gamma \in \mathbb{R}$. In particular we want to deal with A uniformly bounded with respect to t and $\gamma < 0$.

The importance of this quasilinear wave equation with time-dependent potential comes from the special scale invariant wave equation. Let $\mu \in \mathbb{R}$. The equation

$$z_{tt} - \Delta z = (1+t)^{-\frac{\mu}{2}(p-1)}|z|^p$$

is equivalent to

$$u_{tt} - \Delta u + \frac{\mu}{1+t}u_t + \frac{\mu(\mu-2)}{4(1+t)^2}u = |u|^p \tag{2}$$

after the transformation $u(t,x) = (1+t)^{\frac{\mu}{2}}z(t,x)$. Similarly, let $\alpha \in \mathbb{R}$, the equation

$$z_{tt} - \Delta z = (1+t)^{\alpha-\frac{\mu}{2}(p-1)}|z_t|^p$$

is equivalent to

$$u_{tt} - \Delta u + \frac{\mu}{1+t}u_t + \frac{\mu(\mu-2)}{4(1+t)^2}u = (1+t)^\alpha \left|\left(\partial_t + \frac{\mu}{2(1+t)}\right)u\right|^p . \tag{3}$$

The existence theory for initial value problems associated with (2) has been intensively studied. The case $\mu = 2$ has been firstly analyzed in [3], for $\mu \neq 2$ the interested reader can see [7] and the reference therein. Equation (3), with $\alpha = 0$, has been considered only in Girardi-Lucente [4]. The study of the quasilinear scale invariant wave equation is still incomplete, for example,

$$u_{tt} - \Delta u + \frac{\mu}{1+t}u_t + \frac{\mu(\mu-2)}{4(1+t)^2}u = \left|\left(\partial_t + \frac{\mu}{2(1+t)}\right)^2 u\right|^q \tag{4}$$

is equivalent to

$$z_{tt} - \Delta z = (1+t)^{-\frac{\mu}{2}(q-1)}|z_{tt}|^q$$

but, up to our knowledge, no result on this equation is known. This is the inspired motivation of the present paper.

While studying the more general setting (1), we want to show how a decreasing potential $(1+t)^\gamma$, with $\gamma < 0$ interacts with the growth of nonlinear term A in the variables z, z_t, z_{tt}.

On the other hand, applying such result to (3), we can describe the same phenomenon as an interaction between the potential and linear part of the equation. More precisely we will have a blow up result under a condition which relate α, p, μ. We obtain a modified Strauss exponent. In a similar way we deal with (4).

In this paper, we start proving a blow up result for smooth solutions of (1) in Sect. 2. Following [5] we use an averaging method. Then, in Sect. 3, we apply such result to the special scale invariant wave operator. We leave the global existence counterpart of the paper for a further coming paper, except a very simple case given in Sect. 4.

Starting from these examples we can come back to the question of the influence of the lower order terms of the linear operator on the global existence/blow up of the solution. When these terms depend on time, they may become dominant with respect to higher order terms and might cause change of the critical exponents. For this reason, the paper tries to determine null condition for wave equation with time-dependent coefficients hoping that this analysis shall be useful to obtain global existence result in the future.

2 Quasilinear Wave Equations

2.1 Statement of the Main Results

Let us consider the following 3D Cauchy Problem:

$$\begin{cases} z_{tt} - \Delta z = (1+t)^{\gamma} A(t, x, z, z_t, z_{tt}) \,, & x \in \mathbb{R}^3, t \geq 0, \\ z(0, x) = f(x) \,, \\ z_t(0, x) = g(x) \,, \end{cases} \tag{5}$$

with $f, g \in C^2(\mathbb{R}^3)$ having compact support. In the special case, when $A = A(t, x, z_t, z_{tt})$ is independent of z we can set

$$y(t, x) = z_t(t, x) \,,$$

so that the problem takes the form

$$\begin{cases} y_{tt} - \Delta y = \partial_t((1+t)^{\gamma} B(t, x, y, y_t)) \,, \\ y(0, x) = g(x) \,, \\ y_t(0, x) = h(x) \,, \end{cases} \tag{6}$$

with suitable h and B. Some results on (6) can be found in the seminal paper by Fritz John [5]; in particular, reading that paper we can deduce the following:

Proposition 1 *If $\gamma \geq 0$, suppose $B \in C^3$ satisfies*

$$B(t, x, y, y_t) \geq (ay + by_t)^2 \text{ with } a^2 + b^2 > 0.$$

Assume in addition that $B(t, x, 0, 0) = 0$*,* g, h *are compactly supported,* $(g, h) \neq 0$ *and*

$$\int_{\mathbb{R}^3} h(x) - B(0, x, g(x), h(x))\, dx \geq 0\,. \tag{7}$$

Then the smooth maximal solution of (6) *blows up: let the* $T > 0$ *the largest value such that* $y(t, x) \in C^2([0, T) \times \mathbb{R}^3)$ *exists, then* $T < +\infty$*.*

Now we can explain how to relate (5)–(6) in the general case, when A depends also on z. Indeed, if z is a solution of (5), then we can set $y = z_t$, and find z as an integral operator $z(t, x) = f(x) + \int_0^t y(s, x)ds$ acting on y. In this way

$$B(t, x, y, y_t) = A\left(t, x, f(x) + \int_0^t y(s, x)\, ds, y, y_t\right) \tag{8}$$

can be interpreted as a non-local nonlinearity depending on t, x, y, y_t. The initial data $y(0, x) = g(x)$ is automatically satisfied. The other data $y_t(0, x) = h(x)$ means that we need

$$z_{tt}(0, x) = h(x)$$

so using the equation for z we get

$$h(x) = \Delta f + A(0, x, f(x), g(x), h(x))\,. \tag{9}$$

Therefore, we can make the reduction from (5) to (6) is we require that for given f, g Eq. (9) has a unique solution $h(x)$ for any $x \in \mathbb{R}^3$.

In particular if A satisfies

$$A(t, x, 0, 0, \xi) = 0 \quad \forall x \in \mathbb{R}^3\,, \xi \in \mathbb{R}\,, t \geq 0\,, \tag{10}$$

then the information on the support of initial data is preserved, indeed for any $x \in \mathbb{R}^3$ such that $f(x) = g(x) = 0$, we have $h(x) = 0$.

One can try to see how (6) is related to (5). Indeed, setting

$$\eta_0 = t, (\eta_1, \eta_2, \eta_3) = x,\ (\eta_4, \eta_5) = (y_t, y_{tt}),$$

we obtain

$$A = \gamma(1 + t)^{\gamma - 1} B + \frac{\partial B}{\partial \eta_0} + \frac{\partial B}{\partial \eta_4} z_{tt}\,,$$

provided

$$\frac{\partial B}{\partial \eta_5} = 0\,.$$

Our next step is to rewrite Fritz John's result for (5).

Proposition 2 *Let* $T \geq 0$. *If* $\gamma \geq 0$, *suppose* $A \in C^3$ *satisfies*

$$A(t, x, z, z_t, z_{tt}) \geq (az_t + bz_{tt})^2 \text{ with } a^2 + b^2 > 0.$$

Assume in addition (10) *and*

$$A(t, x, f(x), 0, 0) = 0 \quad \forall x \in \mathbb{R}^3\,,\ t \geq 0\,. \tag{11}$$

Let f, g *are compactly supported and* $(f, g) \neq (0, 0)$ *such that* (9) *has unique solution* $h(x)$ *for any* $x \in \mathbb{R}^3$. *Let* $z(t, x) \in C^2([0, T) \times \mathbb{R}^3)$ *be the maximal smooth solution of* (5), *then it blows up:* $T < +\infty$.

We note that it is not necessary to assume (7), indeed it reduces to $\int_{\mathbb{R}^3} \Delta f(x)\, dx = 0$ which is trivially satisfied.

In the present paper we want to deal with

$$B(t, x, y, y_t) \geq a^2|y|^p + b^2|y_t|^q\,, \quad p > 1, q > 1, a^2 + b^2 > 0\,. \tag{12}$$

Our aim is to establish that the smooth solution of (6) blows up for any $\gamma \geq \gamma_0$ with a suitable $\gamma_0 = \gamma_0(p, q) \in \mathbb{R}$. In particular we are looking for negative γ_0 not included in [5] even if $p = q = 2$.

Theorem 1 *Let* $y(t, x) : [0, T) \to \mathbb{R} \times \mathbb{R}^3$ *be a non-trivial* C^2 *solution of* (6) *with* $g, h \in C^2(\mathbb{R}^3)$ *compactly supported with* $(g, h) \neq (0, 0)$ *and*

$$\int_{\mathbb{R}^3} h(x) - B(0, x, g(x), h(x))dx \geq 0\,. \tag{13}$$

Suppose that $B(t, x, 0, 0) = 0$ *and that* (12) *is satisfied. Then we have* $T < \infty$, *provided one of the following:*

1. $\gamma \geq 0$ *and* $p \leq 2$ *or* $q \leq 2$;
2. $1 - \frac{2}{p} < \gamma < 0$ *(except the case* $a = 0$*)*;
3. $1 - \frac{2}{q} < \gamma < 0$ *(except the case* $b = 0$*)*.

Having in mind the relation between (5) and (6), we can deduce the following result for (5) .

Theorem 2 *Let* $z(t, x) : [0, T) \to \mathbb{R} \times \mathbb{R}^3$ *be a non-trivial* C^2 *solution of* (5) *with* $f, g \in C^2(\mathbb{R}^3)$ *compactly supported with* $g \neq 0$. *Assume* (10), (11) *and that Eq.* (9) *has a unique solution* $h(x)$ *for any* $x \in \mathbb{R}^3$. *Suppose*

$$A(t, x, z, z_t, z_{tt}) \geq a^2|z_t|^p + b^2|z_{tt}|^q\,, \quad p > 1, q > 1, a^2 + b^2 > 0\,. \tag{14}$$

Then $T < \infty$ provided one of the following:

1. $\gamma \geq 0$ *and* $p \leq 2$ *or* $q \leq 2$;
2. $1 - \frac{2}{p} < \gamma < 0$ *(except the case* $a = 0$*)*;
3. $1 - \frac{2}{q} < \gamma < 0$ *(except the case* $b = 0$*)*.

Remark 1 Our results do not hold for both $p > 2$ and $q > 2$.

2.2 *Proof of Theorem 1*

We set

$$v(t,x) = \int_0^t y(s,x)ds\,.$$

Let $R > 0$ such that

$$g, h \text{ are compactly supported in } B_R(0) \quad R > 0\,, \tag{15}$$

hence

$$y(t,x) \text{ is compactly supported in } B_{R+t}(0)\,,$$

that is

$$v(t,x) = 0 \text{ for } |x| > t + R\,.$$

We can deduce that

$$\begin{aligned}
&\partial_t(v_{tt} - \Delta v) = y_{tt} - \Delta y = \partial_t((1+t)^\gamma B(t,x,y,y_t))\,,\\
&v(0,x) = 0\,,\\
&v_t(0,x) = y(0,x) = g(x)\,,\\
&(v_{tt} - \Delta v)(0,x) = y_t(0,x) = h(x)\,.
\end{aligned}$$

We gain

$$\partial_t(v_{tt} - \Delta v - (1+t)^\gamma B(t,x,y,y_t)) = 0\,.$$

So that, for any $t > 0$ we have

$$\begin{aligned}
&v_{tt} - \Delta v - (1+t)^\gamma B(t,x,y,y_t) =\\
&= (v_{tt} - \Delta v)(0,x) - B(0,x,g(x),h(x)) = h(x) - B(0,x,g(x),h(x))\,.
\end{aligned}$$

Summarizing we have the Cauchy Problem

$$\begin{cases} v_{tt} - \Delta v = (1+t)^{\gamma} B(t, x, v_t, v_{tt}) + h(x) - B(0, x, g, h)\,, \\ v(0, x) = 0\,, \\ v_t(0, x) = g(x)\,. \end{cases} \tag{16}$$

Then we arrive at

$$v_{tt} - \Delta v \geq (1+t)^{\gamma} (a^2 |v_t|^p + b^2 |v_{tt}|^q) + h(x) - B(0, x, g, h)\,. \tag{17}$$

Let

$$w(t, x) = v_{tt}(t, x) - \Delta v(t, x)\,,$$

so that

$$w(t, x) \geq (1+t)^{\gamma} (a^2 |v_t(t, x)|^p + b^2 |v_{tt}(t, x)|^q) + h(x) - B(0, x, g(x), h(x))\,. \tag{18}$$

We consider the spherical means

$$\bar{w}(t, r) = \frac{1}{4\pi} \int_{|\xi|=1} w(t, r\xi) d\sigma_{\xi} \quad r > 0\,.$$

We have

$$\bar{w} \geq (1+t)^{\gamma} \bar{B} + \overline{h(x) - B(0, x, g, h)}$$

and hence

$$\bar{v}(t, r) = \frac{1}{2r} \int_{r-t}^{r+t} \rho \bar{g}(\rho) d\rho + \int \int_{\mathbf{T_{r,t}}} \frac{\rho}{2r} \bar{w}(\rho, \tau) d\rho d\tau, \tag{19}$$

where (see triangle ABC in Fig. 1)

$$\mathbf{T_{r,t}} = \{(\rho, \tau) \,|\, \tau + \rho \leq t + r\,;\, \tau - \rho \leq t - r\,;\, \tau \geq 0\}.$$

Consider

$$\Sigma = \{(r, t) \,|\, r + R < t < 2r\}\,.$$

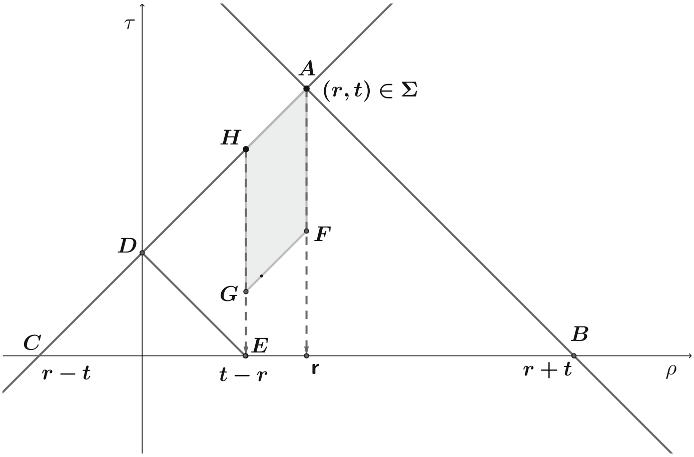

Fig. 1 Domains of integration for $(r, t) \in \Sigma$

Since we are assuming (15), the first term in (19) is zero in Σ, since $\rho\bar{g}(\rho)$ is odd. For a similar reason we can restrict the integration domain of the second term to the trapezoid $ABED$ on Fig. 1:

$$\mathbf{T}^*_{\mathbf{r},\mathbf{t}} = \{(\rho, \tau) \,|\, t - r \le \tau + \rho \le t + r\,;\, \tau - \rho \le t - r\,;\, \tau \ge 0\}.$$

For any $(r, t) \in \Sigma$ we get

$$\begin{aligned}\bar{v}(t, r) &= \iint_{\mathbf{T}^*_{\mathbf{r},\mathbf{t}}} \frac{\rho}{2r} \bar{w}(\rho, \tau) d\rho d\tau \\ &\ge \iint_{\mathbf{T}^*_{\mathbf{r},\mathbf{t}}} \frac{\rho}{2r} (1 + \tau)^{\gamma} \bar{B} d\rho d\tau + 2 \int_0^R \frac{\rho^2}{2r} \overline{h - B(0, x, g, h)}(\rho) d\rho\,.\end{aligned}$$

Due to (13), we conclude

$$\bar{v}(t, r) \ge \iint_{\mathbf{T}^*_{\mathbf{r},\mathbf{t}}} \frac{\rho}{2r} (1 + \tau)^{\gamma} \bar{B} d\rho d\tau \quad (r, t) \in \Sigma\,. \tag{20}$$

Here we used (18). For any $(r, t) \in \Sigma$, we restrict the integration domain to the parallelogram $AFGH$ on Fig. 1:

$$\mathbf{S}_{\mathbf{r},\mathbf{t}} = \{(\rho, \tau) \,|\, t - r < \rho < r\,;\, \rho - R < \tau < \rho + t - r\} \subset \mathbf{T}^*_{\mathbf{r},\mathbf{t}}\,.$$

Applying Jensen inequality to (17), we arrive at

$$\bar{v}(t,r) \geq \frac{1}{2r}\int_{t-r}^{r} \rho d\rho \int_{\rho-R}^{\rho+t-r} (1+\tau)^{\gamma}(a^2|\bar{v}_\tau|^p + b^2|\bar{v}_{\tau\tau}|^q)d\tau\,. \tag{21}$$

Having in mind the location of the support of $\bar{v}(t,r)$, we can write

$$\bar{v}(\rho+t-r,\rho) = \int_{\rho-R}^{\rho+t-r} \bar{v}_\tau(\tau,\rho)d\tau \tag{22}$$

and also

$$\bar{v}(\rho+t-r,\rho) = \int_{\rho-R}^{\rho+t-r} (\rho+t-r-\tau)\bar{v}_{\tau\tau}(\tau,\rho)d\tau\,. \tag{23}$$

The idea is now to slice Σ into half-lines:

$$\sigma_c = \{(r,t)\,|\,t = c+r\,;\,r > c\}\,,\quad \Sigma = \bigcup_{c>R}\sigma_c\,.$$

Let us denote by α the restriction of $\bar{v}$ on these half-lines:

$$\alpha(r) = |\bar{v}(r+c,r)| \quad r > c > R\,.$$

Our aim is to prove that

$$\alpha(r) = 0 \text{ for } r > c > R\,, \tag{24}$$

so that

$$\bar{v}(t,r) = 0 \text{ on } \Sigma\,. \tag{25}$$

Let

$$\beta(r) = \int_c^r \rho\alpha^p(\rho)\,d\rho + \int_c^r \rho\alpha^q(\rho)\,d\rho := \beta_1(r)+\beta_2(r)\,.$$

If $\beta(r) = 0$, then we get (24).

