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Dirichlet Forms and Related Topics

In Honor of Masatoshi Fukushima's Beiju, IWDFRT 2022, Osaka, Japan, August 22–26

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Preface

Professor Masatoshi Fukushima is turning 88 years old this year in Japanese counting tradition. In Japanese culture, this is a milestone birthday called Yoneju (rice longevity or Beiju). This conference proceedings contains 27 peer-reviewed invited papers from leading experts as well as young researchers all over the world in the related fields that Professor Fukushima has made important contributions to. These 27 papers cover a wide range of topics in probability theory, ranging from Dirichlet form theory, Markov processes, heat kernel estimates, entropy on Wiener spaces, analysis on fractal spaces, random spanning tree and Poissonian loop ensemble, random Riemannian geometry, SLE, space-time partial differential equations of higher order, infinite particle systems, Dyson model, functional inequalities, branching processes, to machine learning and Hermitizable problems for complex matrices.

Professor Fukushima is well known for his fundamental contributions to the theory of Dirichlet forms and symmetric Markov processes. In one of his first publications in the late 1960s, Fukushima studied the boundary value problem of Brownian motion on any bounded connected open set using the Dirichlet form method. In particular, he [R5] succeeded in constructing reflected Brownian motions on the Kuramochi compactification of bounded connected open sets. This method is later extended in [R9] to the regular Dirichlet forms via a method of transformation of the underlying spaces, which makes it possible to construct the corresponding symmetric strong Markov processes. More precisely, the approach in his pioneering work [R9] can be summarized as follows.

(i) For any regular symmetric Dirichlet space $(\mathcal{E}, \mathcal{F})$ on a locally compact separable metric space E, there exists an equivalent strongly regular symmetric Dirichlet space $(\mathcal{E}', \mathcal{F}')$ on another locally compact separable metric space E' and there is a capacity-preserving quasi-homeomorphism ϕ between E and E'. Here, a Dirichlet form is said to be regular if the domain of the Dirichlet form contains sufficiently many continuous functions, while strongly regular means it is regular and the Dirichlet form generates Ray resolvent kernels.

(ii) Since the strongly regular Dirichlet form $(\mathcal{E}', \mathcal{F}')$ possesses Ray resolvents, it determines a Ray process X' on E'. The pull-back process $X = \phi^{-1}(X')$ by the quasi-homeomorphism ϕ is a symmetric Hunt process on E associated with the regular Diriclet form $(\mathcal{E}, \mathcal{F})$.

Subsequently, Professor Martin L. Silverstein [8] gave a new method in constructing the Hunt process from a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ by using the quasi-continuous modifications of the functions in the domain $\mathcal{F}$ of Dirichlet form. Fukushima felt that Silverstein's construction is more direct, and thus, he put a modified version of Silverstein's construction in his 1980s book [MT3] as well as in his book [MT4, MT7] with Oshima and Takeda instead of his original one.

In 1959, Beurling and Deny have shown that symmetric Markov semigroups on L^2-spaces are generated by symmetric Dirichlet forms. Fukushima strengthened this result as follows. If a Dirichlet form is regular, then the semigroup is represented by the transition probability kernel of a strong Markov process. More importantly, the strong Markov process generated by a regular Dirichlet form is a Hunt process whose sample paths are quasi-left continuous. In the Markov process theory, quasi-left continuity is very important for the analysis of the sample path properties and for the probabilistic potential theory. The association of the Hunt process to a regular Dirichlet form opened up a door for using probabilistic methods to analyze and study various properties of the Dirichlet forms. For instance, it yields a probabilistic interpretation of A. Beurling and J. Deny's potential theory of Dirichlet spaces. Fukushima's fundamental result on the association of a Hunt process with a regular Dirichlet form has later been extended by S. Albeverio and Z.-M. Ma [1] in 1991 to quasi-regular symmetric Dirichlet forms. The quasi-regularity is in fact a necessary and sufficient condition for a Dirichlet form to associate a nice strong Markov process and works on infinite dimensional spaces as well. It is further extended in Z.-M. Ma and M. Röckner [6] to nearly symmetric Dirichlet forms. Z.-Q. Chen, Z.-M. Ma and M. Röckner [2] showed in 1994 that any quasi-regular Dirichlet form is quasi-homeomorphic to a regular Dirichlet form on a locally compact separable metric space. Hence, essentially all the results that have been established for regular Dirchlet spaces can be carried over to quasi-regular Dirichlet forms through such a quasi-homeomorphism. This point of view has been taken in the monograph [MT8] by Chen and Fukushima.

Recently, we had an opportunity to conduct an interview with Professor Fukushima. We learned the following historic remarks during this interview. Professor Fukushima said that J. Deny had hoped to define the Dirichlet space independently of the basic measure used in the L^2-space. However, since Professor Fukushima considered the speed measure in the one-dimensional diffusion process as the basic measure to construct the process, he took it for granted from the beginning to rely on the speed measure as the symmetrizing measure to define the Dirichlet space. Martin L. Silverstein [8] introduced the concept of extended Dirichlet space that is independent of the underlying symmetrizing measure. This notion plays a crucial role in the development of the Dirichlet form theory, and it is invariant under time change by positive continuous additive functionals having full support. As a consequence, Dirichlet form theory becomes a big leap from Feller's theory for one-dimensional diffusion processes.

The Hunt process $X = \{X_t,\ t \geq 0;\ \mathbb{P}_x,\ x \in E\}$ generated by a regular Dirichlet form is in general not a semimartingale. As a substitution of Ito's formula, Professor Fukushima introduced the notion of continuous additive functional of zero energy and showed that for any u in the domain of a regular Dirichlet form, $u(X_t)$ can be decomposed as the sum of a square integrable martingale additive functional and a continuous additive functional of zero energy of X. This decomposition has many important implications and is nowadays called Fukushima decomposition. Dirichlet form theory is an effective tool in many areas of probability theory and mathematical physics, including diffusion processes in infinite dimensional spaces, analysis on fractals and metric measure spaces, optimal transports, and random walks and Markov processes in random media. Fukushima's decomposition has played an important role in these areas.

Professor Fukushima appreciates Professor Hiroshi Kunita's formulation [5] of semi-Dirichlet forms. He wrote a joint paper with Uemura [R90] in this connection and suggested Professor Yoichi Oshima (Kumamoto University) to work under the setting of semi-Dirichlet forms rather than non-symmetric Dirichlet forms in his book [7].

Professor Fukushima has been active in research throughout his mathematical career. He has even started new lines of research after his retirement from University. For example, he has studied stochastic Komatu-Löwner's equations in multiply connected domains in joint work [R94, R95, R97, R92] with Z.-Q. Chen, S. Rohde, H. Suzuki as well as with H. Kaneko. In these works, Brownian motion with darning that arises in his study of the boundary theory of Markov processes with Chen [R84, MT8] played an important role. Very recently, in his solo paper [R99] and in his joint work [R100, R101] with Oshima, Professor Fukushima has studied Gaussian fields, multiplicative chaos and the Markov property of the Gaussian fields parametrized functions in extended Dirichlet spaces. For recurrent Dirichlet forms, treatment of extended Dirichlet spaces becomes more difficult. By employing the recurrent potential theory developed by Oshima in the framework of regular Dirichlet forms, however, they could overcome the difficulty.

Additional information about Professor Fukushima's mathematical work and his important contributions can be found in the article by N. Jacob in [3] and an essay by Chen, Jacob, Takeda and Uemura in [4].

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References

1. S. Albeverio and Z. M. Ma, Necessary and sufficient conditions for the existence of m-perfect processes associated with Dirichlet forms. *Séminaire de Probabilités, XXV*, 374-406, Lecture Notes in Math., **1485**, Springer, Berlin, 1991.

2. Z.-Q. Chen, Z. M. Ma and M. Röckner, Quasi-homeomorphisms of Dirichlet forms. *Nagoya Math. J.* **136** (1994), 1–15.
3. *Fukushima Masatoshi Selecta.* Edited by Niels Jacob, Yoichi Oshima and Masayoshi Takeda, Walter de Gruyter, 2010
4. *Festschrift Masatoshi Fukushima, In honor of Masatoshi Fukushima's Sanju.* Edited by Zhen-Qing Chen, Niels Jacob, Masayoshi Takeda and Toshihiro Uemura, Interdisciplinary, World Scientific, 2015
5. H. Kunita, Sub-Markov semi-group in Banach-lattice. In *Proc. International Conference on Functional Analysis and Related Topics (Tokyo, 1969)*, pp. 332–343 Univ. Tokyo Press, 1970
6. Z. M. Ma and M. Röckner, *Introduction to the Theory of (Non-symmetric) Dirichlet Forms.* Universitext. Springer-Verlag, Berlin, 1992. vi+209 pp.
7. Y. Oshima, *Semi-Dirichlet forms and Markov Processes.* Walter de Gruyter, 2013
8. M. L. Silverstein, *Symmetric Markov Processes.* Lecture Notes in Math. **426**, Springer, 1974.

List of Masatoshi Fukushima's Publications

Research Papers (English)

[R1] On Feller's kernel and the Dirichlet norm, *Nagoya Math. J.*, **24** (1964) 167–175

[R2] Resolvent kernels on a Martin space, *Proc. Japan Acad.*, **41** (1965), 260–263

[R3] On spectral functions related to birth and death processes, *J. Math. Kyoto Univ.*, **5** (1966), 151–161

[R4] On a class of Markov processes taking values on lines and the central limit theorem, (with M. Hitsuda), *Nagoya Math. J.*, **30** (1967), 47–56

[R5] A construction of reflecting barrier Brownian motions for bounded domains, *Osaka J. Math.*, **4** (1967), 183–215

[R6] On boundary conditions for multi-dimensional Brownian motions with symmetric resolvent densities, *J. Math. Soc. Japan*, **21** (1969), 58–93

[R7] On Dirichlet spaces and Dirichlet rings, *Proc. Japan Acad.*, **45** (1969), 433–436

[R8] Regular representations of Dirichlet spaces, *Trans. Amer. Math. Soc.*, **155** (1971), 455–473

[R9] Dirichlet spaces and strong Markov processes, *Trans. Amer. Math. Soc.*, **162** (1971), 185–224

[R10] On transition probabilities of symmetric strong Markov processes, *J. Math. Kyoto Univ.*, **12** (1972), 431–450

[R11] On the generation of Markov processes by symmetric forms, in *Proceedings of the Second Japan-USSR Symposium on Probability Theory*, (Kyoto, 1972), 46–79, Lecture Notes in Math., **330**, Springer, Berlin, 1973

[R12] Almost polar sets and an ergodic theorem, *J. Math. Soc. Japan*, **26** (1974), 17–32

[R13] On the spectral distribution of a disordered system and the range of a random walk, *Osaka J. Math.*, **11** (1974), 73–85

[R14] Local property of Dirichlet forms and continuity of sample paths, *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **29** (1974), 1–6

[R15] On an asymptotic property of spectra of a random difference operator (with H Nagai and S. Nakao), *Proc. Japan Acad.*, **51** (1975), 100–102

[R16] Asymptotic properties of the spectral distributions of disordered systems, in *International Symposium on Mathematical Problems in Theoretical Physics*, (Kyoto Univ., Kyoto, 1975), 224–227, Lecture Notes in Phys., **39**, Springer, Berlin, 1975

[R17] On spectra of the Schrödinger operator with a white Gaussian noise potential (with S. Nakao), *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, **37** (1976/77), 267–274

[R18] Potential theory of symmetric Markov processes and its applications, in *Proceedings of the Third Japan-USSR Symposium on Probability Theory*, (Tashkent, 1975), 119–133. Lecture Notes in Math., **550**, Springer, Berlin, 1976

[R19] On an L^p-estimate of resolvents of Markov processes, *Publ. Res. Inst. Math. Sci.*, **13** (1977/78), 277–284

[R20] A decomposition of additive functionals of finite energy, *Nagoya Math. J.*, **74** (1979), 137–168

[R21] On additive functionals admitting exceptional sets, *J. Math. Kyoto Univ.*, **19** (1979), 191–202

[R22] Dirichlet spaces and additive functionals of finite energy, in *Proceedings of the International Congress of Mathematicians*, (Helsinki, 1978), 741–747, Acad. Sci. Fennica, Helsinki, 1980

[R23] A generalized stochastic calculus in homogenization, in *Quantum fields–algebras, processes*, (Proc. Sympos., Univ. Bielefeld, Bielefeld, 1978), 41–51, Springer, Vienna, 1980

[R24] On a stochastic calculus related to Dirichlet forms and distorted Brownian motions, in *New stochasitic methods in physics, Phys. Rep.*, **77** (1981), 255–262

[R25] On a representation of local martingale additive functionals of symmetric diffusions, in *Stochastic integrals*, (Proc. Sympos. Durham, 1980), 110–118, Lecture Notes in Math., **851**, Springer, Berlin, 1981

[R26] Capacity and quantum mechanical tunneling (with S. Albeverio, W. Karwowski and L. Streit), *Comm. Math. Phys.*, **81** (1981), 501–513

[R27] On asymptotics of spectra of Schrödinger operators, in *Statistical and physical aspects of Gaussian processes*, (Saint-Flour, 1980), 335–347, Colloq. Internat. CNRS, **307**, CNRS, Paris, 1981

[R28] On absolute continuity of multidimensional symmetrizable diffusions, in *Functional analysis in Markov processes*, (Katata/Kyoto, 1981), 146–176, Lecture Notes in Math., **923**, Springer, Berlin-New York, 1982

[R29] A note on irreducibility and ergodicity of symmetric Markov processes, in *Stochastic processes in quantum theory and statistical physics*, (Marseille, 1981), 200–207, Lecture Notes in Phys., **173**, Springer, Berlin, 1982

[R30] Markov processes and functional analysis, in *Proc. International Math. Conf. Singapore,* 187–202, Eds. L.H. Chen, Y.B. Ng, M.J. Wicks, North Holland, 1982

[R31] Capacitary maximal inequalities and an ergodic theorem, in *Probability theory and mathematical statistics*, (Tbilisi, 1982), 130–136, Lecture Notes in Math., **1021**, Springer, Berlin, 1983

[R32] Basic properties of Brownian motion and a capacity on the Wiener space, *J. Math. Soc. Japan* **36**(1984), 161–176

[R33] On conformal martingale diffusions and pluripolar sets (with M. Okada), *J. Funct. Anal.*, **55** (1984), 377–388

[R34] A transformation of a symmetric Markov process and the Donsker-Varadhan theory (with M. Takeda), *Osaka J. Math.*, **21** (1984), 311–326

[R35] A Dirichlet form on the Wiener space and properties on Brownian motion, in *Théorie du potentiel*, (Orsay, 1983), 290–300, Lecture Notes in Math., **1096**, Springer, Berlin, 1984

[R36] Energy forms and diffusion processes, in *Mathematics* + *physics.* Vol. **1**, 65–97, World Sci. Publishing, Singapore, 1985

[R37] (r, p)-capacities for general Markovian semigroups (with H. Kaneko), in *Infinite-dimensional analysis and stochastic processes*, (Bielefeld, 1983), 41–47, Res. Notes in Math., **124**, Pitman, Boston, MA, 1985

[R38] On the continuity of plurisubharmonic functions along conformal diffusions, *Osaka J. Math.*, **23** (1986), 69–75

[R39] A stochastic approach to the minimum principle for the complex Monge-Ampère operator, in *Stochastic processes and their applications*, (Nagoya, 1985), 38–50, Lecture Notes in Math., **1203**, Springer, Berlin, 1986

[R40] Reversibility of solutions to martingale problems (with D. W. Stroock), in *Probability, statistical mechanics, and number theory*, 107–123, Adv. Math. Suppl. Stud., **9**, Academic Press, Orlando, FL, 1986

[R41] On recurrence criteria in the Dirichlet space theory, in *From local times to global geometry, control and physics*, (Coventry, 1984/85), 100–110, Pitman Res. Notes Math. Ser., **150**, Longman Sci. Tech., Harlow, 1986

[R42] On Dirichlet forms with random data-recurrence and homogenization (with S. Nakao and M. Takeda), in *Stochastic processes-mathematics and physics, II*, (Bielefeld, 1985), 87–97, Lecture Notes in Math., **1250**, Springer, Berlin, 1987

[R43] On Dirichlet forms for plurisubharmonic functions (with M. Okada), *Acta Math.*, **159** (1987), 171–213

[R44] A note on capacities in infinite dimensions, in *Probability theory and mathematical statistics*, (Kyoto, 1986), 80–85, Lecture Notes in Math., **1299**, Springer, Berlin, 1988

[R45] On holomorphic diffusions and plurisubharmonic functions, in *Geometry of random motion*, (Ithaca, N.Y., 1987), 65–78, Contemp. Math., **73**, Amer. Math. Soc., Providence, RI, 1988

[R46] On two classes of smooth measures for symmetric Markov processes, in *Stochastic analysis*, (Paris, 1987), 17–27, Lecture Notes in Math., **1322**, Springer, Berlin, 1988

[R47] On the skew product of symmetric diffusion processes (with Y. Oshima), *Forum Math.*, **1** (1989), 103–142

[R48] Capacities on Wiener space: tightness and invariance (with S. Albeverio, W. Hansen, Z.-M. Ma and M. Roc̈kner), *C. R. Acad. Sci. Paris Se’r. I Math.*, **312** (1991), 931–935

[R49] On quasi-supports of smooth measures and closability of pre-Dirichlet forms (with Y. LeJan), *Osaka J. Math.*, **28** (1991), no. 4, 837–845

[R50] On the closable parts of pre-Dirichlet forms and the fine supports of underlying measures (with K. Sato and S. Taniguchi), *Osaka J. Math.*, **28** (1991), 517–535

[R51] An invariance result for capacities on Wiener space (with S. Albeverio, W. Hansen, Z.-M. Ma and M. Roc̈kner), *J. Functional Analysis*, **106** (1992), 35–49

[R52] On $(r, 2)$-capacities for a class of elliptic pseudo differential operators (with N. Jacob and H. Kaneko), *Math. Ann.*, **293** (1992), 343–348

[R53] Dirichlet forms, diffusion processes and spectral dimensions for nested fractals, in *Ideas and methods in mathematical analysis, stochastics, and applications*, (Oslo, 1988), 151–161, Cambridge Univ. Press, Cambridge, 1992

[R54] (r, p)-capacities and Hunt processes in infinite dimensions, in *Probability theory and mathematical statistics*, (Kiev, 1991), 96–103, World Sci. Publishing, River Edge, NJ, 1992

[R55] On a spectral analysis for the Sierpin′ski gasket (with T. Shima), *Potential Analysis*, **1** (1992), 1–35

[R56] Two topics related to Dirichlet forms: quasi-everywhere convergences and additive functionals, in *Dirichlet forms* (Varenna, 1992), 21–53, G. Dell’Antonio and U. Mosco (Eds), Lecture Notes in Math., **1563**, Springer, Berlin, 1993

[R57] On discontinuity and tail behaviours of the integrated density of states for nested pre-fractals (with T. Shima), *Comm. Math. Phys.*, **163** (1994), 461–471

[R58] On a strict decomposition of additive functionals for symmetric diffusion processes, *Proc. Japan Acad. Ser. A Math. Sci.*, **70** (1994), 277–281

[R59] On a decomposition of additive functionals in the strict sense for a symmetric Markov processes, in *Dirichlet forms and stochastic processes,* Z. Ma, M. Roeckner, J. Yan (Eds), Walter de Gruyter, (1995), 155–169

[R60] Reflecting diffusions on Lipschitz domains with cusps-analytic construction and Skorohod representation (with M. Tomisaki), in *Potential theory and degenerate partial differential operators* (Parma), *Potential Analysis.* **4** (1995), 377–408

[R61] Construction and decomposition of reflecting diffusions on Lipschitz domains with Hölder cusps (with M. Tomisaki), *Probab. Theory Related Fields*, **106** (1996), 521–557

[R62] On decomposition of additive functionals of reflecting Brownian motions (with M. Tomisaki), in *Ito's stochastic calculus and probability theory,* N. Ikeda, S. Watanabe, M. Fukushima, H. Kunita (Eds), Springer, Tokyo, 1996, 51–61

[R63] Dirichlet forms, Caccioppoli sets and the Skorohod equation, in *Stochastic Differential and Difference equations,* Csiszar Michaletzky (Eds.), Birkhauser 1997, 56–66

[R64] Distorted Brownian motions and BV functions, in *Trends in probability and related analysis*, (Taipei, 1996), 143–150, N.-R. Shieh (Ed), World Sci. Publishing, River Edge, NJ, 1997

[R65] Large deviations and related LIL's for Brownian motions on nested fractals (with T. Shima and M. Takeda), *Osaka J. Math.*, **36** (1999), 497–537

[R66] On semi-martingale characterizations of functionals of symmetric Markov processes, *Electron. J. Probab.*, **4** (1999), Paper 18, 1–32, http://www.math.washington.edu/~ejpecp

[R67] BV functions and distorted Ornstein Uhlenbeck processes over the abstract Wiener space, *J. Funct. Analysis.*, **174** (2000), 227–249

[R68] On limit theorems for Brownian motions on unbounded fractal sets, in *Fractal geometry and stochastics, II* (Greifswald/Koserow, 1998), 227–237, *Progr. Probab.*, S. Graf (Ed), **46**, Birkhäuser, Basel, 2000

[R69] On Ito's formulae for additive functionals of symmetric diffusion processes, in *Stochastic processes, physics and geometry: new interplays, I* (Leipzig, 1999), 201–211, F. Getsztesy (Ed), *CMS Conf. Proc.*, **28**, Amer. Math. Soc., Providence, RI, 2000

[R70] On the space of BV functions and a related stochastic calculus in infinite dimensions (with M. Hino), *J. Funct. Analysis*, **183** (2001), 245–268

[R71] Dynkin games via Dirichlet forms and singular control of one-dimensional diffusions (with M, Taksar), *SIAM J. Control Optim.*, **41** (2002), 682–699

[R72] On Sobolev and capacitary inequalities for contractive Besov spaces over d-sets (with T. Uemura), *Potential Analysis*, **18** (2003), 59–77

[R73] Capacitary bounds of measures and ultracontractivity of time changed processes (with T. Uemura), *J. Math. Pures Appl.*, **82** (2003), 553–572

[R74] A note on regular Dirichlet subspaces (with J. Ying), *Proc. Am. Math. Soc.*, **131** (2003) 1607-1610 (2003); erratum ibid. **132** (2004), 1559

[R75] On spectral synthesis for contractive p-norms and Besov spaces (with T. Uemura), *Potential Analysis*, **20** (2004), 195–206

[R76] Function spaces and symmetric Markov processes, in *Stochastic analysis and related topics in Kyoto, Advanced Studies in Pure Mathematics*, **41** (2004), 75–89, Math. Soc. Japan

[R77] Time changes of symmetric diffusions and Feller measures (with P. He and J. Ying), *Ann. Probab.*, **32** (2004) 3138–3166

[R78] On regular Dirichlet subspaces of $H^1(I)$ and associated linear diffusions (with X. Fang and J. Ying), *Osaka J. Math.*, **42** (2005), 27–41

[R79] Poisson point processes attached to symmetric diffusions (with H. Tanaka), *Ann. Inst. Henri Poincaré Probab. Stat.*, **41** (2005), 419–459

[R80] Traces of symmetric Markov processes and their characterizations (with Z.-Q. Chen and J. Ying), *Ann. Probab.*, **34** (2006), 1052–1102

[R81] Entrance law, exit system and Lévy system of time changed processes (with Z.-Q. Chen and J. Ying), *Illinois J. Math.*, **50** (2006), 269–312

[R82] On Feller's boundary problem for Markov processes in weak duality (with Z.-Q. Chen), *J. Func. Anal.*, **252** (2007), 710–733

[R83] Extending Markov processes in weak duality by Poisson point processes of excursions, (with Z.-Q. Chen and J. Ying), in *The Abel Symposium 2005, Stochastic Analysis and Applications, A Symposium in Honor of Kiyosi Itô* (Oslo), F.E. Benth, G.Di Nunno, T. Lindstrom, B. Oksendal, T. Zhang (Eds), 153–196, Springer, 2007

[R84] One-point extensions of Markov processes by darning (with Z.-Q. Chen), *Probab. Th. Rel. Fields*, **141** (2008), 61–112

[R85] Flux and lateral conditions for symmetric Markov processes (with Z.-Q. Chen), *Potential Anal.*, **29** (2008), 241–269

[R86] On unique extension of time changed reflecting Brownian motions (with Z.-Q. Chen), *Ann. Inst. Henri Poincare, Probab. Statist.,* **45** (2009), 864–875

[R87] On extended Dirichlet spaces and the space of BL functions, *Potential Theory and Stochastics in Albac, Aurel Cornea Memorial Volume,* Eds. D. Bakry, L.Beznea, N. Boboc, G. Bucur, M. Roeckner, Theta Foundation, Bucharest, AMS distribution, 2009, 101–110

[R88] From one dimensional diffusions to symmetric Markov processes, *Stoch. Proc. Appl.,* **210** (2010), 590–604

[R89] A localization formula in Dirichlet form theory (with Z.-Q. Chen), *Proc. Amer. Math. Soc.*, **140** (2012), 1815–1822

[R90] Jump-type Hunt processes generated by lower bounded semi-Dirichlet forms (with T. Uemura), *Ann. Probab.*, **40** (2012), 858–889

[R91] On general boundary conditions for one-dimensional diffusions with symmetry, *J. Math. Soc. Japan*, **66** (2014), 289–316

[R92] On Villat's kernels and BMD Schwarz kernels in Komatu-Loewner equations (with H. Kaneko), in: *Stochastic Analysis and Applications 2014*, Springer Proceedings in Mathematics and Statistics, **100**, (Eds) D. Crisan, B. Hambly, T. Zariphopoulous, 2014, pp 327–348

[R93] One-point reflection (with Z.-Q. Chen), *Stochastic Process Appl.*, **125** (2015), 1368–1393

[R94] Chordal Komatu-Loewner equation and Brownian motion with darning in multiply connected domains (with Z.-Q. Chen and S. Rohde), *Trans. Amer. Math. Soc.*, **368** (2016), 4065–4114

[R95] Stochastic Komatu-Loewner evolution and BMD domain constant (with Z.-Q. Chen and H. Suzuki), *Stochastic Process. Appl.*, **127** (2017), 2068–2087

[R96] Reflections at infinity of time changed RBMs on a domain with Liouville branches (with Z.-Q. Chen), *J. Math. Soc. Japan*, **70** (2018), 833–852

[R97] Stochastic Komatu-Loewner evolutions and BMD domain constant (with Z.-Q. Chen), *Stochastic Process. Appl.*, **128** (2018), 545–594

[R98] Liouville property of harmonic functions with finite energy for Dirichlet forms, in: *Stochastic Partial Differential Equations and Relaed Fields*, Springer Proceedings in Mathematics and Statistics, **229**, (Eds) A.Eberle, M. Grothaus, W. Hoh, M. Kassmann, W. Stannat, G. Trutnau, 2018, pp. 25–42

[R99] Logarithmic and linear potentials of signed measures and Markov property of associated Gaussian fields, *Potential Analysis*, **49** (2018), 359–379

[R100] Recurrent Dirichlet forms and Markov property of associated Gaussian fields (with Y. Oshima), *Potential Analysis*, **49** (2018), 609–633

[R101] Gaussian fields, equilibrium potentials and multiplicative chaos for Dirichlet forms (with Y. Oshima), *Potential Analysis*, **55** (2021), 285–337

Expository Writing

[E1] Boundary problems of Brownian motions and the Dirichlet spaces, Sūgaku, **20** (1968), 211–221 (in Japanese)

[E2] On the Theory of Markov Processes, *BUTSURI* (The Physical Society of Japan), **25** (1970), 37–41 (in Japanese)

[E3] On Random Spectra, *BUTSURI* (The Physical Society of Japan), **34** (1979), 153–159 (in Japanese)

[E4] On Spectral Analysis on Fractal, Manufacturing and Technology, **48** no. 1 (1996), 55–59 (in Japanese)

[E5] Fractal and Random walk—Toward the function and the shape of nature, in *Shizennoshikumi to Ningennochie*, Osaka University Press, 1997, 211–222 (in Japanese)

[E6] Decompositions of symmetric diffusion processes and related topics in analysis, *Sugaku Expositions*, **14** (2001), 1–13. AMS

[E7] Stochastic Control Problem and Dynkin's Game Theory, Gien (Organization for Research and Development of Innovative Science and Technology (ORDIST)), Kansai University, **111** (2001), 35–39 (in Japanese)

[E8] Refined solutions of optimal stopping games for symmetric Markov processes, Technology Reports of Kansai University (with K. Menda), 48 (2006), 101–110

[E9] On the works of Kyosi Itô and stochastic analysis, *Japanese J. Math.,* **2** (2007), 45–53

[E10] A brief survey on stochastic calculus in Markov processes, *RIMS Kôkyûroku*, 1672 (2010), 191–197
[E11] On general boundary conditions for one-dimensional diffusions and symmetry, MinnHoKee Lecture at Seoul National University, 2012 June
[E12] Feller's Contributions to the One-Dimensional Diffusion Theory and Beyond, In: *William Feller—Selected Papers II*, Eds. R. Schilling, Z. Vondracek, W. Woyczynski, Springer Verlag, 2015, 63–73
[E13] Komatu-Loewner differential equations, Sūgaku, **69** (2017), 137–156 (in Japanese)
[E13'] Komatu-Loewner differential equations (translation of [E13]), *Sugaku Expositions*, **33** (2020), 239–260
[E14] On the works of Hiroshi Kunita in the sixties, *J. Stoch. Anal.*, **2** (2021), article 3

Seminar on Probability (in Japanese)

[S1] (with Kiyosi Itô and Shinzo Watanabe) On Diffusion processes, Seminar on Probability **3**, 1960
[S2] (with Ken-ichi Sato and Masao Nagasawa) Transformation of Markov processes and boundary problems, Seminar on Probability **16**, 1960
[S3] Dirichlet space and its representations, Seminar on Probability **31**, 1969
[S4] (with Hiroshi Kunita) Studies on Markov processes, Seminar on Probability **40**, 1969
[S5] (with Shintaro Nakao and Sin-ichi Kotani) On Random spectra, Seminar on Probability **45**, 1977

Monographs and Textbooks

[MT1] Dirichlet Forms and Markov Processes (Japanese), KINOKUNIYA Co. Ltd., 1975
[MT2] (with Kazunari Ishii) Natural Phenomena and Stochastic Processes (Japanese), Nippon-Hyoron-Sha Co., Ltd., 1980, (enlarged ed.) 1996
[MT3] Dirichlet Forms and Markov Processes, North-Holland mathematical library **23**, North-Holland, Amsterdam-New York/ Kodansha, Tokyo, 1980
[MT4] (with Yōichi Ōshima and Masayoshi Takeda) Dirichlet Forms and Symmetric Markov Processes, de Gruyter studies in mathematics **19**, de Gruyter, Berlin, 1994
[MT5] Probability Theory (Japanese), SHOKABO Pub. Co., Ltd., 1998
[MT6] (with Masayoshi Takeda) Markov Processes (Japanese), BAIFUKAN Pub. Co., Ltd, 2008

[MT7] (with Yōichi Ōshima and Masayoshi Takeda) Dirichlet Forms and Symmetric Markov Processes, Second revised and extended editions, de Gruyter, Berlin, 2011

[MT8] (with Z.-Q. Chen) Symmetric Markov Processes, Time Change, and Boundary Theory, Princeton University Press, Princeton NJ, 2012

Contents

Markov Uniqueness and Fokker-Planck-Kolmogorov Equations

Sergio Albeverio, Vladimir I. Bogachev, and Michael Röckner

Abstract In this paper we show that Markov uniqueness for symmetric pre-Dirichlet operators L follows from the uniqueness of the corresponding Fokker-Planck-Kolmogorov equation (FPKE). Since in recent years a considerable number of uniqueness results for FPKE's have been achieved, we obtain new Markov unique-

It is a great honour and a great pleasure for us to have the possibility to dedicate this article to Professor Masatoshi Fukushima on the occasion of his 88th birthday. He has influenced our scientific development in a very strong and enduring way. We have had the great luck to meet him on several occasions, through his and our longer common stays in Germany and Japan, and at conferences there and other countries, including the UK and Italy. We greatly enjoyed every conversation with him, both of scientific and personal nature. His deep insights, knowledge and wisdom have had a very strong scientific impact upon us. Even before meeting him in person, we have learned from his publications how to approach problems at the cutting edge of analysis and probability theory in a natural way. His books are jewels of clarity, rigour and deep insights. We enjoyed most pleasant collaborations with him, his coworkers and students, and developed strong ties with the Japanese mathematical community at large. The first named author got in contact with Masatoshi already back in 1973, when he was reading his pioneering paper "On generation of Markov processes by symmetric forms". With Raphael Höegh- Krohn he elaborated a first infinite dimensional version of Masatoshi's approach suitable for applications in quantum field theory. In the late 70's Masatoshi was invited for a longer stay at Bielefeld University, where the first named author was working and the last named author was a student at the time. From then on, an intensive collaboration and fruitful exchange of ideas with Masatoshi and his School started. From the second half of the 80's this also involved the second named author, who visited conferences organized by Masatoshi in Osaka and Tokyo during that time period. We are enormously indebted to Masatoshi for his guidance he always provided for us in various ways. We wish him from the bottom of our hearts many more years in good health, happiness and with success in all his activities!

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ness results in concrete cases. A selection of such will be presented in this paper. They include cases with killing and with degenerate diffusion coefficients.

Keywords Dirichlet from · Dirichlet operator · Fokker-Planck-Kolmogorov equation · Markov uniqueness

1 Introduction and Framework

In this paper we fix a σ-finite measure space $(E, \mathcal{B}, m)$. Let $L^p := L^p(m) = L^p(E, m)$, $p \in [1, \infty]$ be the corresponding (real) L^p-spaces with their usual norms $\|\cdot\|_p$ and inner product $(\ ,\ )_2$ if $p = 2$. On $L^p(m)$ we shall consider linear operators

$$L\colon D(L) \subset L^p(m) \to L^p(m)$$

with their usual partial order defined by

$$L_1 \subset L_2 \underset{Def.}{\Leftrightarrow} \Gamma(L_1) \subset \Gamma(L_2),$$

where $D(L)$ is a linear subspace of L^p, called domain of L, and

$$\Gamma(L) := \{(u, Lu) \in L^p \times L^p\colon u \in D(L)\}$$

is the graph of L. In particular, we shall consider those L which generate a (unique) strongly continuous semigroup of (everywhere defined) continuous linear operators on L^p, denoted by e^{tL}, $t \geq 0$. Henceforth such L will be shortly called generator (on L^P). We refer to [17] for the notions and the well-known characterization of such generators. We recall that a generator L is always closed, i.e. $\Gamma(L)$ is a closed subset of $L^p \times L^p$, with domain $D(L)$ dense in L^p and that for $p \in (1, \infty)$, its adjoint operator $(L^*, D(L^*))$ on $L^{p'}$, with $p' := \frac{p}{p-1}$, generates a strongly continuous semigroup of linear operators, e^{tL^*}, $t \geq 0$, on $L^{p'}$. This satisfies

$$e^{tL^*} = (e^{tL})^*, \quad t \geq 0, \tag{1.1}$$

(see [17, Chap. 1, Corollary 10.6]). We consider three cases of sets of generators on L^p for $p \in (1, \infty)$:

(1) Let D_0^* be a dense linear subspace of $L^{p'}$ and $L_0^* : D_0^* \subset L^{p'} \to L^{p'}$ a linear operator. Define $\mathcal{M} := \mathcal{M}(L_0^*, D_0^*)$ to be the set of all linear operators $L\colon D(L) \subset L^p \to L^p$ such that $L_0^* \subset L^*$ and L is a generator on L^p.
(2) Let D_0 be a dense linear subspace of L^2 and $L_0\colon D_0 \subset L^2 \to L^2$ a symmetric linear operator, i.e., $L_0 \subset L_0^*$, which is upper bounded, i.e.

$$\sup_{u \in D_0 \setminus \{0\}} (L_0 u, u)_2 ||u||_2^{-2} < \infty.$$

Define $\mathcal{M}_{sym} := \mathcal{M}_{sym}(L_0, D_0)$ to be the set of all linear operators $L\colon D(L) \subset L^2 \to L^2$ such that $L_0 \subset L$, L is a generator on L^2 and L is symmetric, i.e., $L \subset L^*$.

(3) Let (L_0, D_0) be as in (2) and define $\mathcal{M}_{sym,M} := \mathcal{M}_{sym,M}(L_0, D_0)$ to be the subset of all $(L, D(L))$ in $\mathcal{M}_{sym}(L_0, D_0)$ such that each e^{tL}, $t \geq 0$, is sub-Markovian, i.e., if $u \in L^2$ such that $0 \leq u \leq 1$, then $0 \leq e^{tL}u \leq 1$.

Concerning (2) we note that by Ref. [17, Theorems 4.2 and 5.3] it obviously follows that $\mathcal{M}_{sym}(L_0, D_0)$ coincides with the set of all linear operators $L\colon D(L) \subset L^2 \to L^2$ such that $L_0 \subset L$ and L is upper bounded and self-adjoint, i.e., $L = L^*$. Furthermore, $\mathcal{M}_{sym}(L_0, D_0)$ is not empty, because the Friedrichs extension of (L_0, D_0) is self-adjoint and upper bounded (see e.g. [12, p. 131]).

Concerning (3) we refer to [12] and [16] for more details on such sub-Markovian operator semigroups.

The first aim of this paper is to derive a "parabolic" condition in each of the cases (1), (2), (3) which implies that the respective sets $\mathcal{M}$, $\mathcal{M}_{sym}$, $\mathcal{M}_{sym,M}$ contain at most one element. Here, "parabolic" means in terms of the corresponding Fokker-Planck-Kolmogorov equation (FPKE). The second aim of this paper is (by refining this "parabolic condition") to use uniqueness results from [8] to obtain new results on "Markov uniqueness" in the sense of the following definition:

Definition 1.1 Let (L_0, D_0) be as in (2) above. (L_0, D_0) is called Markov unique if $\mathcal{M}_{sym,M}$ contains exactly one element.

Let us note that our notion of "Markov uniqueness" is in fact stronger than the one extensively studied in the literature, since there, uniqueness is studied in the subset of all linear operators $(L, D(L))$ in $\mathcal{M}_{sym,M}(L_0, D_0)$, which are nonpositive definite, i.e., $\sup_{u \in D(L)}(Lu, u)_2 \leq 0$, while also assuming that (L_0, D_0) is nonpositive definite.

The literature on Markov uniqueness is quite extensive and a number of types of state spaces E, as e.g. $\mathbb{R}^d$ or infinite dimensional vector spaces or manifolds have been considered . To the best of our knowledge the first paper on this subject is [22] by Masayoshi Takeda. To give an overview of the entire literature is beyond the scope of this paper. Instead, we refer to the references in [4, 5, 10–12, 18] and the more recent papers [3, 19].

It seems, however, that the method to prove Markov uniqueness proposed in this paper, i.e., by using the corresponding FPKE, is new, though it is very natural. Furthermore our applications and examples in Sect. 3, even though they are all in the classical case $E := \mathbb{R}^d$, appear to be not covered by the existing literature, in particular, since they include cases with degenerate diffusion coefficients and we can allow "killing", more precisely in our applications, where L is a partial differential operator on $\mathbb{R}^d$, this operator is allowed to have a (negative) zero order coefficient.

Finally, we would like to recall the notion of "strong uniqueness" which is different from Markov uniqueness. In our context here it means that the larger set

$\mathcal{M}_{sym}(L_0, D_0)$ contains exactly one element which is equivalent to the fact that the closure of (L_0, D_0) is self adjoint on L^2. For more details we refer to [2, 10] and as a very recent paper to [1], in particular to the lists of references in them.

2 The Main Idea and a Parabolic Condition for Uniqueness

For a set $\mathcal{F}$ of real-valued functions on E and $T \in (0, \infty)$ we define $\mathcal{F}_{,T}$ to be the set of all functions of the form

$$[0, T] \times E \ni (t, x) \mapsto f(t)\varphi(x) =: (f \otimes \varphi)(t, x),$$

where $\varphi \in \mathcal{F}$ and $f \in C^1([0, T]; \mathbb{R})$ with $f(T) = 0$.

Let us start with case (1) from the introduction and consider $(L, D(L)) \in \mathcal{M}(L_0^*, D_0^*)$.

For $t \geq 0$ we set

$$T_t^L := e^{tL}, \; T_t^{L^*} = e^{tL^*}$$

[cf. (1.1)]. Then for all $\varphi \in D_0^*$, $u \in L^p$ and $t \geq 0$ we have

$$\begin{aligned} \int \varphi \, T_t^L u \, dm &= \int T_t^{L^*} \varphi \, u \, dm \\ &= \int \varphi \, u \, dm + \int_0^t \int T_s^{L^*} L^* \varphi \, u \, dm \, ds \\ &= \int \varphi \, u \, dm + \int_0^t \int L_0^* \varphi \, T_s^L u \, dm \, ds. \end{aligned} \tag{2.1}$$

Hence defining the (signed) measure $\mu_t(dx) := T_t^L u(x) \, m(dx), \; t \geq 0$, by the (integral) product rule for all $f \otimes \varphi \in D_{0,T}^*$ (defined as above with $\mathcal{F} := D_0^*$) we have

$$\begin{aligned} \int (f \otimes \varphi)(t, x) \, \mu_t(dx) &= f(t) \int \varphi(x) \, \mu_t(dx) \\ &= f(0) \int \varphi \, d\mu_0 + \int_0^t f(s) \int L_0^* \varphi \, d\mu_s \, ds \\ &\quad + \int_0^t f'(s) \int \varphi \, d\mu_s \, ds \\ &= \int (f \otimes \varphi)(0, x) \, \mu_0(dx) \\ &\quad + \int_0^t \int (\frac{\partial}{\partial s} + L_0^*)(f \otimes \varphi) \, d\mu_s \, ds. \end{aligned}$$

In particular, for $t = T$

$$\int_0^T \int (\frac{\partial}{\partial s} + L_0^*)(f \otimes \varphi)\, d\mu_s\, ds = - \int (f \otimes \varphi)(0, x)\, \mu_0(dx). \tag{2.2}$$

(2.1) [equivalently (2.2)] means that $\mu_t = T_t^L u \cdot m,\ u \in L^p,\ t \geq 0$, solves the FPKE [up to time T for every $T \in (0, \infty)$] corresponding to $(L_0^*, D(L_0^*))$ (see [8]).

Now it is very easy to prove the following "parabolic condition" that ensures that

$$\#\mathcal{M}(L_0^*, D_0^*) \leq 1$$

(where as usual # is an abbreviation for cardinality).

Proposition 2.1 *Assume that for every $T \in (0, \infty)$*

$$(\frac{\partial}{\partial s} + L_0^*) D_{0,T}^* \text{ is dense in } L^{p'}([0, T] \times E, dt \otimes m). \tag{2.3}$$

Then $\mathcal{M}(L_0^*, D_0^*)$ contains at most one element.

Proof Let $(\tilde{L}, D(\tilde{L})) \in \mathcal{M}_{sym}(L_0^*, D_0^*)$. Then, as seen above, $\tilde{\mu}_t := T_t^{\tilde{L}} u\, m, u \in L^p$, $t \geq 0$, also satisfies (2.1), hence (2.2). So, (by subtracting) for $g(t, \cdot) := T_t^L u - T_t^{\tilde{L}} u,\ t \geq 0$, we obtain for all $T \in (0, \infty)$

$$\int_0^T \int (\frac{\partial}{\partial s} + L_0^*)(f \otimes \varphi)\, g(s, \cdot)\, dm\, ds = 0$$

for all $f \otimes \varphi \in D_{0,T}^*$. Since $g \in L^p([0, T] \times E,\ dt \otimes m)$, by (2.3) it follows that $g = 0$, and the assertion follows, since $u \in L^p$ was arbitrary. □

Now let us consider case (2) from the introduction. So, let $(L, D(L)) \in \mathcal{M}_{sym}(L_0, D_0)$. Then using the same notation as in case (1) we analogously obtain for all $\varphi \in D_0,\ u \in L^2$ and $t \geq 0$

$$\int \varphi\, T_t^L u\, dm = \int \varphi\, u\, dm + \int_0^t \int L_0 \varphi\, T_s^L u\, dm\, ds, \tag{2.4}$$

hence for $\mu_t := T_t^L u\, m$ and for all $f \otimes \varphi \in D_{0,T}$, $T \in (0, \infty)$ we have

$$\int_0^T (\frac{\partial}{\partial s} + L_0^*)(f \otimes \varphi)\, d\mu_s\, ds = - \int (f \otimes \varphi)(0, x)\, \mu_0(dx), \tag{2.5}$$

i.e., $\mu_t,\ t \geq 0$, solves the FPKE corresponding to (L_0, D_0).
Analogously to Proposition 2.1 we then prove the following result.

Proposition 2.2 *Assume that for every $T \in (0, \infty)$*

$$(\frac{\partial}{\partial s} + L_0)D_{0,T} \text{ is dense in } L^2([0,T] \times E, dt \otimes m). \tag{2.6}$$

Then $\mathcal{M}_{sym}(L_0, D_0)$ consists of exactly one element.
In case (3) if $(L, D(L)) \in \mathcal{M}_{sym,M}(L_0, D_0)$, then obviously $T_t^L(L^2 \cap L^\infty) \subset L^2 \cap L^\infty$, and since $L^2 \cap L^\infty$ is dense in L^2, T_t^L is uniquely determined on $L^2 \cap L^\infty$.

Proposition 2.3 *Suppose that $L_0(D_0) \subset L^1$ (which automatically holds if $m(E) < \infty$) and that for every $T \in (0, \infty)$*

$$(\frac{\partial}{\partial s} + L_0)D_{0,T} \text{ is dense in } L^1([0,T] \times E, dt \otimes m). \tag{2.7}$$

Then $\mathcal{M}_{sym,M}(L_0, D_0)$ consists of at most one element. If the semigroup generated by the Friedrichs extension of (L_0, D_0) is sub-Markovian, then this extension is the unique element in $\mathcal{M}_{sym,M}(L_0, D_0)$.

Proof We repeat the proof of Proposition 2.2 (respectively, 2.1) with $u \in L^2 \cap L^\infty$. Then for all $T \in (0, \infty)$ we have for g as in the proof of Proposition 2.3 that $g \in L^\infty([0,T] \times E, dt \otimes m)$. Then by (2.7) we conclude again that $g = 0$. Since T_t^L is uniquely determined on $L^2 \cap L^\infty$, the assertion follows. □

Clearly, conditions (2.3), (2.6) and (2.7) are not easy to check in applications and certainly too strong, at least in case (3). So, let us discuss a weaker condition in this case.

Let $(L, D(L)) \in \mathcal{M}_{sym,M}(L_0, D_0)$ and fix $u \in L^\infty$ such that $u \geq 0$ and $\int u \, dm = 1$. Then $\mu_t^L := T_t^L u \, m,\ t \geq 0$, are subprobability measures on $(E, \mathcal{B})$, i.e., $\mu_t^L(E) \leq 1$ for all $t \geq 0$. We note that obviously each T_t^L is uniquely determined by its values on such u. We have seen that $\mu_t := \mu_t^L, t \geq 0$, solves the corresponding FPKE

$$\int \varphi \, d\mu_t = \int \varphi \, d\mu_0 + \int_0^t \int L_0 \varphi \, d\mu_s ds, \quad t \geq 0, \forall \varphi \in D_0, \tag{2.8}$$

hence for all $T \in (0, \infty)$, $f \otimes \varphi \in D_{0,T}$

$$\int_0^T \int (\frac{\partial}{\partial s} + L_0)(f \otimes \varphi) \, d\mu_s \, ds = -\int (f \otimes \varphi)(0, x)\mu_0(dx). \tag{2.9}$$

(2.8) and (2.9) are equations for paths $(\mu_t)_{t \geq 0}$ of subprobability measures on $(E, \mathcal{B})$ such that $[0, \infty) \ni t \to \mu_t(A)$ is Lebesgue measurable for all $A \in \mathcal{B}$. Define $\mathcal{SP}$ to be the set of all such paths, and $\mathcal{SP}$(T) the set of their restrictions to [0,T].

Now the following result is obvious.

Theorem 2.1 *If, for every probability density $u \in L^1 \cap L^\infty$, (2.8) [or (2.9)] has a unique solution $(\mu_t)_{t\geq 0} \in \mathcal{SP}$ such that each μ_t, $t \geq 0$, is absolutely continuous with respect to m and such that $\mu_0 = u \cdot m$, then $\mathcal{M}_{sym,M}(L_0, D_0)$ consists of at most one element. If, in addition, the semigroup generated by the Friedrichs extension of $(L_0, D(L_0))$ is sub-Markovian, then this extension is the unique element in $\mathcal{M}_{sym,M}(L_0, D_0)$.*

In Theorem 2.1, it is enough to prove uniqueness for (2.9) [or (2.10)] in the subclass of all $(\mu_t)_{t\geq 0} \in \mathcal{SP}$ for which each μ_t, $t \geq 0$, is absolutely continuous with respect to m with bounded density, i.e., one only needs uniqueness in a convex subset of $\mathcal{SP}$. Therefore, the following result, which was first observed in [6], is useful and goes beyond absolutely continuous solutions.

Proposition 2.4 *Let $T \in (0, \infty)$ and ζ be a subprobability measure on $(E, \mathcal{B})$ and let $\mathcal{K}_\zeta \subset \mathcal{SP}(T)$ be a non-empty convex set such that each $(\mu)_{t\in[0,T]} \in \mathcal{K}_\zeta$ is a solution to (2.8) [hence to (2.9)] with $\mu_0 = \zeta$. Suppose that for every $(\mu_t)_{t\in[0,T]} \in \mathcal{K}_\zeta$*

$$(\frac{\partial}{\partial s} + L_0)(D_{0,T}) \text{ is dense in } L^1([0,T] \times E, \mu_t dt). \tag{2.10}$$

Let $(\mu_t)_{t\in[0,T]}, (\tilde{\mu}_t)_{t\in[0,T]} \in \mathcal{K}_\zeta$. Then $\mu_t = \tilde{\mu}_t$ for dt-a.e. $t \in [0, T]$.

Remark 2.1 Clearly for $(L, D(L)) \in \mathcal{M}_{sym,M}(L_0, D_0)$ and the corresponding solutions $(\mu_t^L)_{t\geq 0}$ defined above, condition (2.10) is weaker than condition (2.7) in Proposition 2.3, since $\sup_{t\in[0,T]} \|T_t^L u\|_\infty < \infty$.

Proof of Proposition 2.4 Since $\mathcal{K}_\zeta$ is convex, we have that $\nu_t := \frac{1}{2}\mu_t + \frac{1}{2}\tilde{\mu}_t$, $t \geq 0$, is again in $\mathcal{K}_\zeta$ and

$$\mu_t dt = g\nu_t dt, \;\; \tilde{\mu}_t dt = \tilde{g}\nu_t dt \tag{2.11}$$

for some $g, \tilde{g} \in L^\infty([0,T] \times E, \nu_t dt)$.

Furthermore, by (2.9) it follows that for all $f \otimes \varphi \in D_{0,T}$

$$\int_0^T \int (\frac{\partial}{\partial s} + L_0)(f \otimes \varphi)(g - \tilde{g})\, d\nu_s\, ds = 0. \tag{2.12}$$

Hence by (2.10) this implies that $g = \tilde{g}$ and the assertion follows. □

Proposition 2.4 and the observation that (at least in many cases) it suffices to check (2.10) for just one solution in $\mathcal{K}_\zeta$, are the core of the proof of many results on uniqueness of solutions in $\mathcal{SP}$ to concrete FPKEs in Chap. 9 of [8], which thus can be applied to prove Markov uniqueness for many examples of given operators (L_0, D_0) on $L^2(m)$ as above. We shall present a selection of such in the next section. We shall restrict ourselves to the symmetric case, i.e., $p = 2$ and $L_0 \subset L_0^*$, though also nonsymmetric cases (as in case (1) from Sect. 1) can be treated if one has enough knowledge about the dual operator (L_0^*, D_0^*) on $(L^p)'$ for $p \in (1, \infty)$ (see Remark 4.2 below).

3 Some Uniqueness Results for FPKEs

In the rest of the paper we shall concentrate on the case where the state space E is equal to $\mathbb{R}^d$. By the same ideas it is, however, possible to obtain Markov uniqueness from uniqueness results of FPKEs on more general state spaces, including infinite dimensional vector spaces or manifolds. This will be done in future work.

3.1 Fokker-Planck-Kolmogorov Equations

As already mentioned we shall use the uniqueness results on FPKEs from [8, Chap. 9]. So, let us briefly recall the framework there, but for simplicity restricting to solutions in $\mathcal{SP}$, since we shall only use these in our applications below.

Below $(E, \mathcal{B})$ from the previous sections will always be $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $d \in \mathbb{N}$, where $\mathcal{B}(\mathbb{R}^d)$ denotes the Borel σ-algebra of $\mathbb{R}^d$. Consider a partial differential operator of the form

$$L_0\varphi = a^{ij}\partial_{x_i}\partial_{x_j}\varphi + b^i\partial_{x_i}\varphi + c\varphi, \quad \varphi \in D_0 := C_0^\infty(\mathbb{R}^d), \tag{3.1}$$

where we use Einstein's summation convention, $\partial_{x_i} := \frac{\partial}{\partial x_i}$, $1 \le i \le d$, a^{ij}, b^i, $c : [0, T] \times \mathbb{R}^d \to \mathbb{R}$, with $c \le 0$, are $\mathcal{B}(\mathbb{R}^d)$-measurable functions, $A(t, x) := (a^{ij}(t, x))_{1 \le i, j \le d}$ is a nonnegative definite matrix for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and $T \in (0, \infty)$ is fixed. For some of the results below we need to assume local boundedness and local strict ellipticity of A, i.e. :

(H1) For each ball $U \subset \mathbb{R}^d$ there exist $\gamma(U)$, $M(U) \in (0, \infty)$ such that

$$\gamma(U) \cdot I \le A(t, x) \le M(U) \cdot I \quad \forall\, (t, x) \in [0, T] \times \mathbb{R}^d,$$

where I denotes the $d \times d$ identity matrix.

Let $\mathcal{SP}$ be defined as in Sect. 2. We say that $(\mu_t)_{t \ge 0} \in \mathcal{SP}$ satisfies the FPKE (up to time T for L_0) if a^{ij}, b^i, $c \in L^1_{loc}([0, T] \times \mathbb{R}^d, \mu_t\, dt)$ and for every $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$\int \varphi\, d\mu_t = \int \varphi\, d\mu_0 + \int_0^t \int L_0\varphi\, d\mu_s\, ds \quad \text{for } dt\text{-a.e. } t \in [0, T]. \tag{3.2}$$

In Sects. 3.2–3.4 below we shall only be interested in the so-called subprobability solutions to (3.2), i.e., we a priori restrict to a class $\mathcal{SP}_\nu \subset \mathcal{SP}$ in which we search for a (hopefully unique) solution to (3.2). So, given a subprobability measure ν on $\mathcal{B}(\mathbb{R}^d)$ (i.e., $\nu \geq 0$ and $\nu(\mathbb{R}^d) \leq 1$), $\mathcal{SP}_\nu$ is defined to be the set of all $(\mu_t)_{t\in[0,T]} \in \mathcal{SP}(T)$ with the following properties:

$$(\mu_t)_{t\in[0,T]} \text{ solves (3.2)}, \tag{3.3}$$

$$c \in L^1([0,T]\times\mathbb{R}^d, \mu_t\, dt), \tag{3.4}$$

$$b \in L^2([0,T]\times U, \mu_t\, dt; \mathbb{R}^d) \text{ for all balls } U \subset \mathbb{R}^d, \tag{3.5}$$

$$\mu_0 = \nu \text{ and } \mu_t(\mathbb{R}^d) \leq \nu(\mathbb{R}^d) + \int_0^t \int c(x,s)\, \mu_s(dx)\, ds \text{ for } dt\text{-a.e. } t \in [0,T]. \tag{3.6}$$

Clearly, if $\nu \neq 0$, by dividing by $\nu(\mathbb{R}^d)$, we may assume, without loss of generality concerning the uniqueness of solutions in $\mathcal{SP}_\nu$ for (3.2), that $\nu(\mathbb{R}^d) = 1$. Below we fix a probability measure ν on $\mathcal{B}(\mathbb{R}^d)$.
Now let us recall several uniqueness results for (3.2) from [8, Chap. 9]. Below let dx denote Lebesgue measure on $\mathbb{R}^d$.

3.2 Nondegenerate VMO Diffusion Coefficients

Let us recall the definition of the VMO(=vanishing mean oscillation)-property of a function (see [14] and the references therein), which is a vast generalization of local Lipschitzianity.

Let g be a bounded Borel-measurable function on $\mathbb{R}^{d+1}$. Set

$$O(g,R) := \sup_{(x,t)\in\mathbb{R}^{d+1}} \sup_{r\leq R} r^{-2}|U(x,r)|^{-2} \times \int_t^{t+r^2} \int \int_{y,z\in U(x,r)} |g(s,y)-g(s,z)|\, dy\, dz\, ds.$$

If $\lim_{R\to 0} O(g,R)=0$, then we say that the function g belongs to the class $VMO_x(\mathbb{R}^{d+1})$.

Suppose that a Borel-measurable function g is defined on $[0,T]\times\mathbb{R}^d$ and bounded on $[0,T]\times U$ for every ball U. We extend g by zero to the whole space $\mathbb{R}^{d+1}$. If for every function $\zeta \in C_0^\infty(R^d)$ the function $g\zeta$ belongs to the class $VMO_x(\mathbb{R}^{d+1})$, then we say that g belongs to the class $VMO_{x,loc}([0,T]\times\mathbb{R}^d)$.

Theorem 3.1 *Let (H1) hold and assume that*

$$a^{ij} \in VMO_{x,loc}([0,T]\times\mathbb{R}^d),\ 1 \leq i,j \leq d.$$

Then the set

$$\{(\mu_t)_{t\in[0,T]} \in \mathcal{SP}_\nu \ : \ a^{ij},\ b^i \in L^1([0,T]\times\mathbb{R}^d, \mu_t\, dt)\} \tag{3.7}$$

contains at most one element.

Proof See ([8, Theorem 9.3.6]). □

3.3 Nondegenerate Locally Lipschitz Diffusion Coefficients

In this subsection and the next one we use the following condition:

(H2) For every ball $U \subset \mathbb{R}^d$ there exists $\Lambda(U) \in (0,\infty)$ such that for all $1 \le i, j \le d$

$$|a^{ij}(t,x) - a^{ij}t, y| \le \Lambda(U)|x-y| \quad \forall t \in [0,T],\ x, y \in U.$$

Theorem 3.2 *Suppose that conditions (H1) and (H2) hold, that $c \le 0$ and that $b \in L^p_{loc}([0,T]\times\mathbb{R}^d, dt\ dx; \mathbb{R}^d)$, $c \in L^{\frac{p}{2}}_{loc}([0,T]\times\mathbb{R}^d, dt\ dx)$ for some $p > d+2$. Assume also that there exists $(\mu_t)_{t\ge 0} \in \mathcal{SP}_\nu$ satisfying the condition*

$$|a^{ij}|/(1+|x|^2) + |b^i|/(1+|x|) \in L^1([0,T]\times\mathbb{R}^d, \mu_t dt), \quad 1 \le i, j \ge d.$$

Then the set $\mathcal{SP}_\nu$ consists of exactly one element.

Proof See ([8, Theorem 9.4.3]). □

3.4 Nondegenerate Diffusion Coefficients and the Lyapunov Function Condition

The function V in the following theorem is called a Lyapunov function.

Theorem 3.3 *Suppose that conditions (H1) and (H2) hold, $c \le 0$ and that $b \in L^p_{loc}([0,T]\times\mathbb{R}^d, dt\ dx; \mathbb{R}^d)$, $c \in L^{\frac{p}{2}}_{loc}([0,T]\times\mathbb{R}^d, dt\ dx)$ for some $p > d+2$. Suppose also that there exists a positive function $V \in C^2(\mathbb{R}^d)$ such that $V(x) \to +\infty$ as $|x| \to +\infty$ and for some $C \in (0,\infty)$ and all $(t,x) \in [0,T]\times\mathbb{R}^d$ we have*

$$L_0 V(t,x) \le C + CV(x).$$

Then the set $\mathcal{SP}_\nu$ contains at most one element.

Proof See ([8, Theorem 9.4.6]). □

Example 3.1 Let $V(x) = \ln(|x|^2 + 1)$ if $|x| > 1$. Then the condition $L_0 V \leq C + CV$ is equivalent to the inequality

$$2\, tr A(t,x) - 4\frac{\langle A(t,x)x, x\rangle}{|x|^2+1} + c(t,x)(|x|^2+1)\ln(|x|^2+1) + 2\langle b(t,x), x\rangle \leq C(|x|^2+1) + C(|x|^2+1)\ln(|x|^2+1). \tag{3.8}$$

Proof See ([8, Theorem 9.4.7]). □

3.5 Degenerate Diffusion Coefficients

3.5.1 A Uniqueness Result of LeBris/Lions

Here we assume that $c = 0$ in (3.1), i.e., we consider a partial differential operator of the form

$$L_0\varphi = a^{ij}\partial_{x_i}\partial_{x_j}\varphi + b^i\partial_{x_i}\varphi, \quad \varphi \in D_0 := C_0^\infty(\mathbb{R}^d), \tag{3.9}$$

where $a^{ij}, b^i, 1 \leq i, j \leq d$, are as in (3.1), and its corresponding FPKE (3.2).

Let $\sigma^{ij} : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be $\mathcal{B}(\mathbb{R}^d)$-measurable functions such that $A = \sigma\sigma^*$, where $\sigma := (\sigma^{ij})_{1\geq i,j\geq d}$. Set

$$\beta^i := b^i - \partial_{x_j} a^{ij}, \quad 1 \leq i, j \leq d.$$

The following result is due to C. LeBris and P. L. Lions (see [15, Proposition 5], and also [8, Theorem 9.8.1]).

Theorem 3.4 *Suppose that in the natural notation*

$$\sigma^{ij} \in L^2([0,T]; W^{1,2}_{loc}(\mathbb{R}^d, dx)), \quad \beta^i \in L^1([0,T]; W^{1,1}_{loc}(\mathbb{R}^d, dx)),$$

$$div\ \beta \in L^1([0,T]; L^\infty(\mathbb{R}^d, dx)),\ \frac{|\beta|}{1+|x|} \in L^1([0,T]; L^1(\mathbb{R}^d, dx)) + L^1([0,T]; L^\infty(\mathbb{R}^d, dx)),$$

$$\frac{\sigma^{ij}}{1+|x|} \in L^2([0,T];\ L^2(\mathbb{R}^d,dx)) + L^2([0,T];L^\infty(\mathbb{R}^d,dx)).$$

Then, for every initial condition given by density ρ_0 from $L^1(\mathbb{R}^d,dx)\cap L^\infty(\mathbb{R}^d,dx)$ there exists a unique solution to (3.2) with $\mu_0 := \rho_0 dx$ in the class

$$\{\rho\colon \rho\in L^\infty([0,T];L^1(\mathbb{R}^d,dx)\cap L^\infty(\mathbb{R}^d,dx)),\\ \sigma^*\nabla\rho\in L^2([0,T];L^2(\mathbb{R}^d,dx)).\}$$

3.5.2 Uniqueness in the Class of Absolutely Continuous Paths of Probability Measures

Here we assume

(H3) (H1) is satisfied with $\gamma=\gamma(U),\ M=M(U)$, independent of the ball U and $(t,x)\mapsto A(t,x)$ is Lipschitz in t and x on $[0,T]\times\mathbb{R}^d$, $T>0$.
(H4) $b\in L^\infty([0,\infty)\times\mathbb{R}^d,dt\,dx;\mathbb{R}^d)$.

Furthermore, we fix a $\mathcal{B}([0,T]\times\mathbb{R}^d)$-measurable non-negative function $\tilde\rho\colon[0,\infty)\times\mathbb{R}^d\to[0,\infty)$.

Consider the operator

$$L_0\varphi=\tilde\rho\ div(A\nabla\varphi)+\sqrt{\tilde\rho}\langle b,\nabla\varphi\rangle,\ \varphi\in D_0:=C_0^\infty(\mathbb{R}^d). \tag{3.10}$$

and its corresponding FPKE (3.2).

Define $\mathcal{Z}_\nu$ to be the set of all $(\mu_t)_{t\in[0,T]}\in\mathcal{SP}(T)$ such that $\mu_0=\nu$ and μ_t dt is absolutely continuous w.r.t. dxdt with density $z:=\frac{d(\mu_t\ dt)}{dx\ dt}$ satisfying the following properties:

$$(\mu_t)_{t\in[0,T]} \text{ solves the FPKE corresponding to (3.10).} \tag{3.11}$$

$$\mu_t(\mathbb{R}^d)=1 \text{ for } dt\text{-a.e. } t\in[0,T]. \tag{3.12}$$

$$\tilde\rho z\in L^2([0,T]\times U,dt\ dx) \text{ for all balls } U\subset\mathbb{R}^d. \tag{3.13}$$

$$\lim_{N\to\infty}\int_0^T\int_{N\le|x|\le 2N}\left[\frac{\sqrt{\tilde\rho(t,x)}+\tilde\rho(t,x)}{1+|x|}z(t,x)+\frac{\tilde\rho^2(t,x)}{1+|x|^2}z^2(t,x)\right]dx\ dt=0. \tag{3.14}$$

Theorem 3.5 *Suppose that (H3) and (H4) hold. Then $\mathcal{Z}_\nu$ contains at most one element.*

Proof This follows from ([8, Theorem 9.8.2]). □

4 Applications to the Markov Uniqueness Problem

4.1 The Framework

Also in this section we take $(E, \mathcal{B}) := (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and $m := \rho\, dx$, where

$$\rho \in L^1_{loc}(\mathbb{R}^d, dx), \quad \rho > 0\ dx\text{-a.e.}$$

We consider the following partial differential operator:

$$L_0\varphi = \frac{1}{\rho}\partial_{x_i}(\rho\, a^{ij}\partial_{x_j}\varphi) + c\,\varphi, \quad \varphi \in D_0 := C_0^\infty(\mathbb{R}^d), \tag{4.1}$$

where a^{ij}, $1 \le i, j \le d$, and c satisfy assumption (A) below, which we assume to hold throughout this section:

(A) $a^{ij}, c : \mathbb{R}^d \to \mathbb{R}$ are $\mathcal{B}(\mathbb{R}^d)$ measurable, $c \le 0$, and $A(x) := (a^{ij}(x))_{1\le i,j\le d}$ is a nonnegative definite matrix for all $x \in \mathbb{R}^d$. Furthermore,

$$a^{ij} \in W^{1,1}_{loc}(\mathbb{R}^d, dx) \cap L^2_{loc}(\mathbb{R}^d, \rho\, dx);$$
$$c, \partial_{x_i}a^{ij} \in L^2_{loc}(\mathbb{R}^d, \rho\, dx), \rho^{\frac{1}{2}} \in W^{1,1}_{loc}(\mathbb{R}^d, dx)$$

such that

$$a^{ij}\rho^{-\frac{1}{2}}\partial_{x_i}\rho^{\frac{1}{2}} \in L^2_{loc}(\mathbb{R}^d, \rho\, dx)$$

for all $1 \le i, j \le d$.

Remark 4.1 We note that (A) is a standard a priori assumption on L_0 in (4.1), because it implies the following:

(i) for every $\varphi \in C_0^\infty(\mathbb{R}^d)$

$$L_0\varphi = a^{ij}\partial_{x_i}\partial_{x_j}\varphi + (\partial_{x_i}a^{ij})\partial_{x_j}\varphi + 2\rho^{-\frac{1}{2}}\partial_{x_i}\rho^{\frac{1}{2}}a^{ij}\partial_{x_j}\varphi + c\varphi, \tag{4.2}$$

and $(L_0, C_0^\infty(\mathbb{R}^d))$ is symmetric on $L^2(\mathbb{R}^d, \rho\, dx)$, i.e., $L_0 \subset L_0^*$, where the adjoint is taken in $L^2(\mathbb{R}^d, \rho\, dx)$.

(ii) The nonnegative definite symmetric bilinear from

$$\begin{aligned}\mathcal{E}_0(\psi, \varphi) : &= -\int \psi\, L_0\,\varphi\,\rho\, dx \\ &= \int \langle A\nabla\psi, \nabla\varphi\rangle_{\mathbb{R}^d}\,\rho\, dx - \int c\,\psi\,\varphi\,\rho\, dx; \quad \psi, \varphi \in C_0^\infty(\mathbb{R}^d),\end{aligned}$$

is a symmetric pre-Dirichlet form, hence its closure $(\mathcal{E}_F, D(\mathcal{E}_F))$ is a symmetric Dirichlet form, whose corresponding generator $(-L_F, D(L_F))$ is just the

Friedrichs extension of $(L_0, C_0^\infty(\mathbb{R}^d))$. In particular, $T_t^{L_F} := e^{tL_F}$, $t \geq 0$, is sub-Markovian. We refer to ([12, Sect. 3.3] and [16, Chap. II, Sect. (1a) and (1c)]) for details on the standard proofs for the above claims. In particular, for (L_0, D_0) as above

$$\mathcal{M}_{sym,M}(L_0, D_0) \neq \emptyset.$$

Below we shall present various sets of additional assumptions on a^{ij}, $1 \leq i, j \leq d$, and c so that a respective theorem from the previous section will apply to imply

$$\# \, \mathcal{M}_{sym,M}(L_0, D_0) = 1,$$

i.e., to imply Markov uniqueness for $(L_0, C_0^\infty(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d, \rho \, dx)$. We briefly repeat the set-up in each subsection to ease selective reading.

Remark 4.2 As mentioned above, we only consider time-independent coefficients for the operator in (4.1) and assume symmetry of L_0 on some weighted L^2-space over $\mathbb{R}^d$. As shown in Sect. 2, however, our approach is much more general and could be applied also to non-symmetric cases and for more general state spaces than merely $E = \mathbb{R}^d$. By time-space homogenization one can also find applications of the theorems in Sect. 3 to the cases of time-dependent coefficients (and the associated generalized Dirichlet forms; see [20] and [23]). A starting point for the nonsymmetric case could be the case of an operator L_0 as in (3.1) with time-independent coefficients and with $c \equiv 0$, which has an infinitesimally invariant measure μ, or equivalently has a stationary solution μ to its corresponding FPKE (3.2). This case has been studied intensively in [8] in Chaps. 1–5. In particular, it has been shown there that under broad conditions μ has a reasonably regular density with respect to Lebesgue measure and L_0 can be written as the sum of a symmetric operator L_{sym} on $L^2(\mathbb{R}^d, \mu)$ and a vector field b which has divergence zero with respect to μ. In this case L_0^* on $L^2(\mathbb{R}^d, \mu)$, calculated on $D_0 (= C_0^\infty(\mathbb{R}^d))$, is just given by $L_{sym} - \langle b, \nabla \rangle_{\mathbb{R}^d}$ and then one can proceed analogously as in the symmetric case to obtain Markov uniqueness results in this nonsymmetric case, which falls into the class (1) introduced in the Introduction.

4.2 Nondegenerate VMO Diffusion Coefficients

Let (L_0, D_0) be as in (4.1) [respectively, (4.2)] and assume that assumption (A) holds. Let

$$\mathcal{M}_{sym,M} := \mathcal{M}_{sym,M}(L_0, D_0)$$

be as defined in Sect. 1.

Theorem 4.1 *Suppose (A) and (H1) hold and that* $a^{ij} \in VMO_{x,loc}([0,T] \times \mathbb{R}^d)$, $1 \leq i, j \leq d$. *Additionally, assume that for* $1 \leq i, j \leq d$

$$a^{ij},\ \partial_{x_i} a^{ij} + a^{ij} \rho^{-\frac{1}{2}} \partial_{x_i} \rho^{\frac{1}{2}},\ \ c \in L^1(\mathbb{R}^d, \rho\, dx) + L^\infty(\mathbb{R}^d, \rho\, dx). \tag{4.3}$$

Then

$$\mathcal{M}_{sym,M} = \{L_F\},$$

i.e. Markov uniqueness holds for $(L_0, C_0^\infty(\mathbb{R}^d))$ *on* $L^2(\mathbb{R}^2, \rho\, dx)$.

Proof Let $L \in \mathcal{M}_{sym,M}$ and $T_t^L := e^{tL}$, $t \geq 0$. Let $u \in L^\infty(\mathbb{R}^d, \rho\, dx)$, $u \geq 0$, $\int u\ \rho\ dx = 1$ and $\mu_t^L := T_t^L u\ \rho\ dx$, $t \geq 0$. Then $(\mu_t^L)_{t \geq 0} \in \mathcal{SP}$ for all $t \geq 0$ and $\mu_0 = u\rho\, dx =: \nu$. Now let us check that $(\mu_t^L)_{t \geq 0} \in \mathcal{SP}_\nu$, i.e., satisfies (3.3)–(3.6). We have seen in (2.8) that $(\mu_t^L)_{t \geq 0}$ solves the FPKE (3.2), hence (3.3) holds.
From (4.2) it follows that L_0 in this section is of type (3.1) with

$$b^j := \partial_{x_i} a^{ij} + 2a^{ij} \rho^{-\frac{1}{2}} \partial_{x_i} \rho^{\frac{1}{2}}, 1 \leq j \leq d. \tag{4.4}$$

Since $T_t^L u \in (L^1 \cap L^\infty)(\rho\, dx)$, it follows from (A) and condition (4.3) that also (3.4), (3.5) holds, and additionally we have that

$$a^{ij}, b^j \in L^1([0,T] \times \mathbb{R}^d; \mu_t^L dt) \quad 1 \leq i, j \leq d. \tag{4.5}$$

So, it remains to check the second half of (3.6). To this end let $\chi_n \in C_0^\infty(\mathbb{R}^d)$, $n \in \mathbb{N}$, such that $\mathbb{1}_{B_n} \leq \chi_n \leq \mathbb{1}_{B_{n+1}}$ for all $n \in \mathbb{N}$, $\sup_{n\in\mathbb{N}} \|\chi_n^{'}\|_\infty$, $\sup_{n\in\mathbb{N}} \|\chi_n^{''}\|_\infty < \infty$, and $\chi_n \nearrow$ in n, where B_n denotes the ball in $\mathbb{R}^d$ with center 0 and radius n. Then by (4.3) for all $t \geq 0$

$$\begin{aligned}
\nu(\mathbb{R}^d) - \mu_t^L(\mathbb{R}^d) &= \int u\rho\, dx - \lim_{n\to\infty} \int \chi_n\ T_t^L u\ \rho\ dx \\
&= \lim_{n\to\infty} \int (1 - T_t^L \chi_n) u\ \rho\ dx) \\
&= \lim_{n\to\infty} \int (1 - \chi_n - \int_0^t T_s^L L_0 \chi_n\ ds) u\ \rho\ dx \\
&= - \lim_{n\to\infty} \int_0^t \int L_0\ \chi_n\ T_s^L\ u\ \rho\, dx\, ds \\
&= - \int_0^t \int c\ d\mu_s^L\ ds
\end{aligned} \tag{4.6}$$

and the second part of (3.6) follows even with equality sign. Hence $(\mu_t^L)_{t \geq 0} \in \mathcal{SP}_\nu$. By (4.5) it thus follows that $(\mu_t^L)_{t \geq 0}$ also lies in the set defined in (3.7). Since

T_t^L, $t \geq 0$, is uniquely determined by its values on all functions u as above and $L \in \mathcal{M}_{sym,M}$ was arbitrary, Theorem 3.1 implies that

$$\#\mathcal{M}_{sym,M} \leq 1.$$

Now the assertion follows by Remark 4.1(ii). □

4.3 Nondegenerate Locally Lipschitz Diffusion Coefficients

Let (L_0, D_0) be as in (4.1) [respectively, (4.2)] such that assumption (A) holds and let $\mathcal{M}_{sym,M} := \mathcal{M}_{sym,M}(L_0, D_0)$ be defined as in Sect. 1. In the following result we shall assume (H2) for our a^{ij}, $1 \leq i, j \leq d$, which is stronger than the local VMO-condition in Theorem 4.1. As a reward we can relax the global conditions in (4.3). We need, however, to restrict to the case $c \equiv 0$.

Theorem 4.2 *Suppose that $c \equiv 0$ and that conditions (A), (H1) and (H2) hold. Additionally, assume that for $1 \geq i, j \geq d$, and some $p > d+2$*

$$\rho^{-\frac{1}{2}} \partial_{x_i} \rho^{\frac{1}{2}} \in L^p_{loc}(\mathbb{R}^d, dx), \tag{4.7}$$

and that

$$\frac{|a^{ij}|}{(1+|x|^2)} + \frac{|\partial_{x_i} a^{ij} + a^{ij} \rho^{-\frac{1}{2}} \partial_{x_i} \rho^{\frac{1}{2}}|}{(1+|x|)} \in L^1(\mathbb{R}^d, \rho\, dx) + L^\infty(\mathbb{R}^d, \rho\, dx) \tag{4.8}$$

Then

$$\mathcal{M}_{sym,M} = \{L_F\},$$

i.e., Markov uniqueness holds for $(L_0, C_0^\infty(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d, \rho\, dx)$.

Proof Define $b = (b^j)_{1 \leq i \leq d}$ as in (4.4). We note that by (H2) we have $\partial_{x_i} a^{ij} \in L^\infty_{loc}(\mathbb{R}^d, dx)$ for $1 \leq i, j \leq d$. Let $L \in \mathcal{M}_{sym,M}$ and let $(\mu_t^L)_{t \geq 0}$, ν, χ_n, $n \in \mathbb{N}$, be as defined in the proof of Theorem 4.1. Then for every $t \geq 0$, since T_t^L is sub-Markovian, we have

$$\begin{aligned}
\mu_t^L(\mathbb{R}^d) &= \int T_t^L u\, \rho\, dx \\
&= \lim_{n\to\infty} \int \chi_n\, T_t^L u\, \rho\, dx \\
&= \lim_{n\to\infty} \int T_t^L \chi_n\, u\, \rho\, dx \\
&\leq \int u\, \rho\, dx = \nu(\mathbb{R}^d).
\end{aligned}$$

Hence (3.6) holds and then exactly as in the proof of Theorem 4.1 on checks [without using (4.8)] that by assumption (A) also (3.3)–(3.5) hold to conclude that $(\mu_t^L)_{t\geq 0} \in \mathcal{SP}_\nu$. Furthermore, since $T_t^L u \in (L^1 \cap L^\infty)(\mathbb{R}^d, \rho\, dx)$, the left-hand side of (4.8) is also an element of $L^1([0,T] \times \mathbb{R}^d, \mu_t^L dt)$, hence by (4.7) all assumptions of Theorem 3.2 are fulfilled. So, $\#\mathcal{M}_{sym,M} \leq 1$, and Remark 4.1(ii) implies the assertion. □

Remark 4.3 Let us mention the uniqueness problem studied [13] for the one-dimensional Fokker-Planck-Kolmogorov equation. For simplicity we consider the case of the unit diffusion coefficient (note that in [13] the opposite notation is used, the drift is denoted by a, but we follow our notation). The problem posed in [13, §8, p. 116] (in the case of the equation on the whole real line) is this: to find necessary and sufficient conditions in order that for every function $h \in L^1(\mathbb{R})$ with $Lh = h'' - (bh)' \in L^1(\mathbb{R})$ there is a unique solution $T(x,t,h)$ of the equation $\partial_t u = \partial_x^2 u - \partial_x(ub)$ with initial condition h in the sense of the relation $\|T(\cdot,t,h) - h\|_{L^1} \to 0$ as $t \to 0$. This setting is called Problem L_0, and in Problem L it is required in addition that the solutions with probability initial densities from the domain of definition of the operator L must be probabilistic. According to [13, Theorems 8.5 and 8.7], where the drift coefficient is assumed to be continuous, a necessary and sufficient condition for the solvability of Problem L_0 is the divergence of the integral

$$\int_0^x \exp B(y) \int_0^y \exp(-B(u))\, du\, dy, \quad \text{where} \quad B(y) = \int_0^y b(s)\, ds$$

at $-\infty$ and $+\infty$, and for the solvability of Problem L the divergence of the integral

$$\int_0^x \exp(-B(y)) \int_0^y \exp(B(u))\, du\, dy$$

at $-\infty$ and $+\infty$ is additionally required. This is the previous condition for the drift $-b$, which makes the conditions for b and $-b$ the same. In both cited theorems of Hille the closure of the operator L generates a semigroup on $L^1(\mathbb{R})$. It is proved in [7] that a probability solution is always unique in the one-dimensional case (under the stated assumptions about a and b). However, an example constructed in [7] shows that the situation is possible where for an initial condition that is a probability measure there exists a unique probability solution of the Cauchy problem, but there are also other solutions. It is worth noting that it is asserted in Remark 4.6 in [7] that if Hille's condition is violated, then for some initial condition there is no solution, but this does follow from the results in [7], because they ensure uniqueness only for probability solutions, so that one cannot rule out the possibility that existence holds for all initial solutions, but uniqueness fails in the class of signed solutions.

4.4 Nondegenerate Diffusion Coefficients and Lyapunov Function Conditions

Let (L_0, D_0) be as in (4.1) [respectively, (4.2)] and assume that assumption (A) holds. Let $\mathcal{M}_{sym,M} := \mathcal{M}_{sym,M}(L_0, D_0)$ be as defined in Sect. 1.

Theorem 4.3 *Suppose that $c \equiv 0$ and that conditions (A), (H1) and (H2) hold. Additionally, assume that (4.7) holds and that (3.8) holds with $b = (b^j)_{1\le j\le d}$ defined as in (4.4). Then*

$$\mathcal{M}_{sym,M} = \{L_F\},$$

i.e., Markov uniqueness holds for $(L_0, C_0^\infty(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d, \rho\, dx)$.

Proof The proof is completely analogous to the proof of Theorem 4.2 except for applying Theorem 3.3 and Example 3.1 instead of Theorem 3.2 and replacing condition (4.8) by (3.8). □

Remark 4.4 We would like to point out that Theorem 4.3 is close to Corollary 2.3 in [21] and to Proposition 2.9.4 in [9]. However, it is not covered by them, since ρ is not a probability density. The function ρ is not even assumed to be in $L^1(\mathbb{R}^d, dx)$ here.

4.5 Degenerate Diffusion Coefficients

4.5.1 Markov Uniqueness as a Consequence of the Results of Le Bris and Lions

Theorem 4.4 *Let $\sigma := (\sigma^{ij})_{1\le i,j\le d}$, $A := \sigma\sigma^*$ and $a^{ij} = (\sigma\sigma^*)^{ij}, 1 \le i, j \le d$, where*

$$\sigma^{ij} \in W^{1,2}_{loc}(\mathbb{R}^d, dx),\ \partial_{x_i} a^{ij} \in W^{1,2}_{loc}(\mathbb{R}^d, dx) \tag{4.9}$$

and

$$\sigma^{ij}, \partial_{x_j}\partial_{x_i} a^{ij} \in L^\infty(\mathbb{R}^d, dx),\ \frac{\partial_{x_i} a^{ij}}{1+|x|} \in L^1(\mathbb{R}^d, dx) + L^\infty(\mathbb{R}^d, dx). \tag{4.10}$$

Then condition (A) holds for $\rho \equiv 1, c \equiv 0$, and the corresponding operator $(L_0.D_0)$ from (4.2) is symmetric on $L^2(\mathbb{R}^d, dx)$. Let $\mathcal{M}_{sym,M} := \mathcal{M}_{sym,M}(L_0, D_0)$ be as defined in Sect. 1. Then

$$\mathcal{M}_{sym,M} = \{L_F\},$$

i.e., Markov uniqueness holds for $(L_0, C_0^\infty(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d, dx)$.

Proof Let $L \in \mathcal{M}_{sym,M}$ and $\mu_t^L := T_t^L u\ dx, t \geq 0$, with u as in the proof of Theorem 4.1. Then by assumptions (4.9), (4.10), we can apply Theorem 3.4 with $\rho_0 := u$, since $T_t^L u \in (L^1 \cap L^\infty)(\mathbb{R}^d, dx)$ and $\sigma^* \nabla T_t^L u \in L^2(\mathbb{R}^d, dx;\ \mathbb{R}^d)$, because $\nabla T_t^L u \in L^2(\mathbb{R}^d, dx;\ \mathbb{R}^d)$ and $\sigma^{ij} \in L^\infty(\mathbb{R}^d, dx)$, $1 \leq i, j \leq d$. Hence $\#\mathcal{M}_{sym,M} \leq 1$ and by Remark 4.1(ii) the assertion follows. □

4.5.2 Markov Uniqueness in Another Degenerate Case

Let $\rho \in (L^1 \cap L^3)(\mathbb{R}^d, dx)$ such that

$$\rho > 0,\ \int \rho dx = 1,\ \rho^{\frac{1}{2}} \in W^{1,1}_{loc}(\mathbb{R}^d, dx)$$

and $\nabla \rho^{\frac{1}{2}} \in L^\infty(\mathbb{R}^d, dx;\ \mathbb{R}^d)$, and assume that (H3) holds. Consider the operator

$$L_0\varphi := \rho\ div(A\nabla\varphi) + \sqrt{\rho}\langle A\nabla\sqrt{\rho}, \nabla\varphi\rangle_{\mathbb{R}^d}, \varphi \in D_0 := C_0^\infty(\mathbb{R}^d), \tag{4.11}$$

and its corresponding FPKE (3.2). Note that by our assumptions on A and ρ we have that $L_0 \colon D_0 \subset L^2(\mathbb{R}^d, \rho\ dx) \to L^2(\mathbb{R}^d, \rho\ dx)$ and $L_0\varphi = \frac{1}{\rho}\text{div}(\rho^2 A\nabla\varphi)$ for all $\varphi \in D_0 = C_0^\infty(\mathbb{R}^d)$, hence (L_0, D_0) is symmetric on $L^2(\mathbb{R}^d, \rho\ dx)$. Let $\mathcal{M}_{sym,M} := \mathcal{M}_{sym,M}(L_0, D_0)$ be as defined in Sect. 1.

Theorem 4.5 *Assume that (H3) holds and let ρ satisfy the assumptions specified above. Then*

$$\mathcal{M}_{sym,M} = \{L_F\},$$

i.e., Markov uniqueness holds for $(L_0, C_0^\infty(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d, \rho\ dx)$.

Proof Let $L \in \mathcal{M}_{sym,M}$ and $\mu_t^L := T_t^L u\ \rho\ dx, t \geq 0$, with u as in the proof of Theorem 4.1. We have seen in (2.8) that $(\mu_t^L)_{t\geq 0}$ solves the FPKE associated with (L_0, D_0) in (4.11). To show that it is the only such solution we are going to apply Theorem 3.5. So, let us check its assumptions for $z(t, \cdot) := T_t^L u\ \rho$ and $\tilde{\rho} := \rho$. First of all, (3.11) holds as just seen. So, let us show (3.12). As in (4.6) we have for every $t \geq 0$

$$\mu_t^L(\mathbb{R}^d) = \int u\rho\ dx + \lim_{n\to\infty} \int_0^t \int L_0\chi_n\ T_s^L u\ \rho\ dxds.$$

By our assumptions about A and since $\nabla\sqrt{\rho} \in L^\infty(\mathbb{R}^d, dx;\ \mathbb{R}^d)$, we have that for some $C \in (0, \infty)$ and all $s \geq 0$

$$\sup_n |L_0\chi_n\ T_s^L u| \leq C\|u\|_\infty(\rho + 1), \quad dx - \text{a.e.}$$

Since $\rho \in (L^1 \cap L^2)(\mathbb{R}^d, dx)$ and $L_0 \chi_n \to 0$ dx-a.e. as $n \to \infty$, we conclude that

$$\mu_t^L(\mathbb{R}^d) = \int u\rho \, dx = 1 \text{ for all } t \geq 0.$$

Next, (3.13) is clear, since $T_t^L u \in L^\infty(\mathbb{R}^d, dx)$ and $\rho \in L^\infty_{loc}(\mathbb{R}^d, dx)$, because $\nabla \rho^{\frac{1}{2}} \in L^\infty(\mathbb{R}^d, dx)$.

Finally, let us show (3.14). It suffices to show that all functions under the integral in (3.14) are in $L^1(\mathbb{R}^d, dx)$ in our case, due to our assumptions. For the first summand this is immediate, since

$$\begin{aligned}(\rho^{\frac{1}{2}} + \rho)\, z(t, \cdot) &= (\rho^{\frac{1}{2}} + \rho)\rho \; T_t^L u \\ &\leq (1 + 2\rho)\rho \|u\|_\infty \in L^1(\mathbb{R}^d, dx),\end{aligned}$$

since $\rho \in (L^1 \cap L^2)(\mathbb{R}^d, dx)$ by assumption. For the second summand we note that $\rho^{\frac{1}{2}}$ has a Lipschitz dx-version on $\mathbb{R}^d$, since $\nabla\sqrt{\rho} \in L^\infty(\mathbb{R}^d, dx;\ \mathbb{R}^d)$ by assumption. Hence $\rho^{\frac{1}{2}}$ is of at most linear growth and thus ρ of at most quadratic growth. Hence, since $\rho \in L^3(\mathbb{R}^d, dx)$, for some $C \in (0, \infty)$ and all $t \geq 0$ we have

$$\frac{\rho^2(x)}{1 + |x|^2}\, z^2(t, \cdot) \leq C\rho^3 \|u\|_\infty \in L^1(\mathbb{R}^d, dx),$$

and altogether (3.14) follows. Since (H4) also holds by our assumptions about A and $\nabla\sqrt{\rho} \in L^\infty(\mathbb{R}^d, dx;\ \mathbb{R}^d)$, we can apply Theorem 3.5 and conclude that $\#\mathcal{M}_{sym,M} \leq 1$ and again by Remark 4.1(ii) the assertion follows. □

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References

1. S. Albeverio, H. Kawabi, S.-R. Mihalache, M. Röckner, Strong uniqueness for Dirichlet operators related to stochastic quantization under exponential/trigonometric interactions on the two-dimensional torus. http://arxiv.org/abs/2004.12383 to appear in Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5), 28 p (2021)
2. S. Albeverio, Yu.G. Kondratiev, M. Röckner, Dirichlet operators via stochastic analysis. J. Funct. Anal. **128**(1), 102–138 (1995)
3. S. Albeverio, Z.-M. Ma, M. Röckner, Quasi regular Dirichlet forms and the stochastic quantization problem, in *Festschrift Masatoshi Fukushima* (World Scientific Publishing, Hackensack, NJ, 2015), pp. 27–58
4. S. Albeverio, M. Röckner, Dirichlet form methods for uniqueness of martingale problems and applications, in *Stochastic Analysis. Proceedings of Symposia in Pure Mathematics (Provi-*

dence, Rhode Island), vol. 57, ed. by M.C. Cranston, M.A. Pinsky (American Mathematical Society, 1995), pp. 513–528
5. S. Albeverio, M. Röckner, T.S. Zhang, Markov uniqueness for a class of infinite dimensional Dirichlet operators, in *Stochastic Processes and Optimal Control, Stochastic Monographs*, vol. 7, ed. by H.J. Engelbert et al. (Gordon & Breach, 1993), pp. 1–26
6. V.I. Bogachev, G. Da Prato, W. Stannat, Uniqueness of solutions to weak parabolic equations for measures. Bull. London Math. Soc. **39**(4), 631–640 (2007)
7. V.I. Bogachev, T.I. Krasovitskii, S.V. Shaposhnikov, On uniqueness of probability solutions of the Fokker-Planck-Kolmogorov equation. Sbornik Math. **212**(6), 745–781 (2021)
8. V.I. Bogachev, N.V. Krylov, M. Röckner, S.V. Shaposhnikov, *Fokker-Planck-Kolmogorov Equations* (American Mathematical Society, Providence, RI, 2015), pp. xii+479
9. V.I. Bogachev, M. Röckner, W. Stannat, Uniqueness of invariant measures and essential m-dissipativity of diffusion operators on L^1, in *Infinite Dimensional Stochastic Analysis*, ed. by P. Clément et al. (Royal Netherlands Academy of Arts and Sciences, Amsterdam, 2000), pp. 39–54
10. A. Eberle, Uniqueness and non-uniqueness of singular diffusion operators, in *Lecture Notes in Mathematics*, vol. 1718 (Springer, Berlin, 1999)
11. K.D. Elworthy, X.M. Li, Interwining and the Markov uniqueness problem on path spaces, in *Stochastic Partial Differential Equations and Applications, Lecture Notes in Pure and Applied Mathematics-VII* (Chapman & Hall/CRC, Boca Raton, FL, 2006), pp. 89–95
12. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, 2nd ed. (Walter de Gruyter, Berlin, New York, 2011), x+489 pp
13. E. Hille, The abstract Cauchy problem and Cauchy's problem for parabolic differential equations. J. Anal. Math. **3**, 81–196 (1954)
14. N.V. Krylov, Parabolic and elliptic equations with VMO coefficients. Comm. Part. Differ. Equ. **32**, 453–475 (2007)
15. C. Le Bris, P.L. Lions, Existence and uniqueness of solutions to Fokker-Planck type equations with irregular coefficients. Comm. Partial Differ. Equ. **33**, 1272–1317 (2008)
16. Z.M. Ma, M. Röckner, *Introduction to the Theory of (Non-symmetric) Dirichlet Forms* (Springer, Berlin, 1992), viii+209 pp
17. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations* (Springer, New York, 1983), viii+279 pp
18. D.W. Robinson, A. Sikora, Markov uniqness of degenerate elliptic operators. Ann. Sc. Norm. Super. Pisa Cl. Sci. **10**(3), 683–710 (2011)
19. M. Röckner, R. Zhu, X. Zhu, Restricted Markov uniqueness for the stochastic quantization of $P(\varphi)_2$ and its applications. J. Funct. Anal. **272**(10), 4263–4303 (2017)
20. W. Stannat, The theory of generalized Dirichlet forms and its applications in analysis and stochastics. Mem. Am. Math. Soc. **142**(678), viii+101 pp (1999)
21. W. Stannat, (Nonsymmetric) Dirichlet operators on L^1: existence, uniqueness and associated Markov processes. Ann. Sc. Norm. Sup. Pisa Cl. Sci. **28**(1), 99–140 (1999)
22. M. Takeda, On the uniqueness of Markovian-self-adjoint extension of diffusion operators on infinite dimensional spaces. Osaka J. Math. **22**, 733–742 (1985)
23. G. Trutnau, Stochastic calculus of generalized Dirichlet forms and applications to stochastic differential equations in infinite dimensions. Osaka J. Math. **37**(2), 315–343 (2000)

A Chip-Firing and a Riemann-Roch Theorem on an Ultrametric Space

Atsushi Atsuji and Hiroshi Kaneko

Abstract A Riemann-Roch theorem on an edge-weighted infinite graph with local finiteness was established by the present authors in [1], where the spectral gap of Laplacian associated determined by the edge-weight was investigated as the corner stone of the proof. On the other hand, as for non-archimedean metric space, the Laplacians in the construction of Hunt processes such as in [3, 5] based on the Dirichlet space theory can be highlighted. However, in those studies, a positive edge-weight was given substantially between each pair of balls with an identical diameter with respect to the ultrametric and the spectral gap is infeasible. In the present article, we rethink the notion of chip-firing and show an upper bound of function given by accumulation of chip-firing to materialize a counterpart of the dimension of linear system in ultrametric space. In the final section of this article, we establish a Riemann-Roch theorem on an ultrametric space.

Keywords Dirichlet space · Laplacian · Riemann-Roch theorem · Ultrametric space · Weighted graph

1 Introduction

A Riemann-Roch theorem on connected finite graph was initiated by Baker and Norine in [2]. Originally, on the complex plane, the exponent of lowest degree in the Laurent series around a pole admit an interpretation as multiplicity of the single pole. We note that it is also regarded as a coefficient of divisor at the pole. In accordance

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with this fact, when unit weight is given at each vertex of a graph, the notion of the divisor is justified by assignment of integer multiples of the unit weight given at each vertex of the graph. In their article, a theoretical scheme for a Riemann-Roch theorem on a locally finite graph with uniform edge-weight was proposed and a proof of a Riemann-Roch theorem was shown in [2].

Baker and Norine suggested in [2] a possibility of establishment of a Riemann-Roch theorem on infinite graph in [3]. Recently, the authors of the present article proposed a scheme for the proof of a Riemann-Roch theorem on an infinite graph. The method relies on the spectral gap of the Laplacian for deriving a boundedness of $\mathbb{Z}$-valued chip-firing function, which was crucial for establishment of dimension of the linear system determined by a divisor on infinite graph with local finiteness. On the other hand, as for infinite state space with non-archimedean metric, typical Hunt processes were constructed by Kigami in [5] and the second author in [3] based on the complete orthonormal system associated with non-local Laplacian, i.e., associated with non-local Dirichlet space. However, in the formulations in those articles, the absence of spectral gap is observed and since any pair of balls with an identical diameter admits an edge with positive weight, the set of balls with an identical diameter constitutes a set of vertices of a complete graph. This means that if there are infinitely many balls with an identical diameter in such a ultrametric space, inevitably a complete graph is brought by the set of vertices consisting of all ball with the identical diameter is not locally finite, In this article, in order to overcome the difficulty in dealing with infinite graph without local finiteness, we rethink the notion of chip-firing on the basis of orthogonal system on a space of square integrable functions with respect to a measure on the graph consistently given by edge-weight, instead of an observation on spectral gap arising from the Laplacian. After this shift of our focus, we demonstrate the proof of a Riemann-Roch theorem on ultrametric space.

In Sect. 2, we look back at the methods to proof a Riemann-Roch theorem in [2] and give a refinement of their discussion with the shift of our focus from existing chip-firing to the one given by a function in an orthogonal family of functions with respect to a measure given reasonably by the edge-weights. In Sect. 3, we focus on the situation where an initial graph with an ultrametric is given and extra graphs are added to the originally given graph to obtain an enlarged graph. We derive an upper estimate on L^2-norm of function which is given as an accumulation of chip-firings in our scheme to determine the dimension of the linear system so that the estimate is valid even after our procedure of enlargement of originally given graph. In the final section, by taking procedures to obtain a family of consistent total orders on subgraphs of the infinite graph equipped with an ultrametric, we establish a Riemann-Roch theorem on the space.

2 Laplacian and Riemann-Roch Theorem on Ultrametric Space with Finite Vertices

We take a finite graph $G = (V, E)$ consisting of a finite set V of vertices and a finite set E of edges without loops. To be more precise, E is given as a subset of $(V \times V \setminus \{\{x, x\} \mid x \in V\})/\sim$, where $\{x, y\} \sim \{y, x\} \in E$ for any pair $x, y \in V$. Accordingly, $\{x, y\}$ and $\{y, x\}$ are regarded as an identical element in E. We make a basic assumption on G as follows:

(A0) The graph G admits a finite family $\mathcal{G} = \{f_k \mid k \in \{0, 1, \ldots, N\}\}$ of non-constant $\mathbb{Q}$-valued functions on V, a Radon measure μ on V and a finite subset $\mathcal{R} \subset (0, \infty)$ such that

- (i) for each $k \in \{0, 1, \ldots, N\}$, $\int_V f_k d\mu = 0$,
- (ii) each $k \in \{0, 1, \ldots, N\}$ assigns an element $r_k \in \mathcal{R}$ such that, for any pair $k', k'' \in \{0, 1, \ldots, N\}$, if $f_{k'}$ is constant on $\{x \in V \mid f_{k''}(x) \neq 0\}$ then $r_{k'} > r_{k''}$,
- (iii) the space of all linear combinations of $\{f_k \mid k \in \{1, 2, \ldots, N\}\}$ with rational coefficients coincides with the potential space $\ell(V) = \{f : V \to \mathbb{Q}$ with $\int_V f d\mu = 0\}$.

For any pair v, w of distinct vertices in V, we define $d(v, w) = \inf\{r_k \in \mathcal{R} \mid r_k$ is associated with some f_k in the condition (ii) satisfying either $f_k(v) > 0$ and $f_k(w) < 0$ or $f_k(v) < 0$ and $f_k(w) > 0\}$ and then we assume that d is a metric on V by letting $d(v, v) = 0$ for any $v \in V$. In what follows, for any function on V, $\{x \in V \mid f(x) \neq 0\}$ is denoted by $\mathrm{supp}[f]$ and we denote the function taking $\mu(\{x\})$ at each $x \in V$ by μ.

We introduce a rational positive edge-weight $C_{x,y}$ for each edge $\{x, y\} \in E$ and the Laplacian Δ defined by $\Delta\phi(x) = \sum_{\{y,x\}\in E} C_{x,y}(\phi(x) - \phi(y))$ at each $x \in V$ for any function ϕ on V.

Example 1 Let p be a prime number. We note that $\mathbb{Z}_p$ admits the normalized Haar measure μ and denote the radius of ball B in $\mathbb{Z}_p$ with respect to the p-adic valuation by $\mathrm{r}(B)$. For a positive integer M, we take the family $\{B \mid \mathrm{r}(B) = p^{-M}\}$ of balls in $\mathbb{Z}_p$ and each element in the family is viewed as a vertex, i.e., element in the set V. As in the articles [3, 5], we assume that, for each pair $B, B' \in V$, the edge $\{B, B'\}$ admits a positive edge-weight $C_{B,B'}$ so that a Hunt process is constructed on the basis of the Dirichlet space theory assigned by the edge-weights. Consistently with a such construction of Hunt process on an orthonormal system on $L^2(\mu)$, we note that the orthonormal system can be taken the family $\{f_k \mid k \in \{1, 2, \ldots, N\}\}$ is given as follows:

When $M = 1$, $\mathbb{Z}_p$ is represented as the union of the disjoint balls $B(0, 1/p)$, $B(1, 1/p), \ldots, B(p-1, 1/p)$ and we can take an example of $\{f_k \mid k \in \{1, 2, \ldots, N\}\}$ with $N = p - 1$:

$$f_1 = (p-1)1_{B(0,1/p)} - 1_{B(1,1/p)} - \cdots - 1_{B(p-1,1/p)},$$
$$f_2 = (p-2)1_{B(1,1/p)} - 1_{B(2,1/p)} - \cdots - 1_{B(p-1,1/p)}.$$
$$\ldots$$
$$f_{p-1} = 1_{B(p-2,1/p)} - 1_{B(p-1,1/p)},$$

and then $\mathcal{R} = \{1\}$ induces the same metric d as the ordinary p-adic metric due to the definition of our metric. By taking a uniform edge-weight $C_{B,B'} = \dfrac{1}{p(p-1)}$ regardless of the choice of edge $\{B, B'\}$, we can redefine the normalized Haar measure μ on V by $\mu(\{B\}) = \sum_{B' \neq B} C_{B,B'} = 1/p$ for any $B \in V$.

When $M = 2, \mathbb{Z}_p$ is represented as the union of disjoint balls $B(0, 1/p), B(1, 1/p), \ldots, B(p-1, 1/p)$, each $B(l, 1/p)$ of which is given as the union of $B(l, 1/p^2) = \varphi_{p,l}(B(0, 1/p)), B(p+l, 1/p^2) = \varphi_{p,l}(B(1, 1/p)), \ldots, B(p(p-1)+l, 1/p^2) = \varphi_{p,l}(B(p-1, 1/p))$, where $\varphi_{p,l}(x) = px + l,\ l \in \{0, \ldots, p-1\}$. We can take an example of $\{f_k \mid k \in \{1, 2, \ldots, N\}\}$ with $N = p^2 - 1$. In fact, in addition to

$$f_1 = (p-1)1_{B(0,1/p)} - 1_{B(1,1/p)} - \cdots - 1_{B(p-1,1/p)}$$
$$f_2 = (p-2)1_{B(1,1/p)} - 1_{B(2,1/p)} - \cdots - 1_{B(p-1,1/p)}.$$
$$\ldots$$
$$f_{p-1} = 1_{B(p-2,1/p)} - 1_{B(p-1,1/p)},$$

we employ also

$$f_p = f_1 \circ \varphi_{p,0}^{-1}, f_{p+1} = f_2 \circ \varphi_{p,0}^{-1}, \ldots, f_{p-1+(p-1)l+m} = f_m \circ \varphi_{p,l}^{-1}, \ldots, f_{p^2-1} = f_{p-1} \circ \varphi_{p,p-1}^{-1}.$$

as the functions in $\mathcal{G}$ and $\mathcal{R} = \{1/p, 1\}$ induces the same metric d as the ordinary p-adic metric due to the definition of our metric. By taking edge-weight

$$C_{B,B'} = \begin{cases} \dfrac{t}{p^3(p-1)} & \text{if } d(B, B') = 1/p, \\ \dfrac{1-t}{p^2(p-1)} & \text{if } d(B, B') = 1 \end{cases}$$

with some $t \in \mathbb{Q} \cap (0, 1)$, we can redefine a normalized Haar measure μ on V by $\mu(\{B\}) = \sum_{B' \neq B} C_{B,B'} = 1/p^2$ for any $B \in V$.

We can classify the functions in $\mathcal{G}$ in the previous example with $M = 2$, according to the diameter of the support of function with respect to the metric d: $\mathcal{G}_{1/p} = \{f \in \mathcal{G} \mid$ the diameter of supp$[f]$ is $1/p\} = \{\varphi_{p,k'}^* f_k \mid k, k' \in \{0, 1, \ldots, p-1\}\}$ and $\mathcal{G}_1 = \{f \in \mathcal{G} \mid$ the diameter of supp$[f]$ is $1\} = \{f_k \mid k \in \{0, 1, \ldots, p-1\}\}$.

In what follows, for any finite graph, we define $\Delta f(x) = \sum_{y \in V \setminus \{x\}} C_{x,y}(f(x) - f(y))$ for any $\mathbb{Z}$-valued function f and $i(x) = \min\{|\Delta f(x)| \mid f : V \to \mathbb{Z}$ satisfying $f(x) = 0$ and $\Delta f(x) \neq 0\}$, which is viewed as the minimum positive absolute value of feasible current flows at the grounded vertex x given by integer valued voltage $f(y)$ with $y \in V \setminus \{x\}$.

A divisor on the graph G is given by $D = \sum_{x \in V} \ell(x) i(x) 1_{\{x\}}$ and its degree $\deg(D)$ is defined by $\deg(D) = \sum_{x \in V} \ell(x) i(x)$. A divisor $D = \sum_{x \in V} \ell(x) i(x) 1_{\{x\}}$ is said to be effective if $\ell(x) \geq 0$ for all $x \in V$. Since f is $\mathbb{Z}$-valued, $\Delta f(x)$ is given as an integer multiple of $i(x)$ at each $x \in V$ and the Laplacian Δf will be identified with the divisor $\sum_{x \in V} \Delta f(x) 1_{\{x\}}$. This identification makes it possible to add the Laplacian Δf to any divisor. The degree $\deg(D) = \sum_{x \in V} \ell(x) i(x)$ enjoys the following:

Lemma 2.1 *For any $\mathbb{Z}$-valued function f on V,* $\deg(\Delta f) = 0$ *and* $\deg(D) = \deg(D + \Delta f)$.

Proof It suffices to show the first identity. It is follows from the trivial identity $C_{x,y}(f(x) - f(y)) = -C_{y,x}(f(y) - f(x))$. □

In the sequel, we define the sphere centered at v with radius s by $S(v, s) = \{x \in V \mid d(x, v) = s\}$ and the ball centered at v with radius r by $B(v, r)$.

For $v \in V$ and $E \subset S(v, s)$, a $\mathbb{Q}$-valued function g on V satisfying $\int_V g d\mu = 0$ is called an inward chip-firing of E toward v if g takes positive constant on some $F \subset B(v, r)$ with $r < s$ and negative constant on E. Without specifying E, we may call g an inward chip-firing toward v. A $\mathbb{Q}$-valued function g on V satisfying $\int_V g d\mu = 0$ is called an outward chip-firing with respect to v if g takes a positive constant on some $B \subset B(v, r)^c$ with $r \in \mathcal{R}$ and $g(v) < 0$. A chip-firing $\frac{1}{\mu} \Delta f$ with $f : V \to \mathbb{Z}$ is said to be unit if f/k can not be $\mathbb{Z}$-valued for any integer $k \geq 2$.

Here after, added to (A0) above, we impose the following as well:

(A0) (iv) There exists an orthogonal family $\mathcal{G}$ of $\mathbb{Q}$-valued functions with respect to some positive rational valued measure μ and each function in $\mathcal{G}$ is orthogonal to constant functions and that, for any $g \in \mathcal{G}$, there exists a $\mathbb{Z}$-valued function f on V such that $g = \frac{1}{\mu} \Delta f$ and supp$[g]$ =supp$[f]$.
(v) There exists a reference vertex v_0 such that any subset $E \subset S(v_0, r)$ with $r \in \mathcal{R}$ admits an inward chip-firing of E toward v_0.
(vi) For any pair $v, v' \in V$, there is an outward chip-firing $g \in \mathcal{G}$ with respect to v such that $g(v') > 0$.

The choice of unit chip-firing as in (v) is unique under the maximal choice of the support of chip-firing, i.e., there exists a unique unit g of chip-firing which is an inward chip-firing of E with respect to v_0 such that supp$[g'] \subset$ supp$[g]$ is satisfied as long as g' is inward chip-firing of E toward v_0 other than g.

Lemma 2.2 *For any $E \subset V$,* $\frac{1}{\mu} \Delta 1_E = \sum_{f_i \in \mathcal{G}, \mathrm{supp}[f_i] \cap B(v_0, r_j) \neq \emptyset} c_i f_i$ *with some* $c_1, \ldots, c_N \in \mathbb{Q}$.

Proof Since each $g_* \in \mathcal{G}$ admits expression

$$g_* = \sum_{i=1}^{\nu} \frac{t_i}{\mu(F)} 1_{E_i} - \sum_{j=1}^{\kappa} \frac{s_j}{\mu(E)} 1_{F_j}$$

with rational numbers $t_1, \ldots, t_\nu, s_1, \ldots, s_\kappa$ satisfying $\sum_i^\nu t_i = \sum_j^\kappa s_j$, disjoint subsets $E_1, \ldots, E_\nu$ of E satisfying $\cup_i E_i = E$ and disjoint subsets $F_1, \ldots, F_\kappa$ of F satisfying $\cup_j F_j = F$, we see that

$$\|g_*\|_{L^2(\mu)} = \left(\sum_{i=1}^{\nu} \Big(\frac{t_i}{\mu(F)}\Big)^2 \mu(E_i) + \sum_{j=1}^{\kappa} \Big(\frac{s_j}{\mu(E)}\Big)^2 \mu(F_j) \right)^{1/2}.$$

We note that this sort of expression on L^2-norm is valid for each $f_i \in \mathcal{G}$.

Since $\int_V \frac{1}{\mu} \Delta 1_E d\mu = 0$ and assumption (iii), $\frac{1}{\mu}\Delta 1_E = \sum_{f_i \in \mathcal{G}, \mathrm{supp}[f_i] \cap B(v_0, r_j) \neq \emptyset} c_i f_i$ with some $c_1, \ldots, c_N \in \mathbb{R}$. On the other hand, the coefficient c_i is given as $\left(\frac{1}{\mu}\Delta 1_E, f_i\right)_{L^2(\mu)} / \|f_i\|^2_{L^2(\mu)}$. By combining this with the explicit expression on $\|f_i\|_{L^2(\mu)}$, we can conclude that $c_1, \ldots, c_N \in \mathbb{Q}$. □

We denote the set consisting of all total orders on V by $\mathcal{O}$ or more specifically by $\mathcal{O}_V$ and for a divisor $D = \sum_{x \in V} \ell(x) i(x)$, $\sum_{x \in V, \ell(x) > 0} \ell(x) i(x)$ by $\deg^+(D) = \sum_{x \in V} \ell(x) i(x)$. Similarly to observations by Baker and Norine's article [2], for a Riemann-Roch theorem on finite graph, the minimization of

$$\min\{\deg^+(D + \Delta f - \nu_O) \mid f : V \to \mathbb{Z} \text{ and } O \in \mathcal{O}_V\}$$

with $\nu_O = \sum_{x \in V} (\sum_{y \in V \setminus \{x\}, y < x} C_{x,y} - i(x)) 1_{\{x\}}$ is required. In fact, $r(D)$ called the dimension of the linear system assigned by D is given by the minimized value of $\deg^+(D + \Delta f - \nu_O) - i_{(G,C)}$, where $i_{(G,C)} = \min\{|\sum_{x \in V} \ell(x) i(x)| \in (0, \infty) \mid \ell : V \to \mathbb{Z}\}$.

Take an arbitrarily fixed $v_0 \in V$ and $r_{j_G} = \max\{r \in \mathcal{R} \mid \mathrm{supp}[f] \subset B(v_0, r)$ for any $f \in \mathcal{G}\}$. For given $D = \sum_{x \in V} \ell(x) i(x) 1_{\{x\}}$, a j_G-dimensional vector

$$\mathbf{V}_1(D) = \Big(\sum_{z \in S_{j_G}, \ell(z) < 0} \ell(z) i(z), \ldots, \sum_{z \in S_1, \ell(z) < 0} \ell(z) i(z) \Big)$$

and a $j_G + 1$-dimensional vector

$$\mathbf{V}_2(D) = \Big(\sum_{z \in S_0} \ell(z) i(z), \sum_{z \in S_1} \ell(z) i(z), \ldots, \sum_{z \in S_{j_G}} \ell(z) i(z) \Big),$$

are introduced, where $S_j = \{v \in V \mid d(v_0, v) = r_j\}$.

Now we take the subfamily

$$\mathcal{D}' = \left\{D' \mid D' \text{ attains the maximum } \max_{D'\sim D} \mathbf{V}_1(D') \text{ in the sense of the lexicographical order}\right\},$$

of all divisors on G, we subsequently take the subfamily $\mathcal{D}''$ of $\mathcal{D}'$ given by

$$\mathcal{D}'' = \left\{D'' \in \mathcal{D}' \mid D'' \text{ attains the maximum } \max_{D''\sim D} \mathbf{V}_2(D'') \text{ in the sense of the lexicographical order}\right\}.$$

First, to attain $\max_{D'\sim D} \mathbf{V}_1(D')$ in the sense of the lexicographical order, we can use $f : V \to \mathbb{Z}$ given as a linear combination of the functions in $\mathcal{G}$ with a non-zero rational coefficient of the inward chip-firing. This is because the discussion for the proof of Proposition 3.1 in [2] is completed even in our case by performing such non-zero rational multiples of the inward chip-firing toward v_0. Second, we take a similar procedure to attain $\max_{D''\in\mathcal{D}'} \mathbf{V}_2(D'')$ in the sense of the lexicographical order. The maximal element $D'' = \sum_{x\in V} \ell''(z)i(z)$ in $\mathcal{D}''$ is called a v_0-reduced divisor. The procedures to attain those maxima are detailed as follows:

Proposition 2.3 *For any divisor $D = \sum_{v\in V} \ell(v)i(v)1_{\{v\}}$ given on G with the set V of the vertices, there exists a v_0-reduced divisor D' which is equivalent to D.*

Proof If $D(v) < 0$ for some vertex $v \neq v_0$, then there is a vertex v' such that v' is contained in the support of some outward chip-firing g with respect to v_0 satisfying $g(v) > 0$ and $g(v') < 0$. Then we can take a $\mathbb{Z}$-valued function f such that $g = \frac{1}{\mu}\Delta f$. In particular $d(v_0, v') < d(v_0, v)$ and $D' = D + \Delta f$ satisfies $D'(v) > D(v)$ and $D'(w) \geq D(w)$ for every w in supp$[g]$ with $d(v_0, w) \geq d(v_0, v)$. It follows that $\mathbf{V}_1(D') > \mathbf{V}_1(D)$, which contradicts the choice of D. Therefore $D(v) \geq 0$ for every $v \in V(G)$ with $v \neq v_0$.

Suppose now that some non-empty subset $A \subset V(G) \setminus \{v_0\}$ satisfies $D(v) \geq \text{outdeg}_A(v)$ for every $v \in A$. Here and in the sequel, $\text{outdeg}_A(v) = \sum_{y\in A^c} C_{x,y}$. Then, $d(v_0, A) = \min_{v\in A} d(v_0, v) > 0$ and, for the divisor $D' = D - \Delta(1_A)$, we have $D'(v) \geq D(v)$ for all $v \in V \setminus A$ and $D'(v) = D(v) - \text{outdeg}_A(v) \geq 0$ for every $v \in A$. Therefore $\mathbf{V}_1(D')$ coincides with $\mathbf{V}_1(D)$ as the zero vector. For $r = d(v_0, A)$, $-\frac{1}{\mu}\Delta(1_{A\cap S(v_0,r)})$ is a chip-firing represented as a linear combination of elements in $\mathcal{G}$ with rational coefficients in which a positive rational multiple of an inward chip-firing toward v_0 is contained. Accordingly, there is a vertex v' in the support of the chip-firing with $d(v_0, v') < r$ satisfying $-\Delta(1_{A\cap S(v_0,r)})(v') > 0$. It follows that $D'(v') = D(v') - \Delta 1_A(v') \geq D(v') - \Delta 1_{A\cap S(v_0,r)}(v') > D(v')$, and consequently $\mathbf{V}_2(D') > \mathbf{V}_2(D)$, again which contradicts the choice of D. This finishes the proof of the existence. □

We can determine a sequence of elements in V as follows:

(S0) v_0 is taken as the smallest element with respect to the total order,
(S1) the second smallest v_1 is given as $\ell''(v_1)i(v_1) < \sum_{y \in V \setminus \{v_0\}} C_{v_1,y}$,
(S2) the third smallest v_2 is given as $\ell''(v_2)i(v_2) < \sum_{y \in V \setminus \{v_0,v_1\}} C_{v_2,y}$, etc.

Consequently, the total order $\mathcal{O}$ is assigned by $v_0 \le v_1 \le v_2 \le \cdots$.

After that, we can take a minimization of $\deg^+(D + \Delta f - \nu_O)$ subject to $f : V \to \mathbb{Z}$ and the dimension $r(D)$ of linear system given by D is determined by

$$r(D) = \deg^+(D + \Delta f - \nu_O) - i_{(G,C)}$$

with the minimized value in the right-hand side. We note also that $K_G = \sum_{x \in V} \{\sum_{y \in V \setminus \{x\}} C_{x,y} - 2i(x)\} 1_{\{x\}}$ is taken as a counterpart of the canonical divisor introduced in Baker and Norine's article [2].

We can derive a Riemann-Roch theorem on ultrametric space with finite vertices, which will be utilized in the following sections:

Theorem 2.4

$$r(D) - r(K_G - D) = \deg(D) + \mathfrak{e}_{(G,C)},$$

where $\mathfrak{e}_{(G,C)} = \deg(K_G) = \sum_{x \in V} i(x) - \sum_{x,y \in V, x \neq y} C_{x,y}$.

We close this section with the following fundamental properties of $r(D)$ which are utilized later.

Lemma 2.5 (i) *If D' is effective, then $r(D) + \deg(D') \ge r(D + D')$,*
(ii) *If $-D''$ is effective, then $r(D) + \deg(D'') \le r(D + D'')$.*

Proof (i) Since there exists an effective divisor E with $\deg(E) > r(D)$ such that $D - E$ does not admit any equivalent effective divisor. It turns out that the effective divisor $D' + E$ satisfies $\deg(E + D') > r(D) + \deg(D')$ and $(D + D') - (E + D')$ does not admit any equivalent effective divisor. Accordingly, $r(D + D')$ does not exceed $r(D) + \deg(D')$.
(ii) It suffices to show that $r(D) \le r(D + D'') - \deg(D'')$. This follows from (i) by regarding $D + D''$ and $-D''$ as D and D' respectively in the identity of (i). □

3 Unification of Ultrametric Space with Finite Verteces

In this section, we deal with the case that $G = (V, E)$ is a complete graph equipped with an ultrametric d determined by the family $\mathcal{G}$ of orthonormal functions with respect to a measure μ_V on V as in the last section, the diameter of G is given by $s \in \mathcal{R}$ and G is divided into $G_0 = (V_0, E_0)$, $G_1 = (V_1, E_1), \ldots, G_m = (V_m, E_m)$ with $V_0 = V \cap B(v_0^{V_0}, r), E_0 = E \cap B(v_0^{V_0}, r), V_1 = V \cap B(v_0^{V_1}, r), E_1 = E \cap B(v_0^{V_1}, r)$, $\ldots, V_m = V \cap B(v_0^{V_m}, m)$, $E_n = E \cap B(v_0^{V_m}, r)$, where $v_0^{V_0}, \ldots, v_0^{V_m} \in V$ and $r = \max\{t \in \mathcal{R} \cap (0, s)\}$.

We consider the situation where each G_i of $G_0, \ldots, G_m$ admits a divisor D_i, the family $\mathcal{G}_i$ of functions as in the last section with $\mathcal{G}_i \subset \mathcal{G}$ and a vertex $v_0^{G_i}$ such that D_i admits a $v_0^{G_i}$-reduced equivalent divisor. Thus divisor D is determined by $D = D_0 + D_1 + \cdots + D_m$, i.e., $D_0 = D|_{V_0}, D_1 = D|_{V_1}, \ldots, D_m = D|_{V_m}$.

We give an algorithmic procedure to obtain an appropriate vertex v_0 in V for the dimension of the divisor D and for precedently required v_0-reduced divisor which is equivalent to D. For the purpose, we note that we can divide the current situation into the following four cases:

(I) there is at least one $i \in \{0, 1, \ldots, m\}$ such that D_i is equivalent to an effective divisor and so is $D_0 + \cdots + D_{i-1} + D_{i+1} + \cdots + D_m$,
(II) $D_0 + D_1 + \cdots + D_m$ is equivalent to an effective divisor and, for all $i \in \{0, 1, \ldots, m\}$, either D_i is not equivalent to any effective divisors or $D_0 + D_1 + \cdots + D_{i-1} + D_{i+1} + \cdots + D_m$ is not equivalent to any effective divisors,
(III) $D_0 + D_1 + \cdots + D_m$ is not equivalent to any effective divisors and, for all $i \in \{0, 1, \ldots, m\}$, either D_i is not equivalent to any effective divisors or $D_0 + D_1 + \cdots + D_{i-1} + D_{i+1} + \cdots + D_m$ is not equivalent to any effective divisors,
(IV) there is at least one $i \in \{0, 1, \ldots, m\}$ such that either D_i nor $D_0 + D_1 + \cdots + D_{i-1} + D_{i+1} + \cdots + D_m$ is not equivalent to any effective divisor,

In the case (I), since $D = D_0 + \cdots + D_m$ is an effective divisor, similar to the discussion of the proof of Theorem 3.3 in [2], we may take either of $v_0^{V_0}, \ldots, v_m^{V_m}$ as the reference vertex v_0 to obtain v_0-reduced divisor. In fact, we can obtain a $v_0^{V_i}$-reduced divisor of $D_0 + \cdots + D_m$ for any $i = 0, \ldots, m$ on G.

In the case (II), since $D_0 + D_1 + \cdots + D_m = \dfrac{1}{m} \sum_{i=0}^{m} (D_0 + D_1 + \cdots + D_{i-1} + D_{i+1} + \cdots + D_m)$, at least one of $\{D_0 + D_1 + \cdots + D_{i-1} + D_{i+1} + \cdots + D_m \mid i \in \{0, \ldots, m\}\}$ is equivalent to an effective divisor, (II) says that we may assume D_i is not equivalent to any effective divisors. We see that any $v_0^{V_i}$-reduced divisor D_i on G_i takes a negative value at $v_0^{V_i}$. On the other hand, since $D_0 + D_1 + \cdots + D_m$ is equivalent to an effective divisor, by adding D_i to $D_0 + \cdots + D_{i-1} + D_{i+1} + \cdots + D_m$, a divisor equivalent to an effective divisor on G is obtained. Then, $v_0^{V_i}$-reduced divisor $(D_0 + \cdots + D_m)'$ of $(D_0 + \cdots + D_m)$ takes a non-negative value at $v_0^{V_i}$. In this case, we may take $v_0^{V_i}$ as the reference vertex v_0 of G.

(III) Since $D_0 + D_1 + \cdots + D_m = \dfrac{1}{m} \sum_{i=0}^{m} (D_0 + D_1 + \cdots + D_{i-1} + D_{i+1} + \cdots + D_m)$, at least one of $\{D_0 + \cdots + D_{i-1} + D_{i+1} \cdots + D_m \mid i \in \{0, \ldots, m\}\}$ can not be equivalent to any effective divisors. The condition (III) says we can find some D_i which is equivalent to an effective divisor. On the other hand, the divisor obtained by adding D_i to $D_0 + D_1 + \cdots + D_{i-1} + D_{i+1} + \cdots + D_m$ can not be equivalent to any effective divisor. By performing a chip-firing shown in the following (P3) (In (P3), $v_0^{V_i}$ is regarded already as the reference vertex of G and denoted by $v_0^{V_0}$), we can obtain a divisor $(D_1 + \cdots + D_{i-1} + D_{i+1} \cdots + D_m)'$ taking non-negative value on $V_1 \cup \cdots \cup V_{i-1} \cup V_{i+1} \cup \cdots \cup V_m$. Then, $v_0^{V_i}$-reduced divisor $(D_0 + \cdots + D_m)'$ takes a negative value at $v_0^{V_i}$. In this case, we rename G_i as G_0 and take $v_0^{V_i}$ as the reference vertex v_0 of G.

In the case (IV), similarly to the case (I), we may take any of $v_0^{V_0}, \dots, v_m^{V_m}$ as v_0.

Corresponding to the splitting $G = (V, E)$ into $G_1 = (V_1, E_1), \dots G_n = (V_n, E_n)$, we introduce the graph $\overline{G} = (\overline{V}, \overline{E})$ determined by $\overline{V} = \{V_0, V_1, \dots, V_m\}$, $\overline{E} = \{\{V_i, V_j\}\}_{i>j}$, and define conductance between V_i, V_j by $C_{V_i,V_j} = \sum_{x \in V_i, y \in V_j} C_{x,y}$. We define $i^{\overline{V}}(V_j) = \min\{|\Delta^{\overline{V}} f(V_j)| \mid f : \overline{V} \to \mathbb{Z}$ satisfying $f(V_j) = 0$ and $\Delta^{\overline{V}} f(V_j) \neq 0\}$, where $\Delta^{\overline{V}}$ is determined by the edge-weights C_{V_i,V_j} with $i, j \in \{0, \dots, m\}$ and introduce the notation $i^{V_i}(x) = \min\{|\Delta^{V_i} f(x)| \mid f : V_i \to \mathbb{Z}$ satisfying $f(x) = 0$ and $\Delta^{V_i} f(x) \neq 0\}$, where $\Delta^{V_i} f(x) = \sum_{y \in V_i \setminus \{x\}} C_{x,y}(f(x) - f(y))$ for any $i \in \{0, 1, \dots, m\}$.

Here, we assume that

(A1) $i^{V_0}(x), i^{V_1}(x), \dots, i^{V_m}(x)$ are all integer-multiples of $i^{\overline{V}}(V_j)$ for any pair x, V_j with $x \in V = V_0 \cup V_1 \cup \dots \cup V_m$ and $V_j \in \overline{V}$, and

(A2) when each V^i has diameter r and $\overline{V}$ has diameter s with $s = \min(\mathcal{R} \cap (r, \infty))$, $\mathcal{G}_s = \{g \in \mathcal{G} \mid \operatorname{supp}[g] \subset B(v_0, s)$, g is positive on some $B(w, r)$ and negative on some $B(w', r)$ with $w' \neq w\}$ consists of linearly independent m functions including a unique unit chip-firing function g such that g is positive on $B(v_0, r)$ and negative on $S(v_0, s)$.

