Scattering Lengths for Additive Functionals and Their Semi-classical Asymptotics

Daehong Kim and Masakuni Matsuura

Abstract Scattering lengths for positive additive functionals of symmetric Markov processes are studied. The additive functionals considered here are not necessarily continuous. After giving a systematic presentation of the fundamentals of the scattering length, we study the problems of semi-classical asymptotics for scattering length under relativistic stable processes, which extend previous results for the case of positive continuous additive functionals.

Keywords Additive functionals · Scattering length · Semi-classical asymptotics · Relativistic stable processes

Mathematics Subject Classification Primary: 60J45 · Secondary: 31C15 · 34L25

1 Introduction

There is a notion of scattering length of a positive integrable function V on $\mathbb{R}^3$, one of the important quantities in scattering theory. It is the limit of the scattering amplitude $-f_k(e_y)$ given by

$$f_k(e_y) = -\frac{1}{2\pi}\int_{\mathbb{R}^3} e^{ik\sqrt{2}x\cdot e_y} h_k(x)V(x)\mathrm{d}x, \quad e_y = y/|y|, \; y \in \mathbb{R}^3$$

D. Kim (✉)
Faculty of Advanced Science and Technology, Kumamoto University, Kumamoto 860-8555, Japan
e-mail: daehong@gpo.kumamoto-u.ac.jp

M. Matsuura
National Institute of Technology, Kagoshima College, Kirishima 899-5193, Japan
e-mail: matsuura@kagoshima-ct.ac.jp

Z.-Q. Chen et al. (eds.), *Dirichlet Forms and Related Topics*, Springer Proceedings in Mathematics & Statistics 394, https://doi.org/10.1007/978-981-19-4672-1_14

as the wave number k tends to 0, where h_k is the solution of the scattering problem for V, that is, the solution of the equation $-(1/2)\Delta u + Vu = k^2 u$ having a certain asymptotic behaviour at infinity [8].

In [7, 8], Kac and Luttinger gave some applications of the probabilistic method to scattering problems. As one of such applications, the authors studied the problem of semi-classical asymptotics for scattering length of finite range potentials. To do this, they used a probabilistic expression for the scattering length of V in terms of Brownian motion $\mathbf{X} = (B_t, \mathbf{P}_x)$ on $\mathbb{R}^3$,

$$\Gamma(V) = \frac{1}{2\pi} \lim_{t\to\infty} \frac{1}{t} \int_{\mathbb{R}^3} \mathbf{E}_x \left[1 - e^{-\int_0^t V(B_s)\mathrm{d}s} \right] \mathrm{d}x, \tag{1}$$

and proved the following semi-classical limit: if $V = \mathbf{1}_K$ for a compact subset $K \subset \mathbb{R}^3$ satisfying the so-called Kac's regularity (see Sect. 4 for the definition), then $\uparrow \lim_{p\to\infty} \Gamma(pV) = \mathrm{Cap}(K)$, where Cap denotes the electrostatic capacity. Further, they conjectured that

$$\lim_{p\to\infty} \Gamma(pV) = \mathrm{Cap}(\mathrm{supp}[V]) \tag{2}$$

for *any* positive integrable function V with compact support in $\mathbb{R}^3$ satisfying the regularity as above. The Kac-Luttinger's conjecture (2) was confirmed by Taylor [18, 19] (also by Tamura [16] in an analytic way) who developed the notion of scattering length further into a tool for studying the effectiveness of potential as a perturbation of $-\Delta$ on $\mathbb{R}^d$. For more general framework of symmetric Markov processes, Takahashi [14] gave a new probabilistic representation of the scattering length of a continuous potential which makes the limit (2) quite transparent. For symmetric Markov processes again, Takeda [15] considered the behaviour of the scattering length of a positive smooth measure potential by using the random time change argument for Dirichlet forms and gave a simple elegant proof of the analog of (2) without Kac's regularity. The result in [15] was extended to a non-symmetric case by He [6]. For general right Markov processes, Fitzsimmons, He and Ying [4] extended Takahashi's result by using the tool of Kutznetsov measure and proved the analog of (2) for a positive continuous additive functional.

Scattering lengths cited so far were considered only for positive continuous additive functionals. But, there are many discontinuous additive functionals admitted to Markov processes. Hence, it is a natural question of how to understand the notion of scattering length of additive functionals that are not necessarily continuous. The objective of the present paper is to provide a partial answer to this question. Let E be a locally compact separable metric space and $\mathfrak{m}$ is a positive Radon measure on E with full topological support. Let $\mathbf{X} = (X_t, \mathbf{P}_x)$ be an $\mathfrak{m}$-symmetric Markov process on E. It is natural to consider the following additive functional of the form

$$A_t^{\mu} + \sum_{0<s\le t} F(X_{s-}, X_s) \tag{3}$$

which is not necessarily continuous. Here A_t^{μ} is the positive continuous additive functional of $\mathbf{X}$ with a positive smooth measure μ on E as its Revuz measure and F is a positive bounded Borel function on $E \times E$ vanishing on the diagonal. Let $(N(x, \mathrm{d}y), H_t)$ be a Lévy system for $\mathbf{X}$. For $p \ge 1$, let $\mathbf{F}^{(p)}$ be a non-local linear operator defined by

$$\mathbf{F}^{(p)} f(x) = \int_E \left(1 - e^{-pF(x,y)}\right) f(y) N(x, \mathrm{d}y), \quad x \in E$$

for any bounded measurable function f on E. Put $\mathbf{F}f := \mathbf{F}^{(1)} f$. We assume that $\mathbf{F}^{(p)} 1 \in L^1(E; \mu_H)$ for any $p \ge 1$. Let $U_{\mu+F}$ be the capacitary potential relative to the additive functional (3) defined by

$$U_{\mu+F}(x) := \mathbf{E}_x \left[1 - e^{-A_\infty^{\mu} - \sum_{t>0} F(X_{t-}, X_t)}\right].$$

In this paper, we define the scattering length $\Gamma(\mu + F)$ relative to (3) by

$$\Gamma(\mu + F) := \int_E (1 - U_{\mu+F})(x)\mu(\mathrm{d}x) + \int_E \mathbf{F}(1 - U_{\mu+F})(x)\mu_H(\mathrm{d}x),$$

where μ_H is the Revuz measure of H_t. In Sect. 2, we explain why the expression above is natural for the definition of the scattering length relative to (3). We will also give another expression for the scattering length above, which plays a crucial role throughout this paper (see Lemma 3).

Section 3 is devoted to studying the semi-classical limit of the scattering length. We investigate the behaviour of the scattering length $\Gamma(p\mu + pF)$ when $p \to \infty$. More precisely, let τ_t be the right continuous inverse of the positive continuous additive functional $A_t^{\mu} + \int_0^t \mathbf{F}1(X_s)\mathrm{d}H_s$. Denote by $\mathsf{S}_{\mu+\mu_H\mathbf{F}1}$ the fine support of $A_t^{\mu} + \int_0^t \mathbf{F}1(X_s)\mathrm{d}H_s$,

$$\mathsf{S}_{\mu+\mathbf{F}1\mu_H} = \left\{x \in E \,\middle|\, \mathbf{P}_x(\tau_0 = 0) = 1\right\}.$$

Our first result of this paper is as follows:

Theorem 1 *Suppose that F is symmetric. Further, assume that there exists a positive function $\psi(p)$ satisfying $\psi(p) \le p$, $\psi(p) \to \infty$ as $p \to \infty$ and the non-local operator $\mathbf{F}^{(p)}$ induced by F satisfies the following condition: for large $p \ge 1$ and a constant $C > 0$*

$$\mathbf{F}^{(p)} 1(x) \ge C\psi(p)\mathbf{F}1(x) \quad \textit{for} \;\; x \in E. \tag{4}$$

Then we have

$$\lim_{p\to\infty} \Gamma(p\mu + pF) = \mathrm{Cap}(\mathsf{S}_{\mu+\mathbf{F}1\mu_H}).$$

Note that Theorem 1 is already stated in [10, Theorem 1.1] in the framework of symmetric stable processes. However, we will show that its proof remains valid under general symmetric Markov processes, with the help of Lemmas 3 and 5. In this sense, Theorem 1 can be regarded as a generalization of the result in [15]. We also give some concrete examples of Fs under relativistic stable processes on $\mathbb{R}^d$ (Example 6).

In Sect. 4, we study the problem of semi-classical asymptotics for the scattering length of positive potentials with infinite range. It was proved analytically by Tamura [17] that the scattering length $\Gamma(V)$ of a positive integrable function V induced by 3-dimensional Brownian motion obeys

$$\Gamma(\lambda^{-2}V) \sim \lambda^{-2/(\rho-2)} \tag{5}$$

in the semi-classical limit $\lambda \to 0$, if $V(x)$ behaves like the Hardy type's potential $|x|^{-\rho}$, $\rho > 3$ at infinity. As an application of the result obtained in Sect. 3, we will extend the result (5) probabilistically for the scattering length of positive potentials including a jumping function in the framework of relativistic stable processes. Our second result of the present paper is as follows: let $\mathbf{X}^m = (X_t, \mathbf{P}_x^m)$ be a Lévy process on $\mathbb{R}^d$ with

$$\mathbf{E}_0^m\left[e^{\sqrt{-1}\langle\xi, X_t\rangle}\right] = e^{-t\left((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m\right)}, \quad 0 < \alpha \le 2,\ m \ge 0. \tag{6}$$

The limiting case $\mathbf{X}^0$, corresponding to $m = 0$, is nothing but the usual (rotationally) symmetric α-stable process. Let $\mathbf{F}_m^{(p)}$ be the non-local operator induced by F and $\Gamma_m^{(1)}$ the scattering length with respect to the 1-subprocess $\mathbf{X}^{m,(1)}$ of $\mathbf{X}^m$, respectively.