Assume by contradiction that there exists $r_0 > 0$ such that $\beta(r_0) \neq 0$.

By using (22), we have

$$\begin{aligned} a^2\beta_1(r) &\leq a^2\int_c^r \rho\left|\int_{\rho-R}^{\rho+c} a^{-\frac{2}{p}}a^{\frac{2}{p}}\bar{v}_\tau(\rho,\tau)(1+\tau)^{\gamma/p}(1+\tau)^{-\gamma/p}d\tau\right|^p d\rho \\ &\leq \int_c^r \rho\left(\int_{\rho-R}^{\rho+c}(1+\tau)^{-\gamma p'/p}d\tau\right)^{p'/p}\left(\int_{\rho-R}^{\rho+c}(a^2\bar{|}v_\tau|^p(1+\tau)^{\gamma}d\tau\right)d\rho\,. \end{aligned}$$

Let us recall that $p/p' = p-1$. Setting

$$\Gamma_1(r) = \sup_{\rho\in[c,r]} \left(\int_{\rho-R}^{\rho+c} (1+\tau)^{-\frac{\gamma}{p-1}} d\tau \right)^{p-1},$$

Having in mind (21), we can conclude that

$$a^2\beta_1(r) \le 2r\Gamma_1(r)\alpha(r).$$

Similarly, we can estimate

$$\begin{aligned} b^2\beta_2(r) &\le \int_c^r \rho \left| \int_{\rho-R}^{\rho+c} (\rho+c-\tau) b^{\frac{2}{q}} \bar{v}_{\tau\tau}(\rho,\tau)(1+\tau)^{\gamma/q}(1+\tau)^{-\gamma/q} d\tau \right|^q d\rho \\ &\le \int_c^r \rho \left(\int_{\rho-R}^{\rho+c} (\rho+c-\tau)^{q'} (1+\tau)^{-\gamma q'/p} d\tau \right)^{q'/q} \\ &\qquad \left(\int_{\rho-R}^{\rho+c} b^2 |\bar{v}_{\tau\tau}|^q (1+\tau)^{\gamma} d\tau \right) d\rho. \end{aligned}$$

We can conclude that

$$b^2\beta_2(r) \le 2r\Gamma_2(r)\alpha(r)$$

with

$$\Gamma_2(r) = \sup_{\rho\in[c,r]} \left(\int_{\rho-R}^{\rho+c} (\rho+c-\tau)^{q'} (1+\tau)^{-\frac{\gamma}{q-1}} d\tau \right)^{q-1}.$$

On the other hand

$$\beta'(r) = r\alpha^p(r) + r\alpha^q(r) \ge \frac{a^{2p}\beta^p(r)}{2^p r^{p-1}\Gamma_1^p(r)} + \frac{b^{2q}\beta^q(r)}{2^q r^{q-1}\Gamma_1^q(r)}.$$

We can deduce that β is increasing and for $r > r_0$ we get

$$(\beta(r_0))^{1-p} \ge \frac{(p-1)a^{2p}}{2^p} \int_{r_0}^r \frac{1}{\xi^{p-1}\Gamma_1^p(\xi)} d\xi$$

and

$$(\beta(r_0))^{1-q} \ge \frac{(q-1)b^{2q}}{2^q} \int_{r_0}^r \frac{1}{\xi^{q-1}\Gamma_2^q(\xi)} d\xi.$$

In order to have contradiction, it remains to find (p, q, γ) such that

$$\int_{r_0}^{+\infty} \frac{1}{\xi^{p-1}\Gamma_1^p(\xi)}\,d\xi = +\infty \quad \text{or} \quad \int_{r_0}^{+\infty} \frac{1}{\xi^{q-1}\Gamma_2^q(\xi)}\,d\xi = +\infty\,. \tag{26}$$

In the case $b = 0$ and $a \neq 0$ we only require that the first integral is divergent. In the case $a = 0$ and $b \neq 0$ we only require that the second integral is divergent.

We observe that there exist $s_1 \in [-R, c]$ and $s_2 \in [-R, c]$ such that

$$\left(\int_{\rho-R}^{\rho+c} (1+\tau)^{-\frac{\gamma}{p-1}}\,d\tau\right)^{p-1} = (1+s_1+\rho)^{-\gamma}$$

and

$$\left(\int_{\rho-R}^{\rho+c} (\rho+c-\tau)^{q'}(1+\tau)^{-\frac{\gamma}{q-1}}\,d\tau\right)^{q-1} = (c-s_2)^{q'}(1+s_2+\rho)^{-\gamma}\,.$$

For $\gamma > 0$ we find $\Gamma_1(r) \leq 1$. It follows that (26) is satisfied for any $p \leq 2$. Similarly, for $\gamma > 0$ we find $\Gamma_2(r) \leq 1$ and (26) is satisfied for any $q \leq 2$. If $\gamma < 0$, then (26) is equivalent to

$$\int_{r_0}^{+\infty} \frac{1}{\xi^{p-1-\gamma p}}\,d\xi = +\infty \text{ or } \int_{r_0}^{+\infty} \frac{1}{\xi^{q-1-\gamma q}}\,d\xi = +\infty\,.$$

Again for $b = 0$ and $a \neq 0$ we only require that the first integral is divergent. In the case $a = 0$ and $b \neq 0$ we only require that the second integral is divergent. We can conclude that (26) is satisfied in one of the following cases

1. $\gamma > 0$ and $p \leq 2$ or $q \leq 2$;
2. $1 - \frac{2}{p} < \gamma < 0$ (except the case $a = 0$);
3. $1 - \frac{2}{q} < \gamma < 0$ (except the case $b = 0$).

Coming back to the proof of the blow up of the solution of (5), through the solution of (16) and (6), from $\bar{v} = 0$ on Σ we can deduce that

$$y(x,t) = 0 \qquad \text{for } x \in \mathbb{R}^3, t > R\,.$$

First we notice that combining (25) with (20) we have $\bar{B} = 0$ on $T^*_{r,t}$ with $(r, t) \in \Sigma$. But this trapezoids cover the region $\{(\rho, t) \mid \rho + t > R\}$, hence, being $B \geq 0$ we have $B = 0$ in the region $|x| + t > 0$. This implies

$$a^2|v_t|^p + b^2|v_{tt}|^q = 0 \text{ for } |x| + t > 0\,, t > 0\,.$$

We can deduce

$$v_t(x,t) = 0\,,\, v_{tt}(x,t) = 0\,,\ \text{for } |x| + t > 0\,,\, t > 0\,.$$

Recalling that $y(x,t) = v_t(x,t)$ we get

$$y_t(x,t) = 0\,,\ \text{for } |x| + t > 0\,,\, t > 0\,. \tag{27}$$

In turn this implies that

$$y(x,t) = y(x, t+R) = 0\,,\ \text{for } x \in \mathbb{R}^3\,,\, t > 0\,.$$

This gives the conclusion. Indeed, this and (27) are impossible for $(g,h) \neq (0,0)$.

2.3 Proof of Theorem 2

We assume that $z(t,x) : [0,T) \to \mathbb{R} \times \mathbb{R}^3$ is a non-trivial C^2 solution of (5) with $f, g \in C^2(\mathbb{R}^3)$ compactly supported with $g \neq 0$. Make the substitution $v(x,t) = z(t,x) - f(x)$. The function h is determined as the unique solution of (9). Then we can write $h(x) = v_{tt}(0,x)$ and moreover we have

$$\begin{cases} v_{tt} - \Delta v = (1+t)^\gamma B(t,x,v_t,v_{tt}) + \Delta f\,, \\ v(0,x) = 0\,, \\ v_t(0,x) = g(x)\,, \end{cases} \tag{28}$$

with $B(t,x,v_t,v_{tt}) = A(t,x,v+f,v_t,v_{tt})$. It is not necessary to assume (13) since Δf has vanishing mean. Assumption (14) guarantees that v satisfies (21) and we arrive at the same absurd as before. We can conclude that blow up holds.

3 Applications

A trivial application of Theorem 1 is the blow up of compactly supported classical solution of

$$y_{tt} - \Delta y = \partial_t \left((1+t)^\gamma |y|^p \right)\,,$$

provided initial data $(g,h) \in C^3 \times C^2$ satisfies

$$\int h(x)\,dx \geq \int |g(x)|^p\,dx$$

and $p \leq 2$ with $\gamma > 1 - 2/p$ or $p = 2$ and $\gamma \geq 0$. This example is deeply different from the results in [5]. Indeed the right side can be written as

$$\partial_t \left((1+t)^\gamma |y|^p\right) = \gamma(1+t)^{\gamma-1}|y|^p + p(1+t)^\gamma |y|^{p-2} y y_t$$

and we do not know any sign assumption on y and y_t, so that John's result is not directly available.

Now we turn to other applications of Theorem 2. Our starting point is a scale invariant damping wave equations that can be reduced to (5) by means of a suitable transformation. Let us consider the covariant time derivative

$$\partial_{(\mu),t} = \partial_t + \frac{\mu}{2(1+t)} \quad \mu \geq 0.$$

We can write

$$u_{tt} + \frac{\mu}{1+t}u_t + \frac{\mu(\mu-2)}{4(1+t)^2}u = \partial_{(\mu),t}\partial_{(\mu),t}u - \Delta u \tag{29}$$

and hence a meaningful nonlinear term for this equation is $\left|\partial_{(\mu),t}u\right|^p$. On the other hand, the relation between this covariant derivative and the standard derivative is given by the transformation $u = (1+t)^{-\frac{\mu}{2}}z$, indeed $\partial_{(\mu),t}u = (1+t)^{-\frac{\mu}{2}}\partial_t z$. For this reason the equation

$$u_{tt} - \Delta u + \frac{\mu}{1+t}u_t + \frac{\mu(\mu-2)}{4(1+t)^2}u = (1+t)^\alpha \left(a^2\left|\partial_{(\mu),t}u\right|^p + b^2\left|\partial_{(\mu),t}\partial_{(\mu),t}u\right|^q\right) \tag{30}$$

becomes

$$z_{tt} - \Delta z = a^2(1+t)^{\alpha-\frac{\mu}{2}(p-1)}|z_t|^p + b^2(1+t)^{\alpha-\frac{\mu}{2}(q-1)}|z_{tt}|^q \tag{31}$$

and this is a special case of (5) with

$$\gamma = \alpha - \frac{\mu}{2}(\max\{p,q\} - 1).$$

In (30) the linear zero-order term can be seen as a positive time-dependent mass only for $\mu \geq 2$. As seen in the Introduction, many papers deal with the scale invariant damping wave equation

$$u_{tt} + \frac{\mu}{1+t}u_t + \frac{\mu(\mu-2)}{4(1+t)^2}u = F(t,u,u_t,u_{tt}) \tag{32}$$

for $F = |u|^p$. The case $F(u_t) = |u_t|$ has been analyzed in [8]. With the choice of a different nonlinear term in (30), we add another step to understand the interplay between the lower order terms of the wave equation and some *admissible* nonlinear terms.

Let us start considering (30) with $b = 0$.

Corollary 1 *Let us consider the Cauchy problem*

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t} u_t + \frac{\mu(\mu-2)}{4(1+t)^2} u = (1+t)^\alpha \left| \left(\partial_t + \frac{\mu}{2(1+t)} \right) u \right|^p, & x \in \mathbb{R}^3, t \geq 0, \\ u(0,x) = f(x), \\ u_t(0,x) = g(x). \end{cases}$$

Let $u(t,x) : [0,T) \to \mathbb{R}$ *be the corresponding maximal solution with* $f, g \in C^2(\mathbb{R}^3)$ *compactly supported. Let* $1 < p < 2$ *and*

$$\alpha > 1 - \frac{2}{p} + \frac{\mu}{2}(p-1), \tag{33}$$

or $p = 2$ *and* $\alpha \geq \frac{\mu}{2}$. *Then* $T < +\infty$.

Proof After the transformation $z = (1+t)^{-\frac{\mu}{2}} u$ the previous Cauchy problem becomes

$$\begin{cases} z_{tt} - \Delta z = (1+t)^{\alpha - \frac{\mu}{2}(p-1)} |z_t|^p, & x \in \mathbb{R}^3, t \geq 0, \\ z(0,x) = f(x), \\ z_t(0,x) = -\frac{\mu}{2} f(x) + g(x). \end{cases}$$

Since $A(t, x, z, z_t, z_{tt}) = |z_t|^p$ satisfies (10) and (11), setting $h(x) = \Delta f + |g(x)|^p$, the result is a direct application of Theorem 2. □

Remark 2 In [4] the case $\alpha = 0$, $\mu > 0$ $f = 0$ is considered. A blow up result for radial solution is established, provided

$$p < \min\left\{ 1 + \frac{2}{\mu}, 1 + \frac{2}{2k + \mu} \right\}, \tag{34}$$

where $k > 0$ is such that $g(|x|) \gtrsim (1 + |x|)^{-k}$. A similar result for the semilinear case is contained in [2]. Let us compare our result with the one in [4]. Though we are considering smooth solution with compact support, our result improves [4], since we do not assume radial solution and we can also treat some $\mu < 0$, for example, for $p = 2$. Moreover our admissible exponents satisfy

$$\frac{\mu}{2} p^2 + \left(1 - \frac{\mu}{2} \right) p - 2 < 0$$

and $1 < p < 2$. At least for $0 < \mu < 3/2$ this range is larger than (34).

Remark 3 The expression (33) shows also the interaction between the potential, the linear operator (29) and nonlinear term. More precisely, following [1], if we describe as Strauss-type exponent a positive solution of an equation like

$$\beta p^2 + (\delta - \beta)p - 2 = 0\,, \quad \beta > 0, \delta > \beta$$

then for $\beta = \frac{\mu}{2}$ and $\delta = 1 - \alpha$ our result provides a subcritical blow up behavior. The word *subcritical* refers to a critical Strauss-type exponent.

The analogous result for (30) with $a = 0$ is the following

Corollary 2 *Let us consider the Cauchy problem*

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t}u_t + \frac{\mu(\mu-2)}{4(1+t)^2}u = (1+t)^\alpha \left| \left(\partial_t + \frac{\mu}{2(1+t)}\right)^2 u \right|^q\,, \quad x \in \mathbb{R}^3, t \geq 0, \\ u(0,x) = f(x)\,, \\ u_t(0,x) = g(x)\,. \end{cases}$$

Let $u(t,x) : [0,T) \to \mathbb{R}$ *be the corresponding maximal solution with* $f, g \in C^2(\mathbb{R}^3)$ *compactly supported. Let* $1 < q < 2$ *and*

$$\alpha > 1 - \frac{2}{q} + \frac{\mu}{2}(q-1)\,, \tag{35}$$

or $q = 2$ *and* $\alpha \geq \frac{\mu}{2}$*. Then* $T < +\infty$*.*

In a similar way we can treat combined nonlinearity involving time-covariant derivatives. We get a blow up result for classical solution of (30) with smooth and compactly supported initial data for $\mu \leq 0$, $p \leq 2$ or $q \leq 2$ or $\mu > 0$ and

$$\alpha > \frac{\mu}{2}(\max\{p,q\} - 1) + 1 - \frac{2}{\max\{p,q\}}\,.$$

Since (30) is equivalent to (31) we expected such interaction between p and q.

Theorem 2 is very general. Firstly, it gives the possibility to consider second order time derivatives in the nonlinear term. Moreover we are requiring the positivity of the entire nonlinear term, not of any terms which appears in A. For example, take

$$\Box z = N(z)$$

with $N = A_1$ written as

$$A_1 = \alpha_1(1+t)^{\gamma_1}|z_t|^{p_1} + \alpha_2(1+t)^{\gamma_2}|z_t|^{p_2}\,,$$

with $\alpha_i > 0$, $p_i \in (1, 2)$ and $1 - \frac{2}{p_i} < \gamma_i$. Then any classical solution z blows up. But also we have blow up if $N = A_1 + A_0$ or $N = A_1 + A_2$ or $N = A_1 + A_0 + A_2$ being

$$A_0 = \alpha_0 (1+t)^{\gamma_0} |z|^{p_0} \quad \alpha_0 \geq 0,\, p_0 > 1,\, \gamma_0 \in \mathbb{R}$$

and

$$A_2 = |z_t|^{\ell} + |z_{tt}|^{m} - z_t z_{tt} \quad \frac{1}{\ell} + \frac{1}{m} = 1$$

which is positive due to Young inequality.

Finally this idea can be applied for other scale invariant operators. For example, we can consider

$$u_{tt} - \Delta u + 2b(t)u_t + (b' + b^2)u = (\partial_t + b(t))(\partial_t + b(t))u$$

hence one can put $\partial_{(b),t} = (\partial_t + b(t))$ and study

$$\partial_{(b),t}\partial_{(b),t}u - \Delta u = |\partial_{(b),t}u|^p + |\partial_{(b),t}\partial_{(b),t}u|^q\,.$$

Let

$$B(t) = \int_0^t b(s)ds\,,$$

since $\partial_{(b),t)}(\exp(-B(t))u) = \exp(-B(t))\partial_t u$, setting $u = \exp(-B(t))z$ previous equation becomes

$$z_{tt} - \Delta z = \exp((1-p)B(t))|z_t|^p + \exp((1-q)B(t))|z_{tt}|^q\,.$$

Suitable assumptions on b gives the possibility to apply Theorem 2. For example, negative $b(t)$ leads to the case without potential, while

$$b(t) \leq \frac{C}{1+t}$$

leads to Corollary 1.