Here, we note that a uniquely given function g in this assumption gives an outward chip-firing with respect to v_0. In the sequel, we denote $\mathcal{G}_s \setminus \{f\}$ by $\mathcal{H}_s$.

For a divisor $D = \sum_{x \in V} \ell(x) i(x) 1_{\{x\}}$ on G, we introduce effective divisors D^+ and D^- given respectively by $D^+ = \sum_{x \in V, \ell(x)>0} \ell(x) i(x) 1_{\{x\}}$ and $D^- = -\sum_{x \in V, \ell(x)<0} \ell(x) i(x) 1_{\{x\}}$ and we denote $\sum_{x \in V, \ell(x)>0} \ell(x) i(x)$ and $\sum_{x \in V, \ell(x)<0} \ell(x) i(x)$ by $\deg^+(D)$ and $\deg^-(D)$, respectively.

Under these assumptions and notations, we see that the following proposition shows the existence of v_0-reduced divisor of any divisor D on the complete graph $G = (V, E)$.

We denote $D' \overset{\mathcal{H}_s}{\sim} D$, if $D' = D + \Delta h$ with some $\mathbb{Z}$-valued h with $\frac{1}{\mu_V} \Delta h$ written as a linear combination of functions in $\mathcal{H}_s$ with rational coefficients. Let us define ℓ by

$$\ell = \max \left\{ \min_{x \in V \setminus V_0, V_i \ni x} \left\lfloor \frac{D'(x)}{i^{\overline{V}}(V_i)} \right\rfloor \mid D' \overset{\mathcal{H}_s}{\sim} D \text{ under the minimizations of } \mathbf{V}_1(D') \right.$$
$$\left. \text{and the subsequent minimization of } \mathbf{V}_2(D') \right\}.$$

In the case (I) or (II), either of the following procedure (P1) or (P2) is taken and in the case (III) or (IV), the following procedure (P3) is taken:

(P1) If ℓ is positive, then we perform a chip-firing g including positive rational multiple of a unit inward chip-firing $g' \in \mathcal{G}_s$ toward $v_0^{V_0}$ for $D = D_0 + D_1 + \dots + D_m$ so that $(D + \Delta f)|_{V \setminus V_0}$ is effective and $\deg^+((D + \Delta f)|_{V \setminus V_0})$ is minimum, if necessary, linear combination of the functions in $\mathcal{H}_s$ with rational coefficients can be contained in the chip-firing g to minimize $\deg^+((D + \Delta f)|_{V \setminus V_0})$.

(P2) If ℓ is zero, then $(D_0 + D_1 + \cdots + D_m)|_{V\setminus V_0}$ admits an $\mathcal{H}_s$-equivalent effective divisor $D'|_{V\setminus V_0}$. We may assume that $\deg^+(D'|_{V\setminus V_0})$ is minimal. In fact, if necessary, a chip-firing h written as a linear combination of the functions in $\mathcal{H}_s$ with rational coefficients can be performed to minimize $\deg^+((D + \Delta h)|_{V\setminus V_0})$.

(P3) If ℓ is negative, then $(D_0 + D_1 + \cdots + D_m)|_{V\setminus V_0}$ does not admit any $\mathcal{H}_s$-equivalent effective divisor unless any chip-firing including negative rational multiple of the unit outward chip-firing g is performed. Accordingly, it is required to perform a chip-firing g including negative rational multiple of a unit outward chip-firing g' with respect to $v_0^{V_0}$ for $D = D_0 + D_1 + \cdots + D_m$ to obtain effective divisor on $V \setminus V_0$. Here, we note that, if necessary, a linear combination of functions in $\mathcal{H}_s$ with rational coefficients other than g' can be contained in the chip-firing g to minimize $\deg^+((D + \Delta f)|_{V\setminus V_0})$ with $\frac{1}{\mu_V}\Delta f = g$.

After taking these procedures, we can take maximization procedures on $\mathbf{V}_1(D)$ and $\mathbf{V}_2(D)$ by lexicographical order in [2] and obtain the following proposition:

Proposition 3.1 *If a divisor D_i is given on graph G_i for any $i \in \{0, 1, \ldots, m\}$, then there exists a permutation $\sigma : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, m\}$ such that G_{σ_0} admits a vertex $v_0^{V_{\sigma_0}}$ and $D_0 + \cdots + D_m$ admits $v_0^{V_{\sigma_0}}$-reduced divisor $(D_{\sigma_1} + \cdots + D_{\sigma_m})'$.*

In what follows, for any $i \in \{0, 1, \ldots, m\}$, we renumber σ_i to i for a shorter notation. We have just obtained those renumbered graphs $G_0, G_1, \ldots, G_m$ each of which admits the renumbered divisor D_i. Thus by denoting m by m_0, G_1 by $G_1^0, \ldots$ and G_m by $G_{m_0}^0$, we take an initial graph $G^{(0)} = G_0$ and by adding graphs $G_1^0, \ldots, G_{m_0}^0$ to the initial graph $G^{(0)}$ we obtain a complete graph $G^{(1)}$ determined by the set $\cup_i V_i$ of vertices and $v_0^{V_0}$-reduced divisor $D^{(1)}$ by applying the proposition. This is the end of the first step in our algorithmic procedures. Hereafter, the family $\mathcal{G}_0$ of functions inheritedly obtained through this renumbering procedures is denoted by $\mathcal{G}^{(0)}$ without confusion and the family $\mathcal{G}$ of functions is denoted by $\mathcal{G}^{(1)}$. We obtain a measure μ_0 on the renumbered set $V^{(0)}$ of vertices so that $\mathcal{G}^{(0)}$ constitutes an orthogonal family of functions with respect to μ_0 and obtain μ_1 on $V^{(1)}$ so that $\mathcal{G}^{(1)}$ constitutes an orthogonal family of functions with respect to μ_1 where μ_0 is given as the restriction of μ_1 to $V^{(0)}$.

In the second step, by adding graphs $G_1^1, \ldots, G_{m_1}^1$ to the enlarged graph $G^{(1)}$ we obtain $G^{(2)}$ by taking similar procedures including renumberings. By iterating similar procedures again and again, we can construct a sequence $G^{(0)}, G^{(1)}, \cdots$ satisfying $G^{(0)} \subset G^{(1)} \subset \cdots G^{(n)} \subset \cdots$.

We now focus on the situation where just $G_1^j, \ldots, G_{m_j}^j$ has been added to $G^{(j)}$ and $G^{(j+1)}$ is obtained. We denote the measure supported on $V^{(j+1)}$ by μ_{j+1} and assume that

(A3) there exists some positive constant $C_G > 0$ such that

$$\max\{\mu_{j+1}(V_i^j) \mid i \in \{1, \ldots, m_j\}\} \le C_G \min\{C_{V^{(j)}, V_i^j} \mid i \in \{1, \ldots, m_j\}\},$$

and we assume that

(A4) there exists a positive integer M_G such that $\sup_j m_j \leq M_G$.

We take the subfamily $\mathcal{G}_{r_j}$ of $\mathcal{G}$ determined by $\mathcal{G}_{r_j} = \{g \in \mathcal{G} \mid$ the diameter of supp$[g]$ is $r_j\}$. The family of functions represented as a linear combinations of functions in $\cup_{k=1}^{j} \mathcal{G}_{r_k}$ with rational coefficients is denoted by $\mathcal{L}[\mathcal{G}_{r_1}, \ldots, \mathcal{G}_{r_j}]$. In particular, the family of functions represented as the linear combination of functions in $\mathcal{G}_{r_j}$ with rational coefficients is denoted by $\mathcal{L}[\mathcal{G}_{r_j}]$. We denote the Laplacian on the graph $G^{(j)}$ is denoted by Δ_j. Due to Lemma 2.1, as long as the validity of $\deg^+(D|_{V^{(j+1)}} + \Delta_{j+1} f) < \deg^+(D|_{V^{(j+1)}})$ with some accumulation f of chip-firings is concerned, we may replace Δ_{j+1} with Δ_{j+k}, where $k \in \{2, 3, 4, \ldots\}$. This is because $\deg^+(D|_{V^{(j+1)}} + \Delta_{j+k} f) \geq \deg^+(D|_{V^{(j+1)}})$ for any $\frac{1}{\mu_{j+k'}} \Delta f \in \mathcal{G}_{r_{j+k'}}$ with k' satisfying $k \geq k' \geq 2$. In this respect, in this section we denote Δ_{j+1} simply by Δ. We introduce the family of functions $\mathcal{M}[\mathcal{G}_{r_j}] = \{rg \mid g \in \mathcal{G}_{r_j}, r \in \mathbb{Q}\}$.

Lemma 3.2 *If* $\frac{1}{\mu_{j+1}} \Delta f \in \mathcal{M}[\mathcal{G}_{r_{j+1}}]$,

$$deg^+(D|_{V^{(j+1)}} + \Delta f) \geq deg^+(D|_{V^{(j+1)}}) - \max\left\{ - \sum_{x \in \cup_i V_i^j, \ell(x) < 0} \ell(x) i(x), \sum_{x \in \cup_i V_i^j, \ell(x) > 0} \ell(x) i(x) \right\}.$$

Proof We recall that any function in $\mathcal{M}[\mathcal{G}_{r_{j+1}}]$ takes either only positive value or only negative value on each ball with radius r_j, unless the function vanishes on the ball. This means that $V^{(j)}$ can not intersect both of two subsets $\{x \mid \Delta f(x) > 0\}$ and $\{x \mid \Delta f(x) < 0\}$. In other words, whichever chip-firing is performed, $V^{(j)}$ can not be a source of outflow and a receptor of inflow at the same time . Accordingly, Δf induces increase either in $\deg^+(D|_{V^{(j)}})$ or in $\deg^-(D|_{V^{(j)}})$. As a result, this shows that $\deg^+(D)$ may decrease at most $\max\left\{ - \sum_{x \in \cup_i V_i^j, \ell(x) < 0} \ell(x) i(x), \sum_{x \in \cup_i V_i^j, \ell(x) > 0} \ell(x) i(x) \right\}$, by replacing D with $D + \Delta f$, where $\frac{1}{\mu_{j+1}} \Delta f \in \mathcal{M}[\mathcal{G}_{r_{j+1}}]$. □

Lemma 3.3 *If* $\frac{1}{\mu_{j+1}} \Delta f \in \mathcal{M}[\mathcal{G}_{r_{j+1}}]$ *and* $\deg^+(D|_{V^{(j+1)}} + \Delta f) < \deg^+(D|_{V^{(j+1)}})$, *then* $\|f\|_{L^2(\mu_{j+1})} \leq C_G M_G (\deg^+(D|_{V^{(j+1)} \setminus V^{(j)}}) + \deg^-(D|_{V^{(j+1)} \setminus V^{(j)}}))$ *holds.*

Proof It suffices to consider the case $- \sum_{x \in \cup_i V_i^j, \ell(x) < 0} \ell(x) i(x) > \sum_{x \in \cup_i V_i^j, \ell(x) > 0} \ell(x) i(x)$ and the case $j = 0$. For any function $g_* \in \mathcal{M}[\mathcal{G}_{r_1}]$, we already know that $\mathrm{diam}(\mathrm{supp}[g_*]) = \mathrm{diam}(V^{(1)})$. Even if a function $\frac{1}{\mu_1} \Delta f_* \in \mathcal{M}[\mathcal{G}_{r_1}]$ satisfies $\deg^+(D|_{V^{(1)}} + \Delta f_*) < \deg^+(D|_{V^{(1)}})$, the last lemma shows that $\deg^+(D|_{V^{(1)}} + \Delta f_*) \geq \deg^+(D|_{V^{(1)}}) + \sum_{x \in \cup_i V_i^1, \ell(x) < 0} \ell(x) i(x)$. We may assume that $V^{(0)}$ is grounded, i.e., $f_*(V^{(0)}) = 0$ and f_* is non-negative on $V^{(1)} \setminus V^{(0)}$ for the function f_* satisfying $\frac{1}{\mu_1} \Delta f_* = g_*$ and we have the following estimate:

$$0 \geq \sum_{i=1}^{m_0} C_{V^{(0)},V_i^0}(f_*(V^{(0)}) - f_*(V_i^0)) \geq \sum_{x\in\cup_i V_i^0, \ell(x)<0} \ell(x)i(x).$$

Consequently, we see that

$$\begin{aligned}
&\min\{C_{V^{(0)},V_i^0} \mid i \in \{1,\dots,m_0\}\} \max\{f_*(V_i^0) \mid i \in \{1,\dots,m_0\}\} \\
&\leq \sum_{i=1}^{m_0} C_{V^{(0)},V_i^0} f_*(V_i^0) \\
&\leq - \sum_{x\in\cup_j V_j^0, \ell(x)<0} \ell(x)i(x)
\end{aligned}$$

and

$$\begin{aligned}
\max\{f_*(V_i^0) \mid k \in \{1,\dots,m_0\}\} &\leq \frac{\sum_{i=1}^{m_0} C_{V^{(0)},V_i^0} f_*(V_i^0)}{\min\{C_{V^{(0)},V_i^0} \mid i \in \{1,\dots,m_0\}\}} \\
&\leq -\frac{\sum_{x\in\cup_j V_j^0, \ell(x)<0} \ell(x)i(x)}{\min\{C_{V^{(0)},V_i^0} \mid i \in \{1,\dots,m_0\}\}}.
\end{aligned}$$

This, (A.3) and (A.4) imply

$$\begin{aligned}
\sqrt{\sum_{i=1}^{m_0} \mu_1(V_i^0) f_*(V_i^0)^2} &\leq \sqrt{\frac{\sum_{i=1}^{m_0} \mu_1(V_i^0)}{\min\{C_{V^{(0)},V_i^0} \mid i \in \{1,\dots,m_0\}\}}} \\
&\quad \times \sqrt{\frac{\sum_{i=1}^{m_0} \mu_1(V_i^0)\big(-\sum_{x\in\cup_j V_j^0, \ell(x)<0} \ell(x)i(x)\big)^2}{\min\{C_{V^{(0)},V_i^0} \mid i \in \{1,\dots,m_0\}\}}} \\
&\leq C_G M_G \Bigg(-\sum_{x\in\cup_j V_j^0, \ell(x)<0} \ell(x)i(x)\Bigg).
\end{aligned}$$

This means that $\|f_*\|_{L^2(\mu_1)} \leq C_G M_G \deg^-(D|_{V^{(1)}\setminus V^{(0)}})$. On the other hand, for the case that the reversed inequality $-\sum_{x\in\cup_i V_i^j, \ell(x)<0} \ell(x)i(x) \leq \sum_{x\in\cup_i V_i^j, \ell(x)>0} \ell(x)i(x)$ is satisfied, we need a discussion on the function $\frac{1}{\mu_1}\Delta f_{**} \in \mathcal{M}[\mathcal{G}_{r_1}]$ satisfies $\deg^+ (D|_{V^{(1)}} + \Delta f_{**}) \geq \deg^+(D|_{V^{(1)}}) - \sum_{x\in\cup_i V_i^1, \ell(x)>0} \ell(x)i(x)$. We see now that f_{**} admits a similar estimate to the one for f_*, as a whole we have just assured the inequality in the assertion. □

Lemma 3.4 *If* $\frac{1}{\mu_{j+1}}\Delta f \in \mathcal{L}[\mathcal{G}_{r_{j+1}}]$ *and* $\deg^+(D|_{V^{(j+1)}} + \Delta f) < \deg^+(D|_{V^{(j+1)}})$, $\|f\|_{L^2(\mu_{j+1})} \leq C_G M_G^2(\deg^+(D|_{V^{(j+1)}\setminus V^{(j)}}) + \deg^-(D|_{V^{(j+1)}\setminus V^{(j)}}))$.

Proof Since $\mathcal{G}_{r_{j+1}}$ constitutes an orthogonal family of at most M_G functions and each function in $\mathcal{M}[\mathcal{G}_{r_{j+1}}]$ satisfies the inequality in the last lemma, we see that

$\|f\|_{L^2(\mu_{j+1})} \le C_G M_G^2(\deg^+(D|_{V^{(j+1)}\setminus V^{(j)}}) + \deg^-(D|_{V^{(j+1)}\setminus V^{(j)}}))$ for any $\frac{1}{\mu_{j+1}}\Delta f \in \mathcal{L}[\mathcal{G}_{r_{j+1}}]$. □

Remark 3.5 For any positive integers j, k, each function $u + c$ vanishing on $B(v_0, r_{j-1})$ given by $u \in \mathcal{G}_{r_j}$ and $c \in \mathbb{R}$ is orthogonal to any element $v \in \mathcal{G}_{r_{j-k}}$ with respect to the ordinary inner product of $L^2(\mu)$.

We now focus on the graph $G_0 = (V_0, E_0)$ with divisor D_0 obtained by Proposition 3.1. We note that a measure μ_0 on V_0 is given as $\mu|_{V_0}$.

Lemma 3.6 *There exists a positive constant L_{G_0} depending only on the total mass $\mu_0(V_0)$ of μ_0 and Λ_0 such that $\|f\|_{L^2(\mu_0)} \le \dfrac{1}{\Lambda_0} L_{G_0}(\deg^+(D_0) + \deg^-(D_0))$ for any minimizing element f of the minimum $\min\{\deg^+(D_0 + \Delta_0 f) \mid f : V \to \mathbb{Z}\}$, where the Laplacian Δ_0 is given by $\Delta_0\phi(x) = \sum_{\{x,y\}\in E_0} C_{x,y}(\phi(x) - \phi(y))$ for real valued function ϕ on V_0 and Λ_0 stand for the smallest positive eigenvalue of Δ_0.*

Proof It is clear that positive eigenvalues of the Laplacian Δ_0 has the minimum Λ_0. We note that any minimizing element f in the assertion admits the estimate $|\Delta_0 f(x)| \le (\deg^+(D_0) + \deg^-(D_0))$ for any $x \in V_0$. Consequently, from $\|\Delta_0 f\|_{L^2(\mu_0)} \le \sqrt{\mu_0(V_0)}(\deg^+(D_0) + \deg^-(D_0))$, we can derive that $\|f\|_{L^2(\mu_0)} \le \dfrac{1}{\Lambda_0}\sqrt{\mu_0(V_0)}(\deg^+(D_0) + \deg^-(D_0)))$. By taking L_{G_0} as $\sqrt{\mu_0(V_0)}$, we obtain the conclusion. □

In the algorithmic procedures previously shown, when $G_1^{j'}, \dots, G_{m_j}^{j'}$ are added to $G^{(j')}$, the divisors $D_1^{j'}, \dots, D_{m_j}^{j'}$ are involved in our observation. If $\sum_{k=1}^{m_{j'}}$ $(\deg^+(D_k^{j'}) + \deg^-(D_k^{j'}))$ is sufficiently small, $D^{(1)}, \dots, D^{(n)}$ with $n < j'$ do not alter even after the renumbering procedure is taken. Hereafter, we may deal only with $D^{(1)}, D^{(2)}, \dots$ with such persistence under the renumbering procedures.

A pair of divisors $D^{(n')}$ on $G^{(n')}$ and $D^{(n'')}$ on $G^{(n'')}$ with $n'' > n'$ is said to be consistent, if $D^{(n')} = \sum_{x\in V^{(n')}} \ell_{n'}(x) i^{V^{(n')}}(x)1_{\{x\}}$, $D^{(n'')} = \sum_{x\in V^{(n'')}} \ell_{n''}(x) i^{V^{(n'')}}(x)1_{\{x\}}$ and $\ell_{n'}(x) i^{V^{(n')}}(x) = \ell_{n''}(x) i^{V^{(n'')}}(x)$ for any $x \in V^{(n')}$. We start with the whole graph G which is determined as the complete graph with $V = \cup_n V^{(n)}$ and a divisor D determined by $D|_{V^{(n)}} = D^{(n)}$ with a family $\{D^{(n)}\}$ of consistent divisors in which each $D^{(n)}$ is given on $G^{(n)}$ with $n \in \{0, 1, \dots\}$.

Proposition 3.7 *Assume that $\frac{1}{\mu_{j+1}}\Delta f \in \mathcal{L}[\mathcal{G}_{r_1}, \dots, \mathcal{G}_{r_{j+1}}]$ and $\deg^+(D|_{V^{(j+1)}} + \Delta f) < \deg^+(D|_{V^{(j+1)}})$. There exists a constant L_G such that $\|f\|_{L^2(\mu_{j+1})} \le L_G C_G M_G^2(\deg^+(D|_{V^{(j+1)}}) + \deg^-(D|_{V^{(j+1)}}))$.*

Proof The proof is performed by an induction. First, from the last lemma, we see that there exists a positive constant L_G such that $\|f\|_{L^2(\mu_0)} \le L_G C_G M_G^2(\deg^+(D) + \deg^-(D))$.

Second, we assume that $\|f\|_{L^2(\mu_{j+1})} \le L_G C_G M_G^2(\deg^+(D|_{V^{(j+1)}}) + \deg^-(D|_{V^{(j+1)}}))$ for any $\frac{1}{\mu_{j+1}}\Delta f \in \mathcal{L}[\mathcal{G}_{r_1}, \dots, \mathcal{G}_{r_{j+1}}]$ satisfying $\deg^+(D|_{V^{(j+1)}} + \Delta f) < \deg^+(D|_{V^{(j+1)}})$.

Then, by Lemma 3.4, for any $\Delta f' \in \mathcal{L}[\mathcal{G}_{r_{j+2}}]$ with $\deg^+(D|_{V^{(j+2)}} + \Delta f + \Delta f') < \deg^+(D|_{V^{(j+2)}} + \Delta f)$, we see that $\|f'\|_{L^2(\mu_{j+2})} \le L_G C_G M_G^2(\deg^+(D|_{V^{(j+2)}\setminus V^{(j+1)}}) + \deg^-(D|_{V^{(j+2)}\setminus V^{(j+1)}}))$. This is because we may assume $L_G \ge 1$. By an induction on j, we conclude that the assertion is justified. □

For the dimension of the linear system assigned by divisor $D|_{V^{(j)}}$, the equivalence $D' \overset{j}{\sim} D''$ defined by the identity $D'' = D' + \Delta_j f$ with some $f : V^{(j)} \to \mathbb{Z}$ is required and the minimization of $\min_{D' \overset{j}{\sim} D|_{V^{(j)}}, O_j \in \mathcal{O}^{(j)}} (\deg^+(D' - \nu_{O_j}) - i_{(G^{(j)}, C^{(j)})})$ should be performed. Consequently, the divisor D in the previous assertion should be replaced with $D - \nu_{O_j}$ in this section, along with an appropriate choice $O_j \in \mathcal{O}^{(j)}$ for the minimization, where $\mathcal{O}^{(j)}$ stands for the set of total orders on $V^{(j)}$. The dimension $r_j(D)$ of linear system assigned by the divisor D is determined by $r_j(D) = \min_{D' \overset{j}{\sim} D|_{V^{(j)}}, O_j \in \mathcal{O}^{(j)}} (\deg^+(D' - \nu_{O_j}) - i_{(G^{(j)}, C^{(j)})})$. A function $f : V^{(j)} \to \mathbb{Z}$ is called a minimizer for $r_j(D)$, if the minimum is attained by $D' = D|_{V^{(j)}} + \Delta_j f$. Since $\deg^+(\nu_O|_{V^{(j+1)}\setminus V^{(j)}} 1_{\{x\}}) + \deg^-(\nu_O|_{V^{(j+1)}\setminus V^{(j)}} 1_{\{x\}}) \le \mu(x)$, we have the following proposition:

Proposition 3.8 *Let f be a minimizer for* $\min_{D' \overset{j+1}{\sim} D|_{V^{(j+1)}}{}_j, O_{j+1} \in \mathcal{O}^{(j+1)}} (\deg^+(D' - \nu_{O_{j+1}}) - i_{(G^{(j+1)}, C^{(j+1)})})$ *with* $\frac{1}{\mu_{j+1}} \Delta f \in \mathcal{L}[\mathcal{G}_{r_1}, \ldots, \mathcal{G}_{r_{j+1}}]$.
Then $\|f\|_{L^2(\mu_{j+1})} \le L_G C_G M_G^2(\deg^+(D|_{V^{(j+1)}}) + \deg^-(D|_{V^{(j+1)}}) + \mu(V^{(j+1)}))$.

4 Riemann-Roch Theorem on Ultrametric Space with Countably Many Vertices

In this section, we start with the situation where a sequence of graphs $G^{(1)}, G^{(2)}, \ldots$ is obtained as the result of the last section. To be more precise, we assume that each $G^{(i)} = (V^{(i)}, E^{(i)})$ admits a family of function $\mathcal{G}^{(i)}$, a measure μ_i on $V^{(i)}$ and a divisor $D^{(i)}$ such that $\mathcal{G}^{(i)}$ is an orthogonal family of functions with respect to μ_i and the sequence $\{D^{(i)}\}$ of consistent divisors determines a divisor D on G.

We take the complete graph G determined by the set $\cup_i V^{(i)}$ of vertices and then we see also that $\deg^+(D) + \deg^-(D)$ is given as the limit of $\deg^+(D^{(i)}) + \deg^-(D^{(i)})$ as $i \to \infty$.

In what follows, we assume that

(A.5) $\sum_{y \in V\setminus\{x\}} C_{x,y} \le \mu(\{x\})$ for any $x \in V$ and $\mu(V) \le 1$,

(A.6) for any pair j, n with $j > n$, any function f on the complete graph $V^{(n)}$ obtained in the third section admits an asymptotically harmonic extension g to $V^{(j)}$, in the sense that there exists constants I_G and J_{G^i} such that $\Delta_n f = \Delta_j g$ on $V^{(n)}$, $|\Delta_j g| \le \frac{I_G}{J_G^i}$ on $V^{(i+1)} \setminus V^{(i)}$ for any $i \ge n$ with $J_{G^i} > 1$ and $\|g\|_{L^2(\mu; V^{(j)})} \le I_G(\|f\|_{L^2(\mu; V^{(j)})} + 1)$.

Remark 4.1 We can build the example which fulfills the condition (A.6) by taking an advantage of scheme for a construction of Hunt process on an ultrametric space in [4]. In the article, when $\frac{1}{\mu_{n+1}}\Delta_{n+1} f$ with $\mathrm{diam}(\mathrm{supp}[f]) = r_{n+1}$ is represented as a linear combination of functions in $\cup_{k=1}^{n+1}\mathcal{G}_{r_k}$, the linear combination may contain a non-zero real constant times of some function in $\mathcal{L}[\mathcal{G}_{r_n}]$. In fact, if the Λ in the article is sufficiently small, we can construct a harmonic extension $\tilde{f}$ of f to $V^{(n+1)}$ in the sense that $\tilde{f} = f$ on $V^{(n)}$ and $\Delta_{n+1}\tilde{f} = 0$ on $V^{(n+1)}\backslash V^{(n)}$. For finding such a function $\tilde{f}$, we take $r_{n+1} = \min\{s \in \mathcal{R} \mid s > r_n = \mathrm{diam}(\mathrm{supp}[g])\}$ and determine $\tilde{f}$ as the solution $\sum_{i\in I} t_i f_i$ of the minimizing problem

$$\min\left\{\mathcal{E}_{n+1}\left(\sum_{i\in I} t_i f_i + f, \sum_{i\in I} t_i f_i + f\right) \mid t_i \in \mathbb{R} \text{ with } i \in I\right\},$$

where $I = \{i \mid f_i \in \mathcal{G}_{r_{n+1}}\}$. Since the case where $\max_{i\in I} |\mathcal{E}_{n+1}(f_i, f)/\mathcal{E}_{n+1}(f_i, f_i)|$ are small enough and $\mathcal{E}_{n+1}(f_i, f_j) = 0$ for all distinct pair i, j with $i, j \in I$ is covered in the article, a fundamental theory on the solutions of quadratic equation shows that the all absolute values of t_i's which attain the minimized value are so small that (A.6) is fulfilled. To ensure the fulfillment of (A.6) with some function f, we note that $\frac{1}{\mu_{n+1}}\Delta_{n+1}\tilde{f}$ may differ from $\frac{1}{\mu_n}\Delta_n f$ up to constant times of a function in $\mathcal{L}[\mathcal{G}_{r_1}, \ldots, \mathcal{G}_{r_n}]$. To be more precise, there exists a function $g \in \mathcal{L}[\mathcal{G}_{r_1}, \ldots, \mathcal{G}_{r_n}]$ with $\mathrm{supp}[g] \subset V^{(n)}$ such that $(\Delta_{n+1}\tilde{f} - \Delta_n f)|_{V^{(n)}} = g\mu_n$. Accordingly, we replace $\tilde{f}$ with the function $\tilde{f} - \tilde{f}'$ by taking the function $\tilde{f}'$ determined by $\mathrm{supp}[\tilde{f}'] \subset V^{(n)}$ and $\Delta_{n+1}\tilde{f}' = g\mu_{n+1}$ on $V^{(n)}$ to achieve the condition with $f = \tilde{f} - \tilde{f}'$. In fact, we see that the smallness of $\|g\|_{L^2(\mu)}$ shows the fulfillment of (A.6) with $f = \tilde{f} - \tilde{f}'$.

Lemma 4.2 *Let f_n be the minimizer for $r_n(D)$ with $\dfrac{1}{\mu_n}\Delta f_n \in \mathcal{L}[\mathcal{G}_{r_1}, \ldots, \mathcal{G}_{r_n}]$. For each n, f_n admits a sequence $\{f_j^{(n)}\}_{j>n}$ of functions such that*

(i) $f_n = f_j^{(n)}$ *on* $V^{(n)}$ *for any* $j > n$,
(ii) $\lim_{n\to\infty} \sup_{j>n} \|\mu_j(\cdot)^{-1}(\Delta_j f_j^{(n)} - D)\|_{L^1(V^{(j)}\backslash V^{(n)};\mu_j)} = 0.$

Proof We take the harmonic function $h_j^{(n)}$ on $V^{(j)} \setminus V^{(n-1)}$ with the same value as f_n on S_n. The function $g_j^{(n)}$ defined as the integer part of $h_j^{(n)}$ admits the estimate

$$\frac{|\Delta_j g_j^{(n)}(x)|}{\mu_j(x)}\Big|_{V^{(j)}\backslash V^{(n)}} \le 2 1_{V^{(j)}\backslash V^{(n)}}$$

on $\Delta_j g_j^{(n)}(x) = \sum_{y\in V^{(j)}\backslash\{x\}} C_{x,y}(g_j^{(n)}(x) - g_j^{(n)}(y))$.

In fact, since $\max_{x\in V^{(j)}\backslash V^{(n)}} |h_j^{(n)}(x) - g_j^{(n)}(x)| \le 1$, we see

$$C_{x,y}|g_j^{(n)}(x) - h_j^{(n)}(x) - g_j^{(n)}(y) - h_j^{(n)}(y)| \le 2C_{x,y}$$

for any $x \in V^{(j)} \setminus V^{(n)}$ and $y \in V^{(j)} \setminus \{x\}$. By combining this with the harmonicity of $h_j^{(n)}$ on $V^{(j)} \setminus V^{(n)}$, equivalently the fact that $x \in V^{(j)} \setminus V^{(n)}$ implies $\sum_{y \in V^{(j)} \setminus \{x\}} C_{x,y} (h_j^{(n)}(x) - h_j^{(n)}(y)) = 0$, we see that $\frac{|\sum_{y \in V^{(j)} \setminus \{x\}} C_{x,y}(g_j^{(n)}(x) - g_j^{(n)}(y))|}{\mu_j(x)} \leq 2$ for any $x \in V^{(j)} \setminus V^{(n)}$.

Accordingly, we see that the function $f_j^{(n)}$ taking f_n on $V^{(n)}$ and $g_j^{(n)}$ on $V^{(n)c}$ enjoys

$$\left\| \frac{1}{\mu_j} \Delta_j f_j^{(n)} 1_{V^{(n)c}} \right\|_{L^1(\mu_j)} \leq 2\mu_j(V^{(j)} \setminus V^{(n)}).$$

In other words, $f_j^{(n)}$ meets the conditions (i) and (ii) in the assertion. □

Lemma 4.3 *For any $\varepsilon > 0$, there exists a sequence $\{O_{N(\varepsilon/2^j)}\}$ of total orders satisfying $O_{N(\varepsilon/2^j)} \in \mathcal{O}^{(N(\varepsilon/2^j))}$ with $\mu(V^{(N(\varepsilon/2^j))^c}) < \varepsilon/2^j$ for any non-negative integer j and a sequence $\{n_j\}$ satisfying $n_1 < n_2 < \cdots$ and $n_{j+1} \geq N(\varepsilon/2^j)$ for any non-negative integer j such that*

$$r_{n_k}(D^{(n_l)}) = \Big(\min_{D' \overset{n_k}{\sim} D^{(n_l)}, O_{N(\varepsilon/2^l)} = O_{n_k}|_{V(N(\varepsilon/2^l))}, O_{n_k} \in \mathcal{O}^{(n_k)}} \deg^+(D' - \nu_{O_{n_k}}) \Big) - i(G^{(n_k)}, C^{(n_k)}), \tag{1}$$

whenever $k > l$. In particular, $k > l$ implies $O_{N(\varepsilon/2^l)} = O_{n_k}|_{V^{(N(\varepsilon/2^l))}}$ and

$$\deg^+(\nu_{O_{n_l}} - \nu_{O_{n_k}}) + \deg^-(\nu_{O_{n_l}} - \nu_{O_{n_k}}) < \mu(V^{N(\varepsilon/2^{\min\{k,l\}})^c}) < \varepsilon/2^{\min\{k,l\}} \tag{2}$$

for any positive integers k and l.

Proof We take a divisor $D^{(n_0(\varepsilon))}$ satisfying $\deg^+(D - D^{(n_0(\varepsilon))}) + \deg^-(D - D^{(n_0(\varepsilon))}) < \varepsilon$. Since the finiteness $\mu(V) < \infty$ implies $\lim_{n \to \infty} \mu(V^{(n)}) = \mu(V)$, there exists a positive integer $N(\varepsilon)$ such that $\mu(V^{(N(\varepsilon))^c}) < \varepsilon$. We may assume that $n_0(\varepsilon) \geq N(\varepsilon)$. Since the cardinality of $\mathcal{O}^{(N(\varepsilon))}$ is finite, the sequence $O_{N(\varepsilon)+1} \in \mathcal{O}^{(N(\varepsilon)+1)}$, $O_{N(\varepsilon)+2} \in \mathcal{O}^{(N(\varepsilon)+2)}, \ldots$ admits a subsequence $O_{n_1(\varepsilon)}, O_{n_2(\varepsilon)}, \ldots$ with $N(\varepsilon) \leq n_0(\varepsilon) \leq n_1(\varepsilon) < n_2(\varepsilon) < \cdots$ such that $O_{N(\varepsilon)} = O_{n_k(\varepsilon)}|_{V^{(N(\varepsilon))}}$ and

$$r_{n_k(\varepsilon)}(D^{(n_0(\varepsilon))}) = \Big(\min_{D' \overset{n_k(\varepsilon)}{\sim} D^{(n_0(\varepsilon))}, O_{N(\varepsilon)} = O_{n_k(\varepsilon)}|_{V(N(\varepsilon))}, O_{n_k(\varepsilon)} \in \mathcal{O}^{(n_k(\varepsilon))}} \deg^+(D' - \nu_{O_{n_k(\varepsilon)}}) \Big) - i(G^{(n_k(\varepsilon))}, C^{(n_k(\varepsilon))})$$

for any $k \geq 1$. We may assume that $\deg^+(D - D^{(n_1(\varepsilon))}) + \deg^-(D - D^{(n_1(\varepsilon))}) < \varepsilon/2$.

By taking sufficiently large $n_2(\varepsilon)$, we may concentrate our attention to the case $n_2(\varepsilon) \geq N(\varepsilon/2)$. Since the cardinality of $\mathcal{O}^{(N(\varepsilon/2)}$ is finite, the sequence $O_{n_2(\varepsilon)}, O_{n_3(\varepsilon)}, O_{n_4(\varepsilon)}, \ldots$ admits a subsequence $O_{n_2(\varepsilon/2)}, O_{n_3(\varepsilon/2)}, \ldots$ with $N(\varepsilon/2) \leq n_2(\varepsilon/2) < n_3(\varepsilon/2) < \cdots$ such that $O_{N(\varepsilon/2)} = O_{n_k(\varepsilon/2)}|_{V^{(N(\varepsilon/2))}}$ and

$$\begin{aligned} &r_{n_k(\varepsilon/2)}(D^{(n_1(\varepsilon))}) \\ &= \Big(\min_{D' \overset{n_k(\varepsilon/2)}{\sim} D^{(n_1(\varepsilon))}, O_{N(\varepsilon/2)} = O_{n_k(\varepsilon/2)}|_{V^{(N(\varepsilon/2))}}, O_{n_k(\varepsilon/2)} \in \mathcal{O}^{(n_k(\varepsilon/2))}} \deg^+(D' - \nu_{O_{n_k(\varepsilon/2)}}) \Big) \\ &- i(G^{(n_k(\varepsilon/2))}, C^{(n_k(\varepsilon/2))}) \end{aligned}$$

for any $k \geq 2$. We may assume that $\deg^+(D - D^{(n_2(\varepsilon/2))}) + \deg^-(D - D^{(n_2(\varepsilon/2))}) < \varepsilon/4$.

By repeating this procedure, we obtain a subsequence $n_{j+1}(\varepsilon/2^j), n_{j+2}(\varepsilon/2^j), \dots$ of $n_{j+1}(\varepsilon/2^{j-1}), n_{j+2}(\varepsilon/2^{j-1}), \dots$ with $N(\varepsilon/2^j) \leq n_{j+1}(\varepsilon/2^j) < n_{j+2}(\varepsilon/2^j) < \cdots$ such that $O_{N(\varepsilon/2^j)} = O_{n_k(\varepsilon/2^j)}|_{V^{(N(\varepsilon/2^j)}}$ and

$$\begin{aligned} &r_{n_k(\varepsilon/2^j)}(D^{(n_j(\varepsilon/2^{j-1}))}) \\ &= \Big(\min_{D' \overset{n_k(\varepsilon/2^j)}{\sim} D^{(n_j(\varepsilon/2^{j-1}))}, O_{N(\varepsilon/2^j)} = O_{n_k(\varepsilon/2^j)}|_{V^{(N(\varepsilon/2^j))}}, O_{n_k(\varepsilon/2^j)} \in \mathcal{O}^{(n_k(\varepsilon/2^j))}} \deg^+(D' - \nu_{O_{n_k(\varepsilon/2^j)}}) \Big) \\ &- i(G^{(n_k(\varepsilon/2^j))}, C^{(n_k(\varepsilon/2^j))}) \end{aligned}$$

for any $k > j$. We may assume that $\deg^+(D - D^{(n_{j+1}(\varepsilon/2^j))}) + \deg^-(D - D^{(n_{j+1}(\varepsilon/2^j))}) < \varepsilon/2^j$.

As a result, by taking $n_1 = n_1(\varepsilon), n_2 = n_2(\varepsilon/2), \dots$, we obtain the sequence $n_1, n_2, \dots$ which meets all conditions in the assertion.

Equation (2) follows from the straightforward estimate $|\nu_{O_n}(x) - \nu_{O_{n'}}(x)| \leq \mu(x)$ for any $x \in V^n \cap V^{n'}$ with $O_n|_{V^{(N(\varepsilon))}} = O_{n'}|_{V^{(N(\varepsilon))}}$. □

Proposition 4.4 *If*

$$\max_{x \in V^{(n)}} \sum_{y \notin V^{(n)}} C_{x,y} / \min_{x \in V_n} \mu(\{x\}) \to 0 \text{ as } n \to \infty, \tag{3}$$

then $\{r_{n_k}(D^{(n_l)})\}_{k=l+1}^{\infty}$ *converges as* $k \to \infty$*, for any fixed non-negative integer* l*, where* $n_1, n_2, \dots$ *is the subsequence satisfying (1) associated with a sequence* $\{O_{N(\varepsilon/2^j)}\}$ *of total orders in Lemma 4.3.*

Proof Let $\{f_j\}_{j=n}^{\infty}$ be the sequence consisting of $\mathbb{Z}$-valued functions each of which is the minimizer for $r_j(D)$ taken in Lemma 4.2 with $j \geq n$.

We take the sequence $n_1, n_2, \dots$ satisfying (1) associated with the sequence $\{O_{N(\varepsilon/2^l)}\}$ of total orders in Lemma 4.3 and take integers ℓ, k with $\ell > k$. By applying the identity

$$\Delta_{n_\ell}\phi(x) = \begin{cases} \Delta_{n_k}\phi(x) + \Big(\sum_{x \in B(v_0, r_{n_k})} \Big(\sum_{y \in B(v_0, r_{n_k})^c} (\phi(x) - \phi(y)) C_{x,y} \Big) 1_{\{x\}} \Big) & (x \in V^{(n_k)}), \\ \Delta_{n_\ell}\phi(x) & (x \in V^{(n_k)^c}), \end{cases}$$

For the function $f_{n_\ell}^{(n_k)}$ taken in Lemma 4.2 by starting with the minimizer f_{n_k}, we have

$$\begin{aligned} r_{n_\ell}(D) &= \deg^+(D'_{n_\ell} - \nu_{O_{n_\ell}}) - i(G^{(n_\ell)}, C^{(n_\ell)}) \\ &= \deg^+(D + \Delta_{n_\ell} f_{n_\ell} - \nu_{O_{n_\ell}}) - i(G^{(n_\ell)}, C^{(n_\ell)}) \\ &\le \deg^+(D + \Delta_{n_\ell} f_{n_\ell}^{(n_k)} - \nu_{O_{n_\ell}}) - i(G^{(n_k)}, C^{(n_k)}) \\ &\quad + |i(G^{(n_\ell)}, C^{(n_\ell)}) - i(G^{(n_k)}, C^{(n_k)})| \\ &= \deg^+(D + \Delta_{n_k} f_{n_k} + \Big(\sum_{y \in V^{(n_k)c}} C_{x,y}(f_{n_\ell}^{(n_k)}(x) - f_{n_\ell}^{(n_k)}(y))\Big) 1_{V^{(n_k)}} \\ &\quad + \Delta_{n_\ell} f_{n_\ell}^{(n_k)} 1_{V^{(n_k)c}} - \nu_{O_{n_\ell}}) - i(G^{(n_k)}, C^{(n_k)}) \\ &\quad + |i(G^{(n_\ell)}, C^{(n_\ell)}) - i(G^{(n_k)}, C^{(n_k)})|. \end{aligned}$$

Lemma 4.2 shows that

$$\begin{aligned} &\deg^+\Big(\sum_{x \in V^{(n_k)}} \sum_{y \in V^{(n_k)c}} C_{x,y}(f_{n_\ell}^{(n_k)}(x) - f_{n_\ell}^{(n_k)}(y))\Big) \\ &\quad + \deg^-\Big(\sum_{x \in V^{(n_k)}} \sum_{y \in V^{(n_k)c}} C_{x,y}(f_{n_\ell}^{(n_k)}(x) - f_{n_\ell}^{(n_k)}(y))\Big) \\ &\le \sum_{x \in V^{(n_k)}} \sum_{y \in V^{(n_k)c}} C_{x,y}(|f_{n_\ell}^{(n_k)}(x)| + |f_{n_\ell}^{(n_k)}(y)|) \\ &\le 4\big(\sqrt{\mu_{n_k}(V)K}(\deg^+(D) + \deg^-(D) + 1)/\min\{\mu(\{x\}) \mid x \in V^{(n_k)}\} + 1\big) \\ &\quad \times \sum_{x \in V^{(n_k)}} \sum_{y \in V^{(n_k)c}} C_{x,y} \\ &\le \rho_{n_k} \mu_{n_k}(V^{(n_k)}) \\ &\quad \times 4\big(\sqrt{\mu_{n_k}(V)K}(\deg^+(D) + \deg^-(D) + 1)/\min\{\mu_{n_k}(\{x\}) \mid x \in V^{(n_k)}\} + 1\big). \end{aligned}$$

Since our estimate on $\Delta_j f_j^{(n)}$ obtained in (ii) of Lemma 4.2 is valid for $\Delta_{n_\ell} f_{n_\ell}^{(n_k)}$, we see

$$\lim_{k\to\infty} \sup_{\ell > k} \deg^\pm \Delta_{n_\ell} f_{n_\ell}^{(n_k)} 1_{V^{(n_\ell)} \setminus V^{(n_k)}} = 0.$$

Hence,

$$\begin{aligned} r_{n_\ell}(D) \le r_{n_k}(D) &+ \deg^+(\nu_{O_{n_\ell}} - \nu_{O_{n_k}}) + \deg^-(\nu_{O_{n_\ell}} - \nu_{O_{n_k}}) \\ &+ |i(G^{(n_\ell)}, C^{(n_\ell)}) - i(G^{(n_k)}, C^{(n_k)})| + o(1). \end{aligned}$$

Combining this with $\deg^+(\nu_{O_{n_\ell}} - \nu_{O_{n_k}}) + \deg^-(\nu_{O_{n_\ell}} - \nu_{O_{n_k}}) < \varepsilon/2^k$ as obtained in (2) and $|i(G^{(n_\ell)}, C^{(n_\ell)}) - i(G^{(n_k)}, C^{(n_k)})| \to 0$ (as $k, \ell \to \infty$), it turns out that

$r_{n_\ell}(D) \leq r_{n_k}(D) + \varepsilon/2^k + o(1)$ as $\ell \to \infty$ for any k, which implies $\limsup_{\ell\to\infty} r_{n_\ell}(D) \leq \liminf_{k\to\infty} r_{n_k}(D)$, in other words, $r_{n_k}(D)$ converges as $k \to \infty$. □

The limit in this proposition depends on the choice of the sequence $\{O_{n_k}\}$ of total orders. However, as long as a divisor D is supported by a finite graph, we can define

$$r_{\{O_{n_k}\}}(D) = \lim_{k\to\infty} r_{n_k}(D).$$

Corollary 4.5 *For any divisor* $D = \sum_{x\in V_G} \ell(x)i(x)1_{\{x\}}$ *on* G *satisfying* $\sum_{x\in V_G} |\ell(x)|i(x) < \infty$, $\{r_{\{O_{n_k}\}}(D^{(n_l)})\}$ *is a Cauchy sequence, where* $\{D^{(n_l)}\}$ *stands for a sequence of divisors in Lemma 4.3.*

This and Lemma 2.5 imply the convergence of the sequence $\{r_{\{O_{n_k}\}}(D^{(n_l)})\}_{l=1}^{\infty}$ and allows us to define $r(D) = \inf_{\{O_{n_k}\}} \lim_{l\to\infty} r_{\{O_{n_k}\}}(D^{(n_l)})$ for any divisor D satisfying $\sum_{x\in V_G} |\ell(x)|i(x) < \infty$.

Thanks to the finiteness of the total volume $\mu(V)$, by the definition of the characteristic $\mathfrak{e}_{(G^{(n)},C^{(n)})}$ for each finite graph $G^{(n)}$ with edge-weights $C^{(n)} = \{C_{x,y} \mid \{x, y\} \in E_n\}$ introduced in the second section, by the assumption (A.5), it is easy to see that the sequence $\{\mathfrak{e}_{(G_n,C_n)}\}$ converges as $n \to \infty$.

For the graph G and divisor D on G given in the third section, we can assert the following Riemann-Roch theorem:

Theorem 4.6 *Let* $G = (V, E)$ *be complete graph with finite volume satisfying (3). For any divisor* D *with* $\deg^+(D) + \deg^-(D) < \infty$, *the Riemann Roch theorem holds on* G:

$$r(D) - r(K_G - D) = \deg(D) + \mathfrak{e}_{(G,C)},$$

where $\mathfrak{e}_{(G,C)} = \lim_{n\to\infty} \mathfrak{e}_{(G^{(n)},C^{(n)})}$.

Proof For a divisor D with $\deg^+(D) + \deg^-(D) < \infty$, we take a sequence $\{D^{(n_l)}\}$ associated with D in the sense of Lemma 4.2 Riemann-Roch theorem on finite weighted graphs shows that $k \geq l$ implies

$$r_{n_k}(D^{(n_l)}) - r_{n_k}(K_{G^{(n_k)}} - D^{(n_l)}) = \deg(D^{(n_l)}) + \mathfrak{e}_{(G^{(n_k)},C^{(n_k)})}.$$

From Corollary 4.5 and Proposition 4.4 starting with $D^{(n_l)}$, by letting $k \to \infty$, we can derive

$$r_{\{O_{n_k}\}}(D^{(n_l)}) - r_{\{O_{n_k}\}}(K_G - D^{(n_l)}) = \deg(D^{(n_l)}) + \mathfrak{e}_{(G,C)}.$$

By passing the limit as $l \to \infty$ and taking the infimum of the first term of the left-hand side over all of sequences $\{O_{n_k}\}$ of total orders in Lemma 4.3, it turns out that

$$r(D) - r(K_G - D) = \deg(D) + \mathfrak{e}_{(G,C)}. \quad \square$$

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References

1. A. Atsuji, H. Kaneko, A Riemann-Roch theorem on a weighted infinite graph (2022). arXiv:2201.07710 [math.CO]
2. M. Baker, S. Norine, Riemann-Roch and Abel-Jacobi theory on a finite graph. Adv. Math. **215**, 766–788 (2007)
3. H. Kaneko, A Dirichlet space on ends of tree and Dirichlet forms with a nodewise orthogonal property. Potential Anal. **41**, 245–268 (2014). https://projecteuclid.org/euclid.aihp/1380718739
4. H. Kaneko, A Dirichlet space on ends of tree and superposition of nodewise given Dirichlet forms with tier linkage, in *Festschrift Masatoshi Fukushima: In Honor of Masatoshi Fukushima's Sanju (Interdisciplinary Mathematical Sciences)* (World Scientific Pub. Co. Inc., 2015), pp. 237–263
5. J. Kigami, Transitions on a noncompact Cantor set and random walks on its defining tree. Ann. Inst. H. Poincaré Probab. Statist. **49**, 1090–1129 (2013)

Hermitizable, Isospectral Matrices or Differential Operators

Mu-Fa Chen

Abstract This paper reports the study on Hermitizable problem for complex matrix or second order differential operator. That is the existence and construction of a positive measure such that the operator becomes Hermitian on the space of complex square-integrable functions with respect to the measure. In which case, the spectrum are real, and the corresponding isospectral matrix/differntial operators are described. The problems have a deep connection to computational mathematics, stochastics, and quantum mechanics.

Keywords Hermitizable · Matrix · Differential operators · Isospectrum

Mathematics Subject Classification 15A18 · 34L05 · 35P05 · 37A30 · 60J27

According to the different objects: matrix and differential operator, the report is divided into two sections, with emphasis on the first one.

1 Hermitizable, Isospectral Matrices

Let us start at the countable state space $E = \{k \in \mathbb{Z}_+ : 0 \le k < N+1\}$ $(N \le \infty)$. Consider the tridiagonal matrix T or Q as follows:

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$$\begin{matrix} T \\ Q \end{matrix} = \begin{pmatrix} -c_0 & b_0 & & & 0 \\ a_1 & -c_1 & b_1 & & \\ & a_2 & -c_2 & b_2 & \\ & & \ddots & \ddots & b_{N-1} \\ 0 & & & a_N & -c_N \end{pmatrix},$$

where for matrix T: the three sequences (a_k), (b_k), (c_k) are complex; and for (birth-death, abbrev. BD-) matrix Q: the subdiagonal sequences (a_k) and (b_k) are positive, and the diagonal one satisfies $c_k = a_k + b_k$ for each $k < N$, except $c_N \geq a_N$ if $N < \infty$. For short, we often write T (or Q) $\sim (a_k, -c_k, b_k)$ to denote the tridiagonal matrix. It is well known that the matrix Q possesses the following property:

$$\mu_n a_n = \mu_{n-1} b_{n-1}, \quad 1 \leq n < N+1 \tag{1}$$

for a positive sequence $(\mu_k)_{k\in E}$. Actually, property (1) is equivalent to

$$\mu_n = \mu_{n-1} \frac{b_{n-1}}{a_n}, \quad 1 \leq n < N+1 \ \text{ with initial } \ \mu_0 = 1. \tag{2}$$

In other words, at the present simple situation, one can write down (μ_k) quite easily: starting from $\mu_0 = 1$, and then compute $\{\mu_k\}_{k=1}^N$ step by step (one-step iteration) along the path

$$0 \to 1 \to 2 \to \cdots.$$

At the moment, it is somehow strange to write T and Q together, since they are rather different. For T, three complex sequences are determined by 6 real sequences and Q is mainly determined by two positive sequences, or equivalently, only one real sequence. However, it will be clear later, these two sequences have some special "blood kinship", a fact discovered only three years ago [6, Sect. 3].

Clearly, for Q, property (1) is equivalent to

$$\mu_i a_{ij} = \mu_j a_{ji}, \quad i, j \in E, \tag{3}$$

provided we re-express the matrix Q as $(a_{ij} : i, j \in E)$ since except the symmetric pair (a_n, b_{n-1}) given in (1), for the other i, j, the equality (3) is trivial. However, for general real $A = (a_{ij} : i, j \in E)$, property (3) is certainly not trivial.

Definition 1 A real matrix $A = (a_{ij} : i, j \in E)$ is called *symmetrizable* if there exists a positive measure $(\mu_k : k \in E)$ such that (3) holds.

The meaning of (3) is as follows. Even though A itself is not symmetric, but once it is evoked by a suitable measure (μ_k), the new matrix $(\mu_i a_{ij} : i, j \in E)$ becomes symmetric. Every one knows that the symmetry is very important not only in nature, but also in mathematics. Now how far away is it from symmetric matrix to the symmetrizable one? Consider $N = \infty$ in particular. Then symmetry means that $\mu_k \equiv$ a nonzero positive constant, and so as a measure, μ_k can not be normalized as a probability one. Hence, there is no equilibrium statistical physics since for which, the equilibrium measure should be a Gibbs measure (a probability measure). Next, in this case, the most part of stochastics is not useful since the system should die out.

A systemic symmetrizable theory was presented by Hou and Chen in [13] in Chinese (note that it was too hard to obtain necessary references and so the paper was done without knowing what happened earlier out of China). The English abstract appeared in [14]. Having this tool at hand, our research group was able to go to the equilibrium statistical physics, as shown in [2, Chaps. 7, 11 and Sect. 14.5].

One of the advantage of the symmetric matrix is that it possesses the real spectrum. This is kept for the symmetrizable one. When we go to complex matrix, the symmetric matrix should be replaced by the Hermitian one for keeping the real spectrum. This leads to the following definition.

Definition 2 A complex matrix $A = (a_{ij} : i, j \in E)$ is called *Hermitizable* if there exists a positive measure $(\mu_k : k \in E)$ such that

$$\mu_i a_{ij} = \mu_j \bar{a}_{ji}, \quad i, j \in E, \tag{4}$$

where $\bar{a}$ is the conjugate of a.

Clearly, in parallel to the real case, even though A itself is not Hermitian, but once it is evoked by a suitable measure (μ_k), the new matrix $(\mu_i a_{ij} : i, j \in E)$ becomes Hermitian. Both A and $(\mu_i a_{ij} : i, j \in E)$ have real spectrum.

From (4), we obtain the following simple result.

Lemma 3 *In order the complex $A = (a_{ij})$ to be Hermitizable, the following conditions are necessary.*

- *The diagonal elements $\{a_{ii}\}$ must be real.*
- *Co-zero property: $a_{ij} = 0$ iff $a_{ji} = 0$ for all i, j.*
- *Positive ratio: $\dfrac{\bar{a}_{ij}}{a_{ji}} = \dfrac{a_{ij}}{\bar{a}_{ji}} > 0$ or equivalently, positive product: $a_{ij}a_{ji} > 0$.*

Proof The last assertion of the lemma comes from the following identity:

$$\frac{\alpha}{\bar{\beta}} = \frac{\alpha\beta}{|\beta|^2}, \qquad \beta \neq 0. \quad \square$$

Combining the lemma with the result on BD-matrix, we obtain the following conclusion.

Theorem 4 (Chen [6, Corollary 6]) *The complex T is Hermitizable iff the following two conditions hold simultaneously.*

- *(c_k) is real.*
- *Either $a_{k+1} = 0 = b_k$ or $a_{k+1}b_k > 0$ for each k: $0 \le k < N$.*

Then, we have

$$\mu_0 = 1, \quad \mu_n = \mu_{n-1}\frac{b_{n-1}}{\bar{a}_n} = \mu_0 \prod_{k=1}^{n} \frac{b_{k-1}}{\bar{a}_k}.$$

In practice, we often ignore the part "$a_{k+1} = 0 = b_k$" since otherwise, the matrix T can be separated into two independent blocks.

We now come to the general setup. First, we write $i \to j$ once $a_{ij} \neq 0$. Next, a given path $i_0 \to i_1 \to \cdots \to i_n$ is said to be closed if $i_n = i_0$. A closed one is said to be smallest if it contains no-cross or no round-way closed path. A round-way path means $i_0 \to i_1 \to i_2 \to i_1 \to i_0$ for example. In particular, each closed path for T must be round-way.

Theorem 5 (Chen [6, Theorem 5]) *The complex $A = (a_{ij})$ is Hermitizable iff the following two conditions hold simultaneously.*

- *Co-zero property. For each pair i, j, either $a_{ij} = 0 = a_{ji}$ or $a_{ij}a_{ji} > 0$ (which implies that (a_{kk}) is real).*
- *Circle condition. For each smallest (no-cross-) closed path $i_0 \to i_1 \to \cdots \to i_n = i_0$, the circle condition holds*

$$a_{i_0 i_1} a_{i_1 i_2} \cdots a_{i_{n-1} i_n} = \bar{a}_{i_n i_{n-1}} \cdots \bar{a}_{i_2 i_1} \bar{a}_{i_1 i_0}.$$

In words, the product of $a_{i_k i_{k+1}}$ along the path equals to the one of product of $\bar{a}_{i_{k+1} i_k}$ along the inversive direction of the path.

Proof One may check that our Hermitizability is equivalent to A being Hermitian on the space $L^2(\mu)$ of square-integrable complex function with the standard inner product

$$(f, g) = \int f\bar{g}\,\mathrm{d}\mu.$$

Hence the Hermitizability seems not new for us. However, the author does not know up to now any book tells us how to find out the measure μ. Hence, our main task is to find such a measure if possible. Here we introduce a very natural proof of Theorem 5, which is published here for the first time.

Next, in view of the construction of μ for BD-matrix Q or T, one can find out the measure step by step along a path. We now fix a path as follows.

$$i_0 \to i_1 \to \cdots \to i_{n-1} \to i_n, \qquad a_{i_k i_{k+1}} \neq 0.$$

Comparing the jumps and their rates for BD-matrix and the present A:

$$\begin{aligned} &k-1 \to k: b_{k-1}, \quad i_{k-1} \to i_k: a_{i_{k-1} i_k}, \\ &k \to k-1: \ \bar{a}_k, \quad i_k \to i_{k-1}: \bar{a}_{i_k i_{k-1}}. \end{aligned}$$

From the iteration for BD-matrix

$$\mu_n = \mu_{n-1} \frac{b_{n-1}}{\bar{a}_n},$$

it follows that for the matrix A along the fixed path above, we should have

$$\mu_{i_n} = \mu_{i_{n-1}} \frac{a_{i_{n-1} i_n}}{\bar{a}_{i_n i_{n-1}}}.$$

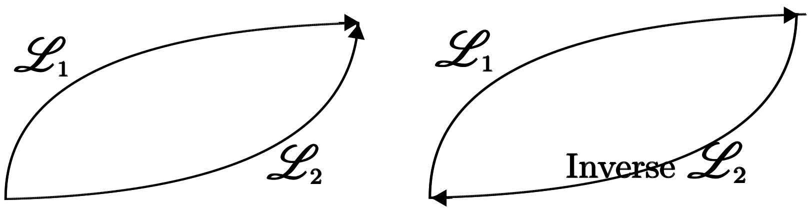
Therefore, we obtain

$$\prod_{k=1}^{n} \frac{a_{i_{k-1} i_k}}{\bar{a}_{i_k i_{k-1}}} = \frac{\mu_{i_n}}{\mu_{i_0}}. \tag{5}$$

Thus, if we fixed i_0 to be a reference point, then we can compute μ_{i_k} $(k = 1, 2, \cdots, n)$ successively by using this formula. The essential point appears now, in the present general situation, there may exist several paths from the same $j_0 = i_0$ to the same $j_m = i_n$, as shown in the left figure below. We have to show that along these two paths, we obtain the same $\mu_{i_n} = \mu_{j_m}$. That is the so-called path-independence. This suggests us to use the conservative field theory in analysis. The path-independence is equivalent to the following conclusion: the work done by the field along each closed path equals zero. This was the main idea we adopted in [13]. To see it explicitly, from (5), it follows that

$$w(\mathscr{L}_1) := \sum_{k=1}^{n} \log \frac{a_{i_{k-1} i_k}}{\bar{a}_{i_k i_{k-1}}} = \log \mu_{i_n} - \log \mu_{i_0}.$$

The left-hand side is the work done by the conservative field along the path $\mathscr{L}_1$: $i_0 \to \cdots \to i_{n-1} \to i_n$, and the right-hand side is the difference of potential of the field at positions i_n and i_0. Clearly, once $i_n = i_0$, the right-hand side equals zero (let call it the conservativeness for a moment).



Left figure: two paths from i_0 to $i_\#$: $\mathscr{L}_1$ and $\mathscr{L}_2$. **Right figure**: combining $\mathscr{L}_1$ and inversive $\mathscr{L}_2$ together, we get a closed path.

For the reader's convenience, we check the equivalence of the path-independence

$$w(\mathscr{L}_1) = w(\mathscr{L}_2)$$

and the conservativeness of the field in terms of the right figure

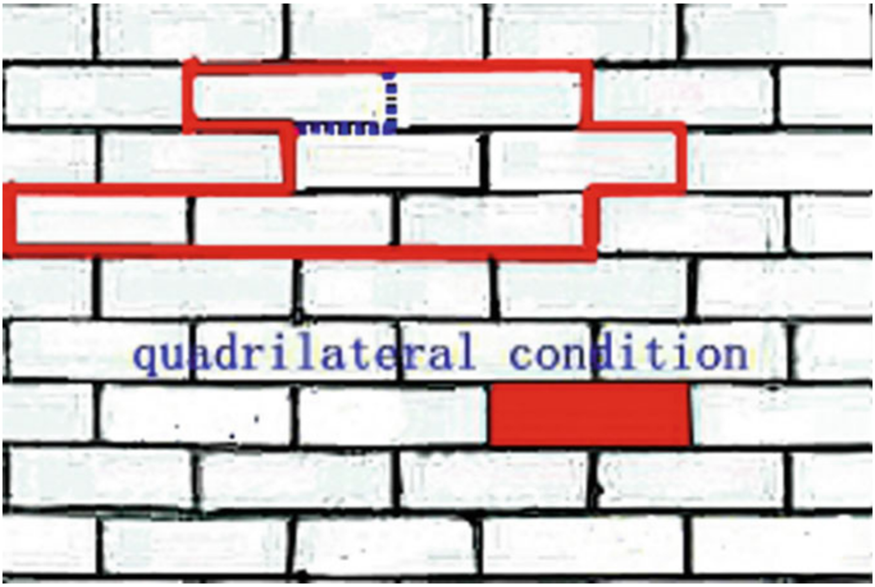
$$w(\mathscr{L}_1) + w(\text{Inverse}\,\mathscr{L}_2) = 0.$$

The conclusion is obvious by using the third assertion of Lemma 3:

$$w(\text{Inverse}\,\mathscr{L}_2) = -w(\mathscr{L}_2).$$

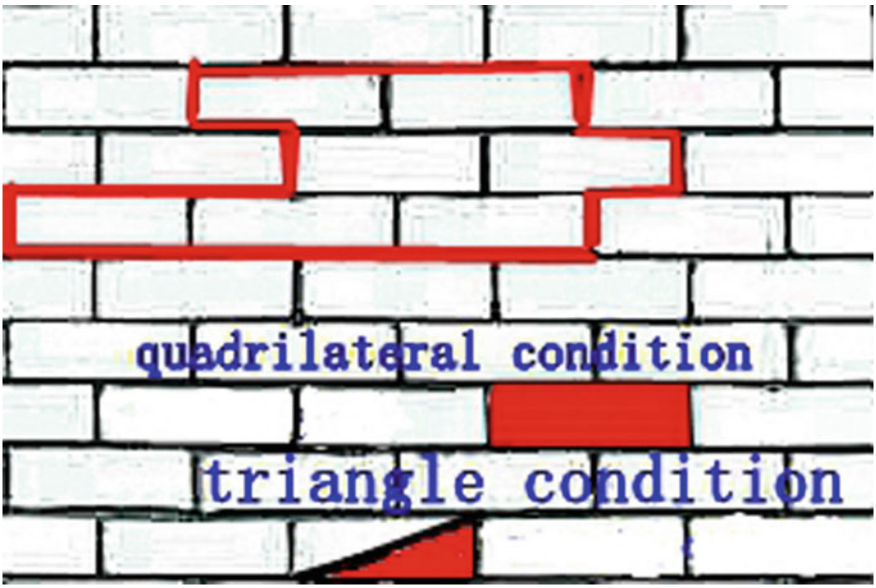
The last property is exactly the circle condition given in the theorem, and so the proof is finished. □

In the special case that A is a transition probability of a finite time-discrete Markov chain, the circle condition was obtained by Kolmogorov [15], as a criterion of the reversibility of the Markov chain. It is also interesting that at the beginning and at the end of [15], the paper by Schrödinger [17] was cited. Moreover, Kolmogorov studied the reversible diffusion in 1937 [16]. These two papers [15, 16] begun the research direction of reversible Markov processes (and also the modern Dirichlet form theory). It also indicates the tight relation between the real symmetrizable operators and equilibrium statistical physics. Nevertheless, the interacting subjects "random fields" and "interacting particle systems" were only born in 1960s. Even though there are some publications along this line, the "Schrödinger diffusion" for instance [1], we are not sure how a distance now to the original aim of Schrödinger who was looking for an equation derived from classical probability, which is as much close as possible to his wave equation in quantum mechanics.



It is regretted that the author had a chance to read [15, 16] only a few years ago when "Selected Works of A. N. Kolmogorov" appeared. Hence, the author did not know anything earlier about Kolmogorov's [15, 16]. There is a Chinese proverb that says "the ignorant are fearless". For this reason, we were brave enough to make a restriction "smallest closed path" instead of "every closed one" in the theorem and then we had gone for much far away, since the total number of the closed paths may be infinite, even not countable. To illustrate the idea, let us consider a random chosen wall above. One sees that there are a lot of closed paths. However, the smallest one is quadrilateral. Hence, one has to check only the "quadrilateral condition". To see this, look at the closed path on the top, and it consists of 7 quadrilaterals. The short path with dash line on the top separates the whole closed path into two smaller ones. To prove that it sufficient to check the "quadrilateral condition" for this model, we use induction. The idea goes as follows. We can make first the union of these two smaller closed paths (choose the clockwise direction for one of the closed path and choose anti-clockwise for the other one). Then remove the round-way path with dash line. Thus, once the work done by the field along each of the smaller closed paths equals zero, then so is the one along the original closed path since the work done by the field along the round-way path equals zero.

However, for the second wall below, the smallest closed path, except the quadrilateral, there is also triangle, so we have the "triangle condition". It is interesting, in [2, Chaps. 7 and 11], we use only these two conditions; and in [2, Sect. 14.5], we use only the triangle condition. The main reason is that for infinite-dimensional objects, their local structures are often regular and simple. Besides, in general we have an algorithm to justify the Hermitizability by computer, refer to [10, Algorithm 1].



We are now arrive at the core part of the paper: describing the spectrum of the Hermitizable matrix, which is also the core part of the so-called matrix mechanics. The next result explains the meaning of "blood kinship" mentioned at the beginning of this section.

Theorem 6 (Chen [6, Corollary 21]) *Up a shift if necessary, each irreducible Hermitizable tridiagonal matrix T is isospectral to a BD-matrix $\widetilde{Q}$ which can be expressed by the known sequences (c_k) and $(a_{k+1}b_k)$.*

The main condition we need for the above result is $c_k \geq |a_k| + |b_k|$ for every $k \in E$. For finite E, the condition is trivial since one may replace (c_k) by a shift $(c_k + m)$ for a large enough constant m. For infinite E, one may require this assumption up to a shift.

We now state the construction of $\widetilde{Q} \sim \big(\tilde{a}, -\tilde{c}_k, \tilde{b}_k\big)$. The essential point is the sequence $\big(\tilde{b}_k\big)$:

$$\tilde{b}_k = c_k - \frac{u_k}{\tilde{b}_{k-1}}, \qquad \tilde{b}_0 = c_0,$$

where $u_k := a_k b_{k-1} > 0$. This is one-step iteration, and we have the explicit expression

$$\tilde{b}_k = c_k - \cfrac{u_k}{c_{k-1} - \cfrac{u_{k-1}}{c_{k-2} - \cfrac{u_{k-2}}{\ddots\, c_2 - \cfrac{u_2}{c_1 - \cfrac{u_1}{c_0}}}}}.$$

Note that here two sequences (c_k) and (u_k) are explicit known. Having $\big(\tilde{b}_k\big)$ at hand, it is easy to write down $\tilde{a}_k = \tilde{c}_k - \tilde{b}_k$ with $\tilde{c}_k = c_k$ for $k < N$, and $\tilde{a}_N = u_N/\tilde{b}_N$ if $N < \infty$. The solution of $(\tilde{a}_k)$ and $(\tilde{c}_k)$ are automatic so that $\widetilde{Q}$ becomes a BD-matrix.

The resulting matrix $\widetilde{Q}$ looks very simple, but it contains a deep intrinsic feature. For instance, the reason is not obvious why the sequences $\big(\tilde{b}_k\big)$ and $(\tilde{a}_k)$ are positive even though so are (c_k) and (u_k). With simple description but deep intrinsic feature is indeed a characteristic of a good mathematical result.

To see the importance of the above theorem, let compare the difference of the principal eigenvector of these two matrices. First, for BD-matrix with four different boundaries, the principal eigenvectors are all monotone, except in one case, it is concave. This enables us to obtain a quite satisfactory theory of the principal eigenvalues (refer to [4]). However, since the Hermitizable T has real spectrum, form the eigenequation

$$Tg = \lambda g,$$

one sees immediately, the eigenvector g must be complex, too far away to be monotone. Thus, the principal eigenvectors of these two operators are essentially different.

It shows that we now have a new spectral theory for the Hermitizable tridiagonal matrices.

Because the intuition is not so clear why Theorem 6 should be true, two alternative proofs are presented in [7].

Theorem 7 (Chen [6, Theorem 24]) *The spectrum of a finite Hermitizable matrix A is equal to the union of the spectrums of m BD-matrices, where m is the largest multiplicity of eigenvalues of A.*

Refer to ([10, Proofs in §4]) for details. The proof is based on Theorem 6 and the "Householder transformation" which is one of the 10 top algorithms in the twentieth century. The restriction to the finite matrix is due to the use of the transformation. The number m is newly added here which comes from the fact that the eigenvalues of BD-matrices must be distinct and simple, as illustrated by [10, Example 9].

Theorem 7 provides us a new architecture for the study on matrix mechanics (and then for quantum mechanics) since we have a unified setup (BD-matrix) to describe its spectrum. This leads clearly to a new spectrum theory, as illustrated by [7] for tridiagonal matrix and by [11] for one-dimensional diffusions. It also leads to some new algorithms for computational mathematics, as illustrated by [9, 10].

2 Hermitizable, Isospectral Differential Operators

Two Approaches for Studying the Schrödinger Operator

(1) The most popular approach to study the Schrödinger operator

$$L = \frac{1}{2}\Delta + V$$

is the Feynman-Kac semigroup $\{T_t\}_{t\ge 0}$:

$$T_t f(x) = \mathbb{E}_x\left\{ f(w_t)\exp\left[\int_0^t V(w_s)\mathrm{d}s\right]\right\},$$

where (w_t) is the standard Brownian motion. This is often an unbounded semigroup. The Schrödinger operator was born for quantum mechanics, and it is 95 years older this year. In the past hundred years or so, there are a huge number of publications devoted to the Schrödinger operator. However, for the discrete spectrum which is the most important problem in quantum mechanics, the useful results are still very limited as far as we know. In particular, even in dimension one, we have not seen the results which are comparable with [5].

(2) As in the first section, this paper introduces a new method to study the spectrum of Schrödinger operator. That is, replacing the operator L above by

$$\tilde{L} = \frac{1}{2}\Delta + \tilde{b}^h \nabla,$$

where h is a harmonic function: $Lh = 0$, $h \neq 0$ (a.e.). Then, the operator L on $L^2(\mathrm{d}x)$ is isospectral to the operator $\widetilde{L}$ on $L^2(\tilde{\mu}) := L^2(|h|^2 \mathrm{d}x)$.

We now consider a general operator. Let $a_{ij}, b_i, c : \mathbb{R}^d \to \mathbb{C}$, $V : \mathbb{R}^d \to \mathbb{R}$, and set $a = (a_{ij})_{i,j=1}^d$, $b = (b_i)_{i=1}^d$. Define $\mathrm{d}\mu = e^V \mathrm{d}x$ and

$$L = \nabla(a\nabla) + b \cdot \nabla - c.$$

Here is the result on the Hermitizability. Denote by a^H the transpose (a^*) and conjugate ($\bar{a}$) of the matrix a.

Theorem 8 (Chen and Li [11]) *Under the Dirichlet boundary condition, the operator L is Hermitizable with respect to μ iff $a^H = a$ and*

$$\begin{aligned} \mathrm{Re}\, \mathrm{b} &= (\mathrm{Re}\, \mathrm{a})(\nabla \mathrm{V}), \\ 2\, \mathrm{Im}\, \mathrm{c} &= -\big((\nabla V)^* + \nabla^*\big)\big((\mathrm{Im}\, \mathrm{a})(\nabla \mathrm{V}) + \mathrm{Im}\, \mathrm{b}\big). \end{aligned}$$

Recall that a key point in the isospectral transform of T and $\widetilde{Q}$ is that the resulting matrix $\widetilde{Q}$ obeys the condition $\tilde{c}_k = \tilde{a}_k + \tilde{b}_k$ for each $k < N$, and there is no killing/potential term at the diagonal (maybe except only one at the endpoint if $N < \infty$). In the next result, we also remove the potential term c in L. Since the isospectal property is described by using the quadratic forms, we do not require much of the regularity of h and Lh in the next result.

Theorem 9 (Chen and Li [11]) *Denote by $\mathscr{D}(L)$ the domain of L on $L^2(\mu)$ and let h be harmonic: $Lh = 0$, $h \neq 0$ (a.e.). Then $(L, \mathscr{D}(L))$ is isospectral to the operator $\big(\widetilde{L},\ \mathscr{D}(\widetilde{L})\big)$:*

$$\begin{cases} \widetilde{L} = \nabla(a\nabla) + \tilde{b} \cdot \nabla, \\ \mathscr{D}(\widetilde{L}) = \big\{\tilde{f} \in L^2(\tilde{\mu}) : \tilde{f}h \in \mathscr{D}(L)\big\}; \end{cases}$$

where

$$\tilde{b} = b + 2\, \mathrm{Re}(a)\, \mathbb{1}_{[h \neq 0]} \frac{\nabla h}{h}, \qquad \tilde{\mu} := |h|^2 \mu.$$

The discrete spectrum for one-dimensional elliptic differential operator is also illustrated in [11]. Certainly, much of the research work should be done in the near future. For instance, Hermitizable operator is clearly the Hermitian operator on the complex space $L^2(\mu)$. It naturally corresponds to a Dirichlet form. Hence there should be a complex process corresponding to the operator. It seems that this is still a quite open area, except a few of papers, Fukushima and Okada [12] for instance.

In conclusion, the paper [13] published 42 years ago opened a door for us to go to the equilibruim/nonequilibrium statistical physics (cf. [2, 3]); the paper [6] that appeared 3 years ago enables us to go to quantum mechanics. The motivation of the present study from quantum mechanics was presented in details in [8] but omitted here.

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References

1. R. Aebi, *Schrödinger Diffusion Processes* (Birkhäuser Verlag, 1996)
2. M.F. Chen, *From Markov Chains to Nonequilibrium Particle Systems*, 2nd ed. (World Scientific Press, Singapore 2004)
3. M.F. Chen, *Eigenvalues, Inequalities, and Ergodic Theory* (Springer, Berlin, 2005)
4. M.F. Chen, Speed of stability for birth-death processes. Front. Math. China **5**(3), 379–515 (2010)
5. M.F. Chen, Basic estimates of estability rate for one-dimensional diffusions, in *Probability Approximations and Beyond*. Lecture Notes in Statistics 205, ed. by A. Barbour, H.P. Chan, D. Siegmund (2011), pp. 75–99
6. M.F. Chen, Hermitizable, isospectral complex matrices or differential operators. Front. Math. China **13**(6), 1267–1311 (2018)
7. M.F. Chen, On spectrum of Hermitizable tridiagonal matrices. Front. Math. China **15**(2), 285–303 (2020)
8. M.F. Chen, A new mathematical perspective of quantum mechanics (in Chinese). Adv. Math. (China) **50**(3), 321–334 (2021)
9. M.F. Chen, R.R. Chen, Top eigenpairs of large dimensional matrix. CSIAM Trans. Appl. Math. 3(1), 1–25 (2022)
10. M.F. Chen, Z.G. Jia, H.K. Pang, Computing top eigenpairs of Hermitizable matrix. Front. Math. China **16**(2), 345–379 (2021)
11. M.F. Chen, J.Y. Li, Hermitizable, isospectral complex second-order differential operators. Front. Math. China **15**(5), 867–889 (2020)
12. M. Fukushima, M. Okada, On Dirichlet forms for plurisubharmonic functions. Acta Math. **159**(1), 171–213 (1987)
13. Z.T. Hou, M.F. Chen, Markov processes and field theory, in *Reversible Markov Processes* (in Chinese), ed. by M. Qian, Z.T. Hou (Hunan Science Press, 1979), pp. 194–242
14. Z.T. Hou, M.F. Chen, Markov processes and field theory (Abstract). Kuoxue Tongbao **25**(10), 807–811 (1980)
15. A.N. Kolmogorov, Zur Theorie der Markoffschen Ketten. Math. Ann. **112**, 155–160 (1936). English translation: On the theory of Markov chains. Article 21 in Selected Works of A.N. Kolmogorov, Vol. II: Probability Theory and Mathematical Statistics, 182–187, edited by A.N. Shiryayev. Nauka, Moscow (1986). Translated by G. Undquist (Springer, 1992)
16. A.N. Kolmogorov, Zur Umkehrbarkeit der statistischen Naturgesetze. Math. Ann. **113**, 766–772. English translation: On the reversibility of the statistical laws of nature. Article 24 in "Selected Works of A.N. Kolmogorov", Vol. II, pp. 209–215
17. E. Schrödinger, Über die Umkehrung der Naturgesetze. Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. KI. 12 März. 1931, 144–153

On Strongly Continuous Markovian Semigroups

Zhen-Qing Chen

Abstract In this short paper, we establish a sufficient condition for a symmetric Markovian semigroup to be strongly continuous in the L^2-space.

Keywords Semigroup · Markovian kernel · Strongly continuous

Mathematics Subject Classification 47D60 · 60J46 · 31C25

Given a topological space E, its Borel σ-field $\mathcal{B}(E)$ is the σ-field generated by open subsets of E, while its universally measurable σ-field $\mathcal{B}^*(E)$ is $\cap_\mu \overline{\mathcal{B}(E)}^\mu$, where the intersection is over all finite measures μ on $(E, \mathcal{B}(E))$ and $\overline{\mathcal{B}(E)}^\mu$ is the completion of $\mathcal{B}(E)$ with respect to the measure μ. Recall that a topological space E is said to be a Lusin space (resp. Radon space) if it is homeomorphic to a Borel (resp. universally measurable) subset of a compact metric space F. For a topological space E, a measure m on $(E, \mathcal{B}(E))$ is said to be regular if for any $A \in \mathcal{B}(E)$,

$$m(A) = \inf\{m(U) : U \text{ is open and } U \supset A\}$$

and

$$m(A) = \sup\{m(K) : K \text{ is compact and } K \subset A\}.$$

It is well known that any finite measure on a Lusin or Radon space is regular.

The following provides a sufficient condition for generating a strongly continuous m-symmetric Markovian semigroup and thus a corresponding symmetric Dirichlet form on $L^2(E; m)$.

Theorem 1 *Let E be a Lusin space equipped with the Borel σ-field $\mathcal{B}(E)$ (resp. a Radon space equipped with the $\sigma^*(E)$ of its universally measurable σ-field). Suppose*

Dedicated to Masatoshi Fukushima on the Occasion of his Beiju.

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that $\{P_t(x, dy); t > 0, x \in E\}$ *is a family of Markovian kernels on* $(E, \mathcal{B}(E))$ *(resp.* $(E, \mathcal{B}^*(E))$*) that is* m*-symmetric with respect to a* σ*-finite measure* m *on* E *and satisfies the following two conditions:*

(t.1) $P_s P_t f = P_{t+s} f$ *for every* $s, t > 0$ *and* $f \in \mathcal{B}_b(E)$ *(resp.* $f \in \mathcal{B}_b^*(E)$*), where* $P_t f(x) := \int_E f(y) P_t(x, dy)$*,*
(t.4) $\lim_{t \downarrow 0} P_t f(x) = f(x)$ m*-a.e. on* E *for every* $f \in C_b(E)$*.*

For each $t > 0$*, let* T_t *be the symmetric linear operator on* $L^2(E; m)$ *uniquely determined by* $\{P_t(x, dy); x \in E\}$*; that is,* $T_t f(x) := \int_E f(y) P_t(x, dy)$*. Then* $\{T_t; t \geq 0\}$ *is a strongly continuous* m*-symmetric contraction semigroup on* $L^2(E; m)$ *after taking* T_0 *to be the identity map. Moreover,* $C_b(E) \cap L^2(E; m)$ *is dense in* $L^2(E; m)$*.*

This result was essentially stated as Lemma 1.1.14(ii) in [1] except the last statement and condition **(t.4)** there is assumed to hold for every $x \in E$ instead of m-a.e. on E. However, there is a gap in the proof of $C_b(F) \cap L^2(F; m)$ being dense in $L^2(F; m)$, which was brought up to our attention by Naotaka Kajino. The proof in [1] works only when $m(E) < \infty$. The purpose of this notes is to provide a corrected proof by showing directly that $\{T_t; t \geq 0\}$ is a strongly continuous contraction semigroup on $L^2(E; m)$ under a slightly weaker condition **(t.4)** than that of [1, p.13]. The proof at the same time also establishes that $C_b(F) \cap L^2(F; m)$ is dense in $L^2(F; m)$. Clearly, the operator T_t is Markovian in the sense that if $0 \leq f \leq 1$, then $0 \leq T_t f \leq 1$. The main point of Theorem 1 is that $\{P_t(x, dy); t > 0, x \in E\}$ is not assumed to be the transition probability function of an m-symmetric right process on E, nor do we assume a priori that a subclass of $C_b(E)$ is dense in $L^2(E; m)$. See [3, Proposition IV.2.3] and [2, Lemma 1.4.3] for the corresponding results under the assumption that E is a Hausdorff space satisfying $\mathcal{B}(E) = \sigma(C(E))$, m is a σ-finite measure on E and $\{P_t(x, dy); t > 0, x \in E\}$ is the transition probability function of an m-symmetric right process on E.

***Proof of Theorem* 1** Note that $P_t(x, dy)m(dx) = P_t(y, dx)m(dy)$ on $E \times E$ by the m-symmetry of the Markovian kernel $P_t(x, dy)$. Using the Cauchy-Schwarz inequality, we have for any $f \in L^2(E; m)$,

$$\begin{aligned} \|T_t f\|^2_{L^2(E;m)} &= \int_E \left(\int_E f(y) P_t(x, dy) \right)^2 m(dx) \\ &\leq \int_E \left(\int_E f(y)^2 P_t(x, dy) \right) m(dx) \\ &= \int_E f(y)^2 P_t(y, E) m(dy) \leq \|f\|^2_{L^2(E;m)}. \end{aligned}$$

So for each $t > 0$, T_t is a contraction operator on $L^2(E; m)$. For any $f, g \in L^2(E; m)$, we have by the m-symmetry of $P_t(x, dy)$ again that

$$\int_E g(x)T_t f(x)m(dx) = \int_E g(x)\left(\int_E f(y)P_t(x, dy)\right)m(dx)$$
$$= \int_E f(y)\left(\int_E g(x)P_t(y, dx)\right)m(dy)$$
$$= \int_E f(y)T_t g(y)m(dy).$$

Clearly it follows from condition **(t.1)** that $T_tT_s = T_{t+s}$ on $L^2(F; m)$ for any $t, s > 0$. This establish that $\{T_t; t \geq 0\}$ is a symmetric contraction semigroup on $L^2(E; m)$.

We next show that the semigroup $\{T_t; t \geq 0\}$ is strongly continuous on $L^2(E; m)$. Without loss of generality, we may and do assume in the following that E is a Borel subset of a compact metric space (F, d) and identify $L^2(E; m)$ with $L^2(F; m)$ by setting $m(F \setminus E) = 0$. As m is σ-finite, there is a partition $\{E_k; k \geq 1\}$ of F so that $m(E_k) < \infty$ for every $k \geq 1$. Since every $f \in L^2(F; m)$ can be L^2-approximated by a sequence of simple functions in $L^2(F; m)$, it suffices to show that for any $A \subset F$ having $m(A) < \infty$, $T_t 1_A$ converges to 1_A in $L^2(F; m)$ as $t \to 0$ and 1_A can be approximated in $L^2(F; m)$ by functions from $C_b(F) \cap L^2(F; m)$. For simplicity, denote $m|_{E_j}$ by m_j. Since each m_j is a regular measure, for any $\varepsilon > 0$, there a compact set $K_j \subset A$ and an open set $U_j \supset A$ so that $m_j(A \setminus K_j) < \varepsilon/2^j$ and $m_j(U_j \setminus A) < \varepsilon/2^j$. As $m(A) = \sum_{j=1}^{\infty} m_j(A)$, there is some $N \geq 1$ so that $\sum_{j=N+1}^{\infty} m_j(A) < \varepsilon/2$. Define $K = \cup_{j=1}^{N} K_j$. Then K is a compact subset of A,

$$m(A \setminus K) \leq \sum_{j=1}^{N} m_j(A \setminus K_j) + \sum_{j=N+1}^{\infty} m_j(A) < \varepsilon, \tag{1}$$

and

$$m(\cap_{j=1}^{\infty} U_j \setminus A) \leq \sum_{j=1}^{\infty} m_j(U_j \setminus A) < \varepsilon. \tag{2}$$

For each $j \geq N$, define

$$g_j(x) = \frac{d(x, (\cap_{k=1}^{j} U_k)^c)}{d(x, (\cap_{k=1}^{j} U_k)^c)) + d(x, K)}.$$

Clearly $g_j \in C_b(F)$ with $0 \leq g_j \leq 1$ on F, $g_j = 1$ on K, and $g_j = 0$ on $(\cap_{k=1}^{j} U_k)^c$. Note that g_j is decreasing in j and $g_\infty(x) := \lim_{j\to\infty} g_j(x)$ vanishes on $(\cap_{k=1}^{\infty} U_k)^c$.

Hence by (2),

$$\int_F 1_{A^c}(x) g_\infty(x)^2 m(dx) \le m(\cap_{k=1}^\infty (U_k \cap A^c) \le \sum_{k=1}^\infty m_k(U_k \setminus A) < \varepsilon.$$

Thus by the monotone convergence theorem, there is some $N_1 \ge N$ so that

$$\int_F 1_{A^c}(x) g_{N_1}(x)^2 m(dx) < \varepsilon. \tag{3}$$

Hence by the L^2-contractiveness of $\{T_t; t \ge 0\}$, condition **(t.4)**, the dominated convergence theorem and the Cauchy-Schwartz inequality,

$$\begin{aligned}
&\limsup_{t\to 0} \|1_A g_{N_1} - T_t(1_A g_{N_1})\|^2_{L^2(F;m)} \\
&\le 2\|1_A g_{N_1}\|^2_{L^2(F;m)} - 2\liminf_{t\to 0} \int_F 1_A g_{N_1} T_t(1_A g_{N_1}) m(dx) \\
&= 2\|1_A g_{N_1}\|^2_{L^2(F;m)} - 2\liminf_{t\to 0} \Big(\int_F 1_A(x) g_{N_1}(x) P_t g_{N_1}(x) m(dx) \\
&- \int_F 1_A g_{N_1} T_t(1_{A^c} g_{N_1}) m(dx) \Big) \\
&\le 2\|1_A g_{N_1}\|_{L^2(F;m)} \|1_{A^c} g_{N_1}\|_{L^2(F;m)} \\
&< 2\sqrt{m(A)\varepsilon}.
\end{aligned}$$

On the other hand, as by (1),

$$\|1_A - 1_A g_{N_1}\|_{L^2(F;m)} \le m(A \setminus K)^{1/2} \le \varepsilon^{1/2}, \tag{4}$$

we have by the contraction property of T_t in $L^2(F;m)$

$$\begin{aligned}
&\limsup_{t\to 0} \|1_A - T_t 1_A\|_{L^2(F;m)} \\
&\le \limsup_{t\to 0} \big(\|1_A - 1_A g_{N_1}\|_{L^2(F;m)} + \|1_A g_{N_1} - T_t(1_A g_{N_1})\|_{L^2(F;m)} \\
&+ \|T_t(1_A g_{N_1} - 1_A)\|_{L^2(F;m)} \big) \\
&\le 2\varepsilon^{1/2} + \limsup_{t\to 0} \|1_A g_{N_1} - T_t(1_A g_{N_1})\|_{L^2(F;m)} \\
&\le 2\varepsilon^{1/2} + 2(m(A)\varepsilon)^{1/4}.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $\lim_{t\to 0} \|1_A - T_t 1_A\|_{L^2(F;m)} = 0$. This establishes that the semigroup $\{T_t; t \ge 0\}$ is strongly continuous on $L^2(F;m)$. On the other hand, it follows from (3) and (4) that

$$\|g_{N_1}\|_{L^2(F;m)} \leq \|1_A g_{N_1}\|_{L^2(F;m)} + \|1_{A^c} g_{N_1}\|_{L^2(F;m)} \leq \sqrt{m(A)} + \sqrt{\varepsilon} < \infty$$

and

$$\|1_A - g_{N_1}\|_{L^2(F;m)} \leq \|1_A - 1_A g_{N_1}\|_{L^2(F;m)} + \|1_{A^c} g_{N_1}\|_{L^2(F;m)} \leq 2\varepsilon^{1/2}.$$

This shows that 1_A can be approximated in $L^2(F;m)$ by functions in $C_b(F) \cap L^2(F;m)$. Consequently, $C_b(F) \cap L^2(F;m)$ is dense in $L^2(F;m)$. This completes the proof of the theorem. □

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References

1. Z.-Q. Chen, M. Fukushima, *Symmetric Markov Processes, Time Change, and Boundary Theory* (Princeton University Press, Princeton, 2012)
2. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, 2nd rev. and ext. ed. (de Gruyter, Berlin, 2011)
3. Z.-M. Ma, M. Röckner, *Introduction to the Theory of (Non-symmetric) Dirichlet Forms, Universitext* (Springer, Berlin, 1992)

Two-Sided Heat Kernel Estimates for Symmetric Diffusion Processes with Jumps: Recent Results

Zhen-Qing Chen, Panki Kim, Takashi Kumagai, and Jian Wang

Abstract This article gives an overview of some recent progress in the study of sharp two-sided estimates for the transition density of a large class of Markov processes having both diffusive and jumping components in metric measure spaces. We summarize some of the main results obtained recently in [11, 18] and provide several examples. We also discuss new ideas of the proof for the off-diagonal upper bounds of transition densities which are based on a generalized Davies' method developed in [10].

Keywords Diffusion process with jumps · Symmetric Dirichlet form · Heat kernel estimate · Parabolic Harnack inequality · Inner uniform domain

Mathematics Subject Classification Primary: 60J35 · 35K08 · 60J76 · Secondary: 31C25 · 35K08 · 60J45

1 Introduction

Transition density function $p(t, x, y)$ of a Markov process X, if exists, satisfies the Kolmogorov backward equation, which is a parabolic equation involving the infinitesimal generator $\mathcal{L}$ of X. Thus $p(t, x, y)$ is also called a heat kernel of $\mathcal{L}$ or a

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fundamental solution of $\partial_t u = \mathcal{L}u$. Analysis of heat kernels is an important research topic both in analysis and in probability theory. Most of the studies on sharp two-sided estimates of the heat kernel concentrate on cases when X is a diffusion or a pure jump Markov process; that is, when the infinitesimal generator $\mathcal{L}$ is local or purely non-local. However, there are classes of Markov processes that can have both diffusive and jumping components. Discontinuous Lévy processes having Gaussian parts are such typical examples.

Markov processes having both diffusive (continuous) and jumping components have interesting features. Such processes run on two different scales: on the small scale one expects the continuous component to be dominant, while on the large scale the jumping component of the process should be the dominant one. In fact, there are even ranges of times and sizes of distances where both components appear together (in a short-time and short-distance region). See Figs. 1 and 2. Therefore, both components play essential roles. These are also the source of challenges in the study of such processes.

The literature on the potential theory of Markov processes having both continuous and jumping components is scarce. Elliptic Harnack inequalities for some of these processes are studied in [20, 25, 26]. Two-sided heat kernel estimates for a family of Lévy processes having Gaussian components with variable drifts are derived in [8]. The first work that establishes sharp two-sided bounds for a large class of symmetric diffusions with jumps is [15]. More precisely, consider the following regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^d; dx)$ given by

$$\begin{cases} \mathcal{E}(u,v) = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^d} \nabla u(x)\cdot A(x)\nabla v(x)\,dx \\ \qquad\qquad + \displaystyle\int_{\mathbb{R}^d\times\mathbb{R}^d\setminus\mathrm{diag}} (u(x)-u(y))(v(x)-v(y))J(x,y)\,dx\,dy, \\ \mathcal{F} = \overline{C_c^1(\mathbb{R}^d)}^{\mathcal{E}_1}, \end{cases} \tag{1}$$

where $C_c^1(\mathbb{R}^d)$ is the space of C^1-functions on $\mathbb{R}^d$ with compact support and $\mathcal{E}_1(u,u) := \mathcal{E}(u,u) + \int_{\mathbb{R}^d} |u(x)|^2 dx$. Here $A(x) = (a_{ij}(x))_{1\le i,j\le d}$ is a measurable symmetric $d\times d$ matrix-valued function on $\mathbb{R}^d$ that is uniform elliptic and bounded in the sense that there exists a constant $c \ge 1$ such that

$$c^{-1}\sum_{i=1}^d \xi_i^2 \le \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \le c\sum_{i=1}^d \xi_i^2 \quad \text{for every } x \text{ and } \xi = (\xi_1,\dots,\xi_d)\in\mathbb{R}^d,$$

and $J(x,y)$ is a symmetric non-negative measurable kernel on $\mathbb{R}^d\times\mathbb{R}^d\setminus\mathrm{diag}$ that satisfies condition

$$\frac{1}{c_1|x-y|^d\phi_j(|x-y|)} \le J(x,y) \le \frac{c_1}{|x-y|^d\phi_j(|x-y|)}$$

for all $x, y \in \mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}$, where ϕ_j is a strictly increasing function on $(0, \infty)$ satisfying

$$c_2^{-1}\Big(\frac{R}{r}\Big)^{\alpha_*} \le \frac{\phi_j(R)}{\phi_j(r)} \le c_2\Big(\frac{R}{r}\Big)^{\alpha^*} \quad \text{for all } 0 < r \le R \tag{2}$$

with $0 < \alpha_* \le \alpha^* < 2$ and $c_1, c_2 \ge 1$. Here and in what follows, diag is the diagonal of a given state space $\mathcal{X}$; that is, diag $:= \{(x, x) : x \in \mathcal{X}\}$. It is shown in [15] that the symmetric strong Markov process X associated with the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^d; dx)$ is conservative and has a jointly Hölder continuous transition density function $p(t, x, y)$ that enjoys

$$\begin{aligned} c_1 &\left(t^{-d/2} \wedge \phi_j^{-1}(t)^{-d}\right) \wedge \left(p^{(c)}(t, c_2|x-y|) + p^{(j)}(t, |x-y|)\right) \le p(t, x, y) \\ &\le c_3 \left(t^{-d/2} \wedge \phi_j^{-1}(t)^{-d}\right) \wedge \left(p^{(c)}(t, c_4|x-y|) + p^{(j)}(t, |x-y|)\right), \end{aligned} \tag{3}$$

for every $t > 0$ and $x, y \in \mathbb{R}^d$. Here

$$p^{(c)}(t, r) := t^{-d/2} \exp(-r^2/t) \quad \text{and} \quad p^{(j)}(t, r) := (\phi_j^{-1}(t))^{-d} \wedge \frac{t}{r^d \phi_j(r)}.$$

Note that there are two scaling functions involved in the transition density function $p(t, x, y)$ of diffusions with jumps on $\mathbb{R}^d$; namely the diffusive scaling function $\phi_c(r) := r^2$ and the scaling function ϕ_j for the pure jump part of the process. In the special case when X is the independent sum of a Brownian motion B and an isotropic stable process α-stable process Z, the transition density function $p(t, x, y) = p(t, x - y)$ for the Lévy process X is the convolution of those of B and Z from which the estimates on $p(t, x, y)$ can be derived. Indeed, in this case estimates on $p(t, x)$ have been derived in [27] by computing the convolution; however the upper and lower bounds obtained there do not match for the case of $|x|^2 < t < |x|^\alpha \le 1$.

The study of heat kernel for symmetric diffusion with jumps has been conducted further in two directions. One is to establish analytic characterizations of heat kernels of the form (3) for symmetric diffusions with jumps on general metric measure doubling spaces that are stable under bounded perturbation; see the last paragraph of Sect. 2.1 for its precise meaning. This has been carried out in [18]. In addition, stability results for upper bound heat kernel estimates and parabolic Harnack inequalities are also established in [18]. The other direction is to obtain sufficient conditions on the jumping kernels $J(x, y)$ under which sharp two-sided heat kernel estimates for symmetric diffusions with jumps can be obtained, and to investigate how the shape of the jumps influence the behavior of the heat kernels. Our recent work [11] is in this direction, in which the ideas and techniques from [16–18] have played an essential role.

The purpose of this paper is to survey recent results on sharp two-sided heat kernel estimates for symmetric diffusions with jumps obtained in [11, 18]. In this paper, we focus on symmetric diffusions with jumps on inner uniform domains of complete measure metric spaces. We mention that recently in [9], heat kernel estimates are established for quite general non-symmetric time-dependent diffusions with jumps in $\mathbb{R}^d$. Estimates for Dirichlet heat kernels of the Lévy processes that are the independent sum of a Brownian motion and an isotropic stable process have been obtained in [12, 13]. Dirichlet heat kernel estimates for more general subordinate Brownian motions with Gaussian components can be found in [2, 14].

The rest of the article is organized as follows. In Sect. 2, we present the stability result for heat kernel estimates (3) from [18]. In Sect. 3, we present the main results from [11], where the jumping kernel can have exponential decays at infinity. In both papers [11, 18], the state spaces satisfy the volume doubling condition but the volumes of balls with same radius may not be comparable. Two different arguments are used in [11, 18] for upper bound estimates of the heat kernels. In Sect. 3.4, we give a brief explanation of the argument used in [11] for off-diagonal heat kernel upper bound estimates.

Notations. We write $f(s, x) \simeq g(s, x)$, if there exist constants $c_1, c_2 > 0$ such that $c_1 g(s, x) \le f(s, x) \le c_2 g(s, x)$ for the specified range of the argument (s, x). Similarly, we write $f(s, x) \asymp g(s, x)$, if there exist constants $c_k > 0$, $k = 1, \dots, 4$, such that $c_1 g(c_2 s, x) \le f(s, x) \le c_3 f(c_4 s, x)$ for the specified range of (s, x). We write $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$ for $a, b \in \mathbb{R}$.

Denote $\log^+(x) := \log(x \vee 1)$. For a given metric measure space $(\mathcal{X}, \rho, \mu)$ and for any $p \in [1, \infty]$, we will use $\|f\|_p$ to denote the L^p-norm in $L^p(\mathcal{X}; \mu)$. For any $x \in \mathcal{X}$ and $r > 0$, we use $B(x, r)$ to denote the open ball of radius r under the metric ρ centered at x. For a function space $H = H(U)$ on an open set U in $\mathcal{X}$, we let $H_c(U) := \{f \in H(U) : f \text{ has compact support in U}\}$ and $H_b := \{f \in H : f \text{ is bounded}\}$.

2 Stability of Heat Kernel Estimates for Symmetric Diffusions with Jumps

In this section, we discuss stability of heat kernel estimates for symmetric processes that contain both diffusive and jumping components on general metric measure spaces, obtained recently in [18]. See also [19].

Let $(\mathcal{X}, \rho)$ be a locally compact separable metric space equipped with a positive Radon measure μ with full support. We assume that all balls are relatively compact, and that $\mu(\mathcal{X}) = \infty$. (Note that we do not assume $(\mathcal{X}, \rho)$ to be neither connected nor geodesic.) Denote the ball centered at x with radius r by $B(x, r)$, and set $V(x, r) = \mu(B(x, r))$.

Definition 1 (i) We say that $(\mathcal{X}, \rho, \mu)$ satisfies the *volume doubling property* (VD), if there exists $C_\mu \ge 1$ such that

$$V(x, 2r) \le C_\mu V(x, r) \quad \text{for all } x \in \mathcal{X},\ r > 0.$$

(ii) We say that $(\mathcal{X}, \rho, \mu)$ satisfies the *reverse volume doubling property* (RVD), if there exist $l_\mu, c_\mu > 1$ such that,

$$V(x, l_\mu r) \geq c_\mu V(x, r) \quad \text{for all } x \in \mathcal{X}, r > 0,$$

Note that under RVD, $\mu(\mathcal{X}) = \infty$ if and only if $\mathcal{X}$ has infinite diameter; and if $\mathcal{X}$ is connected and unbounded, then VD implies RVD.

Suppose that we have a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; \mu)$. By the Beurling-Deny formula, such a form can be decomposed into the strongly local term, the pure-jump term and the killing term; see [7, 21]. In this section, we consider the Dirichlet form $(\mathcal{E}, \mathcal{F})$ having no killing term, namely

$$\begin{aligned}\mathcal{E}(f, g) = \ & \mathcal{E}^{(c)}(f, g) + \int_{\mathcal{X}\times\mathcal{X}\setminus\mathrm{diag}} (f(x) - f(y)(g(x) - g(y))\, J(dx, dy) \\ =: \ & \mathcal{E}^{(c)}(f, g) + \mathcal{E}^{(j)}(f, g), \qquad f, g \in \mathcal{F},\end{aligned}$$

where $(\mathcal{E}^{(c)}, \mathcal{F})$ is the strongly local part of $(\mathcal{E}, \mathcal{F})$ and $J(\cdot, \cdot)$ is a symmetric Radon measure on $\mathcal{X} \times \mathcal{X} \setminus \mathrm{diag}$. Here and in what follows, we always take a quasi-continuous version of a function in $\mathcal{F}$. We assume that neither $\mathcal{E}^{(c)}(\cdot, \cdot)$ nor $J(\cdot, \cdot)$ is identically zero.

Given the regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; \mu)$, there is an associated μ-symmetric *Hunt process* $X := \{X_t, t \geq 0; \mathbb{P}^x, x \in \mathcal{X} \setminus \mathcal{N}\}$ that is unique up to a properly exceptional set, where $\mathcal{N} \subset \mathcal{X}$ is a properly exceptional set for $(\mathcal{E}, \mathcal{F})$; see [7, 21]. In this case, X is a symmetric diffusion with jumps. We fix X and $\mathcal{N}$, and write $\mathcal{X}_0 = \mathcal{X} \setminus \mathcal{N}$. Define

$$P_t f(x) = \mathbb{E}_x f(X_t), \quad x \in \mathcal{X}_0$$

for bounded Borel measurable function f on $\mathcal{X}$. The *heat kernel* associated with the semigroup $\{P_t\}_{t\geq 0}$ (if it exists) is a jointly measurable function $p(t, x, y) : (0, \infty) \times \mathcal{X}_0 \times \mathcal{X}_0 \to (0, \infty)$ so that

$$\mathbb{E}_x f(X_t) = P_t f(x) = \int p(t, x, y) f(y)\, \mu(dy) \quad \text{for all } x \in \mathcal{X}_0,\ f \in L^\infty(\mathcal{X}; \mu).$$

Let $\phi_c : \mathbb{R}_+ \to \mathbb{R}_+$ (resp. $\phi_j : \mathbb{R}_+ \to \mathbb{R}_+$) be a strictly increasing continuous function with $\phi_c(0) = 0$, $\phi_c(1) = 1$ (resp. $\phi_j(0) = 0$, $\phi_j(1) = 1$) such that there exist constants $C_1 \geq 1$ and $1 < \gamma_* \leq \gamma^*$ (resp. $C_2 \geq 1$ and $0 < \alpha_* \leq \alpha^*$) so that

$$C_1^{-1}\left(\frac{R}{r}\right)^{\gamma_*} \leq \frac{\phi_c(R)}{\phi_c(r)} \leq C_1\left(\frac{R}{r}\right)^{\gamma^*} \quad \text{for all } 0 < r \leq R, \tag{4}$$

(resp. (2)). The function ϕ will serve as the diffusive scaling, while ϕ_j corresponds to the scaling function for the pure jump part. We assume that

$$\phi_c(r) \le \phi_j(r) \quad \text{for } r \in (0,1] \quad \text{and} \quad \phi_c(r) \ge \phi_j(r) \quad \text{for } r \in [1,\infty). \tag{5}$$

This assumption is natural in view of (3) where $\phi_c(r) := r^2$ and $\phi_j(r)$ satisfies (2). Set

$$\phi(r) := \phi_c(r) \wedge \phi_j(r) = \begin{cases} \phi_c(r), & r \in (0,1], \\ \phi_j(r), & r \in [1,\infty). \end{cases}$$

It is well known that for any $f \in \mathcal{F}_b$, there exist unique positive Radon measures $\mu_{\langle f\rangle}$ and $\mu^c_{\langle f\rangle}$ (called the *energy measures* of f for the Dirichlet forms $(\mathcal{E},\mathcal{F})$ and $(\mathcal{E}^{(c)},\mathcal{F})$) on $\mathcal{X}$ so that for every $g \in \mathcal{F}$,

$$\int_{\mathcal{X}} g\, d\mu_{\langle f\rangle} = \mathcal{E}(f, fg) - \frac{1}{2}\mathcal{E}(f^2, g) \quad \text{and}$$
$$\int_{\mathcal{X}} g\, d\mu^c_{\langle f\rangle} = \mathcal{E}^{(c)}(f, fg) - \frac{1}{2}\mathcal{E}^{(c)}(f^2, g).$$

Let $U \subset V$ be open sets of $\mathcal{X}$ with $U \subset \overline{U} \subset V$. We say a non-negative bounded measurable function φ is a cut-off function for $U \subset V$, if $\varphi \ge 1$ on U, $\varphi = 0$ on V^c and $0 \le \varphi \le 1$ on $\mathcal{X}$.

Definition 2 (i) We say that J_{ϕ_j} holds if there exists a non-negative symmetric function $J(x,y)$ such that for $\mu \times \mu$-almost all $x, y \in \mathcal{X}$,

$$J(dx, dy) = J(x,y)\, \mu(dx)\, \mu(dy),$$

and

$$J(x,y) \asymp \frac{1}{V(x, \rho(x,y))\phi_j(\rho(x,y))}.$$

(ii) We say that the *(weak) Poincaré inequality* $\mathrm{PI}(\phi)$ holds (for $\mathcal{E}$) if there exist $C > 0$ and $\kappa \ge 1$ such that for any ball $B_r = B(x,r)$ with $x \in \mathcal{X}$, $r > 0$ and for any $f \in \mathcal{F}_b$,

$$\int_{B_r} (f - \overline{f}_{B_r})^2\, d\mu \le C\phi(r) \left(\mu^c_{\langle f\rangle}(B_{\kappa r}) + \int_{B_{\kappa r} \times B_{\kappa r} \setminus \mathrm{diag}} (f(y) - f(x))^2\, J(dx, dy) \right),$$

where $\overline{f}_{B_r} = \frac{1}{\mu(B_r)} \int_{B_r} f\, d\mu$.

When $\phi(r)$ is a power function r^{d_w} with $d_w > 1$, we write $\mathrm{PI}(d_w)$ for $\mathrm{PI}(\phi)$.

(iii) We say that the cut-off Sobolev inequality $\mathrm{CS}(\phi)$ holds if there exist $\delta_0 \in [1/2, 1)$ and $C_1, C_2 > 0$ such that the following holds: for any $0 < r \le R, x_0 \in \mathcal{X}$ and any $f \in \mathcal{F}$, there exists a cut-off function $\varphi \in \mathcal{F}_b$ for $B(x_0, R) \subset B(x_0, R + \delta_0 r)$ so that

$$\begin{aligned}\int_{B(x_0,R+r)} f^2\, d\mu_{\langle\varphi\rangle} \le C_1 \Bigg(&\int_{B(x_0,R+r)} \varphi^2\, d\mu^c_{\langle f \rangle} \\ &+ \int_{B(x_0,R+r)\times B(x_0,R+r)\setminus \mathrm{diag}} \varphi^2(x)(f(x)-f(y))^2\, J(dx,dy)\Bigg) \\ &+ \frac{C_2}{\phi(r)} \int_{B(x_0,R+r)} f^2\, d\mu.\end{aligned}$$

2.1 Two-Sided Heat Kernel Estimates

In the following, we write $\phi_c^{-1}(t)$ (resp. $\phi_j^{-1}(t)$) to denote the inverse function of the strictly increasing function $t \mapsto \phi_c(t)$ (resp. $t \mapsto \phi_j(t)$). Define

$$p^{(c)}(t, x, y) := \frac{1}{V(x, \phi_c^{-1}(t))} \exp\left(-\sup_{s>0}\left\{\frac{\rho(x,y)}{s} - \frac{t}{\phi_c(s)}\right\}\right), \quad t > 0, x, y \in \mathcal{X}, \tag{6}$$

which arises in the two-sided estimates of heat kernels for strongly local Dirichlet forms; see, e.g., [1]. There is another expression of heat kernels for strongly local Dirichlet forms, which is given by

$$p^{(c)}(t, x, y) = \frac{1}{V(x, \phi_c^{-1}(t))} \exp\left(-\frac{\rho(x,y)}{\bar{\phi}_c^{-1}(t/\rho(x,y))}\right), \quad t > 0, x, y \in \mathcal{X}, \tag{7}$$

where $\bar{\phi}_c(r) : \mathbb{R}_+ \to \mathbb{R}_+$ is a strictly increasing continuous function such that

$$c_1 \frac{\phi_c(r)}{r} \le \bar{\phi}_c(r) \le c_2 \frac{\phi_c(r)}{r} \quad \text{for all } r > 0$$

with some $c_2 \ge c_1 > 0$. When $\phi_c(r) = r^{d_w}$ with $d_w \ge 2$, $p^{(c)}(t, x, y)$ is reduced into Gaussian ($d_w = 2$) and sub-Gaussian ($d_w > 2$) estimates. Set

$$p^{(j)}(t, x, y) := \frac{1}{V(x, \phi_j^{-1}(t))} \wedge \frac{t}{V(x, \rho(x,y))\phi_j(\rho(x,y))}.$$

It can be verified that under mild conditions the expressions (6) and (7) are equivalent; see [18, Corollary 2.3].

Definition 3 Let $\phi := \phi_c \wedge \phi_j$.

(i) We say that $\mathrm{HK}(\phi_c, \phi_j)$ holds if there exists a heat kernel $p(t, x, y)$ for the semigroup $\{P_t\}_{t\geq 0}$ associated with $(\mathcal{E}, \mathcal{F})$ such that the following holds for all $t > 0$ and all $x, y \in \mathcal{X}_0$,

$$\begin{aligned} &c_1\Big(\frac{1}{V(x, \phi_c^{-1}(t))} \wedge \frac{1}{V(x, \phi_j^{-1}(t))} \wedge \big(p^{(c)}(c_2 t, x, y) + p^{(j)}(t, x, y)\big)\Big) \\ &\leq\ p(t, x, y) \qquad (8) \\ &\leq c_3\Big(\frac{1}{V(x, \phi_c^{-1}(t))} \wedge \frac{1}{V(x, \phi_j^{-1}(t))} \wedge \big(p^{(c)}(c_4 t, x, y) + p^{(j)}(t, x, y)\big)\Big), \end{aligned}$$

where $c_k > 0$, $k = 1, \ldots, 4$, are constants independent of $x, y \in \mathcal{X}_0$ and $t > 0$. Below, we abbreviate the two-sided estimate (8) as

$$p(t, x, y) \asymp \frac{1}{V(x, \phi_c^{-1}(t))} \wedge \frac{1}{V(x, \phi_j^{-1}(t))} \wedge \Big(p^{(c)}(t, x, y) + p^{(j)}(t, x, y)\Big).$$

(ii) We say $\mathrm{HK}_-(\phi_c, \phi_j)$ holds if the upper bound in (8) holds but the lower bound is replaced by the following: there are $c_0, c_1 > 0$ so that

$$\begin{aligned} p(t, x, y) \geq c_0\bigg(&\frac{t}{V(x, \rho(x, y))\phi_j(\rho(x, y))}\mathbf{1}_{\{\rho(x,y)>c_1\phi^{-1}(t)\}} \\ &+ \frac{1}{V(x, \phi^{-1}(t))}\mathbf{1}_{\{\rho(x,y)\leq c_1\phi^{-1}(t)\}}\bigg), \quad \forall t > 0,\ \forall x, y \in \mathcal{X}_0. \end{aligned}$$

With the notations above, we now state the following stable characterizations of two-sided heat kernel estimates for symmetric diffusions with jump from [18].

Theorem 1 *Suppose that the metric measure space $(\mathcal{X}, \rho, \mu)$ satisfies* VD *and* RVD*, and that the scale functions ϕ_c and ϕ_j satisfy* (2)*,* (4) *and* (5)*. Let $\phi := \phi_c \wedge \phi_j$. Then the following are equivalent:*

(i) $\mathrm{HK}_-(\phi_c, \phi_j)$.
(ii) J_{ϕ_j}, $\mathrm{PI}(\phi)$ *and* $\mathrm{CS}(\phi)$.
If in addition, $(\mathcal{X}, \rho, \mu)$ is connected and ρ is geodesic, then all the conditions above are equivalent to:
(iii) $\mathrm{HK}(\phi_c, \phi_j)$.

Note that statement (ii) in Theorem 1 is stable under *bounded perturbation* in the sense that if it holds for the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; \mu)$, then it holds for any other Dirichlet form $(\mathcal{E}', \mathcal{F})$ on $L^2(\mathcal{X}; \mu)$ with jumping kernel J' as long as there is a constant $c > 1$ so that $c^{-1}\mathcal{E}^{(c)}(f, f) \leq \mathcal{E}'^{,(c)}(f, f) \leq c\mathcal{E}^{(c)}(f, f)$ for all $f \in \mathcal{F}$ and $c^{-1}J(x, y) \leq J'(x, y) \leq cJ(x, y)$ for all $x \neq y \in \mathcal{X}$. We refer [18, Theorem 1.13] for more equivalent characterizations of $\mathrm{HK}_-(\phi_c, \phi_j)$. We note that the connectedness and the geodesic condition (in fact, so-called chain condition suffices) of the underlying metric measure space $(\mathcal{X}, \rho, \mu)$ are only used to derive optimal lower bounds off-diagonal estimates for the heat kernel when the time is small (i.e., from $\mathrm{HK}_-(\phi_c, \phi_j)$ to $\mathrm{HK}(\phi_c, \phi_j)$).

2.2 Example

In this section, we give an example to illustrate a typical application of Theorem 1.

Example 1 (Transferring Method on d-Set) Let $(\mathcal{X}, \rho, \mu)$ be an Alfhors d-regular set and suppose that there is a strongly local Dirichlet form $(\bar{\mathcal{E}}, \bar{\mathcal{F}})$ on $L^2(\mathcal{X}; \mu)$ such that there is a transition density function $q(t, x, y)$ with respect to the measure μ that has the following two-sided estimates:

$$q(t, x, y) \asymp t^{-d/d_w} \exp\left(-\left(\frac{\rho(x, y)^{d_w}}{t}\right)^{1/(d_w-1)}\right), \quad t > 0, x, y \in \mathcal{X}$$

for some $d_w \geq 2$. Let $\{Z_t, t \geq 0; \mathbb{P}_x, x \in \mathcal{X}\}$ be the corresponding μ-symmetric diffusion on $\mathcal{X}$. A typical example is a Brownian motion on the n-dimensional unbounded Sierpiński gasket; see for instance [4]. In this case, $d = \log(n + 1)/\log 2$ and $d_w = \log(n + 3)/\log 2$.

For any $\alpha \in (0, d_w)$, let $s = \alpha/d_w$ and $\xi_t = t + \eta_t$, where η_t is the s-subordinator independent of Z. Then one can verify by direct computations that the subordinated process $X_t := Z_{\xi_t}$ has a transition density function that enjoys $\mathrm{HK}(\phi_c, \phi_j)$ with $\phi_c(r) = r^{d_w}$ and $\phi_j(r) = r^\alpha$.

Now consider the following symmetric regular Dirichlet form $(\mathcal{E}, \bar{\mathcal{F}})$ in $L^2(\mathcal{X}; \mu)$:

$$\mathcal{E}(u, v) = \mathcal{E}^{(c)}(u, v) + \int_{\mathcal{X}\times\mathcal{X}\setminus\mathrm{diag}} (u(x) - u(y))(v(x) - v(y)) \times \frac{c(x, y)}{\rho(x, y)^{d+\alpha}}\, \mu(dx)\, \mu(dy),$$

where $(\mathcal{E}^{(c)}, \bar{\mathcal{F}})$ is a strongly local regular Dirichlet form on $L^2(\mathcal{X}; \mu)$ such that $\mathcal{E}^{(c)}(f, f) \asymp \bar{\mathcal{E}}(f, f)$ for all $f \in \bar{\mathcal{F}}$, and $c(x, y)$ is a symmetric measurable function on $\mathcal{X} \times \mathcal{X} \setminus \mathrm{diag}$ that is bounded between two positive constants. Clearly, J_{ϕ_j} and $\mathrm{PI}(\phi)$ hold for $(\mathcal{E}, \bar{\mathcal{F}})$. $\mathrm{CS}(\phi)$ also holds for $(\mathcal{E}, \bar{\mathcal{F}})$ because it holds for the subordinated process $\{X_t\}_{t\geq 0}$. Hence, by Theorem 1 we obtain $\mathrm{HK}(\phi_c, \phi_j)$.

This type of argument (i.e. first establishing heat kernel estimates for a particular process and then use the stability results to obtain heat kernel estimates for more general processes) is sometimes called "transferring method".

In [18], relations between heat kernel estimates and parabolic Harnack inequalities are also established. Unlike the cases of local operators/diffusions, for pure-jump processes, parabolic Harnack inequalities are no longer equivalent to (in fact weaker than) the two-sided heat kernel estimates—see [3, 17]. For the cases of diffusions with jumps, it is even more complex. We refer the readers to [18, Theorem 1.18], for more details and for further characterizations of parabolic Harnack inequalities.

3 Symmetric Reflected Diffusions with Jumps in Inner Uniform Domains

In this section, we consider the case that $\mathcal{X}$ is an inner uniform domain D on a Harnack-type space E. In this framework, there exists a reflected diffusion on D whose heat kernel enjoys two-sided Gaussian estimates (see Theorem 2). We will consider this reflected diffusion perturbed by jumps which may decay exponentially (even super-exponentially). Thus, this setting does not belong to that studied in [18], where the jumps will decay at most polynomially; see (4). This section is a survey of the recent paper [11]. In contrast with the previous section, this section as well as [11] is concerned with sufficient conditions under which we have two-sided sharp heat kernel estimates rather than stable characterization of the heat kernel estimates. However, as mentioned earlier, the ideas and techniques developed from the study of the stability results for heat kernel estimates and parabolic Harnack inequalities in [16–18] play an essential role in the work [11].

3.1 Reflected Diffusions on Inner Uniform Domains

Let E be a locally compact separable metric space, and m a σ-finite Radon measure with full support on E. Suppose that there is a strongly local regular Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$ on $L^2(E; m)$, and let $\mu^0_{\langle u\rangle}$ be the ($\mathcal{E}^0$-) energy measure of $u \in \mathcal{F}^0$ so that $\mathcal{E}^0(u, u) = \frac{1}{2}\mu^0_{\langle u\rangle}(E)$. Then the intrinsic metric ρ of $(\mathcal{E}^0, \mathcal{F}^0)$ is defined by

$$\rho(x, y) = \sup\left\{ f(x) - f(y) : \ f \in \mathcal{F}^0 \cap C_c(E) \quad \text{with}\, \mu^0_{\langle f\rangle}(dz) \le m(dz)\right\}.$$

We assume that $\rho(x, y) < \infty$ for any $x, y \in E$ and induces the original topology on E, and that (E, ρ) is a complete metric space. It is known (see for example [23, Theorem 2.11]) that (E, ρ) is a geodesic length space; that is, for each $x, y \in E$, there exists a continuous curve $\gamma : [0, 1] \to E$ with $\gamma(0) = x$, $\gamma(1) = y$ such that for every $s, t \in [0, 1]$, $\rho(\gamma(s), \gamma(t)) = |t - s|\, \rho(x, y)$. In the following, we will always use the intrinsic metric ρ for E.

We assume that (E, ρ, m) enjoys (VD) and $(\mathcal{E}^0, \mathcal{F}^0)$ enjoys PI(2); see Definitions 1 and 2(ii) for these definitions. According to [23], such a space is called a *Harnack-type Dirichlet space*. It is known that the state space E for Harnack-type Dirichlet space $(\mathcal{E}^0, \mathcal{F}^0)$ is connected and the diffusion process Z^0 associated with $(\mathcal{E}^0, \mathcal{F}^0)$ is conservative—see [23, Lemma 2.33].

For a domain D of the length metric space (E, ρ), define for $x, y \in D$,

$$\rho_D(x, y) = \inf\{\text{length}(\gamma) : \text{acurve}\, \gamma \text{ in } D \text{ with } \gamma(0) = x \text{ and } \gamma(1) = y\}.$$

The completion of D under the metric ρ_D is denoted by $\bar{D}$. We extend the definition of $m|_D$ to $\bar{D}$ by setting $m|_D(\bar{D} \setminus D) = 0$. For notational simplicity, we will use m to denote this measure $m|_D$.

Definition 4 ([23, Definition 3.6]) We say that D is *inner uniform* if there are constants $C_1, C_2 \in (0, \infty)$ such that, for any $x, y \in D$, there exists a continuous map $\gamma_{x,y} : [0, 1] \to D$ with $\gamma_{x,y}(0) = x$, $\gamma_{x,y}(1) = y$ that satisfies the following:

(i) The length of $\gamma_{x,y}$ is at most $C_1 \rho_D(x, y)$.
(ii) For any $z \in \gamma_{x,y}([0, 1])$, it holds that

$$\rho(z, \partial D) := \inf_{w \in \partial D} \rho(z, w) \geq C_2 \frac{\rho_D(z, x)\rho_D(z, y)}{\rho_D(x, y)}.$$

When D is inner uniform, $(\bar{D}, \rho_D)$ is locally compact—see [23, Lemma 3.9]. It is well known that $(\mathcal{E}^0, \mathcal{F}^0_D)$ is the part Dirichlet form of $(\mathcal{E}^0, \mathcal{F}^0)$ on D, where $\mathcal{F}^0_D = \{f \in \mathcal{F}^0 : f = 0\ \mathcal{E}^0-\text{q.e.on } D^c\}$. In other words, $(\mathcal{E}^0, \mathcal{F}^0_D)$ is the Dirichlet form on $L^2(D; m)$ of the subprocess of the diffusion process Z^0 associated with $(\mathcal{E}^0, \mathcal{F}^0)$ killed upon leaving D. We write $f \in \mathcal{F}^0_{D,\text{loc}}$ if for every relatively compact subset U of D, there is $g \in \mathcal{F}^0_D$ such that $f = g$ m-a.e. on U. By [23, Proposition 2.13], it holds that for $x, y \in D$,

$$\rho_D(x, y) = \sup\left\{f(x) - f(y) :\ f \in \mathcal{F}^0_{D,\text{loc}} \cap C_c(D) \text{ with } \mu^0_{\langle f \rangle}(dz) \leq m(dz)\right\}.$$

Let $\mathcal{F}^{0,\text{ref}}_D := \{f \in \mathcal{F}^0_{D,\text{loc}} : \mu^0_{\langle f \rangle}(D) < \infty\}$ and define $\mathcal{E}^{0,\text{ref}}(f, f) := \frac{1}{2}\mu^0_{\langle f \rangle}(D)$ for $f \in \mathcal{F}^{0,\text{ref}}_D$. $(\mathcal{E}^{0,\text{ref}}, \mathcal{F}^{0,\text{ref}}_D \cap L^2(D; m))$ is the active reflected Dirichlet form of $(\mathcal{E}^0, \mathcal{F}^0_D)$, which is known to be a Dirichlet form on $L^2(\bar{D}; m) = L^2(D; m)$—see [7, Chap. 6]. Let $B_{\bar{D}}(x, r) := \{y \in \bar{D} : \rho_D(x, y) < r\}$, and denote $V_D(x, r) := m(B_{\bar{D}}(x, r))$. Let $\text{Lip}_c(\bar{D})$ be the space of compactly supported Lipschitz functions in $\bar{D}$. Then the following holds.

Theorem 2 ([23, Sect. 3]) *Suppose that* $(\mathcal{E}^0, \mathcal{F}^0)$ *is a strongly local regular Dirichlet form on* $L^2(E; m)$ *which admits a carré du champ operator* Γ_0 *(that is,* $\mu^0_{\langle u \rangle}(dx) = \Gamma_0(u, u)\, m(dx)$ *and* $\Gamma_0(u, u) \in L^1(E; m)$ *for every* $u \in \mathcal{F}$*). Assume that* (VD) *and* (PI(2)) *hold for* $(\mathcal{E}^0, \mathcal{F}^0)$ *on* (E, ρ, m)*, and suppose that* D *is an inner uniform subdomain of* E*. Then* $(\mathcal{E}^{0,\text{ref}}, \mathcal{F}^{0,\text{ref}}_D \cap L^2(D; m))$ *is a strongly local regular Dirichlet form on* $L^2(D; m)$ *with core* $\text{Lip}_c(\bar{D})$*, and the following hold for* $\left(\mathcal{E}^{0,\text{ref}}, \mathcal{F}^{0,\text{ref}}_D \cap L^2(D; m)\right)$ *on* $(\bar{D}, \rho_D, m)$*:*

(VD) (**Volume doubling property on** $\bar{D}$) *There exists* $C_3 > 0$ *such that for every* $x \in \bar{D}$ *and* $r > 0$*,* $V_D(x, 2r) \leq C_3\, V_D(x, r)$*.*

(PI(2)) (**Poincaré inequality on** $\bar{D}$) *There exists* $C_4 > 0$ *such that for every* $x \in \bar{D}$*,* $r > 0$ *and* $f \in \mathcal{F}^{0,\text{ref}}_D \cap L^2(D; m)$*,*

$$\min_{a \in \mathbb{R}} \int_{B_{\bar{D}}(x,r)} (f(y) - a)^2\, m(dy) \leq C_4 r^2\, \mu^0_{\langle f \rangle}(B_{\bar{D}}(x, r)).$$

Consequently, $(\mathcal{E}^{0,\mathrm{ref}}, \mathcal{F}_D^{0,\mathrm{ref}})$ *admits a jointly continuous transition density function* $p_D^N(t, x, y)$ *on* $(0, \infty) \times \bar{D} \times \bar{D}$*, and there exist* $c_1, c_2 \geq 1$ *depending on* C_3, C_4 *such that*

$$\frac{c_1^{-1}}{V_D(x, \sqrt{t})} \exp\left(-\frac{c_2 \rho_D(x, y)^2}{t}\right) \leq p_D^N(t, x, y) \leq \frac{c_1}{V_D(x, \sqrt{t})} \exp\left(-\frac{\rho_D(x, y)^2}{c_2 t}\right)$$

for every $x, y \in \bar{D}$ *and* $t > 0$.