Theorem 2 *Let $\rho > d > \alpha$ and $0 < \lambda \ll 1$. For a compact set $K \subset \mathbb{R}^d$, let $M > 0$ be such that $K \subset B(0, M)$. For some constants $c_1, c_2 > 0$, let V be a positive function on $\mathbb{R}^d$ and F a positive bounded symmetric function on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal satisfying $V(x) \le c_1|x|^{-\rho}$ for $x \in B(0,\ \lambda^{-\alpha/(\rho-\alpha)})^c$ and*

$$F(x, y) \le c_2|x - y|^{\alpha-\rho}\mathbf{1}_{B(x,\lambda^{-\alpha/(\rho-\alpha)})^c \cap \lambda^{-\alpha/(\rho-\alpha)}K}(y)$$

for $x \in B(0, \lambda^{-\alpha/(\rho-\alpha)}M)^c$, respectively. Here $\lambda^{-\alpha/(\rho-\alpha)}K := \{\lambda^{-\alpha/(\rho-\alpha)}x \mid x \in K\}$. If there exists a positive function ψ satisfying $\psi(\sigma) \le \sigma$, $\psi(\sigma) \to \infty$ as $\sigma \to \infty$ and

$$\mathbf{F}_m^{(\lambda^{-\alpha})}1(x) \ge C\psi(\lambda^{-\alpha})\mathbf{F}_m 1(x) \quad \text{for } x \in B(0, \lambda^{-\alpha/(\rho-\alpha)}M),\ C > 0,$$

then we have for any $m \geq 0$

$$C_1 \lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \leq \Gamma_m^{(1)}\left(\lambda^{-\alpha} V + \lambda^{-\alpha} F\right) \leq C_2 \lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}}$$

for some constants $C_2 > C_1 > 0$.

We note that Theorem 2 is not only the extension of the result (5) (or [11, Theorem 1.1]) but also it provides us with a new semi-classical asymptotic order of the scattering length for a jumping potential with infinite range under a purely discontinuous Markov process.

Throughout this paper, we use c, C, c', C', c_i, C_i $(i = 1, 2, \dots)$ as positive constants which may be different at different occurrences. For notational convenience, we let $a \vee b := \max\{a, b\}$ for any $a, b \in \mathbb{R}$.

2 Scattering Length for Additive Functionals

Let E be a locally compact separable metric space and $\mathfrak{m}$ is a positive Radon measure on E with full topological support. Let ∂ be a point added to E so that $E_\partial := E \cup \{\partial\}$ is the one-point compactification of E. The point ∂ also serves as the cemetery point for E. Let $\mathbf{X} = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \mathbf{P}_x, \zeta)$ be an $\mathfrak{m}$-symmetric *transient* Hunt process on E, where ζ is the lifetime of X, $\zeta = \inf\{t > 0 \mid X_t = \partial\}$. We assume that $\mathbf{X}$ is conservative, that is, $\mathbf{P}_x(\zeta = \infty) = 1$ for any $x \in E$. Here and in the sequel, unless mentioned otherwise, we use the convention that a function defined on E takes the value 0 at ∂. Let $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ be the Dirichlet form of $\mathbf{X}$ on $L^2(E; \mathfrak{m})$ which is assumed to be regular.

Let Cap be the (0-)capacity associated with the Dirichlet form $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$ of $\mathbf{X}$, that is, for an open set $O \subset E$ and the extended Dirichlet space $\mathscr{D}_e(\mathcal{E})$ of $\mathscr{D}(\mathcal{E})$,

$$\mathrm{Cap}(O) = \inf\{\mathcal{E}(u, u) \mid u \in \mathscr{D}_e(\mathcal{E}), u \geq 1 \ \mathfrak{m}\text{-a.e. on } O\} \tag{7}$$

and for a Borel set $B \subset E$,

$$\mathrm{Cap}(B) = \inf\{\mathrm{Cap}(O) \mid O \text{ is open}, \ B \subset O\} \tag{8}$$

(see [5, Chap. 2]).

We say that a positive continuous additive functional (PCAF in abbreviation) A_t^ν of $\mathbf{X}$ and a smooth measure ν are in the Revuz correspondence if they satisfy for any $t > 0$,

$$\int_E f(x)\nu(\mathrm{d}x) = \uparrow \lim_{t \downarrow 0} \frac{1}{t} \mathbf{E}_{\mathfrak{m}}\left[\int_0^t f(X_s)\, \mathrm{d}A_s^\nu\right], \quad f \in \mathscr{B}_b(E). \tag{9}$$

Here $\mathbf{E}_{\mathfrak{m}}[\,\cdot\,] = \int_E \mathbf{E}_x[\,\cdot\,]\mathfrak{m}(\mathrm{d}x)$ and $\mathscr{B}_b(E)$ is the space of bounded Borel functions on E. It is known that the family of equivalence classes of the set of PCAFs and the family of smooth measures are in one to one correspondence under the Revuz correspondence [5, Theorem 5.1.4]. Let $(N(x,\mathrm{d}y), H_t)$ be a Lévy system for $\mathbf{X}$, that is, $N(x,\mathrm{d}y)$ is a kernel on $(E,\mathscr{B}(E))$ and H_t is a PCAF with bounded 1-potential such that for any non-negative Borel function ϕ on $E\times E$ vanishing on the diagonal and any $x\in E$,

$$\mathbf{E}_x\left[\sum_{0<s\le t}\phi(X_{s-},X_s)\right] = \mathbf{E}_x\left[\int_0^t\int_E \phi(X_s,y)N(X_s,\mathrm{d}y)\,\mathrm{d}H_s\right].$$

Let μ_H be the Revuz measure of the PCAF H_t. Then the jumping measure J and the killing measure κ of $\mathbf{X}$ are given by $J(\mathrm{d}x\mathrm{d}y) = \frac{1}{2}N(x,\mathrm{d}y)\mu_H(\mathrm{d}x)$ and $\kappa(\mathrm{d}x) = N(x,\{\partial\})\mu_H(\mathrm{d}x)$. These measures feature in the Beurling-Deny decomposition of $\mathcal{E}$ [5, Theorem 3.2.1].

A non-negative Borel measure ν on E (resp. a non-negative symmetric Borel function ϕ on $E\times E$ vanishing on the diagonal) is said to be Green-bounded relative to $\mathbf{X}$ if

$$\sup_{x\in E}\mathbf{E}_x\left[A^{\nu}_{\infty}\right]<\infty,\quad \left(\text{resp.}\quad \sup_{x\in E}\mathbf{E}_x\left[\sum_{t>0}\phi(X_{t-},X_t)\right]<\infty\right).$$

Let μ be a positive smooth measure on E and A^{μ}_t the PCAF of $\mathbf{X}$ with μ as its Revuz measure. Let $F(x,y)$ be a bounded positive Borel function on $E\times E$ vanishing along the diagonal. Then $\sum_{0<s\le t}F(X_{s-},X_s)$ is a positive (discontinuous) additive functional of $\mathbf{X}$. Throughout this section, we assume that F is Green-bounded relative to $\mathbf{X}$. It is natural to consider a combination of the additive functionals of the form

$$A^{\mu}_t + \sum_{0<s\le t}F(X_{s-},X_s) \tag{10}$$

because the process $\mathbf{X}$ admits many discontinuous additive functionals. For $p\ge 1$, let $\mathbf{F}^{(p)}$ be a non-local linear operator defined by

$$\mathbf{F}^{(p)}f(x) = \int_E \left(1-e^{-pF(x,y)}\right)f(y)N(x,\mathrm{d}y),\quad x\in E \tag{11}$$

for any $f\in\mathscr{B}_b(E)$. Put $\mathbf{F}f := \mathbf{F}^{(1)}f$. We assume that $\mathbf{F}^{(p)}1\in L^1(E;\mu_H)$ for any $p\ge 1$. Let $U_{\mu+F}$ be the capacitary potential relative to the additive functional (10) defined by

$$U_{\mu+F}(x) := \mathbf{E}_x\left[1-e^{-A^{\mu}_{\infty}-\sum_{t>0}F(X_{t-},X_t)}\right].$$

We shall define the *scattering length* $\Gamma(\mu + F)$ relative to the additive functional (10) by

$$\Gamma(\mu + F) := \int_E (1 - U_{\mu+F})(x)\mu(\mathrm{d}x) + \int_E \mathbf{F}(1 - U_{\mu+F})(x)\mu_H(\mathrm{d}x). \tag{12}$$

Let us explain intuitively why the expression (12) is natural for the definition of the scattering length relative to (10). Let $\mathcal{L}$ be the the infinitesimal generator associated with $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$: $\mathcal{E}(f, g) = (\sqrt{-\mathcal{L}}f, \sqrt{-\mathcal{L}}g)_{\mathfrak{m}}$ and $\mathscr{D}(\mathcal{E}) = \mathscr{D}(\sqrt{-\mathcal{L}})$. In analogy with classical one, we define $\Gamma(\mu + F)$ by the total mass of $-\mathcal{L}\, U_{\mu+F}$,

$$\Gamma(\mu + F) = -\int_E \mathcal{L}\, U_{\mu+F}\, \mathrm{d}\mathfrak{m}. \tag{13}$$

Note that the capacitary potential $U_{\mu+F}$ satisfies the following formal equation

$$-\mathcal{L}\, U_{\mu+F} = (1 - U_{\mu+F})\mu + \mathbf{F}1\mu_H - \mathbf{F}U_{\mu+F}\mu_H. \tag{14}$$

Indeed, let $\widetilde{\mathbf{X}} = (X_t, \widetilde{\mathbf{P}}_x)$ be the transformed process of $\mathbf{X}$ by the pure jump Girsanov transform

$$Y_t^F := \exp\left(-\sum_{0<s\le t} F(X_{s-}, X_s) + \int_0^t \mathbf{F}1(X_s)\mathrm{d}H_s\right), \quad t \in (0, \infty). \tag{15}$$

The multiplicative functional (15) is a uniformly integrable martingale under the Green-boundedness of F relative to $\mathbf{X}$, because $e^{-F} - 1 \ge \delta - 1$ for some $\delta > 0$ by the boundedness of F and

$$\begin{aligned}
&\sup_{x\in E} \mathbf{E}_x\left[\int_0^\infty\int_E \left(1 - e^{-F(X_s,y)}\right)^2 N(x, \mathrm{d}y)\mathrm{d}H_s\right] \\
&\qquad \le \sup_{x\in E} \mathbf{E}_x\left[\int_0^\infty\int_E F(X_s, y)N(x, \mathrm{d}y)\mathrm{d}H_s\right] \\
&\qquad = \sup_{x\in E} \mathbf{E}_x\left[\sum_{s>0} F(X_{s-}, X_s)\right] < \infty
\end{aligned}$$

(cf. [1, Theorem 3.2]). From this fact, we see that the transformed process $\widetilde{\mathbf{X}}$ is also a transient and conservative Markov process on E. Let $\widetilde{\mathcal{L}}$ be the infinitesimal generator of $\widetilde{\mathbf{X}}$. Then $\widetilde{\mathcal{L}}$ is formally given by

$$-\widetilde{\mathcal{L}} = -\mathcal{L} + \mu_H \mathbf{F} - \mathbf{F}1\mu_H, \tag{16}$$

where $\mu_H \mathbf{F}$ denotes the measure valued operator defined by $\mu_H \mathbf{F} f(x) = \mathbf{F} f(x)\mu_H(\mathrm{d}x)$. It is known that a PCAF of $\mathbf{X}$ can be regarded as a PCAF of $\widetilde{\mathbf{X}}$. Thus we see from [9, Lemma 4.9] that

$$\begin{aligned} U_{\mu+F}(x) &= \mathbf{E}_x\left[1 - e^{-A^{\mu}_{\infty} - \sum_{t>0} F(X_{t-},X_t)}\right] \\ &= \widetilde{\mathbf{E}}_x\left[1 - e^{-A^{\mu}_{\infty} - \int_0^{\infty} \mathbf{F}1(X_t)\mathrm{d}H_t}\right] \\ &= \widetilde{\mathbf{E}}_x\left[\int_0^{\infty} e^{-A^{\mu}_t - \int_0^t \mathbf{F}1(X_s)\mathrm{d}H_s}\left(\mathrm{d}A^{\mu}_t + \mathbf{F}1(X_t)\mathrm{d}H_t\right)\right]. \end{aligned} \tag{17}$$

Equation (17) implies that $U_{\mu+F}$ satisfies the following formal equation

$$\left(\mu + \mathbf{F}1\mu_H - \widetilde{\mathcal{L}}\right) U_{\mu+F} = \mu + \mathbf{F}1\mu_H. \tag{18}$$

Hence we have (14) by applying (16)–(18), in other words, the total mass of $-\mathcal{L}\, U_{\mu+F}$ is given as the right-hand side of (12). We note that the relation (14) is rigorously established whenever $U_{\mu+F} \in L^2(E; \mathfrak{m})$.