4 An Existence Result

First of all, we assert that one can generalize our result, when the nonlinear term in (1) depends also on space-derivatives of the solutions. We leave detailed discussion for a future work, but we shall give a simple example of a suitable variant of (32) so that small data global existence result holds. The example can be considered as a complementary case to our blow up results in Corollary 1 with $p = 2$ and $\alpha = \frac{\mu}{2}$. More precisely, we consider the Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u + \frac{\mu}{1+t} u_t + \frac{\mu(\mu-2)}{4(1+t)^2} u = (1+t)^{\frac{\mu}{2}} \left(\left| \left(\partial_t + \frac{\mu}{2(1+t)} \right) u \right|^2 - |\nabla u|^2 \right) , \\ u(0,x) = f(x) , \\ u_t(0,x) = g(x) , \end{cases}$$

being $x \in \mathbb{R}^3$, $t \geq 0$. As usual it becomes

$$\begin{cases} z_{tt} - \Delta z = \left(|z_t|^2 - |\nabla z|^2 \right) , \quad x \in \mathbb{R}^3, t \geq 0 \\ z(0,x) = f(x) , \\ z_t(0,x) = -\frac{\mu}{2} f(x) + g(x) . \end{cases} .$$

Then we can use the Nirenberg transform, see Klainerman in [6]:

$$w = 1 - \mathrm{e}^{-z} .$$

We get $w_{tt} - \Delta w = 0$ that gives global existence. We underline that also in this case $\mu < 0$ is admissible.

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Hamiltonian Field Theory Close to the Wave Equation: From Fermi-Pasta-Ulam to Water Waves

Matteo Gallone and Antonio Ponno

Abstract In the present work we analyse the structure of the Hamiltonian field theory in the neighbourhood of the wave equation $q_{tt} = q_{xx}$. We show that, restricting to "graded" polynomial perturbations in q_x, p and their space derivatives of higher order, the local field theory is equivalent, in the sense of the Hamiltonian normal form, to that of the Korteweg-de Vries hierarchy of second order. Within this framework, we explain the connection between the theory of water waves and the Fermi-Pasta-Ulam system.

1 Introduction

The present work aims to treat the perturbations of a linear string in the framework of classical Hamiltonian field theory. The unperturbed base model we have in mind, the linear string, is described by the one-dimensional wave equation

$$q_{tt} = c^2 q_{xx} , \tag{1}$$

where $q : \mathbb{R} \times D \to \mathbb{R} : (t, x) \to q(t, x)$ is the unknown, real-valued field, and c is a real, positive parameter, the *speed of the wave*. As usual, partial derivatives are denoted by subscripts, i.e. $q_t = \partial_t q$, $q_x = \partial_x q$ and so on. Concerning the space domain D and the boundary conditions of the field q, we here focus on the 1-periodic case, namely $D = \mathbb{T} := \mathbb{R}/\mathbb{Z}$ (the L-periodic case, with $D = \mathbb{R}/(L\mathbb{Z})$, can always be reduced to the case $L = 1$ by rescaling both the independent variables to $x' = x/L$, $t' = t/L$).

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Solving Eq. (1), for any initial condition $q(0,x)$, $q_t(0,x)$ defined on $\mathbb{T}$ and regular enough, is a standard exercise in Fourier analysis. Indeed, substituting $q(t,x) = \sum_{k\in\mathbb{Z}} \hat{q}_k(t) e^{\imath 2\pi k x}$ ($\imath$ is the imaginary unit) into (1), one gets

$$\frac{d^2\hat{q}_k}{dt^2} = -4\pi^2 c^2 k^2\, \hat{q}_k \ ,$$

which implies $\hat{q}_k(t) = a_k e^{\imath\omega_k t} + \bar{a}_{-k} e^{-\imath\omega_k t}$, where the a_k are complex constants (the bar denoting complex conjugation), and

$$\omega_k := 2\pi c|k| \ ; \quad k \in \mathbb{Z} \ . \tag{2}$$

Observe that $\omega_{-k} = \omega_k$, which implies $\overline{\hat{q}}_k = \hat{q}_{-k}$, i.e. q is real. Relation (2) defines the dispersion relation of the wave equation. A given space periodic system, characterised by a certain dispersion relation $k \to \omega_k$, is said to be *non dispersive* if $\omega_{k+1} - \omega_k$ is piecewise constant, i.e. if ω_k is piecewise linear in k and this is clearly the case for the wave equation. One can check that the solution $q(t,x)$ of the problem is time periodic for all initial conditions, the period being $2\pi/\omega_1 = 1/c$.

It is almost impossible to give a complete account of physical phenomena that, to the first linear approximation, are described by the wave equation. Let us just mention, to have in mind concrete examples that we are going to analyse later, wave propagation in fluids and long-wavelength vibrations of interacting particle chains. In all these problems, the need to go beyond the first approximation arises, in order to take into account the effects of both nonlinearity and dispersion, typically determining whether some interesting form of energy localisation may take place, as opposed to a fast energy spreading among the degrees of freedom of the system. One is thus led to look for a general treatment of the possible perturbations of Eq. (1) regardless of the specific physical problem giving rise to it. This in turn calls for the restriction to a mathematical context where the possible perturbations constitute a well-defined ordered class of objects. We do this within the framework of Hamiltonian field theory, at the price to exclude, among others, all the dissipative effects from the theory (no claim is made here about their irrelevance: the other way around. See, for example, the enlightening discussion made by Nekhoroshev in [33]). Moreover, we consider nonlinear and dispersive perturbations depending on q_x, p and their higher order derivatives, but not on q. Indeed, all systems made of interacting particles, such as solids, fluids and gasses, in absence of external forces, and on a sufficiently large space scale, are described by a certain wave equation at the linear level, with perturbations depending, in principle, only by the space derivatives of the field (and its momentum, possibly). This is due to the fact that interactions in matter depend on *differences* of coordinates, which in the continuum approximation corresponds to *derivatives*.

On the other hand, considering smooth perturbations of the wave equation depending on q (not only through derivatives) would be interesting as well. For example, as shown by Bambusi and Nekhoroshev and by Nekhoroshev [6, 7, 33], the smooth perturbations of the wave equation depending on q only (no derivatives) give rise to very nice, long-lasting localisation phenomena. Whether be possible to include such a class of problems in our treatment, drawing meaningful conclusions, looks unclear, at present.

Although we decided to focus on one-dimensional systems, it is worth mentioning that the techniques presented here can be generalised to study problems in higher space dimension. In this case one can predict, for example, energy localisation for a certain class of anisotropic rectangular lattices [22].

The paper is organised as follows. In Sect. 2 we introduce the Hamiltonian formalism of classical field theory, at the end of which we provide an informal presentation of the main results. Section 3 contains the elements of perturbation theory framed in the more general context of Poisson systems, which is the one appropriate to our purposes. Section 4 contains the formal statements and proofs of the results. The application of such results to the FPU problem and to the water wave problem is treated in Sect. 5. Finally, a short list of open problems is provided in Sect. 6.

2 Outline of the Method and Results

2.1 Hamiltonian Field Theory

For the sake of completeness, we report here a short review on what is meant by *Hamiltonian field theory*. The reader is referred to the monographs [16, 25], and [32], for details and/or a more extensive treatment of the subject.

In Hamiltonian field theory the dynamical variables (e.g. coordinates and conjugate momenta) are points in a certain function space, the phase space of the system, and the observables, including the Hamiltonian, are functionals, admitting a density, defined on the phase space.

In order to specify the notations used below, let us first consider the space of smooth functions, or fields $u : \mathbb{T} \to \mathbb{R}$. A functional $F[u]$, with density $\mathscr{F}$ depending on x and on $u(x)$ and its derivatives up to a given order, is defined as

$$F[u] = \oint \mathscr{F}(x, u, u_x, u_{xx}, \dots)\, dx \ , \tag{3}$$

where here and in the sequel we make use of the short hand notation $\oint := \int_{\mathbb{T}}$. The functional derivative (or variational derivative) of F with respect to u, denoted by $\delta F/\delta u$, is defined by the relation

$$\delta F[u, \delta u] := \frac{d}{d\epsilon} F[u + \epsilon\delta u]\big|_{\epsilon=0} = \oint \frac{\delta F}{\delta u} \delta u\, dx \ , \tag{4}$$

for any smooth finite increment δu defined on $\mathbb{T}$. Through repeated integrations by parts and erasing the boundary terms one finds

$$\frac{\delta F}{\delta u} = \sum_{j\geq 0} (-1)^j \frac{d^j}{dx^j} \frac{\partial \mathscr{F}}{\partial(\partial_x^j u)} = \frac{\partial \mathscr{F}}{\partial u} - \frac{d}{dx}\frac{\partial \mathscr{F}}{\partial u_x} + \frac{d^2}{dx^2}\frac{\partial \mathscr{F}}{\partial u_{xx}} + \cdots \, , \tag{5}$$

the sum above being finite if $\mathscr{F}$ is a polynomial in u and its derivatives up to a given finite order (as will be in our case). Relation (4) defines the Gateaux, or weak differential of the functional F at u with increment δu, which under further requirements coincide with the Fréchet, or strong differential of F; see e.g. [39]. The functional derivative is also referred to, in the mathematical literature, as the L_2-gradient of F with respect to u. Indeed, in the Hilbert space $L_2(\mathbb{T})$ of square integrable functions on $\mathbb{T}$, endowed with the usual scalar product $\langle f, g\rangle := \oint fg\, dx$, one can rewrite (4) as $\delta F = \langle \delta F/\delta u, \delta u\rangle := \langle \nabla F, \delta u\rangle$, identical in form to its finite-dimensional counterpart.

In the *Hamiltonian field theory* considered in the present paper, the phase space Γ of the system is the space of two components, smooth, real-valued fields $(q(x), p(x))$ defined on $\mathbb{T}$. The observables of the theory are the functionals $F : \Gamma \to \mathbb{R}$ admitting a density $\mathscr{F}$ which is a polynomial in $q(x)$, $p(x)$ and their space derivatives up to a finite order, with coefficients possibly depending on x. One then selects, among the observables, the Hamiltonian defining the given system, namely

$$H[q, p] := \oint \mathscr{H}(x, q, p, q_x, p_x, \dots)\, dx \, . \tag{6}$$

The motion of the system, a certain curve $\gamma : [t_1, t_2] \ni t \mapsto (q, p)(t) \in \Gamma$, is then specified by a stationary action principle, as in the finite-dimensional case. Indeed, defining the *action functional* $S[q, p]$ as

$$S[q, p] := \int_{t_1}^{t_2} \big[\langle p, q_t\rangle - H\big]\, dt = \int_{t_1}^{t_2} \oint [p q_t - \mathscr{H}]\, dt\, dx \, , \tag{7}$$

one defines the actual motion of the system as the critical point of S in the space of smooth curves $(q(t, x), p(t, x))$ in Γ with fixed ends on the first component: $q(t_1, x) := q_1(x)$, $q(t_2, x) := q_2(x)$, q_1 and q_2 being two assigned fields on $\mathbb{T}$. The smooth increment curves $(\delta q, \delta p)(t)$ must then satisfy the condition $\delta q(t_1, x) = \delta q(t_2, x) = 0$. With the notation just introduced, and performing simple integrations by parts, one gets the differential δS of the action S, namely

$$\delta S = \int_{t_1}^{t_2} \oint \left[\left(q_t - \frac{\delta H}{\delta p}\right)\delta p - \left(p_t + \frac{\delta H}{\delta q}\right)\delta q\right] dt\, dx \, . \tag{8}$$

This is zero for any increment $(\delta q, \delta p)(t)$ if and only if the following Hamilton equations hold:

$$q_t = \frac{\delta H}{\delta p} \ ; \ p_t = -\frac{\delta H}{\delta q} \ . \tag{9}$$

This is the Hamilton principle of stationary action in classical field theory.

In this work, we restrict our attention to scalar fields q and p defined on the (flat) unit circle $\mathbb{T}$. However, all the above construction and most of the results presented below can be extended to vector fields defined on any multi-dimensional space domain (not necessarily a torus).

Consider now a functional $F[q, p] := \oint \mathscr{F}(x, q, p, q_x, p_x, \dots)dx$. Its time derivative along the solutions of the Hamilton equations (9) associated with H is computed by means of repeated integrations by parts with respect to x. The result can be written as $dF/dt = \{F, H\}_{q,p}$, where

$$\{F, H\}_{q,p} := \oint \left(\frac{\delta F}{\delta q}\frac{\delta H}{\delta p} - \frac{\delta F}{\delta p}\frac{\delta H}{\delta q} \right) dx := \langle \nabla F, \mathsf{J}_2 \nabla H \rangle \tag{10}$$

is the *Poisson bracket* of the functionals F and H. In the second definition above, $\mathsf{J}_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the standard 2×2 symplectic matrix, $\nabla F = \begin{pmatrix} \delta F/\delta q \\ \delta F/\delta p \end{pmatrix}$ and the same for H. The product $\xi^T \mathsf{J}_2 \eta = \xi_1 \eta_2 - \xi_2 \eta_1$, for any pair of vectors $\xi, \eta \in \mathbb{R}^2$, defines the symplectic 2-form. The Poisson bracket (10) defines a bilinear, skew-symmetric product on the algebra of functionals defined on Γ, and it satisfies the Jacobi identity $\{\{F, G\}_{q,p}, H\}_{q,p} + \{\{G, H\}_{q,p}, F\}_{q,p} + \{\{H, F\}_{q,p}, G\}_{q,p} \equiv 0$ and the Leibniz rule $\{FG, H\}_{q,p} = F\{G, H\}_{q,p} + \{F, H\}_{q,p}G$ for any triple of functionals F, G, H. The algebra of functionals on Γ endowed with the Poisson bracket becomes a Poisson algebra and is typically referred to as the algebra of observables.

Remark 1 Given any skew-symmetric bilinear product on an algebra, the Jacobi identity characterises it as a Lie bracket. The latter, by further assuming the Leibniz rule, becomes a Poisson bracket (by definition). Thus, a Poisson algebra is a Lie algebra of Leibniz type.

The fundamental Poisson brackets of the Hamiltonian field theory on $\mathbb{T}$ are

$$\{q(x), p(y)\}_{q,p} = \delta(x - y) \ ; \ \{q(x), q(y)\}_{q,p} = \{p(x), p(y)\}_{q,p} = 0 \ , \tag{11}$$

where $\delta(x)$ is the Dirac delta distribution on $\mathbb{T}$. This is proved by considering the identity $\oint \delta(x - y) f(y) dy = f(x)$, valid for any continuous function on $\mathbb{T}$, from which $\delta f(x)/\delta f(y) = \delta(x - y)$ follows. As a consequence, the Hamilton equations (9) can be written in the form

$$q_t = \{q, H\}_{q,p} \ ; \ p_t = \{p, H\}_{q,p} \ . \tag{12}$$

2.2 Results: Informal Presentation

Within the Hamiltonian formalism just introduced, we study a well-defined class of problems, defined as follows. We introduce a “bookkeeping parameter” λ and give a weight λ^2 to both q_x and p, weighting any successive derivative ∂_x of them by λ. Defining $r := q_x$, this amounts to assume a “grading” (perturbative ordering of the dynamical variables and their derivatives) $r \sim p \ll r_x \sim p_x \ll r_{xx} \sim p_{xx} \dots$, and $(r_x)^2 \sim r^3$, where, in a loose notation, $\sim$ and $\ll$ mean “of the same order of” and “of an order smaller than”, respectively. For the sake of simplicity, we assume the smooth density $\mathscr{H}$ of H to be a function of q_x, p and their space derivatives up to order four. Such a limitation is due to the fact that, in the present paper, we do not consider λ-expansions of the Hamiltonian H to degree higher than four, and with the chosen grading, derivatives of q_x and p of order higher than four enter the perturbative problem from degree five on (in λ). The parameter λ is formal: it is necessary to define the grading and to trace the perturbative ordering, and it can be set to one at the end of the computations.

Definition 1 The class of problems considered in the present work is defined by the family of Hamiltonians of the form

$$H_\lambda := \frac{1}{\lambda^4} \oint \mathscr{H}(\lambda^2 q_x, \lambda^2 p, \lambda^3 q_{xx}, \lambda^3 p_x, \dots, \lambda^6 q_{xxxxx}, \lambda^6 p_{xxxx})\, dx \ , \tag{13}$$

with the condition

$$\left(\frac{\partial^2 \mathscr{H}}{\partial q_x^2}\Big|_{\lambda=0} \right) \left(\frac{\partial^2 \mathscr{H}}{\partial p^2}\Big|_{\lambda=0} \right) > 0 \, . \tag{14}$$

By Taylor expanding $\mathscr{H}$ in powers of λ, close to $\lambda = 0$, and assuming without loss of generality that $\mathscr{H}|_{(q,p)=0} = 0$, one gets a perturbative ordering of the Hamiltonian of the form

$$H_\lambda = H_0 + \lambda H_1 + \lambda^2 H_2 + \lambda^3 H_3 + \lambda^4 H_4 + \cdots \, . \tag{15}$$

We here observe that the absence of a term proportional to $1/\lambda^2$ in the latter expansion is due to the conservation of the total momentum $\oint p \; dx$, which can be always set to zero.

The main results are now presented in an informal way, their precise statements and proofs being provided below. The condition (14), which characterises the elliptic nature of the fixed point $q = p = 0$, implies that there exists a canonical transformation bringing the unperturbed Hamiltonian H_0 into the standard wave form

$$K_0 := \oint \frac{p^2 + (q_x)^2}{2}\, dx \ , \tag{16}$$

and leaving the perturbative expansion (15) unaltered. The equations of motion associated with the latter Hamiltonian are $q_t = p$, $p_t = q_{xx}$, i.e. in second-order form, the wave equation $q_{tt} = q_{xx}$.

Now, in terms of the variables $r := q_x$ and p, the expanded Hamiltonian (15) reads $K_0 + \lambda H_1 + \lambda^2 H_2 + \cdots$, where $K_0 = \frac{1}{2}\oint(p^2 + r^2)dx$, and the H_j are functionals whose density is a homogeneous polynomial of "grade" j in r, p and their derivatives. One then conveniently performs the change of field variables $(r, p) \mapsto (u, v)$ defined by $u = (r + p)/\sqrt{2}$, $v = (r - p)/\sqrt{2}$, in terms of which $K_0 = \frac{1}{2}\oint(u^2 + v^2)dx$, and its flow separates the left from right wave: $u_t = u_x$, $v_t = -v_x$, so that u and v are simply the left and right translation of the corresponding initial datum, respectively.

The key idea is now to decouple the left from the right dynamics to higher orders. To such an end, we build up an explicit transformation of the field variables

$$\mathscr{T}_\lambda : (u, v) \mapsto (\tilde{u}, \tilde{v}) ,$$

λ-close to the identity, which sets the Hamiltonian $H = K_0 + \lambda H_1 + \lambda^2 H_2 + \cdots$ (expressed in the (u, v) variables) into normal form to order $1 \le s \le 4$ with respect to K_0. This means, by definition, that $H \circ \mathscr{T}_\lambda^{-1} = K_0 + \lambda Z_1 + \lambda^2 Z_2 + \cdots$ is such that the Z_j are first integrals of K_0, for $1 \le j \le s$.

The results proved below are the following. In the general case, i.e. no further hypotheses being added to the Definition 1, we show that the normal form Hamiltonian to order $s = 2$ has the form $K_0 + \lambda^2 Z_2 + \cdots$, and the corresponding dynamics of the variables $\tilde{u}$, $\tilde{v}$ reads

$$\begin{cases} \tilde{u}_t = c_l \tilde{u}_x + a_l \kappa_3(\tilde{u}) + \cdots \\ \tilde{v}_t = -c_r \tilde{v}_x - a_r \kappa_3(\tilde{v}) + \cdots \end{cases} . \tag{17}$$

On the other hand, in certain relevant cases, such as the "mechanical" one, where $\mathscr{H} = p^2/2 + \mathscr{U}(q_x, q_{xx}, \ldots, q_{xxxxx})$, or that of the water waves, one has $H_1 = H_3 \equiv 0$. In such situations the normal form Hamiltonian to order $s = 4$ has the form $K_0 + \lambda^2 Z_2 + \lambda^4 Z_4 + \cdots$, whose associated dynamics reads

$$\begin{cases} \tilde{u}_t = c_l \tilde{u}_x + a_l \kappa_3(\tilde{u}) + b_l \kappa_5(\tilde{u}) + \cdots \\ \tilde{v}_t = -c_r \tilde{v}_x - a_r \kappa_3(\tilde{v}) - b_r \kappa_5(\tilde{v}) + \cdots \end{cases} . \tag{18}$$

In systems (17) and (18) $a_{l/r}$, $b_{l/r}$ and $c_{l/r}$ are certain constants (depending on the model, on the parameter λ and on the initial condition), whereas κ_3 and κ_5 are the vector fields of the first and second integral in the KdV hierarchy [1], namely

$$\kappa_3(w) = \gamma w w_x + w_{xxx} = \partial_x \frac{\delta I_3}{\delta w} , \tag{19}$$

$$\kappa_5(w) = \frac{5}{6}\gamma^2 w^2 w_x + \frac{10}{3}\gamma w_x w_{xx} + \frac{5}{3}\gamma w w_{xxx} + u_{xxxxx} = \partial_x \frac{\delta I_5}{\delta w} . \tag{20}$$

Here $\gamma \in \mathbb{R}$ is a parameter, whose value is explicitly determined by the first order normal form transformation, whereas the first two integrals I_3 and I_5 of the KdV hierarchy are given by

$$I_3 = \oint \left(\frac{\gamma}{6} w^3 - \frac{1}{2}(w_x)^2 \right) dx \,, \tag{21}$$

$$I_5 = \oint \left(\frac{5\gamma^2}{72} w^4 + \frac{5\gamma}{12} w^2 w_{xx} + \frac{1}{2}(w_{xx})^2 \right) dx \,. \tag{22}$$

The conclusion is that *both in the general and in the special case, the dynamics of the perturbed wave equation is integrable in the KdV hierarchy sense to the second perturbative order included.*

Remark 2 The standard Hamiltonian normal form construction to leading order always leads to (17). On the other hand, in order to get (18), the second step of Hamiltonian normalisation is not enough, in general. With the aid of Hamiltonian transformations, we generally succeed in decoupling equations of motion for the two independent variables to higher orders but, in general, this is not enough to conjugate the equations of motion to those of the KdV integrable hierarchy. It is remarkable that, at this point, each of the two decoupled equations of motion falls in a class that was analysed by Kodama [26, 29–31] (and whose results have been extended to equations on the torus in [23]). Without entering the details, which could deserve an entire work, the idea is the following. One starts from a PDE of the form

$$u_t \;=\; F(u) \;:=\; F_0(u) + \lambda F_1(u) + \lambda^2 F_2(u) + O(\lambda^3) \tag{23}$$

and one considers the effect of a change of variables $u \mapsto u + \lambda G(u)$. Denoting with $[\cdot, \cdot]$ the commutator of two vector fields, the effect of the transformation on the RHS of the PDE (23) is

$$\begin{aligned} F(u) \mapsto e^{\lambda[G,\cdot]} F(u) =& F_0(u) + \lambda\big(F_1(u) + [G, F_0](u)\big) \\ &+ \lambda^2 \left(F_2(u) + [G, F_1](u) + \frac{1}{2}[G, [G, F_0]](u) \right) + O(\lambda^3) \,. \end{aligned} \tag{24}$$

The latter conjugation of the vector field F holds in general, i.e. for any G. The Kodama transformation consists in making use of the natural grading of the KdV equation in order to choose a G consisting of a finite sum of monomials and satisfying two fundamental requirements. The first one is $[G, F_0] = 0$, which allows to leave F_1 in the KdV hierarchy, as it is given by the normal form construction. The second one consists just in “forcing” $F_2 + [G, F_1]$ to fit the KdV hierarchy, even though F_2 does not. This part of the theory is only sketched in the present review and we refer to [23, 26] for details.

Remark 3 The treatment of the general case to orders $s = 3$ and $s = 4$ requires three and four perturbative steps, respectively, and is currently in progress.

3 Abstract Setting: Perturbation Theory in Poisson Systems

In order to treat our problem, we need to frame our Hamiltonian field theory in the more general context of Poisson systems [32, 36]. Such a short digression is adapted to our present purposes and does not aim at any generality.

3.1 Poisson Formalism

Definition 2 Let Γ be the phase space of the system and let $\mathscr{A}(\Gamma)$ be the algebra of real-valued smooth functions defined on Γ. A binary application, or product, $\{\cdot,\cdot\} : \mathscr{A}(\Gamma) \times \mathscr{A}(\Gamma) \to \mathscr{A}(\Gamma)$ is called a *Poisson bracket* on Γ if it satisfies the following properties

(i) Skew-symmetry: $\{F, G\} = -\{G, F\}$;
(ii) Left-linearity: $\{\alpha F + \beta G, H\} = \alpha\{F, H\} + \beta\{G, H\}$;
(iii) Jacobi identity: $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$;
(iv) Leibniz rule: $\{FG, H\} = F\{G, H\} + \{F, H\}G$,

$\forall F, G, H \in \mathscr{A}(\Gamma)$ and $\alpha, \beta \in \mathbb{R}$. The pair $(\mathscr{A}, \{\ ,\ \})$ is called *Poisson algebra*.

Remark 4 The bracket $\{\cdot,\cdot\}_{q,p}$ defined in (10) satisfies axioms (i)-(iv) in the above definition. Thus, the axiomatic definition above contains both the usual Hamiltonian mechanics and the field theory (as well as quantum mechanics).

For the sake of concreteness, let us consider the case where Γ is the space of two components, smooth, real-valued fields $u(x) = (u_1(x), u_2(x))^T$ defined on $\mathbb{T}$ (what we show can be exported to the case of n components, complex-valued fields on a d-dimensional domain D).

By analogy with the standard case (10), a bilinear, skew-symmetric, Leibniz bracket on such a space is defined by the formula

$$\{F, G\}_J := \langle \nabla F, J\nabla G\rangle := \oint \sum_{i,j=1}^{2} \frac{\delta F[u]}{\delta u_i} J_{ij}[u] \frac{\delta G[u]}{\delta u_j}\, dx\ , \tag{25}$$

where $J_{ij}[u]$ is a tensor valued operator, skew-symmetric with respect to the L_2 scalar product $\langle\ ,\ \rangle$, functionally dependent on u. Notice that with the choice $J = \mathsf{J}_2$, and denoting $u_1 = q$, $u_2 = p$, (25) coincides with (10). On the other hand, the bracket (25) does not satisfy the Jacobi identity (hypothesis (iii) above), in general. We state without proof the following Proposition [32], which characterises the Poisson brackets of the form (25).

Proposition 1 *The bracket (25) satisfies the Jacobi identity, so that it is a Poisson bracket, if and only if the skew-symmetric tensor $J[u]$ satisfies the Schouten identity*

$$\sum_{s=1}^{2} \left(J_{is} D_{u_s} J_{jk} + J_{js} D_{u_s} J_{ki} + J_{ks} D_{u_s} J_{ij} \right) = 0 \tag{26}$$

for all u and all $i, j, k = 1, 2$.

Here D_{u_s} denotes the weak partial derivative with respect to u_s, defined in the usual way:

$$\left(D_{u_s} f \right) h := \frac{d}{d\epsilon} f[u_s + \epsilon h]) \Big|_{\epsilon=0} , \tag{27}$$

for any f functionally dependent on u. Observe that, for example, $D_{u_1} u_1 = 1$, $D_{u_2} \partial_x u_2 = \partial_x$ and so on. Thus, any skew-symmetric tensor $J[u]$ satisfying the identity (26) is a *Poisson tensor*, i.e. it defines through (25) a Poisson bracket. An obvious but fundamental consequence of Proposition 1 is the following

Corollary 1 *Any skew-symmetric tensor J independent of u (i.e. constant on the phase space) is a Poisson tensor.*

Remark 5 One does not require $J[u]$ to be non-degenerate, so that J is allowed to have a nontrivial kernel. The functionals F such that $J\nabla F = 0$ are called *Casimir invariants* of the given Poisson structure and represent constants of motion for all Hamiltonian systems: $\{H, F\} = 0$ for any $H \in \mathscr{A}(\Gamma)$.

Within this framework, fixing a Hamiltonian $H[u]$ in the given Poisson algebra, the associated dynamics is defined in the usual way, namely

$$u_t = \{u, H\}_J = J\nabla_u H , \tag{28}$$

to be read by components, $\nabla_u H$ being the functional gradient of $H[u]$. Of course, any functional F evolves along the solutions of (28) according to $F_t = \{F, H\}_J$. Hamiltonian dynamical systems, in the generalised Poisson sense, have the form (28), which includes the standard (symplectic) case.

The fundamental feature of generalised Hamiltonian systems is their invariant character under any change of variables.

Proposition 2 *Any smooth change of variables $f : u \mapsto \tilde{u} = f[u]$ maps the Hamiltonian system $u_t = J\nabla_u H$ into the Hamiltonian system $\tilde{u}_t = \tilde{J}\nabla_{\tilde{u}} \tilde{H}$, where $\tilde{H} = H \circ f^{-1}$, whereas the transformed Poisson tensor $\tilde{J}$ is given by*

$$\tilde{J}[\tilde{u}] := (D_u f) J (D_u f)^T \Big|_{u=f^{-1}[\tilde{u}]} . \tag{29}$$

The corresponding Poisson brackets are related, for any $F, G \in \mathscr{A}(\Gamma)$, by

$$\{F, G\}_J \circ f^{-1} = \{F \circ f^{-1}, G \circ f^{-1}\}_{\tilde{J}} \, . \tag{30}$$

In the latter formula, D_u denotes the weak Jacobian of u, as defined in (27). The proof of the above Proposition is direct and not reported. The important point is the following: if J is a Poisson tensor, its transformed $\tilde{J}$ under any f is a Poisson tensor. Of course, the Hamilton equations are not invariant in form under f, which happens if and only if $\tilde{J} = J$. *Canonical transformations* are then defined as those transformations f leaving the Poisson tensor invariant. In order to check the canonicity of a transformation f, it is easier to make use of (30) which, with $J = \tilde{J}$, yields $\{F, G\}_J \circ f^{-1} = \{F \circ f^{-1}, G \circ f^{-1}\}_J$.

Remark 6 If $J = \mathsf{J}_2$, the transformation law (29), together with the canonicity condition $\tilde{J} = J$, yields the requirement that the Jacobian $D_u f$ be symplectic.

The equation of motion (28) can be rewritten as $u_t = \mathscr{L}_H u$, where the operator $\mathscr{L}_H \cdot = \{\cdot, G\}_J$, such that $\mathscr{L}_H F = \{F, H\}_J$ for any F, is the Lie derivative of F in the direction of the Hamiltonian vector field $J \nabla H$. One can then formally solve the equation by exponentiation, which defines the flow Φ_H^t of the system, namely

$$u(t) = e^{t\mathscr{L}_H} w := \Phi_H^t(w) \, , \tag{31}$$

where $w = u(0)$ is an arbitrary initial condition. Of course the exponential operator above is defined, as usual, by its formal series

$$e^{t\mathscr{L}_H} = 1 + t\mathscr{L}_H + \frac{t^2}{2}\mathscr{L}_H^2 + O(t^3) \, . \tag{32}$$

Now, since the evolution equation $F_t = \{F, H\}_J = \mathscr{L}_H F$ of any functional F is solved by $e^{t\mathscr{L}_H} F(w)$, which must equal $F[u(t)] = F[\Phi_H^t(w)]$ for any initial condition w, one gets the useful relation

$$e^{t\mathscr{L}_H} F = F \circ \Phi_H^t \, , \tag{33}$$

which is known as the exchange Lemma; we will make use of it below.

The Hamiltonian flow $\Phi_H^t : \Gamma \to \Gamma$ represents a one-parameter family of canonical transformations of Γ into itself (the family is a group if the flow is global).

Proposition 3 *For any t such that Φ_H^t exists, and any pair of functionals F and G, one has*

$$\{F, G\}_J \circ \Phi_H^t = \{F \circ \Phi_H^t, H \circ \Phi_H^t\}_J \, . \tag{34}$$

Proof Define $\Delta(t)$ the difference between the left and the right-hand side of (34), and observe that $\Delta(0) \equiv 0$. Making use of relation (33), and of the Jacobi identity, one gets $d\Delta(t)/dt = \{\Delta(t), H\}_J = \mathscr{L}_H \Delta(t)$, whose solution is $\Delta(t) = e^{t\mathscr{L}_H}\Delta(0) \equiv 0$. □

Remark 7 In the above treatment, the Hamiltonian H is arbitrary. It follows that any functional G, regarded as a Hamiltonian, generates a one-parameter family of canonical transformations, which is given by its flow $\Phi_G^s = e^{s\mathscr{L}_G}$, where $\mathscr{L}_G = \{\ ,G\}_J$. In the jargon, G is called the generating Hamiltonian, and $\mathscr{L}_G = d\Phi_G^s/ds|_{s=0}$ the generator of the transformation.

As a final point of this section, we state a simple version of the Nöther theorem in the Poisson framework.

Theorem 1 *If the Hamiltonian $H[u]$ is invariant with respect to the flow $e^{s\mathscr{L}_K}$ of generator $\mathscr{L}_K = \{\ ,K\}_J$, i.e. $e^{s\mathscr{L}_K}H = H$ for any s close to zero, then $\{H,K\}_J = 0$.*

Proof The derivative of $e^{s\mathscr{L}_K}H = H$ with respect to s, at $s = 0$, gives the result. □

In the practice, one usually "sees" a certain symmetry of H, i.e. one is able to write down a certain transformation Ψ^s such that $\Psi^0 = 1$ and $H \circ \Psi^s = H$ for any s around zero. Then, if Ψ^s is a Hamiltonian flow, its generating Hamiltonian K is a constant of motion of the given system.

3.2 Perturbation Theory

The target of Hamiltonian perturbation theory, which goes back to Poincaré and Birkhoff, is the following. Given a Hamiltonian

$$H = H_0 + \lambda H_1 + \lambda^2 H_2 + O(\lambda^3) , \tag{35}$$

formally ordered with respect to the small parameter λ, one looks for a canonical transformation, λ-close to the identity, erasing completely or in part the perturbation terms $H_{j\geq 1}$ up to a given order (possibly infinite, as in the KAM theory). As is well known, the complete removal of the perturbation terms, even to the first few orders, is not possible, in general. The best one can do is instead to find a canonical transformation setting H in normal form, according to the following definition.

Definition 3 The Hamiltonian $H_0 + \lambda Z_1 + \cdots + \lambda^n Z_n + O(\lambda^{n+1})$ is said to be in normal form to order $n \geq 1$ with respect to H_0 if $\mathscr{L}_{H_0} Z_j = \{Z_j, H_0\} = 0$ for any $j = 1, \ldots, n$.

Observe that $Z_j \equiv 0$ fits the normal form requirement, which means that the definition includes the possibility of complete removal of some perturbation terms.