In the rest of this section we assume that the strongly local Dirichlet form $(\mathcal{E}^0, \mathcal{F}^0)$ on $L^2(E; m)$ and $D \subset E$ satisfy the assumptions of Theorem 2.

Characteristic Constants. Recall that (C_1, C_2) are constants appearing in the definition of the inner uniform domain D, and (C_3, C_4) are constants in (VD) and (PI(2)) of Theorem 2. We will call (C_1, C_2, C_3, C_4) the *characteristic constants* of the domain D.

3.2 Reflected Diffusions with Jumps

Let $(\mathcal{E}, \mathcal{F})$ be a symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(D; m)$, where $\mathcal{F} := \mathcal{F}_D^{0,\mathrm{ref}} \cap L^2(D; m)$, such that for $u \in \mathcal{F}$,

$$\mathcal{E}(u, u) = \mathcal{E}^{0,\mathrm{ref}}(u, u) + \frac{1}{2} \int_{D \times D} (u(x) - u(y))^2 \, J(x, y) \, m(dx) \, m(dy). \tag{9}$$

Here $J(x, y)$ is a non-negative symmetric measurable function on $D \times D \setminus \mathrm{diag}$ satisfying certain conditions to be specified below.

Let ϕ_j be a strictly increasing function on $[0, \infty)$ such that $\phi_j(0) = 0$, $\phi_j(1) = 1$ and (2) holds for $0 < \alpha_* \leq \alpha^* < 2$. Since $\alpha^* < 2$, there exists $c_1 > 0$ such that

$$\int_0^r \frac{s}{\phi_j(s)} \, ds \leq \frac{c_1 r^2}{\phi_j(r)} \qquad \text{for every } r > 0.$$

Definition 5 Let $\beta \in [0, \infty]$ and ϕ_j be a strictly increasing function on $[0, \infty)$ with $\phi_j(0) = 0$ and $\phi_j(1) = 1$ that satisfies the condition (2) (with $0 < \alpha_* \leq \alpha^* < 2$). Let $J(x, y)$ be a non-negative symmetric measurable function on $D \times D \setminus \mathrm{diag}$.

(i) We say condition $(\mathbf{J}_{\phi_j,\beta,\leq})$ holds if there are $\kappa_1, \kappa_2 > 0$ so that

$$J(x, y) \leq \frac{\kappa_1}{V_D(x, \rho_D(x, y)) \phi_j(\rho_D(x, y)) \exp(\kappa_2 \rho_D(x, y)^\beta)}, \tag{$\mathbf{J}_{\phi_j,\beta,\leq}$}$$

for $(x, y) \in D \times D \setminus \mathrm{diag}$. Similarly, we say condition $(\mathbf{J}_{\phi_j,\beta,\geq})$ holds if the opposite inequality holds, and we say condition $(\mathbf{J}_{\phi_j,\beta})$ holds if both $(\mathbf{J}_{\phi_j,\beta,\leq})$ and $(\mathbf{J}_{\phi_j,\beta,\geq})$ hold with possibly different constants κ_i in the upper and lower bounds.

(ii) We say condition $(\mathbf{J}_{\phi_j,0_+,\leq})$ holds if there are $\kappa_3, \kappa_4 > 0$ so that

$$\begin{cases} \sup_{x\in D} \int_{\{y\in D:\rho_D(x,y)>1\}} \rho_D(x, y)^2 J(x, y)\, m(dy) \leq \kappa_3 < \infty, \\ J(x, y) \leq \frac{\kappa_4}{V_D(x,\rho_D(x,y))\phi_*(\rho_D(x,y))} \end{cases} \qquad (\mathbf{J}_{\phi_j,0_+,\leq})$$

for $(x, y) \in D \times D \setminus \mathrm{diag}$, where

$$\phi_*(r) := \phi_j(r)\mathbb{1}_{\{r\leq 1\}} + r^2\mathbb{1}_{\{r>1\}} \quad \text{for } r \geq 0. \tag{10}$$

Clearly, $(\mathbf{J}_{\phi_j,\beta,\leq}) \Longrightarrow (\mathbf{J}_{\phi_j,0_+,\leq}) \Longrightarrow (\mathbf{J}_{\phi_j,0,\leq})$ for any $\beta \in (0, \infty]$. When $\beta = 0$, $(\mathbf{J}_{\phi_j,0})$ coincides with J_{ϕ_j} in Definition 2(i). When $\beta = \infty$, condition $(\mathbf{J}_{\phi_j,\infty,\leq})$ is equivalent to

$$J(x, y) \leq \frac{\widetilde{\kappa}_1}{V_D(x, \rho_D(x, y))\phi_j(\rho_D(x, y))}\mathbb{1}_{\{\rho(x,y)\leq 1\}} \quad \text{for } (x, y) \in D \times D \setminus \mathrm{diag}.$$

It can be easily proved (see [11, Proposition 2.1]) that, under condition $(\mathbf{J}_{\phi_j,0,\leq})$, $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(D; m)$. Moreover, the corresponding process Y is conservative; namely, Y has infinite lifetime almost surely.

For notational convenience, we regard 0_+ as an “added" or “extended" number and declare that it is larger than 0 but smaller than any positive real number. With this notation, we can write, for instance, $(\mathbf{J}_{\phi_j,\beta,\leq})$ for $\beta \in [0, \infty] \cup \{0_+\}$.

In the following, we present results concerning global two-sided sharp estimates on the heat kernel of Dirichlet form $(\mathcal{E}, \mathcal{F})$ under the assumption that $J(x, y)$ satisfies $(\mathbf{J}_{\phi_{j,1},\beta_*,\leq})$ and $(\mathbf{J}_{\phi_{j,2},\beta^*,\geq})$ for some strictly increasing functions $\phi_{j,1}$ and $\phi_{j,2}$ satisfying $\phi_{j,i}(0) = 0$, $\phi_{j,i}(1) = 1$ and (2) (with $\phi_{j,i}$ in place of ϕ_j) for $1 \leq i \leq 2$, and for β_* and β^* in $[0, \infty] \cup \{0_+\}$ (but excluding $\beta_* = \beta^* = 0_+$).

First let us consider the case $\beta_* = \beta^* = 0$ and $\phi_{j,1} = \phi_{j,2} =: \phi_j$. Note that, in the present setting, $\mathrm{diam}(D) = \infty$ is equivalent to $m(D) = \infty$; see [22, Corollary 5.3]. In this case, it is easy to check that, with $\phi_c(r) := r^2$, $\phi_j(r) := r^\alpha$ and $\phi(r) := \phi_c(r) \wedge \phi_j(r)$, J_{ϕ_j}, $\mathrm{PI}(\phi)$ and $\mathrm{CS}(\phi)$ hold; see [18, Example 1.1 and Remark 1.7] for the details. Thus we can apply the stable characterization of Theorem 1 to conclude that $\mathrm{HK}(\phi_c, \phi_j)$ holds. When $\mathrm{diam}(D) < \infty$, according to [18, Theorem 1.13] (noting that the results of the paper [18] continue to hold for bounded state space with obvious localized versions), one can obtain estimates of $p(t, x, y)$ for $t \in (0, 1]$. Now, when D is bounded, it holds that $V_D(x, \sqrt{t}) \simeq 1$ for all $x \in D$ and $t \geq 1$, and the large time estimates of $\mathrm{HK}(\phi_c, \phi_j)$ are simply

$$p(t, x, y) \simeq 1 \quad \text{for } x, y \in \bar{D} \text{ and } t \geq 1,$$

which is a consequence of the strong ergodicity of the Markov process Y. Hence $\mathrm{HK}(\phi_c, \phi_j)$ is the desired estimates for $\mathrm{diam}(D) < \infty$ as well.

The main contribution of [11] is to obtain two-sided heat kernel estimates for $0_+ \le \beta_* \le \beta^* \le \infty$ excluding $\beta_* = \beta^* = 0_+$ when D is unbounded. Note that when D is bounded, $(\mathbf{J}_{\phi_j,\beta,\le})$ and $(\mathbf{J}_{\phi_j,\beta,\ge})$ with $\beta \in \{0_+\} \cup (0,\infty]$ are reduced to $(\mathbf{J}_{\phi_j,0,\le})$ and $(\mathbf{J}_{\phi_j,0,\ge})$, respectively. We present the precise statement in the next subsection.

3.3 *Heat Kernel Estimates for the* $\beta_* \le \beta^* \le \infty$ *in* $\{0_+\} \cup (0, \infty]$ *Case*

We need some notations. Let

$$p^{(c)}(t,x,r) := \frac{\exp(-r^2/t)}{V_D(x,\sqrt{t})}, \quad p^{(j)}_{\phi}(t,x,r) := \frac{1}{V_D(x,\phi^{-1}(t))} \wedge \frac{t}{V_D(x,r)\phi(r)}.$$

For $\beta \in [0,\infty]$ and a strictly increasing function ϕ_j on $[0,\infty)$ with $\phi_j(0) = 0$ and $\phi_j(1) = 1$, set for $x \in \bar{D}$, $t > 0$ and $r \ge 0$,

$$p^{(j)}_{\phi_j,\beta}(t,x,r) := \frac{1}{V_D(x,\phi_j^{-1}(t))} \wedge \frac{t}{V(x,r)\phi_j(r)\exp(r^\beta)}.$$

In particular, $p^{(j)}_{\phi_j,0}(t,x,r) \simeq p^{(j)}_{\phi_j}(t,x,r)$. Define for $\beta \in (0,1]$,

$$H_{\phi_j,\beta}(t,x,r) := \begin{cases} \dfrac{1}{V_D(x,\sqrt{t})} \wedge \left(p^{(c)}(t,x,r) + p^{(j)}_{\phi_j,\beta}(t,x,r)\right) & \text{if } t \in (0,1], \\ \dfrac{1}{V_D(x,\sqrt{t})} \exp\left(-\left(r^\beta \wedge (r^2/t)\right)\right) & \text{if } t \in (1,\infty); \end{cases}$$

for $\beta \in (1,\infty)$,

$$H_{\phi_j,\beta}(t,x,r) := \begin{cases} \dfrac{1}{V_D(x,\sqrt{t})} \wedge \left(p^{(c)}(t,x,r) + p^{(j)}_{\phi_j,\beta}(t,x,r)\right) \\ \qquad \text{if } t \in (0,1], r \le 1, \\ \dfrac{t}{V_D(x,r)\phi_j(r)} \exp\left(-\left(r(1+\log^+(r/t))^{(\beta-1)/\beta}\right) \wedge r^\beta\right) \\ \qquad \text{if } t \in (0,1], r > 1, \\ \dfrac{1}{V_D(x,\sqrt{t})} \exp\left(-\left(r\,(1+\log^+(r/t))^{(\beta-1)/\beta}\right) \wedge (r^2/t)\right) \\ \qquad \text{if } t \in (1,\infty); \end{cases}$$

where $H_{\phi_j,\infty}(t,x,r) := \lim_{\beta\to\infty} H_{\phi_j,\beta}(t,x,r)$ for $\beta = \infty$, that is,

$$H_{\phi_j,\infty}(t,x,r) := \begin{cases} \dfrac{1}{V_D(x,\sqrt{t})} \wedge \left(p^{(c)}(t,x,r) + p^{(j)}_{\phi_j,\beta}(t,x,r)\right) & \text{if } t \in (0,1], r \le 1, \\ \dfrac{t}{V_D(x,r)\phi_j(r)} \exp\left(-r(1+\log^+(r/t))\right) & \text{if } t \in (0,1], r > 1, \\ \dfrac{1}{V_D(x,\sqrt{t})} \exp\left(-\left(r\left(1+\log^+(r/t)\right)\right) \wedge (r^2/t)\right) & \text{if } t \in (1,\infty). \end{cases}$$

See Figs. 1 and 2 for a more explicit expression on the dominate terms in $H_{\phi_j,\beta}(t,x,r)$. The following is the main result on the two-sided heat kernel estimates of Y.

Theorem 3 ([11, Theorem 1.6]) *Suppose that D is unbounded. Assume that $J(x,y)$ satisfies $(\mathbf{J}_{\phi_{j,1},\beta_*,\le})$ and $(\mathbf{J}_{\phi_{j,2},\beta^*,\ge})$ for some strictly increasing functions $\phi_{j,1}, \phi_{j,2}$ satisfying $\phi_{j,i}(0)=0$, $\phi_{j,i}(1)=1$ and (2) (with $\phi_{j,i}$ in place of ϕ_j) for $i=1,2$, and for $\beta_* \le \beta^*$ in $\{0_+\} \cup (0,\infty]$ excluding $\beta_*=\beta^*=0_+$. Then the transition density function $p(t,x,y)$ of the conservative Feller process Y associated with $(\mathcal{E},\mathcal{F})$ has the following estimates: for every $t>0$ and $x,y \in \bar{D}$,*

$$c_1 H_{\phi_{j,2},\beta^*}(t,x,c_2\rho_D(x,y)) \le p(t,x,y) \le c_3 H_{\phi_{j,1},\beta_*}(t,x,c_4\rho_D(x,y)),$$

where $c_i>0$, $1\le i\le 4$, depend only on the characteristic constants (C_1,C_2,C_3,C_4) of D and the constant parameters in $(\mathbf{J}_{\phi_{j,1},\beta_*,\le})$ and $(\mathbf{J}_{\phi_{j,2},\beta^*,\ge})$ as well as in (2) for $\phi_{j,1}$ and $\phi_{j,2}$, respectively.

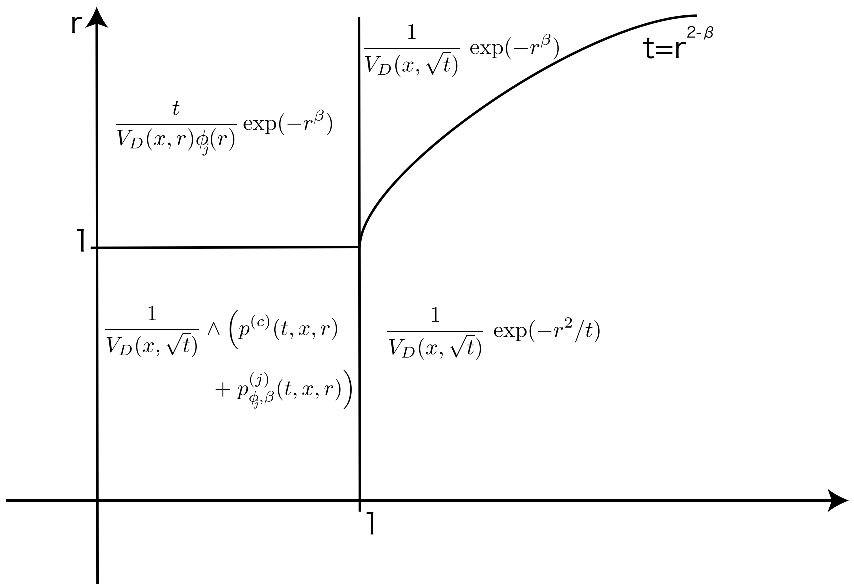

Fig. 1 $\beta \in (0,1]$: dominant terms in the heat kernel estimates $H_{\phi_j,\beta}(t,x,r)$ for $p(t,x,y)$

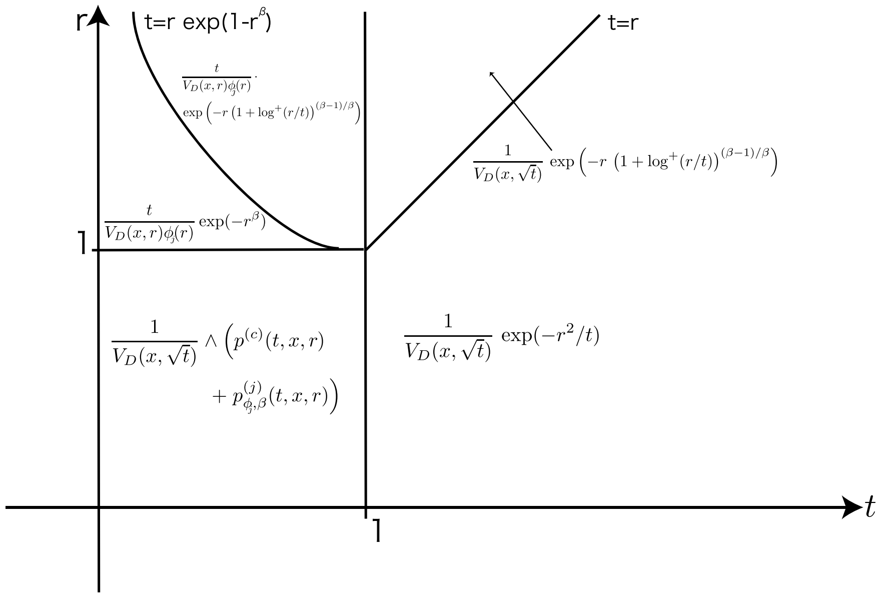

Fig. 2 $\beta \in (1, \infty]$: dominant terms in the heat kernel estimates $H_{\phi_j,\beta}(t, x, r)$ for $p(t, x, y)$

3.4 Discussion on Off-Diagonal Heat Kernel Upper Bound

In this subsection, we present results on the heat kernel upper bound under milder condition and give a brief explanation of the argument for the off-diagonal upper bound of heat kernels.

Assume that $\mathrm{diam}(D) = \infty$. Then, from the volume doubling and the reverse volume doubling property of D, we have that there exist positive constants c_1, c_2, d_1, d_2 such that

$$c_1\left(\frac{R}{r}\right)^{d_1} \le \frac{V_D(x, R)}{V_D(x, r)} \le c_2\left(\frac{R}{r}\right)^{d_2} \quad \text{for } R \ge r > 0. \tag{11}$$

Since one can verify the localized version of Faber-Krahn inequality and the cut-off Sobolev inequality with order 2 for the Dirichlet form $(\mathcal{E}, \mathcal{F})$, under the condition $(\mathbf{J}_{\phi_j,0_+,\le})$, the heat kernel upper bound in the next result essentially follows from (the local version of) [18, Theorem 1.14] and a modification of Doeblin's result (see [5, p. 365, Theorem 3.1]).

Proposition 1 ([11, Theorem 1.5]) *Suppose that condition* $(\mathbf{J}_{\phi_j,0_+,\le})$ *holds for some strictly increasing function* ϕ_j *on* $[0, \infty)$ *satisfying* (2). *Then* $(\mathcal{E}, \mathcal{F})$ *in* (9) *is regular on* $L^2(D; m)$ *and the corresponding process* Y *on* $\bar{D}$ *is a conservative Feller process that starts from every point in* $\bar{D}$. *Moreover,* Y *has a jointly Hölder continuous transition density function* $p(t, x, y)$ *on* $(0, \infty) \times \bar{D} \times \bar{D}$ *with respect to the measure* m*, and there exist constants* $c_1, c_2 > 0$ *such that*

$$p(t,x,y) \le \frac{c_1}{V_D(x,\sqrt{t})} \wedge \Big(p^{(c)}(t,x,c_2\rho_D(x,y)) + p^{(j)}_{\phi_*}(t,x,c_2\rho_D(x,y)) \Big)$$

for all $x, y \in \bar{D}$ *and* $t > 0$*, where* ϕ_* *is given by* (10)*. The positive constants* c_1, c_2 *depend only on the characteristic constants* (C_1, C_2, C_3, C_4) *of* D *and on the constant parameters in* $(\mathbf{J}_{\phi_j,0_+,\le})$ *and* (2) *for the function* ϕ_j*.*

The Meyer's construction [24] is very useful to obtain off-diagonal upper bounds for $p(t,x,y)$. Based on this, the main part of proving the off-diagonal upper bounds is to obtain the correct off-diagonal upper bounds of $q^{\langle\lambda\rangle}(t,x,y)$, the transition density of truncated process $Y^{(\lambda)}$ obtained from Y by removing jumps of size larger than λ. In order to deal with the general VD setting (11), we first consider off-diagonal upper bounds for Dirichlet heat kernel of the truncated process $Y^{(\lambda)}$. For an open set $U \subset \bar{D}$, let $q^{\langle\lambda\rangle,U}(t,x,y)$ be the (Dirichlet) heat kernel of the subprocess $Y^{(\lambda),U}$ of $Y^{(\lambda)}$ killed up exiting U.

Very recently, in [10] we have established the equivalences between on-diagonal heat kernel upper bounds and off-diagonal heat kernel upper bounds for a large class of symmetric Markov processes, which are generalizations of the results in [6]. The results in [10] are applicable for $q^{\langle\lambda\rangle,U}(t,x,y)$ in the present setting. For the remainder of this subsection, we provide the outline of the proof of the upper bound of $q^{\langle\lambda\rangle}(t,x,y)$.

In the following, suppose that $(\mathbf{J}_{\phi_j,\beta_*,\le})$ holds with $\beta_* \in (0,\infty]$. Using [10, Theorem 5.1], we can check that for any $\beta_* \in (0,\infty]$ and $l \ge 2$, there exists a constant $c_0 > 0$ such that for any $x_0 \in \bar{D}$, $\lambda > 0$, any $f \in \mathrm{Lip}_c(\bar{D})$, any $t > 0$ and any $x, y \in B_{\bar{D}}(x_0, l\lambda)$,

$$\begin{aligned} &q^{\langle\lambda\rangle, B_{\bar{D}}(x_0,l\lambda)}(t,x,y) \\ &\quad \le \frac{c_0}{V_D(x_0,\lambda)} \left(\left(\frac{\lambda}{\sqrt{t}}\right)^{d_1} \vee \left(\frac{\lambda}{\sqrt{t}}\right)^{d_2} \right) \exp\Big(-|f(y)-f(x)| + 2\Lambda^{\langle\lambda\rangle}(f)^2 t \Big), \end{aligned} \tag{12}$$

where $d_1, d_2 > 0$ are the constants in (11) and

$$\Lambda^{\langle\lambda\rangle}(f)^2 = \|e^{-2f}\Gamma_{\langle\lambda\rangle}(e^f)\|_\infty \vee \|e^{2f}\Gamma_{\langle\lambda\rangle}(e^{-f})\|_\infty$$

with

$$\Gamma_{\langle\lambda\rangle}(f)(\xi) = \Gamma_0(f,f)(\xi) + \int_{B_{\bar{D}}(\xi,\lambda)} (f(\xi)-f(\eta))^2\, J(\xi,\eta)\, m(d\eta).$$

For fixed $x, y \in B_{\bar{D}}(x_0, l\lambda)$, by taking $f(\xi) = s\,(\rho_D(\xi,x) \wedge \rho_D(x,y))$ with $s > 0$, we see that $|f(y)-f(x)| = s\rho_D(x,y)$ and, thanks to $(\mathbf{J}_{\phi_j,\beta_*,\le})$,

$$\begin{aligned}
& e^{-2f(\xi)}\Gamma_{\langle\lambda\rangle}(e^f)(\xi) \\
&= e^{-2f(\xi)}\Gamma_0(e^f, e^f)(\xi) + \int_{B_{\bar{D}}(\xi,\lambda)} (e^{f(\xi)-f(\eta)}-1)^2 J(\xi,\eta)\, m(d\eta) \\
&\le \Gamma_0(f,f)(\xi) + s^2 \int_{B_{\bar{D}}(\xi,\lambda)} \rho_D(\xi,\eta)^2 e^{2s\rho_D(\xi,\eta)} J(\xi,\eta)\, m(d\eta) \qquad (13)\\
&\le s^2 + c_1 s^2 \int_D \rho_D(\xi,\eta)^2 \frac{e^{2s\rho_D(\xi,\eta)-\kappa_2\rho_D(\xi,\eta)^{\beta_*}}}{V_D(\xi,\rho_D(\xi,\eta))\phi_j(\rho_D(\xi,\eta))}\, m(d\eta).
\end{aligned}$$

In [11], we consider the cases $\beta_* \in (0,1]$, $\beta_* \in (1,\infty)$ and $\beta_* = \infty$ separately and find a proper s for each case to bound (12) optimally.

Let $\tau_B^{(\lambda)}$ by the first exit time from the ball B by $Y^{(\lambda)}$ and $\tau_B^{(\lambda),U}$ be the first exit time from the ball B of the process $Y^{(\lambda),U}$. Since the size of jumps of $Y^{(\lambda)}$ is less than λ, we see that

$$\mathbb{P}_x\big(\tau^{(\lambda)}_{B(x,r)} \le t\big) = \mathbb{P}_x\big(\tau^{(\lambda),B(x,\lambda+r)}_{B(x,r)} \le t\big), \quad x \in \bar{D},\ \lambda, t, r > 0.$$

Using this and the strong Markov property, we have that for any $x \in \bar{D}$ and $\lambda, t, r>0$,

$$\begin{aligned}
& \int_{B_{\bar{D}}(x,r)^c} q^{\langle\lambda\rangle}(t,x,y)\, m(dy) \\
&\le \mathbb{P}_x\big(\tau^{(\lambda)}_{B(x,r)} \le t\big) = \mathbb{P}_x\big(\tau^{(\lambda),B(x,r+\lambda)}_{B(x,r)} \le t\big) \\
&\le \mathbb{P}_x\big(\rho_D(Y_{2t}^{(\lambda),B(x,r+\lambda)}, x) \ge r/2\big) \\
&\quad + \mathbb{P}_x\big(\sup_{0<s\le t} \rho_D(Y_s^{(\lambda),B(x,r+\lambda)}, x) \ge r,\ \rho_D(Y_{2t}^{(\lambda),B(x,r+\lambda)}, x) \le r/2\big) \\
&\le \mathbb{P}_x\big(\rho_D(Y_{2t}^{(\lambda),B(x,r+\lambda)}, x) \ge r/2\big) \\
&\quad + \mathbb{P}_x\Big(\tau^{(\lambda)}_{B(x,r)} \le t,\ \rho_D(Y_{2t}^{(\lambda),B(x,r+\lambda)}, Y^{(\lambda)}_{\tau^{(\lambda)}_{B(x,r)}}) \ge r/2\Big) \\
&\le \mathbb{P}_x\big(\rho_D(Y_{2t}^{(\lambda),B(x,r+\lambda)}, x) \ge r/2\big) \\
&\quad + \sup_{z\in B(x,r+\lambda)} \sup_{t\le s\le 2t} \mathbb{P}_z\big(\rho_D(Y_s^{(\lambda),B(x,r+\lambda)}, z) \ge r/2\big) \\
&\le 2 \sup_{z\in B(x,r+\lambda)} \sup_{t\le s\le 2t} \int_{B_{\bar{D}}(z,r/2)^c} q^{\langle\lambda\rangle,B(x,r+\lambda)}(s,z,y)\, m(dy). \qquad (14)
\end{aligned}$$

We now assume that $\rho_D(x,y) \ge C(\sqrt{t} \vee 1)$ where $C \ge 1$. Let $R = \rho_D(x,y)$ and $\lambda = R/k$ where k will be determined later. By [16, Lemma 7.2(2)] and Proposition 1,

$$q^{\langle\lambda\rangle}(t,x,y) \le \frac{c_2}{V_D(x,\sqrt{t})}.$$

Using this, (11) and (14), we obtain that

$$
\begin{aligned}
& q^{\langle\lambda\rangle}(t,x,y) \\
& \quad = \int_{\bar{D}} q^{\langle\lambda\rangle}(t/2,x,z) q^{\langle\lambda\rangle}(t/2,z,y)\, m(dz) \\
& \quad \le \left(\int_{B_{\bar{D}}(x,R/2)^c} + \int_{B_{\bar{D}}(y,R/2)^c} \right) q^{\langle\lambda\rangle}(t/2,x,z) q^{\langle\lambda\rangle}(t/2,z,y)\, m(dz) \\
& \quad \le \frac{c_2}{V_D(y,\sqrt{t})} \int_{B_{\bar{D}}(x,R/2)^c} q^{\langle\lambda\rangle}(t/2,x,z)\, m(dz) \\
& \qquad + \frac{c_2}{V_D(x,\sqrt{t})} \int_{B_{\bar{D}}(y,R/2)^c} q^{\langle\lambda\rangle}(t/2,y,z)\, m(dz) \\
& \quad \le \frac{c_3}{V_D(x,R)} \left(\frac{R}{\sqrt{t}}\right)^{d_2} \times \\
& \qquad \sup_{w\in\bar{D}} \sup_{z\in B(w,R/2+\lambda)} \sup_{t/2\le s\le t} \int_{B_{\bar{D}}(z,R/4)^c} q^{\langle\lambda\rangle, B(w,R/2+\lambda)}(s,z,u)\, m(du).
\end{aligned}
\tag{15}
$$

Therefore, to obtain upper bounds of $q^{\langle\lambda\rangle}(t,x,y)$, it suffices to bound (15). Recall that, in (12), (13) and the sentence below, we have discussed how to get the upper bounds of $q^{\langle\lambda\rangle, B(w,R/2+\lambda)}(s,z,u)$. Using such upper bounds, with proper C and k, in [11, Proposition 4.3] we have obtained upper bounds of (15) for the cases $\beta_* \in (0,1]$, $\beta_* \in (1,\infty)$ and $\beta_* = \infty$ separately. Finally, we have the following

Proposition 2 ([11, Theorem 4.4]) *Suppose that* $(\mathbf{J}_{\phi_j,\beta_*,\le})$ *holds for some* $\beta_* \in (0,\infty]$. *Then there exist* $c_1, c_2 > 0$ *that depend only on characteristic constants* (C_1, C_2, C_3, C_4) *of* D *and the constant parameters in* $(\mathbf{J}_{\phi_j,\beta_*,\le})$ *and* (2) *for* ϕ_j *so that*

$$
q(t,x,y) \le c_1 H_{\phi_j,\beta_*}(t,x,c_2\rho_D(x,y)) \quad \textit{for every}\ t>0\ \textit{and}\ x,y\in\bar{D}.
$$

3.5 Example

Example 2 A typical example for Theorem 3 is the following. In (1) with D instead of $\mathbb{R}^d$ where D is a Lipschitz domain in $\mathbb{R}^d$, suppose that $J(x,y)$ is a symmetric function on $D \times D \setminus \text{diag}$ defined by

$$
J(x,y) = \int_{[\alpha_1,\alpha_2]} \frac{c(\alpha,x,y)}{|x-y|^{d+\alpha}\, \Phi(|x-y|)}\, \nu(d\alpha),
$$

where ν is a probability measure on $[\alpha_1,\alpha_2] \subset (0,2)$, Φ is an increasing function on $[0,\infty)$ with

$$c_1 e^{c_2 r^\beta} \le \Phi(r) \le c_3 e^{c_4 r^\beta} \quad \text{for some } \beta \in [0, \infty],$$

and $c(\alpha, x, y)$ is a jointly measurable function that is symmetric in (x, y) and is bounded between two positive constants. When $\beta = 0$ and $D = \mathbb{R}^d$, the two sided heat kernel estimates are obtained in [15] as mentioned in the introduction.

Finally, we would like to mention that unde the setting in this section, parabolic Harnack inequalities do not hold for the whole range. In fact under conditions $(\mathbf{J}_{\phi_1,\beta_*,\le})$ and $(\mathbf{J}_{\phi_2,\beta^*,\ge})$ with $\beta_* < \beta^*$ in $\{0_+\} \cup (0, \infty]$, the jumping kernel $J(x, y)$ may not satisfy the **UJS** condition, see [3] and [11, Sect. 6.2] for more details. Thus, it follows from (the proof of) [17, Proposition 3.3], that parabolic Harnack inequalities of full ranges do not hold. Thus, the results of [18] in particular give a family of Feller processes that satisfy global two-sided heat kernel estimates, but the associated parabolic Harnack inequalities for full ranges fail. We further mention that, under condition $(\mathbf{J}_{\phi,0_+,\le})$, we always have the joint Hölder continuity for the heat kernel $q(t, x, y)$ so that we can establish two-sided estimates for $q(t, x, y)$ for every $t > 0$ and $x, y \in \bar{D}$ without introducing any exceptional set.

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References

1. S. Andres, M.T. Barlow, Energy inequalities for cutoff functions and some applications. J. Reine Angew. Math. **699**, 183–215 (2015)
2. J. Bae, P. Kim, On estimates of transition density for subordinate Brownian motions with Gaussian components in $C^{1,1}$-open sets. Potential Anal. **52**, 661–687 (2020)
3. M.T. Barlow, R.F. Bass, T. Kumagai, Parabolic Harnack inequality and heat kernel estimates for random walks with long range jumps. Math. Z. **261**, 297–320 (2009)
4. M.T. Barlow, E.A. Perkins, Brownian motion on the Sierpiński gasket. Probab. Theory Relat. Fields **79**, 543–623 (1988)
5. A. Bensoussan, J.L. Lions, G. Papanicolaou, *Asymptotic Analysis for Periodic Structures* (North-Holland, Amsterdam, 1978)
6. E.A. Carlen, S. Kusuoka, D.W. Stroock, Upper bounds for symmetric Markov transition functions. Ann. Inst. Henri Poincaré Probab. Stat. **23**, 245–287 (1987)
7. Z.-Q. Chen, M. Fukushima, *Symmetric Markov Processes, Time Change, and Boundary Theory* (Princeton University Press, Princeton, NJ, 2012)
8. Z.-Q. Chen, E. Hu, Heat kernel estimates for $\Delta + \Delta^{\alpha/2}$ under gradient perturbation. Stochastic Process. Appl. **125**, 2603–2642 (2015)
9. Z.-Q. Chen, E. Hu, L. Xie, X. Zhang, Heat kernels for non-symmetric diffusion operators with jumps. J. Differential Equations **263**, 6576–6634 (2017)

10. Z.-Q. Chen, P. Kim, T. Kumagai, J. Wang: Heat kernel upper bounds for symmetric Markov semigroups. J. Funct. Anal. **281** (2021), paper 109074
11. Z.-Q. Chen, P. Kim, T. Kumagai, J. Wang: Heat kernels for reflected diffusions with jumps on inner uniform domains. Trans. Amer. Math. Soc. https://doi.org/10.1090/tran/8678
12. Z.-Q. Chen, P. Kim, R. Song, Heat kernel estimates for $\Delta + \Delta^{\alpha/2}$ in $C^{1,1}$ open sets. J. Lond. Math. Soc. **84**, 58–80 (2011)
13. Z.-Q. Chen, P. Kim, R. Song: Global heat kernel estimates for $\Delta + \Delta^{\alpha/2}$ in half-space-like domains. Electron. J. Probab. **17**, paper 32 (2012)
14. Z.-Q. Chen, P. Kim, R. Song, Dirichlet heat kernel estimates for subordinate Brownian motions with Gaussian components. J. Reine Angew. Math. **711**, 111–138 (2016)
15. Z.-Q. Chen, T. Kumagai, A priori Hölder estimate, parabolic Harnack principle and heat kernel estimates for diffusions with jumps. Rev. Mat. Iberoam. **26**, 551–589 (2010)
16. Z.-Q. Chen, T. Kumagai, J. Wang: Stability of heat kernel estimates for symmetric non-local Dirichlet form. Memoirs Amer. Math. Soc. **271**(1330), v+89 (2021)
17. Z.-Q. Chen, T. Kumagai, J. Wang, Stability of parabolic Harnack inequalities for symmetric non-local Dirichlet forms. J. Eur. Math. Soc. **22**, 3747–3803 (2020)
18. Z.-Q. Chen, T. Kumagai, J. Wang: Heat kernel estimates and parabolic Harnack inequalities for symmetric Dirichlet forms. Adv. Math. **374**, paper 107269 (2020)
19. Z.-Q. Chen, T. Kumagai, J. Wang: Stability of heat kernel estimates for symmetric diffusion processes with jumps, in *Proceedings of the International Congress of Chinese Mathematicians* (Beijing, 2019) (to appear)
20. M. Foondun, Harmonic functions for a class of integro-differential operators. Potenial Anal. **31**, 21–44 (2009)
21. M. Fukushima, Y. Oshima, M. Takeda: *Dirichlet Forms and Symmetric Markov Processes*, 2nd rev. and ext. edn. (De Gruyter, Berlin, 2011)
22. A. Grigor'yan, J. Hu, Upper bounds of heat kernels on doubling spaces. Mosco Math. J. **14**, 505–563 (2014)
23. P. Gyrya, L. Saloff-Coste: Neumann and Dirichlet heat kernels in inner uniform domains. Astérisque **336**, viii+144 (2011)
24. P.-A. Meyer, Renaissance, recollements, mélanges, ralentissement de processus de Markov. Ann. Inst. Fourier **25**, 464–497 (1975)
25. M. Rao, R. Song, Z. Vondraček, Green function estimates and Harnack inequalities for subordinate Brownian motion. Potential Anal. **25**, 1–27 (2006)
26. R. Song, Z. Vondraček, Harnack inequality for some discontinuous Markov processes with a diffusion part. Glas. Mat. Ser. **III**(40), 177–187 (2005)
27. R. Song, Z. Vondraček, Parabolic Harnack inequality for the mixture of Brownian motion and stable process. Tohoku Math. J. **59**, 1–19 (2007)

On Non-negative Solutions to Space-Time Partial Differential Equations of Higher Order

Kristian P. Evans and Niels Jacob

Abstract We discuss when certain higher order partial differential operators in space and time admit non-negative solutions which have a semigroup representation as well as a representation by some associated Markov process.

Keywords Non-negative solutions of higher order PDEs · Positivity preserving operator semi-groups · Lévy processes

Mathematics Subject Classification 35B09, 35C99, 35G10, 47D06, 47F05

1 Introduction

Modelling the dependencies of a process with the help of space-time partial differential equations shall lead to solutions which capture typical observed phenomena, e.g. the propagation of singularities, preservation of positivity, etc. The heat or diffusion equation is an example of an equation the solutions of which preserve the positivity (more correctly, the non-negativity) of initial data. It is also an example of an equation whose solution operator exhibits strong smoothing effects, e.g. continuous initial data are turned into C^∞-solutions. In addition, we encounter the semigroup property. These analytic properties do have a probabilistic companion. With the heat equation we can associate a Brownian motion and we can use Brownian motion to represent solutions to the heat equation. Indeed, the Gaussian semigroup $(T_t^G)_{t\geq 0}$ which gives the solution to the initial value problem to the heat equation admits a representation using Brownian motion $(B_t)_{t\geq 0}$ by

Dedicated to Masatoshi Fukushima, Scholar, Mentor, and Friend.

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$$u(t,x) = (T_t^G g)(x) = E^x(g(B_t)). \tag{1}$$

Since we may construct Brownian motion with the help of the fundamental solution to the heat equation, Formula (1) looks rather natural. An obvious question is to find those space-time partial differential operators which allow an analogous treatment. It is well known that this is limited to second order partial differential operators with suitable coefficients of the type

$$\frac{\partial}{\partial t} - \sum_{k,l=1}^{n} a_{kl} \frac{\partial^2}{\partial x_k \partial x_l} + \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j} + c, \tag{2}$$

where $(a_{kl})_{k,l=1,\ldots,n}$ is a non-negative definite (symmetric) matrix. In our paper we are not interested in minimal smoothness assumptions for coefficients, but we are stimulated by the fact that certain higher order (in space and/or in time) partial differential operators still admit certain positive solutions, some of which can even be represented with the help of Markov processes, not necessarily Brownian motion. The simplest and best known example is the Laplace operator $\frac{\partial^2}{\partial t^2} + \Delta_n$ in the half-space $\mathbb{R}_+ \times \mathbb{R}^n$ which is not of the type (2), but which has solutions we can represent with the help of the Cauchy process. Indeed, the Dirichlet problem

$$\frac{\partial^2}{\partial t^2} u(t,x) + \Delta_n u(t,x) = 0 \text{ in } \mathbb{R}_+ \times \mathbb{R}^n, \tag{3}$$

$$\lim_{t\to 0} u(t,x) = g(x) \tag{4}$$

has for a suitable g a (unique) solution which is given by the Poisson integral, i.e.

$$u(t,x) = \int_{\mathbb{R}^n} P_n(t, x-y) g(y)\, \mathrm{d}y. \tag{5}$$

However, this classical Poisson formula in the half-space is clearly related to the Cauchy process $(C_t)_{t\geq 0}$ and the Cauchy semigroup $(T_t^C)_{t\geq 0}$, namely by

$$u(t,x) = (T_t^C g)(x) = E^x(g(C_t)). \tag{6}$$

Note that Eq. (3) is of second order in t, not of first order. Taking in (3) the (partial) Fourier transform with respect to x we arrive at the ordinary differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \hat{u}(t,\xi) - |\xi|^2 \hat{u}(t,\xi) = 0 \tag{7}$$

and the "initial" condition

$$\hat{u}(0,\xi) = \hat{g}(\xi). \tag{8}$$

Note that we have only one "initial" condition for the second order equation. The Ansatz $\hat{u}(t,\xi) = e^{-\lambda t}$, $\lambda = \lambda(\xi)$, leads to the characteristic equation

$$\lambda^2 - |\xi|^2 = 0 \tag{9}$$

with the two solutions $\lambda_{1,2} = \lambda_{1,2}(\xi) = \pm|\xi|$. The solution $\lambda_1(\xi) = |\xi|$ gives

$$u(t,x) = F^{-1}_{\xi\mapsto x}(e^{-t|\cdot|}\hat{g})(x) = (T_t^C g)(x). \tag{10}$$

We may factorise (9) according to

$$(\lambda^2 - |\xi|^2) = (\lambda - |\xi|)(\lambda + |\xi|) \tag{11}$$

and the solution $\lambda(\xi) = |\xi|$ is the one of interest. It is a continuous negative definite function, hence it is associated with a convolution semigroup and therefore with a Lévy process. Guided by this well known example, see [9], we want to discuss the following problem: Let the partial differential equation with constant coefficients

$$\frac{\partial^N}{\partial t^N}u(t,x) - \sum_{j=0}^{N-1}\sum_{|\alpha|\le m} a_{j\alpha}\frac{\partial^j}{\partial t^j}\left(-i\frac{\partial}{\partial x}\right)^\alpha u(t,x) = 0 \tag{12}$$

subject to the initial condition

$$u^{(l-1)}(0,x) = h_l(x), \quad l = 0,\dots,N-1. \tag{13}$$

Is it possible to obtain solutions to (12)/(13) of the type

$$u(t,x) = \sum_{j=1}^{L}(\gamma_j T_t^{(j)} g_j)(x) = \sum_{j=1}^{L}\gamma_j E^x(g_j(X_t^{(j)})), \quad L \le N, \tag{14}$$

where $(T_t^{(j)})_{t\ge 0}$ is a positivity preserving semigroup acting on functions defined on $\mathbb{R}^n$ and which is associated with a Markov process $(X_t^{(j)})_{t\ge 0}$? Clearly, there are quite a few problems such as regularity or domain questions. To handle such question we choose to work in the Hilbert space setting, i.e. we use $L^2(\mathbb{R}^n)$ as underlying space, a restriction which is not as restrictive as it seems, other settings e.g. the Feller setting working with $C_\infty(\mathbb{R}^n)$, the space of all continuous functions vanishing at infinity, is in principle possible. In addition, we are searching only for holomorphic semigroups. Another problem is that we need to associate with (12) a total of N independent "initial" conditions, not necessarily of the form (13), but in (14) we have only $L \le N$ conditions.

In Sect. 1 we look at an abstract version of our problem and push it formally to a stage such that we can derive conditions to solve (12) with the help of (14). We then turn to equations of the form (12) and for this we need to introduce pseudo-differential operators with constant coefficients, but rather general ξ-dependence of their symbols, see Sect. 2. In Sect. 3 we discuss in more detail the case $N = 2$ in order to understand how to transfer our problem to questions posed on the involved symbols. Maybe the most important insight of this section is that our programme to find solutions of the type (14) works in principle well, however only case by case studies will allow us to cope with initial data. The final section is devoted to various classes of examples, by no means covering the full scope of our programme. Indeed, in some sense this paper is more about a programme to obtain positivite solutions of higher order space-time partial differential equations which allow representations with the help of some Markov processes.

Our notions and notation are standard and we refer to [4]. The Fourier transform is given by

$$\hat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, \mathrm{d}x$$

which entails that the constant in Plancherel's theorem is 1 and that in the convolution theorem it is $(2\pi)^{-\frac{n}{2}}$, i.e. $(u \cdot v)^{\wedge} = (2\pi)^{-\frac{n}{2}}(\hat{u} * \hat{v})$. Sometimes we write Fu for $\hat{u}$ and the inverse Fourier transform is denoted by F^{-1}. Note that we mainly use the partial Fourier transform with respect to x, i.e. for $u = u(t, x)$ we denote by Fu or $\hat{u}$ the Fourier transform with respect to x only. We write $L^2_+(\mathbb{R}^n)$ or L^2_+ for the cone $\{u \in L^2(\mathbb{R}^n) | u \geq 0 \text{ a.e.}\}$ and $u \geq 0$ in the sense of $L^2(\mathbb{R}^n)$ means $u \geq 0$ a.e. The term $\left(-i\frac{\partial}{\partial x}\right)^{\alpha}$ means $(-i)^{|\alpha|}\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$. A continuous function $\psi : \mathbb{R}^n \to \mathbb{C}$ is called a continuous negative definite function if $\psi(0) \geq 0$ and for all $t > 0$ the function $\xi \mapsto e^{-t\psi(\xi)}$ is positive definite in the sense of Bochner. Equivalently, a function is continuous and negative definite if it admits a Lévy-Khinchin representation. A Bernstein function $f : (0, \infty) \to \mathbb{R}_+$ is a C^∞-function satisfying $(-1)^k f^{(k)}(s) \leq 0$, $k \in \mathbb{N}$. The most important result for us is that if f is a Bernstein function and ψ a continuous negative definite function, then $f \circ \psi$ is a continuous negative definite function too. The standard reference for Bernstein functions is [7].

2 An Abstract Problem

Let $(A_j, D(A_j))$, $1 \leq j \leq N$, be a finite family of closable operators densely defined on $L^2(\mathbb{R}^n)$, each of which extends to a generator, denoted again by A_j, of a strongly continuous contraction semigroup $(T_t^{(j)})_{t\geq 0}$ on $L^2(\mathbb{R}^n)$. Since

$$D(A_j \circ A_l) = \{g \in D(A_l) | A_l g \in D(A_j)\}$$

it follows that the assumption

$$[A_j, A_l] = A_j A_l - A_l A_j = 0 \quad \text{for all } 1 \leq j, l \leq N \tag{15}$$

implies that for any $j_1, \ldots, j_M$, $1 \leq j_k \leq N$, the operator $A_{j_1} \circ A_{j_2} \circ \cdots \circ A_{j_M}$ is defined on

$$V := D(A_1 \circ \cdots \circ A_N) \tag{16}$$

which we assume to be dense in $L^2(\mathbb{R}^n)$ too. We find for the Yosida approximation $A_{j,\lambda}$ of A_j that $[A_{j,\lambda}, A_{l,\lambda}] = 0$ and it follows that for all $1 \leq j, l \leq N$ we have

$$[A_j, T_t^{(l)}] = 0, \quad t \geq 0. \tag{17}$$

As a further assumption we pose

$$T_t^{(j)} V \subset V \quad \text{for all} 1 \leq j \leq N. \tag{18}$$

Note that in later situations we will replace V in (16) and (18) with a smaller subspace of V. Clearly we have the equalities

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t^{(j)} g = A_j T_j^{(j)} g, \quad g \in D(A_j), \tag{19}$$

as well as

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t^{(j)} g = A_j T_t^{(j)} g \ \text{ on } V. \tag{20}$$

By (17) and (18) we have with $1 \leq j_1, \ldots, j_M \leq N$, $1 \leq l_1, \ldots, l_k \leq N$ that any permutation of the compositions $A_{j_1} \circ \cdots \circ A_{j_M} \circ T_t^{(l_1)} \circ \cdots \circ T_t^{(l_N)}$ is defined on V and these permutations are equal to each other. Consequently we have for each $1 \leq j \leq N$ and for $g \in V$ that

$$\begin{aligned} &\left(\frac{\mathrm{d}}{\mathrm{d}t} - A_1\right) \cdots \left(\frac{\mathrm{d}}{\mathrm{d}t} - A_N\right) T_t^{(j)} g \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t} - A_1\right) \cdots \left(\frac{\mathrm{d}}{\mathrm{d}t} - A_{j-1}\right) \left(\frac{\mathrm{d}}{\mathrm{d}t} - A_{j+1}\right) \cdots \left(\frac{\mathrm{d}}{\mathrm{d}t} - A_N\right) \left(\frac{\mathrm{d}}{\mathrm{d}t} - A_j\right) T_j^{(j)} g = 0 \end{aligned}$$

holds. Thus

$$u_j(t, x) := (T_t^{(j)} g_j)(x) \quad (\text{in } L^2(\mathbb{R}^n)) \tag{21}$$

is a solution to the equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - A_1\right) \cdots \left(\frac{\mathrm{d}}{\mathrm{d}t} - A_N\right) u_j = 0, \quad 1 \leq j \leq N. \tag{22}$$

Hence for any scalars $\gamma_j \in \mathbb{R}$ we find for $g_j \in V$ a solution to (22) by

$$v(t,x) := \sum_{j=1}^{N} (\gamma_j T_j^{(j)} g_j)(x) \quad (\text{in } L^2(\mathbb{R}^n)). \tag{23}$$

By our assumption, $(T_t^{(j)})_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^2(\mathbb{R}^n)$ and therefore we have in $L^2(\mathbb{R}^n)$

$$\lim_{t\to 0} T_t^{(j)} g_j = g_j, \tag{24}$$

and consequently as an identity in $L^2(\mathbb{R}^n)$

$$v(0,x) = \left(\sum_{j=1}^{N} \gamma_j g_j\right)(x). \tag{25}$$

For $t > 0$ we may formally differentiate (23) k-times, $k \in \mathbb{N}$, to find

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k} v(t,x) = \left(\sum_{j=1}^{N} \gamma_j A_j^k T_t^{(j)} g_j\right)(x) \quad (\text{in } L^2(\mathbb{R}^n)), \tag{26}$$

however in order to justify (26) we need to assume $T_t^{(j)} g_j \in D(A_j^k)$. Note that for a holomorphic semigroup $(T_t^{(j)})_{t\geq 0}$ this condition is always satisfied. Passing in (26) formally to the limit as $t \to 0$ we arrive at

$$\frac{\mathrm{d}^{\mathrm{k}}}{\mathrm{d}t^k} v(0,x) = \left(\sum_{j=1}^{N} \gamma_j A_j^k g_j\right)(x) \quad (\text{in } L^2(\mathbb{R}^n)) \tag{27}$$

and once again, if for example $(T_t^{(j)})_{t\geq 0}$ is for each $1 \leq j \leq N$ a holomorphic semigroup, the calculation can be justified.

Now we change our point of view and consider (22) as an ordinary operator-differential equation of order N in $L^2(\mathbb{R}^n)$, i.e. we consider

$$\left(\frac{\mathrm{d}^N}{\mathrm{d}t^N} - \left(\sum_{j=1}^{N} A_j\right) \frac{\mathrm{d}^{N-1}}{\mathrm{d}t^{N-1}} + \cdots + (-1)^N A_1 \circ \cdots \circ A_N\right) u = 0 \tag{28}$$

and for this equation we prescribe the N initial conditions

$$\left.\begin{array}{c} u(0,x) = \tilde{h}_0(x) = h_1(x) \\ \vdots \quad \vdots \quad \vdots \\ u^{(N-1)}(0,x) = \tilde{h}_{N-1}(x) = h_N(x) \end{array}\right\} \tag{29}$$

The function $v(t,x) := \sum_{j=1}^{N}(\gamma_j T_t^{(j)} g_j)(x)$ is of course a special solution to (28) as are $u_j(t,x) := (T_t^{(j)} g_j)(x)$, and we shall not expect that we can always fit the initial conditions using only these solutions. However, under certain (in general, restrictive) conditions on $h_1, \ldots, h_N$ it might become possible to single out solutions to (28) and (29) having special properties, e.g. being positivity preserving.

We want to note that when considering an operator of the type

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - A_1\right) \cdots \left(\frac{\mathrm{d}}{\mathrm{d}t} - A_N\right) \sum_{j=0}^{M} \sum_{l=0}^{m} a_{jl} \frac{\mathrm{d}^j}{\mathrm{d}t^j} B_l, \quad a_{jl} \in \mathbb{R}, \tag{30}$$

where the operators B_l are densely defined on $L^2(\mathbb{R}^n)$ and satisfy certain commutator relations, then under reasonable domain conditions it is still possible to obtain solutions of the corresponding equation

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - A_1\right) \cdots \left(\frac{\mathrm{d}}{\mathrm{d}t} - A_N\right) \sum_{j=0}^{M} \sum_{l=0}^{m} \frac{\mathrm{d}^j}{\mathrm{d}t^j} B_l u(t,x) = 0 \tag{31}$$

with the help of the semigroups $(T_t^{(k)})_{t\geq 0}$, $1 \leq k \leq N$.

It is clear that, in general, no unique solution of (28) and (29) of the type $v(t,x) = \left(\sum_{j=1}^{N} \gamma_j T_t^{(j)} g_j\right)(x)$ with g depending on $h_1, \ldots, h_N$ exists. Indeed, neither the existence nor the uniqueness of such a solution can be taken for granted. In order to get some ideas we now restrict ourselves to the case $N = 2$ and we assume that $(T_t^{(1)})_{t\geq 0}$ is positivity preserving (or sub-Markovian) on $L^2(\mathbb{R}^n)$ whereas $(T_t^{(2)})_{t\geq 0}$ is not. For simplicity we add the assumption that $(T_t^{(1)})_{t\geq 0}$ is holomorphic, which follows for example if $(T_t)_{t\geq 0}$ is symmetric and conservative. Thus for $g \in L^2(\mathbb{R}^n)$, $g \geq 0$ in $L^2(\mathbb{R}^n)$ a non-negative solution to

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - A_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - A_2\right) u(t,x) = 0 \tag{32}$$

is given by $u(t,x) = (T_t^{(1)} g)(x)$. Moreover, we have

$$\lim_{t\to 0} u(t,x) = \lim_{t\to 0} T_t^{(1)} g(x) = g(x) \text{ in } L^2(\mathbb{R}^n)$$

and differentiation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t,x) = A_1 T_t^{(1)} g(x) = (T_t^{(1)} A_1 g)(x)$$

where for the last step we need to assume that $g \in D(A_1)$. Under this assumption we find

$$\lim_{t\to 0} \frac{\mathrm{d}u(t,x)}{\mathrm{d}t} = \lim_{t\to 0} \left(T_t^{(1)} A_1 g\right)(x) = (A_1 g)(x) \text{ in } L^2(\mathbb{R}^n).$$

If we add to (32) the initial condition

$$\left.\begin{aligned} u(0,x) &:= \lim_{t\to 0} u(t,x) = h_1(x) \\ \frac{\mathrm{d}}{\mathrm{d}t}u(0,x) &:= \lim_{t\to 0} \frac{\mathrm{d}}{\mathrm{d}t}u(t,x) = h_2(x) \end{aligned}\right\} \tag{33}$$

we arrive at the relation

$$h_1 = g \quad \text{and} \quad h_2 = A_1 g. \tag{34}$$

Thus, for the initial value problem

$$\left.\begin{array}{c} \left(\frac{\mathrm{d}}{\mathrm{d}t} - A_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t} - A_2\right) u(t,x) = \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - (A_1 + A_2)\frac{\mathrm{d}}{\mathrm{d}t} + A_1 A_2\right) u(t,x) = 0 \\ u(0,x) = g \text{ and } \frac{\mathrm{d}}{\mathrm{d}t}u(0,x) = A_1 g \end{array}\right\} \tag{35}$$

a solution is given by $u(t,x) := (T_t^{(1)} g)(x)$ and this solution is positive in the sense that $g \geq 0$ in $L^2(\mathbb{R}^n)$ implies $u(t,x) \geq 0$. Of course, a uniqueness result for (28) and (29) (with $N = 2$) holds in our situation, but we have to note that the initial data h_1 and h_2 are not independent of each other.

We now want to study the more general case, namely to find positive solutions to (28) and (29) under the assumption that for $M \leq N$ the semigroup $(T_t^{(j)})_{t\geq 0}$, $1 \leq j \leq M$, generated by A_j are positivity preserving in $L^2(\mathbb{R}^n)$. In this case, for $g_j \in L^2(\mathbb{R}^n)$, $g_j \geq 0$, $1 \leq j \leq M$, and coefficients $\gamma_j \geq 0$ each of the functions

$$v(t,x) := \sum_{j=1}^{M} \gamma_j T_j^{(j)} g_j(x) \quad (\text{ in } L^2(\mathbb{R}^n)) \tag{36}$$

gives a non-negative solution to (28) and we need to relate the functions g_j to the initial data $h_1, \ldots, h_N$. Under appropriate conditions on $(T_t^{(j)})_{t\geq 0}$, for example holomorphy, we derive using (27) the equations

$$v^{(k-1)}(0,\cdot) = h_k = \sum_{j=1}^{M} \gamma_j A_j^{k-1} g_j, \quad 1 \leq k \leq N. \tag{37}$$

Thus, in the situation under discussion, given $\gamma_j \geq 0$, $1 \leq j \leq M$, and functions $g_j \in D(A_j^{N-1})$, $1 \leq j \leq M$, for h_k, $1 \leq j \leq M$, determined by (37) we have a non-negative solution to (28) and (29) by (36) provided $g_j \geq 0$. The more interesting question is of course whether we can determine $g_j \in D(A_j^N)$, $g_j \geq 0$, and $\gamma_j \geq 0$, $1 \leq j \leq M$, for given functions h_k, $1 \leq k \leq N$. These are N equations for (essentially) $M < N$ unknown functions, but due to the conditions $g_j \geq 0$, these are non-linear equations. We have to solve for the mapping

$$S : \bigtimes_{j=1}^{M} D_+(A_j^{N-1}) \to \left(L^2\left(\mathbb{R}^n\right)\right)^N \tag{38}$$

$$SG = H, G = (g_1, \ldots, g_M) \mapsto (h_1, \ldots, h_N) = H \tag{39}$$

where h_k is given by (37) and $D_+(A_j^{N-1}) = \{g_j \in D(A_j^{N-1}) | g \geq 0\}$. Clearly $\times_{j=1}^{M} D_+(A_j^{N-1})$ is a convex set in $(L^2(\mathbb{R}^n))^M$ and S maps convex combinations onto convex combinations implying that the image of $\times_{j=1}^{M} D_+(A_j^{N-1})$ under S is a convex subset in $(L^2(\mathbb{R}^n))^N$. For $M = 1$, N fixed and $\gamma_1 = 1$ for simplicity, we have the N equations

$$h_k = A_1^{k-1} g_1, \quad 1 \leq k \leq N, \tag{40}$$

which implies of course $g_1 = h_1 = \tilde{h}_0$. Moreover, for $k = 2$ we get formally

$$g_2 = (A_1)^{-1} h_2 = (A_1)^{-1} \tilde{h}_0. \tag{41}$$

In general, we may try to interpret $(A_1)^{-1}$ as the abstract potential operator in the sense of Yosida associated with $(T_t^{(1)})_{t \geq 0}$. But of course we have to sort out domain problems, and similarly we may try to handle $g_k = (A_1)^{-1} \circ \cdots \circ (A_1)^{-1} h_k$ with k copies of $(A_1)^{-1}$. The case $M \geq 2$ is obviously much more complicated and we will pick it up in forthcoming investigations. In the next section we want to turn our attention to concrete pseudo-differential operators and by this we can reduce our consideration to the level of symbols, i.e. functions which are easier to handle than abstract operators.

3 Some Translation Invariant Pseudo-differential Operators

In order to handle operators such as (30) for concrete operators A_j and B_l we now introduce translation invariant pseudo-differential operators in a quite general manner. Note that any translation invariant operator on $\mathcal{S}'(\mathbb{R}^n)$ is indeed a convolution operator, but its kernel might be rather singular.

Let $q : \mathbb{R}^n \to \mathbb{R}$ be a continuous function of at most polynomial growth, i.e. we have for some $c \geq 0$ and $m \geq 0$ the estimate

$$|q(\xi)| \leq c(1 + |\xi|^2)^{\frac{m}{2}} \quad \text{for all } \xi \in \mathbb{R}^n. \tag{42}$$

On $\mathcal{S}(\mathbb{R}^n)$ we can define the pseudo-differential operator

$$q(D)u(x) := (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(\xi)\hat{u}(\xi)\, \mathrm{d}\xi. \tag{43}$$

From (42) and Plancherel's theorem we deduce immediately that

$$\|q(D)u\|_s \leq c_{q,s}\|u\|_{s+m} \tag{44}$$

for all u belonging to $\mathcal{S}(\mathbb{R}^n)$, or for $u \in H^{s+m}(\mathbb{R}^n)$, where $H^t(\mathbb{R}^n)$, $t \in \mathbb{R}$, denotes the standard Bessel potential space (or Sobolev space of fractional order) with the norm

$$\|u\|_t^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^t |\hat{u}(\xi)|^2\, \mathrm{d}\xi. \tag{45}$$

The operator $q(D)$ has extensions $q(D) : H^{m+s}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$, however, in general, we cannot determine the domain of the closure of $(q(D), \mathcal{S}(\mathbb{R}^n))$ in $L^2(\mathbb{R}^n)$ in terms of classical Sobolev spaces. If q_1 and q_2 are continuous symbols each satisfying (42) with c_j and m_j, then their compositions $q_1(D) \circ q_2(D)$ is given on $\mathcal{S}(\mathbb{R}^n)$ by

$$(q_1(D) \circ q_2(D)u)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q_1(\xi)q_2(\xi)\hat{u}(\xi)\, \mathrm{d}\xi \tag{46}$$

which extends to an operator on $L^2(\mathbb{R}^2)$ with domain $H^{m_1+m_2}(\mathbb{R}^n)$. Moreover, on $H^{m_1+m_2}(\mathbb{R}^n)$ we have $[q_1(D), q_2(D)] = 0$ since all translation invariant operators on $\mathcal{S}'(\mathbb{R}^n)$ commute. Note that $q_j(D)$ maps $H^{m_1+m_2}(\mathbb{R}^n)$ continuously into $H^{m_k}(\mathbb{R}^n)$, $j, k = 1, 2$, $j \neq k$.
Thus, if we restrict in (30) the operators A_j and B_l to be operators of the type (43), translation invariance and hence commutativity can be taken for granted and in addition we can always operate on some space $H^m(\mathbb{R}^n)$, m sufficiently large, in order to handle various compositions of the operators A_j and B_l.

We are interested in the case where some of the operators A_j are generators of translation invariant sub-Markovian semigroups and in this case we know much more.
Let $(\mu_t)_{t\geq 0}$ be a convolution semigroup on $\mathbb{R}^n$ and $\psi : \mathbb{R}^n \to \mathbb{C}$ its associated continuous negative definite function, i.e. we have

$$\hat{\mu}_t(\xi) = (2\pi)^{-\frac{n}{2}} e^{-t\psi(\xi)}, \quad t > 0 \text{ and } \xi \in \mathbb{R}^n. \tag{47}$$

We can associate with $(\mu_t)_{t\geq 0}$ an L^2-sub-Markovian semigroup

$$(T_t^{\psi} g)(x) = (\mu_t * g)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t\psi(\xi)} \hat{g}(\xi)\, \mathrm{d}\xi, \tag{48}$$

i.e. $(T_t^{\psi})_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^2(\mathbb{R}^n)$ satisfying $0 \leq g \leq 1$ in $L^2(\mathbb{R}^n)$, i.e. $\lambda^{(n)}$-almost everywhere, implies $0 \leq T_t^{\psi} g \leq 1$ in $L^2(\mathbb{R}^n)$. Moreover, if ψ is real-valued then $(T_t^{\psi})_{t\geq 0}$ is symmetric, i.e. $(T_t^{\psi} g, h)_0 = (g, T_t^{\psi} h)_0$, and if in addition $\psi(0) = 0$, then $(T_t^{\psi})_{t\geq 0}$ is conservative, hence Markovian, which means that its extension to $L^{\infty}(\mathbb{R}^n)$ has the property that $T_t^{\psi} 1 = 1$ $\lambda^{(n)}$-almost everywhere.

By a theorem of Stein [8] such a semigroup has a holomorphic extension $z \mapsto T_z^{\psi}$ for z in a certain sector of $\mathbb{C}$.

For every continuous negative definite function ψ the function $\xi \mapsto \psi(\xi) - \psi(0)$ is again a continuous negative definite function and if ψ is real-valued then $(T_t^{\psi-\psi(0)})_{t\geq 0}$ is a symmetric Markovian strongly continuous contraction semigroup on $L^2(\mathbb{R}^n)$, hence it has a holomorphic extension. However we have

$$\begin{aligned} T_t^{\psi} g &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t\psi(\xi)} \hat{g}(\xi)\, \mathrm{d}\xi \\ &= (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-t\psi(0)} e^{-t(\psi(\xi)-\psi(0))} \hat{g}(\xi)\, \mathrm{d}\xi \\ &= e^{-t\psi(0)} T_t^{\psi-\psi(0)} g \end{aligned}$$

implying that for every real-valued continuous negative definite function $\psi : \mathbb{R}^n \to \mathbb{R}$ we can consider $(T_t^{\psi})_{t\geq 0}$ as a holomorphic semigroup. We also note that on $L^2(\mathbb{R}^n) \cap C_{\infty}(\mathbb{R}^n)$ this semigroup admits the representation

$$(T_t^{\psi} g)(x) = \int_{\mathbb{R}^n} g(x-y)\mu_t(\mathrm{d}y),\ g \in L^2(\mathbb{R}^2) \cap C_{\infty}(\mathbb{R}^n) \tag{49}$$

which is pointwisely defined and which admits a pointwise extension to $C_b(\mathbb{R}^n)$. Let $(T_t^{\psi})_{t\geq 0}$ be the symmetric L^2-semigroup associated by (48) with ψ. The L^2-generator of $(T_t^{\psi})_{t\geq 0}$ is the operator $(A^{\psi}, H^{\psi,2}(\mathbb{R}^n))$ where

$$H^{\psi,s}(\mathbb{R}^n) := \left\{u \in L^2(\mathbb{R}^n) \,|\, \|u\|_{\psi,s} < \infty\right\}, \quad s \geq 0, \tag{50}$$

and

$$\|u\|_{\psi,s}^2 := \int_{\mathbb{R}^n} (1+\psi(\xi))^s |\hat{u}(\xi)|^2 \mathrm{d}\xi = \|(1+\psi(D))^{\frac{s}{2}} u\|_{L^2}^2, \tag{51}$$

where we denote by $\psi(D)$ and $(1+\psi(D))^{\frac{s}{2}}$ the pseudo-differential operators

$$\psi(D)u(x) = F^{-1}(\psi\hat{u})(x) = (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)\hat{u}(\xi)\,\mathrm{d}\xi \tag{52}$$

and

$$\begin{aligned}(1+\psi(D))^{\frac{s}{2}}u(x) &= F^{-1}((1+\psi(\cdot))^{\frac{s}{2}}\hat{u})(x)\\ &= (2\pi)^{-\frac{n}{2}} \int\limits_{\mathbb{R}^n} e^{ix\cdot\xi}(1+\psi(\xi))^{\frac{s}{2}}\hat{u}(\xi)\,\mathrm{d}\xi,\end{aligned} \tag{53}$$

respectively. These operators are considered as extensions from $\mathcal{S}(\mathbb{R}^n)$ to their natural L^2-domains, i.e. $H^{\psi,2}(\mathbb{R}^n)$ and $H^{\psi,s}(\mathbb{R}^n)$, respectively. An easy calculation shows now that

$$A^{\psi} = -\psi(D), \quad D(A^{\psi}) = H^{\psi,2}(\mathbb{R}^n). \tag{54}$$

(For details we refer to [4] or [2]) In order to cover interesting examples we want to emphasise that if $\varphi : \mathbb{R}^m \to \mathbb{R} = \varphi(\xi_1, \ldots, \xi_m)$, $m \le n$, is a continuous negative definite function on $\mathbb{R}^m$ then $\psi : \mathbb{R}^n \to \mathbb{R}$, $\psi(\xi_1, \ldots, \xi_n) := \varphi(\xi_1, \ldots, \xi_m)$, is a continuous negative definite function on $\mathbb{R}^n$. Moreover, the sum ψ of finitely many continuous negative definite functions $\psi_j : \mathbb{R}^n \to \mathbb{R}$, $1 \le j \le N$, i.e. the function $\psi = \psi_1 + \cdots + \psi_N$, is again a continuous negative definite function as is $\lambda\psi$, $\lambda > 0$, for ψ continuous negative definite. Finally, we note that for every continuous negative definite function we have the estimate

$$|\psi(\xi)| \le c_{\psi}(1+|\xi|^2) \tag{55}$$

which implies that $H^s(\mathbb{R}^n) \subset H^{\psi,s}(\mathbb{R}^n)$ for all $s \ge 0$.

We now suggest to first study problem (22), (29) [or (28), (29)] in the context of generators of the type A^{ψ_j}, $j = 1, 2$, then to investigate the case where $N = 2$ but only A_1 is of the type A^{ψ}. The aim is to come towards an understanding of constraints needed to arrive at certain families of positivity preserving solutions.

4 Some Discussions on the Case $N = 2$

With $N = 2$ we choose $A_1 = -\psi_1(D)$ and $A_2 = -\psi_2(D)$, where $\psi_1, \psi_2 : \mathbb{R}^n \to \mathbb{R}$ are continuous negative definite functions with corresponding operator semigroups $(T_t^{(1)})_{t\ge 0}$ and $(T_t^{(2)})_{t\ge 0}$. For the moment we pretend that all data belong to $\mathcal{S}(\mathbb{R}^n)$, later we will take care on precise domains. The equation we want to solve is

$$\left(\frac{\partial}{\partial t} + \psi_1(D)\right)\left(\frac{\partial}{\partial t} + \psi_2(D)\right)u(t,x) = 0 \tag{56}$$

under the initial conditions

$$\left.\begin{aligned} u(0,x) &= h_1(x) \\ \frac{\partial u}{\partial t}(0,x) &= h_2(x) \end{aligned}\right\} \tag{57}$$

We are looking for solutions of the form

$$v(t,x) = \gamma_1 T_t^{(1)} g_1(x) + \gamma_2 T_t^{(2)} g_2(x). \tag{58}$$

If $\gamma_1, \gamma_2 \geq 0$ and $g_1, g_2 \geq 0$ then v is a non-negative solution to (56). Thus the problem is to find g_1, g_2 (non-negative) for given h_1, h_2. From (57) and the holomorphy of $(T_t^{(1)})_{t\geq 0}$ and $(T_t^{(2)})_{t\geq 0}$ we deduce

$$\left.\begin{aligned} \gamma_1 g_1(x) + \gamma_2 g_2(x) &= h_1(x) \\ (-\psi_1(D)\gamma_1 g_1)(x) + (-\psi_2(D)\gamma_2 g_2)(x) &= h_2(x) \end{aligned}\right\} \tag{59}$$

Using the Fourier transform we arrive at

$$\gamma_1 \hat{g}_1(\xi) + \gamma_2 \hat{g}_2(\xi) = \hat{h}_1(\xi) \tag{60}$$

and

$$\gamma_1 \psi_1(\xi) \hat{g}_1(\xi) + \gamma_2 \psi_2(\xi) \hat{g}_2(\xi) = -\hat{h}_2(\xi). \tag{61}$$

Under the assumption $\gamma_1\gamma_2(\psi_2(\xi) - \psi_1(\xi)) \neq 0$, i.e. $\gamma_1 \neq 0, \gamma_2 \neq 0$ and $\psi_2(\xi) \neq \psi_1(\xi)$ we obtain

$$\hat{g}_1(\xi) = \frac{-\hat{h}_1(\xi)\psi_2(\xi) - \hat{h}_2(\xi)}{\gamma_1(\psi_1(\xi) - \psi_2(\xi))}, \quad \hat{g}_2(\xi) = \frac{\hat{h}_1(\xi)\psi_1(\xi) + \hat{h}_2(\xi)}{\gamma_2(\psi_1(\xi) - \psi_2(\xi))} \tag{62}$$

In order to find g_1 and g_2 we now need some conditions. Even with h_j in $\mathcal{S}(\mathbb{R}^n)$ we cannot expect $\hat{g}_1$ or $\hat{g}_2$ to belong to $\mathcal{S}(\mathbb{R}^n)$, however, $\hat{g}_1$ and $\hat{g}_2$ need only to be in $L^2(\mathbb{R}^n)$ in order to find g_1 and g_2 in $L^2(\mathbb{R}^n)$ too. The holomorphy of $(T_t^{(j)})_{t\geq 0}$ then implies that $T_t^{(j)} g_j \in \bigcap_{k\in\mathbb{N}} D([\psi_j(D)]^k) = \bigcap_{k\in\mathbb{N}} H^{\psi_j,2k}(\mathbb{R}^n)$, hence we can achieve sufficient regularity to obtain a solution of (56). What becomes obvious is that a trade-off between the behaviour of the zeroes of $\psi_1 - \psi_2$ and the zeroes of $\hat{h}_1\psi_j \mp \hat{h}_2$ is now needed to determine g_1 and g_1 uniquely. This shall not surprise us, in general we shall not expect (56) and (57) to have a unique (non-negative) solution of the type (58). Given our initial question, it is natural to change the point of view and to start with $\gamma_1, \gamma_2 \geq 0$ as well as with $g_1, g_2 \geq 0$ and to use (57) to determine conditions for h_1 and h_2. In this case, h_1 is already determined by (59) as is h_2 determined by (60). We introduce the mapping

$$\begin{aligned} S : H^{\psi_1,2}(\mathbb{R}^n) \times H^{\psi_2,2}(\mathbb{R}^n) &\to L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \\ (g_1, g_2) \mapsto S(g_1, g_2) &= (\gamma_1 g_1 + \gamma_2 g_2, -\gamma_1 \psi_1(D) g_1 - \gamma_2 \psi_2(D) g_2) \\ &= (F^{-1}(\gamma_1 \hat{g}_1 + \gamma_2 \hat{g}_2), F^{-1}(-\gamma_1 \psi_1 \hat{g}_1 - \gamma_2 \psi_2 \hat{g}_2)) \end{aligned} \tag{63}$$

and by construction, essentially the range of $S|_{H^{\psi_1,2} \times H^{\psi_2,2} \cap L^2_+ \times L^2_+}$ will consist of exactly those elements (h_1, h_2) for which we can find (g_1, g_2) such that $u(t, x) = \gamma_1 T_t^{(1)} g_1(x) + \gamma_2 T_t^{(2)} g_2(x)$ is a non-negative solution to (56). Since S is linear and $H^{\psi_1,2} \times H^{\psi_2,2} \cap L^2_+ \times L^2_+$ is convex the range of $S|_{H^{\psi_1,2} \times H^{\psi_2,2} \cap L^2_+ \times L^2_+}$ is convex too and it always contains the zero function. The range of $\tilde{S} := S|_{H^{\psi_1,2} \times H^{\psi_2,2} \cap L^2_+ \times L^2_+}$ can be characterised in more detail. Since by assumption $g_j \geq 0$ and $g_j \in L^2(\mathbb{R}^n)$ its Fourier transform $\hat{g}_j$ must be a positive definite distribution belonging to $L^2(\mathbb{R}^n)$. Thus we have

$$\begin{aligned} R(\tilde{S}) = \{(F^{-1}(\gamma_1 w_1 + \gamma_2 w_2), F^{-1}(-\gamma_1 \psi_1 w_1 - \gamma_2 \psi_2 w_2)) | \gamma_1, \gamma_2 \geq 0, \\ w_1, w_2 \in L^2(\mathbb{R}^n) \text{ positive definite}, \psi_j w_j \in L^2(\mathbb{R}^n)\}. \end{aligned} \tag{64}$$

Thus we have

Proposition 1 *For* $(h_1, h_2) \in R(\tilde{S})$ *there exists* $(g_1, g_2) \in (H^{\psi_1,2} \times H^{\psi_2,2}) \cap (L^2_+ \times L^2_+)$ *such that* $v(t, x) = (T_t^{(1)} g_1)(x) + (T_t^{(2)} g_2)(x) \geq 0$*. If in addition* $v(t, \cdot), t > 0$ *belongs to* $\{u \in L^2(\mathbb{R}^n) | (1 + \psi_1(D))(1 + \psi_2(D))u \in L^2(\mathbb{R}^n)\}$ *then* v *is a non-negative solution to* (56) *and* (57)*.*

Remark 1 We may introduce the space $H^{\psi_1,\psi_2,s}(\mathbb{R}^n)$ as the space of all elements in $L^2(\mathbb{R}^n)$ such that

$$\|u\|^2_{\psi_1,\psi_2,s} = \int_{\mathbb{R}^n} (1 + \psi_1(\xi))^s (1 + \psi_2(\xi))^s |\hat{u}(\xi)|^2 \,\mathrm{d}\xi < \infty$$

and replace in Proposition 1 the condition $(g_1, g_2) \in (H^{\psi_1,2} \times H^{\psi_2,2}) \cap (L^2_+ \times L^2_+)$ by $(g_1, g_2) \in (H^{\psi_1,\psi_2,2} \times H^{\psi_1,\psi_2,2}) \cap (L^2_+ \times L^2_+)$. A more practical, but less sharp condition would be $(g_1, g_2) \in (H^4 \times H^4) \cap (L^2_+ \times L^2_+)$, and in the case where ψ_j satisfies $|\psi_j(\xi)| \leq c_{\psi_j}(1 + |\xi|^2)^{m_j}$, $0 < m_j < 1$, instead of the estimate $|\psi_j(\xi)| \leq c_{\psi_j}(1 + |\xi|^2)$ we may require $(g_1, g_2) \in (H^{2(m_1+m_2)} \times H^{2(m_1+m_2)}) \cap (L^2_+ \times L^2_+)$.

We next want to look at the case where $A_1 = -\psi(D)$ is a generator of a symmetric sub-Markovian semigroup, but A_2 is not. We assume that A_2 is of the type $q(D)$ with q satisfying (42). A positive solution to (56) with $\psi_2(D)$ being replaced by $-A_2$ is now sought in the form

$$v(t, x) = T_t^{(1)} g_1(x), \tag{65}$$

since $\gamma_1 \neq 0$ is needed we now may chose $\gamma_1 = 1$, hence we put in (58) $\gamma_2 = 0$ and $\gamma_1 = 1$. This leads to

$$\hat{g}_1(\xi) = \hat{h}_1(\xi) \quad \text{and} \quad \psi_1(\xi)\hat{g}_1(\xi) = -\hat{h}_2(\xi) \tag{66}$$

or $\hat{h}_2(\xi) = -\psi_1(\xi)\hat{h}_1(\xi)$. Thus we may obtain positive solutions to (56) and (57) if $h_1 \in L^2(\mathbb{R}^2)$ is a positive definite distribution such that $\psi_1\hat{h}_1 \in L^2(\mathbb{R}^n)$ and if in addition we have $\hat{h}_2 = -\psi_1\hat{g}_1$, i.e. $h_2 = -\psi_1(D)g_1$. However, as an additional condition we need $\psi_1(D)T_t^{(1)}g \in D(A_2)$, for which $g \in H^{m+2}(\mathbb{R}^n)$ is a sufficient condition.

Eventually we want to switch from (56) and (57) to

$$\left(\frac{\partial}{\partial t} + \psi_1(D)\right)\left(\frac{\partial}{\partial t} + \psi_2(D)\right) Bu(t,x) = 0 \tag{67}$$

or

$$\left(\frac{\partial}{\partial t} + \psi_1(D)\right)\left(\frac{\partial}{\partial t} - A_2\right) Bu(t,x) = 0 \tag{68}$$

under the initial conditions

$$\left.\begin{aligned} u(0,x) &= h_1(x) \\ \frac{\partial u}{\partial t}(0,x) &= h_2(x) \end{aligned}\right\} \tag{69}$$

Here $B = q(D)$ is a pseudo-differential operator with symbol $q(\xi)$ satisfying (42). For $g_1, g_2 \in H^{m+4}(\mathbb{R}^n)$ the operators $\psi_1(D)$, $\psi_2(D)$ and $q(D)$ mutually commute and hence we may search for the solutions of the type

$$v(t,x) = \gamma_1 T_t^{(1)} g_1(x) + \gamma_2 T_t^{(2)} g_2(x) \tag{70}$$

or

$$v(t,x) = T_t^{(1)} g_1(x), \tag{71}$$

respectively. This implies that all of our previous considerations carry over to the new case, however we need to add additional assumptions, i.e. domain conditions. For the case of equation (67) the precise condition is of course

$$\gamma_1\psi_1(D)T_t^{(1)}g_1 + \gamma_2\psi_2(D)T_t^{(2)}g_2 \in D(B), \tag{72}$$

and only if $D(B)$ is better known, say as an anisotropic Bessel potential space, we can say more. In the best case we would expect $D(B) = H^{q,2}(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) | q(D) \in L^2(\mathbb{R}^n)\}$ and then we can give more detailed conditions.

We now consider operators of the type (30) where we assume that for some $L \leq N$ the operators A_j, $1 \leq j \leq L$, have an extension from $\mathcal{S}(\mathbb{R}^n)$ to a generator of a holomorphic sub-Markovian semigroup $(T_t)_{t\geq 0}$ on $L^2(\mathbb{R}^n)$. Our aim is to find solutions to (31) of the type

$$v(t,x) = \sum_{j=1}^{L} (\gamma_j T_t^{(j)} g_j)(x), \quad \gamma_j \geq 0 \text{ and } g_j \geq 0 \text{ in } L^2(\mathbb{R}^n). \tag{73}$$

In addition, we add the initial conditions (29). It is clear that in this generality we cannot obtain existence or uniqueness results. Most of all we need to consider carefully domains of suitable extensions of the operators A_j, $L < j \leq N$, and B_l, $1 \leq l \leq M$, and further, on some suitable common domain we need the commutator relations $[A_j, A_l] = 0$ and $[A_j, B_l]$ to hold. We do not want to follow the general abstract case, but we want to assume that all operators involved are translation invariant pseudo-differential operators of the type (43). More precisely, for $1 \leq j \leq L$ we assume that $A_j = -\psi_j(D)$ where $\psi_j : \mathbb{R}^n \to \mathbb{R}$ is a continuous negative definite function and for $L < j \leq N$, as well as for $1 \leq l \leq M$, we assume that the symbols of the operators A_j and B_l satisfy (42) for some growth exponent depending on m_j and $\tilde{m}_l$ respectively. We put

$$m := 2L + \sum_{j=L+1}^{N} m_j + \sum_{l=1}^{M} \tilde{m}_j, \tag{74}$$

and we consider all operators on $H^m(\mathbb{R}^n)$. It follows that on $H^m(\mathbb{R}^n)$ any composition of operators $A_{j_1} \circ \cdots \circ A_{j_k} \circ B_{l_1} \circ \cdots \circ B_{l_j}$, $1 \leq j_\alpha \leq N$, $1 \leq l_\beta \leq M$, maps $H^m(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$, and any of such a composition for $L < j_\alpha \leq N$ maps $H^m(\mathbb{R}^n)$ into $H^{2L}(\mathbb{R}^n)$. Moreover we have $H^m(\mathbb{R}^n) \subset H^{2L}(\mathbb{R}^n)$ and the compositions do not depend on the ordering of the operators. Since by assumption $(T_t^{(j)})_{t \geq 0}$, $1 \leq j \leq L$, extends to a holomorphic semigroup we have for every $g \in L^2(\mathbb{R}^n)$ that $T_t^{(j)} g \in \bigcap_{k \in \mathbb{N}} H^{2k,\psi_j}(\mathbb{R}^n)$, $t > 0$. In order to guarantee that $T_t^{(j)} g \in H^m(\mathbb{R}^n)$, $t > 0$, and hence that all operators $A_{j_1} \circ \cdots \circ A_{j_k} \circ B_{l_1} \circ \cdots \circ B_{l_i}$ commute with $T_t^{(j)}$, $t > 0$, we add the assumption

$$(1 + \psi_j(\xi)) \geq \kappa_0 (1 + |\xi|^2)^{\frac{m_j'}{2}}, \quad \kappa_0 > 0, m_j' > 0, j = 1, \ldots, L. \tag{75}$$

Now it follows that for every collection $g_j \in L^2(\mathbb{R}^n)$, $1 \leq j \leq L$, a solution to (31) is given by

$$v(t,x) := \sum_{j=1}^{L} (\gamma_j T_t^{(j)} g_j)(x), \quad t > 0, \tag{76}$$

and for $\gamma_j \geq 0$, $g_j \geq 0$ in $L^2(\mathbb{R}^n)$ this solution is non-negative.

Next we want to adjust the initial conditions. For $g_j \in H^{2L}(\mathbb{R}^n)$, or even $g_j \in H^{2(m_1' + \cdots + m_L')}(\mathbb{R}^n)$, due to the holomorphy of the semigroups $(T_t^{(j)})_{t \geq 0}$ we find for $1 \leq k \leq N$

$$v^{(k)}(0,x) = \sum_{j=1}^{L} (\gamma_j A_j^{k-1} g_j)(x) = h_k(x) \quad (\text{in } L^2(\mathbb{R}^n)). \tag{77}$$

Switching to the Fourier transforms we obtain the following system of N equations for the L unknown functions $\hat{g}_j$:

$$\sum_{j=1}^{L} \gamma_j (-\psi_j(\xi))^{k-1} \hat{g}_j(\xi) = \hat{h}_k(\xi), \quad 1 \leq k \leq N. \tag{78}$$

Once more, we change our point of view and we consider (78) as conditions for the initial values $h_1, \ldots, h_N$ to hold in order that (31) under (29) admits a non-negative solution.

We introduce the mappings S and $\tilde{S}$ analogously to (63) by

$$\begin{aligned} &S : (H^m \times \cdots \times H^m) \to L^2 \times \cdots \times L^2 \\ &g := (g_1, \ldots, g_L) \mapsto Sg := (h_1, \ldots, h_N), \end{aligned} \tag{79}$$

where

$$h_k = (Sg)_k := \sum_{j=1}^{L} \gamma_j (-\psi_j(D)))^{k-1} g_j, \quad 1 \leq k \leq N, \tag{80}$$

i.e.

$$\hat{h}_k = (Sg)^{\wedge}{}_k = \sum_{j=1}^{L} \gamma_j (-\psi_j)^{k-1} \hat{g}_j. \tag{81}$$

If by assumption $g_j \geq 0$, $1 \leq j \leq L$, then there exists positive definite distributions $w_j \in L^2(\mathbb{R}^n)$ such that $w_j = \hat{g}_j$ and we find

$$h_k = F^{-1}\left(\sum_{j=1}^{L} \gamma_j (-\psi_j)^{k-1} w_j\right). \tag{82}$$

For the range of $\tilde{S} := S|_{(H^m \times \cdots \times H^m) \cap (L^2_+ \times \cdots \times L^2_+)}$ we derive in analogy to (64)

$$R(\tilde{S}) = \left\{ \left(F^{-1}\left(\sum_{j=1}^{L} \gamma_j w_j\right), F^{-1}\left(\sum_{j=1}^{L} \gamma_j (-\psi_j) w_j\right), \ldots, F^{-1}\left(\sum_{j=1}^{L} \gamma_j (-\psi_j)^{N-1} w_j\right)\right) \Bigg| \right.$$
$$\left. \gamma_1, \ldots, \gamma_L \geq 0, w_1, \ldots, w_L \text{ positive definite}, \psi_j^{k-1} w_j \in L^2(\mathbb{R}^n), 1 \leq k \leq N \right\}. \tag{83}$$

Thus we arrive at

Proposition 2 *For* $(h_1, \ldots, h_N) \in R(\tilde{S})$ *there exists* $(g_1, \ldots, g_L) \in (H^m \times \cdots \times H^m) \cap (L^2_+ \times \cdots \times L^2_+)$ *such that* $v(t, x) := \sum_{j=1}^{L} \gamma_j T_t^{(j)} g_j \geq 0$. *If in addition each* $(T_t^{(j)})_{t \geq 0}$ *is holomorphic and* (75) *is satisfied, then* v *solves* (31) *under the initial condition* (29).

Remark 2 While $R(\tilde{S})$ is in general difficult to determine, we may of course choose some of the parameters γ_j to be 0 and then the situation becomes more transparent. For example, we may choose $\gamma_j = 0$ for all $j \neq j_0$ for a fixed $j_0 \in \{1, \ldots, L\}$ and $\gamma_{j_0} = 1$. In this case the condition (80) reduces to

$$h_1 = F^{-1} w_{j_0}, h_2 = F^{-1}(-\psi_{j_0} w_{j_0}), \ldots, h_N = F^{-1}(-\psi_{j_0}^{N-1} w_{j_0}).$$

In the following chapter we will turn to concrete partial differential operators with constant coefficients and we will try to find families of non-negative solutions for related initial value problems.

However, we first want to extend our considerations by allowing complex-valued continuous negative definite functions $\psi_j : \mathbb{R}^n \to \mathbb{C}$ as symbols of $A_j = -\psi_j(D)$. The only change in our argument is required to justify that the associated operator semigroup $(T_t^{(j)})_{t\geq 0}$ is holomorphic on $L^2(\mathbb{R}^n)$. As the example of the drift, which corresponds to $\psi(\xi) = -i\xi, n = 1$, shows us that we cannot expect for a general complex-valued continuous negative definite function ψ_j the semigroup $(T_t^{(j)})$ to be holomorphic. However, in the case where ψ satisfies the sector condition

$$|\mathrm{Im}\, \psi(\xi)| \leq \kappa_0 \mathrm{Re}\, \psi(\xi), \quad \kappa_0 > 0, |\xi| \geq R \geq 0, \tag{84}$$

it follows that $-\psi(D)$ is a sectorial operator and hence the generator of a holomorphic semigroup on $L^2(\mathbb{R}^n)$, see [6] or [10]. Moreover, since $\mathrm{Re}\, \psi$ is a continuous negative function too, we can form the spaces $H^{\mathrm{Re}\, \psi, s}(\mathbb{R}^n)$. Thus replacing in our previous considerations the real-valued continuous negative definite functions by complex-valued continuous negative definite functions each satisfying the sector condition and using the spaces $H^{\mathrm{Re}\, \psi, s}(\mathbb{R}^n)$ with $\mathrm{Re}\, \psi$ satisfying (where appropriate) additional conditions such as (84), we obtain the previous results in the more general situation. For more details we refer to [1] and [4].

5 Higher Order Partial Differential Equations Admitting Non-negative Solutions

We now turn from operator-valued differential operators to partial differential equations of the type

$$\frac{\partial^N}{\partial t^N} u(t,x) - \sum_{j=0}^{N-1} \sum_{|\alpha| \leq m} a_{j\alpha} \frac{\partial^j}{\partial t^j} \left(-i \frac{\partial}{\partial x}\right)^\alpha u(t,x) = 0, \quad a_{jl} \in \mathbb{R}, \tag{85}$$

and we ask when does such an equation admit a solution given by

$$v(t,x) = \sum_{j=1}^{L} (\gamma_j T_t^{(j)} g_j)(x), \quad \gamma_j \geq 0, g_j \geq 0, L \leq N, \tag{86}$$

where $g_j \in L^2(\mathbb{R}^n)$ and $(T_t^{(j)})_{t\geq 0}$, $1 \leq j \leq L$, is an L^2-sub-Markovian semigroup. When taking in (85) the Fourier transform with respect to x we arrive at the parameter dependent ordinary differential equation

$$\frac{\mathrm{d}^N}{\mathrm{d}t^N}\hat{u}(t,\xi) - \sum_{j=0}^{N-1} \sum_{|\alpha|\leq m} a_{j\alpha}\xi^\alpha \frac{\mathrm{d}^j}{\mathrm{d}t^j}\hat{u}(t,\xi) = 0. \tag{87}$$

We long for solutions of (87) of the form

$$\hat{u}(t,\xi) = \hat{v}_k(t,\xi) = e^{-\psi_k(\xi)t} \tag{88}$$

where $\psi_k : \mathbb{R}^n \to \mathbb{C}$ is a continuous negative definite function satisfying the sector condition and $\operatorname{Re}\psi_k$ satisfies the growth condition (75). From (87) we arrive (with $\lambda(\xi) = \psi_k(\xi)$) at the **characteristic equation**

$$\lambda^N(\xi) + \sum_{j=0}^{N-1} \sum_{|\alpha|\leq m} a_{j\alpha}(-1)^{N-j-1}\xi^\alpha \lambda^j(\xi) = 0 \tag{89}$$

for which we seek solutions $\lambda_k = \lambda_k(\xi)$ which are continuous negative definite, satisfying the sector condition and the real part of which satisfies (75). Every such solution will give rise to a holomorphic sub-Markovian semigroup $(T_t^{(k)})_{t\geq 0}$ associated with λ_k by

$$(T_t^{(k)}u)^\wedge(\xi) = e^{-t\lambda_k(\xi)}\hat{u}(\xi) \tag{90}$$

and we may apply the considerations of the previous chapters to obtain non-negative solutions of the type (86) for the equation (85). The problem is of course to find such solutions λ to (89). Even in the cases where we can obtain solutions with the help of radicals, it is not clear which properties the function $\xi \mapsto \lambda_k(\xi)$ will have. So far we have no general answer to our problem, however the following examples show the scope of our considerations. It is clear that if we obtain solutions of the type (86) the function $\lambda_k(\xi)$ in (90) must be a continuous negative function satisfying the sector condition, provided we assume that $(T_t^{(j)})_{t\geq 0}$ to be holomorphic. We prefer to provide some rather concrete examples, but in each case it is obvious that we can include more general and complicated cases with similar symbol structure.

Example 1 **A.** We take in (85) the dimension $n = 1$ and the values $N = 2$, $m = 4$ and $a_{j\alpha} = \delta_{0,4}$. Then we are dealing with the equation

$$\frac{\partial^2}{\partial t^2}u(t,x) - \frac{\partial^4}{\partial x^4}u(t,x) = 0 \tag{91}$$

which yields

$$\lambda^2 - \xi^4 = 0. \tag{92}$$

Since $\lambda^2 - \xi^4 = (\lambda - \xi^2)(\lambda + \xi^2)$ we have the continuous negative definite function $\lambda_1(\xi) = \xi^2$ as a solution which satisfies all our conditions, and for $g \in L^2(\mathbb{R})$, $g \geq 0$, a non-negative solution to (91) is given by $(x, t) \mapsto (T_t^G g)(x)$, where $(T_t^G)_{t\geq 0}$ is the Gaussian semigroup on $L^2(\mathbb{R})$.

B. Taking next $N = 4$ and $m = 2$ in (85), but again $n = 1$, and further $a_{j\alpha} = \delta_{0,2}$, we get the equation

$$\frac{\partial^4}{\partial t^4} u(t, x) + \frac{\partial^2}{\partial x^2} u(t, x) = 0 \tag{93}$$

which leads to $\lambda^4 - \xi^2 = 0$. Obviously $\lambda(\xi) = |\xi|^{\frac{1}{2}}$ is a solution of this equation and this is a continuous negative definite function which fulfills all of our requirements. The associated semigroup $(T_t^\lambda)_{t\geq 0}$ is the semigroup subordinate to the Gaussian semigroup with the help of the Bernstein function $f(s) = s^{\frac{1}{4}}$. The polynomial $\lambda^4 - \xi^2$ admits the factorisation $\lambda^4 - \xi^2 = (\lambda - |\xi|^{\frac{1}{2}})(\lambda + |\xi|^{\frac{1}{2}})(\lambda - i|\xi|^{\frac{1}{2}})(\lambda + i|\xi|^{\frac{1}{2}})$ and therefore only one solution of $\lambda^4 - \xi^2 = 0$ is a continuous negative definite function as sought.

C. Now we take $a_{j\alpha} = \delta_{0,2m}$ as coefficients for $n = 1$ and $2N$, $2m \in \mathbb{N}$ and hence (89) becomes $\lambda^{2N} = |\xi|^{2m}$. Further, by $\lambda = |\xi|^{\frac{m}{N}}$ we always have for $\frac{m}{N} < 2$, i.e. $m \leq 2N$, a continuous negative definite function as a solution satisfying all of our conditions. We can phrase this differently, namely that for $n = 1$ to every α-stable process $(X_t^{(\alpha)})_{t\geq 0}$ with α rational we can find a partial differential equation of $\frac{\partial^{2N}}{\partial t^{2N}} u(t, x) = \frac{\partial^{2m}}{\partial x^{2m}}, u(t, x) = 0$ such that the transition function of $(X_t^{(\alpha)})_{t\geq 0}$ gives the solution to that equation. We refer to [11] where (fractional) differential equations being solved by transition functions of certain stable processes, i.e. densities of certain convolution semigroups, are discussed.

Our next examples show that there are more than just symmetric stable semigroups which give solutions of the type (86). We still assume $n = 1$.

Example 2 **A.** Consider the differential operator $\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + a\frac{\partial}{\partial x}$, where $a \in \mathbb{R}$ is a parameter. This operator leads to the characteristic equation $\lambda^2 - \xi^2 - ia\xi = 0$ which we can factorise according to $\lambda^2 - \xi^2 - ia\xi = (\lambda - (\xi^2 + ia\xi)^{\frac{1}{2}})(\lambda + (\xi^2 + ia\xi)^{\frac{1}{2}})$. The function $\xi \mapsto \xi^2 + ia\xi$ is a continuous negative definite function for every $a \in \mathbb{R}$. Since $s \mapsto f(s) = s^{\frac{1}{2}}$ is a Bernstein function, it follows that $\xi \mapsto \lambda(\xi) = (\xi^2 + ia\xi)^{\frac{1}{2}}$ is a continuous negative definite function too. Moreover, since $\operatorname{Re} \lambda^2(\xi) = \xi^2$ and $\operatorname{Im} \lambda^2(\xi) = a\xi$, it follows that $\lambda^2(\xi)$ fulfills the sector condition as well as the growth condition (75). Hence the semigroup generated by the differential operator with symbol $\lambda^2(\xi)$ is on $L^2(\mathbb{R})$ holomorphic which is inherited by the semigroup obtained by subordination with the help of the Bernstein function f. In addition, since $|\lambda^{\frac{1}{2}}(\xi)| = (\xi^4 + a\xi^2)^{\frac{1}{2}}$ the growth condition (75) is fulfilled too. Thus for $g \geq 0$, $g \in L^2(\mathbb{R})$, a non-negative solution to $\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} = 0$ is given by $u(t, x) := F^{-1}_{\xi \mapsto x}(e^{-(\xi^2 + ia\xi)^{\frac{1}{2}} t} \hat{g}(\xi))(x)$.

B. The wave operator $\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)$ needs a more careful discussion. The characteristic equation $\lambda^2 + \xi^2 = 0$ admits the factorisation $(\lambda - i\xi)(\lambda + i\xi)$. Although $\xi \mapsto \pm i\xi$ are continuous negative definite functions, we cannot apply our considerations since these functions do not satisfy the sector condition and hence the corresponding pseudo-differential operators are not sectorial, hence do not generate a holomorphic semigroup.
C. We want to investigate the operator $\frac{\partial^2}{\partial t^2} + a\frac{\partial}{\partial t} + b\frac{\partial^2}{\partial x^2}$ with $a, b \in \mathbb{R}$. This gives the characteristic equation $\lambda^2 - a\lambda - b\xi^2 = 0$ with solutions $\lambda_{1,2} = \frac{a}{2} \pm \frac{1}{2}\sqrt{a^2 + 4b\xi^2}$. For $a > 0, b > 0$ the function $\lambda_1(\xi) = \frac{a}{2} + \frac{1}{2}(a^2 + 4b\xi^2)^{\frac{1}{2}}$ is a continuous negative definite function satisfying the sector as well as growth conditions.

Remark 3 It is easy to see that if a continuous negative definite solution to a one-dimensional ($n = 1$) characteristic equation depends only on ξ^2, then we can handle the $n-$dimensional case when replacing ξ^2 by $|\xi|^2$, i.e. $-\frac{\partial^2}{\partial x^2}$ by $-\Delta_n$.

Example 3 In Example 1.C. the operator $\frac{\partial^N}{\partial t^N} - (-1)^m \frac{\partial^{2m}}{\partial x^{2m}}$ was discussed and we want to extend our considerations to the case $\frac{\partial^N}{\partial t^N} - \frac{\partial^m}{\partial x^m}$. This entails the characteristic equation $\lambda^N - (-i\xi)^m = 0$ and we always have a solution $\lambda = (-i\xi)^{\frac{m}{N}}$. For $m \leq N$ this is a continuous negative definite function since $\xi \mapsto -i\xi$ is one and $s \mapsto s^{\frac{m}{N}}$, $s \geq 0$, $m \leq N$, is a Bernstein function. However, for $m, N \in \mathbb{N}$, $m < N$, we find $(-i\xi)^{\frac{m}{N}} = |\xi|^{\frac{m}{N}} e^{-i\frac{m}{N}\pi}$, or

$$(-i\xi)^{\frac{m}{N}} = |\xi|^{\frac{m}{N}}(\cos \tfrac{m}{N}\pi - i \sin \tfrac{m}{N}\pi)$$

which gives

$$|\mathrm{Im}\,(-i\xi)^{\frac{m}{N}}| = |\xi|^{\frac{m}{N}} \sin \tfrac{m}{N}\pi = |\xi|^{\frac{m}{N}} \tan(\tfrac{m}{N}\pi) \cos \tfrac{m}{N}\pi = c_{m,N} \mathrm{Re}\,(-i\xi)^{\frac{m}{N}},$$

where $c_{m,N} > 0$ for $0 < m < N$. Thus $\xi \mapsto (-i\xi)^{\frac{m}{N}}$ fulfills the sector condition as well as the growth condition and for $0 < m < N$ our previous results apply to $\frac{\partial^N}{\partial t^N} - \frac{\partial^m}{\partial x^m}$. Note that Example 1.C. extends by Remark 3 to the case $\frac{\partial^N}{\partial t^N} - (-\Delta_n)^m$, but an extension of the example $\frac{\partial^N}{\partial t^N} - \frac{\partial^m}{\partial x^m}$ to higher dimensions is not obvious. For more properties of the one-dimensional drift operator in relation to fractional derivatives we refer to [5].

We now have a look at the Laplace operator in the half-space $\mathbb{R}_+ \times \mathbb{R}^n$, $n \geq 1$.

Example 4 The operator is of course $\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ and we treat the variable $t \geq 0$ differently to the variable $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The characteristic equation becomes $\lambda^2 - |\xi|^2 = 0$ which we can factorise according to $(\lambda - |\xi|)(\lambda + |\xi|)$. The function $\xi \mapsto |\xi|$ is of course a continuous negative definite function satisfying all of our conditions. The corresponding operator semigroup is the Cauchy semigroup and the result will lead us to the Poisson formula for the Laplacian in the half-space, see [9] and our introduction.

Example 5 We may now use our previous examples to study higher order equations in several space dimensions such as $\frac{\partial^4}{\partial t^4} - \frac{\partial^4}{\partial t^2 \partial y^2} + \frac{\partial^3}{\partial t^2 \partial x} - \frac{\partial^3}{\partial y^3}$ the characteristic equation of which is

$$\lambda^4 - i\lambda^2\eta - \lambda^2|\xi|^2 - i\eta|\xi|^2 = (\lambda - (i\eta)^{\frac{1}{2}})(\lambda - |\xi|)(\lambda + (i\eta)^{\frac{1}{2}})(\lambda + |\xi|).$$

The function $\psi_1(\eta) = (i\eta)^{\frac{1}{2}}$ and $\psi_2(\xi) = |\xi|$ are continuous negative definite functions in $\mathbb{R}$, both satisfying all of our conditions on $\mathbb{R}$ and hence the corresponding semigroup $(T_t^{(j)})_{t\geq 0}$, $j = 1, 2$, are holomorphic sub-Markovian semigroups on $L^2(\mathbb{R})$. However we cannot expect these semigroups to be holomorphic on $L^2(\mathbb{R}^2)$ when associated with $\varphi_1(\xi, \eta) = \psi_1(\eta)$ or $\varphi_2(\xi, \eta) = \psi_2(\xi)$, respectively. Nonetheless, all of our results still apply provided the data $g_1, g_2 \in L^2(\mathbb{R}^2)$ when forming $u(t, x, y) = (T_t^{(1)} g_1)(x, y) + (T_t^{(2)} g_2)(x, y)$, where $(T_t^{(j)} g_j)^\wedge(\cdot, \cdot) = e^{-\varphi_j(\cdot)t} \hat{g}_j(\cdot, \cdot)$, provided the data g_1 and g_2 are sufficiently smooth.

These examples demonstrate the scope of our results as they show how to construct many further ones. However, the central question "How many continuous negative definite solutions does the characteristic equation admit?" is for the general case open, which of course should not be a surprise. In particular, we want to point out that in higher dimensions, i.e. $n \geq 2$, special combinations of terms in the characteristic equation may lead to "unexpected" solutions, similar to the cases where we treat $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ as one variable $|\xi|^2$, or where $(\xi, \eta) \mapsto (\xi^2 - i\eta)^{\frac{1}{2}}$ is treated as one variable when solving the characteristic equation.

In light of the results in [3], handling equations of the type (85) with t-dependent coefficients would be of great interest.

References

1. C. Berg, G. Forst, Non-symmetric translation invariant Dirichlet forms. Inventiones Math. **21**, 199–212 (1973)
2. B. Böttcher, R. Schilling, J. Wang, *Lévy-Type Processes: Construction, Approximation and Sample Path Properties*. LNM2099 (Springer, Cham, 2013)
3. K. Evans, N. Jacob, On adjoint additive processes. Prob. Math. Stat. **40**, 205–223 (2020)
4. N. Jacob, Pseudo differential operators and Markov processes. *Fourier Analysis and Semigroups*, vol. I (Imperial College Press, London, 2001)
5. N. Jacob, R. Schilling, Fractional derivatives, non-symmetric and time-dependent Dirichlet forms and the drift term. Zeitschrift für Analysis und ihre Anwendungen **19**, 801–830 (2000)
6. T. Kato, *Perturbation Theory for Linear Operators* (Springer, Berlin, 1966)
7. R. Schilling, R. Song, Z. Vondraček, *Bernstein Functions*, 2nd ed. (DeGruyter Verlag, Berlin, 2012)
8. E.M. Stein, *Topics in Harmonic Analysis (Related to the Littlewood-Paley Theory)* (Princeton University Press, Princeton NJ, 1970)
9. E.M. Stein, G. Weiss, *Introduction to Fourier Analysis on Euclidean Space* (Princeton University Press, Princeton NJ, 1971)

10. H. Tanabe, *Equations of Evolution* (Pitman Publishing, London, 1979)
11. V.M. Zolotarev, *One-Dimensional Stable Distributions* (American Mathematical Society, Providence RI, 1986)

Monotonicity Properties of Regenerative Sets and Lorden's Inequality

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Abstract Lorden's inequality asserts that the mean return time in a renewal process with (iid) interarrival times $Y_1, Y_2, \ldots$, is bounded above by $2\mathbf{E}[Y_1]/\mathbf{E}[Y_1^2]$. We establish this result in the context of regenerative sets, and remove the factor of 2 when the regenerative set enjoys a certain monotonicity property. This property occurs precisely when the Lévy exponent of the associated subordinator is a *special Bernstein* function. Several equivalent stochastic monotonicity properties of such a regenerative set are demonstrated.

Keywords Renewal process · Regenerative set · Subordinator · Bernstein function

Mathematics Subject Classification 60K05, 60J55, 60J30

1 Introduction

Let $Y_1, Y_2, \ldots$ be i.i.d. positive random variables with finite variance, and use their partial sums $W_n := \sum_{k=1}^n Y_k$, to form a renewal process $W = (W_n)_{n\geq 1}$. For $t > 0$, define $N(t) := \#\{n \geq 1 : W_n \leq t\}$, the number of renewals up to time t, and let $R_t := W_{N(t)+1} - t$ denote the time until the next renewal after time t. Although the distribution of R_t is not particularly simple to express, Lorden [15] has shown that

$$\mathbf{E}[R_t] \leq \frac{\mathbf{E}[Y_1^2]}{\mathbf{E}[Y_1]}, \qquad \forall t > 0. \tag{1}$$

In view of Wald's Identity

$$\mathbf{E}[W_{N(t)+1}] = \mathbf{E}[Y_1] \cdot \mathbf{E}[N(t) + 1], \tag{2}$$

Dedicated to Professor Masatoshi Fukushima.

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the inequality (1) also provides an upper bound on the *renewal function* $\mathbf{E}[N(t)]$. In this paper we examine the analog of (1) in the context of regenerative sets (a continuous analog of renewal processes), and look at a class of such sets in possession of a monotonicity property that leads to an improvement of Lorden's inequality that is sharp in a certain sense.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a filtered probability space. We fix once and for all a *regenerative set* M. This is an $(\mathcal{F}_t)$-progressively measurable set $M \subset \Omega \times [0, \infty[$, with closed sections $M(\omega) := \{t \geq 0 : (\omega, t) \in M\}$, such that $\mathbf{P}[0 \in M] = 1$ and such that for each $(\mathcal{F}_t)$-stopping time T with $T(\omega) \in M(\omega)$ for each $\omega \in \{T < \infty\}$, the (shifted) post-$T$ portion of M, defined by its sections

$$\theta_T M(\omega) := (M(\omega) - T(\omega)) \cap [0, \infty[, \tag{3}$$

is independent of $\mathcal{F}_T$ and has the same distribution as M, on the event $\{T < \infty\}$. The reader is referred to [6] or [13] for more details on such random sets.

It is known that the Lebesgue measure of M is either a.s. strictly positive or a.s. null. This is Kingman's *heavy/light* dichotomy [13, pp. 74–76]. The results presented here are true with slight modifications in the heavy case, but for definiteness we assume M to be light; that is $\int_0^\infty 1_{\{t\in M(\omega)\}}\,dt = 0$ for $\mathbf{P}$-a.e. $\omega \in \Omega$. There is a second dichotomy, according as $T_0 := \inf\,(M\cap\,]0, \infty[)$ satisfies $\mathbf{P}[T_0 = 0] = 0$ or 1. (It must be one or the other because of Blumenthal's 0–1 law.) In the former case, the random set M is discrete; this is the renewal process case. We shall stick to the latter situation (the *unstable* case in Kingman's terminology), in which case M has perfect sections $M(\omega)$ for $\mathbf{P}$-a.e. $\omega \in \Omega$. The generic example of such a regenerative set is the closure of the level set

$$\{t \geq 0 : X_t = x_0\} \tag{4}$$

of a right-continuous strong Markov process $X = (X_t)_{t\geq 0}$ started in a regular point x_0.

There are several stochastic processes associated with M that facilitate its study. First is the *last exit* process $G = (G_t)_{t\geq 0}$,

$$G_t := \sup\,(M \cap [0, t])\,, \qquad t \geq 0, \tag{5}$$

and the associated *age* process $A = (A_t)_{t\geq 0}$,

$$A_t := t - G_t, \qquad t \geq 0. \tag{6}$$

Both A and G are right continuous and adapted to $(\mathcal{F}_t)$, and

$$M = \{t : A_t = 0\}. \tag{7}$$

Moreover A is a time-homogeneous strong Markov process.

Next is the *return time* process

$$D_t := \inf\{s > t : s \in M\} = \inf\left(M \cap]t, \infty[\right), \qquad t \geq 0, (\inf \emptyset := \infty), \tag{8}$$

and the related *remaining life* process

$$R_t := D_t - t, \qquad t \geq 0. \tag{9}$$

These are also right-continuous processes. Notice that each D_t is an $(\mathcal{F}_t)$-stopping time and that R and D are optional with respect to the "advance" filtration $(\mathcal{F}_{D_t})_{t\geq 0}$, and (R_t) is a strong Markov process with respect to this larger filtration, with values in $[0, \infty]$.

Finally, there is the *local time* process $L = L(M) = (L_t)_{t\geq 0}$; this is the unique continuous increasing process adapted to the filtration of A, increasing precisely on M, a.s., normalized so that $\mathbf{E} \int_0^\infty e^{-s}\, dL_s = 1$, and additive in the sense that

$$L_{t+s} = L_t + L_s(\theta_t M), \qquad \forall s, t \geq 0, \text{ a.s.} \tag{10}$$

Here $\theta_t M := (M - t) \cap [0, \infty[$. Thus $L_s(M)$ is a functional of the part $M \cap [0, s]$ of M, while $L_s(\theta_t M)$ is the same functional of $\theta_t M$. One can access L through the general theory of the additive functionals of a Markov process, but Kingman [12] has provided a direct construction that will guide intuition. Before getting to that we need to introduce one more associated process.

The right-continuous inverse process $\tau = (\tau_r)_{r\geq 0}$ defined by

$$\tau_r = \tau(r) := \inf\{t : L_t > r\}, \qquad r \geq 0, \tag{11}$$

is a strictly increasing, pure jump process—the subordinator associated with M. Notice that M coincides with the closure of the range $\{\tau_r : r \geq 0\}$ of τ. The process τ has stationary independent increments (an increasing Lévy process) with Laplace transforms

$$\mathbf{E}[\exp(-\alpha\tau_r)] = \exp(-r\phi(\alpha)], \qquad \alpha > 0, r \geq 0, \tag{12}$$

where the *Lévy exponent* ϕ admits the representation

$$\phi(\alpha) = \int_{]0,\infty]} (1 - e^{-\alpha x})\, \nu(dx), \qquad \alpha > 0, \tag{13}$$

for a Borel measure ν on $]0, \infty]$ satisfying

$$\int_{]0,\infty]} (x \wedge 1)\, \nu(dx) < \infty, \tag{14}$$

which ensures that the integral in (13) is finite for each $\alpha > 0$. We write $h(x)$ for the tail $\nu\left(]x, \infty]\right)$, and note that $\phi(\alpha) = \alpha\hat{h}(\alpha)$, where the hat indicates Laplace transform.

We now turn to Kingman's construction of the local time L: for $\delta > 0$, define

$$M_t(\delta) := \{(\omega, s) \in \Omega \times [0, \infty[: |s - v| < \delta \text{ for some } v \in M(\omega) \cap [0, t]\}, \quad (15)$$

and

$$\ell(\delta) := \int_0^\delta h(s)\, ds, \qquad \delta > 0. \quad (16)$$

Then [12, Thm. 3], there is a constant $c > 0$ such that

$$L_t = c \cdot \lim_{\delta \downarrow 0} \frac{\lambda(M_t(\delta))}{\ell(\delta)}, \qquad \forall t \geq 0, \text{ a.s.}, \quad (17)$$

where λ denotes Lebesgue measure on $[0, \infty[$.

The potential measure U associated with M is the mean occupation time of τ:

$$U(B) := \mathbf{E} \int_0^\infty 1_B(\tau_r)\, dr = \mathbf{E} \int_B dL_t, \qquad \forall B \in \mathbf{B}(\mathbf{R}_+), \quad (18)$$

and the associated distribution function

$$V(t) := \mathbf{E}[L_t], \qquad t \geq 0,$$

plays the role of the renewal function. The Laplace transform of the measure U is given by

$$\hat{U}(\alpha) := \int_0^\infty e^{-\alpha t} U(dt) = \mathbf{E} \int_0^\infty e^{-\alpha \tau(r)}\, dr = \int_0^\infty e^{-r\phi(\alpha)}\, dr = 1/\phi(\alpha), \quad (19)$$

for $\alpha > 0$. For later reference we note that U is related to the potential kernel of the strong Markov process τ: writing $\mathbf{E}^r$ for expectation under the initial condition $\tau_0 = r$, we have

$$\mathbf{E}^r \int_0^\infty f(\tau_s)\, ds = \int_0^\infty f(r + x)\, U(dx), \qquad r \geq 0, \quad (20)$$

for f non-negative and Borel.

2 Lorden's Inequality

By using the regeneration property of M at the stopping time D_t ($t > 0$ fixed) and the fact that L is flat off M, one sees that

$$\mathbf{E}[\exp(-\alpha D_t)] = \phi(\alpha) \cdot \int_t^\infty e^{-\alpha s}\, U(ds). \tag{21}$$

Inverting this we can obtain the distribution of D_t or, equivalently, that of R_t. In fact,

$$\mathbf{E} \int_0^\infty g(x + R_t)\, U(dx) = \int_0^\infty g(y)\, U(dy + t), \tag{22}$$

provided g is a positive Borel function on $[0, \infty]$. This follows immediately from (21) for g of the form $g(x) = e^{-\alpha x}$, and then for general g by Weierstrass's theorem followed by the monotone class theorem. Another direct consequence of (21) is Wald's Identity for regenerative sets:

$$\mathbf{E}[D_t] = \mu \cdot V(t), \tag{23}$$

where $\mu := \int_0^\infty x\, \nu(dx)$.

If the mean μ is finite then (as is well known in the renewal theory context) the random variable R_t converges in distribution, as $t \to \infty$, to a random variable R_∞ whose law has density

$$\frac{h(x)}{\mu}, \qquad x > 0, \tag{24}$$

with respect to Lebesgue measure on $]0, \infty[$; see [3, Thm. 1]. Observe that $\mathbf{E}[R_\infty] = (2\mu)^{-1} \int_0^\infty x^2\, \nu(dx)$.

The following proposition states Lorden's inequality [15] in our context. We reproduce the proof found in [5].

Proposition *Assume that* $\mu := \int_0^\infty x\, \nu(dx) < \infty$. *Then* $\mathbf{E}[R_t] \le 2\mathbf{E}[R_\infty]$ *for all* $t > 0$.

Before turning to the proof we need the following lemma, both parts of which are well known.

Lemma 1 *(a)* V *is subadditive:* $V(t+s) \le V(t) + V(s)$, *for all* $s, t > 0$.

(b) $\mathbf{E}[V(t - R_\infty)] = t/\mu$ *for* $t > 0$, *with the understanding that* $V(s) = 0$ *for* $s \le 0$.

Proof (a) We have, using (21) with $g = 1_{[0,s]}$ for the first equality below,

$$V(t+s) - V(t) = \mathbf{E}[V(s - R_t);\, R_t \le s] \le \mathbf{E}[V(s);\, R_t \le s] \le V(s). \tag{25}$$

(b) The Laplace transform of the left side of this identity is easily seen to be $\hat{U}(\alpha)\hat{h}(\alpha)/(\alpha\mu) = 1/(\alpha^2\mu)$ because of (19). This coincides with the Laplace transform of the right side, so the assertion follows by inversion because both sides are continuous in $t > 0$. □

Proof of the Proposition Let Z_1 and Z_2 by independent random variables with the same distribution as R_∞. The subadditivity asserted in Lemma 1(a) persists for negative values of s, t provided we agree that V vanishes on $]-\infty, 0]$. Thus,

$$V(t) \le V(t + Z_1 - Z_2) + V(Z_2 - Z_1). \tag{26}$$

By Lemma 1(b), the conditional expectation of the first term on the right of (26), given Z_1, is $(t + Z_1)/\mu$. Likewise, the conditional expectation of the second term, given Z_2, is Z_2/μ. It follows that

$$\mu \cdot V(t) \le \mathbf{E}[t + Z_1] + \mathbf{E}[Z_2] = t + 2\mathbf{E}[R_\infty], \tag{27}$$

and the assertion follows because $\mathbf{E}[R_t] = \mathbf{E}[D_t] - t = \mu \cdot V(t) - t$ by (23). □

3 Monotone Potential Density

The exponent ϕ is an example of what is called a *Bernstein function* (non-negative, completely monotone derivative). Such a ϕ is a *special Bernstein function* provided $\phi^* : \alpha \mapsto \alpha/\phi(\alpha)$ is also a Bernstein function. In this case, because $\hat{U}(\alpha)\phi(\alpha) = 1$, the measure U admits a Lebesgue density given by

$$u(x) := \nu^*\left(]x, \infty]\right), \tag{28}$$

where ν^* is the Lévy measure in the representation (13) of ϕ^*. Notice that u is right-continuous, and (more importunely) monotone decreasing. Conversely, if U admits a monotone density with respect to Lebesgue measure, then ϕ is a special Bernstein function. For discussion of special Bernstein functions see Chap. 10 of [17].

Our main result contains further (stochastic) characterizations of the class of special Bernstein functions. For partial results in this vein, in the context of renewal processes, see [4] and [14].

Theorem *For a light unstable regenerative set M the following are equivalent:*
(a) U is absolutely continuous and admits a monotone decreasing density.
(b) $t \mapsto R_t$ is stochastically increasing.
(c) $t \mapsto A_t$ is stochastically increasing.
(d) $t \mapsto \theta_t M$ is stochastically decreasing.
(e) $t \mapsto \theta_t L$ is stochastically decreasing.

[By (d) is meant that for each $0 < s < t$ there is some probability space carrying random sets M^s and M^t such that $M^s \stackrel{d}{=} \theta_s M$, $M^t \stackrel{d}{=} \theta_t M$, and $M^t \subset M^s$ almost surely. Point (e) should be interpreted in an analogous fashion, the local time being thought of as a random measure dL_s, and $(\theta_t L)_b := L_{t+b} - L_t = L_b(\theta_t M)$.]

Proof (a)⇒(b). If U has a monotone density, then the left side of (22), which is nothing but $(\pi_t * U)(g)$ (π_t being the distribution of R_t), is monotone decreasing in t. It follows that if $s < t$ then π_t is "downstream" from π_s in the balayage order of the subordinator τ. By a theorem of Rost [16] there are (randomized) stopping times $T(s)$ and $T(t)$ of τ with $T(s) \le T(t)$ such that $\tau(T(s))$ has the same distribution as R_s and $\tau(T(t))$ has the same distribution as R_t. Since τ is increasing, $\mathbf{P}[R_s > x] \le \mathbf{P}[R_t > x]$ for each $x > 0$; that is, R_t is stochastically larger than R_s.

(b)⇒(a). Conversely, if $t \mapsto R_t$ is stochastically increasing, then from (22) with $g = 1_{[0,b]}$ we see that $t \mapsto U[t, t+b]$ is decreasing for each $b > 0$. In particular, V is midpoint concave, hence concave (because $x \mapsto V(x) = \mathbf{E}[L_x]$ is continuous). This implies that V is concave, so the righthand derivative $u := V'_+$ exists and is decreasing. Moreover, again by the concavity of V, the measure U is absolutely continuous with density u.

(c) ⇔(b). $\mathbf{P}(R_t > x) = P(M \cap]t, t+x] = \emptyset) = \mathbf{P}(A_{t+x} > x)$.

(a) ⇒(d). From the proof of (a) ⇒(b) we know that if $0 < s < t$ then there are (randomized) stopping times $T(s)$ and $T(t)$ of τ such that $T(s) \le T(t)$, $\tau(T(s)) \stackrel{d}{=} R_s$, and $\tau(T(t)) \stackrel{d}{=} R_t$. In particular, $\tau(T(s)) \le \tau(T(t))$. Define $M^s := M \cap [\tau(T(s)), \infty[$ and $M^t := M \cap [\tau(T(t)), \infty[$. Then $M^t \subset M^s$, and the required distributional equalities hold by regeneration at the stopping times $\tau(T(s))$ and $\tau(T(t))$.

(d) ⇒(e). This follows immediately from Kingman's construction (17): For fixed $0 \le s < t$, we have $L_b(M^t) - L_a(M^t) \le L_b(M^s) - L_a(M^s)$, for all $0 \le a < b$, almost surely. This means that the measure with distribution function $b \mapsto L_b(M^t)$ is dominated setwise by the measure with distribution function $b \mapsto L_b(M^s)$, a.s.

(e) ⇒(a). $U(]t, t+b]) = \mathbf{E}[(\theta_t L)_b]$. □

Remarks (a) It is shown in [4, Thm. 3], in the renewal context, that if the tail probability $\mathbf{P}[Y_k > y]$ (the analog of h) is *log-convex* then (a), (b), (c), and the "counting" version of (e) in the Theorem hold true. This log-convexity is equivalent to the *decreasing failure rate* property (DFR). Expressed in the present context this amounts to the statement that

$$y \mapsto \frac{h(x+y)}{h(y)} \tag{29}$$

is non-increasing on the interval where $h(y) > 0$, for each $x > 0$. It was shown by Hawkes [9, Thm. 2.1] that in our context, the log-convexity of the Lévy tail function h implies that U has a decreasing density. For more on this class of subordinators see [17]. Brown conjectured in [4] that the DFR property is equivalent to the concavity of the renewal function; a counterexample was found (after 31 years) by Yu [18].

(b) The use of Rost's theorem (on Skorokhod stopping) in the proof of (a)⇒(b) (and again in (a)⇒(d)) was suggested by an argument of Bertoin [1, p. 568].

Observe that when U has a monotone density, because R_t stochastically increasing in t, each random variable R_t is stochastically dominated by R_∞. This yields the following improvement on Lorden's inequality.

Corollary *Under any of the conditions listed in the Theorem, we have*

$$\mathbf{E}[R_t] \le \frac{\int_0^\infty x^2\, \nu(dx)}{2\mu} = \mathbf{E}[R_\infty], \qquad \forall t > 0, \tag{30}$$

and the inequality is sharp.

Whether the constant 2 in Lorden's original inequality can be improved in the general case is an open question.

4 Concluding Remarks

A regenerative set M is *infinitely divisible* (ID) provided for each positive integer n there are i.i.d. regenerative sets $M_{n,k}$, $1 \le k \le n$, such that $\cap_{k=1}^n M_{n,k}$ has the same distribution as M. A large class of such sets ("Poisson random cutout sets") is discussed and characterized in [7]. It has long been conjectured by the author that this class exhausts (at least among light unstable regenerative sets) all of the ID regenerative sets. In unpublished work the author has shown that an ID regenerative set whose potential measure admits a monotone density is, in fact, a Poisson cutout set. Somewhat irritatingly, this supplementary monotonicity condition *is* satisfied by all Poisson cutout sets. It should be noted that the parallel results for heavy ID sets, and for the discrete-time situation, have been established by Kendall [10, 11]; see also [8] for a detailed discussion of these matters and further references.

From the proof of the Theorem in Sect. 3, we know that when U has a monotone density then for each $t > 0$ we have

$$\theta_t M \stackrel{d}{=} M \setminus I_t, \tag{31}$$

where I_t is a random interval $[0, \tau(T(t))[$ growing stochastically larger as t increases. In the Poisson cutout case, $\theta_t M$ and I_t are independent, because of the independence properties of the Poisson process. Does this independence characterize ID regenerative sets?