The following expressions of the scattering length play a crucial role throughout this paper.

Lemma 3 *Suppose that F is symmetric. Then, the scattering length* (12) *can be rewritten as*

$$\Gamma(\mu + F) = \int_E \mathbf{E}_x\left[e^{-A^{\mu}_{\infty} - \sum_{t>0} F(X_{t-},X_t)}\right](\mu(\mathrm{d}x) + \mathbf{F}1(x)\mu_H(\mathrm{d}x)). \tag{19}$$

$$= \lim_{t\to\infty} \frac{1}{t} \int_E \mathbf{E}_x\left[1 - e^{-A^{\mu}_t - \sum_{0<s\le t} F(X_{s-},X_s)}\right]\mathfrak{m}(\mathrm{d}x). \tag{20}$$

Proof The expression (19) is a consequence of the symmetry of F. Indeed,

$$\begin{aligned} &\int_E \mathbf{F}(1 - U_{\mu+F})(x)\mu_H(\mathrm{d}x) \\ &\quad = \int_E \int_E \left(1 - e^{-F(x,y)}\right) \mathbf{E}_y\left[e^{-A^{\mu}_{\infty} - \sum_{t>0} F(X_{t-},X_t)}\right] N(x, \mathrm{d}y)\mu_H(\mathrm{d}x) \\ &\quad = \int_E \mathbf{E}_x\left[e^{-A^{\mu}_{\infty} - \sum_{t>0} F(X_{t-},X_t)}\right] \mathbf{F}1(x)\mu_H(\mathrm{d}x). \end{aligned}$$

On the other hand, it follows from [9, Lemma 4.9] and (19) that

$$
\begin{aligned}
\Gamma(\mu + F) &= \int_E \widetilde{\mathbf{E}}_x \left[e^{-A^\mu_\infty - \int_0^\infty \mathbf{F}1(X_s)\mathrm{d}H_s} \right] (\mu(\mathrm{d}x) + \mathbf{F}1(x)\mu_H(\mathrm{d}x)) \\
&= \lim_{t\to\infty} \frac{1}{t} \int_E \widetilde{\mathbf{E}}_x \left[1 - e^{-A^\mu_t - \int_0^t \mathbf{F}1(X_s)\mathrm{d}H_s} \right] \mathfrak{m}(\mathrm{d}x) \\
&= \lim_{t\to\infty} \frac{1}{t} \int_E \mathbf{E}_x \left[1 - e^{-A^\mu_t - \sum_{0<s\le t} F(X_{s-}, X_s)} \right] \mathfrak{m}(\mathrm{d}x)
\end{aligned}
$$

which implies the expression (20). In the second equality above, we used the result due to [15, (2.2)] (also [4, Theorem 2.2]). □

In the rest of the paper, we always assume that F is symmetric. It is immediate from the expression (20) that the scattering length $\Gamma(\mu + F)$ has the monotone property: for $i = 1, 2$, let μ_i be a non-negative finite smooth measure on E and F_i be a non-negative symmetric bounded Borel function on $E \times E$ vanishing on the diagonal such that $\mathbf{F}_{(i)}1 \in L^1(E; \mu_H)$, where $\mathbf{F}_{(i)}$ is the non-local operator defined as in (11) for F_i. If $\mu_1 \le \mu_2$ and $F_1 \le F_2$, then

$$
\Gamma(\mu_1 + F_1) \le \Gamma(\mu_2 + F_2). \tag{21}
$$

Moreover, it follows from the elementary inequality $1 - e^{-a-b} \le (1 - e^{-a}) + (1 - e^{-b})$ for $a, b \ge 0$ that the scattering length has the subadditive property:

$$
\Gamma(\mu_1 + \mu_2) \le \Gamma(\mu_1) + \Gamma(\mu_2), \quad \Gamma(F_1 + F_2) \le \Gamma(F_1) + \Gamma(F_2).
$$

Finally, we close this section with the following remark:

Remark 4 The scattering length is trivial when the underlying process $\mathbf{X}$ is not transient. In fact, by virtue of [4, Lemma 2.1], the present scattering length $\Gamma(\mu + F)$ can be represented as

$$
\begin{aligned}
\Gamma(\mu + F) &= \int_{\{U_{\mu+F}=0\}\cup\{U_{\mu+F}=1\}} (1 - U_{\mu+F})(x)(\mu + \mathbf{F}1\mu_H)(\mathrm{d}x) \\
&= \int_E \mathbf{1}_{\{U_{\mu+F}=0\}}(x)(\mu + \mathbf{F}1\mu_H)(\mathrm{d}x)
\end{aligned}
$$

under the non-transience of $\mathbf{X}$. Then, by the Revuz formula (9) and the Markov property

$$\begin{aligned}
&\int_E \mathbf{1}_{\{U_{\mu+F}=0\}}(x)(\mu+\mathbf{F}1\mu_H)(\mathrm{d}x) \\
&=\uparrow \lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_{\mathfrak{m}}\left[\int_0^t \mathbf{1}_{\{U_{\mu+F}=0\}}(X_s)\,\mathrm{d}A_s^{\mu+\mathbf{F}1\mu_H}\right] \\
&=\uparrow \lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_{\mathfrak{m}}\left[\int_0^t \mathbf{E}_{X_s}\left[\mathbf{1}_{\{A_\infty^\mu+\sum_{t>0}F(X_{t-},X_t)=0\}}\right]\mathrm{d}A_s^{\mu+\mathbf{F}1\mu_H}\right] \\
&=\uparrow \lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_{\mathfrak{m}}\left[\int_0^t \mathbf{1}_{\{A_\infty^\mu+\sum_{t>0}F(X_{t-},X_t)=A_s^\mu+\sum_{0<u\le s}F(X_{u-},X_u)\}}\mathrm{d}A_s^{\mu+\mathbf{F}1\mu_H}\right] \\
&=\uparrow \lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_{\mathfrak{m}}\left[\int_0^t \mathbf{1}_{\left\{A_\infty^{\mu+\mathbf{F}1\mu_H}=A_s^{\mu+\mathbf{F}1\mu_H}\right\}}\mathrm{d}A_s^{\mu+\mathbf{F}1\mu_H}\right] \\
&=0.
\end{aligned}$$

This is the reason why we only consider the scattering length for transient processes.

3 Kac's Scattering Length Formula

In this section, we are going to study the behaviour of the scattering length $\Gamma(p\mu+pF)$ when $p\to\infty$. As we mentioned in Sect. 1, this problem was decisively solved in the case $F\equiv 0$ by Takeda [15], through the random time change argument for Dirichlet forms: let ν be a positive finite smooth measure on E and S_ν the fine support of A_t^ν. Then

$$\lim_{p\to\infty}\Gamma(p\nu)=\mathrm{Cap}(\mathsf{S}_\nu). \tag{22}$$

However, we cannot apply time change method to our problem directly because our scattering length contains a discontinuous additive functional.

Let τ_t be the right continuous inverse of the PCAF $A_t^\mu+\int_0^t\mathbf{F}1(X_s)\mathrm{d}H_s$, that is, $\tau_t:=\inf\{s>0\mid A_s^\mu+\int_0^s\mathbf{F}1(X_u)\mathrm{d}H_u>t\}$. Let denote by $\mathsf{S}_{\mu+\mu_H\mathbf{F}1}$ the fine support of $A_t^\mu+\int_0^t\mathbf{F}1(X_s)\mathrm{d}H_s$,

$$\mathsf{S}_{\mu+\mathbf{F}1\mu_H}=\left\{x\in E\,\middle|\,\mathbf{P}_x(\tau_0=0)=1\right\}.$$

To prove Theorem 1, we need to the following lemma.

Lemma 5 *For any* $\varepsilon > 0$

$$\lim_{p\to\infty} \Gamma\left(pF + p^{1+\varepsilon}\mu + p^{1+\varepsilon}\mathbf{F}1\mu_H\right) = \mathrm{Cap}(\mathsf{S}_{\mu+\mathbf{F}1\mu_H}).$$

In particular, $\limsup_{p\to\infty} \Gamma(p\mu + pF) \le \mathrm{Cap}(\mathsf{S}_{\mu+\mathbf{F}1\mu_H})$.

Proof The last assertion easily follows from the first one with the monotonicity of the scattering length. Put $k = 1/(1+\varepsilon)$ and write $A_t^{\mathbf{F}1\mu_H} := \int_0^t \mathbf{F}1(X_s)\mathrm{d}H_s$. From the expression (19), one can easily see that

$$\begin{aligned}\Gamma&\left(p^k F + p\mu + p\mathbf{F}1\mu_H\right)\\ &= \int_E \mathbf{E}_x\left[e^{-p^k\sum_{t>0}F(X_{t-},X_t) - pA_\infty^\mu - pA_\infty^{\mathbf{F}1\mu_H}}\right]\\ &\quad\cdot\left(p\mu(\mathrm{d}x) + \mathbf{F}^{(p^k)}1(x)\mu_H(\mathrm{d}x) + p\mathbf{F}1(x)\mu_H(\mathrm{d}x)\right).\end{aligned}$$

Since $\mathbf{F}^{(q)}1 \le q\mathbf{F}1$ for any $q \ge 1$,

$$\begin{aligned}\Gamma&\left(p^k F + p\mu + p\mathbf{F}1\mu_H\right)\\ &\le \int_E \mathbf{E}_x\left[e^{-pA_\infty^\mu - pA_\infty^{\mathbf{F}1\mu_H}}\right]\left(\left(1+p^{k-1}\right)p\mu(\mathrm{d}x) + \left(p^k\mathbf{F}1 + p\mathbf{F}1\right)(x)\mu_H(\mathrm{d}x)\right)\\ &= \left(1+p^{k-1}\right)\Gamma\left(p\mu + p\mathbf{F}1\mu_H\right).\end{aligned}$$

Therefore we have from the monotonicity of the scattering length that

$$\Gamma\left(p\mu + p\mathbf{F}1\mu_H\right) \le \Gamma\left(p^k F + p\mu + p\mathbf{F}1\mu_H\right) \le \left(1+p^{k-1}\right)\Gamma\left(p\mu + p\mathbf{F}1\mu_H\right).$$

In view of (22), the scattering lengths of both sides of the above converge to $\mathrm{Cap}(\mathsf{S}_{\mu+\mathbf{F}1\mu_H})$ as $p \to \infty$, which implies the first assertion. □

Proof of Theorem 1 Let $\psi(p)$ be the function which appeared in the condition (4). By the monotonicity of the scattering length, we have for some $C > 0$

$$\begin{aligned}\Gamma&\left(\frac{\psi(p)}{n}\mu + \frac{C\psi(p)}{n}\mathbf{F}1\mu_H\right)\\ &\le \Gamma\left(pF + \frac{\psi(p)}{n}\mu + \frac{C\psi(p)}{n}\mathbf{F}1\mu_H\right) \le \Gamma\left(pF + p^{1+\varepsilon}\mu + p^{1+\varepsilon}\mathbf{F}1\mu_H\right)\end{aligned}$$

for any $n \ge 1$ and $\varepsilon > 0$. Then, by Lemma 5 and applying (22) again, one can get that

$$\lim_{p\to\infty} \Gamma\left(pF + \frac{\psi(p)}{n}\mu + \frac{C\psi(p)}{n}\mathbf{F}1\mu_H\right) = \mathrm{Cap}(\mathsf{S}_{\mu+\mathbf{F}1\mu_H}).$$

From this and the condition (4), we see that

$$
\begin{aligned}
&\liminf_{p\to\infty} \Gamma\,(p\mu + pF) \geq \liminf_{p\to\infty} \Gamma\left(\frac{\psi(p)}{n}\mu + pF\right)\\
&= \liminf_{p\to\infty} \int_E \mathbf{E}_x\left[e^{-p\sum_{t>0} F(X_{t-},X_t) - \frac{\psi(p)}{n}A^{\mu}_{\infty}}\right]\left(\frac{\psi(p)}{n}\mu(\mathrm{d}x) + \mathbf{F}^{(p)}1(x)\mu_H(\mathrm{d}x)\right)\\
&= \frac{n}{n+1}\liminf_{p\to\infty} \int_E \mathbf{E}_x\left[e^{-p\sum_{t>0} F(X_{t-},X_t) - \frac{\psi(p)}{n}A^{\mu}_{\infty}}\right]\\
&\qquad\qquad \cdot\left(\frac{n+1}{n}\frac{\psi(p)}{n}\mu(\mathrm{d}x) + \mathbf{F}^{(p)}1(x)\mu_H(\mathrm{d}x) + \frac{1}{n}\mathbf{F}^{(p)}1(x)\mu_H(\mathrm{d}x)\right)\\
&\geq \frac{n}{n+1}\liminf_{p\to\infty} \int_E \mathbf{E}_x\left[e^{-p\sum_{t>0} F(X_{t-},X_t) - \frac{\psi(p)}{n}A^{\mu}_{\infty} - \frac{C\psi(p)}{n}A^{\mathbf{F}1\mu_H}_{\infty}}\right]\\
&\qquad\qquad \cdot\left(\mathbf{F}^{(p)}1(x)\mu_H(\mathrm{d}x) + \frac{\psi(p)}{n}\mu(\mathrm{d}x) + \frac{C\psi(p)}{n}\mathbf{F}1(x)\mu_H(\mathrm{d}x)\right)\\
&= \frac{n}{n+1}\lim_{p\to\infty} \Gamma\left(pF + \frac{\psi(p)}{n}\mu + \frac{C\psi(p)}{n}\mathbf{F}1\mu_H\right)\\
&= \frac{n}{n+1}\mathrm{Cap}(\mathsf{S}_{\mu+\mathbf{F}1\mu_H}).
\end{aligned}
$$

Letting $n \to \infty$, we have

$$
\mathrm{Cap}(\mathsf{S}_{\mu+\mathbf{F}1\mu_H}) \leq \liminf_{p\to\infty} \Gamma(p\mu + pF). \tag{23}
$$

The proof will be finished by the last assertion of Lemma 5 and (23). $\square$

Let $\mathbf{X}^m = (X_t, \mathbf{P}^m_x)$ be a Lévy process on $\mathbb{R}^d$ with

$$
\mathbf{E}^m_0\left[e^{\sqrt{-1}\langle \xi, X_t\rangle}\right] = e^{-t\left((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m\right)}, \quad 0 < \alpha \leq 2,\ m \geq 0. \tag{24}
$$

If $m > 0$ and $0 < \alpha < 2$, it is called the relativistic α-stable process with mass m. In particular, if $m > 0$ and $\alpha = 1$, it is called the relativistic free Hamiltonian process. The limiting case $\mathbf{X}^0$, corresponding to $m = 0$, is nothing but the usual (rotationally) symmetric α-stable process. It is known that $\mathbf{X}^m$ is transient if and only if $d > 2$ under $m > 0$ or $d > \alpha$ under $m = 0$, and it is a doubly Feller conservative process. From (24), one can easily see that $\mathbf{X}^m$ has the following scaling property: for any $r > 0$

$$
(rX_t, \mathbf{P}^m_x) \overset{d}{=} \left(X_{r^{\alpha}t}, \mathbf{P}^{r^{-\alpha}m}_{rx}\right), \tag{25}
$$

where $\stackrel{d}{=}$ means the equality in distribution. Let $(\mathcal{E}^m, \mathscr{D}(\mathcal{E}^m))$ be the Dirichlet form on $L^2(\mathbb{R}^d)$ associated with $\mathbf{X}^m$. It follows from Fourier transform $\hat{f}(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\langle x,y\rangle} f(y)\mathrm{d}y$ that

$$\mathscr{D}(\mathcal{E}^m) := \left\{ f \in L^2(\mathbb{R}^d) \;\middle|\; \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \left((|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m\right) \mathrm{d}\xi < \infty \right\},$$

$$\mathcal{E}^m(f, g) := \int_{\mathbb{R}^d} \hat{f}(\xi)\bar{\hat{g}}(\xi) \left((|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m\right) \mathrm{d}\xi \quad \text{for } f, g \in \mathscr{D}(\mathcal{E}^m)$$

[5, Example 1.4.1]. The Dirichlet form $(\mathcal{E}^0, \mathscr{D}(\mathcal{E}^0))$ for $\mathbf{X}^0$ can also be characterized similarly, only with $|\xi|^\alpha$ in place of $(|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m$ above. Thus, there exist positive constants $c_1 := c_1(m)$ and $c_2 := c_2(m)$ such that

$$c_1 \mathcal{E}_1^0(u, u) \le \mathcal{E}_1^m(u, u) \le c_2 \mathcal{E}_1^0(u, u)$$

and so $\mathscr{D}(\mathcal{E}^m) = \mathscr{D}(\mathcal{E}^0)$. Here $\mathcal{E}_1^*(u, u) := \mathcal{E}^*(u, u) + (u, u)_{\mathfrak{m}}$. From this, we see that for any $m \ge 0$ and a Borel set $B \subset \mathbb{R}^d$

$$c_1 \mathrm{Cap}^{(1)}(B) \le \mathrm{Cap}_m^{(1)}(B) \le c_2 \mathrm{Cap}^{(1)}(B), \tag{26}$$

where $\mathrm{Cap}_m^{(1)}$ (resp. $\mathrm{Cap}^{(1)}$) denotes the 1-capacity associated with $(\mathcal{E}^m, \mathscr{D}(\mathcal{E}^m))$ (resp. $(\mathcal{E}^0, \mathscr{D}(\mathcal{E}^0))$), that is, it is the capacity defined by replacing $\mathscr{D}_e(\mathcal{E}^m)$ and $\mathcal{E}^m$ (resp. $\mathscr{D}_e(\mathcal{E}^0)$ and $\mathcal{E}^0$) in (7) and (8) with $\mathscr{D}(\mathcal{E}^m)$ and $\mathcal{E}_1^m$ (resp. $\mathscr{D}(\mathcal{E}^0)$ and $\mathcal{E}_1^0$). Let denote by $B(a, b)$ the open ball in $\mathbb{R}^d$ with center a and radius b. It is known that

$$\mathrm{Cap}^{(1)}(B(0, r)) = r^{d-\alpha} \mathrm{Cap}^{(1)}(B(0, 1)) \tag{27}$$

(cf. [12, (42.22)]). It is shown in [3] that the corresponding jumping measure J of $(\mathcal{E}^m, \mathcal{F}^m)$ satisfies

$$J(\mathrm{d}x\mathrm{d}y) = J_m(x, y)\mathrm{d}x\mathrm{d}y \quad \text{with} \quad J_m(x, y) = C_{d,\alpha} \frac{\varphi(m^{1/\alpha}|x - y|)}{|x - y|^{d+\alpha}},$$

where $C_{d,\alpha} = \frac{\alpha 2^{d+\alpha} \Gamma(\frac{d+\alpha}{2})}{2^{d+1}\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}$ and

$$\varphi(r) := 2^{-(d+\alpha)} \Gamma\left(\frac{d+\alpha}{2}\right)^{-1} \int_0^\infty s^{\frac{d+\alpha}{2}-1} e^{-\frac{s}{4}-\frac{r^2}{s}} \mathrm{d}s,$$

which is a decreasing function satisfying $\varphi(0) = 1$ and

$$c^{-1}e^{-r}r^{\frac{d+\alpha-1}{2}} \le \varphi(r) \le ce^{-r}r^{\frac{d+\alpha-1}{2}}, \quad r \ge 1 \tag{28}$$

for some constant $c > 1$ (cf. [3]). In particular,

$$\mathscr{D}(\mathcal{E}^m) = \left\{ f \in L^2(\mathbb{R}^d) \;\middle|\; \int_{\mathbb{R}^d \times \mathbb{R}^d} |f(x) - f(y)|^2 J_m(x, y)\mathrm{d}x\mathrm{d}y < \infty \right\},$$
$$\mathcal{E}^m(f, g) := \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (f(x) - f(y))(g(x) - g(y)) J_m(x, y)\mathrm{d}x\mathrm{d}y$$

for $f, g \in \mathscr{D}(\mathcal{E}^m)$. It is known that $\mathbf{X}^m$ has a Lévy system $(N(x, \mathrm{d}y), H_t)$ given by $N(x, \mathrm{d}y) = J_m(x, y)\mathrm{d}y$ and $H_t = t$. In this case, the non-local linear operator (11) for a symmetric positive bounded Borel function $F(x, y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal is given by

$$\mathbf{F}_m^{(p)} f(x) = \int_{\mathbb{R}^d} \frac{(1 - e^{-pF(x,y)})f(y)\varphi(m^{1/\alpha}|x - y|)}{|x - y|^{d+\alpha}} \mathrm{d}y, \quad p \ge 1, \; x \in \mathbb{R}^d.$$

Now, we give some concrete examples of Fs satisfying the condition (4).

Example 6 Let F be the function on $\mathbb{R}^d \times \mathbb{R}^d$ such that for $\beta > \alpha$

$$F(x, y) = \frac{1}{2}|x - y|^\beta \chi_{R,R'}(x, y),$$

where $\chi_{R,R'}(x, y)$ is the indicator function given by

$$\begin{aligned} &\chi_{R,R'}(x, y) \\ &= \big(\mathbf{1}_{B(x,R')}(y)\mathbf{1}_{B(0,R)}(x) + \mathbf{1}_{B(y,R')}(x)\mathbf{1}_{B(0,R)}(y) \\ &\quad + \mathbf{1}_{B(y,R')}(x)\mathbf{1}_{B(x,R')}(y)\mathbf{1}_{B(0,R+R')\setminus B(0,R)}(x)\mathbf{1}_{B(0,R+R')\setminus B(0,R)}(y)\big) \end{aligned}$$

for $R, R' > 0$. Then the condition (4) holds for this F. First, take $x \in B(0, R)$. In this case, F is given by

$$F(x, y) = \begin{cases} |x - y|^\beta & y \in B(x, R') \cap B(0, R) \\ \frac{1}{2}|x - y|^\beta & y \in B(x, R') \cap B(0, R)^c \\ 0 & \text{otherwise} \end{cases}$$

and thus we have

$$
\begin{aligned}
\mathbf{F}_m 1(x) &= C_{d,\alpha} \int\limits_{\mathbb{R}^d} \frac{(1-e^{-F(x,y)})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y \\
&= C_{d,\alpha} \int\limits_{B(x,R')} \frac{(1-e^{-F(x,y)})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y \\
&= C_{d,\alpha} \left\{ \int\limits_{B(x,R')\cap B(0,R)} \frac{(1-e^{-|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y \right. \\
&\qquad \left. + \int\limits_{B(x,R')\cap B(0,R)^c} \frac{(1-e^{-\frac{1}{2}|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y \right\} \\
&\le C_{d,\alpha} \int\limits_{B(x,R')} \frac{(1-e^{-|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y.
\end{aligned}
$$

By using integration by parts, the right-hand side of the above inequality is equal to

$$
\begin{aligned}
& C'_{d,\alpha} \int_0^{R'} \frac{(1-e^{-r^\beta})\varphi(m^{1/\alpha}r)}{r^{1+\alpha}} \mathrm{d}r \\
&= C'_{d,\alpha} \left\{ \frac{(e^{-(R')^\beta}-1)\varphi(m^{1/\alpha}R')}{\alpha(R')^\alpha} \right. \\
&\quad \left. + \frac{1}{\alpha} \int_0^{R'} r^{-\alpha} \left(\beta r^{\beta-1} e^{-r^\beta} \varphi(m^{1/\alpha}r) + (1-e^{-r^\beta}) m^{1/\alpha} \varphi'(m^{1/\alpha}r) \right) \mathrm{d}r \right\} \\
&= C'_{d,\alpha} \left\{ \frac{(e^{-(R')^\beta}-1)\varphi(m^{1/\alpha}R')}{\alpha(R')^\alpha} \right. \\
&\quad \left. + \frac{1}{\alpha} \int_0^{(R')^\beta} t^{-\alpha/\beta} \left(e^{-t}\varphi(m^{1/\alpha}t^{1/\beta}) + m^{1/\alpha}(1-e^{-t}) \frac{\varphi'(m^{1/\alpha}t^{1/\beta})}{\beta t^{(\beta-1)/\beta}} \right) \mathrm{d}t \right\},
\end{aligned}
\tag{29}
$$

where $C'_{d,\alpha}$ is a positive constant depending on d and α. On the other hand, by a similar calculation as above with the inequality $1 - e^{-a-b} \le (1 - e^{-a}) + (1 - e^{-b})$ for any $a, b \ge 0$, we see

$$\begin{aligned}
&\mathbf{F}_m^{(p)}1(x)\\
&= C_{d,\alpha}\left\{\int\limits_{B(x,R')\cap B(0,R)} \frac{(1-e^{-p|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\mathrm{d}y\right.\\
&\qquad\left.+\int\limits_{B(x,R')\cap B(0,R)^c} \frac{(1-e^{-\frac{p}{2}|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\mathrm{d}y\right\}\\
&\geq C_{d,\alpha}\int\limits_{B(x,R')} \frac{(1-e^{-\frac{p}{2}|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\mathrm{d}y\\
&\geq \frac{C_{d,\alpha}}{2}\int\limits_{B(x,R')} \frac{(1-e^{-p|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\mathrm{d}y\\
&= \frac{C'_{d,\alpha}}{2}\left\{\frac{(e^{-p(R')^\beta}-1)\varphi(m^{1/\alpha}R')}{\alpha(R')^\alpha}\right.\\
&\qquad+\frac{1}{\alpha}\int\limits_0^{p(R')^\beta} p^{\alpha/\beta}t^{-\alpha/\beta}\left(e^{-t}\varphi(m^{1/\alpha}(p^{-1}t)^{1/\beta})\right.\\
&\qquad\left.\left.+m^{1/\alpha}(1-e^{-t})\frac{\varphi'(m^{1/\alpha}(p^{-1}t)^{1/\beta})}{\beta(p^{-1}t)^{(\beta-1)/\beta}}\right)\mathrm{d}t\right\}\\
&\geq \frac{p^{\alpha/\beta}}{2}C'_{d,\alpha}\left\{\frac{(e^{-(R')^\beta}-1)\varphi(m^{1/\alpha}R')}{\alpha(R')^\alpha}\right.\\
&\qquad+\frac{1}{\alpha}\int\limits_0^{(R')^\beta} t^{-\alpha/\beta}\left(e^{-t}\varphi(m^{1/\alpha}(p^{-1}t)^{1/\beta})\right.\\
&\qquad\left.\left.+m^{1/\alpha}(1-e^{-t})\frac{\varphi'(m^{1/\alpha}(p^{-1}t)^{1/\beta})}{\beta t^{(\beta-1)/\beta}}\right)\mathrm{d}t\right\}
\end{aligned} \tag{30}$$

for large $p \geq 1$. Since $\varphi(r)$ is decreasing, $\varphi(m^{1/\alpha}(p^{-1}t)^{1/\beta}) \geq \varphi(m^{1/\alpha}t^{1/\beta})$ and $\varphi'(r)$ is a non-positive function on $[0,\infty)$ taking a value close to 0 near $r=0$, that is, $\varphi'(m^{1/\alpha}(p^{-1}t)^{1/\beta}) \geq \varphi'(m^{1/\alpha}t^{1/\beta})$ for large $p \geq 1$. From these facts with (29) and (30), we can confirm that

$$\mathbf{F}_m^{(p)}1(x) \geq \frac{p^{\alpha/\beta}}{2}\mathbf{F}_m 1(x), \quad x \in B(0,R),\ \text{large } p \geq 1. \tag{31}$$

Next, take $x \in B(0, R + R') \setminus B(0, R)$. In this case, F is given by

$$F(x, y) = \begin{cases} \frac{1}{2}|x-y|^\beta & y \in B(x, R') \cap B(0, R) \\ \frac{1}{2}|x-y|^\beta & y \in B(x, R') \cap B(0, R)^c \\ 0 & \text{otherwise.} \end{cases}$$

Then, by the same calculations as above

$$\begin{aligned}
&\mathbf{F}_m 1(x) \\
&\le C_{d,\alpha} \int\limits_{B(x,R')} \frac{(1 - e^{-|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y \\
&= C'_{d,\alpha} \left\{ \frac{(e^{-(R')^\beta} - 1)\varphi(m^{1/\alpha}R')}{\alpha(R')^\alpha} \right. \\
&\qquad \left. + \frac{1}{\alpha} \int\limits_0^{(R')^\beta} t^{-\alpha/\beta} \left(e^{-t}\varphi(m^{1/\alpha}t^{1/\beta}) + m^{1/\alpha}(1 - e^{-t}) \frac{\varphi'(m^{1/\alpha}t^{1/\beta})}{\beta t^{(\beta-1)/\beta}} \right) \mathrm{d}t \right\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{F}_m^{(p)} 1(x) \\
&\ge \frac{C_{d,\alpha}}{2} \int\limits_{B(x,R')} \frac{(1 - e^{-p|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y \\
&\ge \frac{p^{\alpha/\beta}}{2} C'_{d,\alpha} \left\{ \frac{(e^{-(R')^\beta} - 1)\varphi(m^{1/\alpha}R')}{\alpha(R')^\alpha} \right. \\
&\qquad + \frac{1}{\alpha} \int\limits_0^{(R')^\beta} t^{-\alpha/\beta} \big(e^{-t}\varphi(m^{1/\alpha}(p^{-1}t)^{1/\beta}) \\
&\qquad \left. + m^{1/\alpha}(1 - e^{-t}) \frac{\varphi'(m^{1/\alpha}(p^{-1}t)^{1/\beta})}{\beta t^{(\beta-1)/\beta}} \right) \mathrm{d}t \Big\} \\
&\ge \frac{p^{\alpha/\beta}}{2} C'_{d,\alpha} \left\{ \frac{(e^{-(R')^\beta} - 1)\varphi(m^{1/\alpha}R')}{\alpha(R')^\alpha} \right. \\
&\qquad \left. + \frac{1}{\alpha} \int\limits_0^{(R')^\beta} t^{-\alpha/\beta} \left(e^{-t}\varphi(m^{1/\alpha}t^{1/\beta}) + m^{1/\alpha}(1 - e^{-t}) \frac{\varphi'(m^{1/\alpha}t^{1/\beta})}{\beta t^{(\beta-1)/\beta}} \right) \mathrm{d}t \right\} \\
&= \frac{p^{\alpha/\beta}}{2} \mathbf{F}_m 1(x)
\end{aligned}$$

for large $p \geq 1$. Therefore, we can confirm (31) for $x \in B(0, R+R') \setminus B(0, R)$. For $x \in B(0, R+R')^c$, (31) is trivial because $\mathbf{F}_m^{(p)}1(x) = 0$ for any $p \geq 1$. Hence we obtain (31) for any $x \in \mathbb{R}^d$. Moreover, for $x \in B(0, R+R')$

$$\begin{aligned}
\mathbf{F}_m^{(p)}1(x) &= C_{d,\alpha}\left(\int\limits_{B(0,R+R')\cap B(x,1)} \frac{(1-e^{-pF(x,y)})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\mathrm{d}y \right.\\
&\qquad \left. + \int\limits_{B(0,R+R')\cap B(x,1)^c} \frac{(1-e^{-pF(x,y)})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\mathrm{d}y \right)\\
&\leq C_{d,\alpha} \int\limits_{B(x,1)} \frac{1-e^{-\frac{3p}{2}|x-y|^\beta}}{|x-y|^{d+\alpha}}\mathrm{d}y\\
&\qquad + C_{d,\alpha} \int\limits_{\overline{B(0,R+R')}\cap B(x,1)^c} \left(1-e^{-\frac{3p}{2}(R+R')^\beta}\right)\mathrm{d}y\\
&\leq C'_{d,\alpha}\int_0^1 \frac{1-e^{-\frac{3p}{2}r^\beta}}{r^{\alpha+1}}\mathrm{d}r + C_{d,\alpha}\left|\overline{B(0,R+R')}\right|
\end{aligned}$$

and from which it follows that $\mathbf{F}_m^{(p)}1$ is bounded on $B(0, R+R')$ and is zero on $B(0, R+R')^c$ for any $p \geq 1$. This shows that $\mathbf{F}_m^{(p)}1 \in L^1(\mathbb{R}^d)$ for any $p \geq 1$. In fact, by a similar way as in the proof of [2, Proposition 7.10(3)], one can also prove that $\mathbf{F}_m^{(p)}1 \in L^\ell(\mathbb{R}^d)(\ell \geq 1)$ for any $p \geq 1$ and thus F is to be Green-bounded with respect to $\mathbf{X}^{m,(1)}$. Here $\mathbf{X}^{m,(1)}$ is the 1-subprocess of $\mathbf{X}^m$, the killed process by e^{-t}. We omit the details.

Remark 7 There are many functions satisfying the condition (4). In fact, they can be given by the following form:

$$F(x, y) = \frac{1}{2}\phi(|x-y|)\chi_{R,R'}(x, y)$$

with $\phi(t) = t^\beta$, $\phi(t) = t^\beta/(1+t)^\beta$, $\phi(t) := \phi^{(1)}(t) = \log(1+t^\beta)$ and its iterated function $\phi^{(n)}(t) = \phi(\phi^{(n-1)}(t))$ $(n \geq 2)$ for $\beta > \alpha$. Further, we see that Fs induced by these functions are Green-bounded relative to $\mathbf{X}^{m,(1)}$ (cf. [2, Proposition 7.10]) and we can take the function $\psi(p)$ which appeared in (4) as

$$\psi(p) = p^{\alpha/\beta}.$$

Hence, the scattering length $\Gamma_m^{(1)}(p\mu + pF)$ induced by the functions ϕ above converges to $\mathrm{Cap}_m^{(1)}(\mathsf{S}_{\mu+\mathbf{F}_m1})$ as $p \to \infty$, in view of Theorem 1. Here $\Gamma_m^{(1)}$ denotes the scattering length with respect to $\mathbf{X}^{m,(1)}$.

4 Semi-classical Asymptotics for Scattering Length

In this section, we study the semi-classical asymptotics for scattering length by non-negative potentials with infinite ranges. We consider the case that $\mu(\mathrm{d}x) = V(x)\mathrm{d}x$ with V being a non-negative $L^1(\mathbb{R}^d)$-function. Note that the scattering length $\Gamma(pV + pF)$ may diverge as $p \to \infty$ if V or F has a non-compact support. So the question we are interested in is to find the asymptotic order of $\Gamma(pV + pF)$ to infinity as $p \to \infty$.

In the sequel, let $\mathbf{X}^m$, $\mathbf{X}^{m,(1)}$, $\Gamma_m^{(1)}$ and $\mathrm{Cap}_m^{(1)}$ be as in Example 6 and Remark 7. Clearly, $\mathbf{X}^{m,(1)}$ is transient. For $a > 0$ and $b \geq 0$, let V_a^b and F_a^b be the scaling potentials of V and F, respectively, which are defined by

$$V_a^b(x) := a^b V(ax), \quad F_a^b(x, y) := a^b F(ax, ay), \quad x, y \in \mathbb{R}^d.$$

The following simple scaling property of the scattering length plays a role.

Lemma 8 *For any $\beta \geq 0$ and $r > 0$, it holds that*

$$\Gamma_m^{(1)}(V_r^\beta + F_r^{\beta-\alpha}) = r^{\alpha-d}\Gamma_{r^{-\alpha}m}^{(r^{-\alpha})}(r^{\beta-\alpha}V + r^{\beta-\alpha}F).$$

Proof For notational convenience, set

$$A_t^{V_r^\beta} := \int_0^t V_r^\beta(X_s)\mathrm{d}s, \quad A_t^{F_r^{\beta-\alpha}} := \sum_{0<s\leq t} F_r^{\beta-\alpha}(X_{s-}, X_s).$$

It follows from the scaling property (25) that $A_t^{V_r^\beta}$ and $A_t^{F_r^{\beta-\alpha}}$ under $\mathbf{P}_x^m$ are equal to $r^{\beta-\alpha}A_{r^\alpha t}^V$ and $r^{\beta-\alpha}A_{r^\alpha t}^F$ under $\mathbf{P}_{rx}^{r^{-\alpha}m}$, respectively. Then, by the expression of the scattering length (20) and Ito's formula

$$\begin{aligned}
&\Gamma_m^{(1)}(V_r^\beta + F_r^{\beta-\alpha}) \\
&= \lim_{t\to\infty} \frac{1}{t} \int_{\mathbb{R}^d} \mathbf{E}_x^{m,(1)}\left[1 - e^{-A_t^{V_r^\beta} - A_t^{F_r^{\beta-\alpha}}}\right]\mathrm{d}x \\
&= \lim_{t\to\infty} \frac{1}{t} \int_{\mathbb{R}^d} \mathbf{E}_x^m\left[\int_0^t e^{-s-A_s^{V_r^\beta} - A_s^{F_r^{\beta-\alpha}}} \left(V_r^\beta(X_s) + r^\alpha \mathbf{F}_m^{(r^{\beta-\alpha})}1(rX_s)\right)\mathrm{d}s\right]\mathrm{d}x \\
&= \lim_{t\to\infty} \frac{1}{t} \int_{\mathbb{R}^d} \mathbf{E}_{rx}^{r^{-\alpha}m}\left[\int_0^t e^{-s-r^{\beta-\alpha}A_{r^\alpha s}^V - r^{\beta-\alpha}A_{r^\alpha s}^F}\right. \\
&\qquad \left. \cdot \left(r^\beta V(X_{r^\alpha s}) + r^\alpha \mathbf{F}_{r^{-\alpha}m}^{(r^{\beta-\alpha})}1(X_{r^\alpha s})\right)\mathrm{d}s\right]\mathrm{d}x
\end{aligned}$$

$$= r^{\alpha-d} \lim_{t\to\infty} \frac{1}{r^{\alpha}t} \int_{\mathbb{R}^d} \mathbf{E}_y^{r^{-\alpha}m} \left[\int_0^{r^{\alpha}t} e^{-r^{-\alpha}s' - r^{\beta-\alpha}A^V_{s'} - r^{\beta-\alpha}A^F_{s'}} \right.$$

$$\left. \cdot \left(r^{\beta-\alpha} V(X_{s'}) + \mathbf{F}^{(r^{\beta-\alpha})}_{r^{-\alpha}m} 1(X_{s'}) \right) \mathrm{d}s' \right] \mathrm{d}y$$

$$= r^{\alpha-d} \lim_{t\to\infty} \frac{1}{r^{\alpha}t} \int_{\mathbb{R}^d} \mathbf{E}_y^{r^{-\alpha}m,(r^{-\alpha})} \left[1 - e^{-r^{\beta-\alpha}A^V_{r^{\alpha}t} - r^{\beta-\alpha}A^F_{r^{\alpha}t}} \right] \mathrm{d}y$$

$$= r^{\alpha-d} \Gamma^{(r^{-\alpha})}_{r^{-\alpha}m} (r^{\beta-\alpha}V + r^{\beta-\alpha}F),$$

where we used in the second equality above that the non-local operator defined in (11) for F_a^b is given by $a^{\alpha}\mathbf{F}_m^{(a^b)} 1(a\,\cdot)$. □

For an open set $B \subset \mathbb{R}^d$, let $\tau_0^{\mathbf{1}_{B^c}}$ be the first penetrating time of $\mathbf{X}^m$ into B^c,

$$\tau_0^{\mathbf{1}_{B^c}} := \inf \left\{ t > 0 \,\middle|\, \int_0^t \mathbf{1}_{B^c}(X_s)\,\mathrm{d}s > 0 \right\}.$$

We say that B is a Kac's regular set with respect to $\mathbf{X}^m$, if $\tau_0^{\mathbf{1}_{B^c}}$ is the same as $\tau_B := \inf\{t > 0 \mid X_t \in B^c\}$, the first exit time of $\mathbf{X}^m$ from B (with probability one). Note that any open subset of $\mathbb{R}^d$ having a smooth boundary, thus any open ball in $\mathbb{R}^d$, is Kac regular. Let $\mathrm{supp}\,[U]$ be the topological support of a non-negative potential U. Then the set $\mathsf{S}_U \setminus \mathrm{supp}\,[U]$ is of zero capacity, while $\mathrm{supp}\,[U] \setminus \mathsf{S}_U$ is not necessarily of zero capacity. It is known that if $\mathrm{supp}\,[U]$ is a Kac's regular set, then $\mathrm{Cap}(\mathsf{S}_U) = \mathrm{Cap}(\mathrm{supp}\,[U])$ (cf. [15, §3]).

Lemma 9 *Let B be an open ball in $\mathbb{R}^d$. Under the hypotheses in Theorem 1, it holds that*

$$\lim_{p\to\infty} \Gamma_m^{(1)}(pV\mathbf{1}_B + pF\mathbf{1}_B) = \mathrm{Cap}_m^{(1)}(B),$$

where $F\mathbf{1}_B := F(x,y)\mathbf{1}_B(x)$.

Proof Note that we can not obtain the assertion as an immediate consequence of Theorem 1 because $F\mathbf{1}_B$ is not necessarily a symmetric function. Let

$$F\mathbf{1}_{B^2}(x,y) := F(x,y)\mathbf{1}_B(x)\mathbf{1}_B(y).$$

Clearly $F\mathbf{1}_{B^2}$ is symmetric. Denote by $\mathbf{F}_m\mathbf{1}_{B^2}$ the non-local operator induced by $F\mathbf{1}_{B^2}$ which is given as

$$\mathbf{F}_m\mathbf{1}_{B^2}(x) = C_{d,\alpha} \left(\int_B \frac{\left(1 - e^{-F(x,y)}\right)\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\mathrm{d}y \right) \mathbf{1}_B(x).$$

Then, by a similar way as in the proof of Theorem 1, we can see that

$$\liminf_{p\to\infty} \Gamma_m^{(1)}(pV\mathbf{1}_B + pF\mathbf{1}_B) \geq \liminf_{p\to\infty} \Gamma_m^{(1)}(pV\mathbf{1}_B + pF\mathbf{1}_{B^2})$$
$$\geq \mathrm{Cap}_m^{(1)}\left(\mathsf{S}_{V\mathbf{1}_B + \mathbf{F}_m\mathbf{1}_{B^2}}\right) = \mathrm{Cap}_m^{(1)}(B), \tag{32}$$

where we used the equality in (32) that B is a Kac's regular set with respect to $\mathbf{X}^{m,(1)}$.

For an open ball $D \subset \mathbb{R}^d$ such that $B \subset D$, let $U_D^{(1)}$ be its capacitary potential, which is given by $U_D^{(1)}(x) := \mathbf{E}_x^{m,(1)}[1 - e^{-\int_0^\infty \nu_D(X_t)\mathrm{d}t}]$ for $\nu_D = \infty$ on D and 0 off D. Then $U_D^{(1)} = U_1^m \nu_D$, where U_1^m is the 1-potential operator of $\mathbf{X}^m$ and ν_D is the (1-)equilibrium measure on D (cf. [5, 13]). From the definition of the scattering length (13) and the fact that $U_D^{(1)} = 1$ on B, for any $p \geq 1$

$$\begin{aligned}\Gamma_m^{(1)}(pV\mathbf{1}_B + pF\mathbf{1}_B) &= -\int_{\mathbb{R}^d} \mathcal{L}^{m,(1)} U^{(1)}_{pV\mathbf{1}_B + pF\mathbf{1}_B}(x)\,\mathrm{d}x \\ &= -\int_{\mathbb{R}^d} U_D^{(1)}(x)\mathcal{L}^{m,(1)} U^{(1)}_{pV\mathbf{1}_B + pF\mathbf{1}_B}(x)\,\mathrm{d}x \\ &= \mathcal{E}_1^m\left(U_1^m\nu_D,\, U^{(1)}_{pV\mathbf{1}_B + pF\mathbf{1}_B}\right) \\ &= \int_{\mathbb{R}^d} U^{(1)}_{pV\mathbf{1}_B + pF\mathbf{1}_B}(x)\nu_D(\mathrm{d}x),\end{aligned}$$

where $\mathcal{L}^{m,(1)}$ is the infinitesimal generator of $\mathbf{X}^{m,(1)}$ and $U^{(1)}_{pV\mathbf{1}_B + pF\mathbf{1}_B}$ is the capacitary potential relative to $pV\mathbf{1}_B + pF\mathbf{1}_B$ under $\mathbf{X}^{m,(1)}$. On the other hand, since $\mathrm{Cap}_m^{(1)}(B) = \int_{\mathbb{R}^d} U_B^{(1)}(x)\nu_D(\mathrm{d}x)$, we can see that

$$\limsup_{p\to\infty} \Gamma_m^{(1)}(pV\mathbf{1}_B + pF\mathbf{1}_B) \leq \mathrm{Cap}_m^{(1)}(B) \tag{33}$$

because $U^{(1)}_{pV\mathbf{1}_B + pF\mathbf{1}_B} \leq U_B^{(1)}$ for any $p \geq 1$. Here $U_B^{(1)}$ denotes the capacitary potential of B defined as above. The proof will be finished by (32) and (33). □

Proposition 10 *Let $\rho > d > \alpha$ and $0 < \lambda \ll 1$. If a non-negative function V satisfies $V(x) \leq c_1 |x|^{-\rho}$ for $x \in B(0, \lambda^{-\alpha/(\rho-\alpha)})^c$ for some constant $c_1 > 0$, then we have for any $m \geq 0$*

$$C_1\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \leq \Gamma_m^{(1)}\left(\lambda^{-\alpha}V\right) \leq C_2\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \tag{34}$$

for some constants $C_2 \geq C_1 > 0$.

Proof Let W be the function defined by $W(x) = |x|^{-\rho}\mathbf{1}_{B(0,1)^c}(x)$. By applying W with $F \equiv 0$, $\beta = \alpha$ and $r = \lambda^{\alpha/(\rho-\alpha)}$ to Lemma 8, we have

$$\Gamma_m^{(1)}\left(W^{\alpha}_{\lambda^{\alpha/(\rho-\alpha)}}\right) = \lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}}\,\Gamma^{(\lambda^{-\alpha^2/(\rho-\alpha)})}_{\lambda^{-\alpha^2/(\rho-\alpha)}m}(W). \tag{35}$$

Since

$$\begin{aligned} W^{\alpha}_{\lambda^{\alpha/(\rho-\alpha)}}(x) &= \lambda^{\alpha^2/(\rho-\alpha)}|\lambda^{\alpha/(\rho-\alpha)}x|^{-\rho}\mathbf{1}_{B(0,1)^c}(\lambda^{\alpha/(\rho-\alpha)}x) \\ &= \lambda^{-\alpha}|x|^{-\rho}\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})^c}(x), \end{aligned}$$

(35) can be rewritten as

$$\Gamma_m^{(1)}\left(\lambda^{-\alpha}|x|^{-\rho}\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})^c}\right) = \lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}}\,\Gamma^{(\lambda^{-\alpha^2/(\rho-\alpha)})}_{\lambda^{-\alpha^2/(\rho-\alpha)}m}(W). \tag{36}$$

It is clear from the definition of scattering length that for some constant $C > 0$

$$\Gamma^{(\lambda^{-\alpha^2/(\rho-\alpha)})}_{\lambda^{-\alpha^2/(\rho-\alpha)}m}(W) \le \int_{B(0,1)^c} |x|^{-\rho}\mathrm{d}x = \omega_d \int_1^{\infty} r^{d-\rho-1}\mathrm{d}r \le C.$$

So by which and (36), it follows that

$$\Gamma_m^{(1)}\left(\lambda^{-\alpha}|x|^{-\rho}\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})^c}\right) \le C\,\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}}. \tag{37}$$

On the other hand, we see by Lemma 9 with $F = 0$ that for any $\varepsilon > 0$ there exists $0 < \lambda_0 := \lambda_0(\varepsilon) \ll 1$ such that for every $0 < \lambda \le \lambda_0$,

$$\begin{aligned} &C'\,\mathrm{Cap}_m^{(1)}\left(B(0,\lambda^{-\alpha/(\rho-\alpha)})\right) \\ &\quad \le \Gamma_m^{(1)}\left(\lambda^{-\alpha}V\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})}\right) \le \mathrm{Cap}_m^{(1)}(B(0,\lambda^{-\alpha/(\rho-\alpha)})) \end{aligned}$$

for some constant $C' := C'(\varepsilon) > 0$. From which, with (26) and (27), it follows that

$$C_1\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \le \Gamma_m^{(1)}\left(\lambda^{-\alpha}V\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})}\right) \le C_1'\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \tag{38}$$

for some constants $C_1, C_1' > 0$. Now, on account of (37) and (38), the monotonicity and subadditivity of scattering length we can confirm (34). Indeed,

$$\begin{aligned} C_1\,\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} &\le \Gamma_m^{(1)}\left(\lambda^{-\alpha}V\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})}\right) \\ &\le \Gamma_m^{(1)}(\lambda^{-\alpha}V) \\ &\le \Gamma_m^{(1)}\left(\lambda^{-\alpha}V\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})}\right) + \Gamma_m^{(1)}\left(\lambda^{-\alpha}V\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})^c}\right) \\ &\le \Gamma_m^{(1)}\left(\lambda^{-\alpha}V\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})}\right) + c'\Gamma_m^{(1)}\left(\lambda^{-\alpha}|x|^{-\rho}\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})^c}\right) \\ &\le C_2\,\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \end{aligned}$$

where we used in the fourth inequality above that for any non-negative $L^1(\mathbb{R}^d)$-function U and a constant $c > 0$, there exists a constant $c' > 0$ such that $\Gamma_m^{(1)}(cU) \leq c'\Gamma_m^{(1)}(U)$. □

Remark 11 If $m = 0$, $d = 3$ and $\alpha = 2$, then (26) holds for (0-)capacity. As a result, Tamura's result (5) can be easily reproduced from (34).

In the sequel, we let $\alpha \in (0, 2)$.

Proposition 12 *Let $d > \alpha$, $\rho > \frac{d+\alpha-1}{2} \vee \alpha$ and $0 < \lambda \ll 1$. For a compact set $K \subset \mathbb{R}^d$, let $M > 0$ be such that $K \subset B(0, M)$. Assume that for $x \in B(0, \lambda^{-\alpha/(\rho-\alpha)}M)^c$,*

$$F(x, y) \leq c_2|x - y|^{-(\rho-\alpha)}\mathbf{1}_{B(x,\,\lambda^{-\alpha/(\rho-\alpha)})^c\cap\lambda^{-\alpha/(\rho-\alpha)}K}(y) \tag{39}$$

for some constant $c_2 > 0$, where $\lambda^{-\alpha/(\rho-\alpha)}K := \{\lambda^{-\alpha/(\rho-\alpha)}x \mid x \in K\}$. Assume in addition that

$$\mathbf{F}_m^{(\lambda^{-\alpha})}1(x) \geq C\psi(\lambda^{-\alpha})\mathbf{F}_m 1(x) \quad \text{for } x \in B(0, \lambda^{-\alpha/(\rho-\alpha)}M) \tag{40}$$

for a constant $C > 0$ and for some positive function ψ such that $\psi(\sigma) \leq \sigma$ and $\psi(\sigma) \to \infty$ as $\sigma \to \infty$. Then, for any $m \geq 0$

$$C_3\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \leq \Gamma_m^{(1)}(\lambda^{-\alpha}F) \leq C_4\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \tag{41}$$

for some constants $C_4 \geq C_3 > 0$.

Proof Let

$$G(x, y) := |x - y|^{-(\rho-\alpha)}\mathbf{1}_{B(0,M)^c}(x)\mathbf{1}_{B(x,1)^c\cap K}(y)$$

and $\mathbf{G}_m 1(x)$ the associated non-local function defined as in (11). Note that the statement in Lemma 8 with $V \equiv 0$ is valid for any non-negative symmetric bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal such that its non-local operator defined as in (11) being integrable. By applying G with $\beta = \alpha$ and $r = \lambda^{\alpha/(\rho-\alpha)}$ to Lemma 8, one has

$$\Gamma_m^{(1)}\left(G^0_{\lambda^{\alpha/(\rho-\alpha)}}\right) = \lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}}\Gamma_{m_\lambda}^{(\lambda^{-\alpha^2/(\rho-\alpha)})}(G), \tag{42}$$

where $m_\lambda := \lambda^{-\alpha^2/(\rho-\alpha)}m$ and

$$\begin{aligned} &G^0_{\lambda^{\alpha/(\rho-\alpha)}}(x, y) \\ &\quad= \lambda^{-\alpha}|x - y|^{-(\rho-\alpha)}\mathbf{1}_{B(0,\lambda^{-\alpha/(\rho-\alpha)}M)^c}(x)\mathbf{1}_{B(x,\,\lambda^{-\alpha/(\rho-\alpha)})^c\cap\,\lambda^{-\alpha/(\rho-\alpha)}K}(y). \end{aligned}$$

From the definition of scattering length and (28),

$$\begin{aligned}
\Gamma_{m_\lambda}^{(\lambda^{-\alpha^2/(\rho-\alpha)})}(G) &\le \int_{\mathbb{R}^d} \mathbf{G}_{m_\lambda} 1(x)\mathrm{d}x \\
&\le C_{d,\alpha} \int_{B(0,M)^c} \int_K \frac{(1-e^{-|x-y|^{-(\rho-\alpha)}})\varphi(m_\lambda^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\mathrm{d}y\mathrm{d}x \\
&\le C'_{d,\alpha} m_\lambda^{\frac{d+\alpha-1}{2\alpha}} \int_K \int_{B(0,M)^c} \frac{|x-y|^{-(\rho-\alpha)} e^{-m_\lambda^{1/\alpha}|x-y|}|x-y|^{\frac{d+\alpha-1}{2}}}{|x-y|^{d+\alpha}}\mathrm{d}x\mathrm{d}y \\
&= C'_{d,\alpha}\omega_d m_\lambda^{\frac{d+\alpha-1}{2\alpha}} \int_K \int_{d(y,B(0,M)^c)}^{\infty} \frac{e^{-m_\lambda^{1/\alpha} r}}{r^{(2\rho-d-\alpha+3)/2}}\mathrm{d}r\mathrm{d}y \\
&\le C_m \lambda^{-\frac{\alpha(d+\alpha-1)}{2(\rho-\alpha)}} \exp\left(-\lambda^{-\frac{\alpha}{\rho-\alpha}} m^{\frac{1}{\alpha}} \inf_{y\in K} d(y,B(0,M)^c)\right) \\
&\qquad\qquad \cdot \int_K \frac{1}{d(y,B(0,M)^c)^{(2\rho-d-\alpha+1)/2}}\mathrm{d}y \\
&= C'_4 \lambda^{-\frac{\alpha(d+\alpha-1)}{2(\rho-\alpha)}} \exp\left(-c_3\lambda^{-\frac{\alpha}{\rho-\alpha}}\right), \qquad (43)
\end{aligned}$$

where

$$C'_4 = C_m \int_K \frac{1}{d(y,B(0,M)^c)^{(2\rho-d-\alpha+1)/2}}\mathrm{d}y \quad \text{with } C_m := \frac{2C'_{d,\alpha} m^{\frac{d+\alpha-1}{2\alpha}}\omega_d}{2\rho-d-\alpha+1}$$

and $c_3 = m^{1/\alpha}\inf_{y\in K} d(y,B(0,M)^c)$. Thus, it follows from (42) and (43) that

$$\Gamma_m^{(1)}\left(G^0_{\lambda^{\alpha/(\rho-\alpha)}}\right) \le C'_4 \lambda^{-\frac{\alpha(3d-\alpha-1)}{2(\rho-\alpha)}} \exp\left(-c_3\lambda^{-\frac{\alpha}{\rho-\alpha}}\right). \qquad (44)$$

On the other hand, we see by Lemma 9 with $V=0$ that for any $\varepsilon>0$ there exists $0<\lambda_0:=\lambda_0(\varepsilon)\ll 1$ such that for every $0<\lambda\le\lambda_0$,

$$\begin{aligned}
&C'_5\,\mathrm{Cap}_m^{(1)}\left(B(0,\lambda^{-\alpha/(\rho-\alpha)})\right) \\
&\qquad \le \Gamma_m^{(1)}\left(\lambda^{-\alpha}F\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})}\right) \le \mathrm{Cap}_m^{(1)}(B(0,\lambda^{-\alpha/(\rho-\alpha)}))
\end{aligned}$$

for some constant $C'_5 := C'_5(\varepsilon) > 0$. From this fact with (26) and (27), it follows that

$$C_3\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \le \Gamma_m^{(1)}\left(\lambda^{-\alpha}F\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})}\right) \le C'_3\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \qquad (45)$$

for some constants $C'_3 \ge C_3 > 0$. Now, by combining (44) and (45), we have

$$
\begin{aligned}
C_3\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} &\le \Gamma_m^{(1)}\left(\lambda^{-\alpha}F\mathbf{1}_{B(0,\lambda^{-\alpha/(\rho-\alpha)}M)}\right)\\
&\le \Gamma_m^{(1)}\left(\lambda^{-\alpha}F\right)\\
&\le \Gamma_m^{(1)}\left(\lambda^{-\alpha}F\mathbf{1}_{B(0,\lambda^{-\alpha/(\rho-\alpha)}M)}\right)+\Gamma_m^{(1)}\left(\lambda^{-\alpha}F\mathbf{1}_{B(0,\lambda^{-\alpha/(\rho-\alpha)}M)^c}\right)\\
&\le \Gamma_m^{(1)}\left(\lambda^{-\alpha}F\mathbf{1}_{B(0,\lambda^{-\alpha/(\rho-\alpha)}M)}\right)+c'\Gamma_m^{(1)}\left(G^0_{\lambda^{\alpha/(\rho-\alpha)}}\right)\\
&\le C_3'\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}}+c'C_4'\lambda^{-\frac{\alpha(3d-\alpha-1)}{2(\rho-\alpha)}}\exp\left(-c_3\lambda^{-\frac{\alpha}{\rho-\alpha}}\right)\\
&\le \left(C_3'+c'C_4'\lambda^{-\frac{\alpha(d+\alpha-1)}{2(\rho-\alpha)}}\exp\left(-c_3\lambda^{-\frac{\alpha}{\rho-\alpha}}\right)\right)\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}}\\
&\le C_4\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}}
\end{aligned}
$$

for some constant $C_4 > 0$. □

Now, we are ready to prove Theorem 2.

Proof of Theorem 2 The proof is an immediate consequence of Propositions 10 and 12 with the subadditivity of the scattering length. □

Acknowledgements The first named author was partially supported by JSPS KAKENHI Grant number 20K03635.

References

1. Z.-Q. Chen, Uniform integrability of exponential martingales and spectral bounds of non-local Feynman-Kac semigroups, *Stochastic Analysis and Applications to Finance, Essays in Honor of Jia-an Yan*, ed. by T. Zhang, X. Zhou (2012), pp. 55–75
2. Z.-Q. Chen, D. Kim, K. Kuwae, L^p-independence of spectral radius for generalized Feynman-Kac semigroups. Math. Ann. **374**(1–2), 601–652 (2019)
3. Z.-Q. Chen, R. Song, Drift transforms and Green function estimates for discontinuous processes. J. Funct. Anal. **201**(1), 262–281 (2003)
4. P.J. Fitzsimmons, P. He, J. Ying, A remark on Kac's scattering length formula. Sci. China Math. **56**(2), 331–338 (2013)
5. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, second revised and extended edition. de Gruyter Studies in Mathematics, vol. 19 (Walter de Gruyter & Co., Berlin, 2011)
6. P. He, A formula on scattering length of dual Markov processes. Proc. Amer. Math. Soc. **139**(5), 1871–1877 (2011)
7. M. Kac, Probabilistic methods in some problems of scattering theory. Rocky Mt. J. Math. **4**, 511–537 (1974)
8. M. Kac, J. Luttinger, Scattering length and capacity. Ann. Inst. Fourier **25**(3–4), 317–321 (1975)
9. D. Kim, K. Kuwae, Analytic characterizations of gaugeability for generalized Feynman-Kac functionals. Trans. Amer. Math. Soc. **369**(7), 4545–4596 (2017)
10. D. Kim, M. Matsuura, On a scattering length for additive functionals and spectrum of fractional Laplacian with a non-local perturbation. Math. Nachr. **293**(2), 327–345 (2020)

11. D. Kim, M. Matsuura, Semi-classical asymptotics for scattering length of symmetric stable processes. Stat. Probab. Lett. **167**, 108921 (2020)
12. K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Studies in Advanced Mathematics, vol. 68 (Cambridge University Press, 1999)
13. D. Stroock, The Kac approach to potential theory. I. J. Math. Mech. **16**, 829–852 (1967)
14. Y. Takahashi, An integral representation on the path space for scattering length. Osaka J. Math. **7**, 373–379 (1990)
15. M. Takeda, A formula on scattering length of positive smooth measures. Proc. Amer. Math. Soc. **138**(4), 1491–1494 (2010)
16. H. Tamura, Semi-classical limit of scattering length. Lett. Math. Phys. **24**, 205–209 (1992)
17. H. Tamura, Semi-classical asymptotics for scattering length and total cross section at zero energy. Bull. Fac. Sci. Ibaraki Univ. **25**, 1–9 (1993)
18. M. Taylor, Scattering length and perturbations of $-\Delta$ by positive potentials. J. Math. Anal. Appl. **53**, 291–312 (1976)
19. M. Taylor, Scattering length of positive potentials. Houston J. Math. **33**(4), 979–1003 (2007)