The canonical transformation bringing the Hamiltonian (35) into normal form with respect to H_0, to order λ^2 included, is given by composing the flows of two unknown Hamiltonians G_1 and G_2, namely

$$u \mapsto \tilde{u} = e^{-\lambda^2 \mathscr{L}_2} e^{-\lambda \mathscr{L}_1} u \ , \tag{36}$$

where $\mathscr{L}_j := \mathscr{L}_{G_j}$, $j = 1, 2$. The *inverse* transformation maps the Hamiltonian (35) into

$$\begin{aligned} \tilde{H} = e^{\lambda^2 \mathscr{L}_2} e^{\lambda \mathscr{L}_1} H = H_0 + \lambda \left(\mathscr{L}_1 H_0 + H_1 \right) + \\ + \lambda^2 \left(\mathscr{L}_2 H_0 + \mathscr{L}_1 H_1 + \frac{1}{2} \mathscr{L}_1^2 H_0 + H_2 \right) + O(\lambda^3) \ , \end{aligned} \tag{37}$$

which is obtained by expanding the exponentials. The two generating Hamiltonians are then found by imposing that, according to the Definition 3, the quantities

$$\begin{aligned} Z_1 &:= H_1 + \mathscr{L}_1 H_0 \ , \\ Z_2 &:= \mathscr{L}_2 H_0 + \mathscr{L}_1 H_1 + \frac{1}{2} \mathscr{L}_1^2 H_0 + H_2 \end{aligned} \tag{38}$$

be first integrals of H_0. Observing that $\mathscr{L}_j H_0 = -\mathscr{L}_{H_0} G_j$, the latter two equations for the four unknowns Z_j and G_j, can be rewritten in the form

$$\begin{aligned} \mathscr{L}_{H_0} G_1 &:= H_1 - Z_1 \ , \\ \mathscr{L}_{H_0} G_2 &:= \mathscr{L}_1 H_1 + \frac{1}{2} \mathscr{L}_1^2 H_0 + H_2 - Z_2 \ . \end{aligned} \tag{39}$$

These equations have one and the same structure, namely

$$\mathscr{L}_{H_0} G_j = S_j - Z_j \ , \quad (j = 1, 2) \tag{40}$$

with obvious definitions of the S_j.

Remark 8 Looking for a transformation to an arbitrary order n, one finds at any order $j = 1, \dots, n$ an equation of the form (40), where S_j is a known quantity if all the equations up to order $j - 1$ have been solved.

Equation (40) is known as the *homological equation* of order j, which has to be solved determining the unknowns Z_j and G_j under the condition $\mathscr{L}_{H_0} Z_j = 0$.

In what follows we suppose that the flow $\Phi^s_{H_0}$ of H_0 is global (i.e. it exists for all $s \in \mathbb{R}$) and uniformly bounded with respect to s.

Definition 4 The time average of any F along the unperturbed flow of H_0 is denoted by

$$\langle F\rangle_0 := \lim_{t\to\infty}\frac{1}{t}\int_0^t F\circ\Phi_{H_0}^s\,ds\ . \tag{41}$$

If the flow of H_0 is τ-periodic, i.e. $\Phi_{H_0}^\tau = 1$, then $\langle F\rangle_0 = \frac{1}{\tau}\int_0^\tau F\circ\Phi_{H_0}^s ds$.

Lemma 1

$$\mathscr{L}_{H_0}\langle F\rangle_0 = 0\ . \tag{42}$$

Proof Composing the left and right-hand side of (41) with the flow $\Phi_{H_0}^r$, one gets, on the right-hand side, $\lim\frac{1}{t}\int_0^t F\circ\Phi_{H_0}^{s+r}\,ds = \lim\frac{1}{t}\left(\int_r^0+\int_0^t+\int_t^{t+r}\right)F\circ\Phi_{H_0}^a\,da = \lim\frac{1}{t}\int_0^t F\circ\Phi_{H_0}^a\,da$. Thus $\langle F\rangle_0\circ\Phi_{H_0}^r = F$, which implies (42), and vice versa. □

Lemma 2 *The solution of the homological equation (40) is given by*

$$Z_j = \langle S_j\rangle_0\ \ ;\ \ G_j = \langle G_j\rangle_0 + \lim_{t\to\infty}\frac{1}{t}\int_0^t (s-t)e^{s\mathscr{L}_{H_0}}\left(S_j - \langle S_j\rangle_0\right)ds\ . \tag{43}$$

If the flow of H_0 is τ-periodic, $G_j = \langle G_j\rangle_0 + \frac{1}{\tau}\int_0^\tau s\ e^{s\mathscr{L}_{H_0}}\left(S_j - \langle S_j\rangle_0\right)ds$.

Proof Applying $e^{s\mathscr{L}_{H_0}}$ to Eq. (40), taking into account the invariance of Z_j (by the definition of normal form), and taking the time average, one gets the first of (43) in the limit. By the latter result, the homological equation becomes $\mathscr{L}_{H_0}G_j = S_j - \langle S_j\rangle_0$. Applying $(s-t)e^{s\mathscr{L}_{H_0}}$ to the latter equation and time averaging, one gets the second of (43) in the limit. □

Remark 9 The generating Hamiltonians G_j solving the homological equation are defined up to their average along the flow of H_0, i.e. up to an arbitrary constant of motion of H_0. Thus, both the normal form Hamiltonian and the transformation bringing to it are not unique. In the sequel, we make the choice $\langle G_j\rangle_0 \equiv 0$.

Theorem 2 (Averaging Principle) *The canonical transformation*

$$u \mapsto \tilde{u} = e^{-\lambda^2\mathscr{L}_2}e^{-\lambda\mathscr{L}_1}u\ ,$$

generated by

$$
\begin{aligned}
G_1 &= \lim_{t\to\infty}\frac{1}{t}\int_0^t (s-t)e^{s\mathscr{L}_0}\,(H_1-\langle H_1\rangle_0)\,ds\ ;\\
G_2 &= \lim_{t\to\infty}\frac{1}{t}\int_0^t (s-t)e^{s\mathscr{L}_0}\,(S_2-\langle S_2\rangle_0)\,ds\ ;\\
S_2 &:= H_2+\frac{1}{2}\{H_1,G_1\}+\frac{1}{2}\{\langle H_1\rangle_0,G_1\}\ ,
\end{aligned}
\tag{44}
$$

maps the perturbed Hamiltonian $H = H_0 + \lambda H_1 + \lambda^2 H_2 + O(\lambda^3)$ *into the normal form* $\tilde{H} = e^{\lambda^2\mathscr{L}_2}e^{\lambda\mathscr{L}_1}H = H_0 + \lambda Z_1 + \lambda^2 Z_2 + O(\lambda^3)$*, explicitly given by*

$$
\tilde{H} = H_0 + \lambda\langle H_1\rangle_0 + \lambda^2\left(\langle H_2\rangle_0 + \frac{1}{2}\langle\{H_1,G_1\}\rangle_0\right) + O(\lambda^3)\ . \tag{45}
$$

Proof By Lemma 2, solving the first of the homological equations (39) yields Z_1 and G_1. By substituting $\mathscr{L}_1 H_0 = Z_1 - H_1 = \langle H_1\rangle_0 - H_1$ into the right-hand side of the second of the homological equations (39), one gets the latter in the form $\mathscr{L}_{H_0}G_2 = S_2 - Z_2$, with S_2 as in (44). Solving by Lemma 2 again yields Z_2 and G_2. □

Remark 10 As a matter of fact, in order to get the normal form Hamiltonian (45), one does not need to compute G_2. This is a general fact: Z_{j+1} depends on $G_1, \ldots, G_j$.

4 Hamiltonian Field Theory Close to $q_{tt} = q_{xx}$

We now come back to our problem and solve it by applying all the tools introduced in the previous section.

Let us start by considering a Hamiltonian $H = \oint \mathscr{H}\,dx$, whose density $\mathscr{H}$ does not depend explicitly on t and x and is an analytic function of q_x, p and their spatial derivatives up to a certain finite order, in the neighbourhood of the origin. Since $\mathscr{H}$ is invariant under time, space and q translations, Theorem 1 (Nöther) applies.

Proposition 4 $H = \oint \mathscr{H}dx$*,* $I = \oint q_x p\,dx$ *and* $P = \oint p\,dx$ *are the three first integrals corresponding to the symmetries* $t \to t+s$*,* $x \to x+s$ *and* $q \to q+s$*, respectively. Moreover,* $\{I, P\} = 0$*, so that the three first integrals are in involution.*

Proof The conservation of H is obvious. The Hamilton equations for I at time s are: $q_s = q_x$ and $p_s = p_x$, whose solution is $q(t, x+s)$ and $p(t, x+s)$, clearly corresponding to the x-translation. The Hamilton equations for P are $q_s = 1$, $p_s = 0$, solved by $q(t,x) + s$ and $p(t,x)$, corresponding to the q-translation. Finally, observe that $\{I, P\}_{q,p} = \oint(\delta I/\delta q)(\delta P/\delta p)dx = -\oint p_x\,dx = 0$. □

Remark 11 One can always restrict the dynamics to the submanifold $P = \oint p\,dx = 0$ by the canonical transformation $q = q'$, $p = P + p'$.

For the sake of convenience, we repeat below the definition of the class of Hamiltonian functionals considered, with the appropriate grading.

Definition 5 The perturbative ordering of the Hamiltonian H is defined by the following scaling:

$$H_\lambda := \frac{1}{\lambda^4} \oint \mathscr{H}(\lambda^2 q_x, \lambda^2 p, \lambda^3 q_{xx}, \lambda^3 p_x, \ldots, \lambda^6 q_{xxxxxx}, \lambda^6 p_{xxxx})\, dx \, . \tag{46}$$

By Taylor expanding in powers of λ, close to $\lambda = 0$, assuming without loss of generality that $\mathscr{H}|_{(q,p)=0} = 0$, and taking into account Remark 11, one gets

$$H_\lambda = H_0 + \lambda H_1 + \lambda^2 H_2 + \lambda^3 H_3 + \lambda^4 H_4 + \cdots \, , \tag{47}$$

where

$$H_0 = \oint \frac{ap^2 + b(q_x)^2}{2} dx + cI \tag{48}$$

with a, b and c some constants and $I = \oint q_x p\, dx$;

$$H_1 = \oint d_1 q_x p_x \, dx \, ; \tag{49}$$

$$\begin{aligned} H_2 = \oint \; & \Big[e_1 (q_x)^3 + e_2 p^3 + e_3 (q_x)^2 p + e_4 q_x p^2 + e_5 (q_{xx})^2 + \\ + \; & e_6 (p_x)^2 + e_7 q_{xx} p_x\Big] \, dx \, ; \end{aligned} \tag{50}$$

$$H_3 = \oint \Big[f_1 (q_x)^2 p_x + f_2 q_{xx} p^2 + f_3 q_{xx} p_{xx}\Big] \, dx \, ; \tag{51}$$

$$\begin{aligned} H_4 = \oint \; & \big[g_1 (q_x)^4 + g_2 p^4 + g_3 (q_x)^2 p^2 + g_4 (q_x)^3 p + g_5 q_x p^3 \; + \\ + \; & g_6 (q_{xx})^2 q_x + g_7 (q_{xx})^2 p + g_8 (p_x)^2 q_x + g_9 (p_x)^2 p \; + \\ + \; & g_{10} q_{xxx} p^2 + g_{11} (q_x)^2 p_{xx} + g_{12} (q_{xxx})^2 + g_{13} (p_{xx})^2 \; + \\ + \; & g_{14} q_{xxxx} p_x\big] \, dx \, , \end{aligned} \tag{52}$$

and so on. Here $d_1, e_1, \ldots, g_{14}$ are given constants.

Remark 12 Since $\mathscr{H}$ is independent of x, the density of each H_j is independent of x. It follows that $\{I, H_j\} = 0$ for any $j \geq 0$.

Proposition 5 *If the constants $a := \partial^2\mathscr{H}/\partial p^2|_0$ and $b := \partial^2\mathscr{H}/\partial(q_x)^2|_0$ appearing in (48) are different from zero and have the same sign, there exists a time-dependent canonical transformation which brings the Hamiltonian H_0 in the canonical wave equation form $K_0 = \frac{1}{2}\oint[p^2 + (q_x)^2]dx$ and preserves the structure of the perturbations H_j to any order $j \geq 0$.*

Proof Let $a = \sigma|a|$ and $b = \sigma|b|$, with $\sigma = \pm 1$. One first performs the canonical rescaling $q = \sqrt{|a|}\; q'$, $p = \sqrt{|b|}\; p'$, $H = \sigma|ab|H'$, $t = \sigma t'$, which brings H_0 into $K_0 + c'I$, where $c' = \sigma c/\sqrt{|ab|}$. Then one performs the transformation $(q', p') = \Phi^t_{c'I}(q'', p'') = \Phi^{c't}_I(q'', p'')$, where Φ^t_I denotes the flow of $I = \oint q_x p\; dx$. The latter transformation is canonical and erases $c'I$. Clearly, both transformations do not change the structure of any H_j nor the value of the coefficients of the Hamiltonians $H_1, \ldots, H_4$. Observe that the flow of I is the left translation of (q, p), so that it is global and preserves the regularity of the initial condition. □

Remark 13 Consider $K_0 + \lambda H_1 = \frac{1}{2}\oint[p^2 + (q_x)^2 + 2\lambda d_1 q_x p_x]dx$. Its Hamilton equations read

$$q_t = p - \lambda d_1 q_{xx} \;\; ; \;\; p_t = q_{xx} + \lambda d_1 p_{xx} \; .$$

Both q and p satisfy the linear Boussinesq equation

$$u_{tt} = u_{xx} + (\lambda d_1)^2 u_{xxxx} \; .$$

The condition on a and b in the Proposition 5 above identifies the elliptic fixed points in the given class of Hamiltonians. One is then left with the problem of simplifying the dynamics of $K_0 + \lambda H_1 + \lambda^2 H_2 + \cdots$. The perturbations to various order have the structure listed above and no further simplification can be made, in general. However, there is a relevant class of Hamiltonians that display a much simpler structure, namely the class of *mechanical Hamiltonians* of the form $\mathscr{H} = p^2/2 + \mathscr{U}$, where $\mathscr{U}$ depends only on q_x and its derivatives. Such Hamiltonians usually arise as the continuum limit of some lattice system, the notable case being just that of the vibrating string.

Proposition 6 *Suppose that $\mathscr{H} = p^2/2 + \mathscr{U}(q_x, q_{xx}, \ldots, q_{xxxxx})$. Then, if the condition $b := \partial^2\mathscr{U}/\partial(q_x)^2|_0 > 0$ holds, H_0 can be brought in the canonical wave form K_0, $H_1 = H_3 \equiv 0$, and*

$$H_2 = \oint \left[\alpha_1(q_x)^3 + \alpha_2(q_{xx})^2\right]\, dx \; ;$$

$$H_4 = \oint \left[\beta_1 (q_x)^4 + \beta_2 (q_{xx})^2 q_x + \beta_3 (q_{xxx})^2\right] dx \ .$$

Proof The momentum p cannot appear out of H_0, by definition. Notice that in this case there is no term proportional to I in H_0. □

In the latter significant case one can obviously rename $H_2 \to H_1$ and $H_4 \to H_2$, $\lambda^2 \to \lambda$.

4.1 Traveling Waves

The equations of motion associated with $K_0 = \oint \frac{p^2+(q_x^2)}{2}\, dx$ reduce to the wave equation for the field q:

$$q_t = p \ ; \quad p_t = q_{xx} \ , \qquad \Longleftrightarrow \qquad q_{tt} = q_{xx} \ . \tag{53}$$

In order to simplify the analysis of perturbations of the wave equation, it is convenient to perform a change of variables that maps the functions (q, p) into the *Riemann invariants* (u, v):

$$u = \frac{q_x + p}{\sqrt{2}} \ ; \quad v = \frac{q_x - p}{\sqrt{2}} \ . \tag{54}$$

The equations of motion for u and v are the left and right translation equation, respectively:

$$\begin{cases} u_t = u_x \\ v_t = -v_x \end{cases} \ . \tag{55}$$

Indeed, the solution of the above system corresponding to the initial condition $(u_0(x), v_0(x))$ is $(u_0(x+t), v_0(x-t))$, i.e. a rigid translation of the initial profiles. The flow of the wave equation, that is used to compute normal forms, is particularly manageable in these new variables, being a left translation for u and a right translation for v (at positive times).

The change of variables (54) is not canonical and it maps the standard Poisson tensor J_2 into the Gardner tensor [24]

$$J = \begin{pmatrix} \partial_x & 0 \\ 0 & -\partial_x \end{pmatrix} \ . \tag{56}$$

In particular, as can be checked, formula (29) for the transformation (54) reads

$$D_{q,p}(u,v)\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}D^T_{q,p}(u,v)=\begin{pmatrix}\partial_x & 0\\ 0 & -\partial_x\end{pmatrix}\ .$$

The Hamiltonian K_0, expressed in terms of (u,v), reads $K_0=\oint\frac{u^2+v^2}{2}\,dx$, so that the translation equations for u and v are the Hamilton equations associated with K_0 in the Gardner structure.

The explicit expression of the Hamiltonians (48)–(52) in the (u,v) variables is:

$$K_0 \;=\; \oint\frac{u^2+v^2}{2}\,dx\,; \tag{57}$$

$$H_1 \;=\; \oint\frac{d_1}{\sqrt{|ab|}}uv_x\,dx\,; \tag{58}$$

$$\begin{aligned} H_2 \;=\; \oint\Big\{&\frac{1}{2^{3/2}}\Big[\big(\tfrac{e_1}{|b|^{3/2}}+\tfrac{e_2}{|a|^{3/2}}+\tfrac{e_3}{|b|\sqrt{|a|}}+\tfrac{e_4}{|a|\sqrt{|b|}}\big)u^3\\ &+\big(\tfrac{e_1}{|b|^{3/2}}-\tfrac{e_2}{|a|^{3/2}}-\tfrac{e_3}{|b|\sqrt{|a|}}+\tfrac{e_4}{|a|\sqrt{|b|}}\big)v^3\\ &+\big(\tfrac{3e_1}{|b|^{3/2}}-\tfrac{3e_2}{|a|^{3/2}}+\tfrac{e_3}{|b|\sqrt{|a|}}-\tfrac{e_4}{|a|\sqrt{|b|}}\big)u^2v\\ &+\big(\tfrac{3e_1}{|b|^{3/2}}+\tfrac{3e_2}{|a|^{3/2}}-\tfrac{e_3}{|b|\sqrt{|a|}}-\tfrac{e_4}{|a|\sqrt{|b|}}\big)uv^2\Big]+\\ &+\frac{1}{2}\Big[\big(\tfrac{e_5}{|b|}+\tfrac{e_6}{|a|}+\tfrac{e_7}{\sqrt{|ab|}}\big)u_x^2+\\ &+\big(\tfrac{e_5}{|b|}+\tfrac{e_6}{|a|}-\tfrac{e_7}{\sqrt{|ab|}}\big)v_x^2\Big]\Big\}\,dx\,. \end{aligned} \tag{59}$$

4.2 The Generic Case

In order to perform a canonical transformation as stated in Proposition 2, one has to compute time averages, as required in Theorem 2. General formulas applying to the case of an unperturbed flow consisting of left/right translations are provided in the next lemma.

Lemma 3 *Suppose that f and g are continuous functions on $\mathbb{T}$. Then*

$$\oint\oint f(x\pm s)\,dx\,ds=\oint f(x)\,dx\,; \tag{60}$$

$$\oint\oint f(x\pm s)g(x\mp s)\,dx\,ds=\oint f(x)\,dx\oint g(y)\,dy\,; \tag{61}$$

$$\int_0^1 \oint s\, f(x \pm s) g(x \mp s)\, dx\, ds = $$
$$= \frac{1}{2} \oint f(x)\, dx \oint g(y)\, dy \pm \frac{1}{2} \oint g(x)\, \partial_x^{-1} f(x)\, dx \ , \quad (62)$$

where $\partial_x^{-1} f(x)$ *denotes the unique primitive of* f *with zero average on* $\mathbb{T}$.

Proof All these proofs consist of straightforward computations in Fourier space. First, we prove (61):

$$\oint \oint f(x \pm s) g(x \mp s)\, dx\, ds \ = \ \int_0^1 \int_0^1 \sum_{k,k' \in \mathbb{Z}} \hat{f}_k \hat{g}_{k'} e^{2\pi\imath k(x \pm s)} e^{2\pi\imath k'(x \mp s)}\, dx\, ds$$
$$= \sum_{k,k' \in \mathbb{Z}} \hat{f}_k \hat{g}_{k'} \delta_{k+k',0} \delta_{k-k',0} \ = \ \hat{f}_0 \hat{g}_0 \, .$$

From here, (60) follows by choosing $g = 1$. In order to prove (62), we Fourier transform the LHS:

$$\int_0^1 \oint s\, f(x \pm s) g(x \mp s)\, dx\, ds \ = \ \sum_{k \in \mathbb{Z}} \hat{f}_k \hat{g}_{-k} \int_0^1 s e^{\pm 4\pi\imath k s}\, ds \ .$$

It remains to notice that

$$\int_0^1 s e^{\pm 4\pi\imath k s}\, ds \ = \ \delta_{k,0} \int_0^1 s\, ds + (1 - \delta_{k,0}) \int_0^1 s e^{\pm 4\pi\imath k s}\, ds = \frac{1}{2}\delta_{k,0} \pm \frac{1}{2} \frac{1}{2\pi\imath k} (1 - \delta_{k,0})$$

and to recognise that $1/(2\pi\imath k)$ is the Fourier-multiplier corresponding to the operator ∂_x^{-1}. ◻

Proposition 7 *There exists a (formal) near-to-identity, canonical transformation* $(u, v) \mapsto (\tilde{u}, \tilde{v})$ *mapping* H_λ *into*

$$\tilde{H}_\lambda \ = \ K_0 + \lambda^2 Z_2 + O(\lambda^3) \ , \quad (63)$$

where

$$K_0 \ = \ \oint \frac{\tilde{u}^2 + \tilde{v}^2}{2}\, dx \, ; \quad (64)$$

$$
\begin{aligned}
Z_2 \;=\; \oint \Big\{ & \frac{1}{2^{3/2}}\Big[\big(\tfrac{e_1}{|b|^{3/2}} + \tfrac{e_2}{|a|^{3/2}} + \tfrac{e_3}{|b|\sqrt{|a|}} + \tfrac{e_4}{|a|\sqrt{|b|}}\big)\tilde{u}^3 + \\
& + \big(\tfrac{e_1}{|b|^{3/2}} - \tfrac{e_2}{|a|^{3/2}} - \tfrac{e_3}{|b|\sqrt{|a|}} + \tfrac{e_4}{|a|\sqrt{|b|}}\big)\tilde{v}^3\Big] + \\
& + \frac{1}{2}\Big[\big(\tfrac{e_5}{|b|} + \tfrac{e_6}{|a|} + \tfrac{e_7}{\sqrt{|ab|}} - \tfrac{d_1^2}{2|ab|}\big)\tilde{u}_x^2 + \\
& + \big(\tfrac{e_5}{|b|} + \tfrac{e_6}{|a|} - \tfrac{e_7}{\sqrt{|ab|}} - \tfrac{d_1^2}{2|ab|}\big)\tilde{v}_x^2\Big]\Big\}\, dx\, .
\end{aligned}
\tag{65}
$$

Proof *First perturbative step*: Using (45) and (61) one has $Z_1 = 0$:

$$
\begin{aligned}
Z_1 \;&=\; \int_0^1 e^{s\mathscr{L}_{H_0}} H_1\, ds \\
&=\; \int_0^1 \oint \frac{d_1}{\sqrt{|ab|}} u(x+s) v_x(x-s)\, dx\, ds \\
&\overset{(61)}{=}\; \frac{d_1}{\sqrt{|ab|}} \oint u(x)\, dx \oint v_y(y)\, dy \;=\; 0,
\end{aligned}
$$

where in the last step we used that the v_y has zero average.

Additional term at second order: We need the expression of G_1 to compute Z_2. Using (45) and (62) we have

$$
\begin{aligned}
G_1 \;&=\; \int_0^1 s e^{s\mathscr{L}_{H_0}} H_1\, ds \\
&=\; \frac{d_1}{\sqrt{|ab|}} \int_0^1 \oint s\, u(x+s) v_x(x-s)\, dx\, ds \\
&\overset{(62)}{=}\; -\frac{d_1}{2\sqrt{|ab|}} \oint uv\, dx\, .
\end{aligned}
$$

The computation of functional derivatives yields:

$$
\frac{\delta G_1}{\delta u} \;=\; -\frac{d_1}{2\sqrt{|ab|}} v\, ; \qquad \frac{\delta G_1}{\delta v} \;=\; -\frac{d_1}{2\sqrt{|ab|}} u\, ;
$$

$$
\frac{\delta H_1}{\delta u} \;=\; \frac{d_1}{\sqrt{|ab|}} v_x\, ; \qquad \frac{\delta H_1}{\delta v} \;=\; -\frac{d_1}{\sqrt{|ab|}} u_x\, ,
$$

and one finally obtains

$$\{H_1, G_1\} = \oint \left(\frac{\delta H_1}{\delta u} \partial_x \frac{\delta G_1}{\delta u} - \frac{\delta H_1}{\delta v} \partial_x \frac{\delta G_1}{\delta v} \right) dx$$
$$= -\frac{d_1^2}{2|ab|} \oint \left(v_x^2 + u_x^2 \right) dx \, .$$

Computation of the second-order normal form: Using (45), one has to time average (with respect to the unperturbed flow of K_0) the following expression:

$$H_2 + \frac{1}{2}\{H_1 - Z_1, G_1\} = \oint \Big\{ \frac{1}{2^{3/2}} \Big[\big(\tfrac{e_1}{|b|^{3/2}} + \tfrac{e_2}{|a|^{3/2}} + \tfrac{e_3}{|b|\sqrt{|a|}} + \tfrac{e_4}{|a|\sqrt{|b|}} \big) u^3$$
$$+ \big(\tfrac{e_1}{|b|^{3/2}} - \tfrac{e_2}{|a|^{3/2}} - \tfrac{e_3}{|b|\sqrt{|a|}} + \tfrac{e_4}{|a|\sqrt{|b|}} \big) v^3 + \big(\tfrac{3e_1}{|b|^{3/2}} - \tfrac{3e_2}{|a|^{3/2}} + \tfrac{e_3}{|b|\sqrt{|a|}} - \tfrac{e_4}{|a|\sqrt{|b|}} \big) u^2 v$$
$$+ \big(\tfrac{3e_1}{|b|^{3/2}} + \tfrac{3e_2}{|a|^{3/2}} - \tfrac{e_3}{|b|\sqrt{|a|}} - \tfrac{e_4}{|a|\sqrt{|b|}} \big) u v^2 \Big] +$$
$$+ \frac{1}{2} \Big[\big(\tfrac{e_5}{|b|} + \tfrac{e_6}{|a|} + \tfrac{e_7}{\sqrt{|ab|}} - \tfrac{d_1^2}{2|ab|} \big) u_x^2 + \big(\tfrac{e_5}{|b|} + \tfrac{e_6}{|a|} - \tfrac{e_7}{\sqrt{|ab|}} - \tfrac{d_1^2}{2|ab|} \big) v_x^2 \Big] \Big\} \, dx \, .$$

As a consequence of (61) and under the assumption of $\oint u \, dx = \oint v \, dx = 0$:

$$\int_0^1 \oint u^2(x+s) v(x-s) \, dx \, ds = \left(\oint u^2(x) \, dx \right) \left(\oint v(x) \, dx \right) = 0 \, ;$$

$$\int_0^1 \oint u(x+s) v^2(x-s) \, dx \, ds = \left(\oint u(x) \, dx \right) \left(\oint v^2(x) \, dx \right) = 0 \, .$$

Moreover

$$\int_0^1 \oint u^3(x+s) \, dx \, ds = \oint u^3(x) \, dx \, ;$$

$$\int_0^1 \oint v^3(x+s) \, dx \, ds = \oint v^3(x) \, dx \, ;$$

$$\int_0^1 \oint u_x^2(x+s) \, dx \, ds = \oint u_x^2(x) \, dx \, ;$$

$$\int_0^1 \oint v_x^2(x+s) \, dx \, ds = \oint v_x^2(x) \, dx \, ,$$

and this completes the proof. □

Remark 14 $\tilde{H}_\lambda$ is *always* the Hamiltonian of a pair of counter-propagating Korteweg-de Vries equations (up to a small remainder), i.e. its vector field $J\nabla\tilde{H}_\lambda$ is of the form (17). Such a result is somehow expected from, and in agreement with the existing results treating particular cases in the literature, among which those concerning the FPU problem (starting with the seminal work of Zabusky and Kruskal [37]) and the propagation of surface water waves (where the first deduction of the KdV equation goes back to Boussinesq [13]).

4.3 The Mechanical Case

For mechanical Hamiltonians of the form $\mathscr{H} = p^2/2 + \mathscr{U}$, where $\mathscr{U}$ depends on q_x and its derivatives, starting from Proposition 6 and repeating the analysis made in the general case, we perform the change of variables $(q, p) \mapsto (u, v)$, which yields

$$K_0 = \oint \frac{u^2 + v^2}{2}\, dx\,, \tag{66}$$

$$H_2 = \oint \left[\frac{\alpha_1}{2^{3/2}} \left(u^3 + 3u^2 v + 3uv^2 + v^3 \right) + \frac{\alpha_2}{2} \left((u_x)^2 + 2u_x v_x + (v_x)^2 \right) \right] dx\,, \tag{67}$$

$$\begin{aligned} H_4 = \oint \Big\{ &\beta_1 \left[\frac{u^4 + 4u^3 v + 6u^2 v^2 + 4uv^3 + v^4}{4} \right] \\ &+ \beta_2 \left[\frac{(u_x)^2 + 2u_x v_x + (v_x)^2}{2} \right] \frac{u+v}{\sqrt{2}} \\ &+ \frac{\beta_3}{2} [(u_{xx})^2 + 2u_{xx} v_{xx} + (v_{xx})^2] \Big\}\, dx\,. \end{aligned} \tag{68}$$

Proposition 8 *There exists a (formal) near-to-identity, canonical transformation* $(u, v) \mapsto (\tilde{u}, \tilde{v})$ *mapping* H_λ *into*

$$\tilde{H}_\lambda = K_0 + \lambda^2 Z_2 + \lambda^4 Z_4 + O(\lambda^6)\,, \tag{69}$$

where

$$K_0 = \oint \frac{\tilde{u}^2 + \tilde{v}^2}{2}\, dx\,, \tag{70}$$

$$Z_2 = \oint \left[\frac{\alpha_1}{2^{3/2}} (\tilde{u}^3 + \tilde{v}^3) + \frac{\alpha_2}{2} (\tilde{u}_x^2 + \tilde{v}_x^2) \right] dx\,, \tag{71}$$

$$
\begin{aligned}
Z_4 \ &= \oint \Big\{ \Big(\frac{\beta_1}{4} - \frac{9\alpha_1^2}{16}\Big)(\tilde{u}^4 + \tilde{v}^4) + \Big(\frac{\beta_2}{2^{3/2}} - \frac{3\alpha_1\alpha_2}{\sqrt{2}}\Big)\big[\tilde{u}(\tilde{u}_x)^2 + \tilde{v}(\tilde{v}_x)^2\big] + \\
&+ \Big(\frac{\beta_3}{2} - \frac{\alpha_2^2}{2}\Big)\big[(\tilde{u}_{xx})^2 + (\tilde{v}_{xx})^2\big]\Big\}\, dx + \Big(\frac{3\beta_1}{2} - \frac{9\alpha_1^2}{2}\Big)\langle \tilde{u}^2\rangle\langle \tilde{v}^2\rangle + \\
&+ \frac{9\alpha_1^2}{16}\big(\langle \tilde{u}^2\rangle^2 + \langle \tilde{v}^2\rangle^2\big)\,.
\end{aligned} \tag{72}
$$

Proof *First perturbative step:* using Proposition 2 we have

$$
\begin{aligned}
Z_2 \ &= \int_0^1 e^{s\mathscr{L}_{H_0}} H_2\, ds \\
&\overset{(61)}{=} \oint \Big\{ \frac{\alpha_1}{2^{3/2}}(u^3 + v^3) + \frac{\alpha_2}{2}[(u_x)^2 + (v_x)^2]\Big\}\, dx \\
&+ \frac{3\alpha_1}{2^{3/2}}\big(\langle u^2\rangle\langle v\rangle + \langle u\rangle\langle v^2\rangle\big)\ ;
\end{aligned}
$$

here the last term vanishes because $\langle u\rangle = \oint u\, dx = 0$ and $\langle v\rangle = \oint v\, dx = 0$.

Generator of the first order transformation:

$$
\begin{aligned}
G_2 \ &= \int_0^1 s e^{s\mathscr{L}_{H_0}}(H_2 - Z_2)\, ds \\
&= \int_0^1 \oint s \Big\{ \frac{3\alpha_1}{2^{3/2}}\big[u^2(x+s)v(x-s) + u(x+s)v^2(x-s)\big] \\
&\quad + \alpha_2 u_x(x+s)v_x(x-s)\Big\}\, dx\, ds \\
&\overset{(62)}{=} \frac{3\alpha_1}{2^{5/3}}\big(\langle u^2\rangle\langle v\rangle + \langle u\rangle\langle v^2\rangle\big) + \frac{3\alpha_1}{2^{5/2}}\Big(\oint v^2\partial_x^{-1}u\, dx + \oint v\partial_x^{-1}u^2\, dx\Big) \\
&\quad + \frac{\alpha_2}{2}\oint u v_x\, dx \\
&= \frac{3\alpha_1}{2^{5/2}}\Big(\oint v^2\partial_x^{-1}u\, dx + \oint v\partial_x^{-1}u^2\, dx\Big) + \frac{\alpha_2}{2}\oint u v_x\, dx\ ,
\end{aligned}
$$

where in the last step we used $\langle u\rangle = 0$ and $\langle v\rangle = 0$. Making use of the functional derivatives

$$
\begin{aligned}
\frac{\delta G_1}{\delta u} \ &= \frac{3\alpha_1}{2^{3/2}}\Big[-u\partial_x^{-1}v - \frac{1}{2}\partial_x^{-1}v^2\Big] + \frac{\alpha_2}{2}v_x\ ; \\
\frac{\delta G_1}{\delta v} \ &= \frac{3\alpha_1}{2^{3/2}}\Big[\frac{1}{2}\partial_x^{-1}u^2 + v\partial_x^{-1}u\Big] - \frac{\alpha_2}{2}u_x\ ;
\end{aligned}
$$

$$\frac{\delta(H_2 - Z_2)}{\delta u} = \frac{3\alpha_1}{2^{3/2}}\big[2uv + v^2\big] - \alpha_2 v_{xx}\,;$$

$$\frac{\delta(H_2 - Z_2)}{\delta v} = \frac{3\alpha_1}{2^{3/2}}\big[u^2 + 2uv\big] - \alpha_2 u_{xx}\,,$$

we can compute the Poisson bracket

$$\{H_2 - Z_2, G_2\} = \oint \Big[\frac{\delta(H_2 - Z_2)}{\delta u}\partial_x \frac{\delta G_2}{\delta u} - \frac{\delta(H_2 - Z_2)}{\delta v}\partial_x \frac{\delta G_2}{\delta v}\Big]\,dx\,.$$

Since we do not need its full expression, we can use (61) to simplify computations and consider only those terms that do not vanish after taking the average with respect to the flow of K_0. We obtain

$$\Big\langle\{H_2 - Z_2, G_2\}\Big\rangle_0 = \oint \Big[- \frac{9\alpha_1^2}{16}\big(u^4 + v^4\big) - \frac{3\alpha_1\alpha_2}{\sqrt{2}}\big(u_x^2 u + v_x^2 v\big) + \\ - \frac{\alpha_2^2}{2}\big((u_{xx})^2 + (v_{xx})^2\big)\Big]\,dx + \frac{9\alpha_1^2}{16}\big(\langle u^2\rangle^2 + \langle v^2\rangle^2\big) \\ - \frac{9\alpha_1^2}{2}\langle u^2\rangle\langle v^2\rangle\,,$$

whereas

$$\langle H_4\rangle_0 = \oint \Big\{\frac{\beta_1}{4}\big(u^4 + v^4\big) + \frac{\beta_2}{2^{3/2}}\big[u(u_x)^2 + v(v_x)^2\big] + \frac{\beta_3}{2}\big[(u_{xx})^2 + (v_{xx})^2\big]\Big\}\,dx \\ + \frac{3\beta_1}{2}\langle u^2\rangle\langle v^2\rangle\,.$$

Summing the right-hand sides of the two previous equations we get

$$Z_4 = \oint \Big\{\Big(\frac{\beta_1}{4} - \frac{9\alpha_1^2}{16}\Big)(u^4 + v^4) + \Big(\frac{\beta_2}{2^{3/2}} - \frac{3\alpha_1\alpha_2}{\sqrt{2}}\Big)\big[u(u_x)^2 + v(v_x)^2\big] + \\ + \Big(\frac{\beta_3}{2} - \frac{\alpha_2^2}{2}\Big)\big[(u_{xx})^2 + (v_{xx})^2\big]\Big\}\,dx + \Big(\frac{3\beta_1}{2} - \frac{9\alpha_1^2}{2}\Big)\langle u^2\rangle\langle v^2\rangle \\ + \frac{9\alpha_1^2}{16}\big(\langle u^2\rangle^2 + \langle v^2\rangle^2\big)\,.$$

□

Here, as in the generic case, Z_2 is in the KdV hierarchy, i.e. the vector field $J\nabla(\tilde{K}_0 + \lambda^2 Z_2)$ has the form of the right-hand side of (17). On the other hand, Z_4 is not, in general, in the KdV hierarchy: the two components of its vector field $J\nabla Z_4$ are not proportional to κ_5 (as defined in (20)), which is due to the impossibility to fit all the required constraints on its parameters, in general. However, it is still possible to get a dynamics within the KdV hierarchy to order λ^4 by applying the Kodama normalisation procedure to the vector field $J\nabla(\tilde{K}_0 + \lambda^2 Z_2 + \lambda^4 Z_4)$. Although such a normalisation is noncanonical, in principle, it actually yields a system of equations in the form (18). Neglecting the remainder, these equations turn out to be Hamiltonian a fortiori, with the correct Gardner-Poisson tensor (56). The deep reason behind this fact is far from being deeply understood, at present.

Concrete examples are discussed in the next Sect. 5, where we also provide an explicit example of Kodama transformation.

5 Applications

5.1 *The Fermi-Pasta-Ulam Problem*

The Fermi-Pasta-Ulam (FPU) chain consists of N identical (unit) masses connected by nonlinear springs to their nearest neighbours. The dynamics is generated by the Hamiltonian

$$H = \sum_{j\in\mathbb{Z}_N} \left[\frac{p_j^2}{2} + \phi(q_{j+1} - q_j) \right] , \tag{73}$$

where $\mathbb{Z}_N := \mathbb{Z}/(N\mathbb{Z})$, and ϕ is the potential

$$\phi(z) := \frac{z^2}{2} + \alpha\frac{z^3}{3} + \beta\frac{z^4}{4} + O(z^5) , \tag{74}$$

and α,β,... are the parameters measuring the strength of the nonlinear terms. One usually refers to the α*-model* if α is the only non-zero parameter; to the β*-model* if β is the only non-zero parameter; to the $\alpha + \beta$*-model* if both α and β are non-zero, and to the *generalised FPU model* if the lowest degree of the nonlinearity is greater than or equal 5.

When all the parameters in the nonlinearity are set to zero, the Hamiltonian (73) reduces to that of a *harmonic chain*, where particles interact through linear forces only. The latter system is integrable in the sense of Liouville, and the Hamiltonian is diagonalised by the (discrete) Fourier transform

$$p_j = \frac{1}{\sqrt{2N}} \sum_{k=-N}^{N} \hat{p}_k e^{\imath\pi\frac{jk}{N}} , \tag{75}$$

and similarly for q_j. The integrals of motion are the energies of the Fourier modes

$$E_k = \frac{|\hat{p}_k|^2 + \omega_k^2|\hat{q}_k|^2}{2} , \qquad k = -N, \dots, N-1 , \tag{76}$$

where $\omega_k := 2\left|\sin\left(\frac{k\pi}{2N}\right)\right|$ are the proper frequencies of oscillation. Observe that $E_k = E_{-k}$, for all k.

The nonlinear model (73) was introduced by Fermi, Pasta and Ulam (FPU), supported by Tsingou [18], with the purpose of analysing its thermalisation process. The authors expected that the interaction between the Fourier modes due to the nonlinear terms, and the consequent energy sharing between them, would have brought the system to reach the thermal equilibrium on a short time scale. In particular, as a detector of thermal equilibrium, they expected to observe the "equipartition of energy", i.e. a final state of the system where, on time average, all Fourier energies have almost the same value, i.e. $E_k \simeq E/N$, where E is the total energy. Their numerical simulations showed instead a completely different scenario: by initially exciting the lowest frequency mode ($k = 1$), within their available computation time, energy sharing was observed to effectively take place only among the first few modes and, instead of a continuous trend to equipartition, the dynamics showed an almost recurrent behaviour. The first explanation of the latter phenomenon goes back to Zabusky and Kruskal [37], who approximated the traveling wave dynamics of the system by the KdV equation, and based their argument on the recurrent behaviour of its solitons. On the Hamiltonian side, the first correct computation of the resonant normal form of the lattice system, in action angle-variables, is due to [35]. Such a construction was only later recognised to include that of Zabusky and Kruskal [8, 10].

Nowadays, it is well known that a key role in the explanation of the FPU phenomenon, or paradox, is played by the *integrability* of the resonant normal form either of the lattice system or of its infinite-dimensional approximation (we refer to [4, 19] and the references therein). Indeed, the KdV equation admits a complete set of (infinitely many) integrals of motion, whose conservation prevents a fast energy sharing among the Fourier modes. Moreover, the preservation of the analyticity of the initial condition causes an exponential decay of its Fourier energies [28]. These two aspects resemble very much the observations in the FPU experiment.

In fact, the connection FPU-KdV can be made rigorous using the normal form construction of Theorem 2, as follows. As a preliminary step, we perform the canonical change of variables $(q, p) \mapsto (s, r)$ defined by the generating function

$$F(q, s) = \sum_{j\in\mathbb{Z}_N} s_j(q_j - q_{j+1}) \ , \tag{77}$$

which gives

$$\begin{aligned} r_j &= -\frac{\partial F}{\partial s_j} = q_{j+1} - q_j \ , \\ p_j &= \frac{\partial F}{\partial q_j} = s_j - s_{j-1} \ . \end{aligned} \tag{78}$$

In terms of the new variables (s, r) the Hamiltonian (73) reads

$$H = \sum_{j\in\mathbb{Z}_N} \left[\frac{(s_{j+1} - s_j)^2}{2} + \phi(r_j) \right] , \tag{79}$$

whose equations of motion are

$$\begin{aligned} \dot{s}_j &= \frac{\partial H}{\partial r_j} = \phi'(r_j) , \\ \dot{r}_j &= -\frac{\partial H}{\partial s_j} = s_{j+1} + s_{j-1} - 2s_j . \end{aligned} \tag{80}$$

Remark 15 The periodicity of the q_j implies $\sum_{j\in\mathbb{Z}_N} r_j = 0$, whereas the periodicity of the s_j implies $\sum_{j\in\mathbb{Z}_N} p_j = 0$.

We now assume the existence of a pair of analytic functions $R, S : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ such that

$$\begin{aligned} s_j(t) &= \frac{\sqrt{\varepsilon}}{h} S(x,\tau)|_{x=hj,\tau=ht} \\ r_j(t) &= \sqrt{\varepsilon} R(x,\tau)|_{x=hj,\tau=ht} . \end{aligned} \qquad h := \frac{1}{N} . \tag{81}$$

Notice that the choice of the functions R and S is *not* unique. For example, one can add to them any linear combination of the form $\sum_{m\in\mathbb{Z}} c_m \sin(\pi m x/h)$, which vanishes at the lattice sites $x = hj$. Having in mind long-wavelength initial conditions, a natural choice consists in restricting R and S to the Fourier polynomials supported on the first few harmonics at $\tau = 0$, and in regarding the discrete system as a sampling of the continuous one at any $\tau > 0$. This is allowed by the following proposition.

Proposition 9 *Consider the Hamiltonian functional*

$$\mathscr{H}[S, R] = \oint \left[\frac{1}{\varepsilon}\phi(\sqrt{\varepsilon}R(x,\tau)) - \frac{1}{2} S(x,\tau)\Delta_h S(x,\tau) \right] dx , \tag{82}$$

where

$$\Delta_h := \frac{4}{h^2} \sinh^2\left(\frac{h}{2}\partial_x\right) = \partial_x^2 + \frac{h^2}{12}\partial_x^4 + O(h^4) \tag{83}$$

is the discrete Laplacian. Then, its Hamilton equations restricted to the lattice coincide with the FPU equations (80).

Proof Considering (S, R) as a canonical pair coordinate-momentum, one has

$$\begin{aligned} S_\tau &= \frac{\delta \mathscr{H}}{\delta R} = \frac{1}{\sqrt{\varepsilon}}\phi'(\sqrt{\varepsilon}R) \, , \\ R_\tau &= -\frac{\delta \mathscr{H}}{\delta S} = \Delta_h S \, . \end{aligned} \tag{84}$$

The latter equations, restricted to the lattice, i.e. to the points $x = hj$, coincide with those obtained by substituting (81) into (80). □

One thus embeds the dynamics of the FPU lattice (80) within that of the infinite-dimensional Hamiltonian system (84). The latter consists of a system of nonlinear dispersive Hamiltonian PDEs for any expansion to finite order of the discrete Laplacian (83). Moreover, making use of the latter expansion and of the explicit expression (74) of ϕ, one observes that the Hamiltonian (82) has the grading of Definition 5 with

$$\lambda \sim \sqrt{\varepsilon} \sim h^2 \, . \tag{85}$$

Let us see in which sense KdV equation allows us to explain rigorously, in the case of the α-chain, the FPU phenomenon, namely the fact that, if one low-frequency mode is initially excited, then the energy quickly flows to a small packet of modes whose energy, on time average, decreases exponentially with the mode index. The main result is conveniently formulated in terms of the quantities

$$\kappa \ := \ \frac{k}{N} \ ; \ \mathscr{E}_\kappa \ := \ \frac{E_k}{N} \ , \tag{86}$$

denoting the specific mode index (or wave number) and the corresponding specific energy, respectively. We are interested in the evolution of initial data supported on one harmonic mode of long wavelength, i.e. specific index $\kappa_0 = k_0/N \ll 1$.

Theorem 3 (Bambusi-Ponno [8]) *Consider an initial condition of the form*

$$\mathscr{E}_{\kappa_0}(0) \ = \ C_0\mu^4, \qquad \mathscr{E}_\kappa(0) = \ 0, \qquad \forall \kappa \neq \kappa_0 \, , \tag{87}$$

where C_0 is any fixed constant and $\mu := \kappa_0 := k_0/N \ll 1$.

Then, for any fixed time T_f there exist positive constants μ^*, σ, C_1 and C_2 (dependent on C_0 and T_f) such that, for all κ, $\mu < \mu^*$ and $|t| \leq T_f/\mu^3$

(i)

$$\mathscr{E}_\kappa(t) \ \leq \ \mu^4 C_1 e^{-\sigma\kappa/\mu} \ ; \tag{88}$$

(ii) *there exists a sequence of almost periodic functions* $\{F_n(t)\}_{n\in\mathbb{N}}$ *and an associated specific sequence*

$$F_\kappa = \mu^4 F_n \ , \quad if \quad \kappa = n\kappa_0 \ ; \quad F_\kappa = 0 \quad otherwise \ , \tag{89}$$

such that

$$|\mathscr{E}_\kappa(t) - F_\kappa(t)| \ \leq \ C_2\mu^5 \ . \tag{90}$$

The proof of this theorem is based on the fact that a solution of the KdV equation with an analytic initial datum on the torus remains analytic for all times [28]. In particular, analyticity implies the exponential decay of Fourier coefficients, which in turn implies the exponential decay of the Fourier coefficients for the FPU system.

On the other hand, technical difficulties arise when comparing the dynamics of the discrete system with the dynamics of the continuous one, due to the contribution of the *singular* remainder of the discrete Laplacian that contains higher-order derivatives. The latter problem is overcome by a combined use of the analyticity property of the KdV flow, closeness to the identity of the canonical transformation and Grönwall lemma [8].

However, when comparing the above result with the numerical simulations and with the recent results on relating the FPU dynamics to that of the Toda lattice [5], one realises that it is not optimal: the time scale of closeness to the KdV dynamics numerically observed turns out to be longer than $t \sim \mu^{-3} \sim \varepsilon^{-3/4}$. In fact, there is an actual hope to improve the latter result which rests on the fact that the normal form of the FPU problem is in the *KdV hierarchy* not only to the first but also to the second perturbative order. Then, an extension of Theorem 3 could work with a second-order normal form transformation yielding the (presumably) optimal result of localisation of the Fourier spectrum on time scales $\sim \mu^5 \sim \varepsilon^{-5/4}$.

Within this context, we present below the normal form construction of the FPU problem, including the Kodama transformation.

Proposition 10 *The Hamiltonian* (82) *can be mapped into the normal form*

$$\tilde{H} = K_0 + Z_1 + Z_2 + \dots \ , \tag{91}$$

with

$$K_0 = \oint \frac{\tilde{u}^2 + \tilde{v}^2}{2}\, dx \tag{92}$$

$$Z_1 = \frac{h^2}{4!2} \oint \left[\frac{4\alpha\sqrt{2\varepsilon}}{h^2}\left(\tilde{u}^3 + \tilde{v}^3\right) + \tilde{u}\tilde{u}_{xx} + \tilde{v}\tilde{v}_{xx} \right] dx \tag{93}$$

$$Z_2 = \tfrac{3}{20}\tfrac{h^4}{(4!)^2} \oint \Big[\left(\tfrac{\beta}{\alpha^2} - \tfrac{1}{2}\right) \tfrac{240\alpha^2\varepsilon}{h^4}(\tilde{u}^4 + \tilde{v}^4) + \tfrac{20\alpha\sqrt{2\varepsilon}}{h^2}\left(\tilde{u}^2\tilde{u}_{xx} + \tilde{v}^2\tilde{v}_{xx}\right)$$

$$+(\tilde{u}_{xx})^2 + (\tilde{v}_{xx})^2\Big]\,dx + \left(\tfrac{3\beta\varepsilon}{8} - \tfrac{\alpha^2\varepsilon}{4}\right)\left(\oint \tilde{u}^2\,dx\right)\left(\oint \tilde{v}^2\,dx\right) +$$

$$+\frac{\alpha^2\varepsilon}{32}\left(\langle\tilde{u}^2\rangle^2 + \langle\tilde{v}^2\rangle^2\right) \tag{94}$$

Proof By introducing the Riemann variables

$$u := \frac{S_x + R}{\sqrt{2}} \;\; ; \;\; v := \frac{S_x - R}{\sqrt{2}} \,, \tag{95}$$

the result is actually a Corollary of Proposition 8, with the substitutions $\alpha_1 = \frac{\alpha\sqrt{\varepsilon}}{6\sqrt{2}}$, $\alpha_2 = -\frac{h^2}{4!2}$ and $\beta_1 = \frac{\beta\epsilon}{4}$. □

Remark 16 The equations of motion of $K_0 + Z_1$ are those of two counter-propagating KdV equations, i.e. of the form (17), for any α. On the other hand, the equations of motion of $K_0 + Z_1 + Z_2$ are not in the KdV hierarchy, i.e. in the form (18), unless the special condition $\beta = 5\alpha^2/6$ holds.

In order to bring the continuous FPU equations of motion into the KdV hierarchy form (18), one must look for a suitable Kodama transformation, as sketched in Remark 2 [20].

Proposition 11 *The Kodama transformation*

$$\tilde{u} = w + g(w) \;\; ; \;\; \tilde{v} = z + g(z) \,, \tag{96}$$

where

$$g(w) := \frac{h^2}{4!}\left(\frac{7}{2} - \frac{9}{2}\frac{\beta}{\alpha^2}\right) w_{xx} + \frac{\alpha\sqrt{\varepsilon}}{\sqrt{2}}\left(\frac{13}{12} - \frac{3}{2}\frac{\beta}{\alpha^2}\right)\left(w^2 - \oint w^2\,dx\right) +$$
$$-\frac{1}{6}\left(w_x\partial_x^{-1}w - \oint w^2\,dx\right) , \tag{97}$$

maps the equations of motion of the Hamiltonian normal form (91) into the integrable KdV form (18).

Proof The proof consists in a long, though direct computation. Details can be found in [20]. Observe that, according to the grading (85), $g \sim \lambda$, which does not affect the first order normal form. □

A natural question arises now, namely whether it is possible to construct a normal form transformation, including the Kodama procedure, conjugating the continuous FPU equations to those of the KdV hierarchy to perturbative orders higher than the second one. This is an open problem for initial data generically supported on lower modes, but it has recently been addressed for initial data close to the traveling wave. In [23] it is proved that for almost-traveling waves, the conjugation to the third-order works only if the parameters correspond to a curve in the space of parameters containing the Toda lattice.

In general, it is expected that the FPU normal form is in the KdV hierarchy to a finite perturbative order, depending on the model. This is easily seen by considering the family of generalised FPU-systems [9] defined by a Hamiltonian of the form (73) with

$$\phi(z) = \frac{z^2}{2} + \frac{z^p}{p}\,, \qquad p \geq 3\,. \tag{98}$$

Instead of fixing a model and going on with the perturbative order, we here consider how the first order normal form depends on the exponent p. The Hamiltonian (82) with potential (98) reads

$$H \;=\; \oint \left[\frac{R^2}{2} + \gamma\varepsilon^{\frac{p-2}{2}}\frac{R^p}{p} + \frac{1}{2}(S_x)^2 - \frac{h^2}{12}(S_{xx})^2\right] dx + O(h^4)\,. \tag{99}$$

Passing to the (u, v) variables (95), one gets $H = K_0 + H_1$, where

$$K_0 = \oint \frac{u^2+v^2}{2}\, dx\,, \tag{100}$$

$$H_1 = \oint \left[\gamma\varepsilon^{\frac{p-2}{2}}\frac{(u-v)^p}{2^{p/2}p} - \frac{h^2}{24}\Big((u_x)^2 + 2u_x v_x + (v_x)^2\Big)\right] dx\,. \tag{101}$$

Averaging H_1 (using (61)) one computes the normal form $\tilde{H} = K_0 + Z_1 + \cdots$ of the system, where

$$\begin{aligned} Z_1 = \langle H_1\rangle \;=\; &\oint \frac{\gamma\varepsilon^{\frac{p-2}{2}}}{2^{p/2}p}\Big(u^p + (-v)^p\Big) - \frac{h^2}{24}\Big((u_x)^2 + (v_x)^2\Big)\, dx \\ &+ \frac{\gamma\varepsilon^{\frac{p-2}{2}}}{2^{p/2}p}\sum_{j=1}^{p-1}(-1)^j\binom{p}{j}\Big(\oint u^{p-j}\, dx\Big)\Big(\oint v^j\, dx\Big)\,. \end{aligned} \tag{102}$$

For $p = 3$ one finds that $K_0 + Z_1$ is the Hamiltonian of two uncoupled KdV equations, as expected. For $p = 4$, the so-called β-model, $K_0 + Z_1$ is the Hamiltonian of two uncoupled modified KdV (mKdV) equations. Thus, the first order normal form is integrable for $p = 3, 4$. On the other hand, for $p \geq 5$ the first order normal form Hamiltonian is that of two generalised, nonintegrable KdV equations, that are also nonlinearly coupled for $p \geq 6$. For this class of models the integrability of the normal form, and the consequent FPU phenomenon of energy localisation due to closeness to integrability, are lost to first order if the degree of nonlinearity is high enough ($p \geq 5$). More than this, in [9] it is suggested that the blow-up of solutions characterising the nonintegrable KdV equations might play a relevant role in the problem.

As a last point, we stress that the method of infinite-dimensional perturbation theory allows to analyse the FPU system, treated in Proposition 10, in the singular limit $h \to 0$ with fixed, small specific energy ε. Such a limit is justified on the short term, where dispersion is expected to play a minor role with respect to nonlinearity, which explains why the normal modes start to effectively share their energy. Taking the limit $h \to 0$, at fixed ε, of the FPU terms (92), (93) and (94), one finds

$$H = K_0 + Z_1 + Z_2 + \ldots \tag{103}$$

with

$$K_0 = \oint \frac{u^2 + v^2}{2} \, dx \,, \tag{104}$$

$$Z_1 = \frac{\alpha\sqrt{\varepsilon}}{2\sqrt{2}} \oint \frac{u^3 + v^3}{3} \, dx \,, \tag{105}$$

$$Z_2 = \left(\frac{\beta}{\alpha^2} - \frac{1}{2}\right)\frac{\alpha^2\varepsilon}{4} \oint \frac{u^4 + v^4}{4} \, dx \,. \tag{106}$$

The equations of motion associated with this normal form Hamiltonian consist of a pair of uncoupled, generalised Burgers equations, whose solution displays a gradient catastrophe at a finite shock time t_s. It has recently been proved that the Fourier energy spectrum of such a system displays a power law decay characterised by the universal exponent $-8/3$ exactly at t_s. Such a prediction fits very well the numerical spectrum of the FPU system [21]. Of course, the dynamics on times longer than t_s cannot be described in this limit and dispersive effects must be re-included, in agreement with the grading (85).

5.2 Water Waves

Consider an ideal fluid occupying, at rest, the domain

$$\Omega_{0,L} \ := \ \left\{(x,z) \in [0,L] \times \mathbb{R} \ : \ -h < z < 0\right\}, \tag{107}$$

with $L > 0$. We study the evolution of the free surface under the action of gravity, in the irrotational regime. Thus, given a periodic function $\eta : [0,L] \to \mathbb{R}$, we define the domain

$$\Omega_{\eta,L} \ := \ \left\{(x,z) \in [0,L] \times \mathbb{R} \ : \ -h < z < \eta(x)\right\}. \tag{108}$$

Irrotationality makes it possible to describe the velocity of the fluid u as gradient of a function called *velocity potential* by $u = \nabla\phi$. This problem admits a Hamiltonian formulation [14, 15, 38] and the conjugated variables are the wave profile $\eta(x)$ and the trace of the velocity potential at the free surface:

$$\psi(x) \ := \ \phi(x, \eta(x))\,. \tag{109}$$

The Hamiltonian of the system is

$$H(\eta, \psi) \ = \ \oint \left(\frac{1}{2} g\eta^2 + \frac{1}{2}\psi G(\eta)\psi\right) dx, \tag{110}$$

where $G(\eta)$ is the *Dirichlet-to-Neumann operator* defined as follows. Given a function $\psi(x)$ and consider the boundary value problem

$$\Delta\phi = 0\,, \qquad (x, z) \in \Omega_{\eta,L} \tag{111}$$

$$\phi_z\Big|_{z=-h} = 0 \tag{112}$$

$$\phi(0) = \phi(L) \tag{113}$$

$$\phi\Big|_{z=\eta(x)} = \psi \tag{114}$$

and let ϕ be its solution. Then

$$G(\eta)\psi \ := \ \sqrt{1+\eta_x^2}\,\partial_n\phi\big|_{z=\eta(x)} = (\phi_z - \eta_x\phi_x)\big|_{z=\eta(x)}, \tag{115}$$

where ∂_n denotes the derivative in the direction normal to $z = \eta(x)$.

We are interested in solutions of the form

$$\eta(x) \ = \ \mu^2 h^3\sqrt{2}\tilde{\eta}(\mu x)\,, \qquad \psi(x) \ = \ \mu\sqrt{2gh}h^2\tilde{\psi}(\mu x)\,, \qquad \mu = 1/L \ll 1\,, \tag{116}$$

that corresponds to a canonical transformation when rescaling time to

$$\tilde{t} \ = \ \frac{t}{\mu\sqrt{gh}} \tag{117}$$

and the physical space becomes the torus of unitary length.

Note that the dependence on η of the Dirichlet-to-Neumann operator causes the Hamiltonian (110) not to fall within the class of mechanical Hamiltonians of Sect. 4.3.

The small parameter of the theory is $\lambda = (h\mu)^2$. Expanding the Hamiltonian in λ one gets[1]

$$H = H_0 + \lambda H_1 + \lambda^2 H_2 + O(\lambda^3) \tag{118}$$

with H_0 being in the same form of (92) but with renamed variables:

$$H_0 = \oint \frac{\tilde{\eta}^2 + \tilde{\psi}_y^2}{2}\, dy\,, \tag{119}$$

$$H_1 = \frac{1}{2}\oint \Big(-\frac{1}{3}\tilde{\psi}_{yy}^2 + \sqrt{2}\tilde{\eta}\tilde{\psi}_y^2 \Big)\, dy\,, \tag{120}$$

$$H_2 = \frac{1}{2}\oint \Big(\frac{2}{15}\tilde{\psi}_{yyy}^2 - \sqrt{2}\tilde{\eta}\tilde{\psi}_{yy}^2 \Big)\, dy\,. \tag{121}$$

Note that, the Hamiltonian contains terms with the product of $\tilde{\eta}$ and $\tilde{\psi}$ and thus does not fit the definition of *mechanical Hamiltonian* given above. Anyway, as for the FPU problem, it is convenient to use characteristic variables (u, v) defined as

$$\tilde{\eta}(y, t) = \frac{u(y, t) + v(y, t)}{\sqrt{2}}\,, \tag{122}$$

$$\tilde{\psi}_y(y, t) = \frac{u(y, t) - v(y, t)}{\sqrt{2}} \tag{123}$$

we then obtain

$$K_0 = \oint \frac{u^2 + v^2}{2}\, dy\,, \tag{124}$$

$$H_1 = \oint \Big(-\frac{1}{12}(u_y^2 + v_y^2) + \frac{u^3 + v^3}{4} + \frac{u_y v_y}{6} - \frac{u^2 v + u v^2}{4} \Big)\, dy\,, \tag{125}$$

$$H_2 = \oint \Big(\frac{1}{2}\frac{u_{yy}^2 + v_{yy}^2}{15} - \frac{1}{4}(u u_y^2 + v v_y^2) - \frac{1}{15} u_{yy} v_{yy} \tag{126}$$

$$-\frac{1}{4}(u v_y^2 - 2u u_y v_y + v u_y^2 - 2v u_y v_y) \Big)\, dy\,. \tag{127}$$

[1] This step is far from being a trivial Taylor expansion as it involves the asymptotic expansion of the Dirichlet-to-Neumann operator (see [3] for details).

Applying the techniques of Theorem 2 one has

Proposition 12 *Within the normal form procedure outlined above, Hamiltonian* (118) *can be mapped into the normal form*

$$\tilde{H} \;=\; \tilde{K}_0 + \lambda Z_1 + \lambda^2 Z_2 + \dots \tag{128}$$

with

$$Z_1 = \oint \left[\frac{\tilde{u}^3 + \tilde{v}^3}{4} - \frac{1}{12}\left(\tilde{u}_y^2 + \tilde{v}_y^2\right) \right] dy\,, \tag{129}$$

$$Z_2 = \oint \left[\frac{1}{64}\left(\tilde{u}^4 + \tilde{v}^4\right) + \frac{7}{48}\left(\tilde{u}^2 \tilde{u}_{yy} + \tilde{v}^2 \tilde{v}_{yy}\right) + \frac{29}{720}\left(\tilde{u}_{yy}^2 + \tilde{v}_{yy}^2\right) \right] dy + \frac{1}{8}\langle \tilde{u}^2 \rangle \langle \tilde{v}^2 \rangle\,. \tag{130}$$

Proof This result is proved computing normal form Proposition 8.

First perturbative step: We use (61) to average H_1 along the flow of H_0 obtaining the expression for Z_1 in (129). The Hamiltonian generating the canonical transformation can be computed using (62):

$$G_1 \;=\; -\oint \left[\frac{1}{12} v_y u + \frac{1}{8} u^2 \partial_y^{-1} v - \frac{1}{8} v^2 \partial_y^{-1} u \right] dy\,.$$

We can therefore compute the L_2-gradient of G_1 and of $H_1 - Z_1$ obtaining

$$\frac{\delta G_1}{\delta u} \;=\; -\frac{1}{12} v_y - \frac{1}{4} u \partial_y^{-1} v - \frac{1}{8} \partial_y^{-1} v^2\,,$$

$$\frac{\delta G_1}{\delta v} \;=\; \frac{1}{12} u_y + \frac{1}{8} \partial_y^{-1} u^2 + \frac{1}{4} v \partial_y^{-1} u\,,$$

$$\frac{\delta (H_1 - Z_1)}{\delta u} \;=\; -\frac{1}{6} v_{yy} - \frac{1}{2} uv - \frac{1}{4} v^2\,,$$

$$\frac{\delta (H_1 - Z_1)}{\delta v} \;=\; -\frac{1}{6} u_{yy} - \frac{1}{2} uv - \frac{1}{4} u^2\,.$$

Second perturbative step: We use (61) to average H_2 and $\{H_1 - Z_1, G_1\}$ obtaining:

$$\left\langle \{H_1, G_1\} \right\rangle_0 \;=\; \frac{1}{8} \oint \left[\frac{1}{9}(u_{yy}^2 + v_{yy}^2) + \frac{1}{3}(u^2 u_{yy} + v^2 v_{yy}) + \frac{1}{4}(u^4 + v^4) \right] dy$$
$$\frac{1}{4}\langle u^2 \rangle \langle v^2 \rangle\,,$$

$$\langle H_2 \rangle_0 \;=\; \oint \Big[\frac{u_{yy}^2 + v_{yy}^2}{30} + \frac{1}{8}\big(u^2 u_{yy} + v^2 v_{yy}\big) \Big] \, dy \,.$$

We obtain $Z_2 = \langle H_2 \rangle_0 + \frac{1}{2} \langle \{H_1, G_1\} \rangle_0$ that is precisely (130). □

As for equations of the FPU lattice, these Hamiltonians are *not* in the Korteweg-de Vries hierarchy. Exactly as in the previous case, Kodama's theory solves the problem and with a close-to-identity change of variables maps the Hamiltonian into:

$$H_{\rm NF}(u, v) \;=\; K_0(u) + \lambda K_1(u) + \lambda^2 c_2 K_2(u) + K_0(v) + \lambda K_1(v) + \lambda^2 c_2 K_2(v) \tag{131}$$

with c_2 being some explicit constant.

In case μ is a small free parameter not related to L and the water waves are studied on the whole real line (that is, $x \in \mathbb{R}$ and thus, imposing $\lim_{x \to \infty} \phi(x) = 0$ instead of $\phi(0) = \phi(L)$ in (113)), the following result holds

Theorem 4 (Bambusi [3]) *For any s', there exists $\lambda^* > 0$ and s, s'', s.t., if $0 < \lambda < \lambda^*$, then there exists a map $T_\lambda : B_1^s \to W^{s'',1} \times W^{s'',1}$, with the following properties*

(i) $\sup_{(u,v) \in B_1^s} \| T_\lambda(u, v) - (u, v) \|_{W^{s'',1} \times W^{s'',1}} \le C\lambda$,

(ii) Let I_λ be an interval containing the origin and $z(\cdot) = (u(\cdot), v(\cdot)) \in C^1(I_\lambda; B_1^s)$ be a solution of the Hamiltonian system (131) with $c_2 = \frac{299}{389}$ define

$$z_a = (u_a, v_a) \;:=\; T_\lambda(u, v) \,. \tag{132}$$

Then there exists $R \in C^1(I_\lambda, W^{s',2} \times W^{s',2})$ s.t. one has

$$\dot{z}_a \;=\; J \nabla H(z_a(t)) + \lambda^3 R(t) \qquad \forall t \in I_\lambda \,, \tag{133}$$

where H is the Hamiltonian of water waves problem in the variables u and v.

An interesting non-trivial dynamical information one can obtain from this Theorem concerns the goodness of the approximation of the normal form dynamics. That is, for smooth enough initial data, it is possible to go back to the original non-scaled variables and to get the estimate on the wave profile

$$\sup_{|t| \le T^*/\mu^3 \sqrt{gh}} \| \eta(t) - \eta_a(t) \|_{L^\infty} \le C\mu^6 \,. \tag{134}$$

Note that the difference between wave profiles can be proved to be small only for times in which the second perturbative correction is negligible. Thus, as for the FPU system, an interesting open problem is the understanding which results can hold for larger time scales.

We are confident that these two results can be proved also in the periodic setting presented above.

6 Conclusions and Open Problems

In the framework of Hamiltonian field theory, the continuum limit of the FPU chain for long-wavelength excitations and the Hamiltonian of water waves belong to the same wider class of perturbations of the wave equation. This is not the case of other lattice models, such as the Klein-Gordon, for which one has to take into account the presence of the mass term.

Recently, the analysis of lattice model using the machinery of water waves has received a certain interest especially for systems in two spatial dimensions [22, 27] or for the analysis of higher-order normal forms for one-dimensional systems [23].

As a comparison, water waves are now a hot topic in research. The main goals in the field are results on well-posedness as well as regularity result for solutions or existence of quasi periodic or traveling wave solutions (see e.g. the recent results [2, 11, 12, 17]).

In this sense, many open questions remain open and can hopefully be addressed in the next future:

- The analysis at second order performed in Subsec. 5.1 does not allow us to conclude that the dynamics of the integrable system is close to the dynamics of the original system. Actually, it is known how to obtain a result on the dynamics, but only over times over which the effects of the second-order term is invisible. One of the open major problems is to understand how to go beyond the time scale of Theorem 3.
- From the point of view of statistical physics, the regime on which Theorem 3 is proved is not significant as the specific energy of the system $\varepsilon \sim 1/N^4$. The thermodynamic limit would require ε to be constant and independent of the size of the system. This is read, in terms of the normal form construction, as a zero-dispersion limit of the Korteweg-de Vries equation. It would be interesting to study the effect of this limit.
- Last, small attention has been given to the analysis of the FPU model when the dispersion is neglected (see [34]). An interesting question to address would be if Eqs. (103)–(106) can be used to explain some properties of the dynamics, especially for short time scales, low Fourier modes or in the regime of high specific energy.

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