References

1. J. Bertoin, Regenerative embeddings of Markov sets. Prob. Th. Rel. Fields **108**, 559–571
2. J. Bertoin, Subordinators: examples and applications, in *Lectures on Probability Theory and Statistics, Lecture Notes in Mathematics*, vol. 1717 (Springer, Berlin, 1999), pp. 1–91
3. J. Bertoin, K. van Harn, F.W. Steutel, Renewal theory and level passages by subordinators. Stat. Prob. Lett. **45**, 65–69 (1999)
4. M. Brown, Bounds, inequalities, and monotonicity properties for some specialized renewal processes. Ann. Probab. **8**, 227–240 (1980)
5. H. Carlsson, O. Nerman, An alternative proof of Lorden's renewal inequality. Adv. Appl. Prob. **18**, 1015–1016 (1986)
6. P.J. Fitzsimmons, B. Fristedt, B. Maisonneuve, Intersections and limits of regenerative sets. Z. Warsch. Verw. Gebiete **70**, 157–173 (1985)
7. P.J. Fitzsimmons, B. Fristedt, L.A. Shepp, The set of real numbers left uncovered by random covering intervals. Z. Wahrsch. Verw. Gebiete **70**, 175–189 (1985)
8. B. Fristedt, Intersections and limits of regenerative sets, in *Random Discrete Structures*, vol. 76 (Springer, New York, 1996), pp. 121–151
9. J. Hawkes, On the potential theory of subordinators. Z. Warsch. Verw. Gebiete **33**, 113–132 (1975)
10. D.G. Kendall, Renewal sequences and their arithmetic, in *1967 Symposium on Probability Methods in Analysis, Lecture Notes in Mathematics*, vol. 31 (Springer, Berlin, 1967), pp. 147–175 [Reprinted on pp. 47–72 of *Stochastic Analysis*, ed. by D.G. Kendall, E.F. Harding (Wiley, London, 1973)]
11. D.G. Kendall, Delphic semigroups, infinitely divisible regenerative phenomena, and the arithmetic of p-functions. Z. Wahrscheinlichkeitstheorie Verw. Gebiete **9**, 163–195 (1968)
12. J.F.C. Kingman, An intrinsic description of local time. J. Lond. Math. Soc. **2**(6) (1973) 725–731
13. J.F.C. Kingman, Homecomings of Markov processes. Adv. Appl. Prob. **5**, 66–102 (1973)
14. T. Lindvall, On coupling of renewal processes with use of failure rates. Stoch. Proc. Appl. **22**, 1–15 (1986)
15. G. Lorden, On the excess over the boundary. Ann. Math. Stat. **41**, 520–527 (1970)
16. H. Rost, The stopping distributions of a Markov process. Invent. Math. **14**, 1–16 (1971)
17. R. Schilling, R. Song, Z. Vondraček, *Bernstein Functions, Theory and Applications*, 2nd ed. (De Gruyter, Berlin, 2012)
18. Y. Yu, Concave renewal functions do not imply DFR interrenewal times. J. Appl. Prob. **48**, 583–588 (2011)

Doob Decomposition, Dirichlet Processes, and Entropies on Wiener Space

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Abstract As an extension of the Doob-Meyer decomposition of a semimartingale and the Fukushima representation of a Dirichlet process, we introduce a general Doob decomposition in continuous time, where a square-integrable process is represented as the sum of a martingale and a process with "vanishing local risk". For a probability measure Q on Wiener space, we discuss how entropy conditions on Q formulated with respect to Wiener measure P are connected with the Doob decomposition of the coordinate process W under Q. The situation is well understood if the relative entropy $H(Q|P)$ is finite; in this case the decomposition is classical and yields an immediate proof of Talagrand's transport inequality on Wiener space. To go beyond this restriction, we consider the specific relative entropy $h(Q|P)$ on Wiener space that was introduced by Gantert in [11]. We discuss its interplay with the Doob decomposition of W under Q and a corresponding version of Talagrand's inequality, with special emphasis on the case where W is a Dirichlet process under Q.

Keywords Optimal transport · Coupling on Wiener space · Relative entropy · Talagrand's inequality · Dirichlet processes · Doob decomposition · Fukushima decomposition

1 Introduction

Since the Sixties, the interplay between potential theory and the theory of Markov processes has been a rich source of inspiration for the general theory of stochastic processes. In particular, the Riesz decomposition of a superharmonic functions has its general counterpart in the *Doob-Meyer decomposition*

$$X = M + A \tag{1}$$

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of a supermartingale. This has led to the general notion of a semimartingale X, defined by a Doob-Meyer decomposition (1) into a local martingale M and a predictable process A with paths of bounded variation. The canonical role of semimartingales is emphasized by the Bichteler-Dellacherie theorem, where they are characterized as general stochastic integrators.

M. Fukushima was the first to show that there are good reasons to go beyond the conceptual framework provided by the theory of semimartingales. Indeed, for a function F in the Dirichlet space of a symmetric Markov process Z, the process $X = F(Z)$ may not be a semimartingale, and thus may not admit a Doob-Meyer decomposition. However, M. Fukushima showed that X admits a decomposition of the form (1), where M is a square-integrable martingale and A is a process of "zero energy"; cf. [10]. This *Fukushima decomposition* has motivated the general notion of a Dirichlet process X; cf. [2, 7].

But, as shown by Graversen and Rao in [14], representations of the form (1) have an even wider scope. In Sect. 2 we introduce a version that is convenient for our purpose. For a square-integrable adapted process $X = (X_t)_{0 \le t \le 1}$ on a filtered probability space and any $N \ge 1$, we consider its Doob decomposition

$$X = M^N + A^N$$

in discrete time along the N-th dyadic partition of the unit interval. Assuming L^2-convergence of the random variables M_1^N, we are led to a *Doob decomposition in continuous time* of the form (1), where M is a square-integrable martingale and A is "predictable" in the sense that the sum of the local prediction errors along the N-th dyadic partition converges to 0 as N increases to ∞. To avoid confusion with the standard notion of predictability, which is defined as measurability with respect to the predictable σ-field, we will say that A has *vanishing local risk*. Any process with zero energy has this property, and so our Doob decomposition in continuous time may be viewed as an extension of the Fukushima decomposition, and in particular of the Doob-Meyer decomposition. On the other hand, a process A with vanishing local risk is orthogonal to any square-integrable martingale. Thus, the Doob decomposition in continuous time may be viewed as a special case of the general decomposition obtained in [14]; cf. also the discussion of "weak Dirichlet processes" in [3, 13].

In Sect. 3 we consider probability measures Q on the path space $C_0[0, 1]$. We denote by $\mathcal{Q}$ the class of all Q such that the coordinate process W admits a Doob decomposition $W = M + A$ in continuous time under Q. Our aim is to understand the impact of entropy bounds on $Q \in \mathcal{Q}$ with respect to Wiener measure P on the Doob decomposition of W under Q.

If the relative entropy $H(Q|P)$ is finite then Q is absolutely continuous with respect to P, and the Doob decomposition takes the classical form

$$W = W^Q + B^Q,$$

where W^Q is a Wiener process under Q, and where the paths of the process B^Q belong to the Cameron-Martin space $\mathcal{H}$. In this case, the decomposition yields an immediate proof of Talagrand's transport inequality

$$W_{\mathcal{H}}(Q,P) \leq \sqrt{2\,H(Q|P)}, \tag{2}$$

where $W_{\mathcal{H}}$ denotes the Wasserstein distance induced by the Cameron-Martin norm; cf. [9, 16], or Corollary 2.

To go beyond the absolutely continuous case, we consider the *specific relative entropy*

$$h(Q|P) := \liminf_{N\uparrow\infty} 2^{-N} H_N(Q|P),$$

where $H_N(Q|P)$ denotes the relative entropy of Q with respect to P on the σ-field generated by observing the path along the N-th dyadic partition of the unit interval. The notion of specific relative entropy on Wiener space was introduced by N. Gantert in her thesis [11], where it serves as a rate function for large deviations of the quadratic variation from its ergodic behaviour; cf. also [12]. In our context, it allows us to prove a version of Talagrand's inequality of the form

$$W_{\mathcal{D}}(Q,P) \leq \sqrt{2\,h(Q|P)}, \tag{3}$$

where the Wasserstein distance $W_{\mathcal{D}}$ is defined in terms of quadratic variation. This involves a careful analysis of the specific relative entropy, and in particular the inequality

$$2\,h(Q|P) \geq E_Q\big[M_1^2 - 1 + H(\lambda|q(\cdot)) + \langle A\rangle_1\big], \tag{4}$$

where $q(\omega, dt)$ is the random measure on [0, 1] whose distribution function is given by the quadratic variation of the martingale M, λ denotes the Lebesgue measure on [0, 1], and $\langle A\rangle$ is the quadratic variation of A. For a martingale measure Q, inequality (4) with $A = 0$ is proved in [9]; in the special case where $q(\cdot)$ is absolutely continuous, it was already shown in [11]. Here we extend it to a large class of measures $Q \in \mathcal{Q}$. As a corollary we obtain our version (3) of Talagrand's inequality for measures $Q \in \mathcal{Q}$ such that W is a Dirichlet process under Q.

2 Doob Decomposition in Continuous Time

Let $(\Omega, \mathcal{F})$ be a measurable space endowed with a right-continuous filtration $(\mathcal{F}_t)_{0\leq t\leq 1}$, and let Q be a probability measure on $(\Omega, \mathcal{F})$. In the sequel, the measure Q will vary, and so we do not assume that the filtration is completed with respect to Q.

Throughout this paper, the index N will refer to the N-th dyadic partition of the unit interval, that is,

$$D_N = \{i2^{-N} | i = 1, \ldots, 2^N\},$$

and for fixed $N \geq 1$ we use the notation $t_i = i2^N$. For any square-integrable adapted process $Z = (Z_t)$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq 1}, Q)$, defined for $t \in [0, 1]$ or at least for $t \in$

D_N, we denote by

$$\Delta_{N,i} Z = Z_{t_i} - Z_{t_{i-1}}$$

the increments of Z along the N-th dyadic partition, and by

$$\zeta_{N,i}^2 = E_Q[(\Delta_{N,i} Z)^2 | \mathcal{F}_{t_{i-1}}] - \big(E_Q[\Delta_{N,i} Z | \mathcal{F}_{t_{i-1}}]\big)^2 \tag{5}$$

their conditional variances given the past. Note that $\zeta_{N,i}^2$ can be viewed as the conditional prediction error if the increment $\Delta_{N,i} Z$ is predicted by its conditional expectation under Q.

Definition 1 For each $N \geq 1$, the sum

$$R_N(Z) := \sum_{i=1}^{2^N} \zeta_{N,i}^2 \in L^1(Q)$$

will be called the local risk of the process Z along the N-th dyadic partition. We will say that the process Z has vanishing local risk if

$$\lim_{N \uparrow \infty} R_N(Z) = 0 \quad \text{in } L^1(Q). \tag{6}$$

For two square-integrable adapted processes Z and $\tilde{Z}$ we denote by

$$CV_N(Z, \tilde{Z}) := \frac{1}{2}\big(R_N(Z + \tilde{Z}) - R_N(Z) - R_N(\tilde{Z})\big)$$

the sum of the conditional covariances of the increments along the N-th dyadic partition, and we say that Z and $\tilde{Z}$ are orthogonal if

$$\lim_{N \uparrow \infty} CV_N(Z, \tilde{Z}) = 0 \quad \text{in } L^1(Q). \tag{7}$$

Let us now fix an adapted right-continuous process X on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq 1}, Q)$ such that

$$X = (X_t)_{0 \leq t \leq 1} \subset L^2(Q).$$

For any $N \geq 1$, consider its Doob decomposition

$$X_t = M_t^N + A_t^N, \quad t \in D_N \tag{8}$$

in discrete time along the N-th dyadic partition. Thus, the process $A^N = (A_t^N)_{t \in D_N}$ is defined by $A_0^N = 0$ and the increments

$$\Delta_{N,i} A^N = E_Q[\Delta_{N,i} X | \mathcal{F}_{t_{i-1}}],$$

and $M^N = (M_t^N)_{t \in D_N}$ is a square-integrable martingale in discrete time with initial value $M_0^N = X_0$.

Remark 1 The process A^N is "predictable" in discrete time, that is, $A_{t_i}^N$ is $\mathcal{F}_{t_{i-1}}$-measurable for each $t_i \in D_N$. Equivalently, this property can be expressed in terms of actual predictions. Indeed, if we predict the increments of A^N by taking conditional expectations given the past, then the local prediction errors, defined as the conditional variances $(\alpha^N)_{N,i}^2$ of the process A^N, are all equal to 0, i.e.,

$$R_N(A^N) = \sum_{i=1}^{2^N} (\alpha^N)_{N,i}^2 = 0 \quad Q\text{-a.s..}$$

In this sense the process A^N carries no local risk. In fact, the local risk of the process X along the N-th dyadic partition is fully captured by the martingale M^N, that is,

$$R_N(X) = R_N(M^N), \tag{9}$$

since the conditional variances are the same for X and for M^N. This alternative interpretation of "predictability" in discrete time motivates our definition of vanishing local risk and the following version of the Doob decomposition in continuous time.

Theorem 1 (1) *The following two properties of the process X with respect to Q are equivalent:*

(i) *The random variables $(M_1^N)_{N=1,2,\ldots}$ in (8) form a Cauchy sequence in $L^2(Q)$,*
(ii) *X admits a Doob decomposition in continuous time of the form*

$$X = M + A, \tag{10}$$

where $M = (M_t)_{0 \le t \le 1}$ is a square-integrable right-continuous martingale such that $M_0 = X_0$, and where the process $A = (A_t)_{0 \le t \le 1}$ has vanishing local risk.

(2) *The decomposition (10) of X into a square-integrable martingale M and a process A with vanishing local risk is unique.*

Proof (1) Suppose that $(M_1^N)_{N=1,2,\ldots}$ is a Cauchy sequence in $L^2(Q)$, hence convergent in $L^2(Q)$ to a random variable $M_1 \in L^2(Q)$. We denote by $M = (M_t)_{0 \le t \le 1}$ a right-continuous version of the square-integrable martingale given by the conditional expectations $E_Q[M_1|\mathcal{F}_t]$; cf. [6] or [4], Ch. VI.5. Then the process $A = (A_t)_{0 \le t \le 1}$ defined by $A = X - M$ is right-continuous, adapted, and square-integrable. For $N \ge 1$, the increments of A along the N-th dyadic partition satisfy

$$\begin{aligned}\Delta_{N,i}A - E_Q[\Delta_{N,i}A|\mathcal{F}_{t_{i-1}}] &= \Delta_{N,i}X - \Delta_{N,i}M - E_Q[\Delta_{N,i}X|\mathcal{F}_{t_{i-1}}]\\ &= \Delta_{N,i}M^N - \Delta_{N,i}M\\ &= \Delta_{N,i}(M^N - M).\end{aligned}$$

Thus, the conditional variance of $\Delta_{N,i}A$ is given by

$$\alpha^2_{N,i} = E_Q[(\Delta_{N,i}(M^N - M))^2|\mathcal{F}_{t_{i-1}}]. \tag{11}$$

Since $M^N - M$ is a square-integrable martingale along D_N with initial value $(M^N - M)_0 = 0$, we obtain

$$E_Q\left[\sum_{i=1}^{2^N}\alpha^2_{N,i}\right] = E_Q\left[\sum_{i=1}^{2^N}(\Delta_{N,i}(M^N - M))^2\right] = E_Q[(M_1^N - M_1)^2].$$

This implies

$$\lim_{N\uparrow\infty} E_Q\left[\sum_{i=1}^{2^N}\alpha^2_{N,i}\right] = 0, \tag{12}$$

and so we have shown that the process A has vanishing local risk.
Conversely, if X admits a decomposition (10) then the preceding Eq. (12) holds again, and so (6) implies that $(M_1^N)_{N=1,2,...}$ is a Cauchy sequence in $L^2(Q)$.

(2) To check uniqueness of the decomposition (10), suppose that

$$X = M + A = \tilde{M} + \tilde{A},$$

where M and $\tilde{M}$ are square-integrable martingales, and A and $\tilde{A}$ are processes with vanishing local risk. For any $N \geq 1$ we obtain

$$\begin{aligned}E_Q[(M_1 - \tilde{M}_1)^2] &= \sum_{i=1}^{2^N} E_Q[(\Delta_{N,i}(M - \tilde{M}))^2] = \sum_{i=1}^{2^N} E_Q[(\Delta_{N,i}(\tilde{A} - A))^2]\\ &\leq 2\sum_{i=1}^{2^N} E_P[\tilde{\alpha}^2_{N,i} + \alpha^2_{N,i}],\end{aligned} \tag{13}$$

denoting by $\tilde{\alpha}^2_{N,i}$ and $\alpha^2_{N,i}$ the conditional variances of $\tilde{A}$ and A; in the last step we use the fact that

$$E_Q[\Delta_{N,i}\tilde{A}|\mathcal{F}_{t_i}] = E_Q[\Delta_{N,i}A|\mathcal{F}_{t_i}],$$

since both terms are equal to $E_Q[\Delta_{N,i}X|\mathcal{F}_{t_i}]$. For $N\uparrow\infty$ the right hand side of (13) converges to 0, and this implies $M_1=\tilde{M}_1$ Q-a.s., hence $M=\tilde{M}$ and $A=\tilde{A}$. □

Lemma 1 *Let $A=(A_t)_{0\le t\le 1}$ be a square-integrable, right-continuous, and adapted process on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{0\le t\le 1},Q)$, and consider the following properties of A:*

(i) *A has continuous paths of bounded variation, and the total variation process $|A|$ satisfies $|A|_1\in L^2(Q)$.*
(ii) *A has "zero energy" in the sense that*

$$\lim_{N\uparrow\infty} E_Q\left[\sum_{i=1}^{2^N}(\Delta_{N,i}A)^2\right]=0, \tag{14}$$

(iii) *A has vanishing local risk,*
(iv) *A is orthogonal to any square-integrable martingale L.*

Then we have

$$\text{(i)}\Longrightarrow\text{(ii)}\Longrightarrow\text{(iii)}\Longrightarrow\text{(iv)}.$$

Proof Since $\alpha_{N,i}^2\le E_Q\big[(\Delta_{N,i}A)^2|\mathcal{F}_{t_i}\big]$, the process A has vanishing local risk as soon as it has zero energy. As to the first implication, note that

$$E_Q\left[\sum_{i=1}^{2^N}(\Delta_{N,i}A)^2\right]\le E_Q\left[\sum_{i=1}^{2^N}C_N|\Delta_{N,i}A|\right]=E_Q\big[C_N|A|_1\big], \tag{15}$$

where $C_N:=\max_i|\Delta_{N,i}A|\ \le|A|_1$. Property (i) implies $\lim_{N\uparrow\infty}C_N=0$ and $C_N|A|_1\le|A|_1^2\in L^1(Q)$. By Lebesgue's theorem, the right hand side of (15) converges to 0, and so A has energy 0. As to the last implication, note that

$$CV_N(A,L)\le R_N(A)^{1/2}R_N(L)^{1/2},$$

hence

$$E_Q\big[CV_N(A,L)\big]\le E_Q\big[R_N(A)\big]^{1/2}E_Q\big[(L_1-L_0)^2\big]^{1/2}.\ \square$$

Remark 2 Suppose that X admits a continuous Doob decomposition (10).

(1) The preceding implication (iii) $\Longrightarrow$ (iv) shows that this can be viewed as a special case of the decomposition derived in [14]; see also the discussion of "weak Dirichlet processes" in [3, 13].
(2) Applying property (iv) to the martingale M in (10), we see that

$$\lim_{N\uparrow\infty}R_N(X)=\lim_{N\uparrow\infty}R_N(M)\quad\text{in }L^1(Q),$$

that is, in the limit the local risk of X is carried by M. This can be seen as the continuous-time version of Eq. (9).

Definition 2 Let us say that X is a Dirichlet process if it admits a Doob decomposition (10) such that A is a process with zero energy. In this case, (10) is also called the Fukushima decomposition of X.

The preceding lemma shows that the notion of vanishing local risk has a wide scope. Combined with the uniqueness of the decomposition (10), it implies the following corollary.

Corollary 1 *The class of processes X that admit a Doob decomposition of the form* (10) *includes*

(i) *a large class of semimartingales, and in that case* (10) *reduces to the Doob-Meyer decomposition of X,*
(ii) *the class of Dirichlet processes, and in that case* (10) *reduces to the Fukushima decomposition of X.*

Remark 3 Suppose that X admits a Doob decomposition (10) under Q. Then X is a Dirichlet process if and only if

$$\lim_{N\uparrow\infty} E_Q\left[\sum_{i=1}^{2^N}(\varDelta_{N,i}A^N)^2\right] = 0. \tag{16}$$

Indeed, the weaker condition (16) is in fact equivalent to condition (15) as soon as A has vanishing local risk.

The following equivalence was stated in [7], where condition (17) is taken as the definition of a Dirichlet process.

Theorem 2 *The process X is is Dirichlet process if and only if the processes A^N appearing in the discrete Doob decompositions* (8) *satisfy the condition*

$$\lim_{N\uparrow\infty}\ \sup_{K\geq N} E_Q\left[\sum_{i=1}^{2^N}(\varDelta_{N,i}A^K)^2\right] = 0. \tag{17}$$

Proof We include a proof, since the proof in [7] contains several typos.
For each $L \geq N$, we obtain

$$
\begin{aligned}
E_Q\big[(A_1^L - A_1^N)^2\big] &= E_Q\left[\sum_{i=1}^{2^N}(\Delta_{N,i}A^L - \Delta_{N,i}A^N)^2\right] \\
&\le 2\left(E_Q\left[\sum_{i=1}^{2^N}(\Delta_{N,i}A^L)^2\right] + E_Q\left[\sum_{i=1}^{2^N}(\Delta_{N,i}A^N)^2\right]\right) \\
&\le 4 \sup_{K\ge N} E_Q\left[\sum_{i=1}^{2^N}(\Delta_{N,i}A^K)^2\right],
\end{aligned}
$$

since $A^L - A^N = M^N - M^L$ is a martingale in discrete time along D_N. Thus, condition (17) implies that $(A_1^N)_{N=1,2,\ldots}$, and hence $(M_1^N)_{N=1,2,\ldots}$, is a Cauchy sequence in $L^2(Q)$. It also implies condition (16). In view of Theorem 1 and the preceding remark, it follows that X is a Dirichlet process.

Conversely, since

$$
\Delta_{N,i}A^K = \Delta_{N,i}A^N + \Delta_{N,i}(M^N - M^K),
$$

we obtain

$$
\begin{aligned}
E_Q\left[\sum_{i=1}^{2^N}(\Delta_{N,i}A^K)^2\right] &\le 2\left(E_Q\left[\sum_{i=1}^{2^N}(\Delta_{N,i}A^N)^2\right] + E_Q\big[(M_1^N - M_1^K)^2\big]\right) \\
&\le 2\left(E_Q\left[\sum_{i=1}^{2^N}(\Delta_{N,i}A^N)^2\right] + \sup_{L\ge N} E_Q\big[(M_1^N - M_1^L)^2\big]\right)
\end{aligned}
$$

for any $K \ge N$. Thus, condition (17) is satisfied as soon as X is a Dirichlet process. □

3 Entropies and Couplings on Wiener Space

From now on, the underlying measurable space will be the path space

$$
\Omega = C_0[0, 1]
$$

of all continuous functions ω on $[0, 1]$ with initial value $\omega(0) = 0$. We denote by $(\mathcal{F}_t)_{0\le t\le 1}$ the right-continuous filtration on Ω generated by the coordinate process

$$
W = (W_t)_{0\le t\le 1}
$$

defined by $W_t(\omega) = \omega(t)$. We set $\mathcal{F} = \mathcal{F}_1$, and we denote by P the *Wiener measure* on $(\Omega, \mathcal{F})$.

Let $\mathcal{H}$ denote the *Cameron-Martin space* of all absolutely continuous functions $\omega \in \Omega$ such that the derivative $\dot{\omega}$ is square integrable on [0, 1]. For $\omega \in \Omega$ we write

$$||\omega||_{\mathcal{H}} = \begin{cases} \left(\int_0^1 \dot{\omega}^2(t)dt\right)^{1/2} & \text{if } \omega \in \mathcal{H} \\ +\infty & \text{otherwise.} \end{cases}$$

Definition 3 We denote by $\mathcal{Q}$ the class of all probability measures Q on $(\Omega, \mathcal{F})$ such that the process W admits a Doob decomposition (10) under Q with continuous paths, that is,

$$W = M + A,$$

where M is a square-integrable continuous martingale under Q, and where A is a continuous adapted process with vanishing local risk under Q. For $Q \in \mathcal{Q}$ we will write

- $Q \in \mathcal{Q}_{\mathcal{M}}$ if $A = 0$, that is, Q is a martingale measure,
- $Q \in \mathcal{Q}_{\mathcal{H}}$ if A satisfies $E_Q\big[||A||^2_{\mathcal{H}}\big] < \infty$,
- $Q \in \mathcal{Q}_{\mathcal{S}}$ if A has continuous paths of bounded variation, and the total variation process $|A|$ satisfies $|A|_1 \in L^2(Q)$,
- $Q \in \mathcal{Q}_{\mathcal{D}}$ if A has zero energy, that is, W is a Dirichlet process under Q.

Lemma 1 shows that

$$\mathcal{Q}_{\mathcal{M}} \subset \mathcal{Q}_{\mathcal{H}} \subset \mathcal{Q}_{\mathcal{S}} \subset \mathcal{Q}_{\mathcal{D}} \subset \mathcal{Q}. \tag{18}$$

For a given measure $Q \in \mathcal{Q}$, we are now going to study the impact of entropy bounds on the Doob decomposition (10) of the process W under Q. These bounds will be formulated in terms of relative entropies with respect to Wiener measure P.

Remark 4 Recall that, for two probability measures μ and ν on some measurable space $(S, \mathcal{S})$, the *relative entropy* of ν with respect to μ is defined as

$$H(\nu|\mu) = \begin{cases} \int \log \frac{d\nu}{d\mu} d\nu & \text{if } \nu \ll \mu, \\ +\infty & \text{otherwise,} \end{cases}$$

and that $H(\nu|\mu) \geq 0$, with equality if and only if $\mu = \nu$. Moreover,

$$\lim_{n\uparrow\infty} H_n(\nu|\mu) = H(\nu|\mu) \tag{19}$$

if $(\mathcal{S}_n)_{n=1,2...}$ is a sequence of σ-fields increasing to $\mathcal{S}$ and $H_n(\nu|\mu)$ denotes the relative entropy of ν with respect to μ on $(S, \mathcal{S}_n)$. Note also that Eq. (19) extend to the case where ν or μ is a non-negative finite measure.

First we review the case where Q has finite relative entropy $H(Q|P)$ with respect to Wiener measure P. The following proposition is well known; cf., for example, [8, 9].

Proposition 1 *For any probability measure Q on $(\Omega, \mathcal{F})$,*

$$H(Q|P) < \infty \Longleftrightarrow Q \ll P \text{ and } Q \in \mathcal{Q}_{\mathcal{H}}.$$

In this case, the Doob decomposition (10) *takes the form*

$$W = W^Q + B^Q, \tag{20}$$

where W^Q is a Wiener process under Q and the process B^Q has paths in $\mathcal{H}$, and the relative entropy is given by

$$H(Q|P) = \frac{1}{2} E_Q\big[||B^Q||_{\mathcal{H}}^2\big].$$

Remark 5 The process $b^Q := \dot{B}^Q$ will be called the *intrinsic drift* of Q. Note that Eq. (20) can be read as

$$dW_t = dW_t^Q + b_t^Q(W)dt.$$

Thus, any measure Q on path space such that $H(Q|P) < \infty$ can be viewed as a weak solution of the stochastic differential equation

$$dX = dZ + b_t^Q(X)dt,$$

where Z is required to be a Wiener process, and its relative entropy takes the form

$$H(Q|P) = \frac{1}{2} E_Q\left[\int_0^1 (b^Q)_t^2 dt\right].$$

As first observed by Lehec in [16], Proposition 1 yields an immediate proof of *Talagrand's inequality on Wiener space*, which relates the relative entropy $H(Q|P)$ to the Wasserstein distance $W_{\mathcal{H}}(Q, P)$ defined in terms of the Cameron-Martin norm.

Definition 4 For any probability measure Q on $(\Omega, \mathcal{F})$, we define the Wasserstein distance $W_{\mathcal{H}}(Q, P)$ between Q and P as

$$W_{\mathcal{H}}(Q, P) = \inf_{\gamma \in \Gamma(P,Q)} \int ||\omega - \eta||_{\mathcal{H}}^2 \gamma(d\omega, d\eta)^{1/2}, \tag{21}$$

where $\Gamma(P, Q)$ denotes the class of all probability measures γ on the product space $(\Omega \times \Omega, \mathcal{F} \times \mathcal{F})$ with marginals P and Q.

Equivalently, we can write

$$W_{\mathcal{H}}(Q, P) = \inf\ \tilde{E}[||\tilde{X} - \tilde{Y}||^2_{\mathcal{H}}]^{1/2}, \tag{22}$$

where the infimum is taken over all couples $(\tilde{X}, \tilde{Y})$ of Ω-valued random variables on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that $\tilde{X}$ and $\tilde{Y}$ have distributions P and Q, respectively. Such a couple, and also any measure $\gamma \in \Gamma(P, Q)$, will be called a *coupling of* P *and* Q. We refer to [18] for a thorough discussion of Wasserstein distances in various contexts.

Corollary 2 *Any probability measure Q on $(\Omega, \mathcal{F})$ satisfies Talagrand's inequality*

$$W_{\mathcal{H}}(Q, P) \leq \sqrt{2\, H(Q|P)}. \tag{23}$$

Proof If $H(Q|P) < \infty$ then the processes $W^Q = W - B^Q$ and W, defined on the probability space $(\Omega, \mathcal{F}, Q)$, form a coupling of P and Q such that

$$E_Q\big[||W - W^Q||^2_{\mathcal{H}}\big] = 2H(Q|P). \tag{24}$$

Thus, (23) follows from the definition of the Wasserstein distance $W_{\mathcal{H}}(Q, P)$. □

Remark 6 Inequality (23) on Wiener space was first stated by Feyel and Üstunel in [5]. However, using the Lévy-Ciesielski representation of Brownian motion in terms of Schauder functions, it can also be seen as a direct translation, for $n = \infty$, of Talagrand's original inequality in [17], where Q is a probability measure on Euclidean space $\mathbb{R}^n$ with $n \in \{1, \ldots, \infty\}$, P is the product of standard normal distributions, and the Wasserstein distance is defined in terms of the Euclidean norm; cf. [9].

Remark 7 Note that the coupling (W^Q, W) of P and Q, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq 1}, Q)$, is *adaptive* in the sense that both processes are adapted and the first is a Wiener process with respect to the given filtration. As shown by Lassalle in [15], (W^Q, W) is in fact the optimal adaptive coupling of P and Q. Thus, Eq. (24) shows that Talagrand's inequality reduces to the equality

$$W_{\mathcal{H},ad}(Q, P) = \sqrt{2\, H(Q|P)},$$

if the left hand side is defined as in (22), but taking the infimum only over the adaptive couplings of P and Q; cf. [15] or [9]. For a systematic discussion of the optimal transport problem (22) under various constraints we refer to [1].

Let us now go beyond the case where Q has finite entropy with respect to Wiener measure P. For any $N \geq 1$, consider the discretized filtration

$$\mathcal{F}_{N,t} = \sigma(\{W_s | s \in D_N, s \leq t\}), \quad 0 \leq t \leq 1$$

on $\Omega = C_0[0, 1]$. We set $\mathcal{F}_N = \mathcal{F}_{N,1} = \sigma(\{W_s | s \in D_N\})$, and we denote by $H_N(Q|P)$ the relative entropy of Q with respect to P on the σ-field $\mathcal{F}_N$. Since the σ-fields $\mathcal{F}_N$

increase to $\mathcal{F}$, we have

$$H(Q|P) = \lim_{N\uparrow\infty} H_N(Q|P).$$

From now on we assume that the finite-dimensional marginals of Q are such that

$$H_N(Q|P) < \infty, \quad N = 1, 2, \ldots \tag{25}$$

and we focus on the case $H(Q|P) = \infty$. It is then natural to rescale the finite-dimensional entropies $H_N(Q|P)$ in order to obtain meaningful results.

The following concept of specific relative entropy on Wiener space was introduced by N. Gantert in her thesis [11], where it plays the role of a rate function for large deviations of the quadratic variation from its ergodic behaviour; cf. also [12]. In our context, it will allow us to extend Talagrand's inequality on Wiener space beyond the absolutely continuous case $Q \ll P$, and to throw new light on the Doob decomposition in continuous time.

Definition 5 For any probability measure Q on $(\Omega, \mathcal{F})$, the specific relative entropy of Q with respect to Wiener measure P is defined as

$$h(Q|P) = \liminf_{N\uparrow\infty} 2^{-N} H_N(Q|P) \tag{26}$$

To illustrate the role of specific relative entropy $h(Q|P)$, and in particular its connection with Dirichlet processes, we first consider the particularly transparent case where the coordinate process W is square-integrable and has *independent increments* under Q. Then the increments are normally distributed under Q, there are functions a and β in $C_0[0, 1]$ such that

$$E_Q[W_t] = a(t) \quad \text{and} \quad \mathrm{var}_Q(W_t) = \beta(t),$$

and the function β is strictly increasing due to our assumption (25). In this case, let us write

$$Q = Q_{a,\beta}.$$

Note that $Q_{a,\beta} \in \mathcal{Q}$, and that the Doob decomposition (10) under $Q_{a,\beta}$ takes the form $W = M + A$, where M is a Gaussian martingale with quadratic variation

$$\langle M \rangle_t = \beta(t),$$

and where the deterministic process A given by $A_t(\omega) = a(t)$ clearly carries no local risk.

Let q denote the finite measure on $[0, 1]$ with distribution function β, and denote by

$$q(dt) = q_s(dt) + \sigma^2(t)dt$$

its Lebesgue decomposition with respect to Lebesgue measure λ on [0, 1], where q_s denotes the singular part and $\sigma^2(\cdot)$ is the density of the absolutely continuous part.

Proposition 2 *For $Q = Q_{a,\beta}$, the specific relative entropy $h(Q|P)$ is given by*

$$h(Q|P) = \frac{1}{2}\left(\beta(1) - 1 + H(\lambda|q) + \liminf_{N\uparrow\infty} \sum_{i=1}^{2^N} (\Delta_{N,i}a)^2\right). \tag{27}$$

In particular, $H(\lambda|q) < \infty$ implies that $h(Q|P)$ exists as a finite limit if and only if the function a has "finite energy", that is,

$$h(Q|P) = \lim_{N\uparrow\infty} 2^{-N} H_N(Q|P) < \infty \Longleftrightarrow \exists\langle a\rangle_1 := \lim_{N\uparrow\infty} \sum_{i=1}^{2^N} (\Delta_{N,i}a)^2 < \infty.$$

In this case,

$$h(Q|P) = \frac{1}{2} q_s([0,1]) + \int_0^1 f(\sigma^2(t))dt + \frac{1}{2}\langle a\rangle_1, \tag{28}$$

where f is the convex function on [0, 1] *defined by $f(x) = \frac{1}{2}(x - 1 - \log x)$. In particular,*

$$Q \in \mathcal{Q}_{\mathcal{D}} \Longleftrightarrow h(Q|P) = \frac{1}{2} q_s([0,1]) + \int_0^1 f(\sigma^2(t))dt, \tag{29}$$

that is, W is a Dirichlet process under Q iff $h(Q|P)$ only depends on β and not on a.

Proof For two normal distributions $N(m, \sigma^2)$ and $N(\tilde{m}, \tilde{\sigma}^2)$ on $\mathbb{R}^1$, the relative entropy is given by

$$H\big(N(\tilde{m}, \tilde{\sigma}^2)|N(m, \sigma^2)\big) = f(\tilde{\sigma}^2/\sigma^2) + \frac{1}{2}\frac{(\tilde{m} - m)^2}{\sigma^2}. \tag{30}$$

Since the increments $\Delta_{N,i}W$ along the N-th dyadic partition are independent under both Q and P, with distribution $N(\Delta_{N,i}a, \Delta_{N,i}\beta)$ under Q and $N(0, 2^{-N})$ under P, we get

$$\begin{aligned} H_N(Q|P) &= \sum_{i=1}^{2^N} H\big(N(\Delta_{N,i}a, \Delta_{N,i}\beta)|N(0, 2^{-N})\big) \\ &= \sum_{i=1}^{2^N} f(2^N \Delta_{N,i}\beta) + \frac{1}{2}\sum_{i=1}^{2^N} 2^N (\Delta_{N,i}a)^2. \end{aligned}$$

Thus,

$$2^{-N} H_N(Q|P) = \int_0^1 f(\varphi_N(t))dt + \frac{1}{2}\sum_{i=1}^{2^N}(\Delta_{N,i}a)^2,$$

if we denote by φ_N the density of the finite measure q on [0, 1] with respect to Lebesgue measure λ on the discrete σ-field $\mathcal{B}_N$ generated by the N-th dyadic partition. Note that $\varphi_N > 0$ since β is strictly increasing. Denoting by $H_N(\lambda|q) = \int \log \varphi_N^{-1} d\lambda$ the relative entropy of λ with respect to q on $\mathcal{B}_N$, we can write

$$2^{-N} H_N(Q|P) = \frac{1}{2}\left(q([0,1]) - 1 + H_N(\lambda|q) + \sum_{i=1}^{2^N}(\Delta_{N,i}a)^2\right). \tag{31}$$

Since $q([0, 1]) = \beta(1)$, and since $H_N(\lambda|q)$ increases to $H(\lambda|q)$, we obtain Eq. (27). If we assume $H(\lambda|q) < \infty$ and the existence of a finite limit $\langle a\rangle_1$, then Eq. (27) reduces to (28) since

$$\beta(1) = q_s([0,1]) + \int_0^1 \sigma^2(t)dt.$$

In particular, we obtain the characterization (29) of a measure $Q = Q_{a,\beta} \in \mathcal{Q}_{\mathcal{D}}$. □

Let us now consider the general case $Q \in \mathcal{Q}$. Thus, the coordinate process W admits a continuous Doob decomposition

$$W = M + A \tag{32}$$

under Q, where M is a continuous square-integrable martingale and A is a square-integrable, continuous and adapted process with vanishing local risk. Consider the continuous quadratic variation process $\langle M\rangle$ of M and the corresponding finite random measure $q(\omega, dt)$ on [0, 1] with distribution function $\langle M\rangle(\omega)$, and denote by

$$q(\omega, dt) = q_s(\omega, dt) + \sigma^2(\omega, t)dt \tag{33}$$

its Lebesgue decomposition into a singular and an absolutely continuous part with respect to Lebesgue measure λ on [0, 1]; cf. [9] for an explicit construction. Our aim is to show how the specific relative entropy $h(Q|P)$ depends on the Doob decomposition (32), and in particular on the random measure $q(\cdot, dt)$.

For $N \geq 1$ and $i = 1, \ldots, 2^N$, we denote by $\nu_{N,i}(\omega, \cdot)$ the conditional distribution of the increment $\Delta_{N,i}W$ under Q given the σ-field $\mathcal{F}_{N,t_{i-1}}$, by

$$a_{N,i} = E_Q\big[\Delta_{N,i}W|\mathcal{F}_{N,t_{i-1}}\big] = E_Q\big[\Delta_{N,i}A|\mathcal{F}_{N,t_{i-1}}\big]$$

its conditional mean, by

$$\tilde{\sigma}^2_{N,i} = E_Q\big[(\Delta_{N,i}W)^2|\mathcal{F}_{N,t_{i-1}}\big] - a^2_{N,i}$$

its conditional variance, and by

$$\sigma^2_{N,i} = E_Q\big[(\Delta_{N,i}M)^2|\mathcal{F}_{N,t_{i-1}}\big] = E_Q\big[\langle M\rangle_{t_i} - \langle M\rangle_{t_{i-1}}|\mathcal{F}_{N,t_{i-1}}\big] \tag{34}$$

the conditional variance of the martingale increment $\Delta_{N,i}M$.

Lemma 2 *The finite-dimensional entropy $H_N(Q|P)$ can be decomposed as follows:*

$$\begin{aligned} H_N(Q|P) = H_N(Q|Q_N) + E_Q\left[\sum_{i=1}^{2^N} f(2^N\sigma^2_{N,i})\right] + \frac{1}{2}2^N E_Q\left[\sum_{i=1}^{2^N} a^2_{N,i}\right] \\ +\frac{1}{2}E_Q\left[\sum_{i=1}^{2^N}(\tilde{\sigma}^2_{N,i} - \sigma^2_{N,i})(2^N - \sigma^{-2}_{N,i})\right] \end{aligned} \tag{35}$$

where f is the function defined in Proposition 2, and where Q_N denotes the probability measure on $(\Omega, \mathcal{F}_N)$ such that the increments $\Delta_{N,i}W$ have conditional distribution $N(a_{N,i}, \sigma^2_{N,i})$ given the σ-field $\mathcal{F}_{N,t_{i-1}}$.

Proof Since

$$H_N(Q|P) = \sum_{i=1}^{2^N} E_Q\big[H\big(\nu_{N,i}(\omega,\cdot)|N(0,2^{-N})\big)\big],$$

and since

$$\begin{aligned} H\big(\nu_{N,i}|N(0,2^{-N})\big) = H\big(\nu_{N,i}|N(a_{N,i},\sigma^2_{N,i})\big) + f(2^N\sigma^2_{N,i}) \\ +\frac{1}{2}2^N a^2_{N,i} + \frac{1}{2}(\tilde{\sigma}^2_{N,i} - \sigma^2_{N,i})(2^N - \sigma^{-2}_{N,i}), \end{aligned}$$

we obtain Eq. (35). □

Let us first look at the asymptotic behavior of the second term on the right hand side of Eq. (35). We denote by $Q \otimes q$ the finite measure on $\bar{\Omega} = \Omega \times [0,1]$ defined by $(Q \otimes q)(d\omega, dt) = Q(d\omega)q(\omega, dt)$. On the σ-field

$$\mathcal{P}_N := \sigma(\{A_t \times (t,1] \mid t \in D_N, A_t \in \mathcal{F}_{N,t}\}),$$

the measure $Q \otimes q$ is absolutely continuous with respect to the product measure $Q \otimes \lambda$, and the density is given by

$$\sigma^2_N(\omega,t) := \sum_{i=1}^{2^N} 2^N \sigma^2_{N,i}(\omega) I_{(t_{i-1},t_i]}(t).$$

The σ-fields $\mathcal{P}_N$ increase to the predictable σ-field $\mathcal{P}$ on $\bar{\Omega}$, generated by the sets $A_t \times (t, 1]$ with $t \in [0, 1]$ and $A_t \in \mathcal{F}_t$, and we denote by

$$H(Q \otimes \lambda | Q \otimes q) = E_Q[H(\lambda | q(\cdot))]$$

the relative entropy of $Q \otimes \lambda$ with respect to $Q \otimes q$ on $\mathcal{P}$.

Lemma 3

$$\begin{aligned}
&\lim_{N \uparrow \infty} 2^{-N} E_Q\left[\sum_{i=1}^{2^N} f(2^N \sigma^2_{N,i})\right] \\
&= \frac{1}{2}\big(E_Q[q(\cdot, [0, 1])] - 1 + H(Q \otimes \lambda | Q \otimes q)\big) \\
&= \frac{1}{2} E_Q[q_s(\cdot, [0, 1])] + E_Q\left[\int_0^1 f(\sigma^2(\cdot, t))dt\right].
\end{aligned} \tag{36}$$

Proof Since

$$E_Q\left[\int_0^1 \sigma^2_N(\cdot, t)dt\right] = E_Q\big[\langle M\rangle_1\big] = E_Q\big[q(\cdot, [0, 1])\big]$$

for any $N \geq 1$, we can write

$$\begin{aligned}
2^{-N} E_Q\left[\sum_{i=1}^{2^N} f(2^N \sigma^2_{N,i})\right] &= E_Q\left[\int_0^1 f(\sigma^2_N(\cdot, t))dt\right] \\
&= \frac{1}{2}\left(E_Q[q(\cdot, [0, 1])] - 1 - E_Q\left[\int_0^1 \log \sigma^2_N(\cdot, t)dt\right]\right) \\
&= \frac{1}{2}\big(E_Q[q(\cdot, [0, 1])] - 1 + H_N(Q \otimes \lambda | Q \otimes q)\big),
\end{aligned}$$

where $H_N(Q \otimes \lambda | Q \otimes q)$ denotes the relative entropy of $Q \otimes \lambda$ with respect to $Q \otimes q$ on $\mathcal{P}_N$. Since $\mathcal{P}_N$ increases to $\mathcal{P}$, these entropies increase to

$$H(Q \otimes \lambda | Q \otimes q) = E_Q\left[\int_0^1 \log(\sigma_N^{-2}(\cdot, t))dt\right],$$

and this yields Eq. (36). □

For a martingale measure $Q \in \mathcal{Q}_{\mathcal{M}}$ with absolutely continuous quadratic variation, the following proposition is due to N. Gantert in [11]. Here we extend it to the case where the quadration variation may have a singular component; see also [9].

Proposition 3 *For a martingale measure $Q \in \mathcal{Q}_{\mathcal{M}}$,*

$$\begin{aligned} h(Q|P) &\geq \frac{1}{2}\big(E_Q[q(\cdot,[0,1])] - 1 + H(Q \otimes \lambda | Q \otimes q)\big) \\ &= \frac{1}{2} E_Q\big[q_s(\omega,[0,1])\big] + E_Q\left[\int_0^1 f\big(\sigma^2(\omega,t)\big)dt\right]. \end{aligned} \tag{37}$$

If $h(Q|P) < \infty$ then we have $H(Q \otimes \lambda | Q \otimes q) < \infty$, and in particular

$$\sigma^2(\cdot,\cdot) > 0 \quad Q \otimes \lambda\text{-a.s.}. \tag{38}$$

Moreover, equality holds in (37) if and only if Q is "almost locally normal" in the sense that

$$\lim_{N \uparrow \infty} 2^{-N} H_N(Q|Q_N) = 0. \tag{39}$$

Proof For $Q \in \mathcal{Q}_{\mathcal{M}}$ we have $A = 0$, hence $a_{N,i} = 0$ and $\tilde{\sigma}^2_{N,i} = \sigma^2_{N,i}$. Thus, Eq. (35) implies

$$2^{-N} H_N(Q|P) = 2^{-N} H_N(Q|Q_N) + 2^{-N} E_Q\left[\sum_{i=1}^{2^N} f(2^N \sigma^2_{N,i})\right],$$

and so inequality (37) as well as the condition for equality follow from Lemma 3. Due to (37), $h(Q|P) < \infty$ implies $H(Q \otimes \lambda | Q \otimes q)\big) < \infty$, hence $Q \otimes \lambda \ll Q \otimes q$, and in particular (38). □

Let us denote by

$$\tilde{\alpha}^2_{N,i} = E_Q[(\Delta_{N,i} A)^2 | \mathcal{F}_{N,t_{i-1}}] - a^2_{N,i} \tag{40}$$

the conditional variance of $\Delta_{N,i} A$ with respect to $\mathcal{F}_{N,t_{i-1}} \subset \mathcal{F}_{t_{i-1}}$, and recall the definition of $\alpha^2_{N,i}$ in (11). Since $\tilde{\alpha}^2_{N,i}$ is defined with respect to the smaller σ-field, we have

$$E_Q[\alpha^2_{N,i}] \leq E_Q[\tilde{\alpha}^2_{N,i}] \leq E_Q[(\Delta A_{N,i})^2]. \tag{41}$$

This shows that the condition

$$\lim_{N \uparrow \infty} E_Q\left[\sum_{i=1}^{2^N} \tilde{\alpha}^2_{N,i}\right] = 0 \tag{42}$$

strengthens our assumption on $Q \in \mathcal{Q}$ that A has vanishing local risk, and that it is satisfied as soon as A has energy 0 under Q.

Remark 8 Note that we have $\tilde{\alpha}^2_{N,i} = \alpha^2_{N,i}$ as soon as Q is Markovian, that is, if W is a Markov process under Q. Thus, condition (42) is satisfied for any Markovian $Q \in \mathcal{Q}$.

Definition 6 We denote by $\mathcal{Q}_{\mathcal{E}}$ the class of all probability measures $Q \in \mathcal{Q}$ such that

(i) Q satisfies condition (42),
(ii) the process A has "finite energy" under Q, that is,

$$\exists \langle A \rangle_1 := \lim_{N \uparrow \infty} \sum_{i=1}^{2^N} (\Delta_{N,i} A)^2 \qquad \text{in } L^1(Q). \tag{43}$$

Thus, the chain of inclusions in (18) can be extended as follows:

$$\mathcal{Q}_{\mathcal{M}} \subset \mathcal{Q}_{\mathcal{H}} \subset \mathcal{Q}_{\mathcal{S}} \subset \mathcal{Q}_{\mathcal{D}} \subset \mathcal{Q}_{\mathcal{E}} \subset \mathcal{Q}.$$

In [9], Proposition 3 for a martingale measure $Q \in \mathcal{Q}_{\mathcal{M}}$ is extended to a large class of semimartingale measures $Q \in \mathcal{Q}_{\mathcal{S}}$, where the process A appearing in the Doob decomposition of W under Q has paths of bounded variation. Here we go two steps further and consider the case $Q \in \mathcal{Q}_{\mathcal{E}}$, and in particular the case $Q \in \mathcal{Q}_{\mathcal{D}}$ where W is a Dirichlet process under Q.

Theorem 3 *Let $Q \in \mathcal{Q}_{\mathcal{E}}$ be such that the variance $\sigma^2(\cdot, \cdot)$ in (33) is bounded away from* 0. *Then*

$$\begin{aligned} h(Q|P) &\geq \frac{1}{2}\big(E_Q[q(\cdot,[0,1])] - 1 + E_Q\big[H\big(\lambda|q(\cdot)\big)\big] + \frac{1}{2}E_Q[\langle A\rangle_1] \\ &= \frac{1}{2}E_Q\big[q_s(\cdot,[0,1])\big] + E_Q\left[\int_0^1 f\big(\sigma^2(\cdot,t)\big)dt\right] + \frac{1}{2}E_Q[\langle A\rangle_1]. \end{aligned} \tag{44}$$

If $h(Q|P) < \infty$ then equality holds if and only if Q satisfies condition (39). *In that case,*

$$Q \in \mathcal{Q}_{\mathcal{D}} \Longleftrightarrow h(Q|P) = \frac{1}{2}E_Q\big[q_s(\omega,[0,1])\big] + E_Q\left[\int_0^1 f\big(\sigma^2(\omega,t)\big)dt\right]. \tag{45}$$

Proof Equation (35) can be written as

$$2^{-N}H_N(Q|P) = 2^{-N}H_N(Q|Q_N) + E_Q\left[\int_0^1 f(\sigma_N^2(\cdot,t)dt)\right] + \frac{1}{2}I_N$$
$$+\frac{1}{2}E_Q\left[\sum_{i=1}^{2^N}\delta_{N,i}(1-2^{-N}\sigma_{N,i}^{-2})\right], \tag{46}$$

where

$$I_N := E_Q\left[\sum_{i=1}^{2^N} a_{N,i}^2\right] \quad \text{and} \quad \delta_{N,i} := \tilde{\sigma}_{N,i}^2 - \sigma_{N,i}^2.$$

Note that

$$I_N = E_Q\left[\sum_{i=1}^{2^N}(\Delta_{N,i}A)^2\right] - J_N,$$

where

$$J_N := E_Q\left[\sum_{i=1}^{2^N}\tilde{\alpha}_{N,i}^2\right].$$

Since $Q \in \mathcal{Q}_{\mathcal{E}}$, we obtain

$$\lim_{N\uparrow\infty} I_N = E_Q[\langle A\rangle_1],$$

Let us now show that the last term in (46) converges to 0 as $N \uparrow \infty$. Since

$$\delta_{N,i} = \tilde{\alpha}_{N,i}^2 + 2E_Q\big[(\Delta M_{N,i})(\Delta A_{N,i})|\mathcal{F}_{N,t_{i-1}}\big]$$

satisfies

$$|\delta_{N,i}| \le \tilde{\alpha}_{N,i}^2 + 2\sigma_{N,i}\tilde{\alpha}_{N,i}, \tag{47}$$

we get

$$E_Q\left[\sum_{i=1}^{2^N}|\delta_{N,i}|\right] \le E_Q\left[\sum_{i=1}^{2^N}\tilde{\alpha}_{N,i}^2\right] + 2\sum_{i=1}^{2^N}E_Q\big[\sigma_{N,i}^2\big]^{1/2}E_Q\big[\tilde{\alpha}_{N,i}^2\big]^{1/2}$$
$$\le J_N + 2\,E_Q\big[M_1^2\big]^{1/2}J_N^{1/2},$$

hence

$$\lim_{N\uparrow\infty} E_Q\left[\sum_{i=1}^{2^N}|\delta_{N,i}|\right] = 0, \tag{48}$$

due to condition (42). Moreover, if $\sigma^2(\cdot,\cdot) \geq c \quad Q \otimes \lambda$-a.s. for some $c > 0$ then

$$\sum_{i=1}^{2^N} 2^N \sigma_{N,i}^2(\omega) I_{(t_{i-1},t_i]}(t) = \sigma_N^2(\omega, t) \geq E_{Q\otimes\lambda}\big[\sigma^2 | \mathcal{P}_N\big] \geq c \quad Q \otimes \lambda\text{-a.s.},$$

and this implies

$$\lim_{N\uparrow\infty} E_Q\left[\sum_{i=1}^{2^N} |\delta_{N,i}| 2^{-N} \sigma_{N,i}^{-2}\right] \leq c^{-1} \lim_{N\uparrow\infty} E_Q\left[\sum_{i=1}^{2^N} |\delta_{N,i}|\right] = 0. \tag{49}$$

Thus, the last two terms in Eq. (46) converge to 0. In view of Lemma 3, this completes the proof. □

Remark 9 The proof shows that, instead of requiring that $\sigma^2(\cdot,\cdot)$ is bounded away from 0, it is enough to assume that the conditional variances $\sigma_{N,i}^2$ and $\tilde{\alpha}_{N,i}^2$ of M and A under $Q \in \mathcal{Q}_{\mathcal{E}}$ satisfy the condition

$$\lim_{N\uparrow\infty} E_Q\left[2^{-N} \sum_{i=1}^{2^N} \tilde{\alpha}_{N,i}^2 \sigma_{N,i}^{-2}\right] = 0. \tag{50}$$

This includes the case of a martingale measure $Q \in \mathcal{Q}_{\mathcal{M}}$, and also the case where the process A is locally deterministic in the sense that $\tilde{\alpha}_{N,i}^2 = 0$ for large enough N.

Theorem 3 allows us to prove an extension of Talagrand's inequality on Wiener space beyond the absolutely continuous case $Q \ll P$. For $Q \in \mathcal{Q}_{\mathcal{S}}$ we refer to [9] for an extension that covers Talagrand's inequality (2) as a special case. Here we focus on the case $Q \in \mathcal{Q}_{\mathcal{D}}$ and consider the following Wasserstein distance $W_{\mathcal{D}}(Q, P)$, where the cost function is defined in terms of quadratic variation.

Definition 7 The Wasserstein distance $W_{\mathcal{D}}(Q, P)$ between $Q \in \mathcal{Q}_{\mathcal{D}}$ and Wiener measure P is defined as

$$W_{\mathcal{D}}(Q, P) = \inf \big(\tilde{E}\big[\langle \tilde{Y} - \tilde{X}\rangle_1\big]\big)^{1/2}, \tag{51}$$

where the infimum is taken over all adaptive couplings $(\tilde{Y}, \tilde{X})$ of Q and P on some filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{0\leq t\leq 1}, \tilde{P})$ such that $\tilde{Y}$ is a Dirichlet process on $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{0\leq t\leq 1}, \tilde{P})$.

For a martingale measure $Q \in \mathcal{Q}_{\mathcal{M}}$, the following corollary is proved in [9]. For $Q \in \mathcal{Q}_{\mathcal{D}}$, the proof is essentially the same, and so we just sketch the argument and refer to [9] for further details.

Corollary 3 *For a probability measure $Q \in \mathcal{Q}_{\mathcal{D}}$ that satisfies condition (50),*

$$W_{\mathcal{D}}(Q, P) \leq \sqrt{2\, h(Q|P)}. \tag{52}$$

Proof We may assume $h(Q|P) < \infty$. As shown in [9], this implies that there is a Wiener process $W^Q = (W_t^Q)_{0 \le t \le 1}$, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le 1}, Q)$, such that the coupling (W^Q, W) of P and Q is optimal for the Wasserstein distance $W_{\mathcal{D}}$, that is,

$$W_{\mathcal{D}}^2(Q, P) = E_Q\big[\langle W - W^Q \rangle_1\big]. \tag{53}$$

Moreover,

$$E_Q\big[\langle W - W^Q \rangle_1\big] = E_Q\left[\int_0^1 \big(\sigma(\cdot, s) - 1\big)^2 ds + q_s(\cdot, (0, 1])\right].$$

Since

$$(\sigma - 1)^2 \le \sigma^2 - 1 - \log \sigma^2 = 2 f(\sigma^2),$$

we obtain

$$\begin{aligned} E_Q\big[\langle W - W^Q \rangle_1\big] &\le 2\, E_Q\left[\int_0^1 f\big(\sigma^2(\cdot, s)\big) dt + \frac{1}{2} q_s(\cdot, (0, 1])\right] \\ &\le 2\, h(Q|P), \end{aligned} \tag{54}$$

where the second inequality follows from Theorem 3, and so we have shown inequality (52). □

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References

1. B. Acciaio, J. Backhoff Veraguas, A. Zalashko, Causal optimal transport and its links to enlargement of filtrations and continuous-time stochastic optimization. Stoch. Process. Appl. **9**(3), 203–228 (2019)
2. J. Bertoin, Les Processus de Dirichlet en tant qu'Espaces de Banach. Stochastics **18**, 155–188 (1986)
3. F. Coquet, A. Jakubowski, J. Mémin, L. Slominski, Natural decomposition of processes and weak dirichlet processes, in *In Memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX*. Lecture Notes in Mathematics, vol. 1874 (Springer, Berlin, 2006), pp. 81–116

4. C. Dellacherie, P.-A. Meyer, *Probabilités et Potentiel, Ch. V–VIII* (Hermann, Paris, 1980)
5. D. Feyel, A.S. Üstünel, Monge-Kantorovich measure transportation and Monge-Ampere equation on Wiener Space. Probab. Theor. Relat. Fields **128**(3), 347–385 (2004)
6. H. Föllmer, The exit measure of a supermartingale. Z. Wahrscheinlichkeitstheorie Verw. Geb. **21**, 154–166 (1972)
7. H. Föllmer, Dirichlet processes, in *Stochastic Integrals*, ed. by D. Williams. Lecture Notes in Mathematics, vol. 851 (Springer, 1981), pp. 476–478
8. H. Föllmer, Random fields and diffusion processes, in *Ecole d' été de Probabilités de Saint-Flour XV–XVII, 1985–87*. Lecture Notes in Mathematics, vol. 1362 (Springer, Berlin, 1988), pp. 101–203
9. H. Föllmer, Optimal coupling on Wiener space and an extension of Talagrand's transport inequality, in *Stochastic Analysis, Stochastic Control, and Stochastic Optimization—A Commemorative Volume to Honor Professor Mark H. Davis's Contributions*, ed. by G. Yin, T. Zariphopoulou (Springer, 2022)
10. M. Fukushima, *Dirichlet Forms and Markov Processes*, North-Holland Mathematical Library, vol. 23 (North-Holland Publishing Co., Amsterdam, 1980)
11. N. Gantert, Einige grosse Abweichungen der Brownschen Bewegung. Ph.D. thesis. Universität Bonn, 1991
12. N. Gantert, Self-similarity of Brownian motion and a large deviation principle for random fields on a binary tree. Probab. Theory Relat. Fields **98**, 7–20 (1994)
13. F. Gozzi, F. Russo, Weak Dirichlet processes with a stochastic control perspective. Stoch. Process. Appl. **116**(11), 1563–1583 (2006)
14. S.E. Graversen, M. Rao, Quadratic variation and energy. Nagoya Math. J. **100**, 163–180 (1985)
15. R. Lassalle, Causal transference plans and their Monge-Kantorovich problems. Stoch. Anal. Appl. **36**(1), 1–33 (2018)
16. J. Lehec, Representation formula for the entropy and functional inequalities. Ann. Inst. Henri Poincaré Probab. Stat. **49**(3), 885–899 (2013)
17. M. Talagrand, Transportation cost for Gaussian and other product measures. Geom. Funct. Anal. **6**(3), 587–600 (1996)
18. C. Villani, *Optimal Transport: Old and New*. Grundlehren der Mathematischen Wissenschaften, vol. 338 (Springer, 2009)

Analysis on Fractal Spaces and Heat Kernels

Alexander Grigor'yan

Abstract We give overview of heat kernel estimates on fractal spaces in connection with the notion of walk dimension.

Keywords Heat kernel · Fractal · Dirichlet form · Walk dimension

1 Introduction

Since the time of Newton and Leibniz, differentiation and integration have been major concepts of mathematics. The theory of integration has come a long way from Riemann's integration of continuous functions to measure theory, including construction of Hausdorff measures on metric spaces.

In this survey we discuss the notion of *differentiation* in metric spaces, especially in fractals with self-similar structures. The existing theory of the *upper gradient* of Heinonen and Koskela [24] and Cheeger [12] provides an analogue of Rademacher's theorem about differentiability of Lipschitz functions. However, it imposes quite strong assumptions on the metric space in question, including the Poincaré inequality with the quadratic scaling factor. Such assumptions are typically satisfied on the limits of sequences of non-negatively curved manifolds, but never on commonly known fractal spaces.

More specifically, our goal is the notion of a *Laplace-type* operator on general metric measure spaces, in particular, on fractal spaces. The Laplace operator in $\mathbb{R}^n$ is a *second* order differential operator. Hence, unlike the upper gradient that is a generalization of the first order differential operator, we aim at a generalization of a second order differential operator. Our present understanding is that such operators should be carried by a larger family of metric spaces.

Dedicated to Masatoshi Fukushima on the occasion of his 米寿.

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By a Laplace-type operator we mean the generator of a strongly local regular *Dirichlet form*. The theory of Dirichlet forms was developed by M. Fukushima et al., and its detailed account can be found in [15] (see also [31]). Although the original motivation of this theory was to create a universal framework for construction of Markov processes in $\mathbb{R}^n$, it suits perfectly for development of analysis on metric measure spaces.

Strongly local regular Dirichlet forms and associated diffusion processes have been successfully constructed on large families of fractals, in particular, on the Sierpinski gasket by Barlow and Perkins [8], Goldstein [16] and Kusuoka [28], on p.c.f. fractals by Kigami [26, 27], and on the Sierpinski carpet by Barlow and Bass [3] and Kusuoka and Zhou [29].

It has been observed that the quantitative behavior of the diffusion processes on fractals is drastically different from that in $\mathbb{R}^n$. In particular, the expected time needed for the diffusive particle to cover distance r is of the order r^β with some $\beta > 2$, whereas in $\mathbb{R}^n$ it is r^2. In physics such a process is called an *anomalous diffusion*. The parameter β is called the *walk dimension* of the diffusion. It also determines *sub-Gaussian* estimates of the heat kernel.

It was shown in [20] that the walk dimension β is, in fact, an invariant of the *metric space* alone, and it can be characterized in terms of the family of *Besov seminorms*.

In this note we give an overview of some results related to the notion of the walk dimension.

2 Classical Heat Kernel

The *heat kernel* in $\mathbb{R}^n$ is the fundamental solution of the heat equation $\frac{\partial u}{\partial t} = \Delta u$:

$$p_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

This function is also called the Gauss-Weierstrass function. Let us briefly mention some applications of this notion.

1. The Cauchy problem for the heat equation with the initial condition $u|_{t=0} = f$ is solved by $u(t, \cdot) = p_t * f$, under certain restriction on f, for example, for $f \in C_b(\mathbb{R}^n)$ (where $C_b(X)$ stands for the space of bounded continuous functions on X). Since then $p_t * f \to f$ as $t \to 0+$, the smooth function $p_t * f$ can be regarded as a mollification of f. This idea was used by Weierstrass in his proof of the celebrated Weierstrass approximation theorem.
2. It is less known but the heat kernel can be used to prove some Sobolev embedding theorems (see [17, pp. 156–157]).
3. The function $p_{t/2}(x)$ coincides with the transition density of Brownian motion $\{X_t\}$ in $\mathbb{R}^n$ (Fig. 1).

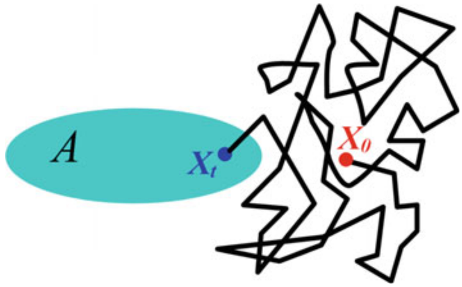


Fig. 1 The probability that $X_t \in A$ is given by integration of the heat kernel $p_{t/2}(X_0 - \cdot)$ over A

4. Approximation of the Dirichlet integral: for any $f \in W^{1,2}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |\nabla f|^2 \, dx = \lim_{t \to 0} \frac{1}{2t} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} p_t(x-y) \, |f(x) - f(y)|^2 \, dx dy.$$

3 Examples of Fractals

Let (M, d) be a locally compact separable metric space and μ be a Radon measure on M with full support. A triple (M, d, μ) will be referred to as a *metric measure space*. A metric measure space (M, d, μ) is called α*-regular* for some $\alpha > 0$ if all metric balls

$$B(x, r) := \{y \in M : d(x, y) < r\}$$

are relatively compact and if for all $x \in M$ and $r < \operatorname{diam} M$ we have

$$\mu(B(x, r)) \simeq r^\alpha. \tag{1}$$

The sign $\simeq$ means that the ratio of the two sides is bounded from above and below by positive constants, and $\operatorname{diam} M = \sup_{x,y \in M} d(x, y)$.

It follows from (1) that

$$\dim_H M = \alpha \quad \text{and} \quad \mathcal{H}_\alpha \simeq \mu$$

where $\dim_H M$ denotes the Hausdorff dimension of M (with respect to the metric d) and $\mathcal{H}_\alpha$ denotes the Hausdorff measure of dimension α. The number α is also referred to as the *fractal dimension* of M. In some sense, α is a numerical characteristic of the integral calculus on M.

The original meaning of the popular term "fractal" refers to α-regular spaces with fractional values of α. Such spaces first appeared in mathematics as curious examples and initially served as counterexamples to various theorems. The most famous example of a fractal set is the *Cantor set* that was introduced by Georg Cantor in 1883. However, Mandelbrot [32] in 1982 put forward a novel point of view according to which fractals are typical of nature rather than exceptional. This point of view is also confirmed from within pure mathematics by the spectacular development of the analysis on fractals and metric measure spaces over the past three decades, which sheds new light on some aspects of classical analysis in $\mathbb{R}^n$. See [1] for a very good introduction to analysis on fractals.

Another example of a fractal is the Vicsek snowflake (VS) shown on Figs. 6 and 7.

There is nowadays no commonly accepted rigorous definition of the term "fractal". Typical fractal sets are obtained by some self-similar constructions as limits of sequences of iterations. Important examples of fractal sets are the *Sierpinski gasket* (SG) and *Sierpinski carpet* (SC) that were introduced by Wacław Sierpiński in 1915. They are shown on Figs. 2, 3 and 4, 5, respectively.

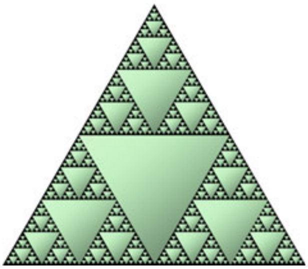


Fig. 2 Sierpinski gasket, $\alpha = \frac{\log 3}{\log 2} \approx 1.58$

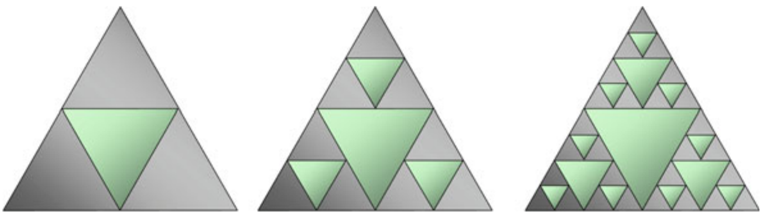


Fig. 3 Three interations of construction of SG

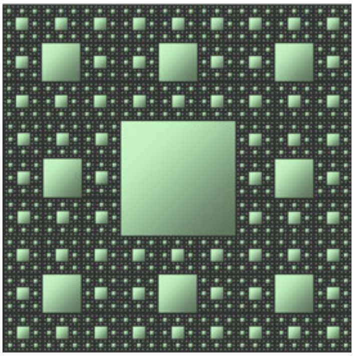


Fig. 4 Sierpinski carpet, $\alpha = \frac{\log 8}{\log 3} \approx 1.89$

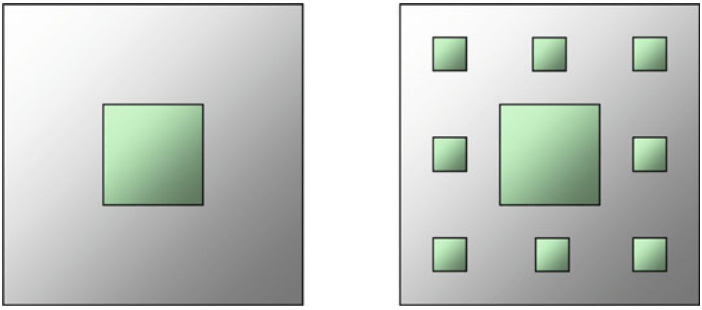
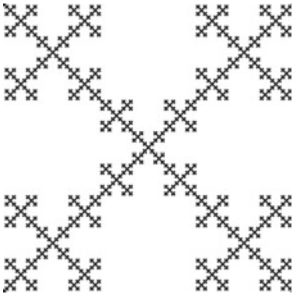


Fig. 5 Two iterations of construction of SC

Fig. 6 Vicsek snowflake



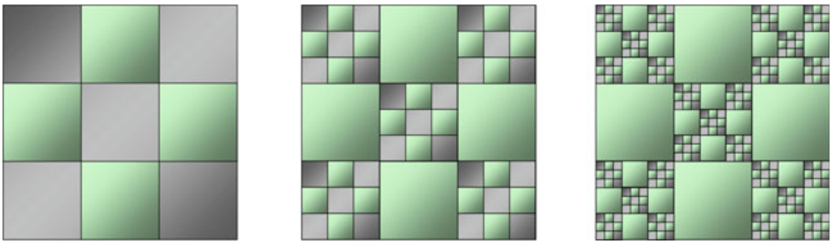


Fig. 7 Three iterations of construction of VS

4 Dirichlet Forms

On certain metric spaces, including fractal spaces, it is possible to construct a *Laplace-type* operator, by means of the theory of Dirichlet forms by Fukushima [15].

A (symmetric) *Dirichlet form* in $L^2(M,\mu)$ is a pair $(\mathcal{E},\mathcal{F})$ where $\mathcal{F}$ is a dense subspace of $L^2(M,\mu)$ and $\mathcal{E}$ is a symmetric bilinear form on $\mathcal{F}$ with the following properties:

1. It is positive definite, that is, $\mathcal{E}(f,f)\geq 0$ for all $f\in\mathcal{F}$.
2. It is closed, that is, $\mathcal{F}$ is complete with respect to the norm

$$\left(\int_M f^2 d\mu + \mathcal{E}(f,f)\right)^{1/2}.$$

3. It is Markovian, that is, if $f\in\mathcal{F}$ then $\widetilde{f}:=\min(f_+,1)\in\mathcal{F}$ and $\mathcal{E}(\widetilde{f},\widetilde{f})\leq\mathcal{E}(f,f)$.

Any Dirichlet form has the generator: a positive definite self-adjoint operator $\mathcal{L}$ in $L^2(M,\mu)$ with domain $\operatorname{dom}(\mathcal{L})\subset\mathcal{F}$ such that

$$(\mathcal{L}f,g)=\mathcal{E}(f,g)\quad\text{for all } f\in\operatorname{dom}(\mathcal{L}) \text{ and } g\in\mathcal{F}.$$

For example, the bilinear form

$$\mathcal{E}(f,g)=\int_{\mathbb{R}^n}\nabla f\cdot\nabla g\,dx \tag{2}$$

in $\mathcal{F}=W^{1,2}(\mathbb{R}^n)$ is a Dirichlet form in $L^2(\mathbb{R}^n,dx)$, whose quadratic part is the Dirichlet integral. Its generator is $\mathcal{L}=-\Delta$ with $\operatorname{dom}(\mathcal{L})=W^{2,2}(\mathbb{R}^n)$.

Another example of a Dirichlet form in $L^2(\mathbb{R}^n,dx)$:

$$\mathcal{E}(f,f)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(f(x)-f(y))^2}{|x-y|^{n+s}}dx\,dy, \tag{3}$$

where $s\in(0,2)$ and $\mathcal{F}=B^{s/2}_{2,2}(\mathbb{R}^n)$. It has the generator $\mathcal{L}=c_{n,s}(-\Delta)^{s/2}$ with a positive constant $c_{n,s}$.

A Dirichlet form $(\mathcal{E},\mathcal{F})$ is called *local* if $\mathcal{E}(f,g)=0$ for any two functions $f,g\in\mathcal{F}$ with disjoint compact supports, and $(\mathcal{E},\mathcal{F})$ is called *strongly local* if $\mathcal{E}(f,g)=0$ whenever $f=\text{const}$ in a neighborhood of $\operatorname{supp} g$. The Dirichlet form $(\mathcal{E},\mathcal{F})$ is called *regular* if $C_0(M)\cap\mathcal{F}$ is dense both in $\mathcal{F}$ and $C_0(M)$, where $C_0(M)$ is the space of continuous functions on M with compact supports endowed with the sup-norm.

For example, both Dirichlet forms (2) and (3) are regular, the form (2) is strongly local, while the form (3) is nonlocal.

The generator of any regular Dirichlet form determines the *heat semigroup* $\left\{e^{-t\mathcal{L}}\right\}_{t\geq 0}$, as well as a Markov process $\{X_t\}_{t\geq 0}$ on M with the transition semigroup $e^{-t\mathcal{L}}$, that is,

$$\mathbb{E}_x f(X_t) = e^{-t\mathcal{L}} f(x) \quad \text{for all } f \in C_0(M).$$

If $(\mathcal{E}, \mathcal{F})$ is local then $\{X_t\}$ is a diffusion while otherwise the process $\{X_t\}$ contains jumps.

For example, the Dirichlet form (2) determines Brownian motion in $\mathbb{R}^n$, whose transition density is exactly the Gauss-Weierstrass function

$$p_t(x) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

The Dirichlet form (3) determines a jump process: a symmetric stable Levy process of the index s. In the case $s = 1$ its transition density is the Cauchy distribution

$$p_t(x) = \frac{c_n t}{\left(t^2 + |x|^2\right)^{\frac{n+1}{2}}} = \frac{c_n}{t^n}\left(1 + \frac{|x|^2}{t^2}\right)^{-\frac{n+1}{2}},$$

where $c_n = \Gamma\left(\frac{n+1}{2}\right)/\pi^{(n+1)/2}$. For an arbitrary $s \in (0, 2)$ we have

$$p_t(x) \simeq \frac{1}{t^{n/s}}\left(1 + \frac{|x|}{t^{1/s}}\right)^{-(n+s)}.$$

If a metric measure space M possesses a strongly local regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ then we consider its generator $\mathcal{L}$ as an analogue of the Laplace operator. In this case the differential calculus is defined on M.

Nontrivial strongly local regular Dirichlet forms have been successfully constructed on large families of fractals, in particular, on SG by Barlow and Perkins [8], Goldstein [16] and Kusuoka [28], on SC by Barlow and Bass [3] and Kusuoka and Zhou [29], on nested fractals (including VS) by Lindstrøm [30], and on p.c.f. fractals by Kigami [26, 27].

In fact, each of these fractals can be regarded as a limit of a sequence of approximating graphs Γ_n (Fig. 8).

Define on each Γ_n a Dirichlet form $\mathcal{E}_n$ by

$$\mathcal{E}_n(f, f) = \sum_{x\sim y} (f(x) - f(y))^2$$

where $x \sim y$ means that the vertices x and y are neighbors, and then consider a scaled limit

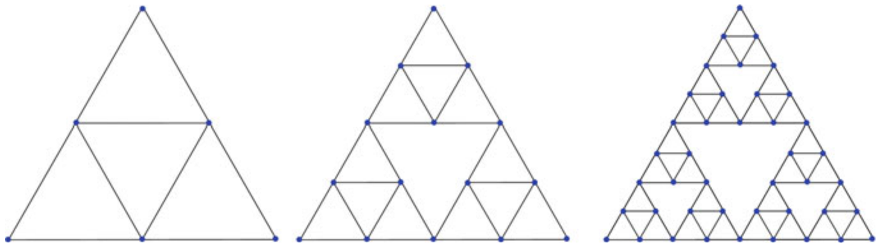


Fig. 8 Approximating graphs $\Gamma_1, \Gamma_2, \Gamma_3$ for SG

$$\mathcal{E}(f, f) = \lim_{n\to\infty} R_n \mathcal{E}_n(f, f) \tag{4}$$

with an appropriately chosen renormalizing sequence $\{R_n\}$. The main difficulty is to ensure the existence of $\{R_n\}$ such that this limit exists and is nontrivial for a dense family of f. For p.c.f. fractals one chooses $R_n = \rho^n$ where, for example, $\rho = \frac{5}{3}$ for SG and $\rho = 3$ for VS, and the limit exists due to monotonicity [27].

For SC the situation is much harder. Initially a strongly local Dirichlet form on SC was constructed by Barlow and Bass [3] in a different way by using a probabilistic approach. After a groundbreaking work of Barlow et al. [6] proving the uniqueness of a canonical Dirichlet form on SC, it became possible to claim that the limit (4) exists for a certain sequence $\{R_n\}$ such that $R_n \simeq \rho^n$, where the exact value of ρ is still unknown. Numerical computation in [7] indicates that $\rho \approx 1.25$. It is also an open question whether the limit $\lim_{n\to\infty} \rho^{-n} R_n$ exists (see [4, Sect. 5, Problem 1]). Other ways of constructing a strongly local Dirichlet form on SC can be found in [29] and [23].

5 Walk Dimension

In all the above examples, the heat semigroup $e^{-t\mathcal{L}}$ of the Dirichlet form $(\mathcal{E}, \mathcal{F})$ is an integral operator:

$$e^{-t\mathcal{L}} f(x) = \int_M p_t(x, y) f(y) d\mu(y),$$

whose integral kernel $p_t(x, y)$ is called the *heat kernel* of $(\mathcal{E}, \mathcal{F})$ or of $\mathcal{L}$. Moreover, in all the above examples of strongly local Dirichlet forms the heat kernel satisfies the following estimates

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x, y)}{t}\right)^{\frac{1}{\beta-1}}\right), \tag{5}$$

for all $x, y \in M$ and $t \in (0, t_0)$ for some $t_0 > 0$ [5, 8]. The sign $\asymp$ means that the both inequalities $\le$ and $\ge$ take place but possibly with different values of the positive constants c, C.

Here α is necessarily the Hausdorff dimension, while β is a new parameter that is called the *walk dimension* of the heat kernel (or that of the Dirichlet form). It can be regarded as a numerical characteristic of the differential calculus on M.

We say that a metric space (M, d) satisfies the *chain condition* (CC) if there exists a constant C such that for all $x, y \in M$ and for all $n \in \mathbb{N}$ there exists a sequence $\{x_k\}_{k=0}^n$ of points in M such that $x_0 = x$, $x_n = y$, and

$$d(x_{k-1}, x_k) \le C\frac{d(x, y)}{n}, \quad \text{for all } k = 1, \ldots, n.$$

For example, if the metric d is geodesic then this condition is satisfied with $C = 1$.

Assume that (5) holds with $t_0 = \infty$. By Ref. [33], (5) implies (CC), while by Ref. [20], (CC) together with (5) yields

$$\alpha \ge 1 \ \text{ and } 2 \le \beta \le \alpha + 1. \tag{6}$$

Conversely, it was shown by Barlow [2], that any pair (α, β) satisfying (6) can be realized in the estimate (5) on a geodesic metric space.

Hence, we obtain a large family of metric measure spaces, each of them being characterized by a pair (α, β) where α is responsible for integration while β is responsible for differentiation. The Euclidean space $\mathbb{R}^n$ belongs to this family with $\alpha = n$ and $\beta = 2$. In the case $\beta = 2$ the estimate (5) is called *Gaussian*, while in the case $\beta > 2$—*sub-Gaussian*.

On fractals the value of β is determined by the scaling parameter ρ. It is known that:

- on SG: $\beta = \frac{\log 5}{\log 2} \approx 2.32$
- on VS: $\beta = \frac{\log 15}{\log 3} \approx 2.46$
- on SC: $\beta = \frac{\log(8\rho)}{\log 3}$ where the exact value of ρ is unknown; the approximation $\rho \approx 1.25$ indicates that $\beta \approx 2.10$.

The walk dimension β has the following probabilistic meaning. Denote by τ_Ω the first exit time of X_t from an open set $\Omega \subset M$, that is,

$$\tau_\Omega = \inf\{t > 0 : X_t \notin \Omega\}$$

(Fig. 9). Then in the above setting, for any ball $B(x, r)$ with $r < \text{const}\, t_0^{1/\beta}$ we have

$$\mathbb{E}_x \tau_{B(x,r)} \simeq r^\beta.$$

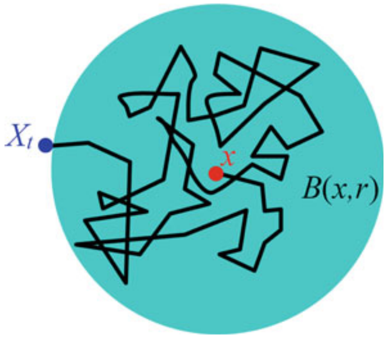


Fig. 9 Exit from a ball $B(x, r)$

6 Besov Spaces and Characterization of β

Given an α-regular metric measure space (M, d, μ), it is possible to define a family $B^{\sigma}_{p,q}$ of *Besov spaces* (see [18]). However, here we need only the following special cases: for any $\sigma > 0$ the space $B^{\sigma}_{2,2}$ consists of functions such that

$$\|f\|^2_{B^{\sigma}_{2,2}} := \|f\|^2_2 + \int\int_{M\times M} \frac{|f(x) - f(y)|^2}{d(x, y)^{\alpha+2\sigma}} d\mu(x)d\mu(y) < \infty$$

and $B^{\sigma}_{2,\infty}$ consists of functions such that

$$\|f\|^2_{B^{\sigma}_{2,\infty}} := \|f\|^2_2 + \sup_{0<r<1} \frac{1}{r^{\alpha+2\sigma}} \int\int_{\{d(x,y)<r\}} |f(x) - f(y)|^2 d\mu(x)d\mu(y) < \infty.$$

It is easy to see that $B^{\sigma}_{2,2}$ shrinks as σ increases and that in the case $\sigma < 1$ the space $B^{\sigma}_{2,2}$ contains the space Lip_0 of compactly supported Lipschitz functions. In $\mathbb{R}^n$ the space $B^{\sigma}_{2,2}$ becomes $\{0\}$ if $\sigma > 1$, so that for $\sigma > 1$ the definition of the Besov spaces in $\mathbb{R}^n$ changes. However, in our setting we are interested in the borderline value of σ at which the space $B^{\sigma}_{2,2}$ degenerates. Hence, define the critical value of the parameter σ by

$$\sigma_{crit} := \sup\left\{\sigma > 0 : B^{\sigma}_{2,2} \text{ is dense in } L^2\right\}. \tag{7}$$

In the next theorem, (M, d, μ) is a metric measure space with relatively compact balls.

Theorem 1 *[20] Let* $(\mathcal{E}, \mathcal{F})$ *be a Dirichlet form on* $L^2(M, \mu)$ *such that its heat kernel exists and satisfies for some* $\alpha > 0$, $\beta > 1$ *the sub-Gaussian estimate*

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x, y)}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{8}$$

for all $t > 0$ *and* μ*-almost all* $x, y \in M$. *Then the following is true:*

1. *the space* (M, d, μ) *is* α*-regular,* $\alpha = \dim_H M$ *and* $\mu \simeq \mathcal{H}_\alpha$*;*
2. $\sigma_{crit} = \beta/2$ *(consequently,* $\beta \geq 2$*);*
3. $\mathcal{F} = B_{2,\infty}^{\beta/2}$ *and* $\mathcal{E}(f, f) \simeq \|f\|^2_{B_{2,\infty}^{\beta/2}}$.

Partial results in this direction were previously obtained by Jonsson [25] and Pietruska-Paluba [34].

Corollary 2 *Both* α *and* β *in (8) are invariants of the metric structure* (M, d) *alone.*

Note that the value σ_{crit} is defined by (7) for any α-regular metric space. In the view of Theorem 1 it makes sense to redefine the notion of the walk dimension simply as $2\sigma_{crit}$. In this way, the walk dimension becomes a second important invariant of any regular metric space, after the Hausdorff dimension.

An open question *Let* (M, d, μ) *be an* α*-regular metric measure space (even self-similar). Assume that* $\sigma_{crit} < \infty$ *and set* $\beta = 2\sigma_{crit}$. *When and how can one construct a strongly local Dirichlet form on* $L^2(M, \mu)$ *with the heat kernel satisfying* (8)?

The result of [11] hints that such a Dirichlet form is not always possible.

7 Dichotomy of Self-similar Heat Kernels

Let (M, d) be a metric space where all metric balls are relatively compact, and let μ be a Radon measure on M with full support. A Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(M, \mu)$ is called *conservative* if its heat semigroup satisfies $e^{-t\mathcal{L}}1 \equiv 1$.

Theorem 3 *[22] Assume that* (M, d) *satisfies in addition the chain condition* (CC) *(see Sect. 5). Let* $(\mathcal{E}, \mathcal{F})$ *be a regular conservative Dirichlet form on* $L^2(M, \mu)$ *and assume that the heat kernel of* $(\mathcal{E}, \mathcal{F})$ *satisfies for all* $t > 0$ *and* $x, y \in M$ *the estimate*

$$p_t(x, y) \asymp \frac{C}{t^{\alpha/\beta}} \Phi\left(c\frac{d(x, y)}{t^{1/\beta}}\right),$$

where $\alpha, \beta > 0$ *and* Φ *is a positive monotone decreasing function on* $[0, \infty)$. *Then* (M, d, μ) *is* α*-regular,* $\beta \leq \alpha + 1$, *and the following dichotomy holds:*

- *either the Dirichlet form* $\mathcal{E}$ *is strongly local,* $\beta \geq 2$*, and*

$$\Phi(s) \asymp C \exp\left(-cs^{\frac{\beta}{\beta-1}}\right),$$

- *or the Dirichlet form $\mathcal{E}$ is non-local and*

$$\Phi(s) \simeq (1+s)^{-(\alpha+\beta)}.$$

That is, in the first case $p_t(x,y)$ satisfies the sub-Gaussian estimate

$$p_t(x,y) \asymp \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{9}$$

while in the second case we obtain a *stable-like estimate*

$$p_t(x,y) \simeq \frac{1}{t^{\alpha/\beta}}\left(1+\frac{d(x,y)}{t^{1/\beta}}\right)^{-(\alpha+\beta)} \simeq \min\left(\frac{1}{t^{\alpha/\beta}}, \frac{t}{d(x,y)^{\alpha+\beta}}\right). \tag{10}$$

8 Estimating Heat Kernels: Strongly Local Case

Let (M,d,μ) be an α-regular metric measure space. Let $(\mathcal{E},\mathcal{F})$ be a strongly local regular Dirichlet form on $L^2(M,\mu)$. For any Borel set $E \subset M$ and any $f \in \mathcal{F}$ denote

$$\mathcal{E}_E(f,f) = \int_E d\nu_{\langle f\rangle},$$

where $\nu_{\langle f\rangle}$ is the energy measure of f (see [15, p. 123]). For example, in $\mathbb{R}^n$ with the classical Dirichlet form (2) we have $d\nu_{\langle f\rangle} = |\nabla f|^2\,dx$.

Definition We say that $(\mathcal{E},\mathcal{F})$ satisfies the *Poincaré inequality* with exponent β if, for any ball $B = B(x,r)$ on M and for any function $f \in \mathcal{F}$,

$$\mathcal{E}_B(f,f) \geq \frac{c}{r^\beta}\int_{\varepsilon B}\left(f-\overline{f}\right)^2 d\mu, \tag{PI}$$

where $\varepsilon B = B(x,\varepsilon r)$, $\overline{f} = \frac{1}{\mu(\varepsilon B)}\int_{\varepsilon B} f\,d\mu$, and c,ε are small positive constants independent of B and f. For example, in $\mathbb{R}^n$ (PI) holds with $\beta = 2$ and $\varepsilon = 1$.

Let $A \Subset B$ be two open subsets of M. Define the capacity of the capacitor (A,B) as follows:

$$\operatorname{cap}(A,B) := \inf\left\{\mathcal{E}(\varphi,\varphi) : \varphi \in \mathcal{F},\ \varphi|_{\overline{A}} = 1,\ \operatorname{supp}\varphi \Subset B\right\}.$$

Here $E \Subset B$ means that the closure $\overline{E}$ of E is a compact set and $\overline{E} \subset B$.

Definition We say that $(\mathcal{E}, \mathcal{F})$ satisfies the *capacity condition* if, for any two concentric balls $B_0 := B(x, R)$ and $B := B(x, R + r)$,

$$\operatorname{cap}(B_0, B) \leq C \frac{\mu(B)}{r^\beta}. \tag{cap}$$

For any function $u \in L^\infty \cap \mathcal{F}$ and a real number $\kappa \geq 1$ define the *generalized capacity* $\operatorname{cap}_u^{(\kappa)}(A, B)$ by

$$\operatorname{cap}_u^{(\kappa)}(A, B) = \inf \left\{ \mathcal{E}\left(u^2\varphi, \varphi\right) : \varphi \in \mathcal{F},\ 0 \leq \varphi \leq \kappa,\ \ \varphi|_{\overline{A}} \geq 1,\ \varphi = 0 \text{ in } B^c \right\}.$$

If $u \equiv 1$ then $\operatorname{cap}_u^{(\kappa)}(A, B) = \operatorname{cap}(A, B)$.

Definition We say that the *generalized capacity condition* (Gcap) holds if there exist $\kappa \geq 1$ and $C > 0$ such that, for any $u \in \mathcal{F} \cap L^\infty$ and for any two concentric balls $B_0 := B(x, R)$ and $B := B(x, R + r)$,

$$\operatorname{cap}_u^{(\kappa)}(B_0, B) \leq \frac{C}{r^\beta} \int_B u^2 d\mu. \tag{Gcap}$$

Theorem 4 *[21] The following equivalence takes place*

$$(CC) + (PI) + (\text{Gcap}) \Leftrightarrow (9). \tag{11}$$

In fact, this result was proved in [21] in a slightly weaker form: assuming the chain condition (CC), we have the equivalence

$$(PI) + (\text{Gcap}) \Leftrightarrow (9).$$

It was later proved by Murugan [33] that

$$(9) \Rightarrow (CC),$$

whence (11) follows. Besides, the condition (Gcap) was formulated in [21] in a different, more complicated form. The present form of (Gcap) was introduced in [19].

The main open question in this field is whether the following conjecture is true.

Conjecture $(CC) + (PI) + (\text{cap}) \Leftrightarrow (9)$.

The implication $\Leftarrow$ clearly is true by Theorem 4, so the main difficulty is in the implication $\Rightarrow$.

9 Estimating Heat Kernels: Jump Case

Let (M, d, μ) be an α-regular metric measure space. Let now $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form of jump type on $L^2(M, \mu)$, that is,

$$\mathcal{E}(f, f) = \iint_{M \times M} (f(x) - f(y))^2 J(x, y) d\mu(x) d\mu(y)$$

for all $f \in \mathcal{F} \cap C_0(M)$. Here $J(x, y)$ is a symmetric non-negative function in $M \times M$ that is called the *jump kernel* of $(\mathcal{E}, \mathcal{F})$.

We use the following condition instead of the Poincaré inequality:

$$J(x, y) \simeq d(x, y)^{-(\alpha+\beta)}. \tag{J}$$

Theorem 5 *[14] and [19]*

$$(J) + (\mathrm{Gcap}) \Leftrightarrow (10).$$

In the case $\beta < 2$ it is easy to show that (J)$\Rightarrow$ (Gcap) so that in this case we obtain the equivalence

$$(J) \Leftrightarrow (10).$$

The latter was also shown by Chen and Kumagai [13], although under some additional assumptions about the metric structure of (M, d).

Conjecture $(J) + (\mathrm{cap}) \Leftrightarrow (10)$.

10 Ultra-metric Spaces

Let (M, d) be a metric space. The metric d is called an *ultra-metric* and (M, d) is called an *ultra-metric space* if, for all $x, y, z \in M$,

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}. \tag{12}$$

A famous example of an ultra-metric space is the field $\mathbb{Q}_p$ of p-adic numbers endowed with the p-adic distance (here p is a prime). Also $\mathbb{Q}_p^n$ is an ultra-metric space with an appropriate choice of a metric. Denoting by μ the Haar measure on $\mathbb{Q}_p^n$, we have $\mu(B(x, r)) \simeq r^n$ so that $\mathbb{Q}_p^n$ is n-regular.

Ultra-metric spaces are totally disconnected and, hence, cannot carry non-trivial strongly local regular Dirichlet forms. However, it is easy to build jump type forms. Let (M, d) be an ultra-metric space where all balls are relatively compact, and let μ be a Radon measure on M with full support. Let us fix a cumulative probability distribution function $\phi(r)$ on $(0, \infty)$ that is strictly monotone increasing and continuous, and consider on $M \times M$ the function

$$J(x, y) = \int_{d(x,y)}^{\infty} \frac{d \log \phi(r)}{\mu(B_r(x))}, \tag{13}$$

where the integration is done over the interval $[d(x, y), \infty)$ against the Lebesgue-Stieltjes measure associated with the function $r \mapsto \log \phi(r)$.

Theorem 6 *[10] The jump kernel (13) determines a regular Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *in* $L^2(M, \mu)$*, and its heat kernel is*

$$p_t(x, y) = \int_{d(x,y)}^{\infty} \frac{d\phi^t(r)}{\mu(B_r(x))}. \tag{14}$$

See also [9] for further heat kernel bounds on ultra-metric spaces.

For example, take $M = \mathbb{Q}_p^n$ and

$$\phi(r) = \exp\left(-\left(\frac{p}{r}\right)^{\beta}\right), \tag{15}$$

where $\beta > 0$ is arbitrary. Then one obtains from (13) by an explicit computation that

$$J(x, y) = c_{p,n,\beta}\, d(x, y)^{-(n+\beta)} \tag{16}$$

and

$$p_t(x, y) \simeq \frac{1}{t^{n/\beta}} \left(1 + \frac{d(x, y)}{t^{1/\beta}}\right)^{-(n+\beta)}.$$

It follows that, for any $\beta > 0$, the space $B_{2,2}^{\beta/2}$ coincides with the domain of the Dirichlet form with the jump kernel (16) and, hence, is dense in L^2. Consequently, we obtain by (7) $\sigma_{crit} = \infty$ so that $\mathbb{Q}_p^n$ has the walk dimension ∞.

On Fig. 10 we represent graphically a classification of regular metric spaces according to the walk dimension $\beta = 2\sigma_{crit}$. Clearly, the Euclidean spaces $\mathbb{R}^n$ and p-adic spaces $\mathbb{Q}_p^n$ form the boundaries of this scale, and the entire interior is filled with fractal spaces.

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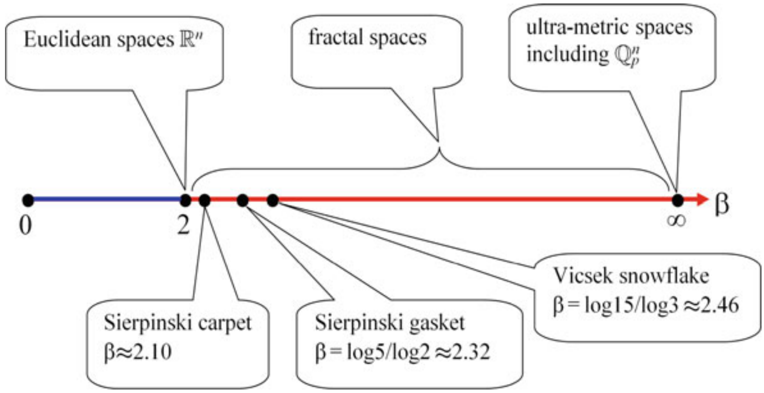

Fig. 10 Classification of regular metric spaces by the walk dimension $\beta = 2\sigma_{crit}$

References

1. M.T. Barlow, Diffusions on fractals, in *Lectures on Probability Theory and Statistics, Ecole d'été de Probabilités*, Lecture Notes in Mathematics 1690 (Springer, Berlin, 1998), pp. 1–121
2. M.T. Barlow, Which values of the volume growth and escape time exponent are possible for a graph? Revista Matemática Iberoamericana **40**, 1–31 (2004)
3. M.T. Barlow, R.F. Bass, The construction of the Brownian motion on the Sierpisnki carpet. Ann. Inst. H. Poincaré **25**, 225–257 (1989)
4. M.T. Barlow, R.F. Bass, On the resistance of the Sierpinski carpet. Proc. Roy. Soc. Lond. Ser. A **431**, 345–360 (1990)
5. M.T. Barlow, R.F. Bass, Transition densities for Brownian motion on the Sierpinski carpet. Probab. Th. Rel. Fields **91**, 307–330 (1992)
6. M.T. Barlow, R.F. Bass, T. Kumagai, A. Teplyaev, Uniqueness of Brownian motion on Sierpinski carpets. J. Eur. Math. Soc. **12**, 655–701 (2010)
7. M.T. Barlow, R.F. Bass, J.D. Sherwood, Resistance and spectral dimension of Sierpinski carpets. J. Phys. A **23**(6), 253–258 (1990)
8. M.T. Barlow, E.A. Perkins, Brownian motion on the Sierpinski gasket. Probab. Th. Rel. Fields **79**, 543–623 (1988)
9. A. Bendikov, A. Grigor'yan, E. Hu, J. Hu, Heat kernels and non-local Dirichlet forms on ultrametric spaces. Ann. Scuola Norm. Sup. Pisa **XXII**, 399–461 (2021)
10. A. Bendikov, A. Grigor'yan, Ch. Pittet, W. Woess, Isotropic Markov semigroups on ultra-metric spaces. Russian Math. Surveys **69**(4), 589–680 (2014)
11. S. Cao, H. Qiu, A Sierpinski carpet like fractal without standard self-similar energy. Preprint *arXiv preprint* arXiv:2109.12760 (2021)
12. J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces. Geom. Funct. Anal. **9**, 428–517 (1999)
13. Z.-Q. Chen, T. Kumagai, Heat kernel estimates for stable-like processes on d-sets. Stochast. Process Appl. **108**, 27–62 (2003)
14. Z.-Q. Chen, T. Kumagai, J. Wang, Stability of heat kernel estimates for symmetric non-local Dirichlet forms. Mem. Am. Math. Soc. **271** (1330) (2021)
15. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, 2nd ed. Studies in Mathematics 19 (De Gruyter, 2011)
16. S. Goldstein, Random walks and diffusion on fractals, in *Percolation Theory and Ergodic Theory on Infinite Particle Systems*, ed. by H. Kesten (Springer, New York, 1987), pp. 121–129

17. A. Grigor'yan, *Heat Kernel and Analysis on Manifolds*. AMS-IP Studies in Advanced Mathematics 47, AMS-IP (2009)
18. A. Grigor'yan, Heat kernels on metric measure spaces with regular volume growth, in *Handbook of Geometric Analysis*, vol. 2, ed. by L. Ji, P. Li, R. Schoen, L. Simon. Advanced Lectures in Mathematics 13 (International Press, 2010), pp. 1–60
19. A. Grigor'yan, E. Hu, J. Hu, Two-sided estimates of heat kernels of jump type Dirichlet forms. Adv. Math. **330**, 433–515 (2018)
20. A. Grigor'yan, J. Hu, K.-S. Lau, Heat kernels on metric measure spaces and an application to semilinear elliptic equations. Trans. Amer. Math. Soc. **355**(5), 2065–2095 (2003)
21. A. Grigor'yan, J. Hu, K.-S. Lau, Generalized capacity, Harnack inequality and heat kernels on metric spaces. J. Math. Soc. Japan **67**, 1485–1549 (2015)
22. A. Grigor'yan, T. Kumagai, On the dichotomy in the heat kernel two sided estimates, in *Proceedings of Symposia in Pure Mathematics*, vol. 77 (2008), pp. 199–210
23. A. Grigor'yan, M. Yang, Local and non-local Dirichlet forms on the Sierpinski carpet. Trans. Am. Math. Soc. **372**(6), 3985–4030 (2019)
24. J. Heinonen, P. Koskela, Quasiconformal maps in metric spaces with controlled geometry. Acta Math. **181**, 1–61 (1998)
25. A. Jonsson, Brownian motion on fractals and function spaces. Math. Zeitschrift **222**, 495–504 (1996)
26. J. Kigami, Harmonic calculus on p.c.f. self-similar sets. Trans. Am. Math. Soc. **335**, 721–755 (1993)
27. J. Kigami, *Analysis on Fractals*. Cambridge Tracts in Mathematics 143 (Cambridge University Press, Cambridge, 2001)
28. S. Kusuoka, A dissusion process on a fractal, in *Probabilistic Methods in Mathematical Physics, Taniguchi Symposium, Katana, 1985*, ed. by K. Ito, N. Ikeda (Kinokuniya-North Holland, Amsterdam, 1987), pp. 251–274
29. S. Kusuoka, X.Y. Zhou, Dirichlet forms on fractals: Poincaré constant and resistance. Probab. Th. Rel. Fields **93**, 169–196 (1992)
30. T. Lindstrøm, Brownian motion on nested fractals. Mem. Am. Math. Soc. **83**(420), 128 p (1990)
31. Z.-M. Ma, M. Röckner, *Introduction to the Theory of (Non-symmetric) Dirichlet Forms* (Universitext, Springer, Berlin, 1992)
32. B.B. Mandelbrot, *The Fractal Geometry of Nature* (W.H. Freeman, San Francisco, 1982)
33. M. Murugan, On the length of chains in a metric space. J. Funct. Anal. **279**(6), 108627 (2020)
34. K. Pietruska-Pałuba, On function spaces related to fractional diffusion on d-sets. Stochast. Stochast. Rep. **70**, 153–164 (2000)

Silverstein Extension and Fukushima Extension

Ping He and Jiangang Ying

Abstract The extension of Dirichlet forms is concerned in this semi-survey paper. The notion of Fukushima extension is introduced and it is proved that a Dirichlet extension of a Dirichlet form can be decomposed uniquely into a Silverstein extension and a Fukushima extension. Some known results on Fukushima extension of 1-dim Brownian motion are illustrated. It will be explained how the algebraic structure on Dirichlet forms plays a role. While Silverstein extension is constructed by changing only the structure of the form on the boundary, Fukushima extension is obtained by changing essentially the whole structure and much more difficult to describe.

Keywords Dirichlet forms · Extension · Silverstein extension · Fukushima extension

Mathematics Subject Classification 31C25 · 60J55

1 Introduction

In 1959, the theory of Dirichlet space was formulated by Beurling and Deny [1], which is an axiomatic extension of classical Dirichlet integrals in the direction of Markovian semigroups, and related to symmetric Markov processes naturally in terms of the regularity condition presented by Fukushima [6] in 1971. This is certainly a beautiful theory built on Hilbert space theory, which provides an alternative approach to construct Markov processes under the conditions weaker than usual methods, e.g., Feller semigroup or SDE, and characterizes the probabilistic structure

This work is dedicated to Professor Masatoshi Fukushima's Beiju.

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of stochastic processes analytically. This theory has been greatly developed since Fukushima's first edition of the book [7] was published in 1980. There are still a few interesting fundamental problems remained open. In this paper, we will explore the extension problem of Dirichlet forms, namely how to construct Dirichlet forms in terms of algebraic structure on a Dirichlet form. The Markov processes corresponding to the extensions are not so intuitive. It is known that there is a one-to-one correspondence between infinitesimal generators and Dirichlet forms. However the extension of Dirichlet forms discussed in this paper is totally different from the extension for self-adjoint operators since there is no non-trivial extension for a self-adjoint operator.

Let $(E, \mathcal{B}, m)$ be a measurable space where E is a locally compact space with countable base, $\mathcal{B}$ is the Borel σ-field on E and m is a fully supported Radon measure on $(E, \mathcal{B})$. A form $(\mathcal{E}, \mathcal{F})$ is called a Dirichlet form on $L^2(E, m)$, if it satisfies the following conditions

1. $\mathcal{E}$ is a non-negative definite symmetric bi-linear form on $\mathcal{F}$;
2. $\mathcal{F}$ is a dense subspace of $L^2(E, m)$;
3. $\mathcal{F}$ is complete with the $\mathcal{E}_1$-norm, which is defined to be

$$\|f\|_{\mathcal{E}_1} := \sqrt{\mathcal{E}(f, f) + \|f\|^2_{L^2}}, \ f \in \mathcal{F};$$

4. Markov property: any normal contraction operates on $(\mathcal{E}, \mathcal{F})$.

It is called a Dirichlet form in wide sense if only 1, 3, 4 are satisfied. The Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E, m)$ is regular if it satisfies the regularity: $C_0(E) \cap \mathcal{F}$ is dense in $C_0(E)$ with uniform norm and in $\mathcal{F}$ with $\mathcal{E}_1$-norm, where $C_0(E)$ is the space of continuous functions on E with compact support. We now fix a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E, m)$. Notice that the regularity will determine E uniquely, while without regularity, E is determined up to an m-null set.

It is due to the Markov property that the set of bounded functions, $\mathcal{F}_b$, is an algebra. In fact, for $f, g \in \mathcal{F}_b$, the product fg is a normal contraction of $\|f\|_\infty g + \|g\|_\infty f$, which is in $\mathcal{F}_b$, and it follows hence that $fg \in \mathcal{F}_b$. This is the basic algebraic structure on Dirichlet forms. It is obvious that $C_0(E) \cap \mathcal{F}$ is also an algebra. The two extensions we will talk about are related to this algebraic structure.

What is an extension? Roughly speaking, it is a Dirichlet form on the same L^2 space with a bigger domain and the same value on the smaller domain. An extension, or a Dirichlet extension more precisely, of a Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E, m)$ is another Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $L^2(E, m)$, satisfying $\widetilde{\mathcal{F}} \supset \mathcal{F}$ and for $f \in \mathcal{F}$, $\widetilde{\mathcal{E}}(f, f) = \mathcal{E}(f, f)$. In this case we dually say $(\mathcal{E}, \mathcal{F})$ is a Dirichlet sub-space of $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$.

Due to Fukushima's regular representation theorem (e.g. Theorem A.4.1 [9]), every Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E, m)$ has an equivalent regular Dirichlet form $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ on $L^2(\widehat{E}, \widehat{m})$. Precisely, there exists an algebraic isomorphism Φ from $\mathcal{F}_b$ to $\widehat{\mathcal{F}}_b$ such that for $u \in \mathcal{F}_b$,

$$\|u\|_\infty = \|\Phi(u)\|_\infty,\ \|u\|_{L^2(m)} = \|\Phi(u)\|_{L^2(\widehat{m})},\ \mathcal{E}(u,u) = \widehat{\mathcal{E}}(\Phi(u),\Phi(u)).$$

Hence we may always assume that $(\mathcal{E}, \mathcal{F})$ is regular if necessary.

Silverstein introduced in his book [18] of 1975 a kind of extension in terms of ideal, which is called Silverstein extension in [2]. In the paper [10] of 2003 another type of extension was introduced and investigated by Fukushima and Ying for the first time. In the current article, the relation between two extensions is explained. Some results on Fukushima extension involving 1-dim Brownian motion, which were obtained recently, are illustrated. It is interesting that the algebraic structure on Dirichlet forms plays a key role. While Silverstein extension is related to ideal, Fukushima extension is related to algebra.

This paper is organized as follows. In Sect. 2, we will briefly introduce Silverstein extension in terms of the structure theorems in [18]. In Sects. 3 and 4, the Fukushima extension, called regular extension before, is defined and several related results concerning this notion are surveyed. Then in Sect. 5, we shall prove that a subalgebra of a Dirichlet form induces a Fukushima subspace, while an ideal induces a Silverstein subspace. Finally in Sect. 6, the reflected Brownian motion is considered as an example in which the problem of Fukushima subspaces is revisited by a different approach.

2 Silverstein Extensions

Silverstein extension is originated from three structure theorems, more and more general in turn, which were formulated by M. Silverstein in Chapter III [18] and also in Theorem A.4.4 [9]. The first structure theorem is about reflected Dirichlet space. The second structure theorem is about ideal and includes the first as a special case. These two were discussed more specifically in Chap. 6 [2]. The third structure theorem corresponds essentially to the killing transform of Markov processes, which was clarified completely in [19, 20]. Actually it can be seen that the key in all three structure theorems is ideal, or Silverstein extension.

Let's now state the second structure explicitly. Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(E,m)$ and $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ a Dirichlet form on $L^2(E,m)$ not necessarily regular, but has a regularizing space $\widetilde{E}$. Assume also that $\mathcal{F} \subset \widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{E}}$ is the same as $\mathcal{E}$ on $\mathcal{F}$. Then $\mathcal{F}_b$ is an ideal of $\widetilde{\mathcal{F}}_b$ if and only if E can be embedded into $\widetilde{E}$ as an open subset and

$$\mathcal{F} = \{f \in \widetilde{\mathcal{F}} : f(x) = 0\ \forall x \notin E\},$$

i.e., $(\mathcal{E}, \mathcal{F})$ is the absorbed space of $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ for E, or the part of $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on the set E, as in page 108, [2]. It follows that if $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is also a regular Dirichlet form on $L^2(E,m)$, then $\widetilde{\mathcal{F}} = \mathcal{F}$. Following §6.6 in [2], we make precise what an extension is and what a Silverstein extension is.

Definition 1 Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(E, m)$.

1. A Dirichlet form $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ on $L^2(E, m)$ is called a Dirichlet extension or simply an extension of $(\mathcal{E}, \mathcal{F})$ if $\mathcal{F} \subset \widetilde{\mathcal{F}}$ and
$$\widetilde{\mathcal{E}}(f, f) = \mathcal{E}(f, f), \ \ f \in \mathcal{F}.$$
2. An extension $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ of $(\mathcal{E}, \mathcal{F})$ is called a Silverstein extension if $\mathcal{F}_b$ is an ideal in $\widetilde{\mathcal{F}}_b$.

3 Fukushima Extensions

The Silverstein extension is certainly an important notion in theory of Dirichlet forms, which characterizes killing a Markov process upon leaving a set. It is interesting to know if there are other type of extensions essentially different from Silverstein's.

Recall that $C_b(E)$ is the set of bounded continuous functions on E. We say that $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ satisfies condition (A) if $C_b(E) \cap \widetilde{\mathcal{F}}$ is dense in $\widetilde{\mathcal{F}}$ with $\widetilde{\mathcal{E}}_1$-norm.

Theorem 1 *Assume now that* $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ *is an extension of* $(\mathcal{E}, \mathcal{F})$ *and that* $(\mathcal{E}, \mathcal{F})$ *is a regular Dirichlet form on* $L^2(E, m)$. *Let* $\mathcal{F}^o$ *be the closure of* $C_0(E) \cap \widetilde{\mathcal{F}}$ *under* $\widetilde{\mathcal{E}}_1$*-norm and*
$$\mathcal{E}^o = \widetilde{\mathcal{E}}|_{\mathcal{F}^o \times \mathcal{F}^o}.$$

The following statements hold.

1. $(\mathcal{E}^o, \mathcal{F}^o)$ *is a regular Dirichlet form on* $L^2(E, m)$.
2. $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ *is an extension of* $(\mathcal{E}^o, \mathcal{F}^o)$.
3. *If* $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ *satisfies condition (A), then* $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ *is a Silverstein extension of* $(\mathcal{E}^o, \mathcal{F}^o)$.

Proof 1. It follows from the fact $C_0(E) \cap \widetilde{\mathcal{F}} \supset C_0(E) \cap \mathcal{F}$ that $C_0(E) \cap \widetilde{\mathcal{F}}$ is dense in $C_0(E)$ and hence $(\mathcal{E}^o, \mathcal{F}^o)$ is a regular Dirichlet form on $L^2(E, m)$. 2 is obvious.

3. Now let us verify that $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is a Silverstein extension of $(\mathcal{E}^o, \mathcal{F}^o)$. Take $f \in \mathcal{F}^o_b$ and $g \in \widetilde{\mathcal{F}}_b$. There exist $f_n \in C_0(E) \cap \widetilde{\mathcal{F}}$ and $g_n \in C_b(E) \cap \widetilde{\mathcal{F}}$ such that
$$f_n \longrightarrow f, \ \text{and} \ g_n \longrightarrow g$$
both in $\widetilde{\mathcal{E}}_1$-norm. Then $f_n g_n \in C_0(E) \cap \widetilde{\mathcal{F}}$ and
$$\|f_n g_n - fg\|_{\widetilde{\mathcal{E}}_1} \le \|f_n\|_\infty \|g_n - g\|_{\widetilde{\mathcal{E}}_1} + \|g\|_\infty \|f_n - f\|_{\widetilde{\mathcal{E}}_1}.$$

Since f is bounded, we may also assume that $\sup_n \|f_n\|_\infty < \infty$ by Theorem 1.4.2(v) [9]. Hence $f_n g_n \longrightarrow fg$ in $\widetilde{\mathcal{E}}_1$-norm. Then $fg \in \mathcal{F}^o_b$, i.e., $\mathcal{F}^o_b$ is an ideal of $\widetilde{\mathcal{F}}_b$.

Remark 1 It is easy to see that if $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ admits an algebra L satisfying (L) and (L.4) in Lemma A.4.6 [9], then it satisfies (A). Conversely, the condition: $L^1(E, m) \cap$

$C_b(E) \cap \widetilde{\mathcal{F}}$ is dense in $\widetilde{\mathcal{F}}$ under $\widetilde{\mathcal{E}}_1$-norm, which is slightly stronger than (A), together with the condition that $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ is an extension of a regular Dirichlet form on $L^2(E, m)$, guarantees that $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ admits an algebra L, for example the closure of $L^1(E, m) \cap C_b(E) \cap \widetilde{\mathcal{F}}$ in $L^\infty(E, m)$, satisfying (L) and (L.4) in Lemma A.4.6 [9].

On the other hand, it is easy to see that $(\mathcal{E}^o, \mathcal{F}^o)$ is still a Dirichlet extension of $(\mathcal{E}, \mathcal{F})$. However, noticing that they both are regular on $L^2(E, m)$, it is no longer a Silverstein extension unless they coincide.

Definition 2 An extension $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ of $(\mathcal{E}, \mathcal{F})$ is called a Fukushima extension (regular extension previously) if it is also a regular Dirichlet form on $L^2(E, m)$. In this case $(\mathcal{E}, \mathcal{F})$ is called a Fukushima subspace (regular subspace previously) of $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$.

It follows that an extension can be decomposed into two kinds of extensions: Silverstein extension ($\widetilde{\mathcal{F}} \supset \mathcal{F}^o$) and Fukushima extension ($\mathcal{F}^o \supset \mathcal{F}$), where the extension $(\mathcal{E}^o, \mathcal{F}^o)$ is uniquely determined. There is no non-trivial Silverstein extension of $(\mathcal{E}, \mathcal{F})$ which is also Fukushima extension, or equivalently the intersection of Silverstein extension and Fukushima extension is trivial.

Roughly speaking, Silverstein extension extends the Dirichlet form through the boundary and does not change the intrinsic structure of the form, while Fukushima extension really changes the structure of the form as we will see in next section. Intuitively, Silverstein and Fukushima extension may be called exterior and interior extension respectively.

4 Examples of Fukushima Extensions

The next problem is whether or not there are non-trivial Fukushima extensions or subspaces for a given regular Dirichlet form. It was shown in [13] that the Dirichlet forms associated to step processes, including compound Poisson processes, has no nontrivial Fukushima subspaces. However the answer is positive for the Dirichlet form $(\mathcal{E}, \mathcal{F})$ associated to one-dimensional Brownian motion, which was first considered and solved by two joint papers [3, 10][1] of the second author and M. Fukushima. Afterwards, the problem in the case of 1-dim diffusions, thanks to the perfect theory (cf. [11]), was solved almost completely in a series papers including [5, 15–17]. Let us briefly recall the related results now.

[1] A remark by J. Ying: In 2000, I visited Professor Fukushima, who was retired from Osaka University and worked at Kansai University. During a discussion, I mentioned this problem which I had thought about since I finished the paper [20] in 1996 which proves the equivalence of strong subordination and killing transform. We decided to study the special case of 1-dim Brownian motion and Professor Fukushima proposed an elegant idea which leads to the breakthrough. But we made an elementary mistake and reached a conclusion that Brownian motion has no non-trivial regular subspaces. After the paper published, I asked a student of mine, Xing Fang, to extend this result to 1-dim diffusions. When he talked about this result to me, we caught up the mistake in [10]. Then the current result in the joint paper [3] was obtained. Therefore it would be suitable to name what we call regular extensions/subspaces in [10] as Fukushima extensions/subspaces.

Let $B = (B_t)$ be one-dimensional Brownian motion with Dirichlet form $(\mathcal{E}, \mathcal{F})$ where

$$\mathcal{F} = H^1(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : f' \in L^2(\mathbb{R})\};$$
$$\mathcal{E}(f, f) = \frac{1}{2}\int (f'(x))^2 dx, \ f \in \mathcal{F},$$

where dx is the Lebesgue measure. We introduce more notations to express Fukushima extensions/subspaces. For an interval $I = \langle a, b\rangle$, where a, b could be included or not, a strictly increasing and continuous function s on I is called a scale function on I, and moreover it is called adapted to I provided

$$\begin{cases} s(a+) = -\infty, & \text{if } a \notin I, a > -\infty, \\ s(b-) = +\infty, & \text{if } b \notin I, b < +\infty. \end{cases}$$

Define a form

$$\mathcal{F}^{(s)} = \{f \in L^2(I) : \ f \ll s, \frac{df}{ds} \in L^2(I, ds)\};$$
$$\mathcal{E}^{(s)}(f, f) = \frac{1}{2}\int_I \left(\frac{df(x)}{ds}\right)^2 ds, \ f \in \mathcal{F}^{(s)}.$$

Actually $(\mathcal{E}^{(s)}, \mathcal{F}^{(s)})$ is a regular Dirichlet form on $L^2(I)$ and the irreducible associated process is the 1-dim diffusion with scale function s and speed measure dx (cf. [5][2]). When s is a linear function on I, it is called natural scale. The following theorem was shown in [3].

Theorem 2 *$(\mathcal{E}', \mathcal{F}')$ is a Fukushima subspace if and only if there exists a strictly increasing and continuous function $s \in \mathcal{F}$ which satisfies $s' = 0$ or 1 a.e. such that*

$$(\mathcal{E}', \mathcal{F}') = (\mathcal{E}^{(s)}, \mathcal{F}^{(s)}).$$

Let us now focus on Fukushima extensions of Brownian motion. At first it was proved in [13] that the jumping measure and killing measure of a regular Dirichlet form will be inherited by its Fukushima subspaces/extensions. Hence a Fukushima extension of Brownian motion is still strongly local. When $I = \mathbb{R}$, could we carefully choose s so that $(\mathcal{E}^{(s)}, \mathcal{F}^{(s)})$ is a Fukushima extension of Brownian motion? If it is true, then for any $f \in \mathcal{F} = H^1(\mathbb{R})$, $f \in \mathcal{F}^{(s)}$ and

$$\int \left(\frac{df(x)}{ds}\right)^2 ds = \int (f'(x))^2 dx.$$

[2] This result first appeared in X. Fang's PhD thesis in 2004. A similar result also appears in [8].

It follows that $dx \ll ds$ and

$$\int \left(f'(x) \cdot \frac{dx}{ds} \right)^2 ds = \int (f'(x))^2 dx.$$

We have

$$\frac{dx}{ds} dx = \left(\frac{dx}{ds} \right)^2 ds = dx = \frac{dx}{ds} ds,$$

and

$$\frac{dx}{ds} = 1, \text{ a.e.-}dx \text{ or equivalently } \frac{dx}{ds} = 0 \text{ or } 1 \text{ a.e.-}ds.$$

Then we have the result which is taken from [5].

Theorem 3 *An irreducible regular Dirichlet form* $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ *on* $L^2(\mathbb{R})$ *is a Fukushima extension of* $(\mathcal{E}, \mathcal{F})$*, Brownian motion, if and only if* $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}) = (\mathcal{E}^{(s)}, \mathcal{F}^{(s)})$ *with the scale function* s *on* $\mathbb{R}$ *satisfying*

$$dx \ll ds, \text{ and } \frac{dx}{ds} = 1 \text{ a.e.-}dx.$$

A continuous and increasing function is called of Cantor's type, if it has zero derivative a.e. It is obvious that a Fukushima extension in theorem above corresponds uniquely to a Cantor's type function c (up to a constant difference) in the form of $s(x) = x + c(x)$. Intuitively we may imagine a particle moving on a line with resistance governed by scale function s. For instance, Brownian motion has uniform resistance. The resistance related to the extension could be infinite on a set of zero measure. It is known that the generator of Fukushima extension $(\mathcal{E}^{(s)}, \mathcal{F}^{(s)})$ is the closure of the operator (with a proper domain)

$$L = \frac{1}{2} \frac{d}{dx} \frac{d}{ds(x)},$$

which differs from the generator of Brownian motion.

However the following two examples will tell us that a Fukushima extension contains indeed much richer structure. At first, a Fukushima extension of Brownian motion may even not be irreducible. In order to characterize all Fukushima extensions for Brownian motion we need first to formulate all strongly local Dirichlet forms. The following theorem, taken from [17], establishes a representation theorem and characterizes general Fukushima extension of Brownian motion, which shows much more structures.

Theorem 4 $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ *is a strongly local and regular Dirichlet form on* $L^2(\mathbb{R})$ *if and only if there exist at most countable disjoint intervals* $\{I_n\}$ *with an adapted scale function* s_n *on each* I_n *such that*

$$\begin{cases} \widetilde{\mathcal{F}} = \{f \in L^2(\mathbb{R}) : f|_{I_n} \in \mathcal{F}^{(s_n)},\ \forall n\}; \\ \widetilde{\mathcal{E}}(f, f) = \sum_n \mathcal{E}^{(s_n)}(f|_{I_n}, f|_{I_n}),\ f \in \widetilde{\mathcal{F}}. \end{cases}$$

Moreover $(\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}})$ *is a Fukushima extension of* $(\mathcal{E}, \mathcal{F})$*, Brownian motion, if and only if*

(a) $\mathbb{R} \setminus (\cup_n I_n)$ *is a null set,*
(b) $dx \ll ds_n$ *and* $\dfrac{dx}{ds_n(x)} = 1$ *a.e.*

The theorem tells us that a Fukushima extension (or its associated process) of Brownian motion may not be irreducible and can be decomposed into irreducible components I_n on which a 1-dim diffusion lives with scale function s_n, while the process stays still elsewhere. Intuitively there are not only infinite resistance but also insulator on a null set. An interesting example is a sequence of mutually disjoint closed intervals $I_n = [a_n, b_n] \cap \mathbb{R}$ with natural scale for each n such that $\mathbb{R} \setminus (\bigcup_{n\geq 1} I_n)$ is null. In this case, the diffusion corresponding to Fukushima extension is a reflected BM on each I_n and stays still elsewhere. For general 1-dim regular local Dirichlet forms, it is discussed in [12] how to obtain a Fukushima extension through a series of routine operations.

Secondly another example shows that a pure jump regular Dirichlet form has a Fukushima extension having strongly local part. Assume that $(\mathcal{E}, \mathcal{F})$ is the Dirichlet form of Brownian motion and $G = \bigcup_n O_n$ is a dense open set with $F = G^c$ being of positive measure, where $O_n = (a_n, b_n)$ is a sequence of disjoint open intervals. Then

$$s(x) = \int_0^x 1_G(y)dy$$

is a scale function such that $(\mathcal{E}^{(s)}, \mathcal{F}^{(s)})$ is a Fukushima subspace of $(\mathcal{E}, \mathcal{F})$, Fukushima extension conversely. We consider now the trace of $(\mathcal{E}, \mathcal{F})$ on F, denoted by $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$, and the trace of $(\mathcal{E}^{(s)}, \mathcal{F}^{(s)})$ also on F, denoted by $(\widehat{\mathcal{E}}^{(s)}, \widehat{\mathcal{F}}^{(s)})$. The following theorem is taken from [15].

Theorem 5 *Let* $\mu(dx) = 1_F(x)dx$.

(a) $(\widehat{\mathcal{E}}, \widehat{\mathcal{F}})$ *is Fukushima extension of* $(\widehat{\mathcal{E}}^{(s)}, \widehat{\mathcal{F}}^{(s)})$*;*
(b) For $f \in \widehat{\mathcal{F}} = L^2(F, \mu) \cap \mathcal{F}_e$*,*

$$\widehat{\mathcal{E}}(f, f) = \frac{1}{2}\int_F (f')^2 dx + \frac{1}{2}\sum_n \frac{(f(a_n) - f(b_n))^2}{b_n - a_n};$$

(c) For $f \in \widehat{\mathcal{F}}^{(s)} = L^2(F, \mu) \cap \mathcal{F}_e^{(s)}$*,*

$$\widehat{\mathcal{E}}^{(s)}(f, f) = \frac{1}{2}\sum_n \frac{(f(a_n) - f(b_n))^2}{b_n - a_n}.$$

5 Fukushima Subspaces

It is seen that ideal defines Silverstein extension. What defines Fukushima extension/subspace? The answer is sub-algebra. Given a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E, m)$. The core $C_0(E) \cap \mathcal{F}$ is an algebra, which is a subalgebra of $\mathcal{F}_b$. We shall prove that a sub-algebra always generates a Dirichlet form in wide sense and a sub-algebra of $C_0(E) \cap \mathcal{F}$ separating points in E always generates a regular Dirichlet form by Stone-Weierstrass theorem.

There are some discussions in [4] about algebraic structure of a Dirichlet form, but not clear enough. We shall re-organize it here. A continuously differentiable function ϕ is called a smooth contraction if $\phi(0) = 0$ and $\|\phi'\|_\infty \le 1$, where $\|\cdot\|_\infty$ is the uniform norm.

Theorem 6 *Let $\mathcal{A}$ be a sub-algebra of $\mathcal{F}_b$ and and $\mathcal{G}$ the closure of $\mathcal{A}$ in $\mathcal{F}$. Then $(\mathcal{E}, \mathcal{G})$ is Markovian.*

Proof Since there is a smooth contraction satisfying (1.1.5) in [7] for any $\varepsilon > 0$, it suffices to prove that every smooth contraction operates on $(\mathcal{E}, \mathcal{G})$, namely, for any $f \in \mathcal{G}$ and a smooth contraction ψ, it holds $\psi(f) \in \mathcal{G}$.

Assume $f \in \mathcal{A}$ first. Since $\mathcal{A}$ is an algebra, for any polynomial p without constant term, $p(f) \in \mathcal{A}$. By mean-value theorem, for any $x, y \in E$,

$$|p(f(x)) - p(f(y))| \le \|p'\|_f |f(x) - f(y)|,$$

where $\|p'\|_f$ is the uniform norm of p' on the range of f. Hence, by Markovian property

$$\mathcal{E}(p(f), p(f)) \le \|p'\|_f^2 \cdot \mathcal{E}(f, f).$$

Then taking a sequence $\{p_n\}$ of polynomials such that

$$\lim_n \|p_n - \psi\|_{f,C^1} = 0,$$

where $\|\cdot\|_{f,C^1}$ denotes the C^1-norm on the range of f. It follows that

$$\mathcal{E}(p_n(f), p_n(f)) \le \|p_n'\|_f^2 \cdot \mathcal{E}(f, f),$$

and that $\{\mathcal{E}(p_n(f), p_n(f)) : n \ge 1\}$ is bounded. Then by Banach-Alaoglu theorem, there exists a subsequence of $\{p_n(f)\}$ convergent to $\psi(f)$ weakly and together with Banach-Saks theorem, we have $\psi(f) \in \mathcal{G}$.

Assume now that $f \in \mathcal{G}$. Then exists $f_n \in \mathcal{A}$ which converges to f in $\mathcal{E}_1$-norm. It follows from above result that $\psi(f_n) \in \mathcal{G}$ and

$$\mathcal{E}(\psi(f_n), \psi(f_n)) \le \mathcal{E}(f_n, f_n).$$

Another argument of Banach-Alaoglu and Banach-Saks proves $\psi(f) \in \mathcal{G}$.

If $\mathcal{A}$ be an ideal of $C_0(E) \cap \mathcal{F}$ with closure $\mathcal{G}$, then there exists an open set $G \subset E$ such that

$$\mathcal{G} = \{f \in \mathcal{F} : f(x) = 0 \; \forall x \in G^c\},$$

which is the part of $(\mathcal{E}, \mathcal{F})$ on G. Hence every ideal corresponds to a unique open subset of E.

In the case of sub-algebra it is known that a subalgebra $\mathcal{A}$ of $C_0(E) \cap \mathcal{F}$ gives birth to a topological space E^* which is obtained intuitively by shrinking those points which can not be separated in $\mathcal{A}$ to one point. Its closure $(\mathcal{E}, \mathcal{G})$ is a Dirichlet form on $L^2(E, m)$ in wide sense and a regular Dirichlet form on $L^2(E^*, m^*)$, where $m^* = m \circ q^{-1}$ where q is the quotient mapping from E to E^*. For example, given a compact subsets $K \subset E$, define

$$\mathcal{A} = \{f \in C_0(E) \cap \mathcal{F} : f|_K \text{ is constant}\}.$$

Then $\mathcal{A}$ is a subalgebra. The following result tells us how to obtain a Fukushima subspace.

Corollary 1 *Assume that $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(E, m)$. Then $(\mathcal{E}, \mathcal{G})$ is a Fukushima subspace of $(\mathcal{E}, \mathcal{F})$ if and only if $\mathcal{G}$ is the closure of a sub-algebra of $C_0(E) \cap \mathcal{F}$, which separates points of E, under $\mathcal{E}_1$-norm.*

Given a regular Dirichlet form, both Fukushima extensions and subspaces are difficult to find. However this result provides us a clue how to find its Fukushima subspaces, while finding Fukushima extensions is surely more challenging, and almost nowhere to start.

6 Example: Revisit

There has been not so much progress on Fukushima extensions and subspaces except in the case of 1-dim diffusions. The next interesting Dirichlet form in our scope is the one associated to 1-dim symmetric stable process, which is very similar to 1-dim Brownian motion in certain sense. As an example, we now revisit the problem how to find Fukushima subspaces for 1-dim Brownian motion and see how the approach of sub-algebra in the previous section plays a role here.

Let $I = [0, 1]$ and $(\mathcal{E}, \mathcal{F})$ be the regular Dirichlet form on $L^2(I)$ associated with reflected Brownian motion on I, i.e.,

$$\mathcal{F} = \{f \in L^2(I) : f' \in L^2(I)\};$$
$$\mathcal{E}(f, f) = \int_I (f'(x))^2 dx, \;\; f \in \mathcal{F}.$$

Take a strictly increasing continuous function $s \in \mathcal{F}$ and let $\mathcal{A}$ be the algebra generated by s, or

$$\mathcal{A} = \{p(s) : \ p \text{ is a polynomial}\}.$$

Assume $s(0) = 0$ for simplicity. Then we claim that the closure of $\mathcal{A}$ is

$$\mathcal{G} = \{f \in \mathcal{F} : f \ll s\},$$

where $f \ll s$ means that there exists a $g \in L^1(ds) = L^1(s'dx)$ such that

$$f(x) - f(0) = \int_0^x g(u)ds(u).$$

In fact assume that $f \in \overline{\mathcal{A}}$, the closure of $\mathcal{A}$, and a sequence of polynomials p_n such that $p_n(s)$ converges to f in $\mathcal{E}_1$. It follows that $f \in \mathcal{F}$ and $\{(p_n(s))'\}$ is L^2-Cauchy. It implies that $\{p_n'(s)\}$ is $L^2((s')^2dx)$-Cauchy and there exists $g \in L^2((s')^2dx)$, such that

$$\int_0^1 |p_n'(s(x)) - g(x)|^2(s'(x))^2dx \longrightarrow 0.$$

This allows us to conclude that

$$\int_0^1 |(p_n(s(x)))' - g(x)s'(x)|^2dx \longrightarrow 0,$$

and $f'(x) = g(x)s'(x), \ x \in I$, i.e., $f \in \mathcal{G}$.

Conversely assume that $f \in \mathcal{G}$. It means that there exists $g \in L^2((s')^2dx)$ such that $f' = gs'$. Then $\mu(dx) = (s'(x))^2dx$ is a finite measure and there exists a sequence of polynomials $\{q_n\}$ such that

$$\int_0^{s(1)} \left(q_n(x) - g(s^{-1}(x))\right)^2 \mu \circ s^{-1}(dx) \longrightarrow 0.$$

Define

$$p_n(x) := \int_0^x q_n(u)du + f(0).$$

We have

$$\mathcal{E}(p_n(s) - f, p_n(s) - f) \longrightarrow 0.$$

It follows from the inequality (2.2.32) in [2] and the fact that $p_n(s(0)) = f(0)$ that

$$\sup_{x\in I}(p_n(s(x)) - f(x))^2 \leq \mathcal{E}(p_n(s) - f, p_n(s) - f).$$

This implies that $p_n(s)$ converges to f uniformly and also in $L^2(I)$. That completes the proof.

When is $\mathcal{G}$ a Fukushima subspace of $(\mathcal{E}, \mathcal{F})$? Let

$$K = \{x \in [0, 1] : s'(x) = 0\}.$$

Then we claim that $(\mathcal{E}, \mathcal{G})$ is a Fukushima subspace of $(\mathcal{E}, \mathcal{F})$ if and only if $|K| > 0$.

Indeed, set $j(x) = x,\ x \in [0, 1]$. If $|K| > 0$, every function in $\mathcal{G}$ has derivative 0 on K and it implies that $\mathcal{G} \neq \mathcal{F}$ because $j \notin \mathcal{G}$. Assume conversely $|K| = 0$. Then $(s')^{-1} \cdot s' = 1$ a.e. dx, and

$$x = \int_0^x du = \int_0^x (s')^{-1} s' du = \int_0^x \frac{1}{s'(u)} ds(u).$$

It follows that j is absolutely continuous with respect to s, i.e., $j \in \mathcal{G}$. This implies that $\mathcal{G} = \mathcal{F}$.

Silverstein extension may be viewed as boundary theory because it is about how to extend a Dirichlet form through its boundary and does not change the basic structure of the form. For example, the reflected Brownian motion on [0, 1] is a Silverstein extension of the Brownian motion absorbed at its boundary. Both are still Brownian motion on the interior of [0, 1]. However we know very little about Fukushima extensions/subspaces, which dramatically change the structure as seen in the previous examples. The problems we may try to answer include the following two basic ones: (1) whether or not are there always non-trivial Fukushima extensions for a given regular Dirichlet form? (2) If the answer for (1) is positive, how to characterize them? Up to now, we have made it clear for 1-dim diffusions. Many problems are still open, for example, the existence and characterization for Fukushima extensions of symmetric α-stable processes and multi-dimensional Brownian motions.

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References

1. A. Beurling, J. Deny, Dirichlet spaces. Proc. Natl. Acad. Sci. USA **45**, 208–215 (1959)
2. Z.-Q. Chen, M. Fukushima, *Symmetric Markov Processes, Time Change, and Boundary Theory* (Princeton University Press, Princeton, NJ, 2012)

3. X. Fang, M. Fukushima, J. Ying, On regular Dirichlet subspaces of $H^1(I)$ and associated linear diffusions. Osaka J. Math. **42**, 27–41 (2005)
4. X. Fang, P. He, J. Ying, Algebraic structure on Dirichlet spaces. Acta Math. Sin. (Engl. Ser.) **22**(3), 723–728 (2006)
5. X. Fang, P. He, J. Ying, Dirichlet forms associated with linear diffusions. Chin. Ann. Math. Ser. B **31**, 507–518 (2010)
6. M. Fukushima, Regular representations of Dirichlet spaces. Trans. Amer. Math. Soc. **155**, 455–473 (1971)
7. M. Fukushima, *Dirichlet Forms and Markov Processes* (North-Holland, 1980)
8. M. Fukushima, From one dimensional diffusions to symmetric Markov processes. Stoch. Process. Appl. **120**, 590–604 (2010)
9. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes* (Walter de Gruyter & Co., Berlin, 2011)
10. M. Fukushima, J. Ying, A note on regular Dirichlet subspaces. Proc. Amer. Math. Soc. **131**(5), 1607–1610 (2003)
11. K. Itô, H.P. McKean Jr., *Diffusion Processes and Their Sample Paths* (Springer, Berlin/New York, 1974)
12. L. Li, W. Sun, J. Ying, Effective intervals and regular Dirichlet subspaces. Stoch. Process. Appl. **130**(10), 6064–6093 (2020)
13. L. Li, J. Ying, Regular subspaces of Dirichlet forms, in *Festschrift Masatoshi Fukushima* (World Sci. Publ., Hackensack, NJ, 2015), pp. 397–420
14. L. Li, J. Ying, Killing transform on regular Dirichlet subspaces. Potential Anal. **46**, 105–118 (2017)
15. L. Li, J. Ying, On structure of regular Dirichlet subspaces for one-dimensional Brownian motion. Ann. Probab. **45**, 2631–2654 (2017)
16. L. Li, J. Ying, Regular Dirichlet extensions of one-dimensional Brownian motion. Ann. Inst. Henri Poincaré Probab. Stat. **4**, 1815–1849 (2019)
17. L. Li, J. Ying, On symmetric linear diffusions. Trans. Amer. Math. Soc. **371**(8), 5841–5874 (2019)
18. M. Silverstein, *Symmetric Markov Processes*, Lecture Notes in Mathematics, vol. 426 (Springer, 1974)
19. J. Ying, Bivariate Revuz measures and the Feynman-Kac formula. Ann. Inst. H. Poincaré Probab. Stat. **32**(2), 250–287 (1996)
20. J. Ying, Killing and subordination. Proc. Amer. Math. Soc. **124**(7), 2215–2222 (1996)

Singularity of Energy Measures on a Class of Inhomogeneous Sierpinski Gaskets

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Abstract We study energy measures of canonical Dirichlet forms on inhomogeneous Sierpinski gaskets. We prove that the energy measures and suitable reference measures are mutually singular under mild assumptions.

Keywords Fractal · Energy measure · Dirichlet form

Mathematics Subject Classification: Primary: 28A80 · Secondary: 31C25 · 60G30 · 60J60

1 Introduction

Energy measures associated with regular Dirichlet forms are fundamental concepts in stochastic analysis and related fields. For example, the intrinsic metric is defined by using energy measures and appears in Gaussian estimates of the transition probabilities. Energy measures are also crucial for describing the conditions for sub-Gaussian behaviors of transition densities. The energy measures are expected to be singular with respect to (canonical) underlying measures for canonical Dirichlet forms on self-similar fractals, which has been confirmed in many cases [4, 9, 10, 13]. Recently, such a singularity was proved under full off-diagonal sub-Gaussian estimates of the transition densities [11].

In this paper, we study a class of inhomogeneous Sierpinski gaskets as examples that have not yet been covered in the previous studies: they do not necessarily have strict self-similar structures or nice sub-Gaussian estimates. We show that the singularity of the energy measures still holds under mild assumptions. The strategy of our proof is based on quantitative estimates of probability measures on shift spaces,

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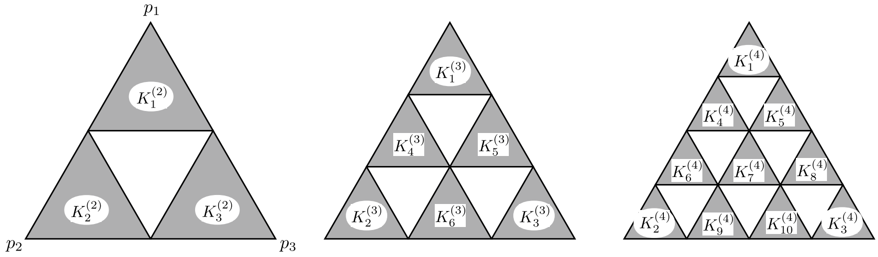

Fig. 1 $K_i^{(\nu)}$, the image of $\tilde{K}$ by the contractive affine map $\psi_i^{(\nu)}$ $(\nu = 2, 3, 4)$

the techniques of which were used in [9, 10]. We expect this study to lead to further progress in stochastic analysis of complicated spaces of this kind.

This paper is organized as follows: In Sect. 2, we introduce a class of inhomogeneous Sierpinski gaskets and canonical Dirichlet forms defined on them, and state the main results. In Sects. 3 and 4, we provide preliminary lemmas and prove the theorems. In Sect. 5, we make some concluding remarks.

2 Framework and Statement of Theorems

We begin by recalling 2-dimensional level-ν Sierpinski gaskets SG(ν) for $\nu \geq 2$. Let $N(\nu) = \nu(\nu + 1)/2$. Let $\tilde{K}$ be an equilateral triangle in $\mathbb{R}^2$ including the interior. Let $K_i^{(\nu)} \subset \tilde{K}, i = 1, 2, \dots, N(\nu)$, be equilateral triangles including the interior that are obtained by dividing the sides of $\tilde{K}$ in ν, joining these points, and removing all the downward-pointing triangles, as in Fig. 1. Let $\psi_i^{(\nu)}, i = 1, 2, \dots, N(\nu)$, be the contractive affine map from $\tilde{K}$ onto $K_i^{(\nu)}$ of type $\psi_i^{(\nu)}(z) = \nu^{-1}z + \alpha_i^{(\nu)}$ for some $\alpha_i^{(\nu)} \in \mathbb{R}^2$. Then, the 2-dimensional level-ν Sierpinski gasket SG(ν) is defined as a unique non-empty compact subset K in $\tilde{K}$ such that

$$K = \bigcup_{i=1}^{N(\nu)} \psi_i^{(\nu)}(K).$$

Let $S_0 = \{1, 2, 3\}$, and let $V_0 = \{p_1, p_2, p_3\}$ be the set of all vertices of $\tilde{K}$. In the definition of SG(ν), the labeling of $K_i^{(\nu)}$ does not matter. For later convenience, we assign $K_i^{(\nu)}$ for $i \in S_0$ to the triangle that contains p_i. As a result, $\psi_i^{(\nu)}$ has a fixed point p_i.

For a general non-empty set X, denote by $l(X)$ the set of all real-valued functions on X. When X is finite, the inner product $(\cdot, \cdot)$ on $l(X)$ is defined by

$$(x, y) = \sum_{p \in X} x(p)y(p), \quad x, y \in l(X).$$

We regard $l(X)$ as the L^2-space on X equipped with the counting measure. Then, the L^2-inner product is identical with $(\cdot,\cdot)$. The induced norm is denoted by $|\cdot|$.

A symmetric linear operator $D = (D_{p,q})_{p,q\in V_0}$ on $l(V_0)$ is defined as

$$D_{p,q} = \begin{cases} -2 & \text{if } p = q, \\ 1 & \text{otherwise.} \end{cases}$$

Let

$$Q(x, y) := (-Dx, y) = -\sum_{p,q\in V_0} D_{p,q}\, x(q) y(p)$$

for $x, y \in l(V_0)$. More explicitly,

$$\begin{aligned} Q(x, y) &= (x(p_1) - x(p_2))(y(p_1) - y(p_2)) + (x(p_2) - x(p_3))(y(p_2) - y(p_3)) \\ &\quad + (x(p_3) - x(p_1))(y(p_3) - y(p_1)). \end{aligned}$$

This is a Dirichlet form on $l(V_0)$. To simplify the notation, we sometimes write $Q(x)$ for $Q(x, x)$.

Let

$$V_1^{(\nu)} = \bigcup_{i=1}^{N(\nu)} \psi_i^{(\nu)}(V_0).$$

Let $r^{(\nu)} > 0$ and $Q^{(\nu)}$ be a symmetric bilinear form on $V_1^{(\nu)}$ that is defined by

$$Q^{(\nu)}(x, y) = \sum_{i=1}^{N(\nu)} \frac{1}{r^{(\nu)}} Q(x \circ \psi_i^{(\nu)}|_{V_0}, y \circ \psi_i^{(\nu)}|_{V_0}), \quad x, y \in l(V_1^{(\nu)}).$$

Then, there exists a unique $r^{(\nu)} > 0$ such that, for every $x \in l(V_0)$,

$$Q(x, x) = \inf\left\{ Q^{(\nu)}(y, y) \;\middle|\; y \in l(V_1^{(\nu)}) \text{ and } y|_{V_0} = x \right\}. \tag{2.1}$$

Hereafter, we fix such $r^{(\nu)}$. For example, $r^{(2)} = 3/5$, $r^{(3)} = 7/15$, and $r^{(4)} = 41/103$, which are confirmed by the concrete calculation.

For each $x \in l(V_0)$, there exists a unique $y \in l(V_1)$ that attains the infimum in (2.1). For $i = 1, 2, \dots, N(\nu)$, the map $l(V_0) \ni x \mapsto y \circ \psi_i^{(\nu)}|_{V_0} \in l(V_0)$ is linear, which is denoted by $A_i^{(\nu)}$. Then, it holds that

$$Q(x, x) = \sum_{i=1}^{N(\nu)} \frac{1}{r^{(\nu)}} Q(A_i^{(\nu)} x, A_i^{(\nu)} x), \quad x \in l(V_0). \tag{2.2}$$

We can construct a Dirichlet form on $\mathrm{SG}(\nu)$ by using such data, but we omit the explanation because we discuss it in more general situations soon.

For reference, we give a quantitative estimate of $r^{(\nu)}$.

Lemma 2.1 $1/\nu < r^{(\nu)} < N(\nu)/\nu^2$.

Proof This kind of inequality should be well-known (see, e.g., [2, Theorem 1]), and see the Proof of [11, Proposition 5.3] (and also [1, Proposition 6.30]) for the second inequality. For the first inequality, let

$$\alpha = \inf\{Q(z,z) \mid z \in l(V_0),\ z(p_1)=1,\ z(p_2)=0\} > 0. \tag{2.3}$$

Then, for general $z \in l(V_0)$,

$$Q(z,z) \geq (z(p_1)-z(p_2))^2\alpha \tag{2.4}$$

by considering $(z - z(p_2))/(z(p_1)-z(p_2))$.

The infimum of (2.3) is attained by $x \in l(V_0)$ given by $x(p_1)=1$, $x(p_2)=0$, $x(p_3)=1/2$ (and $\alpha = 3/2$). Take $y \in l(V_1^{(\nu)})$ attaining the infimum of (2.1). Let $I \subset \{1,2,\dots,N(\nu)\}$ be a ν-points set such that, for each $i \in I$, the intersection of $\psi_i^{(\nu)}(V_0)$ and the segment connecting p_1 and p_2 is a two-points set, say $\{\check{p}_i, \hat{p}_i\}$. Note that $3 \notin I$, and y is not constant on $\psi_3^{(\nu)}(V_0)$, which is confirmed by applying the maximum principle (see, e.g., [12, Proposition 2.1.7]) to the graph whose vertices are all points of $V_1^{(\nu)}$ included in the triangle with p_1, p_3 and the middle point of p_1 and p_2 as the three vertices. Therefore,

$$\begin{aligned}
\alpha &= Q^{(\nu)}(y,y)\\
&> \sum_{i\in I}\frac{1}{r^{(\nu)}}Q(y\circ\psi_i^{(\nu)}|_{V_0}, y\circ\psi_i^{(\nu)}|_{V_0})\\
&\geq \frac{1}{r^{(\nu)}}\sum_{i\in I}(y(\check{p}_i)-y(\hat{p}_i))^2\alpha \quad \text{(from (2.4))}\\
&\geq \frac{\alpha}{r^{(\nu)}}\left(\sum_{i\in I}(y(\check{p}_i)-y(\hat{p}_i))\right)^2\left(\sum_{i\in I}1\right)^{-1}\\
&= \frac{\alpha}{r^{(\nu)}}\cdot 1\cdot \nu^{-1}.
\end{aligned}$$

Thus, $1/\nu < r^{(\nu)}$. □

See also [8] for the asymptotic behavior of $r^{(\nu)}$ as $\nu \to \infty$.

We now introduce 2-dimensional inhomogeneous Sierpinski gaskets. We fix a non-empty finite subset T of $\{\nu \in \mathbb{N} \mid \nu \geq 2\}$. For each $\nu \in T$, let $S^{(\nu)}$ denote the set of the letters i^ν for $i = 1,2,\dots,N(\nu)$. We set $S = \bigcup_{\nu\in T} S^{(\nu)}$ and $\Sigma = S^{\mathbb{N}}$. For example, if $T = \{2,3\}$, then

$$S^{(2)} = \{1^2, 2^2, 3^2\}, \quad S^{(3)} = \{1^3, 2^3, 3^3, 4^3, 5^3, 6^3\},$$

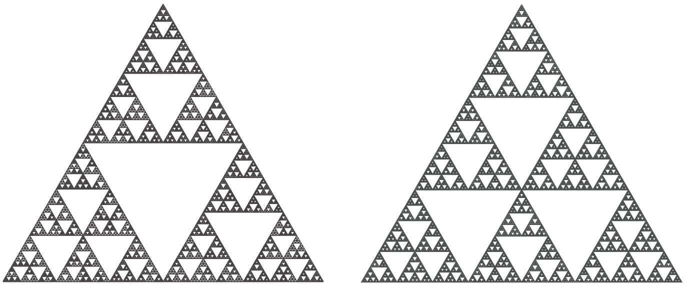

Fig. 2 Examples of inhomogeneous Sierpinski gaskets ($T = \{2, 3\}$)

and $S = S^{(2)} \cup S^{(3)}$ has nine elements. (Note that i^ν *does not mean* $\underbrace{ii\dots i}_{\nu}$, the ν-letter word consisting of only i, in this paper.)

For each $v \in S$, a shift operator $\sigma_v \colon \Sigma \to \Sigma$ is defined by $\sigma_v(\omega_1\omega_2\dots) = v\omega_1\omega_2\dots$. Let $W_0 = \{\emptyset\}$ and $W_m = S^m$ for $m \in \mathbb{N}$, and define $W_{\le n} = \bigcup_{m=0}^{n} W_m$ and $W_* = \bigcup_{m\in\mathbb{Z}_+} W_m$. Here, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. For $w \in W_m$, $|w|$ represents m and is called the length of w. For $w = w_1 \dots w_m \in W_m$ and $w' = w'_1 \dots w'_n \in W_n$, $ww' \in W_{m+n}$ denotes $w_1 \dots w_m w'_1 \dots w'_n$. Also, $\sigma_w \colon \Sigma \to \Sigma$ is defined as $\sigma_w = \sigma_{w_1} \circ \dots \circ \sigma_{w_m}$, and let $\Sigma_w = \sigma_w(\Sigma)$. For $k \le m$, $[w]_k$ denotes $w_1 \dots w_k \in W_k$. Similarly, for $\omega = \omega_1\omega_2 \dots \in \Sigma$ and $n \in \mathbb{N}$, let $[\omega]_n$ denote $\omega_1 \dots \omega_n \in W_n$. By convention, $\sigma_\emptyset \colon \Sigma \to \Sigma$ is the identity map, $[w]_0 := \emptyset \in W_0$ for $w \in W_*$, and $[\omega]_0 := \emptyset \in W_0$ for $\omega \in \Sigma$.

For $i^\nu \in S$, we define $\psi_{i^\nu} := \psi_i^{(\nu)}$ and $A_{i^\nu} := A_i^{(\nu)}$. For $w = w_1 w_2 \dots w_m \in W_m$, ψ_w denotes $\psi_{w_1} \circ \psi_{w_2} \circ \dots \circ \psi_{w_m}$ and A_w denotes $A_{w_m} \dots A_{w_2} A_{w_1}$. Here, $\psi_\emptyset$ and $A_\emptyset$ are the identity maps by definition. For $\omega \in \Sigma$, $\bigcap_{m\in\mathbb{Z}_+} \psi_{[\omega]_m}(\tilde{K})$ is a one-point set $\{p\}$. The map $\Sigma \ni \omega \mapsto p \in \tilde{K}$ is denoted by π. The relation $\psi_v \circ \pi = \pi \circ \sigma_v$ holds for $v \in S$.

Now, we fix $L = \{L_w\}_{w\in W_*} \in T^{W_*}$. That is, we assign each $w \in W_*$ to $L_w \in T$. We set $\tilde{W}_0 = \{\emptyset\}$ and

$$\tilde{W}_m = \bigcup_{w\in\tilde{W}_{m-1}} \left\{ wv \;\middle|\; v \in S^{(L_w)} \right\}$$

for $m \in \mathbb{N}$, inductively. Define $\tilde{W}_* = \bigcup_{m\in\mathbb{Z}_+} \tilde{W}_m \subset W_*$, $\tilde{\Sigma} = \{\omega \in \Sigma \mid [\omega]_m \in \tilde{W}_m$ for all $m \in \mathbb{Z}_+\}$ and $G(L) = \pi(\tilde{\Sigma})$. It holds that

$$G(L) = \bigcap_{m\in\mathbb{Z}_+} \bigcup_{w\in\tilde{W}_m} \psi_w(\tilde{K}).$$

We call $G(L)$ an inhomogeneous Sierpinski gasket generated by L. See Fig. 2 for a few examples. We equip $G(L)$ with the relative topology of $\mathbb{R}^2$. If $L_w = \nu$ for all $w \in W_*$, then $G(L)$ is nothing but $\mathrm{SG}(\nu)$.

For $m \in \mathbb{N}$, let

$$V_m = \bigcup_{w \in \tilde{W}_m} \psi_w(V_0),$$

and let $V_* = \bigcup_{m \in \mathbb{Z}_+} V_m$. The closure of V_* is equal to $G(L)$.

Next, we define reference measures on $G(L)$. Let

$$\mathcal{A}^{(\nu)} = \left\{ q = \{q_v\}_{v \in S^{(\nu)}} \;\middle|\; q_v > 0 \quad \text{for all } v \in S^{(\nu)} \text{ and } \sum_{v \in S^{(\nu)}} q_v = 1 \right\}$$

and

$$\mathcal{A} = \left\{ q = \{q_v\}_{v \in S} \;\middle|\; \text{ for each } \nu \in T, \{q_v\}_{v \in S^{(\nu)}} \in \mathcal{A}^{(\nu)} \right\}.$$

For $q \in \mathcal{A}$, there exists a unique Borel probability measure λ_q on Σ such that

$$\lambda_q(\Sigma_w) = \begin{cases} q_{w_1} \dots q_{w_m} & \text{if } w = w_1 \dots w_m \in \tilde{W}_m, \\ 0 & \text{if } w \notin \tilde{W}_*. \end{cases}$$

We note that

$$\lambda_q(\Sigma \setminus \tilde{\Sigma}) = \lim_{m \to \infty} \lambda_q \left(\Sigma \setminus \bigcup_{w \in \tilde{W}_m} \Sigma_w \right) = 0.$$

In what follows, q_w denotes $q_{w_1} \dots q_{w_m}$ for $w = w_1 \dots w_m \in W_m$. By definition, $q_\emptyset = 1$. The Borel probability measure μ_q on $G(L)$ is defined by $\mu_q = (\pi|_{\tilde{\Sigma}})_* \lambda_q$, that is, the image measure of λ_q by $\pi|_{\tilde{\Sigma}} \colon \tilde{\Sigma} \to G(L)$. It is easy to see that μ_q has full support and does not charge any one points. When $T = \{\nu\}$, μ_q is a self-similar measure on $G(L) = \mathrm{SG}(\nu)$.

We next construct a Dirichlet form on $G(L)$. Let $r_{i^\nu} = r^{(\nu)}$ for $i^\nu \in S$, and $r_w = r_{w_1} \dots r_{w_m}$ for $w = w_1 \dots w_m \in W_m$. By definition, $r_\emptyset = 1$. For $m \in \mathbb{Z}_+$, let

$$\mathcal{E}^{(m)}(x, y) = \sum_{w \in \tilde{W}_m} \frac{1}{r_w} Q(x \circ \psi_w|_{V_0}, y \circ \psi_w|_{V_0}), \quad x, y \in l(V_m).$$

From (2.1) and (2.2), it holds that for every $m \in \mathbb{Z}_+$ and $x \in l(V_m)$,

$$\mathcal{E}^{(m)}(x, x) = \inf\{\mathcal{E}^{(m+1)}(y, y) \mid y \in l(V_{m+1}) \text{ and } y|_{V_m} = x\}.$$

Thus, for any $x \in l(V_*)$, the sequence $\{\mathcal{E}^{(m)}(x|_{V_m}, x|_{V_m})\}_{m=0}^\infty$ is non-decreasing. We define

$$\mathcal{F} = \left\{ f \in C(G(L)) \;\middle|\; \lim_{m \to \infty} \mathcal{E}^{(m)}(f|_{V_m}, f|_{V_m}) < \infty \right\},$$

$$\mathcal{E}(f, g) = \lim_{m \to \infty} \mathcal{E}^{(m)}(f|_{V_m}, g|_{V_m}), \quad f, g \in \mathcal{F},$$

where $C(G(L))$ denotes the set of all real-valued continuous functions on $G(L)$. Then, $(\mathcal{E}, \mathcal{F})$ is a resistance form and also a strongly local regular Dirichlet form on $L^2(G(L), \mu_q)$ for any $q \in \mathcal{A}$ (see [7] and [12, Chapter 2]). Here, $C(G(L))$ is regarded as a subspace of $L^2(G(L), \mu_q)$. We equip $\mathcal{F}$ with the inner product $(f, g)_{\mathcal{F}} := \mathcal{E}(f, g) + \int_{G(L)} f g \, d\mu_q$ as usual.

The energy measure $\mu_{\langle f \rangle}$ of $f \in \mathcal{F}$ is a finite Borel measure on $G(L)$, which is characterized by

$$\int_{G(L)} g \, d\mu_{\langle f \rangle} = 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g), \quad g \in \mathcal{F}.$$

By letting $g \equiv 1$, the total mass of $\mu_{\langle f \rangle}$ is $2\mathcal{E}(f, f)$. Another expression of $\mu_{\langle f \rangle}$ is discussed in Sect. 3.

We introduce the following conditions for $q = \{q_v\}_{v \in S} \in \mathcal{A}$ to describe our main theorem.

(A) $q_{i^\nu} \neq r^{(\nu)}$ for all $i \in S_0$ and $\nu \in T$.
(B) For each $l_0, l_1 \in \mathbb{N}$, there exists $l_2 \in \mathbb{N}$ such that the following $(\star)$ holds for μ_q-a.e. $\omega \in \Sigma$:

$(\star)$ there exist infinitely many $k \in \mathbb{Z}_+$ such that, for every $i, j \in S_0$,

$$\begin{aligned}&[\omega]_k i^{\nu_{k+1}} \dots i^{\nu_{k+l_0}} j^{\nu_{k+l_0+1}} \dots j^{\nu_{k+l_0+l_1}} j^{\nu_{k+l_0+l_1+1}} \dots j^{\nu_{k+l_0+l_1+l_2}} \\ &\in \tilde{W}_{k+l_0+l_1+l_2}\end{aligned} \tag{2.5}$$

implies that

$$\begin{aligned}&\{\nu_m \in T \mid k + l_0 + 1 \leq m \leq k + l_0 + l_1\} \\ &\subset \{\nu_m \in T \mid k + l_0 + l_1 + 1 \leq m \leq k + l_0 + l_1 + l_2\}.\end{aligned} \tag{2.6}$$

Remark 2.2 (1) Condition $(\star)$ is meaningful only for $\omega \in \tilde{\Sigma}$.

(2) For $\omega \in \tilde{\Sigma}$, $k \in \mathbb{Z}_+$, and $i, j \in S_0$, the elements $\nu_{k+1}, \nu_{k+2}, \dots, \nu_{k+l_0+l_1+l_2} \in T$ so that (2.5) holds are uniquely determined. Indeed, $\nu_{k+1} = L_{[\omega]_k}$, $\nu_{k+2} = L_{[\omega]_k i^{\nu_{k+1}}}$, $\nu_{k+3} = L_{[\omega]_k i^{\nu_{k+1}} i^{\nu_{k+2}}}$, and so on.

(3) A simple sufficient condition for (2.6) is

$$\{\nu_m \mid k + l_0 + l_1 + 1 \leq m \leq k + l_0 + l_1 + l_2\} = T. \tag{2.7}$$

Theorem 2.3 *Let $q \in \mathcal{A}$. Suppose that Condition* (A) *or* (B) *holds. Then, $\mu_{\langle f \rangle}$ and μ_q are mutually singular for every $f \in \mathcal{F}$.*

We provide some typical examples.

Example 2.4 Let $\nu \in T$ and define $L = \{L_w\}_{w \in W_*}$ by $L_w = \nu$ for all $w \in W_*$. Then, $G(L)$ is equal to $\mathrm{SG}(\nu)$. In this case, Condition $(\star)$ is trivially satisfied for all $\omega \in \tilde{\Sigma}$

by letting $l_2 = 1$ because both sides of (2.6) are equal to $\{\nu\}$. Thus, by Theorem 2.3, $\mu_{\langle f \rangle} \perp \mu_q$ for every $f \in \mathcal{F}$ and $q \in \mathcal{A}$. This singularity has been proved in [10] already.

Example 2.5 Take any sequence $\{\tau_m\}_{m\in\mathbb{Z}_+} \in T^{\mathbb{Z}_+}$ and let $L_w = \tau_{|w|}$ for $w \in W_*$. The set $G(L)$ associated with $L = \{L_w\}_{w\in W_*}$ has been studied in, e.g., [3, 6, 11], and called a scale irregular Sierpinski gasket.

(1) Let $q = \{q_w\}_{w\in S} \in \mathcal{A}$ be given by $q_v = N(\nu)^{-1}$ for $v \in S^{(\nu)}$. The associated measure μ_q is regarded as a uniform measure on $G(L)$. Since $N(\nu)^{-1} < \nu^{-1}$, Condition (A) holds from Lemma 2.1. Therefore, $\mu_{\langle f \rangle} \perp \mu_q$ for any $f \in \mathcal{F}$ from Theorem 2.3. This case was discussed in [11, Section 5].

(2) (a) Suppose that there exists $l_2 \in \mathbb{N}$ such that $\{\tau_{k+1}, \tau_{k+2}, \ldots, \tau_{k+l_2}\} = T$ for infinitely many $k \in \mathbb{Z}_+$. Then, Condition ($\star$) is satisfied for all $\omega \in \tilde{\Sigma}$, in view of (2.7).

(b) Suppose that for each $l \in \mathbb{N}$ there exists $k \in \mathbb{Z}_+$ such that $\tau_{k+1} = \tau_{k+2} = \cdots = \tau_{k+l}$. Then, Condition ($\star$) with $l_2 = 1$ is satisfied for all $\omega \in \tilde{\Sigma}$ and $l_0, l_1 \in \mathbb{N}$, but (2.7) may fail to hold for any l_2.

In either case, $\mu_{\langle f \rangle} \perp \mu_q$ for any $f \in \mathcal{F}$ and any $q \in \mathcal{A}$ from Theorem 2.3.

Example 2.6 Let ρ be a probability measure on T with full support. We take a family of T-valued i.i.d. random variables $\{L_w(\cdot)\}_{w\in W_*}$ with distribution ρ that are defined on some probability space $(\hat{\Omega}, \hat{\mathcal{B}}, \hat{P})$. For each $\hat{\omega} \in \hat{\Omega}$, we can define an inhomogeneous Sierpinski gasket $G(L(\hat{\omega}))$ associated with $L(\hat{\omega}) := \{L_w(\hat{\omega})\}_{w\in W_*}$. This is called a random recursive Sierpinski gasket [7]. Then, the following holds.

Theorem 2.7 *For $\hat{P}$-a.s. $\hat{\omega}$, $G(L(\hat{\omega}))$ satisfies Condition* (B) *for all $q \in \mathcal{A}$. That is, for $\hat{P}$-a.s. $\hat{\omega}$, the Dirichlet form on $G(L(\hat{\omega}))$ can apply Theorem 2.3 for all $q \in \mathcal{A}$ to conclude that the energy measures and μ_q are mutually singular for all $q \in \mathcal{A}$.*

Theorems 2.3 and 2.7 are proved in Sect. 4.

3 Preliminary Lemmas

In this section, we provide the necessary concepts and lemmas for proving Theorem 2.3. We fix $L = \{L_w\}_{w\in W_*} \in T^{W_*}$ and $q \in \mathcal{A}$ and retain the notation used in the previous section.

For $w \in \tilde{W}_*$, let K_w denote $\pi(\Sigma_w \cap \tilde{\Sigma})$ $(= \psi_w(\tilde{K}) \cap G(L))$.

Let $m \in \mathbb{Z}_+$ and $x \in l(V_m)$. There exists a unique $h \in \mathcal{F}$ that attains

$$\inf\{\mathcal{E}(f, f) \mid f \in \mathcal{F} \text{ and } f|_{V_m} = x\}.$$

We call such h a piecewise harmonic (more precisely, an m-harmonic) function. When $m = 0$, h is called a harmonic function and is denoted by $\iota(x)$.

Lemma 3.1 *For $f \in \mathcal{F}$ and $m \in \mathbb{Z}_+$, let f_m be an m-harmonic function such that $f_m = f$ on V_m. Then, f_m converges to f in $\mathcal{F}$ as $m \to \infty$. In particular, the totality of piecewise harmonic functions is dense in $\mathcal{F}$.*

Proof The proof is standard. From the maximum principle (see, e.g., [12, Lemma 2.2.3]),

$$\min_{K_w} f \le \min_{\psi_w(V_0)} f = \min_{K_w} f_m \le \max_{K_w} f_m = \max_{\psi_w(V_0)} f \le \max_{K_w} f$$

for any $w \in \tilde{W}_m$. Therefore, f_m converges to f uniformly on $G(L)$, in particular, in $L^2(G(L), \mu_q)$ as $m \to \infty$. Because $\{f_m\}_{m\in\mathbb{Z}_+}$ is bounded in $\mathcal{F}$, it converges to f weakly in $\mathcal{F}$. Because $\lim_{m\to\infty}(f_m, f_m)_{\mathcal{F}} = (f, f)_{\mathcal{F}}$, f_m actually converges to f strongly in $\mathcal{F}$. □

Let $v \in W_*$. We define $L^{[v]} = \{L^{[v]}_w\}_{w\in W_*} \in T^{W_*}$ by $L^{[v]}_w = L_{vw}$. Then, we can define a strongly local regular Dirichlet form $(\mathcal{E}^{[v]}, \mathcal{F}^{[v]})$ on $L^2(G(L^{[v]}), \mu^{[v]}_q)$, where $\mu^{[v]}_q$ is defined in the same way as μ_q with L replaced by $L^{[v]}$. The energy measure of $f \in \mathcal{F}^{[v]}$ is denoted by $\mu^{[v]}_{\langle f\rangle}$. The following lemma is proved in a straightforward manner by going back to the above definition.

Lemma 3.2 (1) *Let $f \in \mathcal{F}$ and $m \in \mathbb{N}$. For each $v \in \tilde{W}_m$, $f^{[v]} := f \circ \psi_v|_{G(L^{[v]})}$ belongs to $\mathcal{F}^{[v]}$. Moreover, it holds that*

$$\mathcal{E}(f, f) = \sum_{v\in\tilde{W}_m} \frac{1}{r_v} \mathcal{E}^{[v]}(f^{[v]}, f^{[v]}) \tag{3.1}$$

and

$$\mu_{\langle f\rangle} = \sum_{v\in\tilde{W}_m} \frac{1}{r_v} (\psi_v|_{G(L^{[v]})})_* \mu^{[v]}_{\langle f^{[v]}\rangle}. \tag{3.2}$$

If f is an m-harmonic function, then $f^{[v]}$ is a harmonic function with respect to $(\mathcal{E}^{[v]}, \mathcal{F}^{[v]})$.

(2) It holds that

$$\mu_q = \sum_{v\in\tilde{W}_m} q_v (\psi_v|_{G(L^{[v]})})_* \mu^{[v]}_q. \tag{3.3}$$

By applying (3.1) with $\mathcal{E}$ replaced by $\mathcal{E}^{[\xi]}$ for $\xi \in \tilde{W}_*$ to $f = \iota(x)$ for $x \in l(V_0)$, we obtain the following identity as a special case:

$$r_\xi^{-1}Q(A_\xi x) = \sum_{\zeta\in W_m;\ \xi\zeta\in\tilde{W}_*} r_{\xi\zeta}^{-1}Q(A_{\xi\zeta}x), \quad m\in\mathbb{Z}_+. \tag{3.4}$$

Let $f\in\mathcal{F}$. For each $m\in\mathbb{Z}_+$, let $\lambda_{\langle f\rangle}^{(m)}$ be a measure on W_m defined as

$$\lambda_{\langle f\rangle}^{(m)}(C) = 2\sum_{v\in C\cap\tilde{W}_m} r_v^{-1}\mathcal{E}^{[v]}(f^{[v]}, f^{[v]}), \quad C\subset W_m.$$

Then, we can verify that $\{\lambda_{\langle f\rangle}^{(m)}\}_{m\in\mathbb{Z}_+}$ are consistent in the sense that $\lambda_{\langle f\rangle}^{(m)}(C) = \lambda_{\langle f\rangle}^{(m+1)}(C\times S)$. By the Kolmogorov extension theorem, there exists a unique Borel measure $\lambda_{\langle f\rangle}$ on Σ such that

$$\lambda_{\langle f\rangle}(\Sigma_C) = \lambda_{\langle f\rangle}^{(m)}(C) \quad \text{for any } m\in\mathbb{Z}_+,\ C\subset W_m,$$

where $\Sigma_C = \bigcup_{v\in C}\Sigma_v$. It is easy to see that $\lambda_{\langle f\rangle}(\Sigma\setminus\tilde{\Sigma}) = 0$.

In particular, if $f = \iota(x)$ for $x\in l(V_0)$, we have

$$\lambda_{\langle\iota(x)\rangle}(\Sigma_C) = 2\sum_{v\in C\cap\tilde{W}_m} r_v^{-1}Q(A_v x), \quad C\subset W_m. \tag{3.5}$$

For simplicity, we write $\lambda_{\langle x\rangle}$ for $\lambda_{\langle\iota(x)\rangle}$.

Lemma 3.3 *For $f\in\mathcal{F}$, $(\pi|_{\tilde{\Sigma}})_*\lambda_{\langle f\rangle} = \mu_{\langle f\rangle}$.*

Proof This lemma is proved in [9, Lemma 4.1] when T is a one-point set. In the general case, it suffices to modify the proof line by line by using Lemma 3.2 as a substitution of the self-similar property. We provide a proof here for the reader's convenience.

We define a set function χ_m for $m\in\mathbb{Z}_+$ by

$$\chi_m(A) = \sum_{v\in\tilde{W}_m}\frac{1}{r_v}\mu_{\langle f^{[v]}\rangle}^{[v]}(\pi(\sigma_v^{-1}(A)))$$

for a σ-compact subset A of $\tilde{\Sigma}$.

Let B be a closed subset of $G(L)$. For $v\in\tilde{W}_m$,

$$\begin{aligned}(\psi_v|_{G(L^{[v]})})^{-1}(B) &= \pi\left((\pi|_{\tilde{\Sigma}})^{-1}((\psi_v|_{G(L^{[v]})})^{-1}(B))\right)\\ &= \pi\left(\sigma_v^{-1}((\pi|_{\tilde{\Sigma}})^{-1}(B))\right).\end{aligned}$$

Therefore, $\mu_{\langle f\rangle}(B) = \chi_m\left((\pi|_{\tilde{\Sigma}})^{-1}(B)\right)$ from (3.2).

For $C \subset \tilde{W}_m$,

$$\begin{aligned}\lambda_{\langle f\rangle}(\Sigma_C) &= \lambda^{(m)}_{\langle f\rangle}(C)\\ &= 2\sum_{v\in C} r_v^{-1}\mathcal{E}^{[v]}(f^{[v]}, f^{[v]})\\ &= \sum_{v\in\tilde{W}_m} r_v^{-1}\mu^{[v]}_{\langle f^{[v]}\rangle}\bigl(\pi(\sigma_v^{-1}(\Sigma_C))\bigr)\\ &= \chi_m(\Sigma_C).\end{aligned}$$

Here, in the third equality, we used the identity

$$\pi(\sigma_v^{-1}(\Sigma_C)) = \begin{cases} G(L^{[v]}) & \text{if } v \in C,\\ \emptyset & \text{otherwise.}\end{cases}$$

Let F be a closed subset of $G(L)$. Then, $(\pi|_{\tilde{\Sigma}})^{-1}(F)$ is also closed in $\tilde{\Sigma}$. For $m \in \mathbb{Z}_+$, let $C_m = \bigl\{w \in \tilde{W}_m \bigm| \Sigma_w \cap (\pi|_{\tilde{\Sigma}})^{-1}(F) \neq \emptyset\bigr\}$. Then, $\{\Sigma_{C_m}\}_{m=0}^\infty$ is decreasing in m and $\bigcap_{m\in\mathbb{Z}_+}\Sigma_{C_m} = (\pi|_{\tilde{\Sigma}})^{-1}(F)$. By using the monotonicity of χ_m,

$$\mu_{\langle f\rangle}(F) = \chi_m\bigl((\pi|_{\tilde{\Sigma}})^{-1}(F)\bigr) \le \chi_m(\Sigma_{C_m}) = \lambda_{\langle f\rangle}(\Sigma_{C_m}).$$

Letting $m \to \infty$, we have $\mu_{\langle f\rangle}(F) \le \lambda_{\langle f\rangle}(F)$.

The inner regularity of $\mu_{\langle f\rangle}$ and $\lambda_{\langle f\rangle}$ implies that $\mu_{\langle f\rangle}(B) \le \lambda_{\langle f\rangle}(B)$ for all Borel sets B. Because the total measures of $\mu_{\langle f\rangle}$ and $\lambda_{\langle f\rangle}$ are the same, we also have the reverse inequality by considering $G(L) \setminus B$ in place of B. □

Let $i \in S_0$ and $\nu \in T$. From [12, Proposition A.1.1 and Theorem A.1.2], both 1 and $r^{(\nu)}$ are simple eigenvalues of $A_i^{(\nu)}$, and the modulus of another eigenvalue $s^{(\nu)}$ of $A_i^{(\nu)}$ is less than $r^{(\nu)}$. In our situation, the eigenvectors are explicitly described: the eigenvectors of eigenvalues $1, r^{(\nu)}, s^{(\nu)}$ are constant multiples of

$$\mathbf{1} := \begin{pmatrix}1\\1\\1\end{pmatrix},\quad \tilde{v}_1 := \begin{pmatrix}0\\1\\1\end{pmatrix},\quad y_1 := \begin{pmatrix}0\\1\\-1\end{pmatrix}\quad \text{for } A_1^{(\nu)},$$

$$\mathbf{1},\quad \tilde{v}_2 := \begin{pmatrix}1\\0\\1\end{pmatrix},\quad y_2 := \begin{pmatrix}-1\\0\\1\end{pmatrix}\quad \text{for } A_2^{(\nu)},$$

$$\mathbf{1},\quad \tilde{v}_3 := \begin{pmatrix}1\\1\\0\end{pmatrix},\quad y_3 := \begin{pmatrix}1\\-1\\0\end{pmatrix}\quad \text{for } A_3^{(\nu)},$$

respectively. Here, we identify $x \in l(V_0)$ with $\begin{pmatrix} x(p_1) \\ x(p_2) \\ x(p_3) \end{pmatrix}$. It is crucial for subsequent arguments that the eigenvectors of eigenvalue $r^{(\nu)}$ are independent of ν.

Let $\tilde{l}(V_0)$ be the set of all $x \in l(V_0)$ such that $\sum_{p \in V_0} x(p) = 0$. The orthogonal linear space of $\tilde{l}(V_0)$ in $l(V_0)$ is one-dimensional and spanned by $\mathbf{1}$. The function $\tilde{l}(V_0) \ni x \mapsto Q(x, x)^{1/2} \in \mathbb{R}$ defines a norm on $\tilde{l}(V_0)$. Let P denote the orthogonal projection from $l(V_0)$ onto $\tilde{l}(V_0)$. For each $i \in S_0$, $u_i \in l(V_0)$ denotes the column vector $(D_{p,p_i})_{p \in V_0}$.

Lemma 3.4 (see, e.g., [10, Lemma 5] and [12, Lemma A.1.4]) *For each $i \in S_0$ and $\nu \in T$, u_i is an eigenvector of ${}^t A_i^{(\nu)}$ with respect to the eigenvalue $r^{(\nu)}$. Moreover, $u_i \in \tilde{l}(V_0)$.*

We also note that $(u_i, \mathbf{1}) = (u_i, y_i) = 0$. We take $v_i \in l(V_0)$ such that v_i is a constant multiple of $\tilde{v}_i$ and $(u_i, v_i) = 1$.

Lemma 3.5 *Let $i \in S_0$, $x \in l(V_0)$, and $\boldsymbol{\nu} = \{\nu_k\}_{k \in \mathbb{N}} \in T^{\mathbb{N}}$. Then, it holds that*

$$\lim_{n \to \infty} r^{-1}_{i^{\nu_1} i^{\nu_2} \ldots i^{\nu_n}} P A_{i^{\nu_1} i^{\nu_2} \ldots i^{\nu_n}} x = (u_i, x) P v_i \tag{3.6}$$

and

$$\lim_{n \to \infty} r^{-2}_{i^{\nu_1} i^{\nu_2} \ldots i^{\nu_n}} Q(A_{i^{\nu_1} i^{\nu_2} \ldots i^{\nu_n}} x) = (u_i, x)^2 Q(v_i). \tag{3.7}$$

Moreover, these convergences are uniform in $i \in S_0$, $x \in \mathcal{C}$, and $\boldsymbol{\nu} \in T^{\mathbb{N}}$, where $\mathcal{C}$ is the inverse image of an arbitrary compact set of $l(V_0)$ by P.

Proof Note that $P A_{i^{\nu_1} i^{\nu_2} \ldots i^{\nu_n}} \mathbf{1} = 0$ and $r^{-1}_{i^{\nu_1} i^{\nu_2} \ldots i^{\nu_n}} A_{i^{\nu_1} i^{\nu_2} \ldots i^{\nu_n}} v_i = v_i$ for all n. Moreover, $|r^{-1}_{i^{\nu_1} i^{\nu_2} \ldots i^{\nu_n}} A_{i^{\nu_1} i^{\nu_2} \ldots i^{\nu_n}} y_i| \le \theta^n |y_i|$, where $\theta = \max_{\nu \in T} |s^{(\nu)}/r^{(\nu)}| \in [0, 1)$.

For $x \in l(V_0)$ in general, we can decompose x into $x = x_1 \mathbf{1} + x_2 v_i + x_3 y_i$. By taking the inner product with u_i on both sides, $(u_i, x) = x_2 (u_i, v_i) = x_2$. Therefore, (3.6) holds, and (3.7) follows immediately from (3.6). The uniformity of the convergences is evident from the argument above. □

Although the next lemma can be confirmed by concrete calculation, we provide a proof that is applicable to more general situations.

Lemma 3.6 *The following hold.*

(1) *For every $i, j \in S_0$, $Q(v_i, v_i) = Q(v_j, v_j) > 0$. For $j \in S_0$ and $i, i' \in S_0 \setminus \{j\}$, $(Dv_j)(p_i) = (Dv_j)(p_{i'})$.*
(2) *For every $i, j \in S_0$, $(u_i, v_j) \ne 0$.*
(3) *There exists $\delta_0 > 0$ such that, for each $i \in S_0$, there exists some $i' \in S_0$ satisfying*

$$\big| |(Dv_i)(p_i)| - |(Dv_i)(p_{i'})| \big| \ge \delta_0. \tag{3.8}$$

Proof (1) This is proved in [10, Lemma 10] in more-general situations.

(2) Note that $(u_j, v_j) = 1$. From (1), $(u_i, v_j) = (Dv_j)(p_i)$ is independent of $i \in S_0 \setminus \{j\}$. Moreover, $0 = (Dv_j, \mathbf{1}) = \sum_{i \in S_0}(u_i, v_j)$. Therefore, $(u_i, v_j) = -1/(\#S_0 - 1) = -1/2$ for $i \in S_0 \setminus \{j\}$.
(3) From the proof of (2), we can take $\delta_0 = 1/2$. □

The following are simple estimates used in the proofs of Lemma 4.1 and Theorem 2.3.

Lemma 3.7 *Let $s, t > 0$ and $a > 0$. If $|\log(t/s)| \geq a$, then*

$$|t - s| \geq (1 - e^{-a}) \max\{s, t\}.$$

Proof We may assume that $s \leq t$. Then, $t/s \geq e^a$, which implies $t - s \geq t - te^{-a} = t(1 - e^{-a})$. □

Lemma 3.8 *Let $d \in \mathbb{N}$ and*

$$\mathcal{P}_d = \left\{ a = (a_1, \dots, a_d) \in \mathbb{R}^d \;\middle|\; a_k \geq 0 \;\; \textit{for all } k = 1, \dots, d, \textit{ and } \sum_{k=1}^d a_k = 1 \right\}.$$

For $a = (a_1, \dots, a_d),\ b = (b_1, \dots, b_d) \in \mathcal{P}_d$, it holds that

$$\sum_{k=1}^d \sqrt{a_k b_k} \leq 1 - \frac{|a - b|_{\mathbb{R}^d}^2}{8}.$$

Proof Since all a_k and b_k are dominated by 1,

$$\begin{aligned}\sqrt{a_k b_k} &= \frac{a_k + b_k}{2} - \frac{(a_k - b_k)^2}{2(\sqrt{a_k} + \sqrt{b_k})^2} \\ &\leq \frac{a_k + b_k}{2} - \frac{(a_k - b_k)^2}{8}.\end{aligned}$$

Taking the sum with respect to k on both sides, we arrive at the conclusion. □

At the end of this section, we introduce a general sufficient condition for singularity of two measures. For $z \in \mathbb{R}$, let

$$z^{\oplus} = \begin{cases} 1/z & (z \neq 0) \\ 0 & (z = 0). \end{cases}$$

Theorem 3.9 *Let $(\Omega, \mathcal{B}, \{\mathcal{B}_n\}_{n \in \mathbb{Z}_+})$ be a measurable space equipped with a filtration such that $\mathcal{B} = \bigvee_{n \in \mathbb{Z}_+} \mathcal{B}_n$. Let P_1 and P_2 be two probability measures on $(\Omega, \mathcal{B})$. Suppose that, for each $n \in \mathbb{Z}_+$, $P_2|_{\mathcal{B}_n}$ is absolutely continuous with respect to $P_1|_{\mathcal{B}_n}$.*

Let z_n *be the Radon–Nikodym derivative* $d(P_2|_{\mathcal{B}_n})/d(P_1|_{\mathcal{B}_n})$ *for* $n \in \mathbb{Z}_+$ *and* $\alpha_n = z_n z_{n-2}^{\oplus}$ *for* $n \geq 2$*. If*

$$\sum_{n=2}^{\infty} \left(1 - \mathbb{E}^{P_1}[\sqrt{\alpha_n} \mid \mathcal{B}_{n-2}]\right) = \infty \quad P_1\text{-a.s.} \tag{3.9}$$

holds, then P_1 *and* P_2 *are mutually singular. Here,* $\mathbb{E}^{P_1}[\,\cdot\,|\mathcal{B}_{n-2}]$ *denotes the conditional expectation for* P_1 *given* $\mathcal{B}_{n-2}$*.*

Proof We modify the proof of [9, Theorem 4.1]. By [14, Theorem VII.6.1], $z_\infty := \lim_{n\to\infty} z_n$ exists $(P_1 + P_2)$-a.e. and

$$P_2(A) = \int_A z_\infty \, dP_1 + P_2(A \cap \{z_\infty = \infty\}), \quad A \in \mathcal{B}. \tag{3.10}$$

Moreover, P_1 and $P_2(\,\cdot\, \cap \{z_\infty = \infty\})$ are mutually singular.

Let

$$Z_1 = \left\{ \sum_{k=1}^{\infty} \left(1 - \mathbb{E}^{P_1}[\sqrt{\alpha_{2k}} \mid \mathcal{B}_{2(k-1)}]\right) = \infty \right\},$$
$$Z_2 = \left\{ \sum_{k=1}^{\infty} \left(1 - \mathbb{E}^{P_1}[\sqrt{\alpha_{2k+1}} \mid \mathcal{B}_{2(k-1)+1}]\right) = \infty \right\}.$$

From (3.9), $P_1(Z_1 \cup Z_2) = 1$. Considering the two filtrations $\{\mathcal{B}_{2k}\}_{k\in\mathbb{Z}_+}$ and $\{\mathcal{B}_{2k+1}\}_{k\in\mathbb{Z}_+}$ and following the proof of [14, Theorem VII.6.4], we have $\{z_\infty = \infty\} = Z_1 = Z_2$ up to P_2-null sets. Therefore, $z_\infty = \infty$ P_2-a.e. on $Z_1 \cup Z_2$. Applying (3.10) to $A = \Omega \setminus (Z_1 \cup Z_2)$, which is a P_1-null set, we have $P_2(A) = P_2(A \cap \{z_\infty = \infty\})$, that is, $z_\infty = \infty$ P_2-a.e. on A. Thus, $P_2(z_\infty = \infty) = 1$ and we conclude that P_1 and P_2 are mutually singular. □

4 Proof of the Main Results

We introduce some notation. Let $\mathcal{K}$ be a closed set of $l(V_0)$ that is defined as

$$\mathcal{K} = \{x \in l(V_0) \mid 2Q(x, x) = 1\}.$$

For $l_0 \in \mathbb{Z}_+$ and $l_1, l_2 \in \mathbb{N}$, let

$$L(l_0, l_1, l_2) = \left\{ \boldsymbol{\nu} = \{\nu_k\}_{k=1}^{\infty} \in T^{\mathbb{N}} \;\middle|\; \begin{array}{l} \{\nu_k \mid l_0 + 1 \leq k \leq l_0 + l_1\} \\ \subset \{\nu_k \mid l_0 + l_1 + 1 \leq k \leq l_0 + l_1 + l_2\} \end{array} \right\}.$$

We define several constants as follows:

$$\begin{aligned}
\beta_1 &:= \min\{|(u_i, v_j)| \mid i, j \in S_0\} = \min\{|(Dv_i)(p)| \mid i \in S_0,\ p \in V_0\},\\
\beta_2 &:= \min\{|\log(r_v/q_v)| \mid v \in S,\ r_v \neq q_v\} > 0,\\
\beta_3 &:= \min\{q_v \mid v \in S\} > 0,\\
\beta_4 &:= \min\{r^{(\nu)} \mid \nu \in T\} > 0,\\
\beta_5 &:= 2Q(v_i, v_i) > 0 \quad (i \in S_0).
\end{aligned}$$

By Lemma 3.6(2), $\beta_1 > 0$. In the definition of β_2, $\min\emptyset = 1$ by convention. By Lemma 3.6(1), β_5 is independent of the choice of i.

We fix $q \in \mathcal{A}$. The following is a key lemma for proving Theorem 2.3.

Lemma 4.1 (1) *There exist $N \in \mathbb{N}$ and $N' \in \mathbb{N}$ such that, for any $l \in \mathbb{N}$, there exists $\gamma > 0$ satisfying the following. For all $\boldsymbol{\nu} = \{\nu_k\}_{k=1}^{\infty} \in L(N, N', l)$ and $x \in \mathcal{K}$, there exist*

$$\begin{aligned}
i &= i(x) \in S_0,\\
j &= j(x, \nu_1, \nu_2, \dots, \nu_N) \in S_0,\\
m &= m(l, x, \nu_1, \nu_2, \dots, \nu_{N+N'+l}) \in \{N', N'+1, \dots, N'+l\}
\end{aligned}$$

such that

$$|2r_{\xi}^{-1} Q(A_{\xi} x) - q_{\xi}| \geq \gamma$$

with $\xi = i^{\nu_1} \dots i^{\nu_N} j^{\nu_{N+1}} \dots j^{\nu_{N+m}}$. Here, "$j = j(x, \nu_1, \nu_2, \dots, \nu_N)$" means that "$j$ depends only on $x, \nu_1, \nu_2, \dots, \nu_N$," and so on.

(2) *If Condition* (A) *holds, then the claim of item* (1) *holds with "$\boldsymbol{\nu} = \{\nu_k\}_{k=1}^{\infty} \in L(N, N', l)$" replaced by "$\boldsymbol{\nu} = \{\nu_k\}_{k=1}^{\infty} \in T^{\mathbb{N}}$."*

Proof (1) Let φ be a continuous function on $l(V_0)$ that is defined as

$$\varphi(x) = \sum_{i \in S_0} (u_i, x)^2.$$

Since the range of φ on $\mathcal{K}$ is equal to that on a compact set $P(\mathcal{K})$, φ attains a minimum on $\mathcal{K}$, say β_6. Let $x \in \mathcal{K}$. Because

$$0 < Q(x, x) = (-Dx, x) = -\sum_{i \in S_0} (u_i, x) x(p_i),$$

$(u_i, x) \neq 0$ for some $i \in S_0$. This implies that $\varphi(x) > 0$. (In fact, we can confirm that $\varphi(x) \equiv 3/2$.) Thus, $\beta_6 > 0$. Define $\delta' = \beta_6/\#S_0 = \beta_6/3$ and $\mathcal{K}_i = \{x \in \mathcal{K} \mid (u_i, x)^2 \geq \delta'\}$ for $i \in S_0$. It holds that $\mathcal{K} = \bigcup_{i \in S_0} \mathcal{K}_i$.

We fix $x \in \mathcal{K}$. There exists $i \in S_0$ such that $x \in \mathcal{K}_i$. From Lemma 3.6(3), there exists $i' \in S_0$ such that (3.8) holds. By keeping in mind that $(Dv_i)(p_i) = 1$, it follows that

$$\begin{aligned}\left|(Dv_i)(p_i)^2 - (Dv_i)(p_{i'})^2\right| &= \left|1 + |(Dv_i)(p_{i'})|\right| \left||(Dv_i)(p_i)| - |(Dv_i)(p_{i'})|\right| \\ &\geq \delta_0. \end{aligned} \tag{4.1}$$

Let $\boldsymbol{\nu} = \{\nu_k\}_{k\in\mathbb{N}} \in T^{\mathbb{N}}$ and define $x_n = r_{i^{\nu_1}\dots i^{\nu_n}}^{-1} A_{i^{\nu_1}\dots i^{\nu_n}} x$ for $n \in \mathbb{N}$. From Lemma 3.4,

$$(u_i, x_n) = (r_{i^{\nu_1}\dots i^{\nu_n}}^{-1}\, {}^t A_{i^{\nu_1}\dots i^{\nu_n t}} u_i, x) = (u_i, x). \tag{4.2}$$

Let $\delta_1 = \sqrt{\delta'}\beta_1/2$ and $\delta_2 = \delta'\delta_0/3$. By Lemma 3.5, there exists $N \in \mathbb{N}$ independent of the choice of x, i, and $\boldsymbol{\nu}$ such that, for all $p \in V_0$,

$$\left||(Dx_N)(p)| - |(u_i, x)(Dv_i)(p)|\right| \leq \delta_1 \tag{4.3}$$

and

$$\left|(Dx_N)(p)^2 - (u_i, x)^2(Dv_i)(p)^2\right| \leq \delta_2. \tag{4.4}$$

From (4.2) and (4.3), for any $j \in S_0$,

$$\begin{aligned}|(u_j, x_N)| &= |(Dx_N)(p_j)| \\ &\geq |(u_i, x)(Dv_i)(p_j)| - \delta_1 \\ &\geq \sqrt{\delta'}\beta_1 - \delta_1 = \delta_1.\end{aligned}$$

By Lemma 3.5,

$$\begin{aligned}\lim_{m\to\infty} r_{j^{\nu_{N+1}}\dots j^{\nu_{N+m}}}^{-2} Q(A_{j^{\nu_{N+1}}\dots j^{\nu_{N+m}}} x_N) &= (u_j, x_N)^2 Q(v_j) \\ &\geq \delta_1^2\beta_5/2 > 0.\end{aligned}$$

This convergence is uniform in x, i, j, and $\boldsymbol{\nu}$ because Px_N belongs to some compact set of $\mathcal{K}$ that is independent of them. We take $\delta_3 = \beta_5\delta_2/2$. Then, there exists $N' \in \mathbb{N}$ independent of x, i, j, and $\boldsymbol{\nu}$ such that, for every $n \geq N'$,

$$\left|r_{j^{\nu_{N+1}}\dots j^{\nu_{N+n}}}^{-2} Q(A_{j^{\nu_{N+1}}\dots j^{\nu_{N+n}}} x_N) - (u_j, x_N)^2 Q(v_j)\right| \leq \delta_3/4 \tag{4.5}$$

and

$$\left|\log \frac{r_{j^{\nu_{N+1}}\dots j^{\nu_{N+n-1}}}^{-2} Q(A_{j^{\nu_{N+1}}\dots j^{\nu_{N+n-1}}} x_N)}{r_{j^{\nu_{N+1}}\dots j^{\nu_{N+n}}}^{-2} Q(A_{j^{\nu_{N+1}}\dots j^{\nu_{N+n}}} x_N)}\right| \leq \frac{\beta_2}{2}. \tag{4.6}$$

From (4.1) and (4.4),

$$\begin{aligned}\delta'\delta_0 &\le (u_i, x)^2\left|(Dv_i)(p_i)^2 - (Dv_i)(p_{i'})^2\right| \\ &\le \left|(u_i, x)^2(Dv_i)(p_i)^2 - (Dx_N)(p_i)^2\right| + \left|(Dx_N)(p_i)^2 - (Dx_N)(p_{i'})^2\right| \\ &\quad + \left|(Dx_N)(p_{i'})^2 - (u_i, x)^2(Dv_i)(p_{i'})^2\right| \\ &\le 2\delta_2 + \left|(Dx_N)(p_i)^2 - (Dx_N)(p_{i'})^2\right|,\end{aligned}$$

which implies that

$$\left|(Dx_N)(p_i)^2 - (Dx_N)(p_{i'})^2\right| \ge \delta'\delta_0 - 2\delta_2 = \delta_2.$$

From the identity $(Dx_N)(p_j) = (u_j, x_N)$ $(j \in S_0)$, we have

$$\begin{aligned}2\delta_3 &= \beta_5\delta_2 \\ &\le \beta_5\left|(u_i, x_N)^2 - (u_{i'}, x_N)^2\right| \\ &\le \left|2Q(v_i)(u_i, x_N)^2 - q_w/r_w\right| + \left|2Q(v_{i'})(u_{i'}, x_N)^2 - q_w/r_w\right|,\end{aligned}$$

where we choose $w = i^{\nu_1} \dots i^{\nu_N} \in W_N$. Then, for either $j = i$ or i',

$$\left|2Q(v_j)(u_j, x_N)^2 - q_w/r_w\right| \ge \delta_3. \tag{4.7}$$

We fix such j. Take any $l \in \mathbb{N}$ and suppose $\boldsymbol{\nu} \in L(N, N', l)$. There are two possibilities:

(I) There exists some $k \in \{N'+1, \dots, N'+l\}$ such that $r_{j^{\nu_{N+k}}} \ne q_{j^{\nu_{N+k}}}$.
(II) $r_{j^{\nu_{N+k}}} = q_{j^{\nu_{N+k}}}$ for all $k \in \{N'+1, \dots, N'+l\}$.

Suppose Case (I). Let $w' = j^{\nu_{N+1}} \dots j^{\nu_{N+k-1}} \in W_{k-1}$. From (4.6) with $n = k$,

$$\begin{aligned}\frac{\beta_2}{2} &\ge \left|\log\left(r^2_{j^{\nu_{N+k}}} \times \frac{Q(A_{j^{\nu_{N+1}}\dots j^{\nu_{N+k-1}}}x_N)}{Q(A_{j^{\nu_{N+1}}\dots j^{\nu_{N+k}}}x_N)}\right)\right| \\ &= \left|\log\left(\frac{r_{j^{\nu_{N+k}}}}{q_{j^{\nu_{N+k}}}}\frac{2r^{-1}_{ww'}Q(A_{ww'}x)}{q_{ww'}}\frac{q_{ww'j^{\nu_{N+k}}}}{2r^{-1}_{ww'j^{\nu_{N+k}}}Q(A_{ww'j^{\nu_{N+k}}}x)}\right)\right| \\ &\ge \beta_2 - \left|\log\frac{2r^{-1}_{ww'}Q(A_{ww'}x)}{q_{ww'}}\right| - \left|\log\frac{q_{ww'j^{\nu_{N+k}}}}{2r^{-1}_{ww'j^{\nu_{N+k}}}Q(A_{ww'j^{\nu_{N+k}}}x)}\right|.\end{aligned}$$

Therefore, either

$$\left|\log\frac{2r^{-1}_{ww'}Q(A_{ww'}x)}{q_{ww'}}\right| \ge \frac{\beta_2}{4} \quad \text{or} \quad \left|\log\frac{q_{ww'j^{\nu_{N+k}}}}{2r^{-1}_{ww'j^{\nu_{N+k}}}Q(A_{ww'j^{\nu_{N+k}}}x)}\right| \ge \frac{\beta_2}{4}$$

holds. Since $q_{ww'} \ge q_{ww'j^{\nu_{N+k}}} \ge \beta_3^{N+N'+l}$, Lemma 3.7 implies that either

$$|2r^{-1}_{ww'}Q(A_{ww'}x) - q_{ww'}| \ge (1 - e^{-\beta_2/4})\beta_3^{N+N'+l} \tag{4.8}$$

or

$$|2r_{ww'j^{\nu_{N+k}}}^{-1} Q(A_{ww'j^{\nu_{N+k}}}x) - q_{ww'j^{\nu_{N+k}}}| \geq (1 - e^{-\beta_2/4})\beta_3^{N+N'+l} \tag{4.9}$$

holds.

Next, suppose Case (II). Since $\boldsymbol{\nu} \in L(N, N', l)$, $r_{j^{\nu_{N+k}}} = q_{j^{\nu_{N+k}}}$ for all $k \in \{1, \ldots, N'\}$. Let $\hat{w} = j^{\nu_{N+1}} \ldots j^{\nu_{N+N'}} \in W_{N'}$. Note that $q_{\hat{w}} = r_{\hat{w}}$. From (4.7) and (4.5),

$$\begin{aligned}\delta_3 &\leq |2Q(v_j)(u_j, x_N)^2 - q_w/r_w| \\ &\leq |2Q(v_j)(u_j, x_N)^2 - 2r_{\hat{w}}^{-2}Q(A_{\hat{w}}x_N)| + |2r_{\hat{w}}^{-2}Q(A_{\hat{w}}x_N) - q_{w\hat{w}}/r_{w\hat{w}}| \\ &\leq \delta_3/2 + \beta_4^{-(N+N')}|2r_{w\hat{w}}^{-1}Q(A_{w\hat{w}}x) - q_{w\hat{w}}|.\end{aligned}$$

Therefore,

$$|2r_{w\hat{w}}^{-1}Q(A_{w\hat{w}}x) - q_{w\hat{w}}| \geq \delta_3\beta_4^{N+N'}/2.$$

In conclusion, it suffices to take

$$m = \begin{cases} k-1 & \text{if (4.8) holds in Case (I),} \\ k & \text{if (4.8) fails to hold in Case (I),} \\ N' & \text{in Case (II)} \end{cases}$$

and

$$\gamma = \min\{(1 - e^{-\beta_2/4})\beta_3^{N+N'+l},\ \delta_3\beta_4^{N+N'}/2\}.$$

(2) In the proof of (1), the condition that $\boldsymbol{\nu} \in L(N, N', l)$ is used only in the discussion of Case (II). Under Condition (A), Case (II) never happens. Therefore, the arguments are valid for all $\boldsymbol{\nu} \in T^{\mathbb{N}}$. □

Proof of Theorem 2.3 Let N and N' be natural numbers that are provided in Lemma 4.1. Under Condition (B), take $l_2 \in \mathbb{N}$ associated with $l_0 = N$ and $l_1 = N'$ in (B). Under Condition (A), take $l_2 = 1$.

Let $M = N + N' + l_2$. For $n \in \mathbb{Z}_+$, let $\mathcal{B}_n$ denote the σ-field on Σ that is generated by $\{\Sigma_w \mid w \in W_{Mn}\}$. Then, $\bigvee_{n=0}^{\infty} \mathcal{B}_n$ is equal to the Borel σ-field on Σ.

Take $x \in \mathcal{K}$. We first prove that $\lambda_{\langle x \rangle}$ and λ_q are mutually singular. For each $n \in \mathbb{Z}_+$, $\lambda_{\langle x \rangle}|_{\mathcal{B}_n}$ is absolutely continuous with respect to $\lambda_q|_{\mathcal{B}_n}$. Indeed, if $\lambda_q(\Sigma_w) = 0$ for $w \in W_{Mn}$, then $w \notin \tilde{W}_{Mn}$, which implies $\lambda_{\langle x \rangle}(\Sigma_w) = 0$. Let z_n denote the Radon–Nikodym derivative $d(\lambda_{\langle x \rangle}|_{\mathcal{B}_n})/d(\lambda_q|_{\mathcal{B}_n})$.

Under Condition (B), take $\omega = \omega_1\omega_2 \ldots \in \tilde{\Sigma}$ such that Condition ($\star$) is satisfied, and let $k \in \mathbb{Z}_+$ in ($\star$). Under Condition (A), take $\omega \in \tilde{\Sigma}$ and $k \in \mathbb{Z}_+$ arbitrarily.

There exists a unique natural number $n \geq 2$ such that $M(n-2) \leq k < M(n-1)$. Let $w := [\omega]_{M(n-2)} \in \tilde{W}_{M(n-2)}$ and $\xi \in W_{2M}$. Using (3.5), we have

$$z_{n-2} = \frac{\lambda_{\langle x \rangle}(\Sigma_w)}{\lambda_q(\Sigma_w)} = \frac{2r_w^{-1}Q(A_w x)}{q_w} \quad \text{on } \Sigma_w$$

and

$$z_n = \begin{cases} \dfrac{2r_{w\xi}^{-1} Q(A_{w\xi} x)}{q_{w\xi}} & \text{if } w\xi \in \tilde{W}_{Mn} \\ 0 & \text{if } w\xi \notin \tilde{W}_{Mn} \end{cases} \quad \text{on } \Sigma_{w\xi}.$$

Then, on $\Sigma_{w\xi}$,

$$\alpha_n := z_n z_{n-2}^{\oplus} = \begin{cases} \dfrac{Q(A_{w\xi}x) Q(A_w x)^{\oplus}}{q_\xi r_\xi} & \text{if } w\xi \in \tilde{W}_{Mn}, \\ 0 & \text{if } w\xi \notin \tilde{W}_{Mn}. \end{cases}$$

If $Q(A_w x) = 0$, then $\alpha_n = 0$ on Σ_w, which implies that

$$1 - \mathbb{E}^{\lambda_q}[\sqrt{\alpha_n} \mid \mathcal{B}_{n-2}](\omega) = 1. \tag{4.10}$$

Suppose that $Q(A_w x) \neq 0$. Let $x' = A_w x \big/ \sqrt{2Q(A_w x)} \in \mathcal{K}$. Then,

$$\begin{aligned} \mathbb{E}^{\lambda_q}[\sqrt{\alpha_n} \mid \mathcal{B}_{n-2}](\omega) &= \sum_{\xi \in W_{2M};\ w\xi \in \tilde{W}_{Mn}} \frac{q_{w\xi}}{q_w} \sqrt{\frac{Q(A_{w\xi}x)}{q_\xi r_\xi Q(A_w x)}} \\ &= \sum_{\xi \in W_{2M};\ w\xi \in \tilde{W}_{Mn}} \sqrt{q_\xi \times 2r_\xi^{-1} Q(A_\xi x')} \\ &\leq 1 - \frac{1}{8} \sum_{\xi \in W_{2M};\ w\xi \in \tilde{W}_{Mn}} \big(q_\xi - 2r_\xi^{-1} Q(A_\xi x')\big)^2. \end{aligned} \tag{4.11}$$

Here, the last inequality follows from Lemma 3.8.

Take $\gamma > 0$ in Lemma 4.1 associated with $l = l_2$. Let

$$w' = \omega_{M(n-2)+1}\omega_{M(n-2)+2} \dots \omega_k \in W_{k-M(n-2)} \quad (w' = \emptyset \text{ if } k = M(n-2))$$

and $\gamma' = \min\{\gamma, \beta_3^M\}$. Note that $q_{w'} \geq \beta_3^M \geq \gamma'$. We consider the following two cases:

(i) $|q_{w'} - 2r_{w'}^{-1} Q(A_{w'}x')| \geq \gamma\gamma'/3$;
(ii) $|q_{w'} - 2r_{w'}^{-1} Q(A_{w'}x')| < \gamma\gamma'/3$.

Suppose Case (i). Letting $I = \{\zeta \in W_{Mn-k} \mid ww'\zeta \in \tilde{W}_{Mn}\}$, we have

$$\begin{aligned} &\sum_{\xi \in W_{2M};\ w\xi \in \tilde{W}_{Mn}} \big(q_\xi - 2r_\xi^{-1} Q(A_\xi x')\big)^2 \\ &\geq \sum_{\zeta \in I} \big(q_{w'\zeta} - 2r_{w'\zeta}^{-1} Q(A_{w'\zeta} x')\big)^2 \end{aligned}$$

$$\geq \left\{ \sum_{\zeta \in I} \left(q_{w'\zeta} - 2r_{w'\zeta}^{-1} Q(A_{w'\zeta} x') \right) \right\}^2 \left(\sum_{\zeta \in I} 1 \right)^{-1}$$
$$= \left(q_{w'} - 2r_{w'}^{-1} Q(A_{w'} x') \right)^2 (\# I)^{-1} \qquad \text{(from (3.4))}$$
$$\geq (\gamma \gamma'/3)^2 (\# S)^{-2M}.$$

Next, suppose Case (ii). We have

$$\begin{aligned} 2r_{w'}^{-1} Q(A_{w'} x') &> q_{w'} - \gamma\gamma'/3 \\ &\geq \gamma' - \gamma'/3 = 2\gamma'/3. \end{aligned} \tag{4.12}$$

In particular, $Q(A_{w'} x') \neq 0$. Let $x'' = A_{w'} x' \big/ \sqrt{2Q(A_{w'} x')} \in \mathcal{K}$. We make several choices in order as follows:

- Take $i \in S_0$ associated with $x'' \in \mathcal{K}$ in Lemma 4.1.
- Take $\nu_{k+1}, \nu_{k+2}, \ldots, \nu_{k+N} \in T$ such that $ww' i^{\nu_{k+1}} i^{\nu_{k+2}} \ldots i^{\nu_{k+N}} \in \tilde{W}_{k+N}$; these are uniquely determined.
- Take $j \in S_0$ associated with $x'' \in \mathcal{K}$, $i \in S_0$, and $\{\nu_{k+s}\}_{s=1}^{N}$ in Lemma 4.1.
- Take a unique sequence $\{\nu_s\}_{s=k+N+1}^{\infty} \subset T$ such that

$$ww' i^{\nu_{k+1}} i^{\nu_{k+2}} \ldots i^{\nu_{k+N}} j^{\nu_{k+N+1}} j^{\nu_{k+N+2}} \ldots j^{\nu_{k+N+t}} \in W_{k+N+t}$$

  for every $t \in \mathbb{N}$.
- Take $m \in \{N', N'+1, \ldots, N'+l_2\}$ associated with $x'' \in \mathcal{K}$, $i \in S_0$, $j \in S_0$, and $\{\nu_{k+s}\}_{s=1}^{\infty}$ in Lemma 4.1.

Note that $\{\nu_{k+s}\}_{s=1}^{\infty} \in L(N, N', l_2)$ under Condition (B).

Let

$$\eta = i^{\nu_{k+1}} i^{\nu_{k+2}} \ldots i^{\nu_{k+N}} j^{\nu_{k+N+1}} j^{\nu_{k+N+2}} \ldots j^{\nu_{k+N+m}} \in W_{N+m}.$$

Then, letting $J = \{\eta' \in W_{Mn-k-N-m} \mid ww'\eta\eta' \in \tilde{W}_{Mn}\}$, we have

$$\sum_{\xi \in W_{2M};\ w\xi \in \tilde{W}_{Mn}} \left(q_\xi - 2r_\xi^{-1} Q(A_\xi x') \right)^2$$
$$\geq \sum_{\eta' \in J} \left(q_{w'\eta\eta'} - 2r_{w'\eta\eta'}^{-1} Q(A_{w'\eta\eta'} x') \right)^2$$
$$\geq \left\{ \sum_{\eta' \in J} \left(q_{w'\eta\eta'} - 2r_{w'\eta\eta'}^{-1} Q(A_{w'\eta\eta'} x') \right) \right\}^2 \left(\sum_{\eta' \in J} 1 \right)^{-1}$$
$$= \left(q_{w'\eta} - 2r_{w'\eta}^{-1} Q(A_{w'\eta} x') \right)^2 (\# J)^{-1} \qquad \text{(from (3.4))}$$
$$\geq \left(q_{w'\eta} - 2r_{w'\eta}^{-1} Q(A_{w'\eta} x') \right)^2 (\# S)^{-2M}.$$

Moreover,

$$\begin{aligned}
&\left|q_{w'\eta} - 2r_{w'\eta}^{-1} Q(A_{w'\eta}x')\right| \\
&= \left|q_{w'\eta} - 2r_{\eta}^{-1} Q(A_{\eta}x'') \cdot 2r_{w'}^{-1} Q(A_{w'}x')\right| \\
&\quad \geq 2r_{w'}^{-1} Q(A_{w'}x') \left|q_{\eta} - 2r_{\eta}^{-1} Q(A_{\eta}x'')\right| - \left|q_{w'} - 2r_{w'}^{-1} Q(A_{w'}x')\right| q_{\eta} \\
&\quad \geq \frac{2\gamma'}{3} \cdot \gamma - \frac{\gamma\gamma'}{3} \cdot 1 = \gamma\gamma'/3.
\end{aligned}$$

Here, in the last inequality, we used (4.12) and Lemma 4.1.

Therefore, in both Case (i) and Case (ii),

$$\sum_{\xi \in W_{2M};\ w\xi \in \tilde{W}_{Mn}} \left(q_{\xi} - 2r_{\xi}^{-1} Q(A_{\xi}x')\right)^2 \geq (\gamma\gamma'/3)^2 (\#S)^{-2M}. \tag{4.13}$$

By combining (4.11) with (4.13),

$$1 - \mathbb{E}^{\lambda_q}[\sqrt{\alpha_n} \mid \mathcal{B}_{n-2}](\omega) \geq (\gamma\gamma')^2 (\#S)^{-2M}/72. \tag{4.14}$$

For λ_q-a.s. ω, there are infinitely many n that satisfy (4.10) or (4.14); therefore,

$$\sum_{n=2}^{\infty} \left(1 - \mathbb{E}^{\lambda_q}[\sqrt{\alpha_n} \mid \mathcal{B}_{n-2}]\right) = \infty \quad \lambda_q\text{-a.s.}$$

From Theorem 3.9, we conclude that $\lambda_{\langle x \rangle} \perp \lambda_q$.

Take a σ-compact set B of Σ such that $\lambda_{\langle x \rangle}(B) = 1$ and $\lambda_q(B) = 0$. Recall that

$$V_* \setminus V_0 = \left\{x \in G(L) \;\middle|\; \#(\pi|_{\tilde{\Sigma}})^{-1}(\{x\}) > 1\right\}$$

and $\mu_q(V_* \setminus V_0) = 0$. Let $B' = (\pi|_{\tilde{\Sigma}})^{-1}(V_* \setminus V_0) \cup B$. Because $(\pi|_{\tilde{\Sigma}})^{-1}(\pi(B')) = B'$, from Lemma 3.3

$$\mu_q(\pi(B')) = \lambda_q\left((\pi|_{\tilde{\Sigma}})^{-1}(\pi(B'))\right) = \lambda_q(B') = 0$$

and

$$\mu_{\langle\iota(x)\rangle}(\pi(B')) = \lambda_{\langle x \rangle}\left((\pi|_{\tilde{\Sigma}})^{-1}(\pi(B'))\right) = \lambda_{\langle x \rangle}(B') \geq \lambda_{\langle x \rangle}(B) = 1.$$

Therefore, $\mu_{\langle\iota(x)\rangle} \perp \mu_q$. We have now proved that $\mu_{\langle h \rangle} \perp \mu_q$ for all harmonic functions h.

Next, let f be an arbitrary m-piecewise harmonic function. For $v \in \tilde{W}_m$, we apply the above result to the Dirichlet form $(\mathcal{E}^{[v]}, \mathcal{F}^{[v]})$ on $L^2(G(L^{[v]}), \mu_q^{[v]})$ and $f^{[v]} := f \circ \psi_v|_{G(L^{[v]})}$ to conclude that $\mu^{[v]}_{\langle f^{[v]} \rangle} \perp \mu_q^{[v]}$. Take a σ-compact subset B_v of $G(L^{[v]})$ such that $\mu^{[v]}_{\langle f^{[v]} \rangle}(G(L^{[v]}) \setminus B_v) = 0$ and $\mu_q^{[v]}(B_v) = 0$. Let

$$B = \bigcup_{v \in \tilde{W}_m} \psi_v(B_v) \quad \text{and} \quad \hat{B} = B \setminus (V_* \setminus V_0).$$

From Lemma 3.2 and the property $\mu_q(V_* \setminus V_0) = 0$, we have

$$\begin{aligned} \mu_{\langle f \rangle}(B) &\geq \sum_{v \in \tilde{W}_m} \frac{1}{r_v} \mu^{[v]}_{\langle f^{[v]} \rangle}(B_v) = \sum_{v \in \tilde{W}_m} \frac{2}{r_v} \mathcal{E}^{[v]}(f^{[v]}, f^{[v]}) \\ &= 2\mathcal{E}(f, f) = \mu_{\langle f \rangle}(G(L)) \end{aligned}$$

and

$$\mu_q(B) = \mu_q(\hat{B}) \leq \sum_{v \in \tilde{W}_m} q_v \mu_q^{[v]}(B_v) = 0.$$

Therefore, $\mu_{\langle f \rangle} \perp \mu_q$.

For $f \in \mathcal{F}$ in general, we can take a sequence $\{f_n\}_{n=1}^{\infty}$ of piecewise harmonic functions that converges to f in $\mathcal{F}$ from Lemma 3.1. For each $n \in \mathbb{N}$, take a Borel set B_n of $G(L)$ such that $\mu_q(B_n) = 0$ and $\mu_{\langle f_n \rangle}(G(L) \setminus B_n) = 0$. Let $B = \bigcup_{n=1}^{\infty} B_n$. From a general inequality

$$\left| \sqrt{\mu_{\langle g \rangle}(C)} - \sqrt{\mu_{\langle g' \rangle}(C)} \right| \leq \sqrt{\mu_{\langle g - g' \rangle}(C)}$$

for $g, g' \in \mathcal{F}$ and a Borel set C of $G(L)$ (see, e.g., [5, p. 111]), we obtain

$$\mu_{\langle f \rangle}(G(L) \setminus B) = \lim_{n \to \infty} \mu_{\langle f_n \rangle}(G(L) \setminus B) = 0,$$

while $\mu_q(B) = 0$. Therefore, $\mu_{\langle f \rangle} \perp \mu_q$. □

Lastly, we prove Theorem 2.7.

Proof of Theorem 2.7 Since the assertion obviously holds when $\#T = 1$, we may assume that $\#T \geq 2$.

Let $q = \{q_v\}_{v \in S} \in \mathcal{A}$. Take $l_0, l_1 \in \mathbb{N}$ arbitrarily and let $l_2 = \#T$. For $\hat{\omega} \in \hat{\Omega}$, $\tilde{W}_n(\hat{\omega})$ $(n \in \mathbb{Z}_+)$ and $\mu_q^{(\hat{\omega})}$ denote the set $\tilde{W}_n$ and the measure μ_q associated with $L(\hat{\omega})$, respectively. We define a probability measure $\mathbb{P}$ on $(\Sigma \times \hat{\Omega}, \mathcal{B}(\Sigma) \otimes \hat{\mathcal{B}})$ by

$$\mathbb{P}(A) = \int_{\hat{\Omega}} \mu_q^{(\hat{\omega})}(A_{\hat{\omega}}) \, \hat{P}(d\hat{\omega}), \quad A \in \mathcal{B}(\Sigma) \otimes \hat{\mathcal{B}},$$

where $\mathcal{B}(\Sigma)$ denotes the Borel σ-field on Σ and $A_{\hat{\omega}} = \{\omega \in \Sigma \mid (\omega, \hat{\omega}) \in A\}$. More specifically, if A is expressed as $A = \Sigma_w \times B$ for $w = w_1 w_2 \dots w_m \in W_m$ and $B = \{\hat{\omega} \in \hat{\Omega} \mid L_v(\hat{\omega}) = \tau_v \quad \text{for all } v \in W_{\leq n}\}$ for given $m, n \in \mathbb{Z}_+$ with $m - 1 \leq n$ and $\{\tau_v\}_{v \in W_{\leq n}} \in T^{W_{\leq n}}$, then

$$\mathbb{P}(A) = \int_B \mu_q^{(\hat{\omega})}(\Sigma_w)\, \hat{P}(d\hat{\omega}) = \int_B q_w \mathbf{1}_{\tilde{W}_m(\hat{\omega})}(w)\, \hat{P}(d\hat{\omega})$$
$$= \begin{cases} q_w \prod_{v \in W_{\le n}} \rho(\{\tau_v\}) & \text{if } w_k \in S^{(\tau_{[w]_{k-1}})} \quad \text{for all } k = 1, 2, \ldots, m, \\ 0 & \text{otherwise.} \end{cases}$$

For $k \in \mathbb{Z}_+$, let $\tilde{U}(k)$ denote the set of all elements $(w, \hat{\omega}) \in W_k \times \hat{\Omega}$ such that the following hold:

(i) $w \in \tilde{W}_k(\hat{\omega})$;
(ii) for any $i, j \in S_0$, if we take $\nu_{k+1}, \nu_{k+2}, \ldots, \nu_{k+l_0+l_1+l_2} \in T$ such that

$$w i^{\nu_{k+1}} i^{\nu_{k+2}} \ldots i^{\nu_{k+l_0}} j^{\nu_{k+l_0+1}} j^{\nu_{k+l_0+2}} \ldots j^{\nu_{k+l_0+l_1+l_2}} \in \tilde{W}_{k+l_0+l_1+l_2}(\hat{\omega}),$$

then $\{\nu_{k+l_0+l_1+1}, \nu_{k+l_0+l_1+2}, \ldots, \nu_{k+l_0+l_1+l_2}\} = T$.

Define

$$U(k) = \left\{(\omega, \hat{\omega}) \in \Sigma \times \hat{\Omega} \;\middle|\; ([\omega]_k, \hat{\omega}) \in \tilde{U}(k)\right\}$$

and

$$U_{\hat{\omega}}(k) = \{\omega \in \Sigma \mid (\omega, \hat{\omega}) \in U(k)\}, \quad \hat{\omega} \in \hat{\Omega}.$$

Then,

$$\begin{aligned}
\mathbb{P}(U(k)) &= \int_{\hat{\Omega}} \sum_{w \in W_k} q_w \mathbf{1}_{\tilde{U}(k)}(w, \hat{\omega})\, \hat{P}(d\hat{\omega}) \\
&= \sum_{w \in W_k} q_w \hat{P}\big(\{\hat{\omega} \in \hat{\Omega} \mid w \in \tilde{W}_k(\hat{\omega})\}\big) \left(l_2! \prod_{\nu \in T} \rho(\{\nu\}) \right)^{\#(S_0 \times S_0)} \\
&= p \sum_{\nu_1, \ldots, \nu_k \in T} \sum_{\substack{w_j \in S^{(\nu_j)};\\ j=1,\ldots,k}} \prod_{m=1}^{k} q_{w_m} \prod_{m=1}^{k} \rho(\{\nu_m\}) \\
&\qquad \Big(p := \big(l_2! \prod_{\nu \in T} \rho(\{\nu\}) \big)^9 \in (0, 1) \Big) \\
&= p \left(\sum_{\nu \in T} \sum_{v \in S^{(\nu)}} q_v \rho(\{\nu\}) \right)^k \\
&= p.
\end{aligned}$$

In a similar way, we can confirm that $\{U((l_0 + l_1 + l_2)n)\}_{n \in \mathbb{Z}_+}$ are independent with respect to $\mathbb{P}$.

For $0 \le M < N$, we define

$$F_{M,N} = \bigcap_{n=M+1}^{N} \big((\Sigma \times \hat{\Omega}) \setminus U((l_0 + l_1 + l_2)n)\big),$$

$$F_{M,N,\hat{\omega}} = \{\omega \in \Sigma \mid (\omega, \hat{\omega}) \in F_{M,N}\}, \quad F_{M,\hat{\omega}} = \bigcap_{N=M+1}^{\infty} F_{M,N,\hat{\omega}} \quad (\hat{\omega} \in \hat{\Omega}),$$

$$G_{M,N} = \big\{\hat{\omega} \in \hat{\Omega} \bigm| \mu_q^{(\hat{\omega})}(F_{M,N,\hat{\omega}}) \ge (1-p)^{N/2}\big\}, \quad G_M = \limsup_{N\to\infty} G_{M,N}.$$

Then,

$$\begin{aligned}\hat{P}(G_{M,N}) &\le (1-p)^{-N/2} \int_{\hat{\Omega}} \mu_q^{(\hat{\omega})}(F_{M,N,\hat{\omega}})\, \hat{P}(d\hat{\omega}) \\ &= (1-p)^{-N/2}\mathbb{P}(F_{M,N}) \\ &= (1-p)^{-N/2}(1-p)^{N-M} \\ &= (1-p)^{(N/2)-M}.\end{aligned}$$

From the Borel–Cantelli lemma, $\hat{P}(G_M) = 0$. Let

$$\mathcal{U}_q = \big\{q' = \{q'_v\}_{v\in S} \in \mathcal{A} \bigm| q'_v/q_v < (1-p)^{-1/(4(l_0+l_1+l_2))} \quad \text{for all } v \in S\big\},$$

which is an open neighborhood of q in $\mathcal{A}$. By letting $\mathcal{F}_n = \sigma(\{\Sigma_w \mid w \in W_{(l_0+l_1+l_2)n}\})$ for $n \in \mathbb{Z}_+$, we have

$$\frac{d\big(\mu_{q'}^{(\hat{\omega})}|_{\mathcal{F}_n}\big)}{d\big(\mu_q^{(\hat{\omega})}|_{\mathcal{F}_n}\big)} \le (1-p)^{-n/4} \quad \mu_q^{(\hat{\omega})}\text{-a.e.}$$

for all $q' \in \mathcal{U}_q$ and $\hat{\omega} \in \hat{\Omega}$. Suppose that $q' \in \mathcal{U}_q$ and $\hat{\omega} \notin G_M$. For sufficiently large $N \in \mathbb{N}$, $\hat{\omega} \notin G_{M,N}$. Because $F_{M,N,\hat{\omega}}$ belongs to $\mathcal{F}_N$, we have

$$\mu_{q'}^{(\hat{\omega})}(F_{M,N,\hat{\omega}}) \le (1-p)^{-N/4}\mu_q^{(\hat{\omega})}(F_{M,N,\hat{\omega}}) \le (1-p)^{N/4}$$

for large N, which implies $\mu_{q'}^{(\hat{\omega})}(F_{M,\hat{\omega}}) = 0$. Let $G(q)$ denote $\bigcup_{M\in\mathbb{Z}_+} G_M$. (Here, we specify the dependency of q.) This is a $\hat{P}$-null set. If $\hat{\omega} \notin G(q)$, then $\mu_{q'}^{(\hat{\omega})}\big(\bigcup_{M\in\mathbb{Z}_+} F_{M,\hat{\omega}}\big) = 0$ for $q' \in \mathcal{U}_q$, which means that

$$\mu_{q'}^{(\hat{\omega})}\Big(\limsup_{n\to\infty} U_{\hat{\omega}}((l_0 + l_1 + l_2)n)\Big) = 1, \quad q' \in \mathcal{U}_q.$$

Because $\mathcal{A}$ is σ-compact, we can take a countable subset $\{q_\alpha \mid \alpha \in \mathbb{N}\}$ of $\mathcal{A}$ such that $\bigcup_{\alpha\in\mathbb{N}} \mathcal{U}_{q_\alpha} = \mathcal{A}$. Let $\mathcal{N} = \bigcup_{\alpha\in\mathbb{N}} G(q_\alpha)$. Then, $\hat{P}(\mathcal{N}) = 0$ and for $\hat{\omega} \in \hat{\Omega} \setminus \mathcal{N}$,

$$\mu_q^{(\hat{\omega})}\Bigl(\limsup_{k\to\infty} U_{\hat{\omega}}(k)\Bigr) = 1, \quad q \in \mathcal{A}.$$

This implies that, for $\hat{\omega} \in \hat{\Omega} \setminus \mathcal{N}$, ($\star$) holds with (2.6) replaced by (2.7) for $l_2 = \#T$. □

5 Concluding Remarks

We make some remarks about the main results.

(1) The arguments in this paper are valid for some other inhomogeneous fractals. For example, we can obtain similar results for higher-dimensional inhomogeneous Sierpinski gaskets. A crucial property required here is that the eigenfunctions of $A_i^{(\nu)}$ ($i \in S_0$) associated with the eigenvalues $r^{(\nu)}$ do not depend on ν.
(2) Since Condition (B) in Theorem 2.3 is a rather technical constraint, we focus on arguments that are valid more generally and we do not try to make the assumption as weak as possible by relying on concrete structures of fractals under consideration. Indeed, in Lemma 3.6(3), the part "there exists some $i' \in S_0$" can be strengthened to "any $i' \in S_0 \setminus \{i\}$." As a result, in Condition ($\star$), the part "for every $i, j \in S_0$" can be weakened to "for every $i \in S_0$, for $j = i$ and for some other $j \in S_0$."
(3) We reason that Theorem 2.3 holds true without assuming Condition (A) or (B) in practice.

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References

1. M.T. Barlow, *Diffusions on Fractals, Lectures on Probability Theory and Statistics (Saint-Flour, 1995)*. Lecture Notes in Math, vol. 1690. (Springer, Berlin, 1998), pp. 1–121
2. M.T. Barlow, Which values of the volume growth and escape time exponent are possible for a graph? Rev. Mat. Iberoamericana **20**, 1–31 (2004)
3. M.T. Barlow, B.M. Hambly, Transition density estimates for Brownian motion on scale irregular Sierpinski gaskets. Ann. Inst. H. Poincaré Probab. Statist. **33**, 531–557 (1997)
4. O. Ben-Bassat, R.S. Strichartz, A. Teplyaev, What is not in the domain of the Laplacian on Sierpinski gasket type fractals. J. Funct. Anal. **166**, 197–217 (1999)
5. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, 2nd revised and extended ed., De Gruyter Studies in Mathematics, vol. 19 (Walter de Gruyter, 2011)
6. B.M. Hambly, Brownian motion on a homogeneous random fractal. Probab. Theory Relat. Fields **94**, 1–38 (1992)

7. B.M. Hambly, Brownian motion on a random recursive Sierpinski gasket. Ann. Probab. **25**, 1059–1102 (1997)
8. B.M. Hambly, T. Kumagai, Asymptotics for the spectral and walk dimension as fractals approach Euclidean space. Fractals **10**, 403–412 (2002)
9. M. Hino, On singularity of energy measures on self-similar sets. Probab. Theory Relat. Fields **132**, 265–290 (2005)
10. M. Hino, K. Nakahara, On singularity of energy measures on self-similar sets II. Bull. London Math. Soc. **38**, 1019–1032 (2006)
11. N. Kajino, M. Murugan, On singularity of energy measures for symmetric diffusions with full off-diagonal heat kernel estimates. Ann. Probab. **48**, 2920–2951 (2020)
12. J. Kigami, *Analysis on fractals*. Cambridge Tracts in Mathematics, vol. 143 (Cambridge University Press, 2001)
13. S. Kusuoka, Dirichlet forms on fractals and products of random matrices. Publ. Res. Inst. Math. Sci. **25**, 659–680 (1989)
14. A.N. Shiryaev, *Probability*, 2nd ed., Graduate Texts in Mathematics, vol. 95 (Springer, Berlin, 1996)

On L^p Liouville Theorems for Dirichlet Forms

Bobo Hua, Matthias Keller, Daniel Lenz, and Marcel Schmidt

Abstract We study harmonic functions for general Dirichlet forms. First we review consequences of Fukushima's ergodic theorem for the harmonic functions in the domain of the L^p generator. Secondly we prove analogues of Yau's and Karp's Liouville theorems for weakly harmonic functions. Both say that weakly harmonic functions which satisfy certain L^p growth criteria must be constant. As consequence we give an integral criterion for recurrence.

Keywords Dirichlet forms · Liouville property · Superharmonic functions

1 Introduction

Liouville theorems for harmonic functions have a long tradition and circle around the idea that a harmonic function which satisfies certain boundedness conditions must be constant. The first result of this type for analytic functions was proven by Cauchy in 1844, see [1]. In 1847 Liouville presented the result in a lecture and that is how the name arose. Since then numerous results in various contexts were proven.

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Our setting is the one of Dirichlet forms. This setting includes on the one hand well studied objects such as Laplacians on (sub) Riemannian manifolds but also non-local operators arising from jump processes.

We start by stating the results and refer for the definitions to Sects. 2 and 3. For a general background on Dirichlet forms we refer to [6]. Let $\mathcal{E}$ be a Dirichlet form on $L^2(m)$, where m is a σ-finite measure. The associated self-adjoint operator generates a Markovian C_0-semigroup on $L^2(m)$, which for $p \in [1, \infty)$ extends to a Markovian C_0-semigroup on $L^p(m)$.

We first present a consequence of Fukushima's ergodic theorem [5] for harmonic functions in the domain of the generator of the L^p-semigroup, which we also refer to as the L^p-generator of the Dirichlet form.

Theorem 1 (Basic L^p-Liouville theorem) *For $p \in (1, \infty)$ any harmonic function in the domain of the L^p-generator of an irreducible Dirichlet form is constant. In particular, in case there is a non-trivial harmonic function in the domain of the L^p-generator, then m is a finite measure.*

Next, we restrict ourselves to regular Dirichlet forms where one has an intrinsic metric ρ in the sense of [4] for which all distance balls are precompact. The first result is an analogue of Yau's L^p-Liouville type theorem [16]. For some nonlocal operators on Euclidean space a related result is contained in [12]. Speaking about *(sub)harmonic* functions we refer here to weakly (sub)harmonic functions, see Definition 14.

Theorem 2 (Yau's L^p-Liouville theorem) *Let $\mathcal{E}$ be an irreducible regular Dirichlet form without killing with an intrinsic metric for which all distance balls are precompact. Let $p \in (1, \infty)$ and let $f \in L^p(m)$ be a non-negative subharmonic function. Then f is constant if one of the following additional conditions is satisfied:*

(a) $1 < p \leq 2$.
(b) *m is finite.*
(c) *$p > 2$ and $f \in L^q(m)$ for some $q \in [2p - 2, \infty]$.*
(d) *$p > 2$, the intrinsic metric has finite jump size and $f \in L^{2p-2}_{\text{loc}}(m)$.*

The next result is an analogue to Karp's L^p-Liouville theorem [10], which is found for strongly local regular Dirichlet forms in [15] and for graphs in [7, 8]. Here we additionally need that the intrinsic metric has finite jump size, see Sect. 3. We denote the distance balls with radius $r \geq 0$ about a fixed point $o \in X$ with respect to the intrinsic metric ρ by $B_r = \{x \in X \mid \rho(x, o) \leq r\}$.

Theorem 3 (Karp's L^p-Liouville theorem) *Let $\mathcal{E}$ be an irreducible regular Dirichlet form without killing with an intrinsic metric for which all distance balls are precompact and the jump size is finite. Let $p \in (1, \infty)$ and let $q = \max\{p, 2p - 2\}$. Then every non-negative subharmonic function $f \in L^q_{\text{loc}}(m)$ satisfying*

$$\int_{r_0}^{\infty} \frac{r}{\| f 1_{B_r} \|_p^p} dr = \infty$$

for all $r_0 > 0$ *is constant.*

Other than in the above mentioned references in the case $p > 2$ we need some more integrability than L^p for the analogue of Yau's theorem or L^p_{loc} for the analogue of Karp's theorem. The reason for this is a technical issue concerning the existence of certain integrals in our proof. In many concrete applications L^∞_{loc}-integrability (which is sufficient for our results) of (sub)harmonic functions is known. This follows either from hypoellipticity of the corresponding operators (which yields smoothness of harmonic functions) or, more generally, from local Hölder estimates for nonnegative subharmonic functions deduced by De Giorgi-Nash-Moser iteration. In this sense, the additional integrability assumptions we make can be seen as rather mild assumptions (at least when the jump size is finite).

In many special situations the positive and negative part of harmonic functions are positive subharmonic functions. This is for example the case for strongly local Dirichlet forms or Dirichlet forms on discrete sets. In these cases the results above directly imply the corresponding results for harmonic functions.

We can use Karp's Liouville theorem to prove an integral criterion for recurrence.

Theorem 4 (Recurrence) *Let* $\mathcal{E}$ *be an irreducible regular Dirichlet form without killing with an intrinsic metric for which all distance balls are precompact and the jump size is finite. If*

$$\int_{r_0}^{\infty} \frac{r}{m(B_r)} dr = \infty$$

for some $r_0 > 0$*, then* $\mathcal{E}$ *is recurrent.*

2 Fukushima's Ergodic Theorem

Let m be a σ-finite measure on (a σ-algebra on) X. Let $\mathcal{E}$ be a Dirichlet form on $L^2(m)$ with nonnegative generator L and associated Markovian semigroup $(T_t) = (e^{-tL})$. For $1 \leq p \leq \infty$, let (T_t^p) be the Markovian extension of the semigroup to $L^p(m)$. For $1 \leq p < \infty$, these are C_0-semigroups of contractions and we denote by L_p the corresponding generator [2]. For $p = \infty$, it is also a semigroup of contractions, which is only weak-*-continuous. These extension are compatible in the sense that for $p \neq q$ they agree on $L^p(m) \cap L^q(m)$. Hence, on the semigroup level we often omit the superscript p.

In this section a function $f \in L^p(m)$ is called L_p*-harmonic* if it belongs to the domain of L_p and $L_p f = 0$ holds. For $p = 2$, we also speak about L-harmonic functions instead of L_2-harmonic functions. Clearly, a function f is L-harmonic if and only if $\mathcal{E}(f) = 0$.

As usual we denote the dual pairing between $L^p(m)$ and $L^q(m)$ with $1/q + 1/p = 1$ by $(\cdot, \cdot)$.

Recall that a Dirichlet form $\mathcal{E}$ is called *conservative* if $T_t 1 = 1$. In 1982 Fukushima [5] proved that if $\mathcal{E}$ is conservative (and coming from an m-symmetric Markov transition function), then for $p \in (1, \infty)$ and all $f \in L^p(m)$

$$\lim_{t\to\infty} T_t f = g \quad m\text{-a.e.},$$

where $g \in L^p(m)$ is a (T_t)-invariant function. In particular, if $\mathcal{E}$ is additionally irreducible, then g is constant and equal to $m(X)^{-1} \int_X f dm$ whenever $m(X) < \infty$ and equal to 0 if $m(X) = \infty$. Now, one can argue that L_p-harmonic functions are invariant under the semigroup and therefore all harmonic functions must be constant in the above setting.

Indeed, the setting of [5] starts from an m-symmetric, conservative Markov transition function on a σ-finite measure space. The proof then relies on Rota's ergodic theorem [3]. Here, we give an analytic proof of a related result under weaker assumptions. We only need a (symmetric) Dirichlet form but only show strong convergence in $L^p(m)$ for $p \in (1, \infty)$. This is however enough to establish Theorem 1 along the same lines as discussed above.

Lemma 5 *If $\mathcal{E}$ is irreducible and* 0 *is an eigenvalue of L, then any corresponding eigenfunction must be constant.*

Proof Let φ be an eigenfunction to 0. As $\mathcal{E}$ is a Dirichlet form, we infer, for any normal contraction C,

$$0 \le \mathcal{E}(C\varphi) \le \mathcal{E}(\varphi) = 0$$

and, hence,

$$\mathcal{E}(C\varphi) = 0.$$

Thus, $C\varphi$ is an eigenfunction to 0 for any normal contraction C. Now, as $\mathcal{E}$ is irreducible, we know that the eigenspace to 0 is spanned by a unique function of fixed sign. The preceding reasoning then shows that the span of this function must be invariant under normal contractions. This is only possible if the function is constant. □

Definition 6 (*The ground state* Φ) If $\mathcal{E}$ is irreducible, we define Φ to be zero if 0 is not an eigenvalue of L and to be the unique positive constant eigenfunction to 0 with $\|\Phi\|_2 = 1$ otherwise.

Remark 1 (a) Let us emphasize that Φ is a constant function in $L^2(m)$ (in all situations). Hence, $\Phi \neq 0$ can only occur if $m(X) < \infty$. In this case, constant functions are eigenfunctions to 0 if and only if $\mathcal{E}$ is conservative. Indeed, if $m(X) < \infty$ and $\mathcal{E}$ is conservative, we have by definition $T_t 1 = 1$ for all $t > 0$. This shows

$$\lim_{h\to 0+} h^{-1}(T_h 1 - 1) = 0$$

in $L^2(m)$, so that $1 \in D(L)$ and $L1 = 0$. Conversely, if $1 \in D(L)$ with $L1 = 0$, we have $1 \in D(\mathcal{E})$ and $\mathcal{E}(1) = 0$. This implies conservativeness, see [6, Theorem 1.6.6].

In summary we obtain

$$\Phi = \begin{cases} \frac{1}{\sqrt{m(X)}} & \text{if } m(X) < \infty \text{ and } \mathcal{E} \text{ is conservative,} \\ & \text{else.} \end{cases}$$

(b) Since $\Phi \neq 0$ implies $m(X) < \infty$, the constant function Φ belongs to $L^p(m)$ for any $p \in [1, \infty]$. Thus, in both cases $\Phi = 0$ and $\Phi \neq 0$, the map

$$L^p(m) \to L^p(m), \quad f \mapsto (\Phi, f)\Phi$$

is well defined and continuous. By our discussion above it is given by

$$(\Phi, f)\Phi = \begin{cases} \frac{1}{m(X)} \int_X f\, dm & \text{if } m(X) < \infty \text{ and } \mathcal{E} \text{ is conservative,} \\ & \text{else.} \end{cases}$$

Lemma 7 *If $\mathcal{E}$ is irreducible, then any L-harmonic function is a multiple of Φ.*

Proof This is clear (as any L-harmonic function either vanishes or is an eigenfunction to 0). □

Lemma 8 *If $\mathcal{E}$ is irreducible, then for $p \in (1, \infty)$ and $f \in L^p(m)$*

$$\lim_{t\to\infty} T_t f = (\Phi, f)\Phi \qquad \textit{in } L^p(m).$$

Proof On the $L^2(m)$-level, the spectral theorem implies that $T_t f$ converges to the projection of f to the kernel of L, as $t \to \infty$. In our situation this reads

$$T_t f \xrightarrow{L^2} \langle \Phi, f \rangle \Phi,$$

see e.g. [11, Theorem 1.1]. For $f \in L^2(m) \cap L^\infty(m) \cap L^1(m) \subseteq L^p(m)$, we can estimate by Littlewood's inequality for L^p-spaces (i.e., Hölder inequality with a smart choice of parameters)

$$\begin{aligned} \|T_t f - (\Phi, f)\Phi\|_p &\le \|T_t f - (\Phi, f)\Phi\|_r^{1-\theta} \|T_t f - (\Phi, f)\Phi\|_2^\theta \\ &\le C_r \|f\|_r^{1-\theta} \|T_t f - (\Phi, f)\Phi\|_2^\theta, \end{aligned}$$

with $r = \infty$ and $\theta = 2/p$ if $p \ge 2$, and $r = 1$ and $\theta = 2(p-1)/p$, if $p \le 2$. The second inequality follows from the fact that (T_t) is a contraction on $L^1(m)$ and $L^\infty(m)$ and that the semigroups agree on $L^p(m) \cap L^2(m)$.

This estimate and our discussion on the L^2-case show that the desired convergence holds on a dense subspace of $L^p(m)$. Since the semigroups are uniformly bounded, it extends to all of $L^p(m)$. □

Remark 2 Our discussion after the definition of Φ shows that this is a version of Fukushima's ergodic theorem for semigroups associated with not necessarily conservative Dirichlet forms but with the weaker statement on L^p-convergence instead of m-a.e. convergence.

Lemma 9 *For $p \in [1, \infty)$, let $f \in L^p(m)$ be an L_p-harmonic function. Then, $T_p f = f$ for any $t \geq 0$.*

Proof By abstract theory for C_0-semigroups, for a given $g \in D(L_p)$, the function $[0, \infty) \to L^p(m)$, $t \mapsto T_t g$ is the unique solution in $C^1([0, \infty); L^p(m))$ of the Cauchy problem

$$\begin{cases} \dot{u}_t = L_p u_t & \text{for } t > 0, \\ u_0 = g. \end{cases}$$

Now, next to the solution $[0, \infty) \to L^p(m)$, $t \mapsto T_t f$, the function $[0, \infty) \to L^p(m)$, $t \mapsto f$ is continuously differentiable and solves the problem with initial value f. Thus, $T_t f = f$ for $t > 0$. □

Theorem 1 is a direct consequence of the following theorem.

Theorem 10 *Let $\mathcal{E}$ be an irreducible Dirichlet form and let f be an L_p-harmonic function for some $p \in (1, \infty)$. Then, f is constant, and in fact*

$$f = (\Phi, f)\Phi = \begin{cases} \frac{1}{m(X)} \int\limits_X f\,dm & \text{if } m(X) < \infty \text{ and } \mathcal{E} \text{ is conservative} \\ 0 & \text{else} \end{cases}.$$

Proof For $p \in (1, \infty)$, we infer from the previous two lemmas

$$f = T_t f \to (\Phi, f)\Phi, \text{ as } t \to \infty.$$

Combined with our computation of $(\Phi, f)\Phi$ this yields the result. □

3 Regular Dirichlet Forms and Harmonic Functions

3.1 Basic Notions and Intrinsic Metrics

We use the notation of the previous section. Moreover, from now on we additionally assume that X is a locally compact separable metric space, m is a Radon measure of full support and $\mathcal{E}$ is a regular Dirichlet form on $L^2(m)$, see [6].

By $C(X)$ we denote the space of continuous functions on X and by $C_c(X)$ the space of continuous functions of compact support.

As for continuous functions, we write $D(\mathcal{E})_c$ for the functions in $D(\mathcal{E})$ with compact support. We denote by $D(\mathcal{E})_{\text{loc}}$ the space of functions locally in $D(\mathcal{E})$, that is the set of functions f such that for every open precompact set G there is a function in $D(\mathcal{E})$ which coincides with f on G.

By the Beurling-Deny formula [6, Theorems 3.2.1 and 5.2.1] we have for $f, g \in D(\mathcal{E})$

$$\mathcal{E}(f) = \int_X d\Gamma^{(c)}(f) + \int_{X \times X \setminus d} (\tilde{f}(x) - \tilde{f}(y))^2 dJ(x, y) + \int_X \tilde{f}(x)^2 k(dx),$$

where $\Gamma^{(c)}$ is a measure valued strongly local quadratic form on $D(\mathcal{E})$ (i.e., $\Gamma^{(c)}(f, g) = 0$ if f is constant on a neighborhood of the support of g), J is a Radon measure on $X \times X \setminus d = \{(x, y) \in X \times X \mid x \neq y\}$ and k is a Radon measure on X. Moreover, $\tilde{f}$ is a quasi-continuous representative of f. Such a representative exists for $f \in D(\mathcal{E})$ by [6, Theorem 2.1.7] and the argument given there extends directly to $f \in D(\mathcal{E})_{\text{loc}}$, see e.g. [4, Proposition 3.1]. To simplify notation, below we will always choose a quasi-continuous representative and just write f instead of $\tilde{f}$. Since we exclusively deal with continuous functions or functions from $D(\mathcal{E})_{\text{loc}}$, this is always possible.

Assumption From now on we assume that $\mathcal{E}$ has no killing, i.e., $k = 0$.

Using the measure J from the decomposition we obtain a finite Radon measure valued quadratic form $\Gamma^{(j)}$ on $D(\mathcal{E})$, which is uniquely determined by

$$\int_K d\Gamma^{(j)}(f) = \int_{K \times X \setminus d} (f(x) - f(y))^2 dJ(x, y)$$

for all compact $K \subseteq X$ and $f \in D(\mathcal{E})$.

Next, we discuss how to extend $\Gamma^{(c)}$ and $\Gamma^{(j)}$ to larger classes of functions. We denote by $D(\mathcal{E})^*_{\text{loc}}$ the space of functions in $f \in D(\mathcal{E})_{\text{loc}}$ such that for every compact $K \subseteq X$

$$\int_{K \times X \setminus d} (f(x) - f(y))^2 dJ(x, y) < \infty.$$

This space was introduced in [4]. For later purposes, we note that any $f \in D(\mathcal{E})^*_{\text{loc}}$ with compact support belongs to $D(\mathcal{E})_{\text{c}}$, see [4, Theorem 3.5].

The bilinear forms on $D(\mathcal{E}) \times D(\mathcal{E})$ induced from $\Gamma^{(c)}$ and $\Gamma^{(j)}$ by polarization can be (partially) extended to bilinear forms $D(\mathcal{E})^*_{\text{loc}} \times D(\mathcal{E})_c$ with values in the space of finite signed Radon measures. More precisely, for $f \in D(\mathcal{E})^*_{\text{loc}}$ and $\varphi \in D(\mathcal{E})_c$ these extensions are characterized by

$$\int_K d\Gamma^{(c)}(f, \varphi) = \int_K d\Gamma^{(c)}(f_K, \varphi),$$

and

$$\int_K d\Gamma^{(j)}(f, \varphi) = \int_{K \times X \setminus d} (f(x) - f(y))(\varphi(x) - \varphi(y)) dJ(x, y),$$

for all compact $K \subseteq X$. Here, $f_K \in D(\mathcal{E})$ is a function with $f = f_K$ on a relatively compact open neighborhood of K. Due to the strong locality $\Gamma^{(c)}(f, \varphi)$ is well defined (i.e., independent of the choice of f_K, see the remarks after the proof of [6, Theorem 3.2.2]) and the definition of $D(\mathcal{E})^*_{\text{loc}}$ ensures that $\Gamma^{(j)}(f, \varphi)$ is well defined (i.e., it yields a finite signed Radon measure, see [4, Theorem 3.4]). Similarly, the quadratic forms $\Gamma^{(c)}$ and $\Gamma^{(j)}$ can be extended to quadratic forms on $D(\mathcal{E})^*_{\text{loc}}$ taking values in the set of nonnegative (not necessarily finite) Radon measures, see [4, Proposition 3.3]. We let $\Gamma = \Gamma^{(c)} + \Gamma^{(j)}$ denote both the quadratic form on $D(\mathcal{E})^*_{\text{loc}}$ and the induced bilinear form on $D(\mathcal{E})^*_{\text{loc}} \times D(\mathcal{E})_c$.

Since $\mathcal{E}$ is a Dirichlet form, for any normal contraction $C\colon \mathbb{R} \to \mathbb{R}$ and $f \in D(\mathcal{E})^*_{\text{loc}}$, we have $C \circ f \in D(\mathcal{E})^*_{\text{loc}}$ and

$$\int_X d\Gamma(C \circ f) \leq \int_X d\Gamma(f).$$

Since the killing k vanishes, this inequality extends to 1-Lipschitz functions. Indeed, if $C\colon \mathbb{R} \to \mathbb{R}$ is 1-Lipschitz, then $C \circ f = \tilde{C}(f) + C(0)$ with the normal contraction $\tilde{C} = C - C(0)$. Since the constant function 1 belongs to $D(\mathcal{E})^*_{\text{loc}}$, see [4, Proposition 3.1] and satisfies $\Gamma(1) = 0$, this yields the claim.

The strongly local part satisfies a Leibniz rule. If $f, g \in D(\mathcal{E})^*_{\text{loc}}$ such that $fg \in D(\mathcal{E})^*_{\text{loc}}$, then

$$d\Gamma^{(c)}(fg, \varphi) = f d\Gamma^{(c)}(g, \varphi) + g d\Gamma^{(c)}(f, \varphi)$$

for all $\varphi \in D(\mathcal{E})_c$, see the proof of [4, Theorem 3.9]. Moreover, it satisfies a chain rule. If $C \in C^1(\mathbb{R})$ with bounded derivative, then for $f \in D(\mathcal{E})^*_{\text{loc}}$, we have $C \circ f \in D(\mathcal{E})^*_{\text{loc}}$ and

$$\Gamma^{(c)}(C(f), \varphi) = C'(f)\Gamma^{(c)}\Gamma(f, \varphi)$$

for all $\varphi \in D(\mathcal{E})_c$, see [6, Theorem 3.2.2].

Let $\rho\colon X \times X \to [0, \infty)$ be a pseudo metric, that is a symmetric map with zero diagonal satisfying the triangle inequality. For measurable sets $A \subseteq X$, we write $\rho_A = \inf_{x \in A} \rho(x, \cdot)$. The following definition is a slight modification of the one given in [4].

Definition 11 (*Intrinsic metric*) A pseudo metric ρ on X is called *intrinsic* (for the regular Dirichlet form $\mathcal{E}$) if it is continuous with respect to the topology on X

and there are Radon measures $m^{(c)}$, $m^{(j)}$ with $m^{(c)} + m^{(j)} \leq m$ such that for every measurable $A \subseteq X$ the following holds: $\rho_A \in D(\mathcal{E})^*_{\text{loc}}$ and

$$\Gamma^{(c)}(\rho_A) \leq m^{(c)} \quad \text{and} \quad \Gamma^{(j)}(\rho_A) \leq m^{(j)}.$$

Remark 3 In [4], ρ is allowed to take the value infinity. In this sense our definition is a bit more restrictive.

Definition 12 The *jump size* s of a continuous pseudo metric ρ with respect to the Dirichlet form $\mathcal{E}$ is given by

$$s := \inf\{t \geq 0 \mid J(\{(x, y) \in X \times X \setminus d \mid \rho(x, y) > t\}) = 0\} \in [0, \infty].$$

If ρ is a fixed pseudo-metric, for $U \subseteq X$ and $r \in \mathbb{R}$ we write

$$B_r(U) = \{x \in X \mid \rho(x, y) \leq r \text{ for some } y \in U\}.$$

The relevance of intrinsic metrics comes from the fact that Lipschitz functions with respect to intrinsic metrics induce good cut-off functions. This is discussed next.

Fix $o \in X$ and let $0 \leq r < R$. Below we will make use of

$$\eta = \eta_{r,R} \colon X \to \mathbb{R}, \quad \eta(x) = 1 \wedge \left(\frac{R - \rho(x, o)}{R - r}\right)_+ .$$

It is $(R - r)^{-1}$-Lipschitz and satisfies $\eta = 1$ on B_r and $\eta = 0$ on $X \setminus B_R$.

Lemma 13 *Let ρ be an intrinsic metric for $\mathcal{E}$ with jump size s. Then $\eta \in D(\mathcal{E})^*_{\text{loc}}$ and it satisfies*

$$\Gamma^{(c)}(\eta) \leq \frac{1}{(R - r)^2} 1_{B_R \setminus B_r} m^{(c)} \text{ and } \Gamma^{(j)}(\eta) \leq \frac{1}{(R - r)^2} 1_{B_{R+s} \setminus B_{r-s}} m^{(j)},$$

where $B_{R+s} \setminus B_{r-s} = X$ if $s = \infty$. If, moreover, balls with respect to ρ are precompact, then $\eta \in D(\mathcal{E})_c$.

Proof The first statement and the estimates are the content of [4, Theorem 4.9 and Proposition 8.5]. The last statement follows from the fact that functions in $D(\mathcal{E})^*_{\text{loc}}$ with compact support belong to $D(\mathcal{E})_c$. $\square$

If ρ-balls are precompact, the previous lemma and the definitions of $\Gamma^{(c)}$ and $\Gamma^{(j)}$ yield for $f \in D(\mathcal{E})^*_{\text{loc}}$

$$\int_X d\Gamma^{(c)}(f, \eta) = \int_{B_R \setminus B_r} d\Gamma^{(c)}(f, \eta),$$

and

$$\int_{X\times X\setminus d} (f(x)-f(y))(\eta(x)-\eta(y))dJ(x,y)$$
$$=\int_{U_{r,R}} (f(x)-f(y))(\eta(x)-\eta(y))dJ(x,y)$$

with $U_{r,R} = (B_{R+s} \setminus B_{r-s}) \times (B_{R+s} \setminus B_{r-s}) \setminus d$, whenever the integral makes sense.

3.2 Harmonic Functions

In this subsection we introduce (sub)harmonic functions and discuss their basic properties.

Definition 14 A function $f : X \to \mathbb{R}$ is called *harmonic* (*subharmonic*) if $f \in D(\mathcal{E})^*_{\mathrm{loc}}$ and

$$\int_X d\Gamma(f,\varphi) = 0 \quad (\leq 0),$$

for all $0 \leq \varphi \in D(\mathcal{E})_c$.

Remark 4 Since the domains of Dirichlet forms are lattices, a harmonic function f satisfies $\int_X d\Gamma(f,\varphi) = 0$ for all $\varphi \in D(\mathcal{E})_c$.

Below we will need the following extension of Lemma 5 to $D(\mathcal{E})^*_{\mathrm{loc}}$ functions. It can also be viewed as a Liouville-type theorem.

Lemma 15 *Let $\mathcal{E}$ be irreducible. Then any $f \in D(\mathcal{E})^*_{\mathrm{loc}}$ with $\Gamma(f) = 0$ is constant.*

Proof We show that for each $\alpha > 0$ the set $A_\alpha = \{f > \alpha\}$ is $\mathcal{E}$-invariant. Since $\mathcal{E}$ is irreducible, this is only possible if f is constant. To this end, we show $\mathcal{E}(1_{A_\alpha}\varphi) \leq \mathcal{E}(\varphi)$ which implies the $\mathcal{E}$-invariance, see e.g. [13, Lemma 2.32].

We have

$$1_{A_\alpha} = \lim_{n\to\infty} (n(f-\alpha)_+) \wedge 1$$

pointwise m-a.e. Let $g_n = (n(f-\alpha)_+) \wedge 1$. Up to a constant g_n is a contraction of f and so we have

$$\Gamma(g_n) \leq n^2\Gamma(f) = 0.$$

Now let $\varphi \in D(\mathcal{E}) \cap C_c(X)$ be given. Then $\varphi g_n \in D(\mathcal{E})$ (here we use that φg_n has compact support and belongs to $D(\mathcal{E})^*_{\mathrm{loc}} \cap L^\infty(m)$ by [4, Proposition 3.10]).

The Leibniz rule for $\Gamma^{(c)}$, the "discrete Leibniz rule" (i.e., $f(x)g(x) - f(y)g(y) = f(x)(g(x) - g(y)) + g(y)(f(x) - f(y))$ for functions f, g on X and $x, y \in X$) and the Cauchy-Schwarz inequality yield

$$\begin{aligned}
\mathcal{E}(\varphi g_n) &= \int_X \varphi d\Gamma(g_n, g_n\varphi) + \int_X g_n d\Gamma(\varphi, g_n\varphi) \\
&\le \left(\int_X d\Gamma(g_n)\right)^{1/2} \left(\int_X |\varphi|^2 d\Gamma(\varphi g_n)\right)^{1/2} + \left(\int_X |g_n|^2 d\Gamma(\varphi)\right)^{1/2} \left(\int_X d\Gamma(\varphi g_n)\right)^{1/2} \\
&\le \mathcal{E}(\varphi)^{1/2}\mathcal{E}(\varphi g_n)^{1/2}.
\end{aligned}$$

This shows $\mathcal{E}(\varphi g_n) \le \mathcal{E}(\varphi)$. Since by dominated convergence $1_{A_\alpha}\varphi = \lim_{n\to\infty} \varphi g_n$ in $L^2(m)$, the lower semicontinuity of $\mathcal{E}$ and this inequality yield $1_{A_\alpha}\varphi \in D(\mathcal{E})$ and

$$\mathcal{E}(1_{A_\alpha}\varphi) \le \mathcal{E}(\varphi).$$

Since $D(\mathcal{E}) \cap C_c(X)$ is dense in $D(\mathcal{E})$, this inequality extends to all $\varphi \in D(\mathcal{E})$ (use lower semicontinuity) and we obtain the $\mathcal{E}$-invariance of A_α. □

4 A Caccioppoli Inequality

The following Caccioppoli-type inequality is the key to Yau's L^p-Liouville theorem. A somewhat more subtle estimate is needed for Karp's result. Throughout $\mathcal{E}$ is a regular Dirichlet form without killing and ρ is an intrinsic metric for $\mathcal{E}$.

Theorem 16 (Caccioppoli-type inequality) *Assume the distance balls of ρ are precompact and $p \in (1, \infty)$. Then, there is $C > 0$ such that for every non-negative subharmonic function f and all $0 < r < R$ such that $f 1_{B_{R+s}\setminus B_{r-s}} \in L^q(m)$ for some $q \in [2p-2, \infty]$ we have*

$$\begin{aligned}
&\int_{B_r} f^{p-2} d\Gamma^{(c)}(f) + \int_{X\times B_r\setminus d} (f(x) \vee f(y))^{p-2}(f(x) - f(y))^2 dJ(x, y) \\
&\qquad \le \frac{C}{(R-r)^2} \| f 1_{B_{R+s}\setminus B_{r-s}} \|_p^p,
\end{aligned}$$

where s is the jump size of ρ.

Remark 5 (a) Note that $p \le 2$ if and only if $2p - 2 \le p$. Hence, in this case, $f 1_{B_{R+s}\setminus B_{r-s}} \in L^q(m)$ for some $q \ge 2p - 2$ is already satisfied if the right side of the Caccioppoli-type inequality is finite.

(b) If the jump-size of ρ is finite, the set B_{R+s} is precompact and hence has finite measure. In this case, for $p > 2$ the assumption on the integrability on f is satisfied for all $0 \le r < R$ if and only if $f \in L^{2p-2}_{\mathrm{loc}}(m)$.

Remark 6 For $p \ge 2$ the inequality above yields (with a larger constant C and under suitable conditions on f)

$$\int_X f^{p-2} d\Gamma(f) \le \frac{C}{(R-r)^2} \| f 1_{B_{R+s} \setminus B_{r-s}} \|_p^p.$$

Let $\mathrm{Lip}_c(X)$ be the space of Lipschitz continuous functions with compact support with respect to the intrinsic metric ρ.

Lemma 17 (The key estimate) *Let $p \in (1, \infty)$ and let $n \in \mathbb{N}$. For every non-negative subharmonic function f and $\varphi \in \mathrm{Lip}_c(X)$, we have*

$$\begin{aligned}
&\int_{A_n} f^{p-2} \varphi^2 d\Gamma^{(c)}(f) + \int_{X \times X \setminus d} (f_n(x) \vee f_n(y))^{p-2} \varphi(y)^2 (f_n(x) - f_n(y))^2 dJ(x,y) \\
&\le -C \int_X f_n^{p-1} \varphi d\Gamma^{(c)}(f, \varphi) \\
&\quad - C \int_{X \times X \setminus d} f_n^{p-1}(x) \varphi(y)(f(x) - f(y))(\varphi(x) - \varphi(y)) dJ(x,y) < \infty,
\end{aligned}$$

where $f_n = f \wedge n$, $A_n = \{f < n\}$ and $C = 2/((p-1) \wedge 1)$.

Proof For $j \in \mathbb{N}$, we let $g_j = f_n \vee j^{-1}$. We show the inequality with f_n replaced by g_j and A_n replaced by $\{j^{-1} < f < n\}$. Then the statement follows from Fatou's lemma and Lebegue's dominated convergence theorem after letting $j \to \infty$. Note that the integral on the right hand side of the equation exists due to the boundedness of f_n and the definition of $D(\mathcal{E})^*_{\mathrm{loc}}$.

Since $g_j \in D(\mathcal{E})^*_{\mathrm{loc}}$ is bounded from below by $1/j$ and from above by n, we have $g_j^{p-1} \in D(\mathcal{E})^*_{\mathrm{loc}}$ (here we use that $[j^{-1}, n] \to \mathbb{R}$, $t \mapsto t^{p-1}$ is Lipschitz). Using the boundedness of g_j^{p-1}, for $\varphi \in \mathrm{Lip}_c(X) \subseteq D(\mathcal{E})_c \cap L^\infty(m)$ we obtain $\varphi^2 g_j^{p-1} \in D(\mathcal{E})_c$ as $D(\mathcal{E})_c \cap L^\infty(m)$ is an ideal in the algebra $D(\mathcal{E}) \cap L^\infty(m)$. The subharmonicity and non-negativity of f yields

$$0 \ge \int_X d\Gamma^{(c)}(f, \varphi^2 g_j^{p-1}) + \int_X d\Gamma^{(j)}(f, \varphi^2 g_j^{p-1}).$$

To shorten notation, we write $\nabla_{xy} g = g(x) - g(y)$ for $x, y \in X$ and functions g. We apply the "discrete Leibniz rules" $\nabla_{xy}(fg) = g(x)\nabla_{xy} f + f(y)\nabla_{xy} g$ and $\nabla_{xy} f^2 = 2f(y)\nabla_{xy} f + (\nabla_{xy} f)^2$ to the jump term

$$
\begin{aligned}
&\int_X d\Gamma^{(j)}(f, \varphi^2 g_j^{p-1}) \\
&= \int_{X\times X\setminus d} \nabla_{xy} f \cdot \big(\varphi^2(y)\nabla_{xy} g_j^{p-1} + g_j^{p-1}(x)\nabla_{xy}\varphi^2\big) dJ(x,y) \\
&= \int_{X\times X\setminus d} \nabla_{xy} f \cdot \big(\varphi^2(y)\nabla_{xy} g_j^{p-1} + 2 g_j^{p-1}(x)\varphi(y)\nabla_{xy}\varphi + g_j^{p-1}(x)(\nabla_{xy}\varphi)^2\big) dJ(x,y).
\end{aligned}
$$

We start by estimating the first term, where we apply [8, Lemma 2.9], which states that if $\nabla_{xy} g \geq 0$ for a non-negative function g, then

$$
\nabla_{xy} g^{p-1} \geq C(g(x) \vee g(y))^{p-2} \nabla_{xy} g
$$

for the constant $C = (p-1) \wedge 1$. Since $\nabla_{xy} f \geq 0$ implies $\nabla_{xy} g_j^{p-1} \geq 0$, $\nabla_{xy} f \nabla_{xy} g_j^{p-1} = \nabla_{yx} f \nabla_{yx} g_j^{p-1} \geq 0$ and $\nabla_{xy} f \nabla_{xy} g_j = \nabla_{yx} f \nabla_{yx} g_j \geq |\nabla_{xy} g_j|^2$, this gives

$$
\varphi^2(y)\nabla_{xy} f \cdot \nabla_{xy} g_j^{p-1} \geq C\varphi^2(y)(g_j(x) \vee g_j(y))^{p-2} |\nabla_{xy} g_j|^2
$$

for the first term in the estimate above. We leave the second term as it is for now. For third term we obtain with having $\nabla_{xy} f \nabla_{xy} g_j \geq 0$ in mind

$$
\begin{aligned}
0 &\leq \int_{X\times X\setminus d} \nabla_{xy} g_j^{p-1} \nabla_{xy} f (\nabla_{xy}\varphi)^2 dJ(x,y) \\
&= 2\int_{X\times X\setminus d} g_j^{p-1}(x)\nabla_{xy} f (\nabla_{xy}\varphi)^2 dJ(x,y),
\end{aligned}
$$

where the equality holds as the term on the right hand side converges absolutely by Hölder's inequality. Hence, we obtain

$$
\begin{aligned}
\int_X d\Gamma^{(j)}(f, \varphi^2 g_j^{p-1}) &\geq C \int_{X\times X\setminus d} \varphi^2(y)(g_j(x) \vee g_j(y))^{p-2} |\nabla_{xy} g_j|^2 dJ(x,y) \\
&\quad + 2\int_{X\times X\setminus d} g_j^{p-1}(x)\varphi(y)\nabla_{xy} f \nabla_{xy}\varphi \, dJ(x,y).
\end{aligned}
$$

Applying the Leibniz rule and the chain rule to the strongly local term we get immediately

$$\int_X d\Gamma^{(c)}(f,\varphi^2 g_j^{p-1}) = (p-1)\int_X \varphi^2 g_j^{p-2} d\Gamma^{(c)}(f,g_j) + 2\int_X g_j^{p-1}\varphi d\Gamma^{(c)}(f,\varphi)$$
$$= (p-1)\int_{\{m^{-1}<f<n\}} \varphi^2 f^{p-2} d\Gamma^{(c)}(f) + 2\int_X g_j^{p-1}\varphi d\Gamma^{(c)}(f,\varphi).$$

For the last equality we used the truncation property of $\Gamma^{(c)}$ discussed in [15, Appendix 4.1]. Putting the two estimates for the terms $\int_X d\Gamma^{(j)}(f,\varphi^2 g_j^{p-1})$ and $\int_X d\Gamma^{(c)}(f,\varphi^2 g_j^{p-1})$ into the inequality at the beginning of the proof yields the statement. □

Proof (*Proof of the Caccioppoli-type inequality, Theorem* 16) Let $\eta = \eta_{r,R}$ be the cut-off functions discussed in Lemma 13. We use $\varphi = \eta = \eta_{r,R}$ in our key estimate and deal with the terms on the right-hand side separately. Without loss of generality we can assume $f 1_{B_{R+s}\setminus B_{r-s}} \in L^p(m)$ for otherwise there is nothing to show. We further assume $q < \infty$ as it will become clear in the course of the proof that the case $q = \infty$ is simpler and follows along the same lines.

We use the inequality $|Q(u,v)| \le \varepsilon Q(u,u)^2 + \frac{1}{4\varepsilon} Q(v,v)^2$, which holds for nonnegative bilinear forms Q and $\varepsilon > 0$, and $\eta \le 1$ to estimate

$$C\left|\int_{A_n} f_n^{p-1}\eta d\Gamma^{(c)}(f,\eta)\right| \le \frac{1}{2}\int_{A_n} f_n^{p-2}\eta^2 d\Gamma^{(c)}(f) + C'\int_{A_n} f^p d\Gamma^{(c)}(\eta)$$
$$\le \frac{1}{2}\int_{A_n} f_n^{p-2}\eta^2 d\Gamma^{(c)}(f) + \frac{C'}{(R-r)^2}\|f^p 1_{B_R\setminus B_r}\|_p^p,$$

where $f_n = f \wedge n$ and $A_n = \{f < n\}$. Moreover, $f_n^{p-1} = n^{p-1}$ on $X \setminus A_n$, Cauchy-Schwarz inequality, the cut-off properties of η and Chebyshev's inequality yield

$$\left|\int_{X\setminus A_n} f_n^{p-1}\eta d\Gamma^{(c)}(f,\eta)\right| \le n^{p-1}\left(\int_{X\setminus A_n} d\Gamma^{(c)}(\eta)\right)^{1/2}\left(\int_{X\setminus A_n} \eta^2 d\Gamma^{(c)}(f)\right)^{1/2}$$
$$\le \frac{n^{p-1}}{(R-r)} m(\{f \ge n\}\cap B_R\setminus B_r)^{1/2}\left(\int_{X\setminus A_n} \eta^2 d\Gamma^{(c)}(f)\right)^{1/2}$$
$$\le \frac{n^{p-1-q/2}}{(R-r)}\|f 1_{B_R\setminus B_r}\|^{q/2}\left(\int_{X\setminus A_n} \eta^2 d\Gamma^{(c)}(f)\right)^{1/2}.$$

Since $\infty > q \geq 2p - 2$ and $f 1_{B_R \setminus B_r} \in L^q(m)$, the right side of this inequality converges to 0, as $n \to \infty$. A similar reasoning yields

$$\begin{aligned}
&C \left| \int\limits_{A_n \times X \setminus d} f_n^{p-1}(x)\eta(y)(f(x) - f(y))(\eta(x) - \eta(y))dJ(x, y) \right| \\
&\leq C \int\limits_{A_n \times X \setminus d} (f_n(x) \vee f_n(y))^{p-1}\eta(y)|f(x) - f(y)||\eta(x) - \eta(y)|dJ(x, y) \\
&\leq \frac{1}{2} \int\limits_{A_n \times X \setminus d} (f_n(x) \vee f_n(y))^{p-2}\eta^2(y)(f(x) - f(y))^2 dJ(x, y) \\
&\quad + C' \int\limits_{A_n \times X \setminus d} (f_n(x) \vee f_n(y))^{p}(\eta(x) - \eta(y))^2 dJ(x, y) \\
&\leq \frac{1}{2} \int\limits_{A_n \times X \setminus d} (f_n(x) \vee f_n(y))^{p-2}\eta^2(y)(f(x) - f(y))^2 dJ(x, y) \\
&\quad + \frac{C''}{(R - r)^2} \| f^p 1_{B_{R+s} \setminus B_{r-s}} \|_p^p.
\end{aligned}$$

For the last inequality we used the properties of η and $(a \wedge b)^p \leq a^p + b^p$ as well as the symmetry of J. Moreover, as for the strongly local part we obtain using the cut-off properties of η

$$\begin{aligned}
&\int\limits_{(X \setminus A_n) \times X \setminus d} f_n(x)^{p-1}\eta(y)|f(x) - f(y)||\eta(x) - \eta(y)|dJ(x, y) \\
&\qquad \leq \frac{n^{p-1-q/2}}{R - r} \| f 1_{B_{R+s} \setminus B_{r-s}} \|^{q/2} \left(\int\limits_{X \setminus A_n} \eta^2 d\Gamma^{(j)}(f) \right)^{1/2},
\end{aligned}$$

with the right side converging to 0, as $n \to \infty$.

Plugging these estimates into our key estimate in Lemma 17, using $f = f_n$ on A_n and $\eta = 1$ on B_r, yields a constant $C > 0$ such that for all $n \in \mathbb{N}$

$$\begin{aligned}
&\int\limits_{A_n \cap B_r} f^{p-2} d\Gamma^{(c)}(f) + \int\limits_{A_n \times (A_n \cap B_r) \setminus d} (f(x) \vee f(y))^{p-2}(f(x) - f(y))^2 dJ(x, y) \\
&\leq \frac{C}{(R - r)^2} \| f 1_{B_{R+s} \setminus B_{r-s}} \|_p^p + E_n,
\end{aligned}$$

with $E_n \to 0$, as $n \to \infty$. With this at hand the statement follows after letting $n \to \infty$.

In the case $q = \infty$ we can use the same estimates as above without invoking the limit $n \to \infty$, as in this case $A_n = X$ for some $n \in \mathbb{N}$ and $E_n = 0$ for some $n \in \mathbb{N}$. □

For proving Yau's theorem the Caccioppoli-type inequality is sufficient. For Karp's theorem we need one further estimate, which follows from our key estimate and the Caccioppoli-type inequality. Here, $\eta = \eta_{r,R}$ denotes the cut-off function discussed in Lemma 13.

Lemma 18 *Assume the distance balls of ρ are precompact and the jump-size s is finite. For $p \in (1, \infty)$ there is $C > 0$ such that for every non-negative subharmonic function $f \in L^q_{\mathrm{loc}}(m)$ with $q = \max\{p, 2p-2\}$ and all $0 < r < R$ we have*

$$\left(\int_X f^{p-2}\eta^2 d\Gamma^{(c)}(f) + \int_{X\times X\setminus d} (f(x)\vee f(y))^{p-2}\eta^2(y)(f(x)-f(y))^2 dJ(x,y)\right)^2$$
$$\leq \frac{C}{(R-r)^2}\|f1_{B_{R+s}\setminus B_{r-s}}\|_p^p\left(\int_{B_R\setminus B_r} f^{p-2}\eta^2 d\Gamma^{(c)}(f)\right.$$
$$\left. + \int_{U_{R+s}\setminus U_{r-s}} (f(x)\vee f(y))^{p-2}\eta^2(y)(f(x)-f(y))^2 dJ(x,y)\right),$$

with $\eta = \eta_{r,R}$ and $U_r = B_r \times B_r \setminus d$.

Proof We apply our key estimate Lemma 17 to f with the cut-off function $\eta = \eta_{r,R}$. Then, the Cauchy-Schwarz inequality and the cut-off properties of η yield

$$\left(\int_{A_n} f^{p-2}\eta^2 d\Gamma^{(c)}(f) + \int_{X\times X\setminus d} (f_n(x)\vee f_n(y))^{p-2}\eta(y)^2(f_n(x)-f_n(y))^2 dJ(x,y)\right)^2$$
$$\leq \frac{C}{(R-r)^2}\|f_n 1_{B_{R+s}\setminus B_{r-s}}\|_p^p\left(\int_{B_R\setminus B_r} f_n^{p-2}\eta^2 d\Gamma^{(c)}(f)\right.$$
$$\left. + \int_{U_{R+s}\setminus U_{r-s}} (f_n(x)\vee f_n(y))^{p-2}\eta^2(y)(f(x)-f(y))^2 dJ(x,y)\right).$$

By our Caccioppoli-type inequality Theorem 16 all of the integrals on the right side exist when f_n is replaced by f (here we use that B_{R+s} is precompact, $0 \leq \eta \leq 1$ with $\eta = 0$ on $X \setminus B_R$ and the integrability assumption on f). Hence, we can take the limit $n \to \infty$ to obtain the statement. □

5 Proof of Yau's and Karp's Theorem and Recurrence

In this section we use the results from the previous sections to prove Yau's and Karp's theorem, Theorems 2 and 3. Later in this section we prove the growth test for recurrence, Theorem 4.

5.1 Proof of Yau's and Karp's Theorem

The proofs of our Liouville theorems are based on the following observation.

Lemma 19 *Let $\mathcal{E}$ be irreducible and let $f \in D(\mathcal{E})^*_{\mathrm{loc}}$ be non-negative. If for some $p \in (1, \infty)$*

$$\int_X f^{p-2} d\Gamma^{(c)}(f) + \int_{X\times X\setminus d} (f(x) \vee f(y))^{p-2}(f(x) - f(y))^2 dJ(x, y) = 0,$$

then f is constant.

Proof By the truncation property of the strongly local measure $\Gamma^{(c)}$ and $f \geq 0$ we have

$$\Gamma^{(c)}(f) = \Gamma^{(c)}(f_+) = 1_{\{f>0\}}\Gamma^{(c)}(f),$$

see [15, Appendix 4.1]. Hence, $\int_X f^{p-2} d\Gamma^{(c)}(f) = 0$ implies $\Gamma^{(c)}(f) = 0$. Similarly, $\int_{X\times X\setminus d}(f(x) \vee f(y))^{p-2}(f(x)-f(y))^2 dJ(x, y) = 0$ implies $\int_{X\times X\setminus d}(f(x) - f(y))^2 dJ(x, y) = 0$. These two observations and Lemma 15 show the claim. □

Proof (*Proof of Yau's L^p-Liouville theorem, Theorem* 2) The assumptions we made guarantee that we can apply the Caccioppoli-type inequality Theorem 16 for all $0 < r < R$. In this inequality letting first $R \to \infty$ and then $r \to \infty$ yields

$$\int_X f^{p-2} d\Gamma^{(c)}(f) + \int_{X\times X\setminus d} (f(x) \vee f(y))^{p-2}(f(x) - f(y))^2 dJ(x, y) = 0.$$

Hence, the previous lemma implies that f is constant. □

Proof (*Proof of Karp's theorem, Theorem* 3) With Lemma 18 at hand we basically follow [15]. Suppose $f \neq 0$. Let $R \geq 4s$ be such that $f 1_{B_R} \neq 0$. For $n \in \mathbb{N}$, we define $R_n = 2^n R$, $\eta_n = \eta_{R_{n-1}+s, R_n - s}$, $v_n = \|f 1_{B_{R_n}\setminus B_{R_{n-1}}}\|_p^p$ and

$$Q_n = \int_{B_{R_n}} f^{p-2}\eta_n^2 d\Gamma^{(c)}(f) + \int_{U_{R_n}} (f(x) \vee f(y))^{p-2}\eta_n^2(y)(f(x) - f(y))^2 dJ(x, y),$$

with $U_r = B_r \times B_r \setminus d$. The assumption $\int^{\infty} r/\|f1_{B_r}\|_p^p dr = \infty$ implies

$$\sum_{n=1}^{\infty} \frac{R_n^2}{v_n} = \infty.$$

Once we show $Q_1 = 0$, the assertion follows from Lemma 19 after letting $R \to \infty$.

Lemma 18 (applied to $r = R_{n-1} + s$ and $R = R_n + s$) and $\eta_{n-1} \leq \eta_n$ yield

$$Q_{n-1}Q_n \leq Q_n^2 \leq \frac{Cv_n}{(R_n - R_{n-1} - 2s)^2}(Q_n - Q_{n-1}) \leq \frac{16Cv_n}{R_n^2}(Q_n - Q_{n-1}).$$

For the last inequality we used $R \geq 4s$ and $R_n = 2^n R$. If $Q_1 > 0$, this would imply

$$\frac{R_n^2}{v_n} \leq 16C\left(\frac{1}{Q_{n-1}} - \frac{1}{Q_n}\right)$$

and hence

$$\sum_{n=2}^{\infty} \frac{R_n^2}{v_n} \leq \frac{16C}{Q_1} < \infty,$$

a contradiction. □

5.2 *Proof of the Growth Test for Recurrence*

In this subsection we discuss how our version of Karp's theorem can be used to deduce the volume growth test for recurrence Theorem 4. As in the previous sections we assume that $\mathcal{E}$ is a regular Dirichlet form without killing.

In order to prove the theorem we employ some abstract results from [9], to which we also refer to for the following facts about extended Dirichlet spaces. Recall that a function $f \in L^0(m)$ is said to be in the extended Dirichlet space $D(\mathcal{E}_e)$ of $\mathcal{E}$ if there is an *approximating sequence* for f, i.e., an $\mathcal{E}$-Cauchy sequence (f_n) in $D(\mathcal{E})$ such that m-a.e. $f_n \to f$. Let $f \in D(\mathcal{E}_e)$ and let (f_n) be an approximating sequence. We define

$$\mathcal{E}_e(f) = \lim_{n\to\infty} \mathcal{E}(f_n).$$

This is indeed independent of the choice of the sequence (f_n) such that $\mathcal{E}_e$ is an extension of $\mathcal{E}$. Moreover, it holds that $D(\mathcal{E}) = D(\mathcal{E}_e) \cap L^2(m)$. We need the following observation, which shows that our extension of the form via the measure-valued quadratic form Γ is compatible with the extended Dirichlet space.

Lemma 20 *Let* $f \in D(\mathcal{E}_e) \cap L^\infty(m)$*. Then* $f \in D(\mathcal{E})^*_{\text{loc}}$ *and*

$$\mathcal{E}_e(f) = \int_X d\Gamma(f).$$

Proof Let $f \in D(\mathcal{E}_e) \cap L^\infty(m)$. We first show $f \in D(\mathcal{E})_{\text{loc}}$. For a compact $K \subseteq X$ we choose $\varphi \in D(\mathcal{E}) \cap C_c(X)$ with $0 \le \varphi \le 1$ and $\varphi = 1$ on K. According to [6, Exercise 1.4.1] such a function always exists. Then $\varphi f = f$ on K and $\varphi f \in L^2(m)$. Since $D(\mathcal{E}_e) \cap L^\infty(m)$ is an algebra and $D(\mathcal{E}_e) \cap L^2(m) = D(\mathcal{E})$, we obtain $\varphi f \in D(\mathcal{E})$. This shows $f \in D(\mathcal{E})_{\text{loc}}$.

Now let (f_n) be an approximating sequence for f. Without loss of generality $\|f_n\|_\infty \le \|f\|_\infty$ and according to [6, Theorem 2.1.7] we can additionally assume $f_n \to f$ q.e. (recall that we always choose quasi-continuous representatives). Since the jump measure does not charge sets of capacity 0, Fatou's lemma yields

$$\begin{aligned}
\int_{X \times X \setminus d} (f(x) - f(y))^2 dJ(x,y) &\le \liminf_{n\to\infty} \int_{X \times X \setminus d} (f_n(x) - f_n(y))^2 dJ(x,y) \\
&\le \liminf_{n\to\infty} \mathcal{E}(f_n) \\
&= \mathcal{E}_e(f) < \infty.
\end{aligned}$$

In particular, this implies $f \in D(\mathcal{E})^*_{\text{loc}}$.

We will show $\int_X d\Gamma(f_n) \to \int_X d\Gamma(f)$ as this yields

$$\int_X d\Gamma(f) = \lim_{n\to\infty} \int_X d\Gamma(f_n) = \lim_{n\to\infty} \mathcal{E}(f_n) = \mathcal{E}_e(f).$$

According to [14, Theorem 3.1], for $\varphi \in D(\mathcal{E}) \cap C_c(X)$ with $0 \le \varphi \le 1$ and $g \in D(\mathcal{E}_e) \cap L^\infty(m)$ we have (using $\varphi g, \varphi g^2 \in D(\mathcal{E}) \cap L^\infty(m)$, see first part of the proof)

$$\mathcal{E}(\varphi g) - \mathcal{E}(\varphi g^2, \varphi) \le \mathcal{E}_e(g).$$

Using the Leibniz rule for $\Gamma^{(c)}$ and the discrete Leibniz rule for the integration with respect to J we obtain

$$\int_X \varphi^2 d\Gamma^{(c)}(g) + \int_{X \times X \setminus d} \varphi(x)\varphi(y)(g(x) - g(y))^2 dJ(x,y) = \mathcal{E}(\varphi g) - \mathcal{E}(\varphi g^2, \varphi).$$

Letting $\varphi \nearrow 1$ this implies

$$\int_X d\Gamma(g) \le \mathcal{E}_e(g).$$

In particular, since the LHS is a quadratic form in g, for (g_n) in $D(\mathcal{E}_e) \cap L^\infty(m)$ the convergence $g_n \to g$ with respect to $\mathcal{E}_e$ implies $\int_X d\Gamma(g_n) \to \int_X d\Gamma(g)$.

Since (f_n) is an approximating sequence for f, the lower semicontinuity of $\mathcal{E}_e$ with respect to m-a.e. convergence, see e.g. [14, Lemma 2.3], yields

$$\mathcal{E}_e(f - f_n) \le \liminf_{m\to\infty} \mathcal{E}_e(f_m - f_n) = \liminf_{m\to\infty} \mathcal{E}(f_m - f_n) \to 0, \text{ as } n \to \infty.$$

Hence, we obtain the claimed convergence of $\Gamma(f_n)$ to $\Gamma(f)$. □

Proof (*Proof of the volume growth test for recurrence, Theorem 4*) According to [9, Theorem 1 and Proposition 3] it suffices to show that any non-negative $h \in D(\mathcal{E}_e) \cap L^\infty(m)$ with $\mathcal{E}_e(h, \varphi) \ge 0$ for all nonnegative $\varphi \in D(\mathcal{E})$ is constant (such functions are called excessive). By Lemma 20 such a function h satisfies $h \in D(\mathcal{E})^*_{\text{loc}}$ and

$$\int_X d\Gamma(h, \varphi) \ge 0$$

for all non-negative $\varphi \in D(\mathcal{E})_c$. Now assume without loss of generality $\|h\|_\infty \le 1$. Since $\Gamma(1) = 0$, the function $f = 1 - h$ is nonnegative and subharmonic in our sense. Moreover, $\|f 1_{B_r}\|_p^p \le \|f\|_\infty^p m(B_r)$. Hence, the assumption on $m(B_r)$ and Theorem 3 yield that f is constant. □

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References

1. A.-L. Cauchy, Analyse algébrique. *Cours d'Analyse de l'École Royale Polytechnique* [Course in Analysis of the École Royale Polytechnique] (Éditions Jacques Gabay, Sceaux, 1989). Reprint of the 1821 edition
2. E.B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Tracts in Mathematics, vol. 92 (Cambridge University Press, Cambridge, 1990)
3. C. Dellacherie, P.-A. Meyer, *Probabilités et potentiel. Chapitres V à VIII*. Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], vol. 1385, revised edition (Hermann, Paris, 1980). Théorie des martingales [Martingale theory]
4. R.L. Frank, D. Lenz, D. Wingert, Intrinsic metrics for non-local symmetric Dirichlet forms and applications to spectral theory. J. Funct. Anal. **266**(8), 4765–4808 (2014)
5. M. Fukushima, A note on irreducibility and ergodicity of symmetric Markov processes, in *Stochastic Processes in Quantum Theory and Statistical Physics (Marseille, 1981)*, Lecture Notes in Physics, vol. 173 (Springer, Berlin, 1982), pp. 200–207
6. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, De Gruyter Studies in Mathematics, vol. 19, extended edition (Walter de Gruyter & Co., Berlin, 2011)
7. B. Hua, J. Jost, L^q harmonic functions on graphs. Israel J. Math. **202**(1), 475–490 (2014)

8. Bobo Hua, Matthias Keller, Harmonic functions of general graph Laplacians. Calc. Var. Partial Differ. Equ. **51**(1–2), 343–362 (2014)
9. N. Kajino, Equivalence of recurrence and Liouville property for symmetric Dirichlet forms. Mat. Fiz. Komp' yut. Model. **3**(40), 89–98 (2017)
10. L. Karp, Subharmonic functions on real and complex manifolds. Math. Z. **179**(4), 535–554 (1982)
11. M. Keller, D. Lenz, H. Vogt, R. Wojciechowski, Note on basic features of large time behaviour of heat kernels. J. Reine Angew. Math. **708**, 73–95 (2015)
12. J. Masamune, T. Uemura, L^p-Liouville property for non-local operators. Math. Nachr. **284**(17–18), 2249–2267 (2011)
13. M. Schmidt, Energy forms. Dissertation, arXiv:1703.04883 (2017)
14. M. Schmidt, A note on reflected Dirichlet forms. Potential Anal. **52**(2), 245–279 (2020)
15. K.-T. Sturm, Analysis on local Dirichlet spaces. I. Recurrence, conservativeness and L^p-Liouville properties. J. Reine Angew. Math. **456**, 173–196 (1994)
16. S.-T. Yau, Some function-theoretic properties of complete Riemannian manifold and their applications to geometry. Indiana Univ. Math. J. **25**(7), 659–670 (1976)

On Singularity of Energy Measures for Symmetric Diffusions with Full Off-Diagonal Heat Kernel Estimates II: Some Borderline Examples

Naotaka Kajino

Abstract We present a concrete family of fractals, which we call the *(two-dimensional) thin scale irregular Sierpiński gaskets* and each of which is equipped with a canonical strongly local regular symmetric Dirichlet form. We prove that any fractal K in this family satisfies the full off-diagonal heat kernel estimates with some space-time scale function Ψ_K and the singularity of the associated energy measures with respect to the canonical volume measure (uniform distribution) on K, and also that the decay rate of $r^{-2}\Psi_K(r)$ to 0 as $r \downarrow 0$ can be made arbitrarily slow by suitable choices of K. These results together support the energy measure singularity dichotomy conjecture [*Ann. Probab.* **48** (2020), no. 6, 2920–2951, Conjecture 2.15] stating that, if the full off-diagonal heat kernel estimates with space-time scale function Ψ satisfying $\lim_{r\downarrow 0} r^{-2}\Psi(r) = 0$ hold for a strongly local regular symmetric Dirichlet space with complete metric, then the associated energy measures are singular with respect to the reference measure of the Dirichlet space.

Keywords Thin scale irregular Sierpiński gasket · Singularity of energy measure · Sub-Gaussian heat kernel estimate

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1 Introduction

This paper is a follow-up of the author's recent joint work [26] with Mathav Murugan on singularity of energy measures associated with a strongly local regular symmetric Dirichlet space $(K, d, m, \mathcal{E}, \mathcal{F})$ satisfying full off-diagonal heat kernel estimates. The $\mathcal{E}$*-energy measure* $\mu_{\langle u \rangle}$ of $u \in \mathcal{F}$ is a Borel measure on K which plays, in the theory of regular symmetric Dirichlet forms as presented in [13, 17], the same roles as the classical energy integral measure $|\nabla u|^2\, dx$ on $\mathbb{R}^N$. It is defined for $u \in \mathcal{F} \cap L^\infty(K, m)$

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as the unique Borel measure on K such that

$$\int_K f\,d\mu_{\langle u\rangle} = \mathcal{E}(u, fu) - \frac{1}{2}\mathcal{E}(u^2, f) \quad \text{for any } f \in \mathcal{F} \cap \mathcal{C}_c(K), \tag{1.1}$$

where $\mathcal{C}_c(K)$ denotes the space of $\mathbb{R}$-valued continuous functions on K with compact supports, and then for $u \in \mathcal{F}$ by $\mu_{\langle u\rangle}(A) := \lim_{n\to\infty} \mu_{\langle(-n)\vee(u\wedge n)\rangle}(A)$ for each Borel subset A of K; see [17, Theorem 1.4.2-(ii),(iii), (3.2.13), (3.2.14) and (3.2.15)] for the details of this definition.

The main results of [26] concern the singularity and the absolute continuity of the $\mathcal{E}$-energy measures $\mu_{\langle u\rangle}$ with respect to the reference measure m. While $\mu_{\langle u\rangle}$ is easily identified as $\langle\nabla u, \nabla u\rangle_x\,dm(x)$ if $\mathcal{E} = \int_K \langle\nabla\cdot, \nabla\cdot\rangle_x\,dm(x)$ for some linear differential operator ∇ satisfying the Leibniz rule and some measurable field $\langle\cdot,\cdot\rangle_x$ of non-negative definite symmetric bilinear forms, there is no simple expression of $\mu_{\langle u\rangle}$ and the nature of $\mu_{\langle u\rangle}$ is a deep mystery when K is a fractal. The question of whether $\mu_{\langle u\rangle}$ is singular with respect to m is probably the most fundamental one toward better understanding of $\mu_{\langle u\rangle}$ in such cases, had been answered affirmatively for essentially all known examples of self-similar Dirichlet forms on self-similar fractals in [11, 23, 24, 30, 31], but had been studied only under the assumption of exact self-similarity until [26]. As the main results of [26], it has been now proved that the $\mathcal{E}$-energy measures $\mu_{\langle u\rangle}$ are singular or absolutely continuous with respect to m according to whether the behavior of the associated heat kernel $p_t(x, y)$ in infinitesimal scale is "*sufficiently sub-Gaussian*" or "*Gaussian*", as stated in the following theorem. Recall that a family $\{p_t\}_{t\in(0,\infty)}$ of $[-\infty, \infty]$-valued Borel measurable functions on $K \times K$ is called a *heat kernel* of $(K, d, m, \mathcal{E}, \mathcal{F})$ if and only if the symmetric Markovian semigroup $\{T_t\}_{t\in(0,\infty)}$ on $L^2(K, m)$ associated with $(\mathcal{E}, \mathcal{F})$ is expressed as $T_t u = \int_K p_t(\cdot, y)u(y)\,dm(y)$ m-a.e. for any $t \in (0, \infty)$ and any $u \in L^2(K, m)$. We set $\operatorname{diam}(K, d) := \sup_{x,y\in K} d(x, y)$ and $B_d(x, r) := \{y \in K \mid d(x, y) < r\}$ for $(x, r) \in K \times (0, \infty)$.

Theorem 1.1 (A simplification of [26, Theorem 2.13]) *Let* $(K, d, m, \mathcal{E}, \mathcal{F})$ *be a* metric measure Dirichlet space*, i.e., a strongly local regular symmetric Dirichlet space with* (K, d) *complete and* K *containing at least two elements, so that* $\operatorname{diam}(K, d) \in (0, \infty]$. *Let* $\Psi : [0, \infty) \to [0, \infty)$ *be a homeomorphism satisfying*

$$c_\Psi^{-1}\left(\frac{R}{r}\right)^{\beta_0} \le \frac{\Psi(R)}{\Psi(r)} \le c_\Psi\left(\frac{R}{r}\right)^{\beta_1} \quad \textit{for any } r, R \in (0, \infty) \textit{ with } r \le R \tag{1.2}$$

for some $c_\Psi, \beta_0, \beta_1 \in [1, \infty)$ *with* $1 < \beta_0 \le \beta_1$*, and define* $\Phi_\Psi : [0, \infty) \to [0, \infty)$ *by* $\Phi_\Psi(s) := \sup_{r\in(0,\infty)}(s/r - 1/\Psi(r))$. *Suppose further that* $(K, d, m, \mathcal{E}, \mathcal{F})$ *satisfies the* full off-diagonal heat kernel estimates fHKE(Ψ)*, i.e., that there exist a heat kernel* $\{p_t\}_{t\in(0,\infty)}$ *of* $(K, d, m, \mathcal{E}, \mathcal{F})$ *and* $c_1, c_2, c_3, c_4 \in (0, \infty)$ *such that*

$$\frac{c_1 \exp\bigl(-c_2 t \Phi_\Psi(d(x,y)/t)\bigr)}{m\bigl(B_d(x,\Psi^{-1}(t))\bigr)} \le p_t(x,y) \le \frac{c_3 \exp\bigl(-c_4 t \Phi_\Psi(d(x,y)/t)\bigr)}{m\bigl(B_d(x,\Psi^{-1}(t))\bigr)} \qquad \text{fHKE}(\Psi)$$

for m-a.e. $x, y \in K$ for each $t \in (0,\infty)$. Then the following hold:

(1) (fHKE(Ψ) with "sufficiently sub-Gaussian" Ψ implies singularity) *If*

$$\liminf_{\lambda\to\infty} \liminf_{r\downarrow 0} \frac{\lambda^2 \Psi(r/\lambda)}{\Psi(r)} = 0, \tag{1.3}$$

then $\mu_{\langle u\rangle}$ is singular with respect to m for any $u \in \mathcal{F}$.

(2) (fHKE(Ψ) with "Gaussian" Ψ implies absolute continuity) *If*

$$\limsup_{r\downarrow 0} \frac{\Psi(r)}{r^2} > 0, \tag{1.4}$$

then $m(A) = 0$ if and only if $\sup_{u\in\mathcal{F}} \mu_{\langle u\rangle}(A) = 0$ for each Borel subset A of K, thus in particular $\mu_{\langle u\rangle}$ is absolutely continuous with respect to m for any $u \in \mathcal{F}$, and there exist $r_1 \in (0, \operatorname{diam}(K,d))$ and $c_5 \in [1,\infty)$ such that

$$c_5^{-1} r^2 \le \Psi(r) \le c_5 r^2 \quad \text{for any } r \in (0, r_1). \tag{1.5}$$

Remark 1.2 Let $\Psi : [0,\infty) \to [0,\infty)$ be a homeomorphism satisfying (1.2) for some $c_\Psi, \beta_0, \beta_1 \in [1,\infty)$ with $1 < \beta_0 \le \beta_1$, and let $(K, d, m, \mathcal{E}, \mathcal{F})$ be a metric measure Dirichlet space satisfying fHKE(Ψ).

(1) It is known that in this situation $(K, d, m, \mathcal{E}, \mathcal{F})$ satisfies the assumptions of [26, Theorem 2.13], namely VD, PI(Ψ), CS(Ψ) and the chain condition for (K, d). Indeed, VD follows in the same way as [8, Proof of Lemma 5.1-(i)] by integrating the lower inequality in fHKE(Ψ) on $B_d(x, 2\Psi^{-1}(t))$ with respect to m and applying the upper bound on $\Phi_\Psi(R,t) := t\Phi_\Psi(R/t)$ in [20, (5.13)], (1.2) and the inequality $\int_{B_d(x,2\Psi^{-1}(t))} p_t(x,y)\,dm(y) \le \int_K p_t(x,y)\,dm(y) \le 1$ for m-a.e. $x \in K$. Then VD and fHKE(Ψ) imply PI(Ψ) and CS(Ψ) by the results in [1, 6, 7, 19] as summarized in [32, Theorem 3.2] and [26, Theorem 2.8 and Remark 2.9], and fHKE(Ψ) also implies the chain condition for (K, d) by [33, Theorem 2.11].

(2) *If $\Psi_0 : [0,\infty) \to [0,\infty)$ is a homeomorphism and $\Psi_0(r)/\Psi(r) \in [c_0^{-1}, c_0]$ for any $r \in (0,\infty)$ for some $c_0 \in [1,\infty)$, then $(K, d, m, \mathcal{E}, \mathcal{F})$ satisfies* fHKE(Ψ_0). Indeed, this is immediate from fHKE(Ψ), VD, which is implied by fHKE(Ψ) as noted in (1) above, and the elementary observation based on (1.2) that $\Phi_{\Psi_0}(s)/\Phi_\Psi(s) \in \bigl[(c_0 c_\Psi)^{-\frac{1}{\beta_0 - 1}}, (c_0 c_\Psi)^{\frac{1}{\beta_0-1}}\bigr]$ for any $s \in (0,\infty)$.

Note that, if $\Psi(r) = r^{\beta}$ for any $r \in [0,\infty)$ for some $\beta \in (1,\infty)$, then $\Phi(s) = \beta^{-\frac{\beta}{\beta-1}}(\beta-1)s^{\frac{\beta}{\beta-1}}$ for any $s \in [0,\infty)$, so that fHKE(Ψ) with this Ψ is the typical form of heat kernel estimates known to hold widely; see, e.g., [18, 35–37] and references therein for the studies on the case of $\beta = 2$ and [4, 5, 10, 16, 29] for known results with $\beta > 2$ for self-similar fractals. For this class of Ψ, the classification by (1.3) and (1.4) gives a complete dichotomy between $\beta > 2$ and $\beta \le 2$, with the latter identified further as $\beta = 2$ by (1.5). On the other hand, (1.3) and (1.4) do not give a complete classification of general Ψ since there are examples of Ψ, like $\Psi(r) = r^2/\log(e - 1 + r^{-1})$, satisfying (1.2) but neither (1.3) nor (1.4), and it is not clear under fHKE(Ψ) with such Ψ whether the $\mathcal{E}$-energy measures $\mu_{\langle u\rangle}$ are singular or absolutely continuous with respect to the reference measure m. In view of Theorem 1.1, one might expect the following conjecture to hold.

Conjecture 1.3 (Energy measure singularity dichotomy; a simplification of [26, Conjecture 2.15]) *Theorem* 1.1-(1) *with* (1.3) *replaced by*

$$\lim_{r\downarrow 0} \frac{\Psi(r)}{r^2} = 0 \tag{1.6}$$

(fHKE(Ψ) with "however weakly sub-Gaussian" Ψ implies singularity) *holds.*

As announced already in [26, Remark 2.14], this paper is aimed at giving firm evidence that Conjecture 1.3 should be true, by presenting concrete examples of metric measure Dirichlet spaces satisfying both the singularity of the energy measures and fHKE(Ψ) for some Ψ, *whose decay rate at* 0 *can be made arbitrarily close to* r^2. Their state spaces are certain fractals, which we call the *(two-dimensional) thin scale irregular Sierpiński gaskets* (see Fig. 2), obtained by modifying the construction of the scale irregular (or homogeneous random) Sierpiński gaskets studied in [9, 21, 22] (see also [28, Chap. 24]) so as to make them look very much like one-dimensional frames in infinitesimal scale. An arbitrarily slow decay rate of $\Psi(r)/r^2$ as $r \downarrow 0$ can be then realized by choosing suitably the parameters defining the fractal to make its infinitesimal geometry arbitrarily close to being one-dimensional, which is an idea suggested to the author by Martin T. Barlow in [3]. An important point here is to allow *infinitely* many patterns of cell subdivisions to be present in the construction of the fractal, in contrast to that of the usual scale irregular Sierpiński gaskets considered in [9, 21, 22, 28], each of which involves only finitely many patterns of cell subdivisions and typically falls within the scope of Theorem 1.1-(1) as illustrated in [26, Sect. 5]. We remark that the singularity of the energy measures has been proved also in [25] for a class of (two-dimensional) spatially inhomogeneous Sierpiński gaskets, which typically do not satisfy the volume doubling property VD and are thereby beyond the scope of [26, Theorem 2.13].

The rest of this paper is organized as follows. In Sect. 2 we define the thin scale irregular Sierpiński gaskets and construct the canonical Dirichlet forms (resistance forms) on them, and we verify in Sect. 3 that they satisfy fHKE(Ψ) with Ψ explicit in terms of their defining parameters (Theorem 3.3). In Sect. 4 we prove the singularity of the energy measures for the canonical Dirichlet form on *any* thin scale irregular

Sierpiński gasket (Theorem 4.3), and Sect. 5 is devoted to stating and proving our last main result that an arbitrarily slow decay rate of $\Psi(r)/r^2$ can be realized by some thin scale irregular Sierpiński gasket (Theorem 5.1 and Proposition 5.2).

Notation In this paper, we adopt the following notation and conventions.

(1) The symbols $\subset$ and $\supset$ for set inclusion *allow* the case of the equality.
(2) $\mathbb{N} := \{n \in \mathbb{Z} \mid n > 0\}$, i.e., $0 \notin \mathbb{N}$.
(3) The cardinality (the number of elements) of a set A is denoted by $\#A$.
(4) We set $a \vee b := \max\{a, b\}$, $a \wedge b := \min\{a, b\}$, $a^+ := a \vee 0$, $a^- := -(a \wedge 0)$ and $\lfloor a \rfloor := \max\{n \in \mathbb{Z} \mid n \leq a\}$ for $a, b \in \mathbb{R}$, and we use the same notation also for $\mathbb{R}$-valued functions and equivalence classes of them. All numerical functions in this paper are assumed to be $[-\infty, \infty]$-valued.
(5) The Euclidean inner product and norm on $\mathbb{R}^2$ are denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively.
(6) Let K be a non-empty set. We define $\mathrm{id}_K : K \to K$ by $\mathrm{id}_K(x) := x$, $\mathbb{1}_A = \mathbb{1}_A^K \in \mathbb{R}^K$ for $A \subset K$ by $\mathbb{1}_A(x) := \mathbb{1}_A^K(x) := \left\{\begin{smallmatrix} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A, \end{smallmatrix}\right.$ set $\mathbb{1}_x := \mathbb{1}_x^K := \mathbb{1}_{\{x\}}$ for $x \in K$ and $\|u\|_{\sup} := \|u\|_{\sup,K} := \sup_{x \in K} |u(x)|$ for $u : K \to \mathbb{R}$.
(7) Let K be a topological space. We set $\mathcal{C}(K) := \{u \in \mathbb{R}^K \mid u \text{ is continuous}\}$, and the closure of $K \setminus u^{-1}(0)$ in K is denoted by $\mathrm{supp}_K[u]$ for each $u \in \mathcal{C}(K)$. The Borel σ-algebra of K is denoted by $\mathscr{B}(K)$.
(8) Let (K, d) be a metric space. We set $B_d(x, r) := \{y \in K \mid d(x, y) < r\}$ for $(x, r) \in K \times (0, \infty)$.
(9) Let $(K, \mathscr{B})$ be a measurable space and let μ, ν be measures on $(K, \mathscr{B})$. We write $\nu \ll \mu$ and $\nu \perp \mu$ to mean that ν is absolutely continuous and singular, respectively, with respect to μ.

2 The Examples: Thin Scale Irregular Sierpiński Gaskets

In this section, we introduce the (two-dimensional) thin scale irregular Sierpiński gaskets, and construct the canonical Dirichlet forms (resistance forms) on them by applying the standard method developed in [27, Chaps. 2 and 3]. We closely follow [26, Sect. 5] for the presentation of this section.

To start with, the thin scale irregular Sierpiński gaskets are defined as follows.

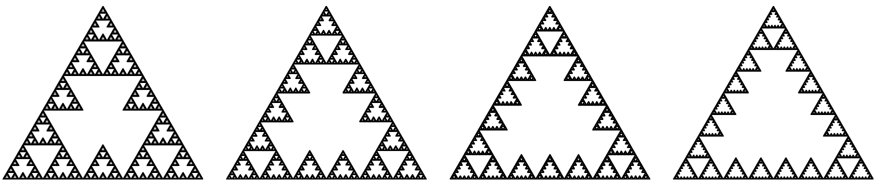

Fig. 1 The level-l (self-similar) thin Sierpiński gaskets K^l ($l = 5, 6, 7, 8$)

Definition 2.1 (Thin scale irregular Sierpiński gasket) Let $q_0, q_1, q_2 \in \mathbb{R}^2$ satisfy $|q_j - q_k| = 1$ for any $j, k \in \{0, 1, 2\}$ with $j \neq k$, so that the convex hull $\triangle$ of $V_0 := \{q_0, q_1, q_2\}$ in $\mathbb{R}^2$ is a closed equilateral triangle with side length 1. For each $l \in \mathbb{N} \setminus \{1, 2, 3, 4\}$, we set

$$S_l := \bigl\{(i_1, i_2) \in (\mathbb{N} \cup \{0\})^2 \bigm| i_1 + i_2 \leq l - 1, i_1 i_2 (l - 1 - i_1 - i_2) = 0\bigr\}, \qquad (2.1)$$

and for each $i = (i_1, i_2) \in S_l$ set $q_i^l := q_0 + \sum_{k=1}^{2} (i_k/l)(q_k - q_0)$ and define $f_i^l : \mathbb{R}^2 \to \mathbb{R}^2$ by $f_i^l(x) := q_i^l + l^{-1}(x - q_0)$. Let $\boldsymbol{l} = (l_n)_{n=1}^{\infty} \in (\mathbb{N} \setminus \{1, 2, 3, 4\})^{\mathbb{N}}$, set $W_n^{\boldsymbol{l}} := \prod_{k=1}^{n} S_{l_k}$ for $n \in \mathbb{N} \cup \{0\}$, $W_*^{\boldsymbol{l}} := \bigcup_{n=0}^{\infty} W_n^{\boldsymbol{l}}$, $|w| := n$ and $f_w^{\boldsymbol{l}} := f_{w_1}^{l_1} \circ \cdots \circ f_{w_n}^{l_n}$ for $n \in \mathbb{N} \cup \{0\}$ and $w = w_1 \ldots w_n \in W_n^{\boldsymbol{l}}$, where $W_0^{\boldsymbol{l}}$ is defined as the singleton $\{\emptyset\}$ of the empty word $\emptyset$ and $f_\emptyset^{\boldsymbol{l}} := \mathrm{id}_{\mathbb{R}^2}$. Noting that $\bigl\{\bigcup_{w \in W_n^{\boldsymbol{l}}} f_w^{\boldsymbol{l}}(\triangle)\bigr\}_{n=0}^{\infty}$ is a strictly decreasing sequence of non-empty compact subsets of $\triangle$, we define the *(two-dimensional) level-$\boldsymbol{l}$ thin scale irregular Sierpiński gasket* $K^{\boldsymbol{l}}$ as the non-empty compact subset of $\triangle$ given by

$$K^{\boldsymbol{l}} := \bigcap_{n=0}^{\infty} \bigcup_{w \in W_n^{\boldsymbol{l}}} f_w^{\boldsymbol{l}}(\triangle) \qquad (2.2)$$

(see Fig. 2), and set $K_w^{\boldsymbol{l}} := K^{\boldsymbol{l}} \cap f_w^{\boldsymbol{l}}(\triangle)$ and $F_w^{\boldsymbol{l}} := f_w^{\boldsymbol{l}}|_{K^{\boldsymbol{l}^{|w|}}}$ for $w \in W_*^{\boldsymbol{l}}$, where $\boldsymbol{l}^k := (l_{n+k})_{n=1}^{\infty}$ for $k \in \mathbb{N} \cup \{0\}$. We also set $V_n^{\boldsymbol{l}} := \bigcup_{w \in W_n^{\boldsymbol{l}}} f_w^{\boldsymbol{l}}(V_0)$ for $n \in \mathbb{N} \cup \{0\}$ and $V_*^{\boldsymbol{l}} := \bigcup_{n=0}^{\infty} V_n^{\boldsymbol{l}}$, so that $V_0^{\boldsymbol{l}} = V_0$, $\{V_n^{\boldsymbol{l}}\}_{n=0}^{\infty}$ is a strictly increasing sequence of finite subsets of $K^{\boldsymbol{l}}$, and $V_*^{\boldsymbol{l}}$ is dense in $K^{\boldsymbol{l}}$.

In particular, for each $l \in \mathbb{N} \setminus \{1, 2, 3, 4\}$ we let $\boldsymbol{l}_l := (l)_{n=1}^{\infty}$ denote the constant sequence with value l, set $K^l := K^{\boldsymbol{l}_l}$ and $V_n^l := V_n^{\boldsymbol{l}_l}$ for $n \in \mathbb{N} \cup \{0\}$, and call K^l the *(two-dimensional) level-l thin Sierpiński gasket*, which is exactly self-similar in the sense that $K^l = \bigcup_{i \in S_l} f_i^l(K^l)$ (see Fig. 1 and, e.g., [27, Sect. 1.1]).

We fix an arbitrary $\boldsymbol{l} = (l_n)_{n=1}^{\infty} \in (\mathbb{N} \setminus \{1, 2, 3, 4\})^{\mathbb{N}}$ in the rest of this section. The following proposition is immediate from Definition 2.1.

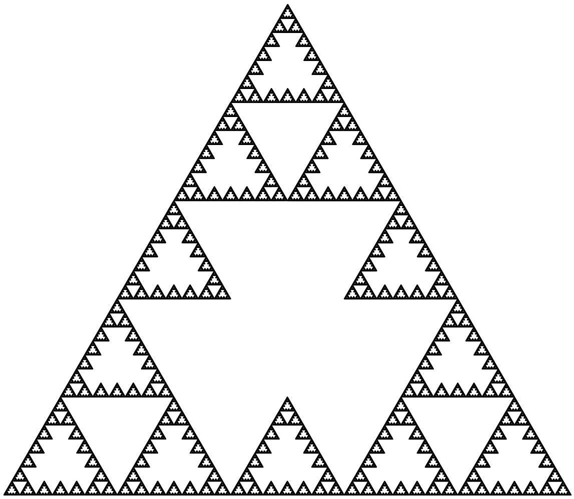

Fig. 2 A level-$\boldsymbol{l}$ thin scale irregular Sierpiński gasket $K^{\boldsymbol{l}}$ ($\boldsymbol{l} = (5, 7, 6, 12, \ldots)$)

Proposition 2.2 (1) *Let* $w = w_1 \ldots w_{|w|}, v = v_1 \ldots v_{|v|} \in W_*^{\boldsymbol{l}} \setminus \{\emptyset\}$ *satisfy* $w_k \neq v_k$ *for some* $k \in \{1, \ldots, |w| \wedge |v|\}$. *Then* $\#(K_w^{\boldsymbol{l}} \cap K_v^{\boldsymbol{l}}) \leq 1$ *and*

$$f_w^{\boldsymbol{l}}(\triangle) \cap f_v^{\boldsymbol{l}}(\triangle) = K_w^{\boldsymbol{l}} \cap K_v^{\boldsymbol{l}} = F_w^{\boldsymbol{l}}(V_0) \cap F_v^{\boldsymbol{l}}(V_0). \tag{2.3}$$

(2) $K^{\boldsymbol{l}} = \bigcup_{w \in W_n^{\boldsymbol{l}}} K_w^{\boldsymbol{l}}$ *for any* $n \in \mathbb{N} \cup \{0\}$, *and* $F_w^{\boldsymbol{l}}(K^{\boldsymbol{l}^{|w|}}) = K_w^{\boldsymbol{l}}$ *for any* $w \in W_*^{\boldsymbol{l}}$.
(3) $V_{n+k}^{\boldsymbol{l}} = \bigcup_{w \in W_n^{\boldsymbol{l}}} F_w^{\boldsymbol{l}}(V_k^{\boldsymbol{l}^n})$ *and* $V_*^{\boldsymbol{l}} = \bigcup_{w \in W_n^{\boldsymbol{l}}} F_w^{\boldsymbol{l}}(V_*^{\boldsymbol{l}^n})$ *for any* $n, k \in \mathbb{N} \cup \{0\}$.

In exactly the same way as in [9, 21, 22] (see also [28, Part 4]), we can define a canonical strongly local regular symmetric Dirichlet space $(K^{\boldsymbol{l}}, d_{\boldsymbol{l}}, m_{\boldsymbol{l}}, \mathcal{E}^{\boldsymbol{l}}, \mathcal{F}_{\boldsymbol{l}})$ over $K^{\boldsymbol{l}}$. First, the metric $d_{\boldsymbol{l}}$ on $K^{\boldsymbol{l}}$ is defined as follows.

Definition 2.3 We define $d_{\boldsymbol{l}} : K^{\boldsymbol{l}} \times K^{\boldsymbol{l}} \to [0, \infty]$ by

$$d_{\boldsymbol{l}}(x, y) := \inf\{\ell_{\mathbb{R}^2}(\gamma) \mid \gamma : [0, 1] \to K^{\boldsymbol{l}}, \gamma \text{ is continuous}, \gamma(0) = x, \gamma(1) = y\}, \tag{2.4}$$

where $\ell_{\mathbb{R}^2}(\gamma)$ denotes the Euclidean length of γ, i.e., the total variation of the $\mathbb{R}^2$-valued map γ with respect to the Euclidean norm $|\cdot|$. We also set $L_n^{\boldsymbol{l}} := l_1 \cdots l_n$ ($L_0^{\boldsymbol{l}} := 1$) for $n \in \mathbb{N} \cup \{0\}$.

Proposition 2.4 *$d_{\boldsymbol{l}}$ is a metric on $K^{\boldsymbol{l}}$, and it is* geodesic, *i.e., for any $x, y \in K^{\boldsymbol{l}}$ there exists $\gamma : [0, 1] \to K^{\boldsymbol{l}}$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $d_{\boldsymbol{l}}(\gamma(s), \gamma(t)) = |s - t| d_{\boldsymbol{l}}(x, y)$ for any $s, t \in [0, 1]$. Moreover,*

$$|x - y| \leq d_{\boldsymbol{l}}(x, y) \leq 6|x - y| \quad \textit{for any } x, y \in K^{\boldsymbol{l}}. \tag{2.5}$$

Proof This proof is similar to [9, Proof of Lemma 2.4], but some additional argument is required to take care of the possible unboundedness of $\boldsymbol{l} = (l_n)_{n=1}^{\infty}$. It is immediate from (2.4) that $|x-y| \le d_{\boldsymbol{l}}(x,y) < \infty$ for any $x, y \in K^{\boldsymbol{l}}$ and thereby that $d_{\boldsymbol{l}}$ is a metric on $K^{\boldsymbol{l}}$, which is also geodesic by [12, Proposition 2.5.19]; indeed, the infimum in (2.4) is easily seen to be attained for each $x, y \in K^{\boldsymbol{l}}$, by choosing a sequence $\{\gamma_n\}_{n=1}^{\infty}$ of continuous maps as in (2.4) with $\lim_{n\to\infty} \ell_{\mathbb{R}^2}(\gamma_n) = d_{\boldsymbol{l}}(x,y)$, reparameterizing them by arc length on the basis of [12, Proposition 2.5.9], and applying to them the Arzelà–Ascoli theorem [12, Theorem 2.5.14] and the lower semi-continuity [12, Proposition 2.3.4-(iv)] of $\ell_{\mathbb{R}^2}$ with respect to pointwise convergence.

Thus it remains to prove the upper inequality in (2.5) for any $x, y \in K^{\boldsymbol{l}}$ with $x \neq y$. First, for any $w \in W_*^{\boldsymbol{l}}$ and any $x \in K_w^{\boldsymbol{l}}$, we easily see that

$$\max_{k\in\{0,1,2\}} d_{\boldsymbol{l}}(F_w^{\boldsymbol{l}}(q_k), x) \le \sum_{n=|w|+1}^{\infty} \frac{\frac{3}{2}l_n - \frac{5}{2}}{L_n^{\boldsymbol{l}}} \le \frac{\frac{3}{2}l_{|w|+1} - \frac{5}{2} + \sum_{n=0}^{\infty} \frac{3}{2}(\frac{1}{5})^n}{L_{|w|}^{\boldsymbol{l}} l_{|w|+1}} < \frac{\frac{3}{2}}{L_{|w|}^{\boldsymbol{l}}}, \tag{2.6}$$

from which it further follows that for any $j, k \in \{0, 1, 2\}$ with $j \neq k$,

$$d_{\boldsymbol{l}}(F_w^{\boldsymbol{l}}(q_k), x) \le 5|\langle x - F_w^{\boldsymbol{l}}(q_k), e_{k,j}\rangle|, \tag{2.7}$$

where $e_{k,j} := q_j - q_k$. Now let $x, y \in K^{\boldsymbol{l}}$ satisfy $x \neq y$ and set $n_0 := \min\{n \in \mathbb{N} \mid \{x,y\} \not\subset K_w^{\boldsymbol{l}} \text{ for any } w \in W_n^{\boldsymbol{l}}\}$, so that $x, y \in K_w^{\boldsymbol{l}}$ for a unique $w \in W_{n_0-1}^{\boldsymbol{l}}$ by Proposition 2.2-(1). If $K_{wi_x}^{\boldsymbol{l}} \cap K_{wi_y}^{\boldsymbol{l}} \neq \emptyset$ for some $i_x, i_y \in S_{l_{n_0}}$ with $x \in K_{wi_x}^{\boldsymbol{l}}$ and $y \in K_{wi_y}^{\boldsymbol{l}}$, then $i_x \neq i_y$ by the definition of n_0, $q_{x,y} = F_{wi_x}^{\boldsymbol{l}}(q_k) = F_{wi_y}^{\boldsymbol{l}}(q_j)$ for the unique element $q_{x,y}$ of $K_{wi_x}^{\boldsymbol{l}} \cap K_{wi_y}^{\boldsymbol{l}}$ and some $j, k \in \{0, 1, 2\}$ with $j \neq k$ by Proposition 2.2-(1), and from (2.7) we obtain

$$\begin{aligned} d_{\boldsymbol{l}}(x,y) &\le d_{\boldsymbol{l}}(x, q_{x,y}) + d_{\boldsymbol{l}}(q_{x,y}, y) \\ &\le 5|\langle x - q_{x,y}, e_{k,j}\rangle| + 5|\langle q_{x,y} - y, e_{k,j}\rangle| = 5|\langle x - y, e_{k,j}\rangle| \le 5|x-y|. \end{aligned}$$

On the other hand, if $K_{wi_x}^{\boldsymbol{l}} \cap K_{wi_y}^{\boldsymbol{l}} = \emptyset$ for any $i_x, i_y \in S_{l_k}$ with $x \in K_{wi_x}^{\boldsymbol{l}}$ and $y \in K_{wi_y}^{\boldsymbol{l}}$, then setting

$$n_1 := \min\left\{ n \in \mathbb{N} \;\middle|\; \begin{array}{l} \text{there exists } \{i_k\}_{k=0}^{n} \subset S_{l_{n_0}} \text{ such that } x \in K_{wi_0}^{\boldsymbol{l}}, y \in K_{wi_n}^{\boldsymbol{l}} \\ \text{and } K_{wi_{k-1}}^{\boldsymbol{l}} \cap K_{wi_k}^{\boldsymbol{l}} \neq \emptyset \text{ for any } k \in \{1, \ldots, n\} \end{array} \right\},$$

we have $2 \le n_1 \le \frac{3}{2}l_{n_0} - \frac{5}{2}$, $L_{n_0}^{\boldsymbol{l}} d_{\boldsymbol{l}}(x,y) \le \frac{3}{2} + (n_1 - 1) + \frac{3}{2} = n_1 + 2 \le \frac{3}{2}l_{n_0}$ by (2.6), $\frac{2}{\sqrt{3}} L_{n_0}^{\boldsymbol{l}}|x-y| \ge (\frac{1}{2}l_{n_0} - 1) \wedge \lfloor \frac{1}{2}n_1 \rfloor$, and thus $d_{\boldsymbol{l}}(x,y)/|x-y| \le \frac{10}{\sqrt{3}} < 6$. □

Next, the canonical volume measure $m_{\boldsymbol{l}}$ on $K^{\boldsymbol{l}}$ is defined as follows.

Definition 2.5 We define $m_{\boldsymbol{l}}$ as the unique Borel measure on $K^{\boldsymbol{l}}$ such that

$$m_{\boldsymbol{l}}(K_w^{\boldsymbol{l}}) = \frac{1}{M_{|w|}^{\boldsymbol{l}}} \quad \text{for any } w \in W_*^{\boldsymbol{l}}, \tag{2.8}$$

where $M^{l}_{n} := (\#S_{l_1}) \cdots (\#S_{l_n}) = \prod_{k=1}^{n}(3l_k - 3)$ ($M^{l}_{0} := 1$) for $n \in \mathbb{N} \cup \{0\}$.

The measure m_{l} can be considered as the "uniform distribution on K^{l}". Its uniqueness stated in Definition 2.5 is immediate from the Dynkin class theorem (see, e.g., [15, Appendixes, Theorem 4.2]). It is also easily seen to be obtained as $m_{l} = \bigl(\prod_{n=1}^{\infty} \mathrm{unif}(S_{l_n})\bigr)(\pi_{l}^{-1}(\cdot))$, where $\mathrm{unif}(S_{l_n})$ denotes the uniform distribution on S_{l_n}, $\prod_{n=1}^{\infty} \mathrm{unif}(S_{l_n})$ their product probability measure on $\prod_{n=1}^{\infty} S_{l_n}$ (see, e.g., [14, Theorem 8.2.2] for its unique existence) and $\pi_{l} : \prod_{n=1}^{\infty} S_{l_n} \to K^{l}$ the continuous surjection given by $\{\pi_{l}((\omega_n)_{n=1}^{\infty})\} := \bigcap_{n=1}^{\infty} K^{l}_{\omega_1\ldots\omega_n}$.

Now we turn to the construction of the canonical Dirichlet form (resistance form) $(\mathcal{E}^{l}, \mathcal{F}_{l})$ on K^{l}, which is achieved by taking the "inductive limit" of a certain canonical sequence of discrete Dirichlet forms on the finite sets $\{V^{l}_{n}\}_{n=0}^{\infty}$ via the standard method presented in [27, Chaps. 2 and 3] (see also [2, Sects. 6 and 7]). The whole construction is based on the following definition and lemma.

Definition 2.6 Recalling that $V^{l}_{0} = V_0$, we define a non-negative definite symmetric bilinear form $\mathcal{E}^{0} : \mathbb{R}^{V_0} \times \mathbb{R}^{V_0} \to \mathbb{R}$ on $\mathbb{R}^{V_0} = \mathbb{R}^{V^{l}_{0}}$ by

$$\mathcal{E}^{0}(u, v) := \frac{1}{2}\sum_{j,k=0}^{2}(u(q_j) - u(q_k))(v(q_j) - v(q_k)), \quad u, v \in \mathbb{R}^{V_0}, \tag{2.9}$$

and set $r_l := (\frac{2}{3}l + \frac{1}{9})^{-1}$ for each $l \in \mathbb{N} \setminus \{1, 2, 3, 4\}$.

The value of r_l is specifically chosen in order for the following lemma to hold.

Lemma 2.7 *Let $l \in \mathbb{N} \setminus \{1, 2, 3, 4\}$. Then for any $u \in \mathbb{R}^{V_0}$,*

$$\min\left\{ \sum_{i \in S_l} \mathcal{E}^{0}\bigl(v \circ F^{l}_{i}|_{V_0}, v \circ F^{l}_{i}|_{V_0}\bigr) \;\middle|\; v \in \mathbb{R}^{V^{l}_{1}}, v|_{V_0} = u \right\} = r_l \mathcal{E}^{0}(u, u). \tag{2.10}$$

Proof This is immediate from a direct calculation using the Δ–Y transform (see, e.g., [27, Lemma 2.1.15]). □

We would like to define a bilinear form $\mathcal{E}^{l,n}$ on $\mathbb{R}^{V^{l}_{n}}$ for each $n \in \mathbb{N}$ as the sum of the copies of (2.9) on $\{F^{l}_{w}(V_0)\}_{w \in W^{l}_{n}}$ and then to take their limit as $n \to \infty$, which is enabled by introducing the scaling factors R^{l}_{n} suggested by Lemma 2.7 as in the following definition.

Definition 2.8 For each $n \in \mathbb{N} \cup \{0\}$, we define a non-negative definite symmetric bilinear form $\mathcal{E}^{l,n} : \mathbb{R}^{V^{l}_{n}} \times \mathbb{R}^{V^{l}_{n}} \to \mathbb{R}$ on $\mathbb{R}^{V^{l}_{n}}$ by

$$\mathcal{E}^{l,n}(u, v) := \frac{1}{R^{l}_{n}} \sum_{w \in W^{l}_{n}} \mathcal{E}^{0}\bigl(u \circ F^{l}_{w}|_{V_0}, v \circ F^{l}_{w}|_{V_0}\bigr), \quad u, v \in \mathbb{R}^{V^{l}_{n}}, \tag{2.11}$$

where $R^{l}_{n} := r_{l_1} \cdots r_{l_n} = \prod_{k=1}^{n}(\frac{2}{3}l_k + \frac{1}{9})^{-1}$ ($R^{l}_{0} := 1$), so that $\mathcal{E}^{l,0} = \mathcal{E}^{0}$.

Proposition 2.9 *The sequence* $\{\mathcal{E}^{l,n}\}_{n=0}^{\infty}$ *of forms is* compatible*, i.e., for any* $n, k \in \mathbb{N} \cup \{0\}$ *and any* $u \in \mathbb{R}^{V_n^l}$*,*

$$\min\bigl\{\mathcal{E}^{l,n+k}(v, v) \bigm| v \in \mathbb{R}^{V_{n+k}^l}, v|_{V_n^l} = u\bigr\} = \mathcal{E}^{l,n}(u, u). \tag{2.12}$$

Proof This is immediate from an induction on k based on Lemma 2.7. □

Proposition 2.9 allows us to take the "inductive limit" of $\{\mathcal{E}^{l,n}\}_{n=0}^{\infty}$ as in the following definition. Note that $\{\mathcal{E}^{l,n}(u|_{V_n^l}, u|_{V_n^l})\}_{n=0}^{\infty} \subset [0, \infty)$ is non-decreasing by (2.12) and hence has a limit in $[0, \infty]$ for any $u \in \mathbb{R}^{V_*^l}$.

Definition 2.10 We define a linear subspace $\mathcal{F}_l$ of $\mathbb{R}^{V_*^l}$ and a non-negative definite symmetric bilinear form $\mathcal{E}^l : \mathcal{F}_l \times \mathcal{F}_l \to \mathbb{R}$ on $\mathcal{F}_l$ by

$$\mathcal{F}_l := \Bigl\{u \in \mathbb{R}^{V_*^l} \Bigm| \lim_{n\to\infty} \mathcal{E}^{l,n}(u|_{V_n^l}, u|_{V_n^l}) < \infty\Bigr\}, \tag{2.13}$$

$$\mathcal{E}^l(u, v) := \lim_{n\to\infty} \mathcal{E}^{l,n}(u|_{V_n^l}, v|_{V_n^l}) \in \mathbb{R}, \quad u, v \in \mathcal{F}_l. \tag{2.14}$$

Then applying [27, Lemma 2.2.2, Proposition 2.2.4, Lemma 2.2.5 and Theorem 2.2.6] on the basis of Proposition 2.9, we obtain the following proposition. See [27, Definition 2.3.1] or [28, Definition 3.1] for the notion of resistance forms.

Proposition 2.11 $(\mathcal{E}^l, \mathcal{F}_l)$ *is a* resistance form *on* V_*^l*, i.e., the following hold:*

(RF1) $\{u \in \mathcal{F}_l \mid \mathcal{E}^l(u, u) = 0\} = \mathbb{R}\mathbb{1}_{V_*^l}$.
(RF2) $(\mathcal{F}_l/\mathbb{R}\mathbb{1}_{V_*^l}, \mathcal{E}^l)$ *is a Hilbert space.*
(RF3) $\{u|_V \mid u \in \mathcal{F}_l\} = \mathbb{R}^V$ *for any non-empty finite subset* V *of* V_*^l.
(RF4) $R_{\mathcal{E}^l}(x, y) := \sup\Bigl\{\frac{|u(x) - u(y)|^2}{\mathcal{E}^l(u, u)} \Bigm| u \in \mathcal{F}_l \setminus \mathbb{R}\mathbb{1}_{V_*^l}\Bigr\} < \infty$ *for any* $x, y \in V_*^l$.
(RF5) $u^+ \wedge 1 \in \mathcal{F}_l$ *and* $\mathcal{E}^l(u^+ \wedge 1, u^+ \wedge 1) \le \mathcal{E}^l(u, u)$ *for any* $u \in \mathcal{F}_l$.

Moreover, $R_{\mathcal{E}^l} : V_*^l \times V_*^l \to [0, \infty)$ *is a metric on* V_*^l*, called the* resistance metric *of* $(\mathcal{E}^l, \mathcal{F}_l)$*, and for any* $u \in \mathcal{F}_l$ *and any* $x, y \in V_*^l$*,*

$$|u(x) - u(y)|^2 \le R_{\mathcal{E}^l}(x, y)\mathcal{E}^l(u, u). \tag{2.15}$$

Recalling Proposition 2.2-(3), we also see from the above construction that the following (non-exact) self-similarity of $(\mathcal{E}^l, \mathcal{F}_l)$ holds.

Proposition 2.12 *Let* $n \in \mathbb{N} \cup \{0\}$*. Then*

$$\mathcal{F}_l = \bigl\{u \in \mathbb{R}^{V_*^l} \bigm| u \circ F_w^l|_{V_*^{l^n}} \in \mathcal{F}_{l^n} \text{ for any } w \in W_n^l\bigr\}, \tag{2.16}$$

$$\mathcal{E}^l(u, v) = \frac{1}{R_n^l} \sum_{w\in W_n^l} \mathcal{E}^{l^n}\bigl(u \circ F_w^l|_{V_*^{l^n}}, v \circ F_w^l|_{V_*^{l^n}}\bigr) \text{ for any } u,v \in \mathcal{F}_l. \tag{2.17}$$

Proof It follows from Proposition 2.2-(3) and (2.11) that for each $n, k \in \mathbb{N} \cup \{0\}$,

$$\mathcal{E}^{\boldsymbol{l},n+k}(u, v) = \frac{1}{R^{\boldsymbol{l}}_n} \sum_{w \in W^{\boldsymbol{l}}_n} \mathcal{E}^{\boldsymbol{l}^n,k}\big(u \circ F^{\boldsymbol{l}}_w|_{V^{\boldsymbol{l}^n}_k}, v \circ F^{\boldsymbol{l}}_w|_{V^{\boldsymbol{l}^n}_k}\big) \quad \text{for any } u, v \in \mathbb{R}^{V^{\boldsymbol{l}}_{n+k}},$$

which together with (2.13) and (2.14) immediately yields (2.16) and (2.17). □

Lemma 2.13 *For any $w \in W^{\boldsymbol{l}}_*$ and any $x, y \in V^{\boldsymbol{l}^{|w|}}_*$,*

$$R_{\mathcal{E}^{\boldsymbol{l}}}(F^{\boldsymbol{l}}_w(x), F^{\boldsymbol{l}}_w(y)) \leq R^{\boldsymbol{l}}_{|w|} R_{\mathcal{E}^{\boldsymbol{l}^{|w|}}}(x, y). \tag{2.18}$$

Proof This is immediate from Proposition 2.12 and Proposition 2.11-(RF4). □

Later we will use the following definition and proposition several times.

Definition 2.14 Let $h \in \mathbb{R}^{V^{\boldsymbol{l}}_*}$ and $n \in \mathbb{N} \cup \{0\}$. We say that h is $\mathcal{E}^{\boldsymbol{l}}$*-harmonic* off $V^{\boldsymbol{l}}_n$ if and only if $h \in \mathcal{F}_{\boldsymbol{l}}$ and

$$\begin{aligned} \mathcal{E}^{\boldsymbol{l}}(h, h) &= \min\{\mathcal{E}^{\boldsymbol{l}}(v, v) \mid v \in \mathcal{F}_{\boldsymbol{l}}, v|_{V^{\boldsymbol{l}}_n} = h|_{V^{\boldsymbol{l}}_n}\}, \\ \text{or equivalently,}\quad \mathcal{E}^{\boldsymbol{l}}(h, v) &= 0 \quad \text{for any } v \in \mathcal{F}_{\boldsymbol{l}} \text{ with } v|_{V^{\boldsymbol{l}}_n} = 0. \end{aligned} \tag{2.19}$$

We set $\mathcal{H}_{\boldsymbol{l},n} := \{h \in \mathbb{R}^{V^{\boldsymbol{l}}_*} \mid h \text{ is } \mathcal{E}^{\boldsymbol{l}}\text{-harmonic off } V^{\boldsymbol{l}}_n\}$, which is a linear subspace of $\mathcal{F}_{\boldsymbol{l}}$.

Proposition 2.15 *Let $n \in \mathbb{N} \cup \{0\}$. Then for each $h \in \mathbb{R}^{V^{\boldsymbol{l}}_*}$, the following four conditions (1), (2), (3) and (4) are equivalent to each other and imply (5) below:*

(1) $h \in \mathcal{H}_{\boldsymbol{l},n}$.
(2) $\sum_{y \in V^{\boldsymbol{l}}_{n+k},\, L^{\boldsymbol{l}}_{n+k} d_{\boldsymbol{l}}(x,y)=1}(h(y) - h(x)) = 0$ *for any $k \in \mathbb{N}$ and any $x \in V^{\boldsymbol{l}}_{n+k} \setminus V^{\boldsymbol{l}}_n$.*
(3) $h \in \mathcal{F}_{\boldsymbol{l}}$ and $\mathcal{E}^{\boldsymbol{l}}(h, h) = \mathcal{E}^{\boldsymbol{l},n}(h|_{V^{\boldsymbol{l}}_n}, h|_{V^{\boldsymbol{l}}_n})$.
(4) $h \circ F^{\boldsymbol{l}}_w|_{V^{\boldsymbol{l}^n}_*} \in \mathcal{H}_{\boldsymbol{l}^n,0}$ *for any $w \in W^{\boldsymbol{l}}_n$.*
(5) (Maximum principle) *For any $w \in W^{\boldsymbol{l}}_n$ and any $x \in F^{\boldsymbol{l}}_w(V^{\boldsymbol{l}^n}_*)$,*

$$\min_{q \in F^{\boldsymbol{l}}_w(V_0)} h(q) \leq h(x) \leq \max_{q \in F^{\boldsymbol{l}}_w(V_0)} h(q). \tag{2.20}$$

Also, for each $u \in \mathbb{R}^{V^{\boldsymbol{l}}_n}$ there exists a unique $h^{\boldsymbol{l}}_n(u) \in \mathcal{H}_{\boldsymbol{l},n}$ with $h^{\boldsymbol{l}}_n(u)|_{V^{\boldsymbol{l}}_n} = u$, and the map $h^{\boldsymbol{l}}_n : \mathbb{R}^{V^{\boldsymbol{l}}_n} \to \mathcal{H}_{\boldsymbol{l},n}$ is a linear isomorphism.

Proof The assertions for $h^{\boldsymbol{l}}_n$ and the equivalence of (1), (2) and (3) follow from Proposition 2.9, [27, Lemma 2.2.2] and (2.11). Moreover, noting that $\mathcal{E}^{\boldsymbol{l}^n}(u, u) \geq \mathcal{E}^0(u|_{V_0}, u|_{V_0})$ for any $u \in \mathcal{F}_{\boldsymbol{l}^n}$, we easily see from (2.16), (2.17) and (2.11) that (3) holds if and only if $h \circ F^{\boldsymbol{l}}_w|_{V^{\boldsymbol{l}^n}_*} \in \mathcal{F}_{\boldsymbol{l}^n}$ and $\mathcal{E}^{\boldsymbol{l}^n}\big(h \circ F^{\boldsymbol{l}}_w|_{V^{\boldsymbol{l}^n}_*}, h \circ F^{\boldsymbol{l}}_w|_{V^{\boldsymbol{l}^n}_*}\big) = \mathcal{E}^0\big(h \circ F^{\boldsymbol{l}}_w|_{V_0}, h \circ F^{\boldsymbol{l}}_w|_{V_0}\big)$ for any $w \in W^{\boldsymbol{l}}_n$, which is equivalent to (4) by the equivalence of (3) and (1) with $h \circ F^{\boldsymbol{l}}_w|_{V^{\boldsymbol{l}^n}_*}, \boldsymbol{l}^n, 0$ in place of $h, \boldsymbol{l}, n$. Lastly, (4) implies (5) by [27, Lemma 2.2.3] applied to $h \circ F^{\boldsymbol{l}}_w|_{V^{\boldsymbol{l}^n}_*}$ for each $w \in W^{\boldsymbol{l}}_n$. □

Note that at this stage the domain $\mathcal{F}_{l}$ of $\mathcal{E}^{l}$ is only a linear subspace of $\mathbb{R}^{V^{l}_{*}}$, unlike that of a regular symmetric Dirichlet form on $L^2(K^{l}, m_{l})$, which is a linear subspace of $L^2(K^{l}, m_{l})$ including a dense subalgebra of $(\mathcal{C}(K^{l}), \|\cdot\|_{\sup})$. As the last step of the construction of the canonical Dirichlet form on K^{l}, we now fill this gap by proving that $\mathrm{id}_{V^{l}_{*}} : (V^{l}_{*}, d_{l}|_{V^{l}_{*}\times V^{l}_{*}}) \to (V^{l}_{*}, R_{\mathcal{E}^{l}})$ is uniformly continuous with uniformly continuous inverse and consequently that each $u \in \mathcal{F}_{l}$ uniquely extends to an element of $\mathcal{C}(K^{l})$ by virtue of (2.15).

Proposition 2.16 *For any $x, y \in V^{l}_{*}$ and any $n \in \mathbb{N}$, the following hold:*

(1) *If $d_{l}(x, y) < 1/L^{l}_{n}$, then $R_{\mathcal{E}^{l}}(x, y) \le 4R^{l}_{n}$.*
(2) *If $R_{\mathcal{E}^{l}}(x, y) < \frac{1}{6}R^{l}_{n}$, then $d_{l}(x, y) \le 3/L^{l}_{n}$.*

In particular, $R_{\mathcal{E}^{l}}$ uniquely extends to $\overline{R}_{\mathcal{E}^{l}} \in \mathcal{C}(K^{l} \times K^{l})$, $\overline{R}_{\mathcal{E}^{l}}$ is a metric on K^{l} compatible with the original (Euclidean) topology of K^{l}, and $\big((K^{l}, \overline{R}_{\mathcal{E}^{l}}), \mathrm{id}_{V^{l}_{*}}\big)$ is the completion of $(V^{l}_{*}, R_{\mathcal{E}^{l}})$.

Proof We essentially follow [28, Chap. 22], but the possible unboundedness of $\boldsymbol{l} = (l_n)_{n=1}^{\infty}$ requires some additional care. First, since

$$R_{\mathcal{E}^{l}}(q_j, q_k) = \big(\min\{\mathcal{E}^0(u, u) \mid u \in \mathbb{R}^{V_0}, u(q_j) = 1, u(q_k) = 0\}\big)^{-1} = \frac{2}{3}$$

for any $j, k \in \{0, 1, 2\}$ with $j \neq k$ by [27, (2.2.3) and Lemma 2.2.5], it follows from Lemma 2.13 that for any $w \in W^{l}_{*}$ and any $j, k \in \{0, 1, 2\}$,

$$R_{\mathcal{E}^{l}}(F^{l}_{w}(q_j), F^{l}_{w}(q_k)) \le \frac{2}{3}R^{l}_{|w|}. \tag{2.21}$$

Recalling that $R_{\mathcal{E}^{l}}$ is a metric on V^{l}_{*} as stated in Proposition 2.11, we easily see from (2.21) and the triangle inequality for $R_{\mathcal{E}^{l}}$ that for any $x \in V^{l}_{*}$,

$$\max_{k\in\{0,1,2\}} R_{\mathcal{E}^{l}}(q_k, x) \le \sum_{n=1}^{\infty} \frac{3l_n - 5}{2} \cdot \frac{2}{3}R^{l}_{n} \le \frac{l_1 - \frac{5}{3} + \sum_{n=2}^{\infty} \frac{3}{2}(\frac{9}{31})^{n-2}}{\frac{2}{3}l_1 + \frac{1}{9}} < 2, \tag{2.22}$$

which together with Lemma 2.13 further implies that for any $w \in W^{l}_{*}$ and any $x \in F^{l}_{w}(V^{\boldsymbol{l}^{|w|}}_{*})$,

$$\max_{k\in\{0,1,2\}} R_{\mathcal{E}^{l}}(F^{l}_{w}(q_k), x) < 2R^{l}_{|w|}. \tag{2.23}$$

To see (1) and (2), let $x, y \in V^{l}_{*}$, $n \in \mathbb{N}$, choose $w \in W^{l}_{n}$ so that $x \in K^{l}_{w}$, and set $\Lambda_{n,w} := \{v \in W^{l}_{n} \mid K^{l}_{w} \cap K^{l}_{v} \neq \emptyset\}$ and $U_{n,w} := \bigcup_{v\in\Lambda_{n,w}} K^{l}_{v}$. It holds that

$$\text{if } y \in U_{n,w}, \quad \text{then } d_{l}(x, y) < \frac{3}{L^{l}_{n}} \text{ and } R_{\mathcal{E}^{l}}(x, y) < 4R^{l}_{n} \tag{2.24}$$

by (2.3), the triangle inequality for d_l and $R_{\mathcal{E}^l}$, (2.6) and (2.23). On the other hand, if $y \notin U_{n,w}$, then clearly $d_l(x, y) \geq 1/L^l_n$ by (2.3) and (2.4), and recalling Proposition 2.15 and setting $h_{n,w} := h^l_n(\mathbb{1}_{F^l_w(V_0)})$, we have $h_{n,w}|_{F^l_w(V^{l^n}_*)} = 1$, $h_{n,w}|_{F^l_v(V^{l^n}_*)} = 0$ for any $v \in W^l_n \setminus \Lambda_{n,w}$, $\mathcal{E}^l(h_{n,w}, h_{n,w}) = \mathcal{E}^{l,n}(\mathbb{1}_{F^l_w(V_0)}, \mathbb{1}_{F^l_w(V_0)})$, and therefore

$$R_{\mathcal{E}^l}(x, y) \geq \frac{|h_{n,w}(x) - h_{n,w}(y)|^2}{\mathcal{E}^l(h_{n,w}, h_{n,w})} = \frac{1}{\mathcal{E}^{l,n}(\mathbb{1}_{F^l_w(V_0)}, \mathbb{1}_{F^l_w(V_0)})} = \frac{\frac{1}{2}R^l_n}{\#\Lambda_{n,w} - 1} \geq \frac{R^l_n}{6}$$

by Proposition 2.11-(RF4), (2.11), (2.9), Proposition 2.2-(1) and $\#\Lambda_{n,w} \leq 4$. It follows that, if either $d_l(x, y) < 1/L^l_n$ or $R_{\mathcal{E}^l}(x, y) < \frac{1}{6}R^l_n$, then $y \in U_{n,w}$, hence $d_l(x, y) < 3/L^l_n$ and $R_{\mathcal{E}^l}(x, y) < 4R^l_n$ by (2.24), proving (1) and (2), which in turn immediately imply the existence and the stated properties of $\overline{R}_{\mathcal{E}^l}$. □

Definition 2.17 Throughout the rest of this paper, we identify $\mathcal{F}_l$ with the linear subspace of $\mathcal{C}(K^l)$ given by

$$\{u \in \mathcal{C}(K^l) \mid u|_{V^l_*} \in \mathcal{F}_l\} = \left\{u \in \mathcal{C}(K^l) \;\middle|\; \lim_{n\to\infty} \mathcal{E}^{l,n}(u|_{V^l_n}, u|_{V^l_n}) < \infty\right\} \tag{2.25}$$

through the mapping $u \mapsto u|_{V^l_*}$, which is a linear isomorphism from (2.25) to $\mathcal{F}_l$ since each $u \in \mathcal{F}_l$ uniquely extends to an element of $\mathcal{C}(K^l)$ by Proposition 2.16 and (2.15). The pair $(\mathcal{E}^l, \mathcal{F}_l)$ is then called the *canonical resistance form* on K^l.

Theorem 2.18 (1) *$(\mathcal{E}^l, \mathcal{F}_l)$ is a resistance form on K^l with resistance metric $\overline{R}_{\mathcal{E}^l}$, which is hereafter denoted as $R_{\mathcal{E}^l}$ for simplicity of the notation.*
(2) *$(\mathcal{E}^l, \mathcal{F}_l)$ is* regular, *i.e., $\mathcal{F}_l$ is a dense subalgebra of $(\mathcal{C}(K^l), \|\cdot\|_{\sup})$.*
(3) *$(\mathcal{E}^l, \mathcal{F}_l)$ is* strongly local, *i.e., $\mathcal{E}^l(u, v) = 0$ for any $u, v \in \mathcal{F}_l$ that satisfy* $\mathrm{supp}_K[u - a\mathbb{1}_{K^l}] \cap \mathrm{supp}_K[v] = \emptyset$ *for some $a \in \mathbb{R}$.*

Proof (1) follows from Propositions 2.11, 2.16, Definition 2.17, [27, Lemma 2.3.9 and Theorem 2.3.10], (2) from (1), the compactness of $(K^l, R_{\mathcal{E}^l})$, [28, Corollary 6.4 and Lemma 6.5], and (3) from $\mathbb{1}_{K^l} \in \mathcal{F}_l$, $\mathcal{E}^l(\mathbb{1}_{K^l}, \mathbb{1}_{K^l}) = 0$ and (2.17). □

Remark 2.19 (1) To be explicit, Theorem 2.18-(1) means the following:
Proposition 2.11-(RF1), (RF2), (RF3), (RF5) *with K^l in place of V^l_* hold and $\overline{R}_{\mathcal{E}^l}(x, y) = \sup\{|u(x) - u(y)|^2/\mathcal{E}^l(u, u) \mid u \in \mathcal{F}_l \setminus \mathbb{R}\mathbb{1}_{K^l}\}$ for any $x, y \in K^l$.*
(2) Under the conventions introduced in Definition 2.17 and Theorem 2.18-(1), we easily get the following, *which we will utilize below without further notice*:

- (2.15) for any $u \in \mathcal{F}_l$ and any $x, y \in K^l$;
- Proposition 2.12 with $\mathcal{C}(K^l)$, F^l_w in place of $\mathbb{R}^{V^l_*}$, $F^l_w|_{V^{l^n}_*}$;
- Lemma 2.13 with $K^{l^{|w|}}$ in place of $V^{l^{|w|}}_*$;
- Proposition 2.15 with $\mathcal{C}(K^l)$, F^l_w, K^l_w in place of $\mathbb{R}^{V^l_*}$, $F^l_w|_{V^{l^n}_*}$, $F^l_w(V^{l^n}_*)$;
- Proposition 2.16-(1), (2) for any $x, y \in K^l$ and any $n \in \mathbb{N}$;
- (2.23) for any $w \in W^l_*$ and any $x \in K^l_w$.

Finally, we can now consider $(\mathcal{E}^{\boldsymbol{l}}, \mathcal{F}_{\boldsymbol{l}})$ as an irreducible, strongly local, regular symmetric Dirichlet form over $K^{\boldsymbol{l}}$ as follows. See [17, Sects. 1.1 and 1.6] or [13, Sects. 1.1, 1.3 and 2.1] for the definitions of the relevant notions.

Theorem 2.20 *Let* μ *be a* Radon measure *on* $K^{\boldsymbol{l}}$ *with* full support*, i.e., a Borel measure on* $K^{\boldsymbol{l}}$ *with* $\mu(K^{\boldsymbol{l}}) < \infty$ *and* $\mu(K^{\boldsymbol{l}}_w) > 0$ *for any* $w \in W^{\boldsymbol{l}}_*$. *Then* $(\mathcal{E}^{\boldsymbol{l}}, \mathcal{F}_{\boldsymbol{l}})$ *is an irreducible, strongly local regular symmetric Dirichlet form on* $L^2(K^{\boldsymbol{l}}, \mu)$.

Proof Since $\mathcal{C}(K^{\boldsymbol{l}})$ is dense in $L^2(K^{\boldsymbol{l}}, \mu)$ by [34, Theorem 3.14], $\mathcal{F}_{\boldsymbol{l}}$ is also dense in $L^2(K^{\boldsymbol{l}}, \mu)$ by Theorem 2.18-(2), and then $(\mathcal{E}^{\boldsymbol{l}}, \mathcal{F}_{\boldsymbol{l}})$ is a regular symmetric Dirichlet form on $L^2(K^{\boldsymbol{l}}, \mu)$ by Proposition 2.11-(RF2), (2.15), Proposition 2.11-(RF5) and Theorem 2.18-(2), strongly local by Theorem 2.18-(3), and irreducible by Proposition 2.11-(RF1) and [13, Theorem 2.1.11]. □

3 Space-Time Scale Function $\Psi_{\boldsymbol{l}}$ and fHKE($\Psi_{\boldsymbol{l}}$)

In this section, we continue to fix an arbitrary $\boldsymbol{l} = (l_n)_{n=1}^{\infty} \in (\mathbb{N} \setminus \{1, 2, 3, 4\})^{\mathbb{N}}$, define a space-time scale function $\Psi_{\boldsymbol{l}}$ explicitly in terms of $\boldsymbol{l} = (l_n)_{n=1}^{\infty}$, and show that $(K^{\boldsymbol{l}}, d_{\boldsymbol{l}}, m_{\boldsymbol{l}}, \mathcal{E}^{\boldsymbol{l}}, \mathcal{F}_{\boldsymbol{l}})$ satisfies fHKE($\Psi_{\boldsymbol{l}}$). First, $\Psi_{\boldsymbol{l}}$ is defined in a way analogous to [26, (5.11)] for the usual scale irregular Sierpiński gaskets but modified so as to take the "asymptotically one-dimensional" nature of $K^{\boldsymbol{l}}$ into account, as follows.

Definition 3.1 We define a homeomorphism $\Psi_{\boldsymbol{l}} : [0, \infty) \to [0, \infty)$ by

$$\begin{aligned}\Psi_{\boldsymbol{l}}(s) &:= \left(\frac{1}{M^{\boldsymbol{l}}_n} + \frac{s - 1/L^{\boldsymbol{l}}_n}{1/L^{\boldsymbol{l}}_{n-1} - 1/L^{\boldsymbol{l}}_n}\left(\frac{1}{M^{\boldsymbol{l}}_{n-1}} - \frac{1}{M^{\boldsymbol{l}}_n}\right)\right) \\ &\qquad\qquad \cdot \left(R^{\boldsymbol{l}}_n + \frac{s - 1/L^{\boldsymbol{l}}_n}{1/L^{\boldsymbol{l}}_{n-1} - 1/L^{\boldsymbol{l}}_n}(R^{\boldsymbol{l}}_{n-1} - R^{\boldsymbol{l}}_n)\right) \\ &= \frac{1}{T^{\boldsymbol{l}}_n}\left(1 + \frac{3l_n - 4}{l_n - 1}(L^{\boldsymbol{l}}_n s - 1)\right)\left(1 + \frac{\frac{2}{3}l_n - \frac{8}{9}}{l_n - 1}(L^{\boldsymbol{l}}_n s - 1)\right) \qquad (3.1)\end{aligned}$$

for $n \in \mathbb{N}$ and $s \in [1/L^{\boldsymbol{l}}_n, 1/L^{\boldsymbol{l}}_{n-1}]$ and $\Psi_{\boldsymbol{l}}(s) := s^{\beta_{\boldsymbol{l},0}}$ for $s \in \{0\} \cup [1, \infty)$, where $T^{\boldsymbol{l}}_n := M^{\boldsymbol{l}}_n / R^{\boldsymbol{l}}_n = (\#S_{l_1}/r_{l_1}) \cdots (\#S_{l_n}/r_{l_n}) = \prod_{k=1}^{n}(2l_k^2 - \frac{5}{3}l_k - \frac{1}{3})$ $(T^{\boldsymbol{l}}_0 := 1)$ and $\beta_{\boldsymbol{l},0} := \inf_{n\in\mathbb{N}} \beta_{l_n}$ with $\beta_l := \log_l(\#S_l/r_l) = \log_l(2l^2 - \frac{5}{3}l - \frac{1}{3}) \in (2, 2 + \log_5 2)$ for $l \in \mathbb{N} \setminus \{1, 2, 3, 4\}$; note that $\{\beta_l\}_{l=5}^{\infty}$ is strictly decreasing and converges to 2. We also set $\beta_{\boldsymbol{l},1} := \max_{n\in\mathbb{N}} \beta_{l_n}$, so that $2 \le \beta_{\boldsymbol{l},0} \le \beta_{\boldsymbol{l},1} \le \beta_5 < 2 + \log_5 2$.

Lemma 3.2 $\Psi_{\boldsymbol{l}}$ *satisfies* (1.2) *with* $c_\Psi = 81$, $\beta_0 = \beta_{\boldsymbol{l},0}$ *and* $\beta_1 = \beta_{\boldsymbol{l},1}$.

Proof Let $r, R \in (0, \infty)$ satisfy $r \le R$. If $r \ge 1$, then $\Psi_{\boldsymbol{l}}(R)/\Psi_{\boldsymbol{l}}(r) = (R/r)^{\beta_{\boldsymbol{l},0}} \le (R/r)^{\beta_{\boldsymbol{l},1}}$. Next, if $r, R \in [1/L^{\boldsymbol{l}}_n, 1/L^{\boldsymbol{l}}_{n-1}]$ for some $n \in \mathbb{N}$, then we easily see from (3.1), $1 \le L^{\boldsymbol{l}}_n r \le L^{\boldsymbol{l}}_n R \le l_n$ and $l_n^{\beta_{l_n} - 2} = 2 - \frac{5}{3}l_n^{-1} - \frac{1}{3}l_n^{-2} < 2$ that

$$\frac{1}{9}\left(\frac{R}{r}\right)^{\beta_{l,0}} < \frac{2}{9} l_n^{2-\beta_{l_n}} \left(\frac{R}{r}\right)^{\beta_{l_n}} \leq \frac{2}{9}\left(\frac{R}{r}\right)^2 \leq \frac{\Psi_l(R)}{\Psi_l(r)} \leq \frac{9}{2}\left(\frac{R}{r}\right)^2 \leq \frac{9}{2}\left(\frac{R}{r}\right)^{\beta_{l,1}}. \quad (3.2)$$

Lastly, if $r < 1$ and no such $n \in \mathbb{N}$ exists, then we can choose $j, k \in \mathbb{N} \cup \{0\}$ with $j \leq k$ so that $r \in [1/L^{\boldsymbol{l}}_{k+1}, 1/L^{\boldsymbol{l}}_k)$ and $R \in [1/L^{\boldsymbol{l}}_j, 1/L^{\boldsymbol{l}}_{j-1})$, where $1/L^{\boldsymbol{l}}_{-1} := \infty$, and by (3.1) and the definitions of $\beta_{l,0}$ and $\beta_{l,1}$ we have

$$\frac{\Psi_l(1/L^{\boldsymbol{l}}_j)}{\Psi_l(1/L^{\boldsymbol{l}}_k)} = \frac{T^{\boldsymbol{l}}_k}{T^{\boldsymbol{l}}_j} = \prod_{n=j+1}^{k} \frac{\#S_{l_n}}{r_{l_n}} = \prod_{n=j+1}^{k} l_n^{\beta_{l_n}} \in \left[\left(\frac{1/L^{\boldsymbol{l}}_j}{1/L^{\boldsymbol{l}}_k}\right)^{\beta_{l,0}}, \left(\frac{1/L^{\boldsymbol{l}}_j}{1/L^{\boldsymbol{l}}_k}\right)^{\beta_{l,1}}\right],$$

which together with (3.2) and the equality

$$\frac{\Psi_l(R)}{\Psi_l(r)} = \frac{\Psi_l(1/L^{\boldsymbol{l}}_k)}{\Psi_l(r)} \frac{\Psi_l(1/L^{\boldsymbol{l}}_j)}{\Psi_l(1/L^{\boldsymbol{l}}_k)} \frac{\Psi_l(R)}{\Psi_l(1/L^{\boldsymbol{l}}_j)}$$

immediately yields (1.2) for Ψ_l with $c_\Psi = 81$, $\beta_0 = \beta_{l,0}$ and $\beta_1 = \beta_{l,1}$. □

The main result of this section is the following theorem.

Theorem 3.3 *$(K^{\boldsymbol{l}}, d_l, m_l, \mathcal{E}^{\boldsymbol{l}}, \mathcal{F}_l)$ satisfies* fHKE(Ψ_l).

The rest of this section is devoted to the proof of Theorem 3.3. We will conclude it from [28, Theorem 15.10] by proving that $(K^{\boldsymbol{l}}, d_l, m_l, \mathcal{E}^{\boldsymbol{l}}, \mathcal{F}_l)$ satisfies the conditions $(\mathrm{DM1})_{\Psi_l,d_l}$ and $(\mathrm{DM2})_{\Psi_l,d_l}$ defined in [28, Definition 15.9-(3),(4)], which are the central assumptions in [28, Theorem 15.10]. A similar argument is given in [28, Chap. 24] for a large class of scale irregular Sierpiński gaskets, but the possible unboundedness of $\boldsymbol{l} = (l_n)_{n=1}^\infty$ requires some additional care.

The core of the proof of Theorem 3.3 is to establish the following proposition, which is an extension of (the proof of) Proposition 2.16 to the case where $n \in \mathbb{N}$, $k \in \{1, \ldots, l_n\}$ and either $d_l(x, y) < k/L^{\boldsymbol{l}}_n$ or $R_{\mathcal{E}^{\boldsymbol{l}}}(x, y) < \frac{1}{12}kR^{\boldsymbol{l}}_n$.

Definition 3.4 Let $n \in \mathbb{N}$ and $w \in W^{\boldsymbol{l}}_n$. For each $k \in \{0, \ldots, l_n\}$, we define

$$\Lambda^{(k)}_{n,w} := \left\{ v \in W^{\boldsymbol{l}}_n \;\middle|\; \begin{array}{l} \text{there exists } \{v^{(j)}\}_{j=0}^k \subset W^{\boldsymbol{l}}_n \text{ such that } v^{(0)} = w,\ v^{(k)} = v \\ \text{and } K^{\boldsymbol{l}}_{v^{(j-1)}} \cap K^{\boldsymbol{l}}_{v^{(j)}} \neq \emptyset \text{ for any } j \in \{1, \ldots, k\} \end{array} \right\} \quad (3.3)$$

($\Lambda^{(0)}_{n,w} := \{w\}$) and $U^{(k)}_{n,w} := \bigcup_{v \in \Lambda^{(k)}_{n,w}} K^{\boldsymbol{l}}_v$, so that $2k + 1 \leq \#\Lambda^{(k)}_{n,w} \leq (6k) \vee 1$.

Proposition 3.5 *Let $n \in \mathbb{N}$, $w \in W^{\boldsymbol{l}}_n$, $x \in K^{\boldsymbol{l}}_w$ and $k \in \{1, \ldots, l_n\}$.*

(1) *If $y \in U^{(k)}_{n,w}$, then $d_l(x, y) < (k+2)/L^{\boldsymbol{l}}_n$ and $R_{\mathcal{E}^{\boldsymbol{l}}}(x, y) < (\frac{2}{3}k + \frac{10}{3})R^{\boldsymbol{l}}_n$.*
(2) *If $y \in K^{\boldsymbol{l}} \setminus U^{(k)}_{n,w}$, then $d_l(x, y) \geq k/L^{\boldsymbol{l}}_n$ and $R_{\mathcal{E}^{\boldsymbol{l}}}(x, y) \geq \frac{1}{12}kR^{\boldsymbol{l}}_n$.*
(3) *$B_{d_l}(x, k/L^{\boldsymbol{l}}_n) \subset U^{(k)}_{n,w} \subset B_{d_l}(x, (k+2)/L^{\boldsymbol{l}}_n)$.*
(4) *$B_{R_{\mathcal{E}^{\boldsymbol{l}}}}(x, \frac{1}{12}kR^{\boldsymbol{l}}_n) \subset U^{(k)}_{n,w} \subset B_{R_{\mathcal{E}^{\boldsymbol{l}}}}(x, (\frac{2}{3}k + \frac{10}{3})R^{\boldsymbol{l}}_n)$.*

Proof (1) is immediate from (2.3), the triangle inequality for d_l and $R_{\mathcal{E}^l}$, (2.6), (2.21) and (2.23). To see (2), let $y \in K^l \setminus U^{(k)}_{n,w}$. For $d_l(x, y)$, by Proposition 2.4 we can take $\gamma : [0, 1] \to K^l$ such that $\gamma(0) = x$, $\gamma(1) = y$ and $d_l(\gamma(s), \gamma(t)) = |s - t| d_l(x, y)$ for any $s, t \in [0, 1]$, and it then follows from (2.3) and $y \notin U^{(k)}_{n,w}$ that $\#(\gamma^{-1}(V^l_n) \cap (0, 1)) \geq k + 1$, which yields $d_l(x, y) = \ell_{\mathbb{R}^2}(\gamma) \geq k/L^l_n$. For $R_{\mathcal{E}^l}(x, y)$, recalling Proposition 2.15, define $u \in \mathbb{R}^{V^l_n}$ by

$$u(z) := \begin{cases} 1 & \text{if } z \in K^l_w = U^{(0)}_{n,w}, \\ 1 - \frac{j}{k} & \text{if } j \in \{1, \ldots, k\} \text{ and } z \in U^{(j)}_{n,w} \setminus U^{(j-1)}_{n,w}, \\ 0 & \text{if } z \in K^l \setminus U^{(k)}_{n,w} \end{cases} \tag{3.4}$$

for each $z \in V^l_n$ and set $h^{(k)}_{n,w} := h^l_n(u)$, so that $\mathcal{E}^l(h^{(k)}_{n,w}, h^{(k)}_{n,w}) = \mathcal{E}^{l,n}(u, u)$, $h^{(k)}_{n,w}|_{K^l_w} = 1$, $h^{(k)}_{n,w}|_{K^l_v} = 0$ for any $v \in W^l_n \setminus \Lambda^{(k)}_{n,w}$, and $u(F^l_v(V_0)) \subset \{1 - \frac{j-1}{k}, 1 - \frac{j}{k}\}$ for any $j \in \{1, \ldots, k\}$ and any $v \in \Lambda^{(j)}_{n,w} \setminus \Lambda^{(j-1)}_{n,w}$. Then combining these properties with Proposition 2.11-(RF4), (2.11), (2.9) and $\#\Lambda^{(k)}_{n,w} \leq 6k$, we obtain

$$R_{\mathcal{E}^l}(x, y) \geq \frac{|h^{(k)}_{n,w}(x) - h^{(k)}_{n,w}(y)|^2}{\mathcal{E}^l(h^{(k)}_{n,w}, h^{(k)}_{n,w})} = \frac{1}{\mathcal{E}^{l,n}(u, u)} \geq \frac{\frac{1}{2}k^2 R^l_n}{\#\Lambda^{(k)}_{n,w} - 1} \geq \frac{kR^l_n}{12},$$

which proves (2). Lastly, we also get (3) and (4) since the conjunction of (3) and (4) is clearly equivalent to that of (1) and (2). □

As an easy consequence of Proposition 3.5, we further obtain the following proposition, which contains $(\mathrm{DM2})_{\Psi_l, d_l}$ as defined in [28, Definition 15.9-(4)].

Proposition 3.6 *Let $x, y \in K^l$, $s \in (0, \infty)$, $n \in \mathbb{N}$ and $k \in \{2, \ldots, l_n\}$.*

(1) *If $s \in [(k-1)/L^l_n, k/L^l_n]$, then*

$$\frac{1}{18}\frac{k^2}{T^l_n} \leq \Psi_l(s) \leq \frac{9}{2}\frac{k^2}{T^l_n} \quad \text{and} \quad \frac{7}{36}\frac{k}{M^l_n} \leq m_l(B_{d_l}(x, s)) \leq 6\frac{k}{M^l_n}, \tag{3.5}$$

whereas if $s \in [1, 3]$, then $1 \leq \Psi_l(s) \leq 14$ and $\frac{7}{12} \leq m_l(B_{d_l}(x, s)) \leq 1$.

(2) *If $d_l(x, y) \in [(k-1)/L^l_n, k/L^l_n)$, then*

$$\frac{1}{48}kR^l_n \leq R_{\mathcal{E}^l}(x, y) \leq \frac{7}{3}kR^l_n, \tag{3.6}$$

whereas $d_l(x, y) < 3$, $R_{\mathcal{E}^l}(x, y) < 4$, and if $d_l(x, y) \geq 1$ then $R_{\mathcal{E}^l}(x, y) \geq \frac{1}{14}$.

(3) *If $x \neq y$, then*

$$6^{-4}\frac{\Psi_l(d_l(x, y))}{m_l\big(B_{d_l}(x, d_l(x, y))\big)} \leq R_{\mathcal{E}^l}(x, y) \leq 2^8\frac{\Psi_l(d_l(x, y))}{m_l\big(B_{d_l}(x, d_l(x, y))\big)}. \tag{3.7}$$

Proof (1) Assume that $s \in [(k-1)/L^{\boldsymbol{l}}_n, k/L^{\boldsymbol{l}}_n]$. By (3.1) and (3.2) we have

$$\begin{aligned}\frac{1}{18}\frac{k^2}{T^{\boldsymbol{l}}_n} \le \frac{2}{9}\frac{(k-1)^2}{T^{\boldsymbol{l}}_n} &= \frac{2}{9}(k-1)^2\Psi_{\boldsymbol{l}}(1/L^{\boldsymbol{l}}_n) \le \Psi_{\boldsymbol{l}}((k-1)/L^{\boldsymbol{l}}_n) \\ &\le \Psi_{\boldsymbol{l}}(s) \le \Psi_{\boldsymbol{l}}(k/L^{\boldsymbol{l}}_n) \le \frac{9}{2}k^2\Psi_{\boldsymbol{l}}(1/L^{\boldsymbol{l}}_n) = \frac{9}{2}\frac{k^2}{T^{\boldsymbol{l}}_n}.\end{aligned} \tag{3.8}$$

For $m_{\boldsymbol{l}}(B_{d_{\boldsymbol{l}}}(x,s))$, choosing $w \in W^{\boldsymbol{l}}_n$ so that $x \in K^{\boldsymbol{l}}_w$, we see from Proposition 3.5-(3), (2.8) and $2j+1 \le \#\Lambda^{(j)}_{n,w} \le 6j$ for $j \in \{1,\dots,l_n\}$ that

$$m_{\boldsymbol{l}}(B_{d_{\boldsymbol{l}}}(x,s)) \le m_{\boldsymbol{l}}(B_{d_{\boldsymbol{l}}}(x,k/L^{\boldsymbol{l}}_n)) \le m_{\boldsymbol{l}}(U^{(k)}_{n,w}) = \frac{\#\Lambda^{(k)}_{n,w}}{M^{\boldsymbol{l}}_n} \le 6\frac{k}{M^{\boldsymbol{l}}_n} \tag{3.9}$$

and that, provided $k \ge 4$,

$$m_{\boldsymbol{l}}(B_{d_{\boldsymbol{l}}}(x,s)) \ge m_{\boldsymbol{l}}(B_{d_{\boldsymbol{l}}}(x,(k-1)/L^{\boldsymbol{l}}_n)) \ge m_{\boldsymbol{l}}(U^{(k-3)}_{n,w}) = \frac{\#\Lambda^{(k-3)}_{n,w}}{M^{\boldsymbol{l}}_n} \ge \frac{2k-5}{M^{\boldsymbol{l}}_n}. \tag{3.10}$$

If $k \in \{2,3\}$, then choosing $v \in W^{\boldsymbol{l}}_{n+1}$ so that $x \in K^{\boldsymbol{l}}_v$, by Proposition 3.5-(3), (2.8) and $\#\Lambda^{(l_{n+1}-2)}_{n+1,v} \ge 2l_{n+1}-3$ we get

$$\begin{aligned}m_{\boldsymbol{l}}(B_{d_{\boldsymbol{l}}}(x,s)) &\ge m_{\boldsymbol{l}}(B_{d_{\boldsymbol{l}}}(x,1/L^{\boldsymbol{l}}_n)) = m_{\boldsymbol{l}}(B_{d_{\boldsymbol{l}}}(x,l_{n+1}/L^{\boldsymbol{l}}_{n+1})) \\ &\ge m_{\boldsymbol{l}}(U^{(l_{n+1}-2)}_{n+1,v}) = \frac{\#\Lambda^{(l_{n+1}-2)}_{n+1,v}}{M^{\boldsymbol{l}}_{n+1}} \ge \frac{2l_{n+1}-3}{(3l_{n+1}-3)M^{\boldsymbol{l}}_n} \ge \frac{7}{12}\frac{1}{M^{\boldsymbol{l}}_n}.\end{aligned} \tag{3.11}$$

(3.8), (3.9), (3.10) and (3.11) together yield (3.5).
On the other hand, if $s \in [1,3]$, then $\Psi_{\boldsymbol{l}}(s) = s^{\beta_{\boldsymbol{l},0}} \in [1,3^{\beta_5}] \subset [1,14]$ and $1 = m_{\boldsymbol{l}}(K^{\boldsymbol{l}}) \ge m_{\boldsymbol{l}}(B_{d_{\boldsymbol{l}}}(x,s)) \ge m_{\boldsymbol{l}}(B_{d_{\boldsymbol{l}}}(x,1)) \ge \frac{7}{12}$ by (3.11) with $n=0$.

(2) Assume that $d_{\boldsymbol{l}}(x,y) \in [(k-1)/L^{\boldsymbol{l}}_n, k/L^{\boldsymbol{l}}_n)$. Then by Proposition 3.5-(3), (4), it follows from $d_{\boldsymbol{l}}(x,y) < k/L^{\boldsymbol{l}}_n$ that $R_{\mathcal{E}^{\boldsymbol{l}}}(x,y) \le (\frac{2}{3}k + \frac{10}{3})R^{\boldsymbol{l}}_n \le \frac{7}{3}kR^{\boldsymbol{l}}_n$, from $d_{\boldsymbol{l}}(x,y) \ge (k-1)/L^{\boldsymbol{l}}_n$ that $R_{\mathcal{E}^{\boldsymbol{l}}}(x,y) \ge \frac{1}{12}(k-3)R^{\boldsymbol{l}}_n \ge \frac{1}{48}kR^{\boldsymbol{l}}_n$ provided $k \ge 4$, and from $d_{\boldsymbol{l}}(x,y) \ge 1/L^{\boldsymbol{l}}_n = l_{n+1}/L^{\boldsymbol{l}}_{n+1}$ that, provided $k \in \{2,3\}$,

$$R_{\mathcal{E}^{\boldsymbol{l}}}(x,y) \ge \frac{1}{12}(l_{n+1}-2)R^{\boldsymbol{l}}_{n+1} = \frac{1}{12}\frac{l_{n+1}-2}{\frac{2}{3}l_{n+1}+\frac{1}{9}}R^{\boldsymbol{l}}_n \ge \frac{9}{124}R^{\boldsymbol{l}}_n > \frac{kR^{\boldsymbol{l}}_n}{48}, \tag{3.12}$$

proving (3.6).
On the other hand, $d_{\boldsymbol{l}}(x,y) < 3$ by (2.6), $R_{\mathcal{E}^{\boldsymbol{l}}}(x,y) < 4$ by (2.23), and if $d_{\boldsymbol{l}}(x,y) \ge 1$ then $R_{\mathcal{E}^{\boldsymbol{l}}}(x,y) \ge \frac{9}{124} > \frac{1}{14}$ by (3.12) with $n=0$.

(3) This is immediate from (1) and (2). □

We need the following definition and lemma for the proof of the other condition $(\mathrm{DM1})_{\Psi_{\boldsymbol{l}},d_{\boldsymbol{l}}}$ required to apply [28, Theorem 15.10].

Definition 3.7 We define homeomorphisms $\Psi_l^{\mathrm{M}}, \Psi_l^{\mathrm{R}} : [0, \infty) \to [0, \infty)$ by

$$\Psi_l^{\mathrm{M}}(s) := \frac{1}{M_n^l} + \frac{(s - 1/L_n^l)(1/M_{n-1}^l - 1/M_n^l)}{1/L_{n-1}^l - 1/L_n^l} = \frac{1}{M_n^l}\left(1 + \frac{3l_n - 4}{l_n - 1}(L_n^l s - 1)\right),$$

$$\Psi_l^{\mathrm{R}}(s) := R_n^l + \frac{(s - 1/L_n^l)(R_{n-1}^l - R_n^l)}{1/L_{n-1}^l - 1/L_n^l} = R_n^l\left(1 + \frac{\frac{2}{3}l_n - \frac{8}{9}}{l_n - 1}(L_n^l s - 1)\right) \tag{3.13}$$

for $n \in \mathbb{N}$ and $s \in [1/L_n^l, 1/L_{n-1}^l]$ and $\Psi_l^{\mathrm{M}}(s) := s^{\beta_{l,0}^{\mathrm{M}}}$ and $\Psi_l^{\mathrm{R}}(s) := s^{\beta_{l,1}^{\mathrm{R}}}$ for $s \in \{0\} \cup [1, \infty)$, where $\beta_{l,0}^{\mathrm{M}} := \inf_{n\in\mathbb{N}} \log_{l_n} \#S_{l_n}$ and $\beta_{l,1}^{\mathrm{R}} := -\inf_{n\in\mathbb{N}} \log_{l_n} r_{l_n}$ (note that $\{\log_l \#S_l\}_{l=5}^{\infty}$ and $\{\log_l r_l\}_{l=5}^{\infty}$ are strictly decreasing), so that $\Psi_l = \Psi_l^{\mathrm{M}}\Psi_l^{\mathrm{R}}$. We also set $\beta_{l,1}^{\mathrm{M}} := \max_{n\in\mathbb{N}} \log_{l_n} \#S_{l_n}$ and $\beta_{l,0}^{\mathrm{R}} := -\max_{n\in\mathbb{N}} \log_{l_n} r_{l_n}$.

Lemma 3.8 (1) *Ψ_l^{M} satisfies (1.2) with $c_\Psi = 81$, $\beta_0 = \beta_{l,0}^{\mathrm{M}}$ and $\beta_1 = \beta_{l,1}^{\mathrm{M}}$.*
(2) *Ψ_l^{R} satisfies (1.2) with $c_\Psi = 6$, $\beta_0 = \beta_{l,0}^{\mathrm{R}}$ and $\beta_1 = \beta_{l,1}^{\mathrm{R}}$.*

Proof These are proved in exactly the same way as Lemma 3.2. □

Finally, $(\mathrm{DM1})_{\Psi_l,d_l}$ as defined in [28, Definition 15.9-(3)] is deduced as follows.

Proposition 3.9 *Let $x, y \in K^l$ and $s \in (0, 3]$. Then*

$$\frac{1}{16}\Psi_l^{\mathrm{M}}(s) \le m_l(B_{d_l}(x, s)) \le 12\Psi_l^{\mathrm{M}}(s), \quad \frac{1}{12}\Psi_l^{\mathrm{R}}(s) \le \frac{\Psi_l(s)}{m_l(B_{d_l}(x, s))} \le 16\Psi_l^{\mathrm{R}}(s), \tag{3.14}$$

$$2^{-14}\Psi_l^{\mathrm{R}}(d_l(x, y)) \le R_{\mathcal{E}^l}(x, y) \le 2^{12}\Psi_l^{\mathrm{R}}(d_l(x, y)). \tag{3.15}$$

In particular, if $x \neq y$, then for any $\lambda \in (0, 1]$,

$$\frac{6^{-4}\lambda^{\beta_{l,1}^{\mathrm{R}}}\Psi_l(d_l(x, y))}{m_l\big(B_{d_l}(x, d_l(x, y))\big)} \le \frac{\Psi_l(\lambda d_l(x, y))}{m_l\big(B_{d_l}(x, \lambda d_l(x, y))\big)} \le \frac{6^4\lambda^{\beta_{l,0}^{\mathrm{R}}}\Psi_l(d_l(x, y))}{m_l\big(B_{d_l}(x, d_l(x, y))\big)}. \tag{3.16}$$

Proof If $n \in \mathbb{N}$, $k \in \{2, \ldots, l_n\}$ and $s \in [(k-1)/L_n^l, k/L_n^l]$ then we easily see from (3.13) that $\frac{1}{2}k/M_n^l \le \Psi_l^{\mathrm{M}}(s) \le 3k/M_n^l$, and if $s \in [1, 3]$ then $\Psi_l^{\mathrm{M}}(s) = s^{\beta_{l,0}^{\mathrm{M}}} \in [1, 3^{\log_5 \#S_5}] \subset [1, 6]$. These facts, Proposition 3.6-(1) and $\Psi_l = \Psi_l^{\mathrm{M}}\Psi_l^{\mathrm{R}}$ together imply (3.14), which in turn in combination with Proposition 3.6-(3) and Lemma 3.8-(2), respectively, yields (3.15) and (3.16) since $d_l(x, y) \in [0, 3)$ by (2.6). □

***Proof of Theorem* 3.3** By Propositions 2.4, 2.16 and Theorem 2.18, $(\mathcal{E}^l, \mathcal{F}_l)$ is a strongly local regular resistance form on K^l whose resistance metric $R_{\mathcal{E}^l}$ gives the same topology as the geodesic metric d_l. We also have (ACC) as defined in [28, Definition 7.4] by [28, Proposition 7.6], $(\mathrm{DM1})_{\Psi_l,d_l}$ by (3.16) and $\beta_{l,0}^{\mathrm{R}} > 0$, and $(\mathrm{DM2})_{\Psi_l,d_l}$ by Proposition 3.6-(3). Thus [28, Theorem 15.10, Cases 1 and 2] are applicable to $(K^l, d_l, m_l, \mathcal{E}^l, \mathcal{F}_l)$ and imply that it satisfies $\mathrm{fHKE}(\Psi_l)$. □

4 Singularity of the Energy Measures

As in the previous two sections, we fix an arbitrary $\boldsymbol{l} = (l_n)_{n=1}^{\infty} \in (\mathbb{N} \setminus \{1, 2, 3, 4\})^{\mathbb{N}}$ throughout this section. We first recall the definition of the $\mathcal{E}^{\boldsymbol{l}}$-energy measures.

Definition 4.1 ($\mathcal{E}^{\boldsymbol{l}}$-energy measure; [17, (3.2.14)]) Let $u \in \mathcal{F}_{\boldsymbol{l}}$. We define the $\mathcal{E}^{\boldsymbol{l}}$-*energy measure* $\mu^{\boldsymbol{l}}_{\langle u \rangle}$ of u as the unique Borel measure on $K^{\boldsymbol{l}}$ such that

$$\int_{K^{\boldsymbol{l}}} f \, d\mu^{\boldsymbol{l}}_{\langle u \rangle} = \mathcal{E}^{\boldsymbol{l}}(u, fu) - \frac{1}{2}\mathcal{E}^{\boldsymbol{l}}(u^2, f) \quad \text{for any } f \in \mathcal{F}_{\boldsymbol{l}}; \tag{4.1}$$

since $\mathcal{F}_{\boldsymbol{l}}$ is a dense subalgebra of $(\mathcal{C}(K^{\boldsymbol{l}}), \|\cdot\|_{\sup})$ by Theorem 2.18-(2) and

$$0 \leq \mathcal{E}^{\boldsymbol{l}}(u, f^+u) - \frac{1}{2}\mathcal{E}^{\boldsymbol{l}}(u^2, f^+) \leq \|f\|_{\sup}\mathcal{E}^{\boldsymbol{l}}(u, u) \qquad \text{for any } f \in \mathcal{F}_{\boldsymbol{l}} \tag{4.2}$$

by (2.9), (2.11) and (2.14), such $\mu^{\boldsymbol{l}}_{\langle u \rangle}$ exists and is unique by the Riesz(–Markov–Kakutani) representation theorem (see, e.g., [34, Theorems 2.14 and 2.18]).

Proposition 2.12 yields the following alternative characterization of $\mu^{\boldsymbol{l}}_{\langle u \rangle}$.

Proposition 4.2 *Let $u \in \mathcal{F}_{\boldsymbol{l}}$. Then $\mu^{\boldsymbol{l}}_{\langle u \rangle}(\{x\}) = 0$ for any $x \in K^{\boldsymbol{l}}$. Moreover, $\mu^{\boldsymbol{l}}_{\langle u \rangle}$ is the unique Borel measure on $K^{\boldsymbol{l}}$ such that*

$$\mu^{\boldsymbol{l}}_{\langle u \rangle}(K^{\boldsymbol{l}}_w) = \frac{1}{R^{\boldsymbol{l}}_{|w|}}\mathcal{E}^{\boldsymbol{l}^{|w|}}(u \circ F^{\boldsymbol{l}}_w, u \circ F^{\boldsymbol{l}}_w) \qquad \textit{for any } w \in W^{\boldsymbol{l}}_*. \tag{4.3}$$

Proof Since $(\mathcal{E}^{\boldsymbol{l}}, \mathcal{F}_{\boldsymbol{l}})$ is a strongly local regular symmetric Dirichlet form on $L^2(K^{\boldsymbol{l}}, m_{\boldsymbol{l}})$ by Theorem 2.20, the Borel measure $\mu^{\boldsymbol{l}}_{\langle u \rangle}(u^{-1}(\cdot))$ on $\mathbb{R}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$ by [13, Theorem 4.3.8], and therefore $\mu^{\boldsymbol{l}}_{\langle u \rangle}(\{x\}) \leq \mu^{\boldsymbol{l}}_{\langle u \rangle}(u^{-1}(u(x))) = 0$ for any $x \in K^{\boldsymbol{l}}$.

The uniqueness of a Borel measure on $K^{\boldsymbol{l}}$ satisfying (4.3) is immediate from (2.17) and the Dynkin class theorem (see, e.g., [15, Appendixes, Theorem 4.2]). To show that $\mu^{\boldsymbol{l}}_{\langle u \rangle}$ has the property (4.3), let $n, k \in \mathbb{N} \cup \{0\}$, $w \in W^{\boldsymbol{l}}_n$ and set $f_k := h^{\boldsymbol{l}}_{n+k}(\mathbb{1}_{K^{\boldsymbol{l}}_w \cap V^{\boldsymbol{l}}_{n+k}})$, so that $\mathbb{1}_{K^{\boldsymbol{l}}_w} \leq f_k \leq \mathbb{1}_{K^{\boldsymbol{l}}_w \cup \bigcup_{q \in F^{\boldsymbol{l}}_w(V_0)} B_{d_{\boldsymbol{l}}}(q, 2/L^{\boldsymbol{l}}_{n+k})}$ by Proposition 2.15 and (2.6). Then from (4.2), (2.17) and (4.1) we obtain

$$\begin{aligned}
\frac{\mathcal{E}^{\boldsymbol{l}^n}(u \circ F^{\boldsymbol{l}}_w, u \circ F^{\boldsymbol{l}}_w)}{R^{\boldsymbol{l}}_n} &= \frac{1}{R^{\boldsymbol{l}}_n}\Big(\mathcal{E}^{\boldsymbol{l}^n}(u \circ F^{\boldsymbol{l}}_w, (f_k u) \circ F^{\boldsymbol{l}}_w) - \frac{1}{2}\mathcal{E}^{\boldsymbol{l}^n}((u^2) \circ F^{\boldsymbol{l}}_w, f_k \circ F^{\boldsymbol{l}}_w)\Big) \\
&\leq \mathcal{E}^{\boldsymbol{l}}(u, f_k u) - \frac{1}{2}\mathcal{E}^{\boldsymbol{l}}(u^2, f_k) = \int_{K^{\boldsymbol{l}}} f_k \, d\mu^{\boldsymbol{l}}_{\langle u \rangle} \\
&\leq \mu^{\boldsymbol{l}}_{\langle u \rangle}\Big(K^{\boldsymbol{l}}_w \cup \bigcup\nolimits_{q \in F^{\boldsymbol{l}}_w(V_0)} B_{d_{\boldsymbol{l}}}(q, 2/L^{\boldsymbol{l}}_{n+k})\Big) \xrightarrow{k \to \infty} \mu^{\boldsymbol{l}}_{\langle u \rangle}(K^{\boldsymbol{l}}_w)
\end{aligned}$$

and hence $\mathcal{E}^{l^n}(u\circ F^l_w, u\circ F^l_w)/R^l_n \le \mu^l_{\langle u\rangle}(K^l_w)$, where the equality necessarily holds since the sum over $w\in W^l_n$ of each side of this inequality is equal to $\mathcal{E}^l(u,u) = \mu^l_{\langle u\rangle}(K^l)$ by (2.17), (2.3), $\mu^l_{\langle u\rangle}(V^l_n)=0$ and (4.1) with $f=\mathbb{1}_{K^l}$. □

The purpose of this section is to prove the following theorem.

Theorem 4.3 *$\mu^l_{\langle u\rangle}\perp m_l$ for any $u\in\mathcal{F}_l$.*

The rest of this section is devoted to the proof of Theorem 4.3. First, we observe that the proof is reduced to the case of $u\in\bigcup_{n=0}^{\infty}\mathcal{H}_{l,n}$ by the following two lemmas.

Lemma 4.4 *Let $u\in\mathcal{F}_l$ and set $u_n := h^l_n(u|_{V^l_n})$ for each $n\in\mathbb{N}\cup\{0\}$* (recall Proposition 2.15). *Then $\mathcal{E}^l(u-u_n,u-u_n)=\mathcal{E}^l(u,u)-\mathcal{E}^{l,n}(u|_{V^l_n},u|_{V^l_n})$ for any $n\in\mathbb{N}\cup$* $\{0\}$. *In particular,* $\lim_{n\to\infty}\mathcal{E}^l(u-u_n,u-u_n)=0$.

Proof We follow [27, Proof of Lemma 3.2.17]. Let $n\in\mathbb{N}\cup\{0\}$. Then $\mathcal{E}^l(u_n,u)=\mathcal{E}^l(u_n,u_n)=\mathcal{E}^{l,n}(u|_{V^l_n},u|_{V^l_n})$ by $u_n\in\mathcal{H}_{l,n}$, (2.19) and Proposition 2.15 and thus $\mathcal{E}^l(u-u_n,u-u_n)=\mathcal{E}^l(u,u)-\mathcal{E}^{l,n}(u|_{V^l_n},u|_{V^l_n})$, which converges to 0 as $n\to\infty$ by (2.14). □

Lemma 4.5 *If a Borel measure μ on K^l, $\{u_n\}_{n=1}^{\infty}\subset\mathcal{F}_l$ and $u\in\mathcal{F}_l$ satisfy* $\lim_{n\to\infty}$ $\mathcal{E}^l(u-u_n,u-u_n)=0$ *and $\mu^l_{\langle u_n\rangle}\perp\mu$ for any $n\in\mathbb{N}$, then $\mu^l_{\langle u\rangle}\perp\mu$.*

Proof This is a special case of [26, Lemma 3.7-(b)], whose proof works for any regular symmetric Dirichlet space. □

To prove that $\mu^l_{\langle h\rangle}\perp m_l$ for any $h\in\bigcup_{n=0}^{\infty}\mathcal{H}_{l,n}$, noting that $\mathcal{H}_{l,0}\subset\mathcal{H}_{l,1}$ and recalling Proposition 2.15, in the following lemma we calculate explicitly the matrix representation of the linear maps $\mathbb{R}^{V_0}\ni u\mapsto h^l_0(u)\circ F^l_i|_{V_0}\in\mathbb{R}^{V_0}$, $i\in W^l_1=S_{l_1}$, which we identify with the linear maps $\mathcal{H}_{l,0}\ni h\mapsto h\circ F^l_i\in\mathcal{H}_{l^1,0}$.

Lemma 4.6 *Set $l:=l_1$, $a_l:=\frac{1}{9}r_l=(6l+1)^{-1}$, and for each $i\in S_l$ let A^l_i denote the matrix representation of the linear map $\mathbb{R}^{V_0}\ni u\mapsto h^l_0(u)\circ F^l_i|_{V_0}\in\mathbb{R}^{V_0}$ with respect to the basis $(\mathbb{1}_{q_0},\mathbb{1}_{q_1},\mathbb{1}_{q_2})$ of $\mathbb{R}^{V_0}$. Then for any $k\in\{2,\dots,l-3\}$,*

$$
\begin{aligned}
A^l_{(k,0)} &= \begin{pmatrix} 1-(6k+3)a_l & (6k-2)a_l & 5a_l\\ 1-(6k+9)a_l & (6k+4)a_l & 5a_l\\ 1-(6k+6)a_l & (6k+1)a_l & 5a_l \end{pmatrix},\\
A^l_{(0,k)} &= \begin{pmatrix} 1-(6k+3)a_l & 5a_l & (6k-2)a_l\\ 1-(6k+6)a_l & 5a_l & (6k+1)a_l\\ 1-(6k+9)a_l & 5a_l & (6k+4)a_l \end{pmatrix},\\
A^l_{(l-1-k,k)} &= \begin{pmatrix} 5a_l & 1-(6k+6)a_l & (6k+1)a_l\\ 5a_l & 1-(6k+3)a_l & (6k-2)a_l\\ 5a_l & 1-(6k+9)a_l & (6k+4)a_l \end{pmatrix}.
\end{aligned}
\tag{4.4}
$$

Proof This follows by solving the linear equation in $(h(x))_{x\in V^l_1\setminus V_0}$ for $h=h^l_0(u)$ from Proposition 2.15-(2) with $(n,k)=(0,1)$ under $h(q)=u(q)$ for $q\in V_0$. □

Our proof that $\mu^{l}_{\langle h\rangle} \perp m_l$ for $h \in \bigcup_{n=0}^{\infty} \mathcal{H}_{l,n}$ is based on the following fact.

Theorem 4.7 ([23, Theorem 4.1]) *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space and let* $\{\mathscr{F}_n\}_{n=0}^{\infty}$ *be a non-decreasing sequence of* σ*-algebras in* Ω *such that* $\bigcup_{n=0}^{\infty} \mathscr{F}_n$ *generates* $\mathscr{F}$. *Let* $\widetilde{\mathbb{P}}$ *be a probability measure on* $(\Omega, \mathscr{F})$ *such that* $\widetilde{\mathbb{P}}|_{\mathscr{F}_n} \ll \mathbb{P}|_{\mathscr{F}_n}$ *for any* $n \in \mathbb{N} \cup \{0\}$, *and for each* $n \in \mathbb{N}$ *define* $\alpha_n \in L^1(\Omega, \mathscr{F}_n, \mathbb{P}|_{\mathscr{F}_n})$ *by*

$$\alpha_n := \begin{cases} \dfrac{d(\widetilde{\mathbb{P}}|_{\mathscr{F}_n})/d(\mathbb{P}|_{\mathscr{F}_n})}{d(\widetilde{\mathbb{P}}|_{\mathscr{F}_{n-1}})/d(\mathbb{P}|_{\mathscr{F}_{n-1}})} & \text{on } \{d(\widetilde{\mathbb{P}}|_{\mathscr{F}_{n-1}})/d(\mathbb{P}|_{\mathscr{F}_{n-1}}) > 0\}, \\ 0 & \text{on } \{d(\widetilde{\mathbb{P}}|_{\mathscr{F}_{n-1}})/d(\mathbb{P}|_{\mathscr{F}_{n-1}}) = 0\}, \end{cases} \tag{4.5}$$

so that $\mathbb{E}[\sqrt{\alpha_n} \mid \mathscr{F}_{n-1}] \le 1$ $\mathbb{P}|_{\mathscr{F}_{n-1}}$*-a.s. by conditional Jensen's inequality, where* $\mathbb{E}[\cdot \mid \mathscr{F}_{n-1}]$ *denotes the conditional expectation given* $\mathscr{F}_{n-1}$ *with respect to* $\mathbb{P}$. *If*

$$\sum_{n=1}^{\infty} (1 - \mathbb{E}[\sqrt{\alpha_n} \mid \mathscr{F}_{n-1}]) = \infty \qquad \mathbb{P}\text{-a.s.}, \tag{4.6}$$

then $\widetilde{\mathbb{P}} \perp \mathbb{P}$.

We will apply Theorem 4.7 under the setting of the following lemma with $\mathbb{P} = m_l$.

Lemma 4.8 *Set* $\Omega := K^l$, $\mathscr{F} := \mathscr{B}(K^l)$ *and let* $\mathbb{P}, \widetilde{\mathbb{P}}$ *be probability measures on* $(\Omega, \mathscr{F})$ *such that* $\mathbb{P}(K^l_w) > 0$ *for any* $w \in W^l_*$ *and* $\mathbb{P}(V^l_*) = \widetilde{\mathbb{P}}(V^l_*) = 0$. *Set* $\mathscr{F}_n := \{A \cup \bigcup_{w\in\Lambda}(K^l_w \setminus V^l_n) \mid \Lambda \subset W^l_n, A \subset V^l_n\}$ *for each* $n \in \mathbb{N} \cup \{0\}$, *so that* $\{\mathscr{F}_n\}_{n=0}^{\infty}$ *is a non-decreasing sequence of* σ*-algebras in* Ω *by* (2.3), $\bigcup_{n=0}^{\infty} \mathscr{F}_n$ *generates* $\mathscr{F}$, *and* $\widetilde{\mathbb{P}}|_{\mathscr{F}_n} \ll \mathbb{P}|_{\mathscr{F}_n}$ *for any* $n \in \mathbb{N} \cup \{0\}$. *Let* $n \in \mathbb{N}$ *and define* $\alpha_n \in L^1(\Omega, \mathscr{F}_n, \mathbb{P}|_{\mathscr{F}_n})$ *by* (4.5). *Then for each* $w \in W^l_{n-1}$,

$$\mathbb{E}[\sqrt{\alpha_n} \mid \mathscr{F}_{n-1}]|_{K^l_w \setminus V^l_{n-1}} = \begin{cases} \displaystyle\sum_{i\in S_{l_n}} \sqrt{\frac{\widetilde{\mathbb{P}}(K^l_{wi})}{\widetilde{\mathbb{P}}(K^l_w)}} \sqrt{\frac{\mathbb{P}(K^l_{wi})}{\mathbb{P}(K^l_w)}} & \text{if } \widetilde{\mathbb{P}}(K^l_w) > 0, \\ 0 & \text{if } \widetilde{\mathbb{P}}(K^l_w) = 0. \end{cases} \tag{4.7}$$

Proof This follows easily by direct calculations based on (4.5) and (2.3). □

The following proposition is the key step of the proof of Theorem 4.3.

Proposition 4.9 *Let* $k \in \mathbb{N} \cup \{0\}$, $h \in \mathcal{H}_{l,k}$, $x \in K^l \setminus V^l_*$, *and let* $\omega^x = (\omega^x_n)_{n=1}^{\infty}$ *be the element of* $\prod_{n=1}^{\infty} S_{l_n}$, *unique by* (2.3), *such that* $\{x\} = \bigcap_{n=1}^{\infty} K^l_{\omega^x_1\ldots\omega^x_n}$. *Let* $n \in \mathbb{N} \cap [k+2, \infty)$ *and assume that* $\mu^l_{\langle h\rangle}\bigl(K^l_{\omega^x_1\ldots\omega^x_{n-1}}\bigr) > 0$ *and that* $\omega^x_{n-1} \in S_{l_{n-1},1}$, *where* $S_{l,1} := \{(i_1, i_2) \in S_l \mid i_1 \vee i_2 \in \{2, \ldots, l-3\}\}$ *for* $l \in \mathbb{N} \setminus \{1, 2, 3, 4\}$. *Then*

$$\sum_{i\in S_{l_n}} \sqrt{\frac{\mu^l_{\langle h\rangle}\bigl(K^l_{\omega^x_1\ldots\omega^x_{n-1}i}\bigr)}{\mu^l_{\langle h\rangle}\bigl(K^l_{\omega^x_1\ldots\omega^x_{n-1}}\bigr)}} \sqrt{\frac{m_l\bigl(K^l_{\omega^x_1\ldots\omega^x_{n-1}i}\bigr)}{m_l\bigl(K^l_{\omega^x_1\ldots\omega^x_{n-1}}\bigr)}} \le \sqrt{\frac{361}{372}}. \tag{4.8}$$

Proof Set $v := \omega_1^x \dots \omega_{n-2}^x$ ($v := \emptyset$ if $n = 2$) and $w := \omega_1^x \dots \omega_{n-1}^x = v\omega_{n-1}^x$. By $h \in \mathcal{H}_{\boldsymbol{l},k} \subset \mathcal{H}_{\boldsymbol{l},n-2}$, Proposition 2.15 and (4.3) we have $h \circ F_v^{\boldsymbol{l}} \in \mathcal{H}_{\boldsymbol{l}^{n-2},0}$, $h \circ F_w^{\boldsymbol{l}} = (h \circ F_v^{\boldsymbol{l}}) \circ F_{\omega_{n-1}^x}^{\boldsymbol{l}^{n-2}} \in \mathcal{H}_{\boldsymbol{l}^{n-1},0}$ and $\mathcal{E}^0(h \circ F_w^{\boldsymbol{l}}|_{V_0}, h \circ F_w^{\boldsymbol{l}}|_{V_0}) = R_{n-1}^{\boldsymbol{l}} \mu_{\langle h \rangle}^{\boldsymbol{l}}(K_w^{\boldsymbol{l}}) > 0$, and therefore $h \circ F_w^{\boldsymbol{l}}(V_0) = \{c - b, c, c + b\}$ for some $b, c \in \mathbb{R}$ with $b > 0$ by Lemma 4.6 with $\boldsymbol{l}^{n-2}$ in place of $\boldsymbol{l}$ applied to $i = \omega_{n-1}^x \in S_{l_{n-1},1}$ and $u = h \circ F_v^{\boldsymbol{l}}|_{V_0}$. Then we see from Lemma 4.6 with $\boldsymbol{l}^{n-1}$ in place of $\boldsymbol{l}$ applied to $i \in S_{l_n,1}$ and $u = h \circ F_w^{\boldsymbol{l}}|_{V_0}$ that $h \circ F_{wi}^{\boldsymbol{l}}(V_0) = (h \circ F_w^{\boldsymbol{l}}) \circ F_i^{\boldsymbol{l}^{n-1}}(V_0)$ is equal to $\{c_i - 3a_{l_n}b, c_i, c_i + 3a_{l_n}b\}$ for some $c_i \in \mathbb{R}$ for $2(l_n - 4)$ elements i of $S_{l_n,1}$ and to $\{c_i - 6a_{l_n}b, c_i, c_i + 6a_{l_n}b\}$ for some $c_i \in \mathbb{R}$ for the other $l_n - 4$ elements i of $S_{l_n,1}$. It follows by combining this fact with (2.8), (4.3), $h \in \mathcal{H}_{\boldsymbol{l},n-1} \subset \mathcal{H}_{\boldsymbol{l},n}$, Proposition 2.15, (2.9) and (2.17) that

$$
\begin{aligned}
&\sum_{i \in S_{l_n}} \sqrt{\frac{\mu_{\langle h \rangle}^{\boldsymbol{l}}(K_{wi}^{\boldsymbol{l}})}{\mu_{\langle h \rangle}^{\boldsymbol{l}}(K_w^{\boldsymbol{l}})}} \sqrt{\frac{m_{\boldsymbol{l}}(K_{wi}^{\boldsymbol{l}})}{m_{\boldsymbol{l}}(K_w^{\boldsymbol{l}})}} = \sum_{i \in S_{l_n}} \sqrt{\frac{\mathcal{E}^{\boldsymbol{l}^n}(h \circ F_{wi}^{\boldsymbol{l}}, h \circ F_{wi}^{\boldsymbol{l}})}{(r_{l_n} \# S_{l_n}) \mathcal{E}^{\boldsymbol{l}^{n-1}}(h \circ F_w^{\boldsymbol{l}}, h \circ F_w^{\boldsymbol{l}})}} \\
&\le (l_n - 4) \frac{2 \cdot 3\sqrt{6} a_{l_n} b + 6\sqrt{6} a_{l_n} b}{\sqrt{r_{l_n} \# S_{l_n}} \cdot \sqrt{6} b} + 3\sqrt{\frac{\sum_{i \in S_{l_n} \setminus S_{l_n,1}} \mathcal{E}^{\boldsymbol{l}^n}(h \circ F_{wi}^{\boldsymbol{l}}, h \circ F_{wi}^{\boldsymbol{l}})}{(r_{l_n} \# S_{l_n}) \cdot 6b^2}} \\
&= \frac{4(l_n - 4)}{\sqrt{\# S_{l_n}/a_{l_n}}} + 3\sqrt{\frac{r_{l_n} \mathcal{E}^{\boldsymbol{l}^{n-1}}(h \circ F_w^{\boldsymbol{l}}, h \circ F_w^{\boldsymbol{l}}) - \sum_{i \in S_{l_n,1}} \mathcal{E}^{\boldsymbol{l}^n}(h \circ F_{wi}^{\boldsymbol{l}}, h \circ F_{wi}^{\boldsymbol{l}})}{6b^2 r_{l_n} \# S_{l_n}}} \\
&= \frac{4(l_n - 4)}{\sqrt{\# S_{l_n}/a_{l_n}}} + 3\sqrt{\frac{6b^2 r_{l_n} - (l_n - 4)(2 \cdot 54 a_{l_n}^2 b^2 + 216 a_{l_n}^2 b^2)}{6b^2 r_{l_n} \# S_{l_n}}} \\
&= \frac{4(l_n - 4)}{\sqrt{\# S_{l_n}/a_{l_n}}} + 3\sqrt{\frac{a_{l_n}^{-1} - 6(l_n - 4)}{\# S_{l_n}/a_{l_n}}} = \frac{4l_n - 1}{\sqrt{(3l_n - 3)(6l_n + 1)}} \le \sqrt{\frac{361}{372}},
\end{aligned}
$$

proving (4.8). □

Proof of Theorem 4.3 Let $k \in \mathbb{N} \cup \{0\}$ and $h \in \mathcal{H}_{\boldsymbol{l},k}$. In view of Lemmas 4.4 and 4.5 it suffices to prove, for any such k and h, that $\mu_{\langle h \rangle}^{\boldsymbol{l}} \perp m_{\boldsymbol{l}}$, which is obvious if $\mu_{\langle h \rangle}^{\boldsymbol{l}}(K^{\boldsymbol{l}}) = 0$. Assume that $\mu_{\langle h \rangle}^{\boldsymbol{l}}(K^{\boldsymbol{l}}) > 0$, set $(\Omega, \mathscr{F}, \mathbb{P}) := (K^{\boldsymbol{l}}, \mathscr{B}(K^{\boldsymbol{l}}), m_{\boldsymbol{l}})$, let $\{\mathscr{F}_n\}_{n=0}^\infty$ denote the non-decreasing sequence of σ-algebras in Ω with $\bigcup_{n=0}^\infty \mathscr{F}_n$ generating $\mathscr{F}$ as defined in Lemma 4.8, and set $\widetilde{\mathbb{P}} := \mu_{\langle h \rangle}^{\boldsymbol{l}}(K^{\boldsymbol{l}})^{-1} \mu_{\langle h \rangle}^{\boldsymbol{l}}$, so that $\mathbb{P}(K_w^{\boldsymbol{l}}) > 0$ for any $w \in W_*^{\boldsymbol{l}}$ and $\mathbb{P}(V_*^{\boldsymbol{l}}) = \widetilde{\mathbb{P}}(V_*^{\boldsymbol{l}}) = 0$ by (2.8) and Proposition 4.2. In particular, $\widetilde{\mathbb{P}}|_{\mathscr{F}_n} \ll \mathbb{P}|_{\mathscr{F}_n}$ for any $n \in \mathbb{N} \cup \{0\}$, and define $\alpha_n \in L^1(\Omega, \mathscr{F}_n, \mathbb{P}|_{\mathscr{F}_n})$ by (4.5) for each $n \in \mathbb{N}$. Now let $\omega^x = (\omega_n^x)_{n=1}^\infty \in \prod_{n=1}^\infty S_{l_n}$ for $x \in K^{\boldsymbol{l}} \setminus V_*^{\boldsymbol{l}}$ and $S_{l,1} \subset S_l$ for $l \in \mathbb{N} \setminus \{1, 2, 3, 4\}$ be as in Proposition 4.9. Then by (2.8), the $\mathbb{P}$-a.s. defined Borel measurable maps $K^{\boldsymbol{l}} \setminus V_*^{\boldsymbol{l}} \ni x \mapsto \omega_n^x \in S_{l_n}$, $n \in \mathbb{N}$, form a sequence of independent random variables on $(\Omega, \mathscr{F}, \mathbb{P})$ and satisfy

$$
\sum_{n=1}^\infty \mathbb{P}\bigl(\{x \in K^{\boldsymbol{l}} \setminus V_*^{\boldsymbol{l}} \mid \omega_n^x \in S_{l_n,1}\}\bigr) = \sum_{n=1}^\infty \frac{\# S_{l_n,1}}{\# S_{l_n}} = \sum_{n=1}^\infty \frac{3l_n - 12}{3l_n - 3} \ge \sum_{n=1}^\infty \frac{1}{4} = \infty,
$$

and hence the second Borel–Cantelli lemma implies that

$$\#\{n \in \mathbb{N} \mid \omega^x_n \in S_{l_n,1}\} = \infty \quad \text{for } \mathbb{P}\text{-a.s., } x \in K^{\boldsymbol{l}} \setminus V^{\boldsymbol{l}}_*. \tag{4.9}$$

On the other hand, for each $x \in K^{\boldsymbol{l}} \setminus V^{\boldsymbol{l}}_*$, Lemma 4.8 and Proposition 4.9 imply that $\mathbb{E}[\sqrt{\alpha_n} \mid \mathscr{F}_{n-1}](x) = 0$ for any $n \in \mathbb{N}$ with $\mu^{\boldsymbol{l}}_{\langle h \rangle}(K^{\boldsymbol{l}}_{\omega^x_1 \ldots \omega^x_{n-1}}) = 0$ and that

$$\mathbb{E}[\sqrt{\alpha_n} \mid \mathscr{F}_{n-1}](x) = \sum_{i \in S_{l_n}} \sqrt{\frac{\mu^{\boldsymbol{l}}_{\langle h \rangle}(K^{\boldsymbol{l}}_{\omega^x_1 \ldots \omega^x_{n-1} i})}{\mu^{\boldsymbol{l}}_{\langle h \rangle}(K^{\boldsymbol{l}}_{\omega^x_1 \ldots \omega^x_{n-1}})}} \sqrt{\frac{m_{\boldsymbol{l}}(K^{\boldsymbol{l}}_{\omega^x_1 \ldots \omega^x_{n-1} i})}{m_{\boldsymbol{l}}(K^{\boldsymbol{l}}_{\omega^x_1 \ldots \omega^x_{n-1}})}} \leq \sqrt{\frac{361}{372}}$$

for any $n \in \mathbb{N} \cap [k+2, \infty)$ with $\mu^{\boldsymbol{l}}_{\langle h \rangle}(K^{\boldsymbol{l}}_{\omega^x_1 \ldots \omega^x_{n-1}}) > 0$ and $\omega^x_{n-1} \in S_{l_{n-1},1}$, whence

$$\sum_{n=1}^{\infty} \bigl(1 - \mathbb{E}[\sqrt{\alpha_n} \mid \mathscr{F}_{n-1}](x)\bigr) \geq \delta \#\{n \in \mathbb{N} \cap [k+2, \infty) \mid \omega^x_{n-1} \in S_{l_{n-1},1}\}, \tag{4.10}$$

where $\delta := 1 - \sqrt{\frac{361}{372}} \in (0, 1)$. Combining (4.9) and (4.10), we obtain (4.6), so that Theorem 4.7 is applicable and yields $\widetilde{\mathbb{P}} \perp \mathbb{P}$, namely $\mu^{\boldsymbol{l}}_{\langle h \rangle} \perp m_{\boldsymbol{l}}$. □

5 Realizing Arbitrarily Slow Decay Rates of $\Psi(r)/r^2$

In this last section, we show that an arbitrarily slow decay rate of $\Psi(r)/r^2$ for a homeomorphism $\Psi : [0, \infty) \to [0, \infty)$ satisfying (1.2) and (1.6) can be realized by $\Psi_{\boldsymbol{l}}$ (recall Definition 3.1) for some $\boldsymbol{l} = (l_n)_{n=1}^{\infty} \in (\mathbb{N} \setminus \{1, 2, 3, 4\})^{\mathbb{N}}$. We achieve this by providing in Theorem 5.1 a simple sufficient condition for Ψ to be comparable to $\Psi_{\boldsymbol{l}}$ for some $\boldsymbol{l} = (l_n)_{n=1}^{\infty} \in (\mathbb{N} \setminus \{1, 2, 3, 4\})^{\mathbb{N}}$ with $\sum_{n=1}^{\infty} l_n^{-1} < \infty$ and proving in Proposition 5.2 that the decay rate of $\Psi(r)/r^2$ for such Ψ can be arbitrarily slow. We also give criteria for verifying this sufficient condition for concrete examples of Ψ in Proposition 5.3 and apply them to the case where $\Psi(r)/r^2$ is a multiple composition of the function $r \mapsto 1/\log(e - 1 + (r \wedge 1)^{-1})$ in Example 5.4.

Theorem 5.1 *Let $\eta : [0, 1] \to [0, 1]$ be a homeomorphism with $\eta(0) = 0$ such that*

$$\sum_{n=1}^{\infty} \frac{\eta^{-1}(2^{-n})}{\eta^{-1}(2^{1-n})} < \infty, \tag{5.1}$$

and define a homeomorphism $\Psi_\eta : [0, \infty) \to [0, \infty)$ by $\Psi_\eta(r) := r^2 \eta(r \wedge 1)$. Then there exists $\boldsymbol{l} = (l_n)_{n=1}^{\infty} \in (\mathbb{N} \setminus \{1, 2, 3, 4\})^{\mathbb{N}}$ with $\sum_{n=1}^{\infty} l_n^{-1} < \infty$ such that $\Psi_\eta(r)/\Psi_{\boldsymbol{l}}(r) \in [c^{-1}, c]$ for any $r \in (0, \infty)$ for some $c \in [1, \infty)$, and consequently, $(K^{\boldsymbol{l}}, d_{\boldsymbol{l}}, m_{\boldsymbol{l}}, \mathcal{E}^{\boldsymbol{l}}, \mathcal{F}_{\boldsymbol{l}})$ satisfies fHKE(Ψ_η).

Proof Set $c_\eta := \inf_{n\in\mathbb{N}} \eta^{-1}(2^{1-n})/\eta^{-1}(2^{-n})$, so that $c_\eta \in (1,\infty)$ since the sequence $\{\eta^{-1}(2^{1-n})/\eta^{-1}(2^{-n})\}_{n=1}^{\infty}$ is $(1,\infty)$-valued and tends to ∞ by (5.1). Then for any $r, R \in (0,1]$ with $r \le R$, taking $j, k \in \mathbb{N}$ such that $\eta(r) \in (2^{-k}, 2^{1-k}]$ and $\eta(R) \in (2^{-j}, 2^{1-j}]$, we have $j \le k$, hence

$$\frac{R}{r} = \frac{\eta^{-1}(\eta(R))}{\eta^{-1}(\eta(r))} \ge \frac{\eta^{-1}(2^{-j})}{\eta^{-1}(2^{1-k})} \vee 1 \ge c_\eta^{k-j-1} = 2^{(k-j-1)/\beta_\eta} \ge \left(\frac{\eta(R)}{4\eta(r)}\right)^{1/\beta_\eta}$$

by the definition of c_η, where $\beta_\eta := (\log_2 c_\eta)^{-1} \in (0,\infty)$, and therefore

$$\frac{\eta(R)}{\eta(r)} \le 4\left(\frac{R}{r}\right)^{\beta_\eta}. \tag{5.2}$$

Recalling that $\lim_{n\to\infty} \eta^{-1}(2^{1-n})/\eta^{-1}(2^{-n}) = \infty$ by (5.1), choose $n_0 \in \mathbb{N}$ so that $\eta^{-1}(2^{1-n})/\eta^{-1}(2^{-n}) \ge 5$ for any $n \in \mathbb{N}$ with $n \ge n_0$, set $l_0 := 1$, and define $\boldsymbol{l} = (l_n)_{n=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ inductively by

$$l_n := \left\lfloor \frac{\eta^{-1}(2^{-n_0})}{(l_0 \cdots l_{n-1})\eta^{-1}(2^{-n-n_0})} \right\rfloor, \qquad n \in \mathbb{N}. \tag{5.3}$$

Then an induction on n based on (5.3) and the choice of n_0 immediately shows that $\boldsymbol{l} = (l_n)_{n=1}^{\infty} \in (\mathbb{N} \setminus \{1,2,3,4\})^{\mathbb{N}}$ and that for any $n \in \mathbb{N} \cup \{0\}$,

$$\frac{\eta^{-1}(2^{-n-n_0})}{\eta^{-1}(2^{-n_0})} \le \frac{1}{L_n^{\boldsymbol{l}}} \le \frac{6}{5}\frac{\eta^{-1}(2^{-n-n_0})}{\eta^{-1}(2^{-n_0})}, \tag{5.4}$$

which together with (5.1) implies in particular that

$$\sum_{n=1}^{\infty} l_n^{-1} = \sum_{n=1}^{\infty} \frac{1/L_n^{\boldsymbol{l}}}{1/L_{n-1}^{\boldsymbol{l}}} \le \frac{6}{5}\sum_{n=1}^{\infty} \frac{\eta^{-1}(2^{-n-n_0})}{\eta^{-1}(2^{1-n-n_0})} < \infty. \tag{5.5}$$

We claim that $\Psi_\eta(r)/\Psi_{\boldsymbol{l}}(r) \in [c^{-1}, c]$ for any $r \in (0,\infty)$ for some $c \in [1,\infty)$. Indeed, recalling Definition 3.1, we have $\beta_{\boldsymbol{l},0} = \inf_{n\in\mathbb{N}} \beta_{l_n} = 2$ by (5.5), hence $\Psi_\eta(r)/\Psi_{\boldsymbol{l}}(r) = r^2/r^{\beta_{\boldsymbol{l},0}} = 1$ for any $r \in [1,\infty)$, and also see for any $n \in \mathbb{N}$ that

$$2^{-n-n_0} \le \eta\left(\frac{\eta^{-1}(2^{-n-n_0})}{\eta^{-1}(2^{-n_0})}\right) \le \eta(1/L_n^{\boldsymbol{l}}) \le \eta\left(\frac{6}{5}\frac{\eta^{-1}(2^{-n-n_0})}{\eta^{-1}(2^{-n_0})}\right) \le c2^{-n} \tag{5.6}$$

by (5.4) and (5.2), where $c := 2^{2-n_0}\left(\frac{6}{5}/\eta^{-1}(2^{-n_0})\right)^{\beta_\eta}$, and thus that

$$\frac{\Psi_\eta(1/L_n^{\boldsymbol{l}})}{\Psi_{\boldsymbol{l}}(1/L_n^{\boldsymbol{l}})} = \frac{T_n^{\boldsymbol{l}}\eta(1/L_n^{\boldsymbol{l}})}{(L_n^{\boldsymbol{l}})^2} = 2^n\eta(1/L_n^{\boldsymbol{l}})\prod_{k=1}^{n}\left(1 - \frac{5}{6}l_k^{-1} - \frac{1}{6}l_k^{-2}\right) \in [c', c], \tag{5.7}$$

where $c' := 2^{-n_0}\prod_{k=1}^{\infty}(1-\frac{5}{6}l_k^{-1}-\frac{1}{6}l_k^{-2}) \in (0,1)$ by (5.5). Now for any $n \in \mathbb{N}$ and any $s \in [1, l_n]$, by (3.1) we have

$$\frac{\Psi_{\boldsymbol{l}}(s/L_n^{\boldsymbol{l}})}{s^2\Psi_{\boldsymbol{l}}(1/L_n^{\boldsymbol{l}})} = \frac{1}{s^2}\left(1+\frac{3l_n-4}{l_n-1}(s-1)\right)\left(1+\frac{\frac{2}{3}l_n-\frac{8}{9}}{l_n-1}(s-1)\right) \in [1,2), \quad (5.8)$$

and it follows from $\eta(s/L_n^{\boldsymbol{l}}) \in \left[\eta(1/L_n^{\boldsymbol{l}}), \eta(1/L_{n-1}^{\boldsymbol{l}})\right]$, (5.6), (5.8) and (5.7) that

$$\begin{aligned}\frac{\Psi_\eta(s/L_n^{\boldsymbol{l}})}{\Psi_{\boldsymbol{l}}(s/L_n^{\boldsymbol{l}})} &= \frac{\Psi_\eta(s/L_n^{\boldsymbol{l}})}{\Psi_\eta(1/L_n^{\boldsymbol{l}})}\frac{\Psi_{\boldsymbol{l}}(1/L_n^{\boldsymbol{l}})}{\Psi_{\boldsymbol{l}}(s/L_n^{\boldsymbol{l}})}\frac{\Psi_\eta(1/L_n^{\boldsymbol{l}})}{\Psi_{\boldsymbol{l}}(1/L_n^{\boldsymbol{l}})}\\ &= \frac{\eta(s/L_n^{\boldsymbol{l}})}{\eta(1/L_n^{\boldsymbol{l}})}\frac{s^2\Psi_{\boldsymbol{l}}(1/L_n^{\boldsymbol{l}})}{\Psi_{\boldsymbol{l}}(s/L_n^{\boldsymbol{l}})}\frac{\Psi_\eta(1/L_n^{\boldsymbol{l}})}{\Psi_{\boldsymbol{l}}(1/L_n^{\boldsymbol{l}})} \in [c'/2, (c^2\vee c)2^{n_0+1}],\end{aligned}$$

proving that $\Psi_\eta(r)/\Psi_{\boldsymbol{l}}(r) \in [c'/2, (c^2\vee c)2^{n_0+1}]$ for any $r \in (0,\infty)$. Lastly, combining this result with Lemma 3.2, Theorem 3.3 and Remark 1.2-(2) shows that $(K^{\boldsymbol{l}}, d_{\boldsymbol{l}}, m_{\boldsymbol{l}}, \mathcal{E}^{\boldsymbol{l}}, \mathcal{F}_{\boldsymbol{l}})$ satisfies fHKE(Ψ_η). □

The decay rate of $\Psi_\eta(r)/r^2 = \eta(r\wedge 1)$ for η as in Theorem 5.1 can be arbitrarily slow in the sense stated in the following proposition.

Proposition 5.2 *Let $\Psi : [0,\infty) \to [0,\infty)$ be a homeomorphism satisfying* (1.6). *Then there exists a homeomorphism $\eta : [0,1] \to [0,1]$ with the properties $\eta(0) = 0$ and* (5.1) *such that $\eta(r) \geq c\Psi(r)/r^2$ for any $r \in (0,1]$ for some $c \in (0,\infty)$.*

Proof Noting (1.6), define $\eta_0 : [0,1] \to [0,\infty)$ by $\eta_0(r) := \sup_{s\in(0,r]}\Psi(s)/s^2$ ($\eta_0(0) := 0$), so that η_0 is continuous and non-decreasing and $\eta_0((0,1]) \subset (0,\infty)$, and set $s_n := \max \eta_0^{-1}(2^{-n}\eta_0(1))$ for $n \in \mathbb{N}\cup\{0\}$, so that $s_0 = 1$, $0 < s_n < s_{n-1}$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} s_n = 0$. Define a homeomorphism $\eta : [0,1] \to [0,1]$ by

$$\eta(r) := \left(1+\frac{r-2^{-n^2}s_n}{2^{-(n-1)^2}s_{n-1}-2^{-n^2}s_n}\right)2^{-n} \quad (5.9)$$

for $n \in \mathbb{N}$ and $r \in [2^{-n^2}s_n, 2^{-(n-1)^2}s_{n-1}]$ and $\eta(0) := 0$. Then since $\eta^{-1}(2^{1-n}) = 2^{-(n-1)^2}s_{n-1}$ and $0 < s_n < s_{n-1}$ for any $n \in \mathbb{N}$,

$$\sum_{n=1}^{\infty}\frac{\eta^{-1}(2^{-n})}{\eta^{-1}(2^{1-n})} = \sum_{n=1}^{\infty}\frac{2^{-n^2}s_n}{2^{-(n-1)^2}s_{n-1}} \leq \sum_{n=1}^{\infty}2^{1-2n} = \frac{2}{3} < \infty,$$

namely η satisfies (5.1), and for any $n \in \mathbb{N}$ and any $r \in [s_n, s_{n-1}]$ we have

$$\eta(r) \geq \eta(s_n) \geq \eta(2^{-n^2}s_n) = 2^{-n} = \frac{\eta_0(s_{n-1})}{2\eta_0(1)} \geq \frac{\eta_0(r)}{2\eta_0(1)} \geq \frac{\Psi(r)/r^2}{2\eta_0(1)},$$

i.e., $\eta(r) \geq c\Psi(r)/r^2$ with $c := (2\eta_0(1))^{-1} \in (0,\infty)$ for any $r \in (0,1]$. □

We conclude this paper with the following proposition, which gives criteria for verifying (5.1) for concrete homeomorphisms $\eta : [0,1] \to [0,1]$ with $\eta(0) = 0$, and some applications of it to $\eta(r) = 1/\log(e - 1 + r^{-1})$ in Example 5.4 below.

Proposition 5.3 *Let* $\eta : [0,1] \to [0,1]$ *be a homeomorphism with* $\eta(0) = 0$, *let* $\delta \in [0,\infty)$, $\alpha, \beta \in (0,\infty)$ *and assume that there exists* $c \in (0,\infty)$ *such that*

$$\frac{\eta(R)}{\eta(r)} \le 1 + \delta + \frac{c(R/r)^{\beta}}{(\log(e-1+R^{-1}))^{\alpha}} \quad \text{for any } r, R \in (0,1] \text{ with } r \le R. \tag{5.10}$$

(1) *If* $\delta < 1$ *and* $\beta < \alpha$, *then* η *satisfies* (5.1).
(2) *Let* $\widetilde{\eta} : [0,1] \to [0,1]$ *be a homeomorphism with* $\widetilde{\eta}(0) = 0$, *let* $\widetilde{\delta} \in [0,1)$ *and assume that there exist* $\widetilde{\alpha}, \widetilde{c} \in (0,\infty)$ *such that* $\widetilde{\eta}$ *satisfies* (5.10) *with* $\widetilde{\delta}, \widetilde{\alpha}, 1, \widetilde{c}$ *in place of* δ, α, β, c. *Then* $\widetilde{\eta} \circ \eta$ *satisfies* (5.10) *with* $\frac{1}{2}(1+\widetilde{\delta}), c'$ *in place of* δ, c *for some* $c' \in (0,\infty)$. *In particular, if* $\beta < \alpha$, *then* $\widetilde{\eta} \circ \eta$ *satisfies* (5.1).

Proof (1) Set $s_n := \eta^{-1}(2^{-n})$ for $n \in \mathbb{N} \cup \{0\}$. For each $n \in \mathbb{N}$, we see from $\eta(s_{n-1})/\eta(s_n) = 2$, (5.10) with $(r,R) = (s_n, s_{n-1})$ and $\delta < 1$ that

$$\frac{s_n}{s_{n-1}} \le \frac{c^{1/\beta}(1-\delta)^{-1/\beta}}{(\log(e-1+s_{n-1}^{-1}))^{\alpha/\beta}}, \tag{5.11}$$

and from $\eta(1)/\eta(s_{n-1}) = 2^{n-1}$, (5.10) with $(r,R) = (s_{n-1}, 1)$ and $\delta < 1$ that $2^{n-1} \le 1 + \delta + c s_{n-1}^{-\beta} \le (2+c) s_{n-1}^{-\beta}$, whence, provided $n \ge 2 + 2\log_2(2+c)$,

$$\log(e-1+s_{n-1}^{-1}) \ge \log(s_{n-1}^{-1}) \ge \frac{\log 2}{\beta}(n - 1 - \log_2(2+c)) \ge \frac{\log 2}{2\beta} n. \tag{5.12}$$

It follows from (5.11), (5.12) and $\alpha/\beta > 1$ that

$$\sum_{n=1}^{\infty} \frac{\eta^{-1}(2^{-n})}{\eta^{-1}(2^{1-n})} = \sum_{n=1}^{\infty} \frac{s_n}{s_{n-1}} \le \sum_{n=1}^{n_c - 1} \frac{s_n}{s_{n-1}} + \sum_{n=n_c}^{\infty} \frac{c^{1/\beta}(2\beta/\log 2)^{\alpha/\beta}}{(1-\delta)^{1/\beta} n^{\alpha/\beta}} < \infty,$$

where $n_c := 3 + \lfloor 2\log_2(2+c) \rfloor$, proving (5.1).
(2) Set $\widetilde{r} := \eta^{-1}\bigl(\exp\bigl(-(2\widetilde{c}(1+\delta)/(1-\widetilde{\delta}))^{1/\widetilde{\alpha}}\bigr)\bigr)$ and let $r, R \in (0,1]$ satisfy $r \le R$. By (5.10) for $\widetilde{\eta}$ and η and $(\log(e-1+\eta(R)^{-1}))^{-\widetilde{\alpha}} \le 1$ we have

$$\begin{aligned}\frac{\widetilde{\eta} \circ \eta(R)}{\widetilde{\eta} \circ \eta(r)} &\le 1 + \widetilde{\delta} + \frac{\widetilde{c}}{(\log(e-1+\eta(R)^{-1}))^{\widetilde{\alpha}}} \frac{\eta(R)}{\eta(r)} \\ &\le 1 + \widetilde{\delta} + \frac{\widetilde{c}(1+\delta)}{(\log(e-1+\eta(R)^{-1}))^{\widetilde{\alpha}}} + \frac{\widetilde{c}c(R/r)^{\beta}}{(\log(e-1+R^{-1}))^{\alpha}}.\end{aligned} \tag{5.13}$$

If $R \le \widetilde{r}$, then $\widetilde{c}(1+\delta)/(\log(e-1+\eta(R)^{-1}))^{\widetilde{\alpha}} \le \frac{1}{2}(1-\widetilde{\delta})$ by the definition of $\widetilde{r}$ and hence (5.13) yields (5.10) with $\widetilde{\eta} \circ \eta$, $\frac{1}{2}(1+\widetilde{\delta})$, $\widetilde{c}c$ in place of η, δ, c, whereas if $R > \widetilde{r}$, then we see from (5.13), $\widetilde{\delta} < 1$ and $(\log(e-1+\eta(R)^{-1}))^{-\widetilde{\alpha}} \leq 1 \leq (\log(e-1+\widetilde{r}^{-1})/\log(e-1+R^{-1}))^{\alpha} \wedge (R/r)^{\beta}$ that (5.10) with $\widetilde{\eta}\circ\eta$, $\frac{1}{2}(1+\widetilde{\delta})$, c' in place of η, δ, c holds, where $c' := \widetilde{c}(1+\delta)(\log(e-1+\widetilde{r}^{-1}))^{\alpha} + \widetilde{c}c$. In particular, if $\beta < \alpha$, then $\widetilde{\eta}\circ\eta$ satisfies (5.1) by $\frac{1}{2}(1+\widetilde{\delta}) < 1$ and (1). □

Example 5.4 Define homeomorphisms $\eta_k : [0,1] \to [0,1]$, $k \in \mathbb{N}$, inductively by

$$\eta_1(r) := \frac{1}{\log(e-1+r^{-1})} \quad (\eta_1(0) := 0) \quad \text{and} \quad \eta_{k+1} := \eta_1 \circ \eta_k, \quad k \in \mathbb{N}. \tag{5.14}$$

Then η_k satisfies (5.10) with $\delta = \frac{1}{2}$ and $\alpha = 1$ for some $c \in (0,\infty)$ for *any* $\beta \in (0,\infty)$ and any $k \in \mathbb{N}$. Indeed, this follows by a straightforward induction on k based on Proposition 5.3-(2), which is applicable with $\eta = \eta_k$ and $\widetilde{\eta} = \eta_1$ since η_1 is easily seen to satisfy (5.10) with $\delta = 0$, $\alpha = 1$ and $c = (e\beta)^{-1}$ for *any* $\beta \in (0,\infty)$ as follows: for any $r, R \in (0,1]$ with $r \leq R$,

$$\begin{aligned}\frac{\eta_1(R)}{\eta_1(r)} &= 1 + \frac{\log \dfrac{e-1+r^{-1}}{e-1+R^{-1}}}{\log(e-1+R^{-1})} = 1 + \frac{\log \dfrac{R}{r} + \log \dfrac{1+(e-1)r}{1+(e-1)R}}{\log(e-1+R^{-1})} \\ &\leq 1 + \frac{\beta^{-1}\log((R/r)^{\beta})}{\log(e-1+R^{-1})} \leq 1 + \frac{(e\beta)^{-1}(R/r)^{\beta}}{\log(e-1+R^{-1})}.\end{aligned} \tag{5.15}$$

As a consequence, for each $k \in \mathbb{N}$, recalling that $\Psi_{\eta_k} : [0,\infty) \to [0,\infty)$ is defined by $\Psi_{\eta_k}(r) := r^2\eta_k(r \wedge 1)$, we conclude from Proposition 5.3-(1) that η_k satisfies (5.1), thus from Theorem 5.1 that there exists $\boldsymbol{l}_k = (l_{k,n})_{n=1}^{\infty} \in (\mathbb{N} \setminus \{1,2,3,4\})^{\mathbb{N}}$ with $\sum_{n=1}^{\infty} l_{k,n}^{-1} < \infty$ such that $\Psi_{\eta_k}(r)/\Psi_{\boldsymbol{l}_k}(r) \in [c_k^{-1}, c_k]$ for any $r \in (0,\infty)$ for some $c_k \in [1,\infty)$, and thereby that $(K^{\boldsymbol{l}_k}, d_{\boldsymbol{l}_k}, m_{\boldsymbol{l}_k}, \mathcal{E}^{\boldsymbol{l}_k}, \mathcal{F}_{\boldsymbol{l}_k})$ satisfies fHKE(Ψ_{η_k}).

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References

1. S. Andres, M.T. Barlow, Energy inequalities for cutoff-functions and some applications. J. Reine Angew. Math. **699** (2015), 183–215. https://www.ams.org/mathscinet-getitem?mr=3305925
2. M.T. Barlow, Diffusions on fractals, in *Lectures on Probability Theory and Statistics (Saint-Flour, 1995)*. Lecture Notes in Mathematics, vol. 1690 (Springer, Berlin, 1998), pp. 1–121. https://www.ams.org/mathscinet-getitem?mr=1668115
3. M.T. Barlow, personal communication, July 17, 2019

4. M.T. Barlow, R.F. Bass, Transition densities for Brownian motion on the Sierpinski carpet. Probab. Theory Related Fields **91**(3–4), 307–330 (1992). https://www.ams.org/mathscinet-getitem?mr=1151799
5. M.T. Barlow, R.F. Bass, Brownian motion and harmonic analysis on Sierpiński carpets. Canad. J. Math. **51**(4), 673–744 (1999). https://www.ams.org/mathscinet-getitem?mr=1701339
6. M.T. Barlow, R.F. Bass, Stability of parabolic Harnack inequalities. Trans. Amer. Math. Soc. **356**(4), 1501–1533 (2004). https://www.ams.org/mathscinet-getitem?mr=2034316
7. M.T. Barlow, R.F. Bass, T. Kumagai, Stability of parabolic Harnack inequalities on metric measure spaces. J. Math. Soc. Japan **58** (2), 485–519 (2006). https://www.ams.org/mathscinet-getitem?mr=2228569
8. M.T. Barlow, A. Grigor'yan, T. Kumagai, On the equivalence of parabolic Harnack inequalities and heat kernel estimates. J. Math. Soc. Japan **64**(4), 1091–1146 (2012). https://www.ams.org/mathscinet-getitem?mr=2998918
9. M.T. Barlow, B.M. Hambly, Transition density estimates for Brownian motion on scale irregular Sierpinski gaskets. Ann. Inst. H. Poincaré Probab. Statist. **33**(5), 531–557 (1997). https://www.ams.org/mathscinet-getitem?mr=1473565
10. M.T. Barlow, E.A. Perkins, Brownian motion on the Sierpinski gasket. Probab. Theory Related Fields **79**(4), 543–623 (1988). https://www.ams.org/mathscinet-getitem?mr=0966175
11. O. Ben-Bassat, R.S. Strichartz, A. Teplyaev, What is not in the domain of the Laplacian on Sierpinski gasket type fractals. J. Funct. Anal. **166**(2), 197–217 (1999). https://www.ams.org/mathscinet-getitem?mr=1707752
12. D. Burago, Y. Burago, S. Ivanov, *A course in metric geometry*. Graduate Studies in Mathematics, vol. 33 (American Mathematical Society, Providence, RI, 2001). https://www.ams.org/mathscinet-getitem?mr=1835418
13. Z.-Q. Chen, M. Fukushima, *Symmetric Markov Processes, Time Change, and Boundary Theory*. London Mathematical Society Monographs Series, vol. 35 (Princeton University Press, Princeton, NJ, 2012). https://www.ams.org/mathscinet-getitem?mr=2849840
14. R. M. Dudley, *Real Analysis and Probability*. Revised reprint of the 1989 original, Cambridge Studies in Advanced Mathematics, vol. 74 (Cambridge University Press, Cambridge, 2002). https://www.ams.org/mathscinet-getitem?mr=1932358
15. S.N. Ethier, T.G. Kurtz, *Markov Processes: Characterization and Convergence*. Wiley Series in Probability and Mathematical Statistics (Wiley, New York, 1986). https://www.ams.org/mathscinet-getitem?mr=0838085
16. P.J. Fitzsimmons, B.M. Hambly, T. Kumagai, Transition density estimates for Brownian motion on affine nested fractals. Comm. Math. Phys. **165**(3), 595–620 (1994). https://www.ams.org/mathscinet-getitem?mr=1301625
17. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*. Second revised and extended edition, de Gruyter Studies in Mathematics, vol. 19 (Walter de Gruyter & Co., Berlin, 2011). https://www.ams.org/mathscinet-getitem?mr=2778606
18. A. Grigor'yan, *Heat Kernel and Analysis on Manifolds*. AMS/IP Studies in Advanced Mathematics, vol. 47 (American Mathematical Society, Providence, RI; International Press, Boston, MA, 2009). https://www.ams.org/mathscinet-getitem?mr=2569498
19. A. Grigor'yan, J. Hu, K.-S. Lau, Generalized capacity, Harnack inequality and heat kernels of Dirichlet forms on metric spaces. J. Math. Soc. Japan **67**, 1485–1549 (2015). https://www.ams.org/mathscinet-getitem?mr=3417504
20. A. Grigor'yan, N. Kajino, Localized upper bounds of heat kernels for diffusions via a multiple Dynkin–Hunt formula. Trans. Amer. Math. Soc. **369**(2), 1025–1060 (2017). https://www.ams.org/mathscinet-getitem?mr=3572263
21. B.M. Hambly, Brownian motion on a homogeneous random fractal. Probab. Theory Related Fields **94**(1), 1–38 (1992). https://www.ams.org/mathscinet-getitem?mr=1189083
22. B.M. Hambly, Heat kernels and spectral asymptotics for some random Sierpinski gaskets, in *Fractal Geometry and Stochastics II*. Progress in Probability, vol. 46, ed. by C. Bandt et al. (Birkhäuser, 2000), pp. 239–267. https://www.ams.org/mathscinet-getitem?mr=1786351

23. M. Hino, On singularity of energy measures on self-similar sets. Probab. Theory Related Fields **132**(2), 265–290 (2005). https://www.ams.org/mathscinet-getitem?mr=2199293
24. M. Hino, K. Nakahara, On singularity of energy measures on self-similar sets. II. Bull. London Math. Soc. **38**(6), 1019–1032 (2006). https://www.ams.org/mathscinet-getitem?mr=2285256
25. M. Hino, M. Yasui, Singularity of energy measures on a class of inhomogeneous Sierpinski gaskets, in *Dirichlet Forms and Related Topics*. Springer Proceedings in Mathematics & Statistics, vol. 394, ed. by Z.-Q. Chen et al. (Springer, 2022), pp. 175–200. https://arxiv.org/abs/2106.06705
26. N. Kajino, M. Murugan, On singularity of energy measures for symmetric diffusions with full off-diagonal heat kernel estimates. Ann. Probab. **48**(6), 2920–2951 (2020). https://www.ams.org/mathscinet-getitem?mr=4164457
27. J. Kigami, *Analysis on Fractals*. Cambridge Tracts in Mathematics, vol. 143 (Cambridge University Press, Cambridge, 2001). https://www.ams.org/mathscinet-getitem?mr=1840042
28. J. Kigami, Resistance forms, quasisymmetric maps and heat kernel estimates. Mem. Amer. Math. Soc. **216**(1015) (2012). https://www.ams.org/mathscinet-getitem?mr=2919892
29. T. Kumagai, Estimates of transition densities for Brownian motion on nested fractals. Probab. Theory Rel. Fields **96**(2), 205–224 (1993). https://www.ams.org/mathscinet-getitem?mr=1227032
30. S. Kusuoka, Dirichlet forms on fractals and products of random matrices. Publ. Res. Inst. Math. Sci. **25**(4), 659–680 (1989). https://www.ams.org/mathscinet-getitem?mr=1025071
31. S. Kusuoka, Lecture on diffusion processes on nested fractals, in *Statistical Mechanics and Fractals*. Lecture Notes in Mathematics, vol. 1567 (Springer, Berlin, 1993), pp. 39–98. https://www.ams.org/mathscinet-getitem?mr=1295841
32. J. Lierl, Scale-invariant boundary Harnack principle on inner uniform domains in fractal-type spaces. Potential Anal. **43**(4), 717–747 (2015). https://www.ams.org/mathscinet-getitem?mr=3432457
33. M. Murugan, On the length of chains in a metric space. J. Funct. Anal. **279**(6), 108627 (2020). https://www.ams.org/mathscinet-getitem?mr=4099475
34. W. Rudin, *Real and Complex Analysis*, 3rd ed. (McGraw-Hill Book Co., New York, 1987). https://www.ams.org/mathscinet-getitem?mr=0924157
35. L. Saloff-Coste, *Aspects of Sobolev-type Inequalities*. London Mathematical Society Lecture Note Series, vol. 289 (Cambridge University Press, Cambridge, 2002). https://www.ams.org/mathscinet-getitem?mr=1872526
36. K.-T. Sturm, Analysis on local Dirichlet spaces—II. Upper Gaussian estimates for the fundamental solutions of parabolic equations. Osaka J. Math. **32**(2), 275–312 (1995). https://www.ams.org/mathscinet-getitem?mr=1355744
37. K.-T. Sturm, Analysis on local Dirichlet spaces—III. The parabolic Harnack inequality. J. Math. Pures Appl. **75**(3), 273–297 (1996). https://www.ams.org/mathscinet-getitem?mr=1387522

Scattering Lengths for Additive Functionals and Their Semi-classical Asymptotics

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Abstract Scattering lengths for positive additive functionals of symmetric Markov processes are studied. The additive functionals considered here are not necessarily continuous. After giving a systematic presentation of the fundamentals of the scattering length, we study the problems of semi-classical asymptotics for scattering length under relativistic stable processes, which extend previous results for the case of positive continuous additive functionals.

Keywords Additive functionals · Scattering length · Semi-classical asymptotics · Relativistic stable processes

Mathematics Subject Classification Primary: 60J45 · Secondary: 31C15 · 34L25

1 Introduction

There is a notion of scattering length of a positive integrable function V on $\mathbb{R}^3$, one of the important quantities in scattering theory. It is the limit of the scattering amplitude $-f_k(e_y)$ given by

$$f_k(e_y) = -\frac{1}{2\pi}\int_{\mathbb{R}^3} e^{ik\sqrt{2}x\cdot e_y} h_k(x)V(x)\mathrm{d}x, \quad e_y = y/|y|,\ y \in \mathbb{R}^3$$

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as the wave number k tends to 0, where h_k is the solution of the scattering problem for V, that is, the solution of the equation $-(1/2)\Delta u + Vu = k^2 u$ having a certain asymptotic behaviour at infinity [8].

In [7, 8], Kac and Luttinger gave some applications of the probabilistic method to scattering problems. As one of such applications, the authors studied the problem of semi-classical asymptotics for scattering length of finite range potentials. To do this, they used a probabilistic expression for the scattering length of V in terms of Brownian motion $\mathbf{X} = (B_t, \mathbf{P}_x)$ on $\mathbb{R}^3$,

$$\Gamma(V) = \frac{1}{2\pi} \lim_{t\to\infty} \frac{1}{t} \int_{\mathbb{R}^3} \mathbf{E}_x \left[1 - e^{-\int_0^t V(B_s)\mathrm{d}s} \right] \mathrm{d}x, \tag{1}$$

and proved the following semi-classical limit: if $V = \mathbf{1}_K$ for a compact subset $K \subset \mathbb{R}^3$ satisfying the so-called Kac's regularity (see Sect. 4 for the definition), then $\uparrow \lim_{p\to\infty} \Gamma(pV) = \mathrm{Cap}(K)$, where Cap denotes the electrostatic capacity. Further, they conjectured that

$$\lim_{p\to\infty} \Gamma(pV) = \mathrm{Cap}(\mathrm{supp}[V]) \tag{2}$$

for *any* positive integrable function V with compact support in $\mathbb{R}^3$ satisfying the regularity as above. The Kac-Luttinger's conjecture (2) was confirmed by Taylor [18, 19] (also by Tamura [16] in an analytic way) who developed the notion of scattering length further into a tool for studying the effectiveness of potential as a perturbation of $-\Delta$ on $\mathbb{R}^d$. For more general framework of symmetric Markov processes, Takahashi [14] gave a new probabilistic representation of the scattering length of a continuous potential which makes the limit (2) quite transparent. For symmetric Markov processes again, Takeda [15] considered the behaviour of the scattering length of a positive smooth measure potential by using the random time change argument for Dirichlet forms and gave a simple elegant proof of the analog of (2) without Kac's regularity. The result in [15] was extended to a non-symmetric case by He [6]. For general right Markov processes, Fitzsimmons, He and Ying [4] extended Takahashi's result by using the tool of Kutznetsov measure and proved the analog of (2) for a positive continuous additive functional.

Scattering lengths cited so far were considered only for positive continuous additive functionals. But, there are many discontinuous additive functionals admitted to Markov processes. Hence, it is a natural question of how to understand the notion of scattering length of additive functionals that are not necessarily continuous. The objective of the present paper is to provide a partial answer to this question. Let E be a locally compact separable metric space and $\mathfrak{m}$ is a positive Radon measure on E with full topological support. Let $\mathbf{X} = (X_t, \mathbf{P}_x)$ be an $\mathfrak{m}$-symmetric Markov process on E. It is natural to consider the following additive functional of the form

$$A_t^\mu + \sum_{0<s\le t} F(X_{s-}, X_s) \tag{3}$$

which is not necessarily continuous. Here A_t^μ is the positive continuous additive functional of $\mathbf{X}$ with a positive smooth measure μ on E as its Revuz measure and F is a positive bounded Borel function on $E \times E$ vanishing on the diagonal. Let $(N(x, dy), H_t)$ be a Lévy system for $\mathbf{X}$. For $p \ge 1$, let $\mathbf{F}^{(p)}$ be a non-local linear operator defined by

$$\mathbf{F}^{(p)} f(x) = \int_E \left(1 - e^{-pF(x,y)}\right) f(y) N(x, dy), \quad x \in E$$

for any bounded measurable function f on E. Put $\mathbf{F}f := \mathbf{F}^{(1)} f$. We assume that $\mathbf{F}^{(p)} 1 \in L^1(E; \mu_H)$ for any $p \ge 1$. Let $U_{\mu+F}$ be the capacitary potential relative to the additive functional (3) defined by

$$U_{\mu+F}(x) := \mathbf{E}_x \left[1 - e^{-A_\infty^\mu - \sum_{t>0} F(X_{t-}, X_t)}\right].$$

In this paper, we define the scattering length $\Gamma(\mu + F)$ relative to (3) by

$$\Gamma(\mu + F) := \int_E (1 - U_{\mu+F})(x) \mu(dx) + \int_E \mathbf{F}(1 - U_{\mu+F})(x) \mu_H(dx),$$

where μ_H is the Revuz measure of H_t. In Sect. 2, we explain why the expression above is natural for the definition of the scattering length relative to (3). We will also give another expression for the scattering length above, which plays a crucial role throughout this paper (see Lemma 3).

Section 3 is devoted to studying the semi-classical limit of the scattering length. We investigate the behaviour of the scattering length $\Gamma(p\mu + pF)$ when $p \to \infty$. More precisely, let τ_t be the right continuous inverse of the positive continuous additive functional $A_t^\mu + \int_0^t \mathbf{F}1(X_s) dH_s$. Denote by $\mathsf{S}_{\mu+\mu_H \mathbf{F}1}$ the fine support of $A_t^\mu + \int_0^t \mathbf{F}1(X_s) dH_s$,

$$\mathsf{S}_{\mu+\mathbf{F}1\mu_H} = \left\{ x \in E \,\middle|\, \mathbf{P}_x(\tau_0 = 0) = 1 \right\}.$$

Our first result of this paper is as follows:

Theorem 1 *Suppose that F is symmetric. Further, assume that there exists a positive function $\psi(p)$ satisfying $\psi(p) \le p$, $\psi(p) \to \infty$ as $p \to \infty$ and the non-local operator $\mathbf{F}^{(p)}$ induced by F satisfies the following condition: for large $p \ge 1$ and a constant $C > 0$*

$$\mathbf{F}^{(p)} 1(x) \ge C\psi(p) \mathbf{F}1(x) \quad \textit{for } \ x \in E. \tag{4}$$

Then we have

$$\lim_{p\to\infty}\Gamma(p\mu+pF)=\mathrm{Cap}(\mathsf{S}_{\mu+\mathbf{F}1\mu_H}).$$

Note that Theorem 1 is already stated in [10, Theorem 1.1] in the framework of symmetric stable processes. However, we will show that its proof remains valid under general symmetric Markov processes, with the help of Lemmas 3 and 5. In this sense, Theorem 1 can be regarded as a generalization of the result in [15]. We also give some concrete examples of Fs under relativistic stable processes on $\mathbb{R}^d$ (Example 6).

In Sect. 4, we study the problem of semi-classical asymptotics for the scattering length of positive potentials with infinite range. It was proved analytically by Tamura [17] that the scattering length $\Gamma(V)$ of a positive integrable function V induced by 3-dimensional Brownian motion obeys

$$\Gamma(\lambda^{-2}V)\sim\lambda^{-2/(\rho-2)} \tag{5}$$

in the semi-classical limit $\lambda\to 0$, if $V(x)$ behaves like the Hardy type's potential $|x|^{-\rho}$, $\rho>3$ at infinity. As an application of the result obtained in Sect. 3, we will extend the result (5) probabilistically for the scattering length of positive potentials including a jumping function in the framework of relativistic stable processes. Our second result of the present paper is as follows: let $\mathbf{X}^m=(X_t,\mathbf{P}_x^m)$ be a Lévy process on $\mathbb{R}^d$ with

$$\mathbf{E}_0^m\left[e^{\sqrt{-1}\langle\xi,X_t\rangle}\right]=e^{-t\left((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m\right)},\quad 0<\alpha\le 2,\ m\ge 0. \tag{6}$$

The limiting case $\mathbf{X}^0$, corresponding to $m=0$, is nothing but the usual (rotationally) symmetric α-stable process. Let $\mathbf{F}_m^{(p)}$ be the non-local operator induced by F and $\Gamma_m^{(1)}$ the scattering length with respect to the 1-subprocess $\mathbf{X}^{m,(1)}$ of $\mathbf{X}^m$, respectively.

Theorem 2 *Let $\rho>d>\alpha$ and $0<\lambda\ll 1$. For a compact set $K\subset\mathbb{R}^d$, let $M>0$ be such that $K\subset B(0,M)$. For some constants $c_1,c_2>0$, let V be a positive function on $\mathbb{R}^d$ and F a positive bounded symmetric function on $\mathbb{R}^d\times\mathbb{R}^d$ vanishing on the diagonal satisfying $V(x)\le c_1|x|^{-\rho}$ for $x\in B(0,\ \lambda^{-\alpha/(\rho-\alpha)})^c$ and*

$$F(x,y)\le c_2|x-y|^{\alpha-\rho}\mathbf{1}_{B(x,\lambda^{-\alpha/(\rho-\alpha)})^c\cap\lambda^{-\alpha/(\rho-\alpha)}K}(y)$$

for $x\in B(0,\lambda^{-\alpha/(\rho-\alpha)}M)^c$, respectively. Here $\lambda^{-\alpha/(\rho-\alpha)}K:=\{\lambda^{-\alpha/(\rho-\alpha)}x\mid x\in K\}$. If there exists a positive function ψ satisfying $\psi(\sigma)\le\sigma$, $\psi(\sigma)\to\infty$ as $\sigma\to\infty$ and

$$\mathbf{F}_m^{(\lambda^{-\alpha})}1(x)\ge C\psi(\lambda^{-\alpha})\mathbf{F}_m1(x)\quad \textit{for } x\in B(0,\lambda^{-\alpha/(\rho-\alpha)}M),\ C>0,$$

then we have for any $m \geq 0$

$$C_1 \lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \leq \Gamma_m^{(1)}\left(\lambda^{-\alpha} V + \lambda^{-\alpha} F\right) \leq C_2 \lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}}$$

for some constants $C_2 > C_1 > 0$.

We note that Theorem 2 is not only the extension of the result (5) (or [11, Theorem 1.1]) but also it provides us with a new semi-classical asymptotic order of the scattering length for a jumping potential with infinite range under a purely discontinuous Markov process.

Throughout this paper, we use c, C, c', C', c_i, C_i $(i = 1, 2, \ldots)$ as positive constants which may be different at different occurrences. For notational convenience, we let $a \vee b := \max\{a, b\}$ for any $a, b \in \mathbb{R}$.

2 Scattering Length for Additive Functionals

Let E be a locally compact separable metric space and $\mathfrak{m}$ is a positive Radon measure on E with full topological support. Let ∂ be a point added to E so that $E_\partial := E \cup \{\partial\}$ is the one-point compactification of E. The point ∂ also serves as the cemetery point for E. Let $\mathbf{X} = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \mathbf{P}_x, \zeta)$ be an $\mathfrak{m}$-symmetric *transient* Hunt process on E, where ζ is the lifetime of X, $\zeta = \inf\{t > 0 \mid X_t = \partial\}$. We assume that $\mathbf{X}$ is conservative, that is, $\mathbf{P}_x(\zeta = \infty) = 1$ for any $x \in E$. Here and in the sequel, unless mentioned otherwise, we use the convention that a function defined on E takes the value 0 at ∂. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the Dirichlet form of $\mathbf{X}$ on $L^2(E; \mathfrak{m})$ which is assumed to be regular.

Let Cap be the (0-)capacity associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ of $\mathbf{X}$, that is, for an open set $O \subset E$ and the extended Dirichlet space $\mathcal{D}_e(\mathcal{E})$ of $\mathcal{D}(\mathcal{E})$,

$$\mathrm{Cap}(O) = \inf\{\mathcal{E}(u, u) \mid u \in \mathcal{D}_e(\mathcal{E}), u \geq 1 \ \mathfrak{m}\text{-a.e. on } O\} \tag{7}$$

and for a Borel set $B \subset E$,

$$\mathrm{Cap}(B) = \inf\{\mathrm{Cap}(O) \mid O \text{ is open}, \ B \subset O\} \tag{8}$$

(see [5, Chap. 2]).

We say that a positive continuous additive functional (PCAF in abbreviation) A_t^ν of $\mathbf{X}$ and a smooth measure ν are in the Revuz correspondence if they satisfy for any $t > 0$,

$$\int_E f(x)\nu(\mathrm{d}x) = \uparrow \lim_{t \downarrow 0} \frac{1}{t} \mathbf{E}_{\mathfrak{m}}\left[\int_0^t f(X_s)\,\mathrm{d}A_s^\nu\right], \quad f \in \mathscr{B}_b(E). \tag{9}$$

Here $\mathbf{E}_{\mathfrak{m}}[\,\cdot\,] = \int_E \mathbf{E}_x[\,\cdot\,]\mathfrak{m}(\mathrm{d}x)$ and $\mathscr{B}_b(E)$ is the space of bounded Borel functions on E. It is known that the family of equivalence classes of the set of PCAFs and the family of smooth measures are in one to one correspondence under the Revuz correspondence [5, Theorem 5.1.4]. Let $(N(x, \mathrm{d}y), H_t)$ be a Lévy system for $\mathbf{X}$, that is, $N(x, \mathrm{d}y)$ is a kernel on $(E, \mathscr{B}(E))$ and H_t is a PCAF with bounded 1-potential such that for any non-negative Borel function ϕ on $E \times E$ vanishing on the diagonal and any $x \in E$,

$$\mathbf{E}_x\left[\sum_{0<s\le t} \phi(X_{s-}, X_s)\right] = \mathbf{E}_x\left[\int_0^t \int_E \phi(X_s, y) N(X_s, \mathrm{d}y)\,\mathrm{d}H_s\right].$$

Let μ_H be the Revuz measure of the PCAF H_t. Then the jumping measure J and the killing measure κ of $\mathbf{X}$ are given by $J(\mathrm{d}x\mathrm{d}y) = \frac{1}{2}N(x, \mathrm{d}y)\mu_H(\mathrm{d}x)$ and $\kappa(\mathrm{d}x) = N(x, \{\partial\})\mu_H(\mathrm{d}x)$. These measures feature in the Beurling-Deny decomposition of $\mathcal{E}$ [5, Theorem 3.2.1].

A non-negative Borel measure ν on E (resp. a non-negative symmetric Borel function ϕ on $E \times E$ vanishing on the diagonal) is said to be Green-bounded relative to $\mathbf{X}$ if

$$\sup_{x\in E} \mathbf{E}_x\left[A_\infty^\nu\right] < \infty, \quad \left(\text{resp.} \quad \sup_{x\in E} \mathbf{E}_x\left[\sum_{t>0} \phi(X_{t-}, X_t)\right] < \infty\right).$$

Let μ be a positive smooth measure on E and A_t^μ the PCAF of $\mathbf{X}$ with μ as its Revuz measure. Let $F(x, y)$ be a bounded positive Borel function on $E \times E$ vanishing along the diagonal. Then $\sum_{0<s\le t} F(X_{s-}, X_s)$ is a positive (discontinuous) additive functional of $\mathbf{X}$. Throughout this section, we assume that F is Green-bounded relative to $\mathbf{X}$. It is natural to consider a combination of the additive functionals of the form

$$A_t^\mu + \sum_{0<s\le t} F(X_{s-}, X_s) \tag{10}$$

because the process $\mathbf{X}$ admits many discontinuous additive functionals. For $p \ge 1$, let $\mathbf{F}^{(p)}$ be a non-local linear operator defined by

$$\mathbf{F}^{(p)} f(x) = \int_E \left(1 - e^{-pF(x,y)}\right) f(y) N(x, \mathrm{d}y), \quad x \in E \tag{11}$$

for any $f \in \mathscr{B}_b(E)$. Put $\mathbf{F}f := \mathbf{F}^{(1)}f$. We assume that $\mathbf{F}^{(p)}1 \in L^1(E; \mu_H)$ for any $p \ge 1$. Let $U_{\mu+F}$ be the capacitary potential relative to the additive functional (10) defined by

$$U_{\mu+F}(x) := \mathbf{E}_x\left[1 - e^{-A_\infty^\mu - \sum_{t>0} F(X_{t-}, X_t)}\right].$$

We shall define the *scattering length* $\Gamma(\mu + F)$ relative to the additive functional (10) by

$$\Gamma(\mu+F) := \int_E (1-U_{\mu+F})(x)\mu(\mathrm{d}x) + \int_E \mathbf{F}(1-U_{\mu+F})(x)\mu_H(\mathrm{d}x). \tag{12}$$

Let us explain intuitively why the expression (12) is natural for the definition of the scattering length relative to (10). Let $\mathcal{L}$ be the the infinitesimal generator associated with $(\mathcal{E}, \mathscr{D}(\mathcal{E}))$: $\mathcal{E}(f,g) = (\sqrt{-\mathcal{L}}f, \sqrt{-\mathcal{L}}g)_{\mathfrak{m}}$ and $\mathscr{D}(\mathcal{E}) = \mathscr{D}(\sqrt{-\mathcal{L}})$. In analogy with classical one, we define $\Gamma(\mu + F)$ by the total mass of $-\mathcal{L}\,U_{\mu+F}$,

$$\Gamma(\mu+F) = -\int_E \mathcal{L}\,U_{\mu+F}\,\mathrm{d}\mathfrak{m}. \tag{13}$$

Note that the capacitary potential $U_{\mu+F}$ satisfies the following formal equation

$$-\mathcal{L}\,U_{\mu+F} = (1-U_{\mu+F})\mu + \mathbf{F}1\mu_H - \mathbf{F}U_{\mu+F}\mu_H. \tag{14}$$

Indeed, let $\widetilde{\mathbf{X}} = (X_t, \widetilde{\mathbf{P}}_x)$ be the transformed process of $\mathbf{X}$ by the pure jump Girsanov transform

$$Y_t^F := \exp\left(-\sum_{0<s\le t} F(X_{s-}, X_s) + \int_0^t \mathbf{F}1(X_s)\mathrm{d}H_s\right), \quad t \in (0,\infty). \tag{15}$$

The multiplicative functional (15) is a uniformly integrable martingale under the Green-boundedness of F relative to $\mathbf{X}$, because $e^{-F} - 1 \ge \delta - 1$ for some $\delta > 0$ by the boundedness of F and

$$\begin{aligned}
&\sup_{x\in E}\mathbf{E}_x\left[\int_0^\infty\int_E \left(1-e^{-F(X_s,y)}\right)^2 N(x,\mathrm{d}y)\mathrm{d}H_s\right]\\
&\qquad \le \sup_{x\in E}\mathbf{E}_x\left[\int_0^\infty\int_E F(X_s,y)N(x,\mathrm{d}y)\mathrm{d}H_s\right]\\
&\qquad = \sup_{x\in E}\mathbf{E}_x\left[\sum_{s>0} F(X_{s-},X_s)\right] < \infty
\end{aligned}$$

(cf. [1, Theorem 3.2]). From this fact, we see that the transformed process $\widetilde{\mathbf{X}}$ is also a transient and conservative Markov process on E. Let $\widetilde{\mathcal{L}}$ be the infinitesimal generator of $\widetilde{\mathbf{X}}$. Then $\widetilde{\mathcal{L}}$ is formally given by

$$-\widetilde{\mathcal{L}} = -\mathcal{L} + \mu_H \mathbf{F} - \mathbf{F}1\mu_H, \tag{16}$$

where $\mu_H\mathbf{F}$ denotes the measure valued operator defined by $\mu_H\mathbf{F}f(x) = \mathbf{F}f(x)\mu_H(\mathrm{d}x)$. It is known that a PCAF of $\mathbf{X}$ can be regarded as a PCAF of $\widetilde{\mathbf{X}}$. Thus we see from [9, Lemma 4.9] that

$$\begin{aligned} U_{\mu+F}(x) &= \mathbf{E}_x\left[1 - e^{-A^\mu_\infty - \sum_{t>0} F(X_{t-},X_t)}\right] \\ &= \widetilde{\mathbf{E}}_x\left[1 - e^{-A^\mu_\infty - \int_0^\infty \mathbf{F}1(X_t)\mathrm{d}H_t}\right] \\ &= \widetilde{\mathbf{E}}_x\left[\int_0^\infty e^{-A^\mu_t - \int_0^t \mathbf{F}1(X_s)\mathrm{d}H_s}\left(\mathrm{d}A^\mu_t + \mathbf{F}1(X_t)\mathrm{d}H_t\right)\right]. \end{aligned} \tag{17}$$

Equation (17) implies that $U_{\mu+F}$ satisfies the following formal equation

$$\left(\mu + \mathbf{F}1\mu_H - \widetilde{\mathcal{L}}\right) U_{\mu+F} = \mu + \mathbf{F}1\mu_H. \tag{18}$$

Hence we have (14) by applying (16)–(18), in other words, the total mass of $-\mathcal{L}\, U_{\mu+F}$ is given as the right-hand side of (12). We note that the relation (14) is rigorously established whenever $U_{\mu+F} \in L^2(E; \mathfrak{m})$.

The following expressions of the scattering length play a crucial role throughout this paper.

Lemma 3 *Suppose that F is symmetric. Then, the scattering length (12) can be rewritten as*

$$\Gamma(\mu + F) = \int_E \mathbf{E}_x\left[e^{-A^\mu_\infty - \sum_{t>0} F(X_{t-},X_t)}\right](\mu(\mathrm{d}x) + \mathbf{F}1(x)\mu_H(\mathrm{d}x)). \tag{19}$$

$$= \lim_{t\to\infty} \frac{1}{t}\int_E \mathbf{E}_x\left[1 - e^{-A^\mu_t - \sum_{0<s\le t} F(X_{s-},X_s)}\right]\mathfrak{m}(\mathrm{d}x). \tag{20}$$

Proof The expression (19) is a consequence of the symmetry of F. Indeed,

$$\begin{aligned} &\int_E \mathbf{F}(1 - U_{\mu+F})(x)\mu_H(\mathrm{d}x) \\ &\quad= \int_E\int_E \left(1 - e^{-F(x,y)}\right)\mathbf{E}_y\left[e^{-A^\mu_\infty - \sum_{t>0} F(X_{t-},X_t)}\right]N(x,\mathrm{d}y)\mu_H(\mathrm{d}x) \\ &\quad= \int_E \mathbf{E}_x\left[e^{-A^\mu_\infty - \sum_{t>0} F(X_{t-},X_t)}\right]\mathbf{F}1(x)\mu_H(\mathrm{d}x). \end{aligned}$$

On the other hand, it follows from [9, Lemma 4.9] and (19) that

$$
\begin{aligned}
\Gamma(\mu+F) &= \int_E \widetilde{\mathbf{E}}_x\left[e^{-A^\mu_\infty-\int_0^\infty \mathbf{F}1(X_s)\mathrm{d}H_s}\right](\mu(\mathrm{d}x)+\mathbf{F}1(x)\mu_H(\mathrm{d}x)) \\
&= \lim_{t\to\infty}\frac{1}{t}\int_E \widetilde{\mathbf{E}}_x\left[1-e^{-A^\mu_t-\int_0^t \mathbf{F}1(X_s)\mathrm{d}H_s}\right]\mathfrak{m}(\mathrm{d}x) \\
&= \lim_{t\to\infty}\frac{1}{t}\int_E \mathbf{E}_x\left[1-e^{-A^\mu_t-\sum_{0<s\le t}F(X_{s-},X_s)}\right]\mathfrak{m}(\mathrm{d}x)
\end{aligned}
$$

which implies the expression (20). In the second equality above, we used the result due to [15, (2.2)] (also [4, Theorem 2.2]). □

In the rest of the paper, we always assume that F is symmetric. It is immediate from the expression (20) that the scattering length $\Gamma(\mu+F)$ has the monotone property: for $i=1,2$, let μ_i be a non-negative finite smooth measure on E and F_i be a non-negative symmetric bounded Borel function on $E\times E$ vanishing on the diagonal such that $\mathbf{F}_{(i)}1\in L^1(E;\mu_H)$, where $\mathbf{F}_{(i)}$ is the non-local operator defined as in (11) for F_i. If $\mu_1\le\mu_2$ and $F_1\le F_2$, then

$$
\Gamma(\mu_1+F_1)\le\Gamma(\mu_2+F_2). \tag{21}
$$

Moreover, it follows from the elementary inequality $1-e^{-a-b}\le(1-e^{-a})+(1-e^{-b})$ for $a,b\ge 0$ that the scattering length has the subadditive property:

$$
\Gamma(\mu_1+\mu_2)\le\Gamma(\mu_1)+\Gamma(\mu_2),\quad \Gamma(F_1+F_2)\le\Gamma(F_1)+\Gamma(F_2).
$$

Finally, we close this section with the following remark:

Remark 4 The scattering length is trivial when the underlying process $\mathbf{X}$ is not transient. In fact, by virtue of [4, Lemma 2.1], the present scattering length $\Gamma(\mu+F)$ can be represented as

$$
\begin{aligned}
\Gamma(\mu+F) &= \int_{\{U_{\mu+F}=0\}\cup\{U_{\mu+F}=1\}}(1-U_{\mu+F})(x)(\mu+\mathbf{F}1\mu_H)(\mathrm{d}x) \\
&= \int_E \mathbf{1}_{\{U_{\mu+F}=0\}}(x)(\mu+\mathbf{F}1\mu_H)(\mathrm{d}x)
\end{aligned}
$$

under the non-transience of $\mathbf{X}$. Then, by the Revuz formula (9) and the Markov property

$$\begin{aligned}
&\int_E \mathbf{1}_{\{U_{\mu+F}=0\}}(x)(\mu+\mathbf{F}1\mu_H)(\mathrm{d}x)\\
&=\uparrow \lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_{\mathfrak{m}}\left[\int_0^t \mathbf{1}_{\{U_{\mu+F}=0\}}(X_s)\,\mathrm{d}A_s^{\mu+\mathbf{F}1\mu_H}\right]\\
&=\uparrow \lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_{\mathfrak{m}}\left[\int_0^t \mathbf{E}_{X_s}\left[\mathbf{1}_{\{A_\infty^\mu+\sum_{t>0}F(X_{t-},X_t)=0\}}\right]\mathrm{d}A_s^{\mu+\mathbf{F}1\mu_H}\right]\\
&=\uparrow \lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_{\mathfrak{m}}\left[\int_0^t \mathbf{1}_{\{A_\infty^\mu+\sum_{t>0}F(X_{t-},X_t)=A_s^\mu+\sum_{0<u\le s}F(X_{u-},X_u)\}}\mathrm{d}A_s^{\mu+\mathbf{F}1\mu_H}\right]\\
&=\uparrow \lim_{t\downarrow 0}\frac{1}{t}\mathbf{E}_{\mathfrak{m}}\left[\int_0^t \mathbf{1}_{\left\{A_\infty^{\mu+\mathbf{F}1\mu_H}=A_s^{\mu+\mathbf{F}1\mu_H}\right\}}\mathrm{d}A_s^{\mu+\mathbf{F}1\mu_H}\right]\\
&=0.
\end{aligned}$$

This is the reason why we only consider the scattering length for transient processes.

3 Kac's Scattering Length Formula

In this section, we are going to study the behaviour of the scattering length $\Gamma(p\mu+pF)$ when $p\to\infty$. As we mentioned in Sect. 1, this problem was decisively solved in the case $F\equiv 0$ by Takeda [15], through the random time change argument for Dirichlet forms: let ν be a positive finite smooth measure on E and S_ν the fine support of A_t^ν. Then

$$\lim_{p\to\infty}\Gamma(p\nu)=\mathrm{Cap}(\mathsf{S}_\nu). \tag{22}$$

However, we cannot apply time change method to our problem directly because our scattering length contains a discontinuous additive functional.

Let τ_t be the right continuous inverse of the PCAF $A_t^\mu+\int_0^t\mathbf{F}1(X_s)\mathrm{d}H_s$, that is, $\tau_t:=\inf\{s>0\mid A_s^\mu+\int_0^s\mathbf{F}1(X_u)\mathrm{d}H_u>t\}$. Let denote by $\mathsf{S}_{\mu+\mu_H\mathbf{F}1}$ the fine support of $A_t^\mu+\int_0^t\mathbf{F}1(X_s)\mathrm{d}H_s$,

$$\mathsf{S}_{\mu+\mathbf{F}1\mu_H}=\left\{x\in E\,\middle|\,\mathbf{P}_x(\tau_0=0)=1\right\}.$$

To prove Theorem 1, we need to the following lemma.

Lemma 5 *For any* $\varepsilon > 0$

$$\lim_{p\to\infty} \Gamma\left(pF + p^{1+\varepsilon}\mu + p^{1+\varepsilon}\mathbf{F}1\mu_H\right) = \mathrm{Cap}(\mathsf{S}_{\mu+\mathbf{F}1\mu_H}).$$

In particular, $\limsup_{p\to\infty} \Gamma(p\mu + pF) \le \mathrm{Cap}(\mathsf{S}_{\mu+\mathbf{F}1\mu_H})$.

Proof The last assertion easily follows from the first one with the monotonicity of the scattering length. Put $k = 1/(1+\varepsilon)$ and write $A_t^{\mathbf{F}1\mu_H} := \int_0^t \mathbf{F}1(X_s)\mathrm{d}H_s$. From the expression (19), one can easily see that

$$\begin{aligned}\Gamma&\left(p^kF + p\mu + p\mathbf{F}1\mu_H\right)\\ &= \int_E \mathbf{E}_x\left[e^{-p^k\sum_{t>0}F(X_{t-},X_t) - pA_\infty^\mu - pA_\infty^{\mathbf{F}1\mu_H}}\right]\\ &\qquad\cdot\left(p\mu(\mathrm{d}x) + \mathbf{F}^{(p^k)}1(x)\mu_H(\mathrm{d}x) + p\mathbf{F}1(x)\mu_H(\mathrm{d}x)\right).\end{aligned}$$

Since $\mathbf{F}^{(q)}1 \le q\mathbf{F}1$ for any $q \ge 1$,

$$\begin{aligned}\Gamma&\left(p^kF + p\mu + p\mathbf{F}1\mu_H\right)\\ &\le \int_E \mathbf{E}_x\left[e^{-pA_\infty^\mu - pA_\infty^{\mathbf{F}1\mu_H}}\right]\left(\left(1+p^{k-1}\right)p\mu(\mathrm{d}x) + \left(p^k\mathbf{F}1 + p\mathbf{F}1\right)(x)\mu_H(\mathrm{d}x)\right)\\ &= \left(1+p^{k-1}\right)\Gamma\left(p\mu + p\mathbf{F}1\mu_H\right).\end{aligned}$$

Therefore we have from the monotonicity of the scattering length that

$$\Gamma\left(p\mu + p\mathbf{F}1\mu_H\right) \le \Gamma\left(p^kF + p\mu + p\mathbf{F}1\mu_H\right) \le \left(1+p^{k-1}\right)\Gamma\left(p\mu + p\mathbf{F}1\mu_H\right).$$

In view of (22), the scattering lengths of both sides of the above converge to $\mathrm{Cap}(\mathsf{S}_{\mu+\mathbf{F}1\mu_H})$ as $p \to \infty$, which implies the first assertion. □

Proof of Theorem 1 Let $\psi(p)$ be the function which appeared in the condition (4). By the monotonicity of the scattering length, we have for some $C > 0$

$$\begin{aligned}\Gamma&\left(\frac{\psi(p)}{n}\mu + \frac{C\psi(p)}{n}\mathbf{F}1\mu_H\right)\\ &\le \Gamma\left(pF + \frac{\psi(p)}{n}\mu + \frac{C\psi(p)}{n}\mathbf{F}1\mu_H\right) \le \Gamma\left(pF + p^{1+\varepsilon}\mu + p^{1+\varepsilon}\mathbf{F}1\mu_H\right)\end{aligned}$$

for any $n \ge 1$ and $\varepsilon > 0$. Then, by Lemma 5 and applying (22) again, one can get that

$$\lim_{p\to\infty} \Gamma\left(pF + \frac{\psi(p)}{n}\mu + \frac{C\psi(p)}{n}\mathbf{F}1\mu_H\right) = \mathrm{Cap}(\mathsf{S}_{\mu+\mathbf{F}1\mu_H}).$$

From this and the condition (4), we see that

$$\begin{aligned}
&\liminf_{p\to\infty} \Gamma\,(p\mu + pF) \geq \liminf_{p\to\infty} \Gamma\left(\frac{\psi(p)}{n}\mu + pF\right)\\
&= \liminf_{p\to\infty} \int_E \mathbf{E}_x\left[e^{-p\sum_{t>0}F(X_{t-},X_t)-\frac{\psi(p)}{n}A^{\mu}_{\infty}}\right]\left(\frac{\psi(p)}{n}\mu(\mathrm{d}x) + \mathbf{F}^{(p)}1(x)\mu_H(\mathrm{d}x)\right)\\
&= \frac{n}{n+1}\liminf_{p\to\infty} \int_E \mathbf{E}_x\left[e^{-p\sum_{t>0}F(X_{t-},X_t)-\frac{\psi(p)}{n}A^{\mu}_{\infty}}\right]\\
&\qquad\qquad \cdot\left(\frac{n+1}{n}\frac{\psi(p)}{n}\mu(\mathrm{d}x) + \mathbf{F}^{(p)}1(x)\mu_H(\mathrm{d}x) + \frac{1}{n}\mathbf{F}^{(p)}1(x)\mu_H(\mathrm{d}x)\right)\\
&\geq \frac{n}{n+1}\liminf_{p\to\infty} \int_E \mathbf{E}_x\left[e^{-p\sum_{t>0}F(X_{t-},X_t)-\frac{\psi(p)}{n}A^{\mu}_{\infty}-\frac{C\psi(p)}{n}A^{\mathbf{F}1\mu_H}_{\infty}}\right]\\
&\qquad\qquad \cdot\left(\mathbf{F}^{(p)}1(x)\mu_H(\mathrm{d}x) + \frac{\psi(p)}{n}\mu(\mathrm{d}x) + \frac{C\psi(p)}{n}\mathbf{F}1(x)\mu_H(\mathrm{d}x)\right)\\
&= \frac{n}{n+1}\lim_{p\to\infty} \Gamma\left(pF + \frac{\psi(p)}{n}\mu + \frac{C\psi(p)}{n}\mathbf{F}1\mu_H\right)\\
&= \frac{n}{n+1}\mathrm{Cap}(\mathsf{S}_{\mu+\mathbf{F}1\mu_H}).
\end{aligned}$$

Letting $n \to \infty$, we have

$$\mathrm{Cap}(\mathsf{S}_{\mu+\mathbf{F}1\mu_H}) \leq \liminf_{p\to\infty} \Gamma(p\mu + pF). \tag{23}$$

The proof will be finished by the last assertion of Lemma 5 and (23). □

Let $\mathbf{X}^m = (X_t, \mathbf{P}^m_x)$ be a Lévy process on $\mathbb{R}^d$ with

$$\mathbf{E}^m_0\left[e^{\sqrt{-1}\langle\xi,X_t\rangle}\right] = e^{-t\left((|\xi|^2+m^{2/\alpha})^{\alpha/2}-m\right)}, \quad 0 < \alpha \leq 2,\ m \geq 0. \tag{24}$$

If $m > 0$ and $0 < \alpha < 2$, it is called the relativistic α-stable process with mass m. In particular, if $m > 0$ and $\alpha = 1$, it is called the relativistic free Hamiltonian process. The limiting case $\mathbf{X}^0$, corresponding to $m = 0$, is nothing but the usual (rotationally) symmetric α-stable process. It is known that $\mathbf{X}^m$ is transient if and only if $d > 2$ under $m > 0$ or $d > \alpha$ under $m = 0$, and it is a doubly Feller conservative process. From (24), one can easily see that $\mathbf{X}^m$ has the following scaling property: for any $r > 0$

$$(rX_t, \mathbf{P}^m_x) \stackrel{d}{=} \left(X_{r^\alpha t}, \mathbf{P}^{r^{-\alpha}m}_{rx}\right), \tag{25}$$

where $\stackrel{d}{=}$ means the equality in distribution. Let $(\mathcal{E}^m, \mathscr{D}(\mathcal{E}^m))$ be the Dirichlet form on $L^2(\mathbb{R}^d)$ associated with $\mathbf{X}^m$. It follows from Fourier transform $\hat{f}(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{i\langle x,y\rangle} f(y)\mathrm{d}y$ that

$$\mathscr{D}(\mathcal{E}^m) := \left\{ f \in L^2(\mathbb{R}^d) \;\middle|\; \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \left((|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m \right) \mathrm{d}\xi < \infty \right\},$$

$$\mathcal{E}^m(f, g) := \int_{\mathbb{R}^d} \hat{f}(\xi)\bar{\hat{g}}(\xi) \left((|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m \right) \mathrm{d}\xi \quad \text{for } f, g \in \mathscr{D}(\mathcal{E}^m)$$

[5, Example 1.4.1]. The Dirichlet form $(\mathcal{E}^0, \mathscr{D}(\mathcal{E}^0))$ for $\mathbf{X}^0$ can also be characterized similarly, only with $|\xi|^\alpha$ in place of $(|\xi|^2 + m^{2/\alpha})^{\alpha/2} - m$ above. Thus, there exist positive constants $c_1 := c_1(m)$ and $c_2 := c_2(m)$ such that

$$c_1 \mathcal{E}_1^0(u, u) \le \mathcal{E}_1^m(u, u) \le c_2 \mathcal{E}_1^0(u, u)$$

and so $\mathscr{D}(\mathcal{E}^m) = \mathscr{D}(\mathcal{E}^0)$. Here $\mathcal{E}_1^*(u, u) := \mathcal{E}^*(u, u) + (u, u)_{\mathfrak{m}}$. From this, we see that for any $m \ge 0$ and a Borel set $B \subset \mathbb{R}^d$

$$c_1 \mathrm{Cap}^{(1)}(B) \le \mathrm{Cap}_m^{(1)}(B) \le c_2 \mathrm{Cap}^{(1)}(B), \tag{26}$$

where $\mathrm{Cap}_m^{(1)}$ (resp. $\mathrm{Cap}^{(1)}$) denotes the 1-capacity associated with $(\mathcal{E}^m, \mathscr{D}(\mathcal{E}^m))$ (resp. $(\mathcal{E}^0, \mathscr{D}(\mathcal{E}^0))$), that is, it is the capacity defined by replacing $\mathscr{D}_e(\mathcal{E}^m)$ and $\mathcal{E}^m$ (resp. $\mathscr{D}_e(\mathcal{E}^0)$ and $\mathcal{E}^0$) in (7) and (8) with $\mathscr{D}(\mathcal{E}^m)$ and $\mathcal{E}_1^m$ (resp. $\mathscr{D}(\mathcal{E}^0)$ and $\mathcal{E}_1^0$). Let denote by $B(a, b)$ the open ball in $\mathbb{R}^d$ with center a and radius b. It is known that

$$\mathrm{Cap}^{(1)}(B(0, r)) = r^{d-\alpha} \mathrm{Cap}^{(1)}(B(0, 1)) \tag{27}$$

(cf. [12, (42.22)]). It is shown in [3] that the corresponding jumping measure J of $(\mathcal{E}^m, \mathcal{F}^m)$ satisfies

$$J(\mathrm{d}x\mathrm{d}y) = J_m(x, y)\mathrm{d}x\mathrm{d}y \quad \text{with} \quad J_m(x, y) = C_{d,\alpha} \frac{\varphi(m^{1/\alpha}|x - y|)}{|x - y|^{d+\alpha}},$$

where $C_{d,\alpha} = \frac{\alpha 2^{d+\alpha}\Gamma(\frac{d+\alpha}{2})}{2^{d+1}\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}$ and

$$\varphi(r) := 2^{-(d+\alpha)} \Gamma\left(\frac{d+\alpha}{2}\right)^{-1} \int_0^\infty s^{\frac{d+\alpha}{2}-1} e^{-\frac{s}{4}-\frac{r^2}{s}} \mathrm{d}s,$$

which is a decreasing function satisfying $\varphi(0) = 1$ and

$$c^{-1}e^{-r}r^{\frac{d+\alpha-1}{2}} \le \varphi(r) \le ce^{-r}r^{\frac{d+\alpha-1}{2}}, \quad r \ge 1 \tag{28}$$

for some constant $c > 1$ (cf. [3]). In particular,

$$\mathscr{D}(\mathcal{E}^m) = \left\{ f \in L^2(\mathbb{R}^d) \;\middle|\; \int_{\mathbb{R}^d\times\mathbb{R}^d} |f(x) - f(y)|^2 J_m(x,y)\mathrm{d}x\mathrm{d}y < \infty \right\},$$
$$\mathcal{E}^m(f,g) := \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} (f(x) - f(y))(g(x) - g(y))J_m(x,y)\mathrm{d}x\mathrm{d}y$$

for $f, g \in \mathscr{D}(\mathcal{E}^m)$. It is known that $\mathbf{X}^m$ has a Lévy system $(N(x, \mathrm{d}y), H_t)$ given by $N(x, \mathrm{d}y) = J_m(x, y)\mathrm{d}y$ and $H_t = t$. In this case, the non-local linear operator (11) for a symmetric positive bounded Borel function $F(x, y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal is given by

$$\mathbf{F}_m^{(p)}f(x) = \int_{\mathbb{R}^d} \frac{(1 - e^{-pF(x,y)})f(y)\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\mathrm{d}y, \quad p \ge 1,\ x \in \mathbb{R}^d.$$

Now, we give some concrete examples of Fs satisfying the condition (4).

Example 6 Let F be the function on $\mathbb{R}^d \times \mathbb{R}^d$ such that for $\beta > \alpha$

$$F(x,y) = \frac{1}{2}|x-y|^\beta \chi_{R,R'}(x,y),$$

where $\chi_{R,R'}(x, y)$ is the indicator function given by

$$\begin{aligned}\chi_{R,R'}&(x,y)\\ &= \big(\mathbf{1}_{B(x,R')}(y)\mathbf{1}_{B(0,R)}(x) + \mathbf{1}_{B(y,R')}(x)\mathbf{1}_{B(0,R)}(y)\\ &\quad + \mathbf{1}_{B(y,R')}(x)\mathbf{1}_{B(x,R')}(y)\mathbf{1}_{B(0,R+R')\setminus B(0,R)}(x)\mathbf{1}_{B(0,R+R')\setminus B(0,R)}(y)\big)\end{aligned}$$

for $R, R' > 0$. Then the condition (4) holds for this F. First, take $x \in B(0, R)$. In this case, F is given by

$$F(x,y) = \begin{cases} |x-y|^\beta & y \in B(x,R') \cap B(0,R) \\ \frac{1}{2}|x-y|^\beta & y \in B(x,R') \cap B(0,R)^c \\ 0 & \text{otherwise} \end{cases}$$

and thus we have

$$
\begin{aligned}
\mathbf{F}_m 1(x) &= C_{d,\alpha} \int\limits_{\mathbb{R}^d} \frac{(1-e^{-F(x,y)})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y \\
&= C_{d,\alpha} \int\limits_{B(x,R')} \frac{(1-e^{-F(x,y)})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y \\
&= C_{d,\alpha} \left\{ \int\limits_{B(x,R')\cap B(0,R)} \frac{(1-e^{-|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y \right. \\
&\qquad \left. + \int\limits_{B(x,R')\cap B(0,R)^c} \frac{(1-e^{-\frac{1}{2}|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y \right\} \\
&\leq C_{d,\alpha} \int\limits_{B(x,R')} \frac{(1-e^{-|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y.
\end{aligned}
$$

By using integration by parts, the right-hand side of the above inequality is equal to

$$
\begin{aligned}
&C'_{d,\alpha} \int_0^{R'} \frac{(1-e^{-r^\beta})\varphi(m^{1/\alpha}r)}{r^{1+\alpha}} \mathrm{d}r \\
&= C'_{d,\alpha} \left\{ \frac{(e^{-(R')^\beta}-1)\varphi(m^{1/\alpha}R')}{\alpha(R')^\alpha} \right. \\
&\quad \left. + \frac{1}{\alpha} \int_0^{R'} r^{-\alpha} \left(\beta r^{\beta-1} e^{-r^\beta} \varphi(m^{1/\alpha}r) + (1-e^{-r^\beta}) m^{1/\alpha} \varphi'(m^{1/\alpha}r) \right) \mathrm{d}r \right\} \\
&= C'_{d,\alpha} \left\{ \frac{(e^{-(R')^\beta}-1)\varphi(m^{1/\alpha}R')}{\alpha(R')^\alpha} \right. \\
&\quad \left. + \frac{1}{\alpha} \int_0^{(R')^\beta} t^{-\alpha/\beta} \left(e^{-t}\varphi(m^{1/\alpha}t^{1/\beta}) + m^{1/\alpha}(1-e^{-t}) \frac{\varphi'(m^{1/\alpha}t^{1/\beta})}{\beta t^{(\beta-1)/\beta}} \right) \mathrm{d}t \right\}, \qquad (29)
\end{aligned}
$$

where $C'_{d,\alpha}$ is a positive constant depending on d and α. On the other hand, by a similar calculation as above with the inequality $1 - e^{-a-b} \leq (1 - e^{-a}) + (1 - e^{-b})$ for any $a, b \geq 0$, we see

$$
\begin{aligned}
&\mathbf{F}_m^{(p)}1(x) \\
&= C_{d,\alpha}\left\{ \int\limits_{B(x,R')\cap B(0,R)} \frac{(1-e^{-p|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\mathrm{d}y \right. \\
&\qquad \left. + \int\limits_{B(x,R')\cap B(0,R)^c} \frac{(1-e^{-\frac{p}{2}|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\mathrm{d}y \right\} \\
&\geq C_{d,\alpha} \int\limits_{B(x,R')} \frac{(1-e^{-\frac{p}{2}|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\mathrm{d}y \\
&\geq \frac{C_{d,\alpha}}{2} \int\limits_{B(x,R')} \frac{(1-e^{-p|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}}\mathrm{d}y \\
&= \frac{C'_{d,\alpha}}{2}\left\{ \frac{(e^{-p(R')^\beta}-1)\varphi(m^{1/\alpha}R')}{\alpha(R')^\alpha} \right. \\
&\qquad + \frac{1}{\alpha}\int_0^{p(R')^\beta} p^{\alpha/\beta}t^{-\alpha/\beta}\left(e^{-t}\varphi(m^{1/\alpha}(p^{-1}t)^{1/\beta})\right. \\
&\qquad \left.\left. + m^{1/\alpha}(1-e^{-t})\frac{\varphi'(m^{1/\alpha}(p^{-1}t)^{1/\beta})}{\beta(p^{-1}t)^{(\beta-1)/\beta}}\right)\mathrm{d}t\right\} \\
&\geq \frac{p^{\alpha/\beta}}{2}C'_{d,\alpha}\left\{ \frac{(e^{-(R')^\beta}-1)\varphi(m^{1/\alpha}R')}{\alpha(R')^\alpha} \right. \\
&\qquad + \frac{1}{\alpha}\int_0^{(R')^\beta} t^{-\alpha/\beta}\left(e^{-t}\varphi(m^{1/\alpha}(p^{-1}t)^{1/\beta})\right. \\
&\qquad \left.\left. + m^{1/\alpha}(1-e^{-t})\frac{\varphi'(m^{1/\alpha}(p^{-1}t)^{1/\beta})}{\beta t^{(\beta-1)/\beta}}\right)\mathrm{d}t\right\}
\end{aligned}
\tag{30}
$$

for large $p \geq 1$. Since $\varphi(r)$ is decreasing, $\varphi(m^{1/\alpha}(p^{-1}t)^{1/\beta}) \geq \varphi(m^{1/\alpha}t^{1/\beta})$ and $\varphi'(r)$ is a non-positive function on $[0,\infty)$ taking a value close to 0 near $r=0$, that is, $\varphi'(m^{1/\alpha}(p^{-1}t)^{1/\beta}) \geq \varphi'(m^{1/\alpha}t^{1/\beta})$ for large $p \geq 1$. From these facts with (29) and (30), we can confirm that

$$
\mathbf{F}_m^{(p)}1(x) \geq \frac{p^{\alpha/\beta}}{2}\mathbf{F}_m 1(x), \quad x \in B(0,R), \text{ large } p \geq 1. \tag{31}
$$

Next, take $x \in B(0, R + R') \setminus B(0, R)$. In this case, F is given by

$$F(x, y) = \begin{cases} \frac{1}{2}|x-y|^\beta & y \in B(x, R') \cap B(0, R) \\ \frac{1}{2}|x-y|^\beta & y \in B(x, R') \cap B(0, R)^c \\ 0 & \text{otherwise.} \end{cases}$$

Then, by the same calculations as above

$$\begin{aligned}
&\mathbf{F}_m 1(x) \\
&\leq C_{d,\alpha} \int_{B(x,R')} \frac{(1 - e^{-|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y \\
&= C'_{d,\alpha} \left\{ \frac{(e^{-(R')^\beta} - 1)\varphi(m^{1/\alpha} R')}{\alpha (R')^\alpha} \right. \\
&\qquad \left. + \frac{1}{\alpha} \int_0^{(R')^\beta} t^{-\alpha/\beta} \left(e^{-t} \varphi(m^{1/\alpha} t^{1/\beta}) + m^{1/\alpha} (1 - e^{-t}) \frac{\varphi'(m^{1/\alpha} t^{1/\beta})}{\beta t^{(\beta-1)/\beta}} \right) \mathrm{d}t \right\}
\end{aligned}$$

and

$$\begin{aligned}
&\mathbf{F}_m^{(p)} 1(x) \\
&\geq \frac{C_{d,\alpha}}{2} \int_{B(x,R')} \frac{(1 - e^{-p|x-y|^\beta})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y \\
&\geq \frac{p^{\alpha/\beta}}{2} C'_{d,\alpha} \left\{ \frac{(e^{-(R')^\beta} - 1)\varphi(m^{1/\alpha} R')}{\alpha (R')^\alpha} \right. \\
&\qquad + \frac{1}{\alpha} \int_0^{(R')^\beta} t^{-\alpha/\beta} \Big(e^{-t} \varphi(m^{1/\alpha} (p^{-1} t)^{1/\beta}) \\
&\qquad \left. + m^{1/\alpha} (1 - e^{-t}) \frac{\varphi'(m^{1/\alpha} (p^{-1} t)^{1/\beta})}{\beta t^{(\beta-1)/\beta}} \Big) \mathrm{d}t \right\} \\
&\geq \frac{p^{\alpha/\beta}}{2} C'_{d,\alpha} \left\{ \frac{(e^{-(R')^\beta} - 1)\varphi(m^{1/\alpha} R')}{\alpha (R')^\alpha} \right. \\
&\qquad \left. + \frac{1}{\alpha} \int_0^{(R')^\beta} t^{-\alpha/\beta} \left(e^{-t} \varphi(m^{1/\alpha} t^{1/\beta}) + m^{1/\alpha} (1 - e^{-t}) \frac{\varphi'(m^{1/\alpha} t^{1/\beta})}{\beta t^{(\beta-1)/\beta}} \right) \mathrm{d}t \right\} \\
&= \frac{p^{\alpha/\beta}}{2} \mathbf{F}_m 1(x)
\end{aligned}$$

for large $p \geq 1$. Therefore, we can confirm (31) for $x \in B(0, R+R') \setminus B(0, R)$. For $x \in B(0, R+R')^c$, (31) is trivial because $\mathbf{F}_m^{(p)} 1(x) = 0$ for any $p \geq 1$. Hence we obtain (31) for any $x \in \mathbb{R}^d$. Moreover, for $x \in B(0, R+R')$

$$\begin{aligned}
\mathbf{F}_m^{(p)} 1(x) &= C_{d,\alpha} \left(\int\limits_{B(0,R+R')\cap B(x,1)} \frac{(1-e^{-pF(x,y)})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y \right. \\
&\quad \left. + \int\limits_{B(0,R+R')\cap B(x,1)^c} \frac{(1-e^{-pF(x,y)})\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y \right) \\
&\leq C_{d,\alpha} \int\limits_{B(x,1)} \frac{1-e^{-\frac{3p}{2}|x-y|^\beta}}{|x-y|^{d+\alpha}} \mathrm{d}y \\
&\quad + C_{d,\alpha} \int\limits_{\overline{B(0,R+R')}\cap B(x,1)^c} \left(1-e^{-\frac{3p}{2}(R+R')^\beta}\right) \mathrm{d}y \\
&\leq C'_{d,\alpha} \int_0^1 \frac{1-e^{-\frac{3p}{2}r^\beta}}{r^{\alpha+1}} \mathrm{d}r + C_{d,\alpha} \left|\overline{B(0,R+R')}\right|
\end{aligned}$$

and from which it follows that $\mathbf{F}_m^{(p)} 1$ is bounded on $B(0, R+R')$ and is zero on $B(0, R+R')^c$ for any $p \geq 1$. This shows that $\mathbf{F}_m^{(p)} 1 \in L^1(\mathbb{R}^d)$ for any $p \geq 1$. In fact, by a similar way as in the proof of [2, Proposition 7.10(3)], one can also prove that $\mathbf{F}_m^{(p)} 1 \in L^\ell(\mathbb{R}^d) (\ell \geq 1)$ for any $p \geq 1$ and thus F is to be Green-bounded with respect to $\mathbf{X}^{m,(1)}$. Here $\mathbf{X}^{m,(1)}$ is the 1-subprocess of $\mathbf{X}^m$, the killed process by e^{-t}. We omit the details.

Remark 7 There are many functions satisfying the condition (4). In fact, they can be given by the following form:

$$F(x, y) = \frac{1}{2}\phi(|x-y|)\chi_{R,R'}(x, y)$$

with $\phi(t) = t^\beta$, $\phi(t) = t^\beta/(1+t)^\beta$, $\phi(t) := \phi^{(1)}(t) = \log(1+t^\beta)$ and its iterated function $\phi^{(n)}(t) = \phi(\phi^{(n-1)}(t))$ $(n \geq 2)$ for $\beta > \alpha$. Further, we see that Fs induced by these functions are Green-bounded relative to $\mathbf{X}^{m,(1)}$ (cf. [2, Proposition 7.10]) and we can take the function $\psi(p)$ which appeared in (4) as

$$\psi(p) = p^{\alpha/\beta}.$$

Hence, the scattering length $\Gamma_m^{(1)}(p\mu + pF)$ induced by the functions ϕ above converges to $\mathrm{Cap}_m^{(1)}(\mathsf{S}_{\mu+\mathbf{F}_m 1})$ as $p \to \infty$, in view of Theorem 1. Here $\Gamma_m^{(1)}$ denotes the scattering length with respect to $\mathbf{X}^{m,(1)}$.

4 Semi-classical Asymptotics for Scattering Length

In this section, we study the semi-classical asymptotics for scattering length by non-negative potentials with infinite ranges. We consider the case that $\mu(\mathrm{d}x) = V(x)\mathrm{d}x$ with V being a non-negative $L^1(\mathbb{R}^d)$-function. Note that the scattering length $\Gamma(pV + pF)$ may diverge as $p \to \infty$ if V or F has a non-compact support. So the question we are interested in is to find the asymptotic order of $\Gamma(pV + pF)$ to infinity as $p \to \infty$.

In the sequel, let $\mathbf{X}^m$, $\mathbf{X}^{m,(1)}$, $\Gamma_m^{(1)}$ and $\mathrm{Cap}_m^{(1)}$ be as in Example 6 and Remark 7. Clearly, $\mathbf{X}^{m,(1)}$ is transient. For $a > 0$ and $b \geq 0$, let V_a^b and F_a^b be the scaling potentials of V and F, respectively, which are defined by

$$V_a^b(x) := a^b V(ax), \quad F_a^b(x, y) := a^b F(ax, ay), \quad x, y \in \mathbb{R}^d.$$

The following simple scaling property of the scattering length plays a role.

Lemma 8 *For any $\beta \geq 0$ and $r > 0$, it holds that*

$$\Gamma_m^{(1)}(V_r^\beta + F_r^{\beta-\alpha}) = r^{\alpha-d}\Gamma_{r^{-\alpha}m}^{(r^{-\alpha})}(r^{\beta-\alpha}V + r^{\beta-\alpha}F).$$

Proof For notational convenience, set

$$A_t^{V_r^\beta} := \int_0^t V_r^\beta(X_s)\mathrm{d}s, \quad A_t^{F_r^{\beta-\alpha}} := \sum_{0<s\leq t} F_r^{\beta-\alpha}(X_{s-}, X_s).$$

It follows from the scaling property (25) that $A_t^{V_r^\beta}$ and $A_t^{F_r^{\beta-\alpha}}$ under $\mathbf{P}_x^m$ are equal to $r^{\beta-\alpha}A_{r^\alpha t}^V$ and $r^{\beta-\alpha}A_{r^\alpha t}^F$ under $\mathbf{P}_{rx}^{r^{-\alpha}m}$, respectively. Then, by the expression of the scattering length (20) and Ito's formula

$$\begin{aligned}
&\Gamma_m^{(1)}(V_r^\beta + F_r^{\beta-\alpha}) \\
&= \lim_{t\to\infty} \frac{1}{t}\int_{\mathbb{R}^d} \mathbf{E}_x^{m,(1)}\left[1 - e^{-A_t^{V_r^\beta} - A_t^{F_r^{\beta-\alpha}}}\right]\mathrm{d}x \\
&= \lim_{t\to\infty} \frac{1}{t}\int_{\mathbb{R}^d} \mathbf{E}_x^{m}\left[\int_0^t e^{-s-A_s^{V_r^\beta} - A_s^{F_r^{\beta-\alpha}}}\left(V_r^\beta(X_s) + r^\alpha \mathbf{F}_m^{(r^{\beta-\alpha})}1(rX_s)\right)\mathrm{d}s\right]\mathrm{d}x \\
&= \lim_{t\to\infty} \frac{1}{t}\int_{\mathbb{R}^d} \mathbf{E}_{rx}^{r^{-\alpha}m}\left[\int_0^t e^{-s-r^{\beta-\alpha}A_{r^\alpha s}^V - r^{\beta-\alpha}A_{r^\alpha s}^F}\right. \\
&\qquad \left.\cdot \left(r^\beta V(X_{r^\alpha s}) + r^\alpha \mathbf{F}_{r^{-\alpha}m}^{(r^{\beta-\alpha})}1(X_{r^\alpha s})\right)\mathrm{d}s\right]\mathrm{d}x
\end{aligned}$$

$$= r^{\alpha-d} \lim_{t\to\infty} \frac{1}{r^\alpha t} \int_{\mathbb{R}^d} \mathbf{E}_y^{r^{-\alpha}m} \left[\int_0^{r^\alpha t} e^{-r^{-\alpha}s' - r^{\beta-\alpha}A^V_{s'} - r^{\beta-\alpha}A^F_{s'}} \right.$$

$$\left. \cdot \left(r^{\beta-\alpha} V(X_{s'}) + \mathbf{F}^{(r^{\beta-\alpha})}_{r^{-\alpha}m} 1(X_{s'}) \right) \mathrm{d}s' \right] \mathrm{d}y$$

$$= r^{\alpha-d} \lim_{t\to\infty} \frac{1}{r^\alpha t} \int_{\mathbb{R}^d} \mathbf{E}_y^{r^{-\alpha}m,(r^{-\alpha})} \left[1 - e^{-r^{\beta-\alpha}A^V_{r^\alpha t} - r^{\beta-\alpha}A^F_{r^\alpha t}} \right] \mathrm{d}y$$

$$= r^{\alpha-d} \Gamma^{(r^{-\alpha})}_{r^{-\alpha}m} (r^{\beta-\alpha}V + r^{\beta-\alpha}F),$$

where we used in the second equality above that the non-local operator defined in (11) for F_a^b is given by $a^\alpha \mathbf{F}_m^{(a^b)} 1(a\,\cdot)$. □

For an open set $B \subset \mathbb{R}^d$, let $\tau_0^{\mathbf{1}_{B^c}}$ be the first penetrating time of $\mathbf{X}^m$ into B^c,

$$\tau_0^{\mathbf{1}_{B^c}} := \inf \left\{ t > 0 \,\middle|\, \int_0^t \mathbf{1}_{B^c}(X_s)\,\mathrm{d}s > 0 \right\}.$$

We say that B is a Kac's regular set with respect to $\mathbf{X}^m$, if $\tau_0^{\mathbf{1}_{B^c}}$ is the same as $\tau_B := \inf\{t > 0 \mid X_t \in B^c\}$, the first exit time of $\mathbf{X}^m$ from B (with probability one). Note that any open subset of $\mathbb{R}^d$ having a smooth boundary, thus any open ball in $\mathbb{R}^d$, is Kac regular. Let $\mathrm{supp}\,[U]$ be the topological support of a non-negative potential U. Then the set $\mathsf{S}_U \setminus \mathrm{supp}\,[U]$ is of zero capacity, while $\mathrm{supp}\,[U] \setminus \mathsf{S}_U$ is not necessarily of zero capacity. It is known that if $\mathrm{supp}\,[U]$ is a Kac's regular set, then $\mathrm{Cap}(\mathsf{S}_U) = \mathrm{Cap}(\mathrm{supp}\,[U])$ (cf. [15, §3]).

Lemma 9 *Let B be an open ball in $\mathbb{R}^d$. Under the hypotheses in Theorem 1, it holds that*

$$\lim_{p\to\infty} \Gamma^{(1)}_m (pV\mathbf{1}_B + pF\mathbf{1}_B) = \mathrm{Cap}^{(1)}_m(B),$$

where $F\mathbf{1}_B := F(x, y)\mathbf{1}_B(x)$.

Proof Note that we can not obtain the assertion as an immediate consequence of Theorem 1 because $F\mathbf{1}_B$ is not necessarily a symmetric function. Let

$$F\mathbf{1}_{B^2}(x, y) := F(x, y)\mathbf{1}_B(x)\mathbf{1}_B(y).$$

Clearly $F\mathbf{1}_{B^2}$ is symmetric. Denote by $\mathbf{F}_m\mathbf{1}_{B^2}$ the non-local operator induced by $F\mathbf{1}_{B^2}$ which is given as

$$\mathbf{F}_m\mathbf{1}_{B^2}(x) = C_{d,\alpha} \left(\int_B \frac{\left(1 - e^{-F(x,y)}\right)\varphi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y \right) \mathbf{1}_B(x).$$

Then, by a similar way as in the proof of Theorem 1, we can see that

$$\begin{aligned}\liminf_{p\to\infty}\Gamma_m^{(1)}(pV\mathbf{1}_B+pF\mathbf{1}_B)&\geq\liminf_{p\to\infty}\Gamma_m^{(1)}(pV\mathbf{1}_B+pF\mathbf{1}_{B^2})\\&\geq \mathrm{Cap}_m^{(1)}\left(\mathsf{S}_{V\mathbf{1}_B+\mathbf{F}_m\mathbf{1}_{B^2}}\right)=\mathrm{Cap}_m^{(1)}(B),\end{aligned}\tag{32}$$

where we used the equality in (32) that B is a Kac's regular set with respect to $\mathbf{X}^{m,(1)}$.

For an open ball $D\subset\mathbb{R}^d$ such that $B\subset D$, let $U_D^{(1)}$ be its capacitary potential, which is given by $U_D^{(1)}(x):=\mathbf{E}_x^{m,(1)}[1-e^{-\int_0^\infty \nu_D(X_t)\mathrm{d}t}]$ for $\nu_D=\infty$ on D and 0 off D. Then $U_D^{(1)}=U_1^m\nu_D$, where U_1^m is the 1-potential operator of $\mathbf{X}^m$ and ν_D is the (1-)equilibrium measure on D (cf. [5, 13]). From the definition of the scattering length (13) and the fact that $U_D^{(1)}=1$ on B, for any $p\geq 1$

$$\begin{aligned}\Gamma_m^{(1)}(pV\mathbf{1}_B+pF\mathbf{1}_B)&=-\int_{\mathbb{R}^d}\mathcal{L}^{m,(1)}U^{(1)}_{pV\mathbf{1}_B+pF\mathbf{1}_B}(x)\,\mathrm{d}x\\&=-\int_{\mathbb{R}^d}U_D^{(1)}(x)\mathcal{L}^{m,(1)}U^{(1)}_{pV\mathbf{1}_B+pF\mathbf{1}_B}(x)\,\mathrm{d}x\\&=\mathcal{E}_1^m\left(U_1^m\nu_D,\,U^{(1)}_{pV\mathbf{1}_B+pF\mathbf{1}_B}\right)\\&=\int_{\mathbb{R}^d}U^{(1)}_{pV\mathbf{1}_B+pF\mathbf{1}_B}(x)\nu_D(\mathrm{d}x),\end{aligned}$$

where $\mathcal{L}^{m,(1)}$ is the infinitesimal generator of $\mathbf{X}^{m,(1)}$ and $U^{(1)}_{pV\mathbf{1}_B+pF\mathbf{1}_B}$ is the capacitary potential relative to $pV\mathbf{1}_B+pF\mathbf{1}_B$ under $\mathbf{X}^{m,(1)}$. On the other hand, since $\mathrm{Cap}_m^{(1)}(B)=\int_{\mathbb{R}^d}U_B^{(1)}(x)\nu_D(\mathrm{d}x)$, we can see that

$$\limsup_{p\to\infty}\Gamma_m^{(1)}(pV\mathbf{1}_B+pF\mathbf{1}_B)\leq\mathrm{Cap}_m^{(1)}(B)\tag{33}$$

because $U^{(1)}_{pV\mathbf{1}_B+pF\mathbf{1}_B}\leq U_B^{(1)}$ for any $p\geq 1$. Here $U_B^{(1)}$ denotes the capacitary potential of B defined as above. The proof will be finished by (32) and (33). □

Proposition 10 *Let $\rho>d>\alpha$ and $0<\lambda\ll 1$. If a non-negative function V satisfies $V(x)\leq c_1|x|^{-\rho}$ for $x\in B(0,\,\lambda^{-\alpha/(\rho-\alpha)})^c$ for some constant $c_1>0$, then we have for any $m\geq 0$*

$$C_1\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}}\leq\Gamma_m^{(1)}\left(\lambda^{-\alpha}V\right)\leq C_2\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}}\tag{34}$$

for some constants $C_2\geq C_1>0$.

Proof Let W be the function defined by $W(x)=|x|^{-\rho}\mathbf{1}_{B(0,1)^c}(x)$. By applying W with $F\equiv 0$, $\beta=\alpha$ and $r=\lambda^{\alpha/(\rho-\alpha)}$ to Lemma 8, we have

$$\Gamma_m^{(1)}\left(W^{\alpha}_{\lambda^{\alpha/(\rho-\alpha)}}\right) = \lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}}\,\Gamma^{(\lambda^{-\alpha^2/(\rho-\alpha)})}_{\lambda^{-\alpha^2/(\rho-\alpha)}m}(W). \tag{35}$$

Since

$$\begin{aligned} W^{\alpha}_{\lambda^{\alpha/(\rho-\alpha)}}(x) &= \lambda^{\alpha^2/(\rho-\alpha)}|\lambda^{\alpha/(\rho-\alpha)}x|^{-\rho}\mathbf{1}_{B(0,1)^c}(\lambda^{\alpha/(\rho-\alpha)}x) \\ &= \lambda^{-\alpha}|x|^{-\rho}\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})^c}(x), \end{aligned}$$

(35) can be rewritten as

$$\Gamma_m^{(1)}\left(\lambda^{-\alpha}|x|^{-\rho}\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})^c}\right) = \lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}}\,\Gamma^{(\lambda^{-\alpha^2/(\rho-\alpha)})}_{\lambda^{-\alpha^2/(\rho-\alpha)}m}(W). \tag{36}$$

It is clear from the definition of scattering length that for some constant $C > 0$

$$\Gamma^{(\lambda^{-\alpha^2/(\rho-\alpha)})}_{\lambda^{-\alpha^2/(\rho-\alpha)}m}(W) \le \int_{B(0,1)^c} |x|^{-\rho}\mathrm{d}x = \omega_d \int_1^{\infty} r^{d-\rho-1}\mathrm{d}r \le C.$$

So by which and (36), it follows that

$$\Gamma_m^{(1)}\left(\lambda^{-\alpha}|x|^{-\rho}\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})^c}\right) \le C\,\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}}. \tag{37}$$

On the other hand, we see by Lemma 9 with $F = 0$ that for any $\varepsilon > 0$ there exists $0 < \lambda_0 := \lambda_0(\varepsilon) \ll 1$ such that for every $0 < \lambda \le \lambda_0$,

$$\begin{aligned} &C'\,\mathrm{Cap}_m^{(1)}\left(B(0,\lambda^{-\alpha/(\rho-\alpha)})\right) \\ &\quad \le \Gamma_m^{(1)}\left(\lambda^{-\alpha}V\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})}\right) \le \mathrm{Cap}_m^{(1)}(B(0,\lambda^{-\alpha/(\rho-\alpha)})) \end{aligned}$$

for some constant $C' := C'(\varepsilon) > 0$. From which, with (26) and (27), it follows that

$$C_1\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \le \Gamma_m^{(1)}\left(\lambda^{-\alpha}V\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})}\right) \le C_1'\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \tag{38}$$

for some constants $C_1, C_1' > 0$. Now, on account of (37) and (38), the monotonicity and subadditivity of scattering length we can confirm (34). Indeed,

$$\begin{aligned} C_1\,\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} &\le \Gamma_m^{(1)}\left(\lambda^{-\alpha}V\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})}\right) \\ &\le \Gamma_m^{(1)}(\lambda^{-\alpha}V) \\ &\le \Gamma_m^{(1)}\left(\lambda^{-\alpha}V\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})}\right) + \Gamma_m^{(1)}\left(\lambda^{-\alpha}V\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})^c}\right) \\ &\le \Gamma_m^{(1)}\left(\lambda^{-\alpha}V\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})}\right) + c'\Gamma_m^{(1)}\left(\lambda^{-\alpha}|x|^{-\rho}\mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})^c}\right) \\ &\le C_2\,\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \end{aligned}$$

where we used in the fourth inequality above that for any non-negative $L^1(\mathbb{R}^d)$-function U and a constant $c > 0$, there exists a constant $c' > 0$ such that $\Gamma_m^{(1)}(c\,U) \le c'\Gamma_m^{(1)}(U)$. □

Remark 11 If $m = 0$, $d = 3$ and $\alpha = 2$, then (26) holds for (0-)capacity. As a result, Tamura's result (5) can be easily reproduced from (34).

In the sequel, we let $\alpha \in (0, 2)$.

Proposition 12 *Let $d > \alpha$, $\rho > \frac{d+\alpha-1}{2} \vee \alpha$ and $0 < \lambda \ll 1$. For a compact set $K \subset \mathbb{R}^d$, let $M > 0$ be such that $K \subset B(0, M)$. Assume that for $x \in B(0, \lambda^{-\alpha/(\rho-\alpha)}M)^c$,*

$$F(x, y) \le c_2|x - y|^{-(\rho-\alpha)}\mathbf{1}_{B(x,\,\lambda^{-\alpha/(\rho-\alpha)})^c\cap\lambda^{-\alpha/(\rho-\alpha)}K}(y) \tag{39}$$

for some constant $c_2 > 0$, where $\lambda^{-\alpha/(\rho-\alpha)}K := \{\lambda^{-\alpha/(\rho-\alpha)}x \mid x \in K\}$. Assume in addition that

$$\mathbf{F}_m^{(\lambda^{-\alpha})}1(x) \ge C\psi(\lambda^{-\alpha})\mathbf{F}_m 1(x) \quad \text{for } x \in B(0, \lambda^{-\alpha/(\rho-\alpha)}M) \tag{40}$$

for a constant $C > 0$ and for some positive function ψ such that $\psi(\sigma) \le \sigma$ and $\psi(\sigma) \to \infty$ as $\sigma \to \infty$. Then, for any $m \ge 0$

$$C_3\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \le \Gamma_m^{(1)}(\lambda^{-\alpha}F) \le C_4\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \tag{41}$$

for some constants $C_4 \ge C_3 > 0$.

Proof Let

$$G(x, y) := |x - y|^{-(\rho-\alpha)}\mathbf{1}_{B(0,M)^c}(x)\mathbf{1}_{B(x,1)^c\cap K}(y)$$

and $\mathbf{G}_m 1(x)$ the associated non-local function defined as in (11). Note that the statement in Lemma 8 with $V \equiv 0$ is valid for any non-negative symmetric bounded function on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal such that its non-local operator defined as in (11) being integrable. By applying G with $\beta = \alpha$ and $r = \lambda^{\alpha/(\rho-\alpha)}$ to Lemma 8, one has

$$\Gamma_m^{(1)}\left(G^0_{\lambda^{\alpha/(\rho-\alpha)}}\right) = \lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}}\Gamma_{m_\lambda}^{(\lambda^{-\alpha^2/(\rho-\alpha)})}(G), \tag{42}$$

where $m_\lambda := \lambda^{-\alpha^2/(\rho-\alpha)}m$ and

$$\begin{aligned} &G^0_{\lambda^{\alpha/(\rho-\alpha)}}(x, y) \\ &\quad = \lambda^{-\alpha}|x - y|^{-(\rho-\alpha)}\mathbf{1}_{B(0,\lambda^{-\alpha/(\rho-\alpha)}M)^c}(x)\mathbf{1}_{B(x,\,\lambda^{-\alpha/(\rho-\alpha)})^c\cap\,\lambda^{-\alpha/(\rho-\alpha)}K}(y). \end{aligned}$$

From the definition of scattering length and (28),

$$
\begin{aligned}
\Gamma_{m_\lambda}^{(\lambda^{-\alpha^2/(\rho-\alpha)})}(G) &\le \int_{\mathbb{R}^d} \mathbf{G}_{m_\lambda} 1(x)\mathrm{d}x \\
&\le C_{d,\alpha} \int_{B(0,M)^c} \int_K \frac{(1-e^{-|x-y|^{-(\rho-\alpha)}})\varphi(m_\lambda^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}} \mathrm{d}y\mathrm{d}x \\
&\le C'_{d,\alpha} m_\lambda^{\frac{d+\alpha-1}{2\alpha}} \int_K \int_{B(0,M)^c} \frac{|x-y|^{-(\rho-\alpha)} e^{-m_\lambda^{1/\alpha}|x-y|} |x-y|^{\frac{d+\alpha-1}{2}}}{|x-y|^{d+\alpha}} \mathrm{d}x\mathrm{d}y \\
&= C'_{d,\alpha} \omega_d m_\lambda^{\frac{d+\alpha-1}{2\alpha}} \int_K \int_{d(y,B(0,M)^c)}^{\infty} \frac{e^{-m_\lambda^{1/\alpha} r}}{r^{(2\rho-d-\alpha+3)/2}} \mathrm{d}r\mathrm{d}y \\
&\le C_m \lambda^{-\frac{\alpha(d+\alpha-1)}{2(\rho-\alpha)}} \exp\left(-\lambda^{-\frac{\alpha}{\rho-\alpha}} m^{\frac{1}{\alpha}} \inf_{y\in K} d(y,B(0,M)^c)\right) \\
&\qquad \cdot \int_K \frac{1}{d(y,B(0,M)^c)^{(2\rho-d-\alpha+1)/2}} \mathrm{d}y \\
&= C'_4 \lambda^{-\frac{\alpha(d+\alpha-1)}{2(\rho-\alpha)}} \exp\left(-c_3 \lambda^{-\frac{\alpha}{\rho-\alpha}}\right),
\end{aligned} \tag{43}
$$

where

$$
C'_4 = C_m \int_K \frac{1}{d(y,B(0,M)^c)^{(2\rho-d-\alpha+1)/2}} \mathrm{d}y \quad \text{with } C_m := \frac{2C'_{d,\alpha} m^{\frac{d+\alpha-1}{2\alpha}} \omega_d}{2\rho-d-\alpha+1}
$$

and $c_3 = m^{1/\alpha} \inf_{y\in K} d(y,B(0,M)^c)$. Thus, it follows from (42) and (43) that

$$
\Gamma_m^{(1)}\left(G^0_{\lambda^{\alpha/(\rho-\alpha)}}\right) \le C'_4 \lambda^{-\frac{\alpha(3d-\alpha-1)}{2(\rho-\alpha)}} \exp\left(-c_3 \lambda^{-\frac{\alpha}{\rho-\alpha}}\right). \tag{44}
$$

On the other hand, we see by Lemma 9 with $V = 0$ that for any $\varepsilon > 0$ there exists $0 < \lambda_0 := \lambda_0(\varepsilon) \ll 1$ such that for every $0 < \lambda \le \lambda_0$,

$$
\begin{aligned}
&C'_5 \,\mathrm{Cap}_m^{(1)}\left(B(0,\lambda^{-\alpha/(\rho-\alpha)})\right) \\
&\qquad \le \Gamma_m^{(1)}\left(\lambda^{-\alpha} F \mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})}\right) \le \mathrm{Cap}_m^{(1)}(B(0,\lambda^{-\alpha/(\rho-\alpha)}))
\end{aligned}
$$

for some constant $C'_5 := C'_5(\varepsilon) > 0$. From this fact with (26) and (27), it follows that

$$
C_3 \lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \le \Gamma_m^{(1)}\left(\lambda^{-\alpha} F \mathbf{1}_{B(0,\,\lambda^{-\alpha/(\rho-\alpha)})}\right) \le C'_3 \lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \tag{45}
$$

for some constants $C'_3 \ge C_3 > 0$. Now, by combining (44) and (45), we have

$$
\begin{aligned}
C_3\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} &\leq \Gamma_m^{(1)}\left(\lambda^{-\alpha}F\mathbf{1}_{B(0,\lambda^{-\alpha/(\rho-\alpha)}M)}\right) \\
&\leq \Gamma_m^{(1)}\left(\lambda^{-\alpha}F\right) \\
&\leq \Gamma_m^{(1)}\left(\lambda^{-\alpha}F\mathbf{1}_{B(0,\lambda^{-\alpha/(\rho-\alpha)}M)}\right) + \Gamma_m^{(1)}\left(\lambda^{-\alpha}F\mathbf{1}_{B(0,\lambda^{-\alpha/(\rho-\alpha)}M)^c}\right) \\
&\leq \Gamma_m^{(1)}\left(\lambda^{-\alpha}F\mathbf{1}_{B(0,\lambda^{-\alpha/(\rho-\alpha)}M)}\right) + c'\Gamma_m^{(1)}\left(G^0_{\lambda^{\alpha/(\rho-\alpha)}}\right) \\
&\leq C_3'\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} + c'C_4'\lambda^{-\frac{\alpha(3d-\alpha-1)}{2(\rho-\alpha)}}\exp\left(-c_3\lambda^{-\frac{\alpha}{\rho-\alpha}}\right) \\
&\leq \left(C_3' + c'C_4'\lambda^{-\frac{\alpha(d+\alpha-1)}{2(\rho-\alpha)}}\exp\left(-c_3\lambda^{-\frac{\alpha}{\rho-\alpha}}\right)\right)\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}} \\
&\leq C_4\lambda^{-\frac{\alpha(d-\alpha)}{\rho-\alpha}}
\end{aligned}
$$

for some constant $C_4 > 0$. □

Now, we are ready to prove Theorem 2.

Proof of Theorem 2 The proof is an immediate consequence of Propositions 10 and 12 with the subadditivity of the scattering length. □

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References

1. Z.-Q. Chen, Uniform integrability of exponential martingales and spectral bounds of non-local Feynman-Kac semigroups, *Stochastic Analysis and Applications to Finance, Essays in Honor of Jia-an Yan*, ed. by T. Zhang, X. Zhou (2012), pp. 55–75
2. Z.-Q. Chen, D. Kim, K. Kuwae, L^p-independence of spectral radius for generalized Feynman-Kac semigroups. Math. Ann. **374**(1–2), 601–652 (2019)
3. Z.-Q. Chen, R. Song, Drift transforms and Green function estimates for discontinuous processes. J. Funct. Anal. **201**(1), 262–281 (2003)
4. P.J. Fitzsimmons, P. He, J. Ying, A remark on Kac's scattering length formula. Sci. China Math. **56**(2), 331–338 (2013)
5. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, second revised and extended edition. de Gruyter Studies in Mathematics, vol. 19 (Walter de Gruyter & Co., Berlin, 2011)
6. P. He, A formula on scattering length of dual Markov processes. Proc. Amer. Math. Soc. **139**(5), 1871–1877 (2011)
7. M. Kac, Probabilistic methods in some problems of scattering theory. Rocky Mt. J. Math. **4**, 511–537 (1974)
8. M. Kac, J. Luttinger, Scattering length and capacity. Ann. Inst. Fourier **25**(3–4), 317–321 (1975)
9. D. Kim, K. Kuwae, Analytic characterizations of gaugeability for generalized Feynman-Kac functionals. Trans. Amer. Math. Soc. **369**(7), 4545–4596 (2017)
10. D. Kim, M. Matsuura, On a scattering length for additive functionals and spectrum of fractional Laplacian with a non-local perturbation. Math. Nachr. **293**(2), 327–345 (2020)

11. D. Kim, M. Matsuura, Semi-classical asymptotics for scattering length of symmetric stable processes. Stat. Probab. Lett. **167**, 108921 (2020)
12. K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Studies in Advanced Mathematics, vol. 68 (Cambridge University Press, 1999)
13. D. Stroock, The Kac approach to potential theory. I. J. Math. Mech. **16**, 829–852 (1967)
14. Y. Takahashi, An integral representation on the path space for scattering length. Osaka J. Math. **7**, 373–379 (1990)
15. M. Takeda, A formula on scattering length of positive smooth measures. Proc. Amer. Math. Soc. **138**(4), 1491–1494 (2010)
16. H. Tamura, Semi-classical limit of scattering length. Lett. Math. Phys. **24**, 205–209 (1992)
17. H. Tamura, Semi-classical asymptotics for scattering length and total cross section at zero energy. Bull. Fac. Sci. Ibaraki Univ. **25**, 1–9 (1993)
18. M. Taylor, Scattering length and perturbations of $-\Delta$ by positive potentials. J. Math. Anal. Appl. **53**, 291–312 (1976)
19. M. Taylor, Scattering length of positive potentials. Houston J. Math. **33**(4), 979–1003 (2007)

Equivalence of the Strong Feller Properties of Analytic Semigroups and Associated Resolvents

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Abstract In this paper, we give sufficient conditions for the equivalence between semigroup strong Feller property and resolvent strong Feller property.

Keywords Feller property of semigroup · semigroup strong Feller property · Resolvent strong Feller property · Analytic semigroup · Sobolev inequality · Ultra contractivity of semigroup

Mathematics Subject Classification 60J46 · 60J45 · 60J35 · 31C25

1 Introduction

The notion of the Feller property was initiated by Feller [14]. In the paper, a pair of one-dimensional parabolic diffusion equations is exhaustively studied through the associated semigroups, where we can observe the origin of the present Feller property. In [13], the Feller property is defined for Markov processes on compact metric spaces, which states that the associated semigroups map the family of continuous functions on the state space into itself. Later, this notion was extended beyond the compactness of the state space. The semigroup $\{P_t\}_{t\geq 0}$ of a Markov process on a locally compact separable metric space E is now said to have the Feller property if each P_t leaves

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invariant the family $C_\infty(E)$ of continuous functions on E vanishing at infinity. It is known that any Feller semigroup generates a Markov process on the state space with strong Markov property and càdlàg path, which is called the *Feller process*. An important subclass is formed by the strong Feller processes initiated by Girsanov [17]. They are generated by transition semigroups with the strong Feller property, i.e., each P_t maps the family $\mathscr{B}_b(E)$ of bounded Borel functions on E into the family $C_b(E)$ of bounded continuous functions on it. In fact, Markov processes other than Feller processes can have such property, and they are also called strong Feller processes. For example, it is known that non-degenerate diffusion processes on Riemannian manifolds are strong Feller processes (see [24]).

The strong Feller property is one of fundamental notions for Markov processes, and has been studied from various sources. For example, the transition semigroups of Markov processes with strong Feller property often possess density functions (with respect to canonical measures), and some potential theoretic aspects of the process are studied. One remarkable result is that the concepts of polar sets and semi-polar sets coincide with each other for such Markov processes associated with semi-Dirichlet forms. See [16, §4, Theorems 4.1.2 and 4.2.7] and [26, §3.5, Theorem 3.5.4] for the precise statement. It is also important to point out that the strong Feller property is used to determine the uniqueness of invariant measures for Markov processes (see, e.g., [9, Sect. 11.3.2]). Hence, the strong Feller property plays a crucial role for the ergodic theory of Markov processes.

Although the definition of the strong Feller property stated above is for semigroups, this is also defined for resolvents in a natural way (Definition 1). It is well-known that the semigroup strong Feller property (SF) implies the resolvent strong Feller property (RSF). As this fact suggests, it is often easier to confirm (RSF) than (SF). Then, the question of under what conditions (RSF) means (SF) naturally arises. This is not obvious. In fact, the uniform motion to the right satisfies (RSF), but not (SF). See [19, Remark 1.1 (1)] for details. This is in contrast to the fact that the Feller property for a resolvent kernel is equivalent to the Feller property for the associated transition semigroup kernel (see [19, Sect. 1]). However, the uniform motion to the right is a little extreme example of non-symmetric Markov processes. Therefore, under an appropriate framework, (RSF) can imply (SF).

In the present paper, we study the question stated above, and provide several sufficient conditions for it. For a given Markov process, we consider a situation in which the semigroup is extended to an analytic semigroup on some L^p-space with $p \in [1, +\infty)$. Then, we utilize the theory of analytic semigroups to describe a general conditions that strengthen (RSF) to (SF), which is the first main result of this paper (Theorem 2). We also show that the semigroup of the uniform motion to the right is not extended to an analytic semigroup on any L^p-space with respect to the invariant measure. Theorem 2 can be applied to Hunt processes associated with lower bounded semi-Dirichlet forms (Theorem 3). As a result, we find that (SF) can be obtained mainly from the assumptions of (RSF) and a kind of ultracontractivity of the resolvent. Even if the ultracontractivity is replaced with the ultracontractivity of the semigroup, the same conclusion holds (Theorem 4). However, we think Theorem 4 does not follow immediately from Theorem 3. These theorems may be restrictive in that the semigroups of Ornstein-Uhlenbeck processes have the strong Feller property, but does not satisfy the ultracontractivity. Because of this fact, we introduce the notion of local ultracontractivity, and extend the Theorems (Theorem 6). Here, it is also necessary to study (RSF) for part processes of a given Markov process.

The organization of this paper is as follows. In Sect. 2, we prepare several related notions, for example, the Feller, and strong Feller properties of transition semigroup and resolvent kernels, analyticity of strongly continuous semigroup on L^p-spaces, lower bounded semi-Dirichlet forms and so on. In Sect. 3, we establish a general criterion from the strong Feller property of resolvents to that for semigroups in terms of the analyticity of the semigroup on L^p-spaces. In Sect. 4, we apply the result in the framework of lower bounded semi-Dirichlet forms. We also provide some examples to clarify when our results are effective.

Notation. The following symbols and conventions are used in the paper.

- For $p \in [1, +\infty]$ and a measure space $(E, \mathscr{E}, \mu)$, we denote by $L^p(E;\ \mu)$ the L^p-space on it. For $f \in L^p(E;\ \mu)$, we set $\|f\|_{L^p(E;\ \mu)} = \{\int_E |f(x)|^p\, \mu(\mathrm{d}x)\}^{1/p}$. For $p, q \in [1, +\infty]$ and a bounded linear operator T from $L^p(E;\ \mu)$ to $L^q(E;\ \mu)$, we denote by $\|T\|_{L^p(E;\ \mu)\to L^q(E;\ \mu)}$ the operator norm. If $(E, \mathscr{E}, \mu)$ is clear from the context, we simply write $\|T\|_{p\to q}$ in stead of $\|T\|_{L^p(E;\ \mu)\to L^q(E;\ \mu)}$.
- For a topological space E, we denote by $\mathscr{B}(E)$ the Borel σ-algebra on E. We set

$$\begin{aligned}
C(E) &:= \{u \mid u \text{ is a real valued continuous function on } E\},\\
C_0(E) &:= \{u \in C(E) \mid \text{the closure of } u^{-1}(\mathbb{R} \setminus \{0\}) \text{ in } E \text{ is compact}\},\\
\mathscr{B}(E) &:= \{u \mid u \text{ is a Borel measurable } [-\infty, +\infty]\text{-valued function on } E\},\\
\mathscr{B}_+(E) &:= \{u \in \mathscr{B}(E) \mid u \text{ is } [0, +\infty]\text{-valued}\},\\
\mathscr{B}_b(E) &:= \{u \in \mathscr{B}(E) \mid \|u\|_\infty < \infty\}.
\end{aligned}$$

  Hereafter, $\|u\|_\infty := \sup_{x\in E} |u(x)|$ for $u\colon E \to \mathbb{R}$. When E is a locally compact separable metric space, we denote by $C_\infty(E)$ the completion of $C_0(E)$ under $\|\cdot\|_\infty$.
- We denote by $\mathrm{i} := \sqrt{-1}$ the imaginary unit. The real and imaginary parts of $z \in \mathbb{C}$ are denoted by $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$, respectively.
- We set $\inf \emptyset = \infty$.
- For $a, b \in \mathbb{R}$, we write $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$.

2 Preliminaries

Let (E, d) be a locally compact separable metric space, and let $E_\partial = E \cup \{\partial\}$ be the one-point compactification. Let $\mathbf{X} = (\{X_t\}_{t\in[0,+\infty]}, \{\mathbf{P}_x\}_{x\in E_\partial})$ be a Hunt process on E. That is, $\mathbf{X}$ is a right continuous process on E with strong Markov property and satisfies the right continuity of sample paths on $[0, +\infty)$ and the existence of left limits in E_∂ of sample paths on $(0, +\infty)$ (see [7, Definition A.1.23]). Define the transition semigroup of $\mathbf{X}$ by

$$P_t f(x) := \mathbf{E}_x[f(X_t)], \quad x \in E;\ t \geq 0,\ f \in \mathscr{B}_b(E),$$

where $\mathbf{E}_x$ denotes the expectation under $\mathbf{P}_x$. The resolvent of $\mathbf{X}$ is defined by

$$R_\alpha f(x) = \int_0^\infty e^{-\alpha t} P_t f(x)\,\mathrm{d}t, \quad x \in E;\ f \in \mathscr{B}_b(E),\ \alpha > 0.$$

We first formulate the strong Feller property of semigroups and resolvents.

Definition 1 (a) The semigroup $\{P_t\}_{t\geq 0}$ is said to have the strong Feller property if for any $f \in \mathscr{B}_b(E)$ and $t > 0$, $P_t f$ is bounded continuous on E.
(b) The resolvent $\{R_\alpha\}_{\alpha>0}$ is said to have the strong Feller property if for any $f \in \mathscr{B}_b(E)$ and $\alpha > 0$, $R_\alpha f$ is bounded continuous on E.

It is easy to see that the semigroup strong Feller property implies the resolvent strong Feller property. Next, we introduce the C_b-Feller property, which is a weaker concept of the strong Feller property.

Definition 2 (a) The semigroup $\{P_t\}_{t\geq 0}$ is said to have the C_b-Feller property if for any $f \in C_b(E)$ and $t > 0$, $P_t f$ is bounded continuous on E.
(b) The resolvent $\{R_\alpha\}_{\alpha>0}$ is said to have the C_b-Feller property if for any $f \in C_b(E)$ and $\alpha > 0$, $R_\alpha f$ is bounded continuous on E.

The Feller properties of semigroups and resolvents are defined as follows.

Definition 3 (a) The semigroup $\{P_t\}_{t\geq 0}$ is said to have the *Feller property* if for any $f \in C_\infty(E)$ and $t > 0$, $P_t f$ belongs to $C_\infty(E)$, and $\lim_{t\to 0} \|P_t f - f\|_\infty = 0$.
(b) The resolvent $\{R_\alpha\}_{\alpha>0}$ is said to have the *Feller property* if for any $f \in C_\infty(E)$ and $\alpha > 0$, $R_\alpha f$ belongs to $C_\infty(E)$, and $\lim_{\alpha\to\infty} \|\alpha R_\alpha f - f\|_\infty = 0$.

By the same argument as in [19, Sect. 1], we see that $\{P_t\}_{t\geq 0}$ possesses the Feller property if and only if so does $\{R_\alpha\}_{\alpha>0}$. We refer the reader to [1, Proposition 3.1] or [32, Proposition 3.1] for probabilistic characterizations of the Feller property under the (resolvent) strong Feller property.

Hereafter, we consider the following two assumptions:

(A1): There exists a positive Radon measure $\mathfrak{m}$ on E with full support such that

$$\int_E P_t f(x)\,\mathfrak{m}(\mathrm{d}x) \leq \int_E f(x)\,\mathfrak{m}(\mathrm{d}x), \quad t \geq 0,\ f \in \mathscr{B}_+(E). \tag{1}$$

(A2): There exists a lower bounded regular semi-Dirichlet form on $L^2(E;\ \mathfrak{m})$ associated with **X**, where $\mathfrak{m}$ is a positive Radon measure having full topological support.

Let $(\mathscr{E}, \mathscr{F})$ be a lower bounded semi-Dirichlet form on $L^2(E;\ \mathfrak{m})$. Here $(\mathscr{E}, \mathscr{F})$ is said to be a *lower bounded semi-Dirichlet form on* $L^2(E;\ \mathfrak{m})$ if $\mathscr{F}$ is a dense linear subspace of $L^2(E;\ \mathfrak{m})$ and $\mathscr{E} : \mathscr{F} \times \mathscr{F} \to \mathbb{R}$ is a closed bilinear form in the following sense $(\mathscr{E}.1)$, $(\mathscr{E}.2)$ and $(\mathscr{E}.3)$, and satisfies the semi-Dirichlet property $(\mathscr{E}.4)$:

$(\mathscr{E}.1)$: There exists a non-negative constant α_0 such that

$$\mathscr{E}_{\alpha_0}(u, u) := \mathscr{E}(u, u) + \alpha_0(u, u)_{L^2(E;\ \mathfrak{m})} \geq 0 \quad \text{for all} \quad u \in \mathscr{F}.$$

$(\mathscr{E}.2)$: $\mathscr{E}$ satisfies the (strong) *sector condition*: there exists a constant $K \geq 1$ such that

$$|\mathscr{E}(u, v)| \leq K \mathscr{E}_{\alpha_0}(u, u)^{1/2} \mathscr{E}_{\alpha_0}(v, v)^{1/2} \quad \text{for all} \quad u, v \in \mathscr{F},$$

where α_0 is the non-negative constant specified in $(\mathscr{E}.1)$.
$(\mathscr{E}.3)$: $\mathscr{F}$ is a Hilbert space relative to the inner product

$$\mathscr{E}^{(s)}_{\alpha}(u, v) := \frac{1}{2}\left(\mathscr{E}_{\alpha}(u, v) + \mathscr{E}_{\alpha}(v, u)\right) \quad \text{for all} \quad \alpha > \alpha_0,$$

where α_0 is the non-negative constant specified in $(\mathscr{E}.1)$.
$(\mathscr{E}.4)$: for all $u \in \mathscr{F}$ and $a \geq 0$, $u \wedge a \in \mathscr{F}$ and $\mathscr{E}(u \wedge a, u - u \wedge a) \geq 0$.

Under $(\mathscr{E}.2)$, we can deduce the following (weak) *sector condition*: For $\alpha > \alpha_0$,

$$|\mathscr{E}_{\alpha}(u, v)| \leq K \mathscr{E}_{\alpha}(u, u)^{1/2} \mathscr{E}_{\alpha}(v, v)^{1/2} \quad \text{for all} \quad u, v \in \mathscr{F}, \tag{2}$$

where $K(\geq 1)$ is the constant appeared in $(\mathscr{E}.2)$. Remark that (2) is a stronger form of the weak sector condition stated in [26, § 1.1, (1.1.3)].

Under $(\mathscr{E}.1)$, $(\mathscr{E}.2)$ and $(\mathscr{E}.3)$, we see from [22, Chapter I, Theorems 1.12 and 2.8] that $(\mathscr{E}, \mathscr{F})$ admits strongly continuous semigroups $\{T_t\}_{t\geq 0}$ and $\{T_t^*\}_{t\geq 0}$ on $L^2(E;\ \mathfrak{m})$ such that $\|T_t\|_{L^2(E;\ \mathfrak{m})\to L^2(E;\ \mathfrak{m})} \leq e^{\alpha_0 t}$, $\|T_t^*\|_{L^2(E;\ \mathfrak{m})\to L^2(E;\ \mathfrak{m})} \leq e^{\alpha_0 t}$,

$$(T_t f, g)_{L^2(E;\ \mathfrak{m})} = (f, T_t^* g)_{L^2(E;\ \mathfrak{m})}.$$

Hereafter, $(\cdot, \cdot)_{L^2(E;\ \mathfrak{m})}$ denotes the L^2 inner product with respect to $\mathfrak{m}$. That is, $\{T_t^*\}_{t\geq 0}$ is the dual semigroup of $\{T_t\}_{t\geq 0}$. For $\alpha > \alpha_0$ and $f \in L^2(E;\ \mathfrak{m})$, we define $G_\alpha f = \int_0^\infty e^{-\alpha t} T_t f \,\mathrm{d}t$ and $G_\alpha^* f = \int_0^\infty e^{-\alpha t} T_t^* f \,\mathrm{d}t$, the integrals being defined as the Bochner integral in $L^2(E;\ \mathfrak{m})$. It then follows from [22, Chapter I, Proposition 1.10 and Theorem 2.13] that

$$\mathscr{E}_{\alpha}(G_\alpha f, u) = (f, u)_{L^2(E;\ \mathfrak{m})} = \mathscr{E}_{\alpha}(u, G_\alpha^* g),$$

for all $f \in L^2(E;\mathfrak{m})$, $u \in \mathscr{F}$, and $\alpha > \alpha_0$. The resolvents $\{G_\alpha\}_{\alpha>\alpha_0}$ and $\{G^*_\alpha\}_{\alpha>\alpha_0}$ are strongly continuous on $L^2(E;\mathfrak{m})$ in the sense that $\lim_{\alpha\to\infty}\alpha G_\alpha f = \lim_{\alpha\to\infty}\alpha G^*_\alpha f = f$ in $L^2(E;\mathfrak{m})$.

The condition $(\mathscr{E}.4)$ is equivalent to the following conditions $(\mathscr{E}.4c)$, or $(\mathscr{E}.4d)$:

$(\mathscr{E}.4c)$: $\{T_t\}_{t\geq 0}$ is sub-Markov: If $f \in L^2(E;\mathfrak{m})$ satisfies $0 \leq f \leq 1$ $\mathfrak{m}$-a.e., then $0 \leq T_t f \leq 1$ $\mathfrak{m}$-a.e.
$(\mathscr{E}.4d)$: $\{T^*_t\}_{t\geq 0}$ is positivity preserving and contractive in $L^1(E;\mathfrak{m})$: If $f \in L^1(E;\mathfrak{m})$ satisfies $f \geq 0$ $\mathfrak{m}$-a.e., then $T^*_t f \geq 0$ $\mathfrak{m}$-a.e. and $\|T^*_t f\|_{L^1(E;\mathfrak{m})} \leq \|f\|_{L^1(E;\mathfrak{m})}$.

Under $(\mathscr{E}.1)$, $(\mathscr{E}.2)$, $(\mathscr{E}.3)$ and $(\mathscr{E}.4)$, $\{T_t\}_{t\geq 0}$ and $\{T^*_t\}_{t\geq 0}$ are positivity preserving in the sense that for $f \in L^2(E;\mathfrak{m})$, $f \geq 0$ $\mathfrak{m}$-a.e. implies $T_t f \geq 0$ $\mathfrak{m}$-a.e. and $T^*_t f \geq 0$ $\mathfrak{m}$-a.e. (see [23, Remark 1.4 (i), (iii) and Theorem 1.5)]). Under $(\mathscr{E}.4)$, $\{T_t\}_{t\geq 0}$ (resp. $\{G_\alpha\}_{\alpha>0}$) can be extended to a bounded linear operator on $L^\infty(E;\mathfrak{m})$ for $t > 0$ (resp. $\alpha > 0$) (see [26, p. 8]. As shown in [26, p. 20], for $t > 0$, $\alpha > 0$, and $f \in L^0_+(E;\mathfrak{m})$, we define $T_t f$ and $G_\alpha f$ (resp. $T^*_t f$ and $G^*_\alpha f$) by $T_t f = \lim_{n\to\infty} T_t(f \wedge nh)$ and $G_\alpha f = \lim_{n\to\infty} G_\alpha(f \wedge nh)$ (resp. $T^*_t f = \lim_{n\to\infty} T^*_t(f \wedge nh)$ and $G^*_\alpha f = \lim_{n\to\infty} G^*_\alpha(f \wedge nh)$). Here, $L^0_+(E;\mathfrak{m})$ denotes the family of all non-negative $\mathfrak{m}$-measurable functions and $h \in L^\infty(E;\mathfrak{m}) \cap L^1(E;\mathfrak{m})$ is a strictly positive function. Then, we have the following generalized duality relation: for non-negative measurable functions f, g,

$$\int_E T_t f\, g\, d\mathfrak{m} = \int_E f\, T^*_t g\, d\mathfrak{m}, \quad \int_E G_\alpha f\, g\, d\mathfrak{m} = \int_E f\, G^*_\alpha g\, d\mathfrak{m}$$

for $t > 0$ and $\alpha > 0$.

The lower bounded semi-Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $L^2(E;\mathfrak{m})$ is said to be *regular* if $\mathscr{F} \cap C_0(E)$ is $\mathscr{E}^{1/2}_{\alpha_0+1}$-dense in $\mathscr{F}$ and $\mathscr{F} \cap C_0(E)$ is uniformly dense in $C_0(E)$. Under the regularity of $(\mathscr{E}, \mathscr{F})$, there exists a Hunt process $\mathbf{X}$ associated with $(\mathscr{E}, \mathscr{F})$ in the sense that for $u \in L^\infty(E;\mathfrak{m}) \cap \mathscr{B}(E)$, $G_\alpha u = R_\alpha u$ $\mathfrak{m}$-a.e. for each $\alpha > 0$ (see [26, §3.3, Theorem 3.3.4]). Moreover, we can prove that the semigroup $P_t u$ with $u \in L^2(E;\mathfrak{m}) \cap \mathscr{B}(E)$ is a quasi-continuous $\mathfrak{m}$-version of $T_t u$ for $t > 0$ by the same way of the proof of [22, Chapter IV, Proposition 2.8] with the help of [26, §2.2, Theorem 2.2.5]. Then, we obtain the next proposition.

Proposition 1 *(a) Suppose that* (**A1**) *is satisfied. For any* $p \in [1, +\infty)$*, the semigroup* $\{P_t\}_{t\geq 0}$ *of* $\mathbf{X}$ *is extended to a strongly continuous contraction semigroup* $\{T_t\}_{t\geq 0}$ *on* $L^p(E;\mathfrak{m})$*.*
(b) Suppose that (**A2**) *is satisfied and let* $-\alpha_0$ *be the lower bound of* $(\mathscr{E}, \mathscr{F})$*. Then, for any* $p \in [2, +\infty)$*, the semigroup* $\{e^{-\alpha_0(2/p)t} P_t\}_{t\geq 0}$ *from* P_t *is extended to a strongly continuous contraction semigroup* $\{e^{-\alpha_0(2/p)t} T_t\}_{t\geq 0}$ *on* $L^p(E;\mathfrak{m})$*. In particular,* $\{T_t\}_{t\geq 0}$ *is strongly continuous on* $L^p(E;\mathfrak{m})$*.*

Proof We first prove (a). Suppose that **(A1)** is satisfied. Jensen's inequality and (1) imply that for $f \in C_0(E)$ and $t \in [0, +\infty)$

$$\|P_t f\|^p_{L^p(E;\,\mathfrak{m})} = \int_E |\mathbf{E}_x[f(X_t)]|^p\,\mathfrak{m}(\mathrm{d}x) \le \int_E \mathbf{E}_x[|f(X_t)|^p]\,\mathfrak{m}(\mathrm{d}x)$$
$$\le \int_E P_t(|f|^p)(x)\,\mathfrak{m}(\mathrm{d}x) \le \int_E |f(x)|^p\,\mathfrak{m}(\mathrm{d}x) \le \|f\|^p_{L^p(E;\,\mathfrak{m})}. \quad (3)$$

Note that $C_0(E)$ is dense in $L^p(E;\ \mathfrak{m})$. Then, (3) implies that $\{P_t\}_{t\ge 0}$ is extended to a contraction semigroup on $L^p(E;\ \mathfrak{m})$, which is denoted by $\{T_t\}_{t\ge 0}$.

Let $f \in L^p(E;\ \mathfrak{m})$ given. Then, for any $\varepsilon > 0$, there exists $g \in C_0(E)$ such that $\|f - g\|_{L^p(E;\,\mathfrak{m})} < \varepsilon/2$. By using (3), we obtain that for any $t > 0$,

$$\|T_t f - f\|_{L^p(E;\,\mathfrak{m})} \le \|T_t f - T_t g\|_{L^p(E;\,\mathfrak{m})} + \|T_t g - g\|_{L^p(E;\,\mathfrak{m})} + \|g - f\|_{L^p(E;\,\mathfrak{m})}$$
$$\le \varepsilon + \|T_t g - g\|_{L^p(E;\,\mathfrak{m})}. \quad (4)$$

It is easy to see that for any $f \in L^p(E;\ \mathfrak{m}) \cap \mathscr{B}_b(E)$, $T_t f(x) = P_t f(x)$ for $\mathfrak{m}$-a.e. $x \in E$. Then, by using the sample path (right-)continuity of $\mathbf{X}$, we have

$$\overline{\lim_{t\downarrow 0}}\, \|T_t g - g\|_{L^p(E;\,\mathfrak{m})} = \overline{\lim_{t\downarrow 0}}\, \|P_t g - g\|_{L^p(E;\,\mathfrak{m})} = 0.$$

Therefore, (4) implies $\overline{\lim}_{t\downarrow 0} \|T_t f - f\|_{L^p(E;\,\mathfrak{m})} \le \varepsilon$. Since $\varepsilon > 0$ is arbitrarily chosen, we complete the proof.

Next we prove (b). Suppose **(A2)** is satisfied. Since $\{T_t^*\}_{t\ge 0}$ is $L^1(E;\ \mathfrak{m})$-contractive and $\{e^{-\alpha_0 t}T_t^*\}_{t\ge 0}$ is $L^2(E;\ \mathfrak{m})$-contractive, we get $\{e^{-\alpha_0(2(q-1)/q)t}T_t^*\}_{t\ge 0}$ is $L^q(E;\ \mathfrak{m})$-contractive for $q \in [1, 2]$ in view of the Riesz-Thorin interpolation theorem (see [10, 1.1.5]). Take a relatively compact open set G. For $f \in C_0(E)$ and $p \in [2, +\infty)$, we see $P_t f \in L^p(G;\ \mathfrak{m})$:

$$\|P_t f\|^p_{L^p(G;\,\mathfrak{m})} \le \int_G P_t|f|^p\,\mathrm{d}\mathfrak{m} = \int_E T_t^*\mathbf{1}_G(x)|f(x)|^p\,\mathfrak{m}(\mathrm{d}x)$$
$$\le \|f\|^p_\infty \int_E T_t^*\mathbf{1}_G(x)\,\mathfrak{m}(\mathrm{d}x) = \|f\|^p_\infty \int_E T_t\mathbf{1}_E(x)\mathbf{1}_G(x)\,\mathfrak{m}(\mathrm{d}x) \quad (5)$$
$$\le \|f\|^p_\infty \mathfrak{m}(G) < \infty.$$

Let $q = p/(p-1) \in (1, 2)$. Since $\{e^{-\alpha_0(2(q-1)/q)t}T_t^*\}_{t\ge 0}$ is $L^q(E;\ \mathfrak{m})$-contractive and $\{T_t^*\}_{t\ge 0}$ is positivity preserving, we obtain that for any $t > 0$,

$$\int_G |P_t f(x)|^p\,\mathfrak{m}(\mathrm{d}x) = \sup_{g\in L^q(G;\,\mathfrak{m}),\|g\|_{L^q(G;\,\mathfrak{m})}=1} \left|\int_G P_t f(x)g(x)\,\mathfrak{m}(\mathrm{d}x)\right|$$
$$= \sup_{g\in L^q(G;\,\mathfrak{m}),\|g\|_{L^q(G;\,\mathfrak{m})}=1} \left|\int_E f(x)T_t^*(\mathbf{1}_G g)(x)\,\mathfrak{m}(\mathrm{d}x)\right|$$

$$
\begin{aligned}
&\leq \sup_{g\in L^q(E;\,\mathfrak{m}),\|g\|_{L^q(E;\,\mathfrak{m})}=1} \int_E \left|f(x)T_t^*g(x)\right| \mathfrak{m}(\mathrm{d}x) \\
&\leq \sup_{g\in L^q(E;\,\mathfrak{m}),\|g\|_{L^q(E;\,\mathfrak{m})}=1} \|f\|_{L^p(E;\,\mathfrak{m})} T_t^* \|g\|_{L^q(E;\,\mathfrak{m})} \\
&\leq e^{\alpha_0(2(q-1)/q)t} \sup_{g\in L^q(E;\,\mathfrak{m}),\|g\|_{L^q(E;\,\mathfrak{m})}=1} \|f\|_{L^p(E;\,\mathfrak{m})}\|g\|_{L^q(E;\,\mathfrak{m})} \\
&= e^{\alpha_0(2/p)t}\|f\|_{L^p(E;\,\mathfrak{m})}.
\end{aligned}
$$

Since G is arbitrary, $\{e^{-\alpha_0(2/p)t}P_t\}_{t\geq 0}$ is extended to a contraction semigroup on $L^p(E;\,\mathfrak{m})$, which is denoted by $\{e^{-\alpha_0(2/p)t}T_t\}_{t\geq 0}$. Let $f\in L^p(E;\,\mathfrak{m})$ given. Then, for any $\varepsilon>0$, there exists $g\in C_0(E)$ such that $\|f-g\|_{L^p(E;\,\mathfrak{m})}<\varepsilon/2$. By using (5), we obtain that for any $t>0$,

$$
\begin{aligned}
&\|T_tf-f\|_{L^p(E;\,\mathfrak{m})} \\
&\quad \leq \|T_tf-T_tg\|_{L^p(E;\,\mathfrak{m})}+\|T_tg-g\|_{L^p(E;\,\mathfrak{m})}+\|g-f\|_{L^p(E;\,\mathfrak{m})} \\
&\quad \leq \varepsilon(1+e^{\alpha_0(2/p)t})+\|T_tg-g\|_{L^p(E;\,\mathfrak{m})}.
\end{aligned}
$$

The rest of the proof is similar to that of (a).

Remark 1 In the proof of Proposition 1, the sample path right-continuity of $\mathbf{X}$ is used. However, the right-continuity of $\{P_t\}_{t\geq 0}$ on $C_0(E)$ and the fact that $C_0(E)$ is a dense subset of $L^p(E;\mathfrak{m})$ play an essential role.

In the sequel, we fix $p\in(1,+\infty)$ under (**A1**) with $\alpha_0:=0$ (resp. $p\in[2,+\infty)$ under (**A2**)), and let $\{T_t\}_{t\geq 0}$ be the strongly continuous contraction semigroup as in Proposition 1. For $f\in L^p(E;\,\mathfrak{m})$ and $\alpha\in(2\alpha_0/p,+\infty)$, we put

$$
G_\alpha f=\int_0^\infty e^{-\alpha t}T_tf\,\mathrm{d}t,
$$

the integral being defined as the Bochner integral in $L^p(E;\,\mathfrak{m})$. Then, $\{G_\alpha\}_{\alpha>2\alpha_0/p}$ becomes a strongly continuous contraction resolvent on $L^p(E;\,\mathfrak{m})$ in the following sense that $\lim_{\alpha\to\infty}\alpha G_\alpha f=f$ for $f\in L^p(E;\,\mathfrak{m})$; $(\alpha-2\alpha_0/p)\|G_\alpha f\|_{L^p(E;\,\mathfrak{m})}\leq \|f\|_{L^p(E;\,\mathfrak{m})}$ for $f\in L^p(E;\,\mathfrak{m})$ and $\alpha>2\alpha_0/p$; $G_\alpha-G_\beta+(\alpha-\beta)G_\alpha G_\beta=0$ for all $\alpha,\beta>2\alpha_0/p$. Moreover, we have $R_\alpha f=G_\alpha f$ $\mathfrak{m}$-a.e. on E for any $f\in L^p(E;\,\mathfrak{m})\cap\mathscr{B}_b(E)$ and $\alpha\in(2\alpha_0/p,+\infty)$. The generator $(A,\mathrm{Dom}(A))$ of $\{T_t\}_{t\geq 0}$ is defined by

$$
\begin{aligned}
\mathrm{Dom}(A) &:= \left\{ f\in L^p(E;\,\mathfrak{m}) \;\middle|\; \lim_{t\to 0}(T_tf-f)/t \text{ exists in } L^p(E;\,\mathfrak{m}) \right\}, \\
Af &:= \lim_{t\to 0}\frac{T_tf-f}{t}, \quad f\in\mathrm{Dom}(A).
\end{aligned}
$$

It is known that $\mathrm{Dom}(A)$ is equal to $G_\alpha(L^p(E;\ \mathfrak{m}))$ for some/any $\alpha \in (2\alpha_0/p, +\infty)$. See [22, Chapter I, Proposition 1.5] for details.

Next, we define the transition kernel of $\mathbf{X}$. For $t > 0$, $x \in E$, and $B \in \mathscr{B}(E)$, we set

$$P_t(x, B) := \mathbf{E}_x[\mathbf{1}_B(X_t)] = \mathbf{P}_x(X_t \in B).$$

The resolvent kernel of $\mathbf{X}$ is defined as

$$R_\alpha(x, B) = \int_0^\infty e^{-\alpha t} P_t(x, B)\,\mathrm{d}t, \quad \alpha > 0,\ x \in E \text{ , and } B \in \mathscr{B}(E).$$

The transition kernel (resp. the resolvent kernel) is often said to be *absolutely continuous with respect to* $\mathfrak{m}$ if $P_t(x, \cdot)$ (resp. $R_\alpha(x, \cdot)$) is absolutely continuous with respect to $\mathfrak{m}$ for any $x \in E$ and $t > 0$ (resp. $\alpha > 0$).

Lemma 1 *(a) If $\{P_t\}_{t\geq 0}$ has the strong Feller property, then $P_t(x, \cdot)$ is absolutely continuous with respect to $\mathfrak{m}$ for all $x \in E$ and $t > 0$.*
(b) If $\{R_\alpha\}_{\alpha>0}$ has the strong Feller property, then $R_\alpha(x, \cdot)$ is absolutely continuous with respect to $\mathfrak{m}$ for all $x \in E$ and $\alpha > 0$.

Proof We prove (a). Fix $t > 0$ and $B \in \mathscr{B}(E)$ with $\mathfrak{m}(B) = 0$. Then, $\mathbf{1}_B \in L^2(E;\ \mathfrak{m})$ and for any $g \in L^2(E;\ \mathfrak{m})$,

$$\int_E g(x) T_t \mathbf{1}_B(x)\,\mathfrak{m}(\mathrm{d}x) = \int_B T_t^* g(x)\,\mathfrak{m}(\mathrm{d}x) = 0.$$

This implies that $T_t \mathbf{1}_B(x) = 0$ for $\mathfrak{m}$-a.e. $x \in E$, and hence $P_t \mathbf{1}_B(x) = 0$ for $\mathfrak{m}$-a.e. $x \in E$. On the other hand, by the strong Feller property of $\{P_t\}_{t\geq 0}$, $P_t \mathbf{1}_B$ is continuous on E. Thus, $P_t \mathbf{1}_B(x) = 0$ for any $x \in E$, and it yields the absolute continuity. The proof of (b) is almost same. So, we omit it.

In what follows, if no confusion will arise, the complexifications of subspaces of any real Banach space and linear operators on them are denoted by the same symbols. So any functions in $L^p(E;\ \mathfrak{m})$ are regarded as complex valued if there is no special remark. With this notation, we give the definitions of analytic semigroups and sectorial operators. For $\theta \in (0, \pi)$, we define

$$S_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| < \theta\}.$$

Definition 4 ([28, Definition 2.5.1]) The semigroup $\{T_t\}_{t\geq 0}$ on $L^p(E;\ \mathfrak{m})$ is said to be *analytic* if there exists $\delta \in (0, \pi/2]$ such that $\{T_t\}_{t\geq 0}$ is extended to a family of operators $\{T_z\}_{z\in S_\delta \cup \{0\}}$ on $L^p(E;\ \mathfrak{m})$ with the following properties:

(a) $T_{z_1+z_2} = T_{z_1} T_{z_2}$ for $z_1, z_2 \in S_\delta$,
(b) $z \mapsto T_z$ is analytic on S_δ,

(c) $\lim_{z\in S_\varepsilon,\, z\to 0} T_z f = f$ for any $f \in L^p(E;\, \mathfrak{m})$ and $\varepsilon \in (0, \delta)$.

The resolvent set and the resolvent operator of the generator $(A, \mathrm{Dom}(A))$ are denoted by $\rho(A)$ and $\{R(z\,;\,A)\}_{z\in\rho(A)}$, respectively. From the definitions, it is obvious that $\rho(A)$ includes the positive half line $(0, +\infty)$.

Definition 5 ([28, (5.1), (5.2)], [21, Definition 2.0.1]) The generator $(A, \mathrm{Dom}(A))$ on $L^p(E;\, \mathfrak{m})$ is said to be *sectorial* if there exist constants $\theta \in (\pi/2, \pi)$ and $M \in (0, +\infty)$ such that $\rho(A) \supset S_\theta$ and

$$\|zR(z\,;\,A)f\|_{L^p(E;\,\mathfrak{m})} \le M\|f\|_{L^p(E;\,\mathfrak{m})}, \quad z \in S_\theta,\ \ f \in L^p(E;\, \mathfrak{m}). \tag{6}$$

We often say that $(A, \mathrm{Dom}(A))$ is a *sectorial operator on* $L^p(E;\, \mathfrak{m})$ *with constants* $\theta \in (\pi/2, \pi)$ *and* $M \in (0, +\infty)$ if it satisfies (6).

The next proposition is a special case of [28, Theorems 1.7.7 and 2.5.2]

Proposition 2 *Let $\{T_t\}_{t\ge 0}$ be the strongly continuous semigroup on $L^p(E;\, \mathfrak{m})$ constructed in Proposition 1. Then, $\{T_t\}_{t\ge 0}$ is analytic if and only if its generator A is sectorial. In this case, $\{T_t\}_{t\ge 0}$ is expressed by the Dunford integral as*

$$T_t = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\theta,\varepsilon}} e^{zt} R(z\,;\,A)\, \mathrm{d}z, \quad t \in (0, +\infty),$$

where θ is the constant as in Definition 5, and

$$\begin{aligned}\Gamma_{\theta,\varepsilon} := &\{\lambda \in \mathbb{C} \setminus \{0\} \mid |\lambda| \ge \varepsilon, |\arg\lambda| = \theta - \varepsilon\}\\ &\cup \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\lambda| = \varepsilon, |\arg\lambda| \le \theta - \varepsilon\}, \quad \varepsilon \in (0, \theta - \pi/2).\end{aligned}$$

Theorem 1 *Let $\{T_t\}_{t\ge 0}$ be the strongly continuous semigroup constructed in Proposition 1. Then we have the following:*

(a) Suppose that (**A1**) *holds and $\{T_t\}_{t\ge 0}$ is analytic on $L^2(E;\, \mathfrak{m})$ with the sector S_δ for some $\delta \in (0, \pi/2]$. Then $\{T_t\}_{t\ge 0}$ is analytic on $L^p(E;\, \mathfrak{m})$ for $p \in (1, +\infty)$ with the sector $S_{\delta'}$, where $\delta' := \delta\,(1 - |(2/p) - 1|)$.*

(b) Suppose that (**A2**) *holds. Then $\{T_t\}_{t\ge 0}$ is analytic on $L^p(E;\, \mathfrak{m})$ for $p \in [2, +\infty)$ with the sector $S_{\delta'}$, where $\delta' := (2/p)\arctan K^{-1}$ and $K \ge 1$ is the constant specified in $(\mathscr{E}.2)$.*

Proof The proof can be done similarly as in the proof of [31, Chapter III, Theorem 1] by using the convexity theorem in [31, p. 69]. Combining the same proof of [22, Chapter I, Corollary 2.21] with (2), (**A2**) implies that $\{e^{-\alpha t}T_t\}_{t\ge 0}$ with $\alpha > \alpha_0$ is analytic on $L^2(E;\, \mathfrak{m})$ with sector $S_{\arctan K^{-1}}$.

Remark 2 (i) Let $\{T_t\}_{t\ge 0}$ be a strongly continuous symmetric semigroup on $L^2(E;\, \mathfrak{m})$. Then $\{T_t\}_{t\ge 0}$ is analytic on $L^2(E;\, \mathfrak{m})$ with the sector $S_{\pi/2}$ as proved in [31, Chapter III, Theorem 1].

(ii) Let $\{T_t\}_{t\geq 0}$ be a strongly continuous contraction semigroup on $L^2(E;\,\mathfrak{m})$ associated with a coercive closed form $(\mathscr{E},\mathscr{F})$ on $L^2(E;\,\mathfrak{m})$ in the sense of [22, Chapter I, Definition 2.4]. Then $\{e^{-t}T_t\}_{t\geq 0}$ (equivalently, $\{T_t\}_{t\geq 0}$) is analytic on $L^2(E;\,\mathfrak{m})$ with the sector S_δ for $\delta := \arctan K^{-1}$, where $K\geq 1$ is the constant appeared in the weak sector condition for $(\mathscr{E},\mathscr{F})$ (see [22, Chapter I, (2.3) and Corollary 2.21], where $K>0$ is only noted, but indeed, $K\geq 1$ holds automatically). If further $(\mathscr{E},\mathscr{F})$ satisfies the strong sector condition with $K\geq 1$ (see [22, Chapter I, (2.4)] for strong sector condition), then $\{T_t\}_{t\geq 0}$ is analytic on $L^2(E;\,\mathfrak{m})$ with the sector S_δ having the same expression as above. This assertion is not so sharp when $\{T_t\}_{t\geq 0}$ is $\mathfrak{m}$-symmetric, in this case $K=1$ so that $\delta=\pi/4$.

(iii) Let $\{T_t\}_{t\geq 0}$ be a strongly continuous contraction semigroup on $L^2(E;\,\mathfrak{m})$ associated with a (non-negative definite) non-symmetric Dirichlet form $(\mathscr{E},\mathscr{F})$ on $L^2(E;\,\mathfrak{m})$ (see [22, Chapter I, Definition 4.5] for non-symmetric Dirichlet form). Then, for $p\in(1,+\infty)$, one can prove that $\{e^{-t}T_t\}_{t\geq 0}$ (equivalently, $\{T_t\}_{t\geq 0}$) is analytic on $L^p(E;\,\mathfrak{m})$ with the sector $S_{\delta'}$ in the same way of the proof of Theorem 1, where $\delta' := (\arctan K^{-1})\,(1-|(2/p)-1|)$ for some $K\geq 1$ derived from the weak sector condition for $(\mathscr{E},\mathscr{F})$. If further $(\mathscr{E},\mathscr{F})$ satisfies the strong sector condition with $K\geq 1$, then $\{T_t\}_{t\geq 0}$ is analytic on $L^p(E;\,\mathfrak{m})$ with the sector $S_{\delta'}$ having the same expression as above.

3 Equivalence of the Strong Feller Properties

In this section, we use the same notation as in Sect. 2. We only consider a Hunt process $\mathbf{X}$ and assume that the semigroup $\{P_t\}_{t\geq 0}$ can be extended to a strongly continuous semigroup $\{T_t\}_{t\geq 0}$ on $L^p(E;\,\mathfrak{m})$ for some $p\in[1,+\infty)$. We fix such a $p\in[1,+\infty)$.

Proposition 3 *Fix $p\in[1,+\infty)$ as above and assume the following conditions are satisfied:*

(i) $(A, Dom(A))$ is a sectorial operator on $L^p(E;\,\mathfrak{m})$ with constants $\theta\in(\pi/2,\pi)$ and $M\in(0,+\infty)$.
(ii) There exists $\varepsilon\in(0,\theta-\pi/2)$ such that for any $z\in\Gamma_{\theta,\varepsilon}$ and $f\in C_b(E)\cap L^p(E;\,\mathfrak{m})$, $R(z\,;A)f$ possesses a bounded continuous $\mathfrak{m}$-version on E.
(iii) $\{R_\alpha\}_{\alpha>0}$ has the strong Feller property.

Then, for any $f\in L^p(E;\,\mathfrak{m})\cap\mathscr{B}_b(E)$, there exist a $\mathbb{C}$-valued Borel measurable function $\overline{Rf}$ on $\Gamma_{\theta,\varepsilon}\times E$ and an $\mathfrak{m}$-null set $N\subset E$ with the following properties:

(a) $\overline{Rf}(z,x)=R(z\,;A)f(x)$ for $|\mathrm{d}z|\otimes\mathfrak{m}$-almost every $(z,x)\in\Gamma_{\theta,\varepsilon}\times E$,
(b) $\overline{Rf}(\cdot,x)$ has a $\{(p-1)/p\}$-Hölder continuous version for any $x\in E$,
(c) for any compact subset $K\subset E$,

$$\lim_{j\to\infty}\sup_{x,y\in K\setminus N;\ d(x,y)<1/j}\left|\overline{Rf}(\cdot,x)-\overline{Rf}(\cdot,y)\right|=0,\quad |\mathrm{d}z|\text{-a.e. on }\Gamma_{\theta,\varepsilon}.$$

Proof Fix $f \in L^p(E;\, \mathfrak{m}) \cap \mathscr{B}_b(E)$. In view of the definitions of $\{R_\alpha\}_{\alpha>0}$ and $\{R(z\,;\,A)\}_{z\in\rho(A)}$,

$$R(1\,;\,A)f(x) = R_1 f(x), \quad \mathfrak{m}\text{-a.e. } x \in E. \tag{7}$$

The resolvent equation implies that

$$R(z\,;\,A) = R(1\,;\,A) + (1-z)R(z\,;\,A)R(1\,;\,A), \quad z \in \rho(A), \tag{8}$$

in the sense of bounded linear operators on $L^p(E;\, \mathfrak{m})$. Hence, by (7) and (8), we have for any $z \in \rho(A)$,

$$R(z\,;\,A)f(x) = R_1 f(x) + (1-z)R(z\,;\,A)R_1 f(x), \quad \mathfrak{m}\text{-a.e. } x \in E. \tag{9}$$

Then, the assumption and (9) imply that for any $z \in \Gamma_{\theta,\varepsilon}$, there exist $V_z f \in C_b(E)$ and an $\mathfrak{m}$-null set $N_1(z)$ (depending on z) such that

$$V_z f(x) = R(z\,;\,A)f(x), \quad x \in E \setminus N_1(z). \tag{10}$$

By [12, Theorem 1.3.2], the mappings $(z,x) \mapsto R(z\,;\,A)f(x)$ and $(z,x) \mapsto V_z f(x)$ possess Borel measurable versions on $\Gamma_{\theta,\varepsilon} \times E$. Then, we see from (10) that

$$\iint_{\Gamma_{\theta,\varepsilon}\times E} |V_z f(x) - R(z\,;\,A)f(x)|^p \, |\mathrm{d}z| \otimes \mathfrak{m}(\mathrm{d}x) = 0. \tag{11}$$

Hence, by Fubini's theorem, there exists an $\mathfrak{m}$-null set $N_1 \subset E$ such that for any $x \in E \setminus N_1$

$$V_z f(x) = R(z\,;\,A)f(x), \quad |\mathrm{d}z|\text{-a.e. } z \in \Gamma_{\theta,\varepsilon}. \tag{12}$$

On the other hand, we see from [28, (5.21)] that $(\mathrm{d}/\mathrm{d}z)R(z\,;\,A) = -R(z\,;\,A)^2$, $z \in \rho(A)$, in the sense of bounded linear operators on $L^p(E;\, \mathfrak{m})$. Therefore, we obtain that for any $z \in S_\theta$,

$$\left\| \frac{\mathrm{d}}{\mathrm{d}z} R(z\,;\,A)f \right\|_{L^p(E;\,\mathfrak{m})} \le \|R(z\,;\,A)\|^2_{L^p\to L^p} \|f\|_{L^p(E;\,\mathfrak{m})} \le M^2 |z|^{-2} \|f\|_{L^p(E;\,\mathfrak{m})}.$$

Hence, we obtain that

$$\int_E \left(\int_{\Gamma_{\theta,\varepsilon}} \left| \frac{\mathrm{d}}{\mathrm{d}z} R(z\,;A)f(x) \right|^p |\mathrm{d}z| \right) \mathfrak{m}(\mathrm{d}x) = \int_{\Gamma_{\theta,\varepsilon}} \left\| \frac{\mathrm{d}}{\mathrm{d}z} R(z\,;A)f \right\|^p_{L^p(E;\,\mathfrak{m})} |\mathrm{d}z|$$
$$\leq M^{2p} \|f\|^p_{L^p(E;\,\mathfrak{m})} \int_{\Gamma_{\theta,\varepsilon}} |z|^{-2p} |\mathrm{d}z| < \infty.$$

This estimate implies

$$R(\,\cdot\,;A)f(x) \in W^{1,p}(\Gamma_{\theta,\varepsilon};\mathbb{C}), \quad \mathfrak{m}\text{-a.e. } x \in E,$$

where we denote by $W^{1,p}(\Gamma_{\theta,\varepsilon};\mathbb{C})$ the $\mathbb{C}$-valued p-th order Sobolev space on $\Gamma_{\theta,\varepsilon}$. Then, by using the Sobolev embedding theorem and the Fubini's theorem, we obtain an $\mathfrak{m}$-null set $N_2 \subset E$ with the following property: for any $x \in E \setminus N_2$, there exists $W_\bullet f(x) \in C^{(p-1)/p}(\Gamma_{\theta,\varepsilon};\mathbb{C})$ such that

$$R(z\,;A)f(x) = W_z f(x), \quad |\mathrm{d}z|\text{-almost every } z \in \Gamma_{\theta,\varepsilon}. \tag{13}$$

Here, we denote by $C^{(p-1)/p}(\Gamma_{\theta,\varepsilon};\mathbb{C})$ the space of $\mathbb{C}$-valued $\{(p-1)/p\}$-Hölder continuous functions on $\Gamma_{\theta,\varepsilon}$.

Finally, we let $N := N_1 \cup N_2$, and define a $\mathbb{C}$-valued function $\overline{Rf}$ on $\Gamma_{\theta,\varepsilon} \times E$ by

$$\overline{Rf}(z,x) := \begin{cases} V_z f(x), & (z,x) \in \Gamma_{\theta,\varepsilon} \times (E \setminus N), \\ 0, & (z,x) \in \Gamma_{\theta,\varepsilon} \times N. \end{cases}$$

Then, properties (a) and (b) follow from (11)–(13). The property (c) is a consequence of the continuity of $V_z f(x)$ in $x \in E$.

We do not know whether the following local uniform estimates can be obtained only from the conditions in Proposition 3: for any $t \in (0,+\infty)$, $f \in L^p(E;\,\mathfrak{m}) \cap \mathscr{B}_b(E)$, and compact subsets $K \subset E$ and $L \subset \mathbb{C}$,

$$\left\| \int_{\Gamma_{\theta,\varepsilon}} e^{t\,\mathrm{Re}\,z} \left| \overline{Rf}(z,\cdot) \right| |\mathrm{d}z| \right\|_{L^\infty(K;\,\mathfrak{m})} < \infty, \tag{14}$$

$$\left\| \overline{Rf}(\cdot,\cdot) \right\|_{L^\infty((L \cap \Gamma_{\theta,\varepsilon}) \times K;\,|\mathrm{d}z| \otimes \mathfrak{m})} < \infty. \tag{15}$$

To confirm (15), we provide the following criterion as a proposition.

Proposition 4 *Fix* $p \in [1,+\infty)$ *as above, and assume that* $(A, Dom(A))$ *is a sectorial operator on* $L^p(E;\,\mathfrak{m})$ *with constants* $\theta \in (\pi/2,\pi)$ *and* $M \in (0,+\infty)$. *Let* $\varepsilon \in (0,\theta-\pi/2)$*, and assume in addition that* $L^p(E;\,\mathfrak{m}) \cap L^\infty(E;\mathfrak{m})$ *is invariant* *under $R(z;A)$ for any $z \in \Gamma_{\theta,\varepsilon}$. Then, for any $f \in L^p(E;\mathfrak{m}) \cap L^\infty(E;\mathfrak{m})$, the map $\Gamma_{\theta,\varepsilon} \ni z \mapsto \|R(z;A)f\|_{L^\infty(E;\mathfrak{m})}$ is continuous.*

Proof For $f \in L^p(E;\mathfrak{m}) \cap L^\infty(E;\mathfrak{m})$, we set $\|f\|_{p,\infty} = \|f\|_{L^p(E;\mathfrak{m})} \vee \|f\|_{L^\infty(E;\mathfrak{m})}$. Then, $L^p(E;\mathfrak{m}) \cap L^\infty(E;\mathfrak{m})$ is a Banach space under $\|\cdot\|_{p,\infty}$. We fix $\varepsilon \in (0, \theta - \pi/2)$ and $z \in \Gamma_{\theta,\varepsilon}$. By noting that $R(z;A)$ is a bounded linear operator on $L^p(E;\mathfrak{m})$ and makes invariant $L^p(E;\mathfrak{m}) \cap L^\infty(E;\mathfrak{m})$, we immediately find that $R(z;A)$ is a closed operator on $L^p(E;\mathfrak{m}) \cap L^\infty(E;\mathfrak{m})$. Hence, the closed graph theorem implies that there exists $C_z > 0$ such that for any $f \in L^p(E;\mathfrak{m}) \cap L^\infty(E;\mathfrak{m})$,

$$\|R(z;A)f\|_{p,\infty} \le C_z \|f\|_{p,\infty}.$$

In particular, we have that for $f \in L^p(E;\mathfrak{m}) \cap L^\infty(E;\mathfrak{m})$,

$$\|R(z;A)f\|_{p,\infty} \le C_z(\|f\|_{L^p(E;\mathfrak{m})} + \|f\|_{L^\infty(E;\mathfrak{m})}). \tag{16}$$

We fix $f \in L^p(E;\mathfrak{m}) \cap L^\infty(E;\mathfrak{m})$ and $z_0 \in \Gamma_{\theta,\varepsilon}$. From the resolvent equation and (16), we obtain that for any $z \in \Gamma_{\theta,\varepsilon}$,

$$\begin{aligned}\|R(z;A)f - R(z_0;A)f\|_{L^\infty(E;\mathfrak{m})} &\le |z - z_0| \|R(z_0;A)R(z;A)f\|_{L^\infty(E;\mathfrak{m})} \\ &\le C_{z_0}|z-z_0|(\|R(z;A)f\|_{L^p(E;\mathfrak{m})} + \|R(z;A)f\|_{L^\infty(E;\mathfrak{m})}) \\ &\le C_{z_0}\frac{M|z-z_0|}{|z|}\|f\|_{L^p(E;\mathfrak{m})} + C_{z_0}|z-z_0|\|R(z;A)f\|_{L^\infty(E;\mathfrak{m})}.\end{aligned} \tag{17}$$

In the last line, we use the fact that A is a sectorial operator. Therefore, for any $z \in \Gamma_{\theta,\varepsilon}$ with $C_{z_0}|z - z_0| < 1/2$,

$$\frac{1}{2}\|R(z;A)f\|_{L^\infty(E;\mathfrak{m})} \le \|R(z_0;A)f\|_{L^\infty(E;\mathfrak{m})} + C_{z_0}\frac{M|z-z_0|}{|z|}\|f\|_{L^p(E;\mathfrak{m})}. \tag{18}$$

By using (17) and (18), we have for any $z \in \Gamma_{\theta,\varepsilon}$ with $C_{z_0}|z - z_0| < 1/2$,

$$\begin{aligned}&\|R(z;A)f - R(z_0;A)f\|_{L^\infty(E;\mathfrak{m})} \\ &\quad \le C_{z_0}\frac{M|z-z_0|}{|z|}\|f\|_{L^p(E;\mathfrak{m})} + 2C_{z_0}|z-z_0||R(z_0;A)f\|_{L^\infty(E;\mathfrak{m})} \\ &\qquad + 2C_{z_0}^2\frac{M|z-z_0|^2}{|z|}\|f\|_{L^p(E;\mathfrak{m})}.\end{aligned}$$

This shows that the map $z \mapsto \|R(z;A)\|_{L^\infty(E;\mathfrak{m})}$ is continuous at z_0.

Theorem 2 *Fix $p \in [1, +\infty)$ as above, and assume the conditions in Proposition 3, (14) and (15). Assume in addition that the transition kernel of $\mathbf{X}$ is absolutely continuous with respect to $\mathfrak{m}$. Then, for any $t > 0$ and $f \in L^p(E;\mathfrak{m}) \cap \mathscr{B}_b(E)$, we have $P_t f \in C_b(E)$. In particular, $\{P_t\}_{t\ge 0}$ has the strong Feller property if $P_t \mathbf{1}_E \in C_b(E)$ for any $t > 0$.*

Proof For the moment, we fix $f \in \mathscr{B}_b(E) \cap L^p(E;\, \mathfrak{m})$ and $t > 0$. For $x \in E$, we define

$$\widehat{T}_t f(x) := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\theta,\varepsilon}} e^{zt} \overline{Rf}(z, x)\, \mathrm{d}z,$$

where $\overline{Rf}$ is the function obtained in Proposition 3. Note that the line integral is well-defined by Proposition 3 (b). Let K be a compact subset of E.

Then, by (14), there exists an $\mathfrak{m}$-null subset N_0 of E such that

$$\sup_{x,y\in K\setminus N_0} \int_{\Gamma_{\theta,\varepsilon}} e^{t\,\mathrm{Re}\, z} \left|\overline{Rf}(z, x) - \overline{Rf}(z, y)\right| |\mathrm{d}z| < \infty.$$

Let N be an $\mathfrak{m}$-null set constructed in Proposition 3. We may assume $N_0 \subset N$. Then, for any $\delta > 0$, there exists a compact subset $L \subset \mathbb{C}$ such that

$$\begin{aligned}
&\frac{1}{2\pi} \sup_{x,y\in K\setminus N} \int_{\Gamma_{\theta,\varepsilon}} e^{t\,\mathrm{Re}\, z} \left|\overline{Rf}(z, x) - \overline{Rf}(z, y)\right| |\mathrm{d}z| \\
&\le \frac{1}{2\pi} \sup_{x,y\in K\setminus N} \int_{\Gamma_{\theta,\varepsilon}\cap L} e^{t\,\mathrm{Re}\, z} \left|\overline{Rf}(z, x) - \overline{Rf}(z, y)\right| |\mathrm{d}z| + \delta.
\end{aligned}$$

For $x, y \in K \setminus N$ and $j \in \mathbb{N}$,

$$\begin{aligned}
\left|\widehat{T}_t f(x) - \widehat{T}_t f(y)\right| &\le \frac{1}{2\pi} \int_{\Gamma_{\theta,\varepsilon}} e^{t\mathrm{Re}\, z} \left|\overline{Rf}(z, x) - \overline{Rf}(z, y)\right| |\mathrm{d}z| \\
&\le \frac{1}{2\pi} \sup_{x,y\in K\setminus N} \int_{\Gamma_{\theta,\varepsilon}\cap L} e^{t\mathrm{Re}\, z} \left|\overline{Rf}(z, x) - \overline{Rf}(z, y)\right| |\mathrm{d}z| + \delta \qquad (19) \\
&\le \frac{\delta|L|}{2\pi} \times e^{t\varepsilon} + \frac{|L \cap \Gamma_{\theta,\varepsilon,\delta,j}|}{\pi} \times e^{t\varepsilon} \times \left\|\overline{Rf}(\cdot,\cdot)\right\|_{L^\infty((L\cap\Gamma_{\theta,\varepsilon})\times K;\, |\mathrm{d}z|\otimes\mathfrak{m})} + \delta,
\end{aligned}$$

where $\Gamma_{\theta,\varepsilon,\delta,j} = \{z \in \Gamma_{\theta,\varepsilon} \mid \sup_{x,y\in K\setminus N,\, d(x,y)<1/j} |\overline{Rf}(z, x) - \overline{Rf}(z, y)| > \delta\}$. By letting $j \to \infty$ in (19), the Markov inequality, (15), and Lebesgue convergence theorem together lead us to

$$\varlimsup_{j\to\infty} \sup_{x,y\in K\setminus N,\, d(x,y)<1/j} \left|\widehat{T}_t f(x) - \widehat{T}_t f(y)\right| \le \frac{\delta|L|}{2\pi} \times e^{t\varepsilon} + \delta.$$

Since δ is arbitrarily chosen, we see that $\widehat{T}_t f$ is uniformly continuous on $K \setminus N$. Because $K \setminus N$ is a dense subset of K, we find that $\widehat{T}_t f$ is extended to a continuous function on K, which is denoted by the same symbol. Since K is arbitrary, $\widehat{T}_t f$ is continuous on E. From Proposition 2 and Proposition 3(a),

$$P_t f(x) = T_t f(x) = \widehat{T}_t f(x) \quad \mathfrak{m}\text{-a.e. } x \in E. \tag{20}$$

From this, $\widehat{T}_t f$ is bounded on E. In particular, if $f \in L^p(E; \mathfrak{m}) \cap C_b(E)$, the absolute continuity and the sample path (right-)continuity imply

$$P_t f = \lim_{s\to 0} P_{s+t} f = \lim_{s\to 0} P_s(P_t f) = \lim_{s\to 0} P_s(\widehat{T}_t f) = \widehat{T}_t f.$$

This implies that P_t maps any function in $L^p(E; \mathfrak{m}) \cap C_b(E)$ to $C_b(E)$. Even if $f \in L^p(E; \mathfrak{m}) \cap \mathscr{B}_b(E)$, we already know $\widehat{T}_t f \in L^p(E; \mathfrak{m}) \cap C_b(E)$. Therefore, by using the C_b-Feller property, (20), and the absolute continuity, we obtain that for any $t > 0$,

$$P_t f = P_{t/2}(P_{t/2} f) = P_{t/2}(\widehat{T}_{t/2} f) \in C_b(E).$$

This implies that P_t maps any function in $L^p(E; \mathfrak{m}) \cap \mathscr{B}_b(E)$ to $C_b(E)$.

"In particular" part is proved as follows. We take a non-decreasing sequence $\{f_n\}_{n=1}^{\infty} \subset L^p(E; \mathfrak{m}) \cap \mathscr{B}_b(E)$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for any $x \in E$. By the non-negativity of f and the monotone convergence theorem, it holds that

$$P_t f(x) = \mathbf{E}_x\left[\lim_{n\to\infty} f_n(X_t)\right] = \lim_{n\to\infty} \mathbf{E}_x[f_n(X_t)] = \lim_{n\to\infty} P_t f_n(x), \quad x \in E.$$

Hence, we have the lower semicontinuity of $P_t f$ from the continuity of $P_t f_n$, $n \in \mathbb{N}$. Since the function $\|f\|_\infty - f$ is also a nonnegative bounded Borel function, we have that $P_t(\|f\|_\infty - f)$ is lower semicontinuous on E. From this fact and the assumption that $P_t \mathbf{1}_E \in C_b(E)$ for any $t > 0$, we find that $P_t f$ is upper semicontinuous. Thus P_t maps any non-negative function in $\mathscr{B}_b(E)$ to $C_b(E)$. For general $f \in \mathscr{B}_b(E)$, decomposing f into $f_+ := f\mathbf{1}_{\{f>0\}}$ and $f_- := -f\mathbf{1}_{\{f\le 0\}}$ and applying the result above to f_+ and f_-, we obtain the semigroup strong Feller property.

In the case that $\{T_t\}_{t\ge 0}$ is not an analytic semigroup on $L^p(E; \mathfrak{m})$, even if $\{R_\alpha\}_{\alpha>0}$ has the strong Feller property, $\{P_t\}_{t\ge 0}$ may not be strong Feller. To see this, let $\{P_t\}_{t\ge 0}$ be the semigroup of the space-time Brownian motion:

$$P_t f(x,\tau) := \mathbf{E}_x^{(1)} \otimes \mathbf{E}_\tau^{(2)}[f(B_t, t)] \quad f \in \mathscr{B}(\mathbb{R}^2),\ (t,x) \in \mathbb{R}^2,\ t > 0.$$

Here, $(\{B_t\}_{t\ge 0}, \{\mathbf{P}_x^{(1)}\}_{x\in\mathbb{R}})$ is a one-dimensional Brownian motion, and $P_\tau^{(2)}$ denotes the law of uniform motion to the right starting from $\tau \in \mathbb{R}$ with unit speed. It is known that the semigroup $\{P_t\}_{t\ge 0}$ does not have the strong Feller property, but the associated resolvent has the strong Feller property [19, Remark 1.1]. Now we let $p \in [1, +\infty)$ and $\mathfrak{m}$ be the Lebesgue measure on $\mathbb{R}^2$, and see the fact that the semigroup $\{T_t\}_{t\ge 0}$ on $L^p(\mathbb{R}^2; \mathfrak{m})$ which is generated by the operator

$$A = \frac{1}{2}\frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_2}, \quad x = (x_1, x_2),$$

is not an analytic semigroup. It is easy to see that $\mathfrak{m}$ is the invariant measure of $\{T_t\}_{t\geq 0}$, and $\{T_t\}_{t\geq 0}$ is the transition semigroup generated by $\{(B_t, t)\}_{t\geq 0}$. Let $a \in \mathbb{R}$ and

$$f_n(x_1, x_2) := \left(\frac{p}{\pi n}\right)^{1/p} \exp\left(\mathrm{i}\, a x_2 - \frac{x_1^2 + x_2^2}{n}\right), \quad (x_1, x_2) \in \mathbb{R}^2$$

for $n \in \mathbb{N}$. Then,

$$\int_{\mathbb{R}^2} |f_n|^p \,\mathrm{d}\mathfrak{m} = \frac{p}{\pi n}\left(\int_{\mathbb{R}} \exp\left(-\frac{p y^2}{n}\right) \mathrm{d}y\right)^2 = 1$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^2} |A f_n - \mathrm{i}\, a f_n|^p \,\mathrm{d}\mathfrak{m} \\
&= \frac{p}{\pi n} \int_{\mathbb{R}^2} \left|\left(\frac{1}{n} + \frac{2x_1^2}{n^2} + \frac{2x_2}{n}\right) \exp\left(\mathrm{i}\, a x_2 - \frac{x_1^2 + x_2^2}{n}\right)\right|^p \mathfrak{m}(\mathrm{d}x) \\
&= \frac{p}{\pi n^{p/2}} \int_{\mathbb{R}^2} \left(\frac{1}{\sqrt{n}} + \frac{2\xi_1^2}{\sqrt{n}} + 2\xi_2\right)^p \exp\left(-p(\xi_1^2 + \xi_2^2)\right) \mathfrak{m}(\mathrm{d}\xi) \\
&\to 0, \quad n \to \infty.
\end{aligned}$$

These imply that $\mathrm{i}\, a \notin \rho(A)$ for all $a \in \mathbb{R}$. Therefore, A is not a sectorial operator, and equivalently, $\{T_t\}_{t\geq 0}$ is not analytic on $L^p(\mathbb{R}^2; \mathfrak{m})$.

4 Application to Markov Processes Associated with Lower Bounded Semi-Dirichlet Forms

Throughout this section, we assume that (E, d) is a locally compact separable metric space with its one point compactification E_∂ and $\mathfrak{m}$ is a positive Radon measure on E with full support, and (**A2**) is satisfied. By Theorem 1 (b), for any $p \in [2, +\infty)$, the semigroup $\{T_t\}_{t\geq 0}$ is analytic on $L^p(E;\ \mathfrak{m})$ with a sector $S_{\delta'}$ for some $\delta' \in (0, \pi/2]$. Therefore, the generator $(A, \mathrm{Dom}(A))$ on $L^p(E;\ \mathfrak{m})$ is also sectorial with some constants $\theta_p \in (\pi/2, \pi)$ and $M_p \in (0, +\infty)$ depending on p. See [10, Theorem 1.4.2] for a quantitative bound of θ_p.

As noted before, the semigroup $P_t u$ with $u \in L^2(E;\ \mathfrak{m}) \cap \mathscr{B}_b(E)$ is a quasi-continuous $\mathfrak{m}$-version of $T_t u$ for $t > 0$. On the basis of this fact, we follow [26, §3.5, Theorem 3.5.4] to obtain the next proposition.

Proposition 5 *If the resolvent kernel of* $\mathbf{X}$ *is absolutely continuous with respect to* $\mathfrak{m}$*, so is the transition kernel,*

$$P_t(x, \mathrm{d}y) = P_t(x, y)\, \mathfrak{m}(\mathrm{d}y), \quad t > 0,\ x \in E.$$

Now, we apply Theorem 2 to $\mathbf{X}$ under (**A2**) for $p \in [2, +\infty)$. In the sequel, we write $R_\alpha(x, y)$ for the resolvent kernel $R_\alpha(x, \cdot)$ of $\mathbf{X}$ with respect to $\mathfrak{m}$ (if it exists).

Theorem 3 *Suppose that* (**A2**) *is satisfied. Assume that* $\{R_\alpha\}_{\alpha>0}$ *has the strong Feller property, and there exist* $C > 0$, $\alpha > \alpha_0$, *and* $p > 2$ *such that*

$$\|G_\alpha g\|_{L^\infty(E;\, \mathfrak{m})} \le C(\|g\|_{L^p(E;\, \mathfrak{m})} + \|g\|_{L^2(E;\, \mathfrak{m})}), \quad g \in L^p(E;\ \mathfrak{m}) \cap L^2(E;\ \mathfrak{m}). \tag{21}$$

In addition, we assume either of the following conditions is satisfied:

(i) $\mathfrak{m}(E)$ *is finite,*
(ii) for any $t > 0$, $P_t \mathbf{1}_E$ *is continuous on* E.

Then, the semigroup of $\mathbf{X}$ *has the strong Feller property.*

Proof In view of Theorem 2 with $p = 2$, Lemma 1, and Proposition 5, it suffices to confirm the conditions in Proposition 3 with $p = 2$, (14) and (15).

We take positive constants $C > 0$, $\alpha > \alpha_0$, and $p > 2$ so that (21) holds. On both $L^p(E;\ \mathfrak{m})$ and $L^2(E;\ \mathfrak{m})$, we see that $(A, \mathrm{Dom}(A))$ is a sectorial operator with constants $\theta = \theta_2 \wedge \theta_p \in (\pi/2, \pi)$ and $M = M_2 \vee M_p$. We fix $z \in S_\theta$ and $f \in L^2(E;\ \mathfrak{m}) \cap C_b(E)$. Then, we have $f \in L^p(E;\ \mathfrak{m}) \cap C_b(E)$. From the resolvent equation and (21),

$$\begin{aligned}
\|R(z\,;A)f\|_{L^\infty(E;\,\mathfrak{m})} &= \|R(\alpha\,;A)f + (\alpha - z)R(\alpha\,;A)R(z\,;A)f\|_{L^\infty(E;\,\mathfrak{m})} \\
&= \|G_\alpha f + (\alpha - z)G_\alpha R(z\,;A)f\|_{L^\infty(E;\,\mathfrak{m})} \\
&\le \|G_\alpha f\|_{L^\infty(E;\,\mathfrak{m})} + \|(\alpha - z)G_\alpha R(z\,;A)f\|_{L^\infty(E;\,\mathfrak{m})} \\
&\le C(\|f\|_{L^p(E;\,\mathfrak{m})} + \|f\|_{L^2(E;\,\mathfrak{m})}) \\
&\quad + C|\alpha - z|(\|R(z\,;A)f\|_{L^p(E;\,\mathfrak{m})} + \|R(z\,;A)f\|_{L^2(E;\,\mathfrak{m})}) \\
&\le C(\|f\|_{L^p(E;\,\mathfrak{m})} + \|f\|_{L^2(E;\,\mathfrak{m})}) \\
&\quad + CM\frac{|\alpha - z|}{|z|}(\|f\|_{L^p(E;\,\mathfrak{m})} + \|f\|_{L^2(E;\,\mathfrak{m})}) < \infty.
\end{aligned} \tag{22}$$

By using the resolvent equation again, we have

$$R(z\,;A)f = R(\alpha\,;A)f + (\alpha - z)R(\alpha\,;A)R(z\,;A)f \quad \text{in} \quad L^2(E;\ \mathfrak{m}). \tag{23}$$

Then, (22) and (23) imply that $R(z\,;A)f$ possesses a continuous $\mathfrak{m}$-version on E. Because $\Gamma_{\theta,\varepsilon} \subset S_\theta$ for any $\varepsilon \in (0, \theta - \pi/2)$, the condition (ii) in Proposition 3 holds. The conditions (14) and (15) immediately follow from (22).

Remark 3 (i) Assume that (**A2**) is satisfied. Then, [15, Corollary] (see also [26, p. 98]) implies that (21) holds for $p > (q/(q-2)) \vee 2$ and $\alpha > \alpha_0$ if there exist $q > 2$, $\alpha \geq \alpha_0$ and $S > 0$ such that $L^q(E;\ \mathfrak{m}) \subset \mathscr{F}$ and

$$\|f\|^2_{L^q(E;\ \mathfrak{m})} \leq S\mathscr{E}_\alpha(f, f), \quad f \in \mathscr{F}. \tag{24}$$

The inequality (24) is often called the *Sobolev type inequality*. Next we assume that $(\mathscr{E}, \mathscr{F})$ is a symmetric Dirichlet form on $L^2(E;\ \mathfrak{m})$. Then (24) implies that the associated semigroup $\{T_t\}_{t>0}$ is ultracontractive, i.e., each T_t is extended to a bounded linear operator from $L^1(E; \mathfrak{m})$ to $L^\infty(E;\ \mathfrak{m})$, and more strongly, we have

$$\|T_t\|_{1\to\infty} \leq Ct^{-q/(q-2)}e^{\alpha t}, \quad t > 0 \tag{25}$$

for some positive constant $C > 0$. We also see from [11, Lemma 2.1.2] that for any $t > 0$, $\|T_{t/2}\|^2_{2\to\infty} = \|T_t\|_{1\to\infty}$ under the symmetry of $\{T_t\}_{t>0}$. It is also known that the ultracontractivity of the semigroup and (25) together imply (24). See [6, Theorems (2.1), (2.9), and (2.16)], [16, Theorem 4.2.7], or [33, Theorem 1] for the proof.

(ii) When $(\mathscr{E}, \mathscr{F})$ is a symmetric Dirichlet form on $L^2(E;\ \mathfrak{m})$, the following condition implies (24) by [25, Theorem 4.1(i)]:

$$\sup_{x\in E} \int_E R_\alpha(x, y)^q\, \mathfrak{m}(dy) < \infty \tag{26}$$

for some $\alpha \in (0 + \infty)$ and $q \in (1, +\infty)$. We also see that (26) for such α, q yields that there exists $C \in (0, +\infty)$ such that $\|G_\alpha f\|_{L^\infty(E;\ \mathfrak{m})} \leq C\|f\|_{L^p(E;\ \mathfrak{m})}$ for any $f \in L^p(E;\ \mathfrak{m})$ with $p = q/(q-1)$.

Even if $(\mathscr{E}, \mathscr{F})$ is a symmetric Dirichlet form, it cannot be expected that (21) follows from the condition that the associated semigroup $\{T_t\}_{t>0}$ is ultracontractive only. In fact, in this case, we do not know a quantitative estimate of $\|T_t\|_{1\to\infty}(= \|T_{t/2}\|^2_{2\to\infty})$ in $t > 0$. However, under the ultracontractivity, we can improve the proof of [4, Proposition 3.4] to obtain the same result as Theorem 3.

Theorem 4 *Suppose that* (**A2**) *is satisfied. Assume that* $\{R_\alpha\}_{\alpha>0}$ *has the strong Feller property, and* $\|T_t\|_{L^2\to L^\infty}$ *is finite for any* $t > 0$. *In addition, we assume either of the following conditions is satisfied:*

(i) $\mathfrak{m}(E)$ *is finite,*
(ii) for any $t > 0$, $P_t\mathbf{1}_E$ *is continuous on* E.

Then, the semigroup of $\mathbf{X}$ *has the strong Feller property.*

Proof We fix $t > 0$. By the duality, we have $\|T_t^*\|_{1\to2} = \|T_t\|_{2\to\infty}$. That is, T_t^* is extended to a bounded linear operator from $L^1(E;\ \mathfrak{m})$ to $L^2(E;\ \mathfrak{m})$. Fix $f \in L^2(E;\ \mathfrak{m}) \cap \mathscr{B}_b(E)$, and let

$$h = (\alpha - A)T_t f. \tag{27}$$

Here, α is a positive number with $\alpha > \alpha_0$. We see from [22, Chapter I, Exercise 1.9] that $T_t f \in \mathrm{Dom}(A)$. Therefore, $h \in \mathrm{Dom}(A) \subset L^2(E;\ \mathfrak{m})$ and

$$AT_t f = AT_{t/2}T_{t/2} f = T_{t/2}(AT_{t/2} f). \tag{28}$$

By noting that $\{e^{-\alpha_0 t}T_t\}_{t\geq 0}$ is an analytic semigroup on $L^2(E;\ \mathfrak{m})$, we have from [28, Chap. 2, Theorem 5.2 (d)] that

$$\|AT_{t/2} f\|_{L^2(E;\ \mathfrak{m})} \leq (C/t)\|f\|_{L^2(E;\ \mathfrak{m})} \tag{29}$$

for some C independent of t and f. Combining (28) and (29), we obtain that for any $g \in L^1(E;\ \mathfrak{m})$,

$$\begin{aligned}
\left|\int_E hg\,\mathrm{d}\mathfrak{m}\right| &= \left|((\alpha - A)T_t f, g)_{L^2(E;\ \mathfrak{m})}\right| = \left|((\alpha - A)T_{t/2} f, T^*_{t/2} g)_{L^2(E;\ \mathfrak{m})}\right| \\
&\leq \left(\alpha e^{\alpha_0 t/2}\|f\|_{L^2(E;\ \mathfrak{m})} + \|AT_{t/2} f\|_{L^2(E;\ \mathfrak{m})}\right)\|T^*_{t/2} g\|_{L^2(E;\ \mathfrak{m})} \\
&\leq \left(\alpha e^{\alpha_0 t/2} + \frac{C}{t}\right)\|f\|_{L^2(E;\ \mathfrak{m})}\|T^*_{t/2} g\|_{L^2(E;\ \mathfrak{m})} \\
&\leq \left(\alpha e^{\alpha_0 t/2} + \frac{C}{t}\right)\|f\|_{L^2(E;\ \mathfrak{m})}\|T^*_{t/2}\|_{L^1\to L^2}\|g\|_{L^1(E;\ \mathfrak{m})}.
\end{aligned}$$

This shows that the functional $L^1(E;\ \mathfrak{m}) \ni g \mapsto \int_E hg\,\mathrm{d}\mathfrak{m}$ is continuous. Therefore, we find that h belongs to $L^2(E;\ \mathfrak{m}) \cap L^\infty(E;\ \mathfrak{m})$, and obtain

$$P_t f = R_\alpha h. \tag{30}$$

From this and the resolvent strong Feller property, we find that $P_t f$ possesses a bounded continuous $\mathfrak{m}$-version. Then, by following the same argument after (20), we know that P_t maps any function in $L^2(E;\mathfrak{m}) \cap \mathscr{B}_b(E)$ to $C_b(E)$. The rest of the proof is exactly similar to that of Theorem 3.

Remark 4 (i) Note that (30) is obtained without the following assumptions: E is locally compact, and $\{T_t\}_{t\geq 0}$ is associated with a lower bounded semi-Dirichlet form. Consider a Markov process on a metric space with right continuous path and the resolvent strong Feller property. Then, if the semigroup is extended to an L^2-space with respect to a suitable measure and has the ultracontractivity, we get the same bounded function as (27), which leads us to (30). Therefore, we obtain the semigroup strong Feller property if either of the same conditions as (i) and (ii) in Theorem 4 holds.

(ii) A similar equation to (30) is also obtained in the proof of [4, Proposition 3.4], where the spectral decomposition theorem is used.

(iii) Let $(\mathscr{E}, \mathscr{F})$ be a lower bounded semi-Dirichlet form on $L^2(E;\, \mathfrak{m})$ which satisfies (24) for some positive constants $q > 2$, $\alpha \geq \alpha_0$ and $S > 0$. Then, for any $f \in L^2(E;\, \mathfrak{m})$ and $t > 0$, we have $U_t f := e^{-\alpha_0 t} T_t f \in \mathrm{Dom}(A) \subset \mathscr{F}$. The analyticity of $\{U_t\}_{t>0}$ implies that for any $t > 0$ and $f \in L^2(E;\, \mathfrak{m})$,

$$\|U_t f\|_{L^q(E;\, \mathfrak{m})} \leq S\mathscr{E}_\alpha(U_t f, U_t f) = -S(AU_t f, U_t f)_{L^2(E;\, \mathfrak{m})} \leq \frac{C}{t}\|f\|_{L^2(E;\, \mathfrak{m})}$$

for some $C > 0$. Then, applying [27, Lemma 6.1] to $\{U_t\}_{t\geq 0}$, we find that $\{T_t\}_{t\geq 0}$ is extended to a bounded linear operator from $L^2(E;\, \mathfrak{m})$ to $L^\infty(E;\, m)$.

Next, we localize conditions in Theorems 3 and 4 in an appropriate framework. Then, we introduce subprocesses of $\mathbf{X}$. For an open subset $U \subset E$, we set $\tau_U = \inf\{t \in [0, +\infty) \mid X_t \notin U\}$. Then, the subprocess of $\mathbf{X}$ killed upon leaving U is defined by

$$X_t^U := \begin{cases} X_t, & \text{if } t < \tau_U, \\ \partial, & \text{if } t \geq \tau_U. \end{cases}$$

We see from [26, §3.5, Theorem 3.5.7] that $\mathbf{X}^U = (\{X_t^U\}_{t\in[0,+\infty]}, \{\mathbf{P}_x\}_{x\in U})$ is associated with the lower bounded semi-Dirichlet form $(\mathscr{E}_U, \mathscr{F}_U)$ on $L^2(U;\, \mathfrak{m})$, and it is a Hunt process on U. We call $\mathbf{X}^U$ the *part process of* $\mathbf{X}$ *on* U. Here, $\mathscr{F}_U$ is identified with the completion of $\{u \in C_0(E) \cap \mathscr{F} \mid \mathrm{supp}[u] \subset U\}$ with respect to $\mathscr{E}_{\alpha_0+1}^{1/2}$, and $\mathscr{E}_U(u, v) := \mathscr{E}(u, v)$ for $u, v \in \mathscr{F}_U$. It is also proved in [26, §3.5, Theorem 3.5.7] that $(\mathscr{E}_U, \mathscr{F}_U)$ is a regular semi-Dirichlet form on $L^2(U;\, \mathfrak{m})$ having the same lower bound $-\alpha_0$ on $L^2(U;\, \mathfrak{m})$. Therefore, by Proposition 1, the semigroup $\{P_t^U\}_{t\geq 0}$ and the resolvent $\{R_\alpha^U\}_{\alpha>0}$ are extended to bounded linear operators on $L^p(E;\, \mathfrak{m})$, $p \in [2, +\infty)$. The extensions are denote by $\{T_t^U\}_{t\geq 0}$ and $\{G_\alpha^U\}_{\alpha>0}$, respectively. Furthermore, Theorem 1 implies that $\{T_t^U\}_{t\geq 0}$ is analytic on $L^p(U;\mathfrak{m})$ for $p \in [2, +\infty)$. For $t, \alpha \in (0, +\infty)$, $x \in U$, $f \in \mathscr{B}_b(U)$, we have

$$P_t^U f(x) = \mathbf{E}_x[f(X_t) : t < \tau_U], \quad R_\alpha^U f(x) = \mathbf{E}_x\left[\int_0^{\tau_U} e^{-\alpha t} f(X_t)\,\mathrm{d}t\right]. \tag{31}$$

Therefore, if the resolvent kernels of $\mathbf{X}$ is absolutely continuous with respect to $\mathfrak{m}$, so is the resolvent kernel of $\mathbf{X}^U$,

$$R_\alpha^U(x, \mathrm{d}y) = R_\alpha^U(x, y)\,\mathfrak{m}(\mathrm{d}y), \quad \alpha > 0,\ x \in U.$$

The following theorem provides a sufficient condition for the resolvent strong Feller property of part processes of $\mathbf{X}$.

Theorem 5 ([19, Theorem 3.1]) *Assume that the resolvent of* $\mathbf{X}$ *has both the strong Feller property and the Feller property. Then, for any open subset* $U \subset E$*, the resolvent of* $\mathbf{X}^U$ *has the strong Feller property.*

Let U be an open subset of E. A bounded Borel measurable function $h\colon U \to \mathbb{R}$ is said to be *harmonic* (with respect to $\mathbf{X}$) if for any relatively compact open set $V \subset U$, $\{h(X_{t\wedge\tau_V})\}_{t\geq 0}$ is a uniformly integrable martingale with respect to $\mathbf{P}_x$, $x \in V$. A Hunt process $\mathbf{X}$ is said to be a *diffusion process without killing inside* if $\mathbf{X}$ is of continuous sample paths and $\mathbf{P}_x(X_{\zeta-} \in E,\ \zeta < +\infty) = 0$ for every $x \in E$. Here ζ denotes the lifetime of $\mathbf{X}$. With these definitions, we give another sufficient condition for the resolvent strong Feller property for part processes of $\mathbf{X}$.

Proposition 6 *Suppose that* $\mathbf{X}$ *is a diffusion process without killing inside, and the resolvent has the strong Feller property. In addition, we assume that any bounded harmonic function* h *on an open subset* $U \subset E$ *is continuous there. Then, the resolvent of* $\mathbf{X}^U$ *has the strong Feller property.*

Proof We fix $\alpha > 0$, $f \in \mathscr{B}_b(U)$, and set $f = 0$ on $E \setminus U$. In view of (31), it suffices to show that $\phi_U^\alpha := \mathbf{E}_\bullet[\int_{\tau_U}^\infty e^{-\alpha t} f(X_t)\,\mathrm{d}t]$ is continuous on K for any compact subset $K \subset U$.

We fix a compact subset $K \subset U$, and let V be a relatively compact open subset of U such that $K \subset V \subset \overline{V} \subset U$. Here we denote by $\overline{V}$ the closure of V in E. For $n \in \mathbb{N}$ and $x \in U$, we define

$$\psi_U^n(x) = \mathbf{E}_x\left[e^{-n\tau_U}\right].$$

The assumption "no killing inside" ensures that $\tau_V < \tau_U$ for $\mathbf{P}_x$-a.s. $x \in V$. It follows that $\tau_U = \tau_V + \tau_U \circ \theta_{\tau_V} \geq \tau_U \circ \theta_{\tau_V}$ for $\mathbf{P}_x$-a.s. $x \in V$. Here, $\{\theta_t\}_{t\in[0,+\infty]}$ denotes the shift operators of X. Then, by using the strong Markov property [7, Theorem A.1.21] of $\mathbf{X}$, we have for any $x \in V$ and $n \in \mathbb{N}$,

$$\psi_U^n(x) = \mathbf{E}_x\left[e^{-n\tau_U} : \tau_V < \tau_U\right] \leq \mathbf{E}_x\left[\mathbf{E}_{X_{\tau_V}}\left[e^{-n\tau_U}\right]\right] =: h_n(x). \tag{32}$$

We see from the strong Markov property and the same argument as in the proof of [7, Lemma 6.1.5] that h_n is a harmonic function on V with respect to $\mathbf{X}$. Noting this fact and the assumption that h_n is bounded continuous on K, we use Dini's theorem to obtain that $\lim_{n\to\infty}\sup_{x\in K} h_n(x) = 0$. In particular, we have from (32) that

$$\lim_{n\to\infty}\sup_{x\in K}\psi_U^n(x) = 0. \tag{33}$$

The same argument as in [19, Theorem 3.1] and (33) imply

$$\lim_{n\to\infty}\sup_{x\in K}\left|\phi_U^\alpha - nR_{n+\alpha}\phi_U^\alpha(x)\right| \leq 2\sup_{x\in U}|f(x)| \times \lim_{n\to\infty}\sup_{x\in K}\psi_U^n(x) = 0.$$

Finally, by using the strong Feller property, we see ϕ_U^α is continuous on U.

Remark 5 By assuming both the Feller and the strong Feller property of the resolvent, we have the continuity of bounded harmonic functions. The proof is almost the same as [30, Theorem 3.4].

Let U be an open subset of E. Then, under both the situations of Theorem 5 and Proposition 6, we see that for any compact subset $K \subset U$,

$$\lim_{s\to 0}\sup_{x\in K}\mathbf{P}_x[\tau_U \le s] = 0. \tag{34}$$

In fact, under the situation of Theorem 5, we use [19, Lemma 2.2] to see that (34) is valid. For the latter situation, we take an open subset $V \subset E$ such that $K \subset V \subset \overline{V} \subset U$. It then holds that $\tau_U \ge \tau_U \circ \theta_{\tau_V}$, $\mathbf{P}_x$-a.s. $x \in V$. This and the strong Markov property [7, Theorem A.1.21] of $\mathbf{X}$ together imply

$$\mathbf{P}_x[\tau_U \le s] \le \mathbf{E}_x\left[\mathbf{P}_{X_{\tau_V}}[\tau_U \le s]\right] =: u_s(x), \quad x \in V,\ s > 0.$$

Since u_s is harmonic on V with respect to $\mathbf{X}$, it is continuous on V. Then, it is straightforward to see that $\lim_{s\to 0}\sup_{x\in K}\mathbf{P}_x[\tau_U \le s] \le \lim_{s\to 0}\sup_{x\in K} u_s(x) = 0$.

In what follows, we say that $\mathbf{X}$ has the *local ultracontracitivity* if either of the following conditions is satisfied:

(a) for any relatively compact open set $U \subset E$, there exist $C > 0$, $\alpha > \alpha_0$, and $p > 2$ such that for any $g \in L^p(U;\ \mathfrak{m}) \cap L^2(U;\ \mathfrak{m})$,

$$\|G_\alpha^U g\|_{L^\infty(U;\,\mathfrak{m})} \le C(\|g\|_{L^p(U;\,\mathfrak{m})} + \|g\|_{L^2(U;\,\mathfrak{m})}). \tag{35}$$

(b) for any relatively compact open set $U \subset E$ and $t > 0$, $\|T_t^U\|_{L^2(U;\,\mathfrak{m})\to L^\infty(U;\,\mathfrak{m})}$ is finite.

The condition (35) is weaker than $\|G_\alpha^U\|_{L^2(U;\,\mathfrak{m})\to L^\infty(U;\,\mathfrak{m})} < \infty$. The localized version of Theorems 3 and 4 are as follows:

Theorem 6 *Assume that the resolvent of* $\mathbf{X}$ *has the strong Feller property, and the local ultracontractivity. In addition, we assume either of the following conditions is satisfied:*

(i) the resolvent of $\mathbf{X}$ *has the Feller property,*
(ii) $\mathbf{X}$ *is a diffusion process without killing inside, and for any relatively compact open subset* $U \subset E$*, any bounded harmonic function on* U *is continuous there.*

Then, the semigroup of $\mathbf{X}$ *has the strong Feller property.*

Proof Let U be a relatively compact open subset of E. From Theorem 5 and Proposition 6, it follows that the resolvent of $\mathbf{X}^U$ has the strong Feller property. Furthermore, Theorems 3 and 4, and the local ultracontractivity together imply that the semigroup of $\mathbf{X}^U$ has the strong Feller property. Let K be a compact subset of U. We obtain from (34) that for any $t > 0$ and $f \in \mathscr{B}_b(E)$,

$$\lim_{s\to 0}\sup_{x\in K}\left|P_t f(x) - P_s^U P_{t-s} f(x)\right| \leq \sup_{x\in E}|f(x)| \times \lim_{s\to 0}\sup_{x\in K}\mathbf{P}_x[\tau_U \leq s] = 0.$$

Thus, $P_t f$ is a continuous function on K. Because K and U are arbitrarily chosen, we see $P_t f$ is continuous on E.

In the proof of Theorem 6, after establishing the semigroup strong Feller property of the part process, (34) is used. However, this can be replaced by the condition that $P_t\mathbf{1}_E$ is continuous on E for any $t > 0$. To clarify this fact, and for future reference, we prove the following lemma.

Lemma 2 *The semigroup of* $\mathbf{X}$ *has the strong Feller property if the following conditions are satisfied:*

- *for any relatively compact open subset* $U \subset E$*, the semigroup of* $\mathbf{X}^U$ *is strong Feller.*
- *for any* $t > 0$*,* $P_t\mathbf{1}_E$ *is continuous on* E*.*

Proof We follow the argument in [8, Theorem 1.4]. Fix $t > 0$ and a compact subset K of E. Let $\{U_n\}_{n=1}^{\infty}$ be a sequence of relatively compact open subsets such that $K \subset U_1$ and $\overline{U}_n \subset U_{n+1}$ for any $n \in \mathbb{N}$. The assumptions imply that for any $n \in \mathbb{N}$,

$$x \mapsto \mathbf{P}_x[\tau_{U_n} \leq t < \zeta](= P_t\mathbf{1}_E(x) - P_t^{U_n}\mathbf{1}_E(x))$$

is continuous on K. Thus, there exists $\{x_n\}_{n=1}^{\infty} \subset K$ such that

$$\mathbf{P}_{x_n}[\tau_{U_n} \leq t < \zeta] = \sup_{x\in K}\{P_t\mathbf{1}_E(x) - P_t^{U_n}\mathbf{1}_E(x)\}, \quad n \in \mathbb{N}.$$

Because K is compact, there exists a subsequence of $\{x_n\}_{n=1}^{\infty}$ which converges to some $x \in K$. We denote the subsequence $\{x_n\}_{n=1}^{\infty}$ again. For any $n, m \in \mathbb{N}$ with $n > m$,

$$\mathbf{P}_{x_n}[\tau_{U_n} \leq t < \zeta] \leq \mathbf{P}_{x_n}[\tau_{U_m} \leq t < \zeta].$$

By using the continuity of the map $x \mapsto \mathbf{P}_x[\tau_{U_m} \leq t < \zeta]$, we obtain that for any $m \in \mathbb{N}$,

$$\overline{\lim_{n\to\infty}}\,\mathbf{P}_{x_n}[\tau_{U_n} \leq t < \zeta] \leq \mathbf{P}_x[\tau_{U_m} \leq t < \zeta].$$

Thus, we arrive at

$$\overline{\lim_{n\to\infty}}\,\mathbf{P}_{x_n}[\tau_{U_n} \leq t < \zeta] \leq \lim_{m\to\infty}\mathbf{P}_x[\tau_{U_m} \leq t < \zeta] = 0,$$

where we use the fact that quasi-left continuity up to ζ of $\mathbf{X}$ implies $\mathbf{P}_x(\lim_{n\to\infty}\tau_{U_n} = \zeta) = 1$. This shows that for any $f \in \mathscr{B}_b(E)$ and $t > 0$,

$$\overline{\lim_{n\to\infty}}\sup_{x\in K}|P_t f(x) - P_t^{U_n} f(x)| \leq \|f\|_\infty \times \overline{\lim_{n\to\infty}}\sup_{x\in K}\mathbf{P}_x[\tau_{U_n} \leq t < \zeta] = 0.$$

Since the semigroups of $\{\mathbf{X}^{U_n}\}_{n=1}^{\infty}$ are strong Feller, so is the semigroup of $\mathbf{X}$.

We close this section to provide some examples.

Example 1 Let $\mathbf{B} = (\{B_t\}_{t\ge0}, \{\mathbf{P}_x\}_{x\in\mathbb{R}})$ be a one-dimensional Brownian motion. For $t \ge 0$, define $A_t = \int_0^t (1 + |B_s|^4)^{-1}\,\mathrm{d}s$. Then, $A = \{A_t\}_{t\ge0}$ is a positive continuous additive functional (PCAF) of $\mathbf{B}$, and the Revuz measure is identified with $\nu(\mathrm{d}x) = (1 + |x|^4)^{-1}\,\mathrm{d}x$ (see [16, Sect. 5.1]) for the definition of Revuz measures and the correspondence with PCAFs). Here, $\mathrm{d}x$ denotes the one-dimensional Lebesgue measure. Let $X_t = B_{A_t^{-1}}$, $t \ge 0$ be the time changed process by $A_t^{-1} := \inf\{s > 0 \mid A_s > t\}$. By [16, Theorem 6.2.1], $\mathbf{X} := (\{X_t\}_{t\ge0}, \{\mathbf{P}_x\}_{x\in\mathbb{R}})$ is a ν-symmetric Hunt process on $\mathbb{R}$, and the Dirichlet form associated with $\mathbf{X}$ is regular on $L^2(\mathbb{R}^d;\ \nu)$. It is straightforward to see that

$$\lim_{t\to0}\sup_{x\in\mathbb{R}} \mathbf{E}_x[A_t] = 0.$$

In other words, ν is of Kato class with respect to $\mathbf{B}$. Then, [18, Lemma 4.1] shows that the resolvent of $\mathbf{X}$ is strong Feller. On the other hand, since the generator of $\mathbf{X}$ is $(1 + x^4)(\mathrm{d}^2/\mathrm{d}x^2)$ $(x \in \mathbb{R})$, the conditions (iii) and (iv) in [29, Theorem 8.4.1] hold, which implies that the resolvent of $\mathbf{X}$ is not Feller. However, $\mathbf{X}$ satisfies condition (b) in Theorem 6 since any harmonic function with respect to $\mathbf{X}$ is harmonic with respect to $\mathbf{B}$. Let U be a bounded open interval, and let $T_U = \inf\{t > 0 \mid B_t \notin U\}$. Noting $\tau_U := \inf\{t > 0 \mid X_t \notin U\} = A_{T_U}$ and using [7, Proposition 4.1.10], we obtain that for any $f \in L^2(\mathbb{R};\ \nu)$, $x \in U$, and $\alpha > 0$,

$$\mathbf{E}_x\left[\int_0^{\tau_U} e^{-\alpha t}|f(X_t)|\,\mathrm{d}t\right] \le \mathbf{E}_x\left[\int_0^{A_{T_U}} |f(X_t)|\,\mathrm{d}t\right] = \mathbf{E}_x\left[\int_0^{T_U} |f(B_t)|\,\mathrm{d}A_t\right]$$
$$= \int_U |f(y)| g_U(x, y)\,\nu(\mathrm{d}y).$$

Here, $g_U(x, y)$ denotes the green function of $\mathbf{B}^U$. Since $\sup_{x,y\in U} g_U(x, y) < \infty$, we see that $\mathbf{X}$ has the local ultracontractivity. Therefore, by Theorem 6, the semigroup of $\mathbf{X}$ is strong Feller.

Example 2 Let $\mathbf{X} = (\{X_t\}_{t\ge0}, \{\mathbf{P}_x\}_{x\in\mathbb{R}^d})$ be an Ornstein–Uhlenbeck process:

$$X_t = e^{-t/2}x + \int_0^t e^{(1/2)(t-s)}\,\mathrm{d}B_s, \quad t \ge 0,\ x \in \mathbb{R}^d.$$

Here, $B = \{B_t\}_{t\ge0}$ is a d-dimensional Brownian motion starting at the origin. Define a Borel measure μ on $\mathbb{R}^d$ by $\mu(\mathrm{d}x) = \exp(-|x|^2/2)\,\mathrm{d}x$, where $\mathrm{d}x$ denotes the d-dimensional Lebesgue measure. $\mathbf{X}$ is a μ-symmetric Hunt process. The Dirichlet form $(\mathscr{E}, \mathscr{F})$ associated with $\mathbf{X}$ is regular on $L^2(\mathbb{R}^d;\ \mu)$, and the core is identified with $C_0^\infty(\mathbb{R}^d)$, the space of smooth functions on $\mathbb{R}^d$ with compact support. For

$u, v \in C_0^\infty(\mathbb{R}^d)$, we have

$$\mathscr{E}(u, v) = \frac{1}{2} \int_{\mathbb{R}^d} \langle \nabla u(x), \nabla v(x) \rangle \, \mu(\mathrm{d}x).$$

Hereafter, ∇ denotes the standard gradient and $\langle \cdot, \cdot \rangle$ the inner product on $\mathbb{R}^d$. The semigroup of $\mathbf{X}$ has both the strong Feller property and Feller property. However, Theorems 3 and 4 are not available for $\mathbf{X}$. Indeed, we have

$$\begin{aligned} \mathbf{E}_x \left[|X_t|^2 \right] &= \mathbf{E}_x \left[|X_t - e^{-t/2} x|^2 \right] + 2\mathbf{E}_x \left[\left\langle e^{-t/2} x, \int_0^t e^{(1/2)(t-s)} \, \mathrm{d}B_s \right\rangle \right] + e^{-t} |x|^2 \\ &\geq 0 + 0 + e^{-t} |x|^2, \quad x \in \mathbb{R}^d. \end{aligned}$$

Hereafter, we define $| \cdot | = \langle \cdot, \cdot \rangle^{1/2}$. Although the map $\mathbb{R}^d \ni x \mapsto |x|^2$ belongs to $L^p(\mathbb{R}^d; \, \mu)$ for any $p > 0$, we see

$$\sup_{x \in \mathbb{R}^d} \mathbf{E}_x [|X_t|^2] = \infty \quad \text{and} \quad \sup_{x \in \mathbb{R}^d} \mathbf{E}_x \left[\int_0^\infty e^{-\alpha t} |X_t|^2 \, \mathrm{d}t \right] = \infty$$

for any $t > 0$ and $\alpha > 0$. This implies that both $\|T_t\|_{L^2 \to L^\infty} < \infty$ and (21) fail. On the other hand, $\mathbf{X}$ has the local ultracontractivity. To see this, let U be an open ball in $\mathbb{R}^d$. Then, [16, Theorem 4.4.3 (i)] shows that the core of the Dirichlet form $(\mathscr{E}^U, \mathscr{F}^U)$ of X^U is identified with $C_0^\infty(U) (= C_0^\infty(\mathbb{R}^d) \cap C_0(U))$ and $\mathscr{E}^U(u, u) = \mathscr{E}(u, u)$, $u \in C_0^\infty(U)$. From [20, Theorems 11.2, 11.23, and 11.34], there exists $C \in (0, +\infty)$ such that

$$\|u\|_{L^q(U; \, \mathrm{d}x)}^2 \leq C \left(\int_U |\nabla u(x)|^2 \, \mathrm{d}x + \|u\|_{L^2(U; \, \mathrm{d}x)}^2 \right), \quad u \in C_0^\infty(U). \tag{36}$$

Here, q is given by

$$q = \begin{cases} 2d/(d-2), & \text{if } d \geq 3, \\ \text{any number in} (2, +\infty), & \text{if } d = 2, \\ \infty, & \text{if } d = 1. \end{cases}$$

By (36) and the boundedness of U, there exists $C > 0$ such that

$$\|u\|_{L^q(U; \, \mu)}^2 \leq C \left(\mathscr{E}^U(u, u) + \|u\|_{L^2(U; \, \mu)}^2 \right), \quad u \in C_0^\infty(U). \tag{37}$$

Since $C_0^\infty(U)$ is a core of $(\mathscr{E}^U, \mathscr{F}^U)$, we see that (37) is valid for any $u \in \mathscr{F}^U$. This and Remark 3 (i) imply that $\mathbf{X}$ has the local ultracontractivity.

The following example appears in [26, §1.5.2, (1.5.17)].

Example 3 Let $\underline{\alpha}, \overline{\alpha}, M$, and δ be positive numbers and let $\alpha\colon \mathbb{R}^d \to \mathbb{R}$ be a continuous function such that $(1/2)(2\overline{\alpha} - \underline{\alpha}) < \delta < 1, \overline{\alpha} < 1 + \underline{\alpha}/2$, and

$$|\alpha(x) - \alpha(y)| \le M|x-y|^\delta, \quad 0 < \underline{\alpha} \le \alpha(x) \le \overline{\alpha} < 2, \quad x, y \in \mathbb{R}^d.$$

Let $C_0^2(\mathbb{R}^d)$ the space of C^2-functions on $\mathbb{R}^d$ with compact support. We define

$$\mathscr{L}u(x) = \int_{\mathbb{R}^d} \{u(x+h) - u(x) - \langle \nabla u(x), h\mathbf{1}_{B_1(0)}(h)\rangle\} \frac{w(x)}{|h|^{d+\alpha(x)}}\,dh, \quad u \in C_0^2(\mathbb{R}^d).$$

Here, $B_1(0)$ denotes the open ball centered at the origin with radius 1, and the weight function $w\colon \mathbb{R}^d \to \mathbb{R}$ is chosen so that $\mathscr{L}e^{-\mathrm{i}\langle h,x\rangle} = -|h|^{\alpha(x)}e^{-\mathrm{i}\langle h,x\rangle}$, $x, h \in \mathbb{R}^d$. Then, we have

$$w(x) = 2^{\alpha(x)-1}\pi^{-(d/2)-1}\Gamma((1+\alpha(x))/2)\Gamma((\alpha(x)+d)/2)\sin(\pi\alpha(x)/2), \quad x \in \mathbb{R}^d.$$

From [26, §1.5.2, (1.5.18)], $(\mathscr{L}, C_0^2(\mathbb{R}^d))$ is associated with a lower bounded semi-Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $L^2(\mathbb{R}^d;\, \mathrm{d}x)$, which is described as

$$\mathscr{E}(u, v) = \lim_{n\to\infty} \iint_{|x-y|>1/n} (u(x) - u(y))v(x)\frac{w(x)}{|x-y|^{d+\alpha(x)}}\,\mathrm{d}x\mathrm{d}y, \quad u, v \in C_0^1(\mathbb{R}^d).$$

Here, $C_0^1(\mathbb{R}^d)$ denotes the space of C^1-functions on $\mathbb{R}^d$ with compact support. Denote by $\mathbf{X} = (\{X_t\}_{t\ge0}, \{\mathbf{P}_x\}_{x\in\mathbb{R}^d})$ the Hunt process associated with $(\mathscr{E}, \mathscr{F})$. For $x, y \in \mathbb{R}^d$, we set $j(x, y) = w(x)/|x-y|^{d+\alpha(x)}$, and suppose that there exists $C > 0$ such that

$$j(x, y) \ge C|x-y|^{-d-1}, \quad x, y \in \mathbb{R}^d \text{ with } 0 < |x-y| < 1.$$

Then, by the argument in [26, § 3.5, Example 3.5.5] and Remark 4(iii), the semigroup $\{T_t\}_{t>0}$ of $(\mathscr{E}, \mathscr{F})$ satisfies $\|T_t\|_{L^2(\mathbb{R}^d;\,\mathrm{d}x)\to L^\infty(\mathbb{R}^d;\,\mathrm{d}x)} < \infty$ for any $t > 0$. Hence, the semigroup of $\mathbf{X}$ is strong Feller if the resolvent is. In [2, 3], the Harnack inequality for bounded harmonic functions on domains with respect to non-local operators with variable orders are obtained. Thus, by using [2, Proposition 3.1] and the argument in [4, Proposition 3.3], we can also give a sufficient condition for the resolvent of $\mathbf{X}$ being Hölder continuous in the spatial variable.

The final example is due to [5], which is a diffusion process on an infinite-dimensional space.

Example 4 Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus, and denote by $\mathbb{T}^\infty$ the product of countably many copies of $\mathbb{T}$. That is, $\mathbb{T}^\infty$ is the infinite-dimensional torus. $\mathbb{T}^\infty$

becomes a compact space by the Tychonoff's theorem. We simply denote by dx the product measure on $\mathbb{T}^\infty$ of the normalized Haar measure on $\mathbb{T}$. Let $\mathscr{A} = \{a_k\}_{k=1}^\infty$ be a sequence of strictly positive numbers, and set

$$\mathscr{E}^{\mathscr{A}}(u, v) = \int_{\mathbb{T}^\infty} \sum_{k=1}^\infty a_k \frac{\partial u}{\partial x_k}(x) \frac{\partial v}{\partial x_k}(x)\, \mathrm{d}x, \quad u, v \in \mathscr{D}.$$

Hereafter, $\mathscr{D}$ denotes the set of cylindrical smooth functions on $\mathbb{T}^\infty$. It is shown in [5, Sect. 1] that $(\mathscr{E}^{\mathscr{A}}, \mathscr{D})$ is well-defined, and closable on $L^2(\mathbb{T}^\infty;\ \mathrm{d}x)$. Let $(\mathscr{E}^{\mathscr{A}}, \mathscr{F}^{\mathscr{A}})$ be the smallest closed extension. Then, $(\mathscr{E}^{\mathscr{A}}, \mathscr{F}^{\mathscr{A}})$ is a regular Dirichlet form on $L^2(\mathbb{T}^\infty;\ \mathrm{d}x)$. It is straightforward to see that $(\mathscr{E}^{\mathscr{A}}, \mathscr{F}^{\mathscr{A}})$ is recurrent. In particular, it is conservative. Under a suitable condition, we see from [5, Lemma 7] that the associated semigroup $\{T_t^{\mathscr{A}}\}_{t>0}$ on $L^2(\mathbb{T}^\infty;\ \mathrm{d}x)$ possesses an integral kernel which is continuous on $(0, +\infty) \times \mathbb{T}^\infty \times \mathbb{T}^\infty$. From this fact, the conservativeness, and the compactness of the state space, $\{T_t^{\mathscr{A}}\}_{t>0}$ generates a Feller process with the semigroup strong Feller property. We also see that $\{T_t^{\mathscr{A}}\}_{t>0}$ is ultracontractive. However, [5, Theorem 6] implies that the Sobolev type inequality (24) does not hold for $(\mathscr{E}^{\mathscr{A}}, \mathscr{F}^{\mathscr{A}})$.

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References

1. R. Azencott, Behavior of diffusion semi-groups at infinity. Bull. Soc. Math. France **102**, 193–240 (1974)
2. R.F. Bass, M. Kassmann, Harnack inequalities for non-local operators of variable order. Trans. Amer. Math. Soc. **357**, 837–850 (2005)
3. R.F. Bass, M. Kassmann, Hölder continuity of harmonic functions with respect to operators of variable order. Commun. Partial Differ. Equ. **30**, 1249–1259 (2005)
4. R.F. Bass, M. Kassmann, T. Kumagai, Symmetric jump processes: localization, heat kernels and convergence. Ann. Inst. Henri Poincaré Probab. Stat. **46**, 59–71 (2010)
5. A.D. Bendikov, Remarks concerning the analysis on local Dirichlet spaces, Dirichlet forms and stochastic processes, Beijing, de Gruyter. Berlin **1995**, 55–64 (1993)
6. E.A. Carlen, S. Kusuoka, D.W. Stroock, Upper bounds for symmetric Markov transition functions. Ann. Inst. H. Poincaré Probab. Stat. **23**, 245–287 (1987)
7. Z.-Q. Chen, M. Fukushima, *Symmetric Markov Processes, Time Change, and Boundary Theory*, London Mathematical Society Monographs Series, vol. 35 (Princeton University Press, Princeton, NJ, 2012)
8. Z.-Q. Chen, K. Kuwae, On doubly Feller property. Osaka J. Math. **46**, 909–930 (2009)
9. G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, 2nd edn. Encyclopedia of Mathematics and its Applications, vol. 152 (Cambridge University Press, Cambridge, 2014)
10. E.B. Davies, *Heat Kernels and Spectral Theory*, Cambridge Tracts in Mathematics, vol. 92. (Cambridge University Press, Cambridge, 1989)

11. E.B. Davies, *Linear Operators and Their Spectra*, Cambridge Studies in Advanced Mathematics, vol. 106 (Cambridge University Press, Cambridge, 2007)
12. N. Dunford, B.J. Pettis, Linear operations on summable functions. Trans. Amer. Math. Soc. **47**, 323–392 (1940)
13. E.B. Dynkin, *Markov Processes I, II* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vols. 121, 122 (Springer, New York, 1965)
14. W. Feller, The parabolic differential equations and the associated semi-groups of transformations. Ann. Math. **55**, 468–519 (1952)
15. M. Fukushima, On an L^p-estimate of resolvent of Markov processes. Publ. Res. Inst. Math. Sci. **13**, 277–284 (1977)
16. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, De Gruyter Studies in Mathematics, vol. 19, second revised and extended (Walter de Gruyter & Co., Berlin, 2011)
17. I.V. Girsanov, On transforming a certain class of stochastic processes by absolutely continuous substitution of measures. Theor. Probab. Appl. **5**, 285–301 (1960)
18. D. Kim, K. Kuwae, Analytic characterizations of gaugeability for generalized Feynman-Kac functionals. Trans. Amer. Math. Soc. **369**, 4545–4596 (2017)
19. M. Kurniawaty, K. Kuwae, K. Tsuchida, On the doubly Feller property of resolvent. Kyoto J. Math. **57**, 637–654 (2017)
20. G. Leoni, *A First Course in Sobolev Spaces*, Graduate Studies in Mathematics, vol. 105. (American Mathematical Society, Providence, RI, 2009)
21. A. Lunardi, Analytic semigroups and optimal regularity in parabolic problems. *Modern Birkhäuser Classics* (Birkhäuser/Springer Basel AG, Basel, 1995)
22. Z.-M. Ma, M. Röckner, Introduction to the theory of (nonsymmetric) Dirichlet forms. Universitext, Springer, Berlin, 1992
23. Z.-M. Ma, M. Röckner, Markov processes associated with positivity preserving coercive forms. Can. J. Math. **47**, 817–840 (1995)
24. S.A. Molchanov, Strong Feller property of diffusion processes on smooth manifolds. Theor. Probab. Appl. **13**, 471–475 (1968)
25. T. Mori, L^p-Kato class measures for symmetric Markov processes under heat kernel estimates. J. Funct. Anal. **281**, 109034 (2021). Available from arXiv:2005.13758
26. Y. Oshima, *Semi-Dirichlet forms and Markov processes*, De Gruyter Studies in Mathematics, vol. 48, 1st edn. (Walter de Gruyter & Co., Berlin, 2013)
27. E.M. Ouhabaz, *Analysis of Heat Equations on Domains*, London Mathematical Society Monographs Series, vol. 31. (Princeton University Press, Princeton, NJ, 2005)
28. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, vol. 44. (Springer, New York, 1983)
29. R.G. Pinsky, *Positive Harmonic Functions and Diffusion*, Cambridge Studies in Advanced Mathematics, vol. 45. (Cambridge University Press, Cambridge, 1995)
30. R.L. Schilling, J. Wang, Strong Feller continuity of Feller processes and semigroups. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **15** (2012)
31. E.A. Stein, *Topics in Harmonic Analysis Related to the Littlewood-Paley Theory*, Annals of Mathematics Studies, vol. 63 (Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1970)
32. Y. Tawara, L^p-independence of spectral bounds of Schrödinger-type operators with non-local potentials. J. Math. Soc. Jpn. **62**, 767–788 (2010)
33. N.T. Varopoulos, Hardy-Littlewood theory for semigroups. J. Funct. Anal. **62**, 240–260 (1985)

Interactions Between Trees and Loops, and Their Representation in Fock Space

Yves Le Jan

Abstract It has been observed that on a weighted graph, an extension of Wilson's algorithm provides an independent pair $(\mathcal{T}, \mathcal{L})$, $\mathcal{T}$ being a spanning tree and $\mathcal{L}$ a Poissonian loop ensemble. This association can be interpreted in the framework of symmetric and skew symmetric Fock spaces. Given a weighted graph, we show how to define a natural interaction between the random spanning tree and the loop ensemble, which corresponds to a local interaction between two Fock spaces.

Keywords Free fields · Markov loops · Spanning trees · Fock space

Mathematics Subject Classification 60J27 · 60G60

1 Framework and Definitions

We first recall some results presented in [2] about loop ensembles and random spanning trees. Consider a system of conductances on a finite connected graph $\mathcal{G} = (\mathcal{X}, \mathcal{E})$ without loop edges nor multiple edges (note that many of the following results are actually valid on infinite graphs, under the transience hypothesis).

After the choice of a root Δ, we denote by E the set of edges not incident to Δ and by E^o the set of oriented such edges. Consider the Dirichlet form defined on $X = \mathcal{X} - \Delta$ by the conductances C on edges of E and the non-vanishing killing measure $\kappa_x = C_{x,\Delta},\ x \in X$:

$$\mathfrak{e}(f, g) = \frac{1}{2} \sum_{x,y \in X} C_{x,y}(f(x) - f(y))(\bar{g}(x) - \bar{g}(y)) + \sum_{x \in X} \kappa_x f(x)\bar{g}(x)$$

Dedicated to Professor Fukushima.

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On X, we define a measure λ by setting $\lambda_x = \sum_y C_{x,y} + \kappa_x$ and denote by M_λ the diagonal matrix representing the multiplication by λ. The Green function on $X \times X$ associated with $\mathfrak{e}$ is $G = [M_\lambda - C]^{-1}$. Recall that $\mathfrak{e}(G_{x,.}, G_{y,.}) = G_{x,y}$. The determinant of G will be denoted by $\mathcal{Z}_\mathfrak{e}$.

An extension of Wilson's algorithm (Cf [2] 8-2, Remark 21) yields an independent pair $(\mathcal{T}, \mathcal{L})$, $\mathcal{T}$ being a spanning tree rooted in Δ and $\mathcal{L}$ a Poissonian loop ensemble on X with intensity given by the loop measure μ defined by the λ-symmetric continