Controller Design and Analysis of Multi-agent Linear Systems Based on H_∞ Index

Tongxing Li(✉), Xiaoyu Tan, and Yongfeng Zhang

School of Mathematics and Statistics, Taishan University, Taian 271000, China
tsultx@126.com

Abstract. This study deals with the multi-agent linear system, which is a more realistic and accurate discrete model with disturbance terms. Based on linear matrix inequality technology, Lyapunov stability theory and H_∞ index, we give the convergence result. Results of simulation show that the proposed method is effective.

Keywords: Linear matrix inequality · Multi-agent linear system · H_∞ index · The Kroneck product

1 Introduction

Linear matrix inequality (LMI) is an effective mathematical tool, and is widely used in the design of control systems with constrained control inputs [1–3]. G. Grimm et al. designed a dynamic compensator based on LMI for the general stability model system, which guaranteed the stability of the system. The system output is guaranteed to have L_2 gain to external interference [4]. H. S. Hu et al. studied the gain characteristics and stable region of L_2 based on LMI for general systems [5]. H. N. Wu et al. used LMI to realize the fuzzy control under control input constraints for the coupled model of ordinary differential and partial differential. S. D. Majumder entailed a finite-element design of a 3d-autopilot missile synthesis of multiple objective controls by solving the inequalities encountered while using a linear matrix [6]. In paper [7], the regression problem is considered. The key point of the paper is finding a matrix approximation of the Koopman operator from data. They formulated the regression problem as a convex optimization problem subject to LMI constraints.

An LMI approach to H_- index and mixed H_-/H_∞ fault detection observer design is proposed in [8]. The finite-horizon H_∞ containment control issue for a general discrete time-varying linear multiagent systems with multileaders is investigated in [9]. A sufficient condition is established to ensure the desired H_∞ containment performance. Then the desired controller and observer parameters are obtained by solving two coupled backward recursive Riccati difference equations [9]. Containment control problems for high-order linear time-invariant multi-agent systems with fixed communication time-delays are investigated in

© The Author(s), under exclusive license to Springer Nature Singapore Pte Ltd. 2022
Y. Jia et al. (Eds.): CISC 2022, LNEE 950, pp. 1–7, 2022.
https://doi.org/10.1007/978-981-19-6203-5_1

[10]. Based on the linear matrix inequality method and Lyapunov-Krasovskii functional method, the feedback gains, sufficient conditions on the communication digraph and the allowed upper bound of the delays are given [10]. The lecture [11] investigates finite-time containment control problem for second-order multi-agent systems with norm-bounded non-linear perturbation. The discontinuous control protocol is designed and the appropriate value range of control parameter is obtained by applying finite-time stability analysis [11]. The H_∞ control problem for Lur'e singular systems with time delays is considered in [12]. By using Lyapunov stability theory, sufficient conditions for the system to be exponentially stable and satisfy the performance index of H_∞ are obtained based on the LMI method.

The main contributions and primary distinctions of this paper with other works can be given as follows.

1. A more accurate and realistic discrete time model is proposed which is relevant for many practical sampled data systems;
2. Sufficient conditions of the convergence result is established based on LMI and the H_∞ index.
3. Results of simulation show that the proposed method is effective.

The organization of this paper is as follows: In Sect. 2, we give the systems definition about the multi-agent linear systems. In Sect. 3, we present the convergence result and give the LMI. In Sect. 4, a simulation example is presented.

2 Systems Definition

We consider the multi-agent linear systems

$$\dot{x}_i = A_i x_i(t) + B_i(u_i + d_i), \ i = 1, \ldots, N, \tag{1}$$

where $x_i = [x_i^1 \ x_i^2]^T$, u_i are the state and control input of the i th agent, d_i is a disturbance term.

$$u_i = K_i x_i, \ i = 1, \ldots, N, \tag{2}$$

where $K_i = [k_i^1 \ k_i^2]$. Our aim is to design LMI to solve the K_i and to achieve that $x_i \to 0$.

3 The Convergence Result and LMI

Design the Lyapunov function as following:

$$V_i = x_i^T P_i x_i, \ i = 1, \ldots, N, \tag{3}$$

where $P_i > 0$, $P_i = P_i^T$.

The convergence effect of x can be effectively adjusted by the design of P_i. It is also beneficial to the solution of LMI. Then

$$\begin{aligned}
\dot{V}_i &= \dot{x}_i^T P_i x_i + x_i P_i \dot{x}_i \\
&= (A_i x_i + B_i u_i + B_i d_i)^T P_i x_i + x_i^T P_i (A_i x_i + B_i u_i + B_i d_i) \\
&= (A_i x_i + B_i u_i + B_i d_i)^T P_i x_i + x_i^T P_i (A_i x_i + B_i k_i x_i + B_i d_i) \\
&= x_i^T (A_i + B_i K_i)^T P_i x_i + x_i^T P_i (A_i + B_i K_i) x_i + (B_i d_i)^T P_i x_i + x_i^T P_i (B_i d_i) \\
&= x_i^T (Q_i^1)^T x_i + x_i^T Q_i^1 x_i + (B_i^T P_i x_i + x_i^T P_i B_i) d_i,
\end{aligned}$$

where $Q_i^1 = P_i(A_i + B_i K_i)$, $Q_i = Q_i^1 + (Q_i^1)^T$.

Let $\eta_i = [x^T \; d_i]^T$, then

$$\eta_i = \begin{bmatrix} x_i \\ d_i \end{bmatrix} = \begin{bmatrix} x_i^1 \\ x_i^2 \\ d_i \end{bmatrix}, \quad \eta_i^T = [x_i^1 \; x_i^2 \; d_i]. \tag{4}$$

Hence, we have

$$\dot{V}_i = x_i^T Q_i x_i + \eta_i^T \begin{bmatrix} 0 & P_i B_i \\ B_i^T P_i & 0 \end{bmatrix} \eta_i = \eta_i^T \begin{bmatrix} Q_i & P_i B_i \\ B_i^T P_i & 0 \end{bmatrix} \eta_i, \tag{5}$$

where

$$x_i^T Q_i x_i = [x_i^T \; d_i] \begin{bmatrix} Q_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_i \\ d_i \end{bmatrix} = \eta_i^T \begin{bmatrix} Q_i & 0 \\ 0 & 0 \end{bmatrix} \eta_i, \tag{6}$$

$$\begin{aligned}
(B_i^T P_i x + x_i^T P_i B_i) d_i &= [d_i B_i^T P_i \quad x^T P_i B_i] \begin{bmatrix} x_i \\ d_i \end{bmatrix} \\
&= [x_i^T \; d_i] \begin{bmatrix} 0 & P_i B_i \\ B_i^T P_i & 0 \end{bmatrix} \begin{bmatrix} x_i \\ d_i \end{bmatrix} \\
&= \eta_i^T \begin{bmatrix} 0 & P_i B_i \\ B_i^T P_i & 0 \end{bmatrix} \eta_i,
\end{aligned}$$

then assume that the output is $Z_i = C_i x_i$ and H_∞ index takes

$$\int_0^t Z_i^T Z_i dt < \int_0^t \xi_i^2 d_i^2(t) dt + V(0), \tag{7}$$

where $\xi_i > 0$ and $C_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Since $Z_i^T Z_i - \xi_i^2 d_i^2 = x_i^T C_i^T C_i x_i - \xi_i^2 d_i^2$, and

$$\begin{aligned}
\eta_i^T \begin{bmatrix} C_i^t C_i & 0 \\ 0 & -\xi_i^2 \end{bmatrix} \eta_i &= [x_i^T \quad d_i] \begin{bmatrix} C_i^t C_i & 0 \\ 0 & -\xi_i^2 \end{bmatrix} \begin{bmatrix} x_i \\ d_i \end{bmatrix} \\
&= [x_i^T C_i^T C_i - \xi_i^2 d_i] \begin{bmatrix} x_i \\ d_i \end{bmatrix} \\
&= x_i^T C_i^T C_i x_i - \xi_i^2 d_i^2,
\end{aligned}$$

then

$$Z_i^T Z_i - \xi_i^2 d_i^2 = \eta_i^T \begin{bmatrix} C_i^T C_i & 0 \\ 0 & -\xi_i^2 \end{bmatrix} \eta_i. \tag{8}$$

Moreover, we find that

$$\dot{V}_i + Z_i^T Z_i - \xi_i^2 d_i^2 = \eta_i^T \begin{bmatrix} C_i^T C_i + Q_i & P_i B_i \\ 0 & -\xi_i^2 \end{bmatrix} \eta_i. \tag{9}$$

Supposed that

$$\Theta = \begin{bmatrix} C_i^T C_i + Q_i & P_i B_i \\ 0 & -\xi_i^2 \end{bmatrix} < 0, \tag{10}$$

then

$$\dot{V}_i + Z_i^T Z_i - \xi_i^2 d_i^2 \leq 0. \tag{11}$$

Integral (11) from 0 to t, we get

$$V_i(t) - V_i(0) + \int_0^t Z_i^T Z_i dt \leq \int_0^t \xi_i^2 d_i^2 dt. \tag{12}$$

Assume that d_i is a decreasing disturbance signal, and we choose that $\int_0^\infty d_i^2 dt \leq (\xi_i)_{\max}^{-2} \alpha_i^{max}$, and the definitions of α_i^{max} and $P_i^{\min}$ are similar to those of ν_{max} and P_{min} in [13]. Due to $0 \leq \int_0^t Z_i^T Z_i dt$. It is clear that

$$V_i(t) - V_i(0) \leq (\xi_i)_{\max}^2 \int_0^t d_i^2 dt \leq \alpha_i^{max}, \tag{13}$$

$$V_i(t) \leq \alpha_i^{max} + V_i(0). \tag{14}$$

Thus, we have

$$P_i^{\min} \|x_i\|^2 \leq x_i^T P_i x_i \leq \alpha_i^{max} + V_i(0), \tag{15}$$

and the convergence result is given by

$$\|x_i\|^2 \leq \frac{\alpha_i^{max} + V_i(0)}{P_i^{\min}}. \tag{16}$$

It is not hard to get following theorem.

Theorem 1. *Assume that H_∞ index satisfying*

$$\int_0^t Z_i^T Z_i dt < \int_0^t \xi_i^2 d_i^2(t) dt + V(0), \tag{17}$$

and the output $Z_i = C_i x_i$, where $\xi_i > 0$ and $C_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then

$$\|x_i\|^2 \leq \frac{\alpha_i^{max} + V_i(0)}{P_i^{\min}}. \tag{18}$$

From (10), we know that

$$\begin{bmatrix} P_iA_i + P_iB_iK_i + A_i^TP_i + K^TB_i^TP_i & P_iB_i \\ 0 & -\xi_i^2 \end{bmatrix} < 0, \tag{19}$$

then multiply $\begin{bmatrix} P_i^{-1} & 0 \\ 0 & I \end{bmatrix}$ to the left and right side of (19), we have

$$\begin{bmatrix} A_iP_i^{-1} + B_iK_iP_i^{-1} + P_i^{-1}A_i^T + P_i^{-1}K^TB_i^T & B_i \\ B_i^T & -\xi_i^2 \end{bmatrix} < 0. \tag{20}$$

Let $F_i = K_iP_i^{-1}$, $N = P^{-1}$, it is easy obtain the first LMI,

$$\begin{bmatrix} A_iN_i + B_iF_i + N_iA^T + F_i^TB_i^T & B_i \\ B_i^T & -\xi_i^2 \end{bmatrix} < 0. \tag{21}$$

According to the definition of P_i, we give the second LMI,

$$N_i = N_i^T, N_i > 0. \tag{22}$$

Through the two LMI (21) and (22), we could get K_i.

4 Simulation

We consider the system

$$\ddot{\theta}_1 = -15\dot{\theta}_1 + 113(u_1(t) + d(t)), \tag{23}$$
$$\ddot{\theta}_2 = -20\dot{\theta}_2 + 123(u_2(t) + d(t)), \tag{24}$$
$$\ddot{\theta}_3 = -25\dot{\theta}_3 + 133(u_3(t) + d(t)), \tag{25}$$

according to (1), choose $d(t) = 0.1e^{-6t}$, $\xi_1 = 3, \xi_2 = 5, \xi_3 = 7$ the initial values are $x_i(0) = [0, 01 \quad 0]$, for $i = 1, 2, 3$. And

$$x_1^1 = \theta_1,\ x_1^2 = \dot{\theta}_1, x_2^1 = \theta_2,\ x_2^2 = \dot{\theta}_2, x_3^1 = \theta_3,\ x_3^2 = \dot{\theta}_3, \tag{26}$$

$$b_1 = \frac{15}{113},\ J_1 = \frac{1}{113}, b_2 = \frac{20}{123},\ J_2 = \frac{1}{123}, b_3 = \frac{25}{133},\ J_3 = \frac{1}{133}. \tag{27}$$

$$A_i = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{b_i}{J_i} \end{bmatrix}, B_i = \begin{bmatrix} 0 \\ -\frac{b_i}{J_i} \end{bmatrix},\ i = 1, 2, 3. \tag{28}$$

By using Matlab to solve LMI (21) and (22), we find that

$$K_1 = [-0.0116 \quad 0.1244], K_2 = [-0.0107 \quad 0.1550], K_3 = [-0.0099 \quad 0.1809]. \tag{29}$$

Furthermore, we give the figures (Figs. 1, 2, 3, 4, 5 and 6) of state responds and control input for the system in the following.

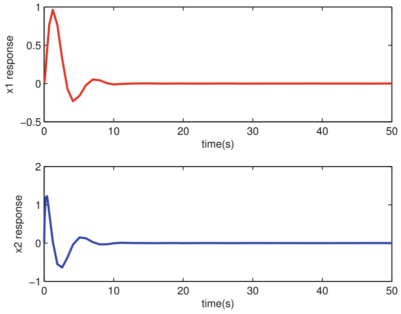

Fig. 1. State responds

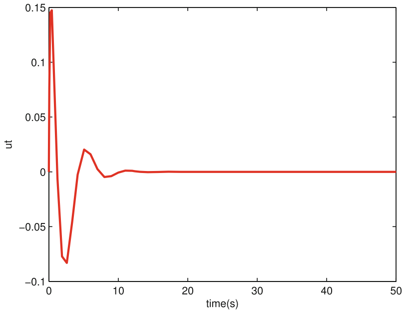

Fig. 2. Control input

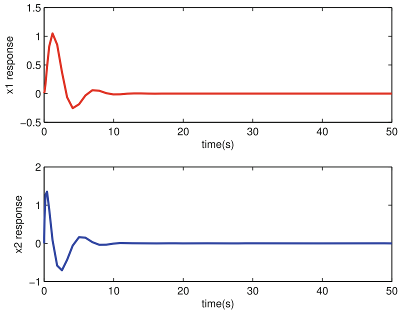

Fig. 3. State responds

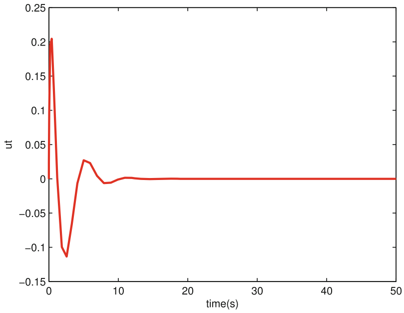

Fig. 4. Control input

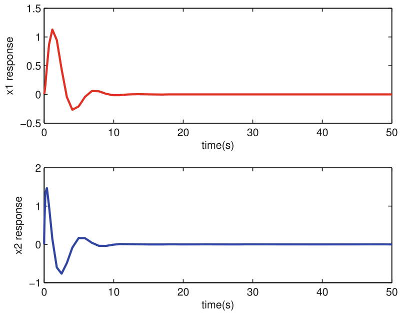

Fig. 5. State responds

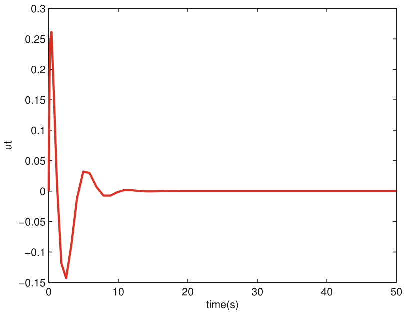

Fig. 6. Control input

5 Conclusion

The establishment of H_∞ control theory has changed the tendency that modern control theory is too mathematical, so it is more convenient for engineers to learn and design. The main mathematical tools has used include modern algebra and operator theory. Whether the design index can be realized or not depends on the accuracy of the mathematical model. However, due to the objective reality, there are various uncertain factors which do not meet the ideal assumptions. Therefore, it is impossible to obtain an accurate mathematical model. This model uncertainty directly affects the application of control theory in practice. H_∞ control is based on system design of uncertain imprecise model. In this paper, by using linear matrix inequality technology, Lyapunov stability theory and H_∞ index, we obtain the convergence result for the multi-agent linear system with disturbance.

References

1. Scherer, C., Weiland, S.: Linear Matrix Inequalities in Control. Lecture Notes, Dutch Institute for Systems and Control, Delft (2000). https://doi.org/10.1201/b10384-28
2. Fadali, M.S., Visioli, A.: Digital Control Engineering. Elsevier, Oxford (2014). https://doi.org/10.1016/B978-0-12-374498-2.00010-2
3. Liu, J.K.: Sliding Mode Control and MATLAB Simulating: The Basic Theory and Design Method. Tsinghua University Press, Beijing (2017)
4. Grimm, G., Hatfield, J., Postlethwaite, I., Teel, A.R., Turner, M.C., Zaccarian, L.: Antiwindup for stable linear systems with input saturation: an LMI-based synthesis. IEEE Trans. Automatic Control **48**(9), 1509–1525 (2003). https://doi.org/10.1109/TAC.2003.816965
5. Hu, T., Teel, A.R., Zaccarian, L.: Stability and performance for saturated systems via quadratic and nonquadratic lyapunov functions. IEEE Trans. Autom. Control **51**(11), 1770–1786 (2006). https://doi.org/10.1109/tac.2006.884942
6. Majumder, S.D.: Flight optimisation of missile using linear matrix inequality (LMI) approach. J. Eng. **2020**(7), 247–250 (2020). https://doi.org/10.1049/joe.2019.0952
7. Dahdah, S., Forbes, J.R.: Linear Matrix Inequality Approaches to Koopman Operator Approximation. arXiv preprint arXiv:2102.03613 (2021)
8. Wang, J., Yang, G., Jian, L.: An LMI approach to H_- index and mixed H_-/H_∞ fault detection observer design. Automatica **43**(9), 1656–1665 (2007). https://doi.org/10.1016/j.automatica.2007.02.019
9. Chen, W., Ding, D., Ge, X., Han, Q., Wei, G.: H_∞ containment control of multi-agent systems under event-triggered communication scheduling: the finite-horizon case. IEEE Trans. Cybern. **99**, 1–11 (2019). https://doi.org/10.1109/TCYB.2018.2885567
10. Li, B., Yang, H.-Y., Chen, Z.-Q., Liu, Z.-X.: Distributed containment control of multi-agent systems with general linear dynamics and time-delays. Int. J. Control Autom. Syst. **16**(6), 2718–2726 (2018). https://doi.org/10.1007/s12555-017-0696-8
11. Yu, D., Ji, X.: Finite-time containment control of perturbed multi-agent systems based on sliding-mode control. Int. J. Syst. Sci. **49**(2), 299–311 (2018). https://doi.org/10.1080/00207721.2017.1406553
12. Zhou, J., Lai, H.L., Men, B.: H_∞ Control for Lur'e singular systems with time delays. Circuits Syst. Signal Process (6), 1–22 (2021). https://doi.org/10.1007/s00034-021-01844-0
13. Liu, J.K.: Robot Control System Design and Matlab Simulation: The Advanced Design Method. Tsinghua University Press, Beijing (2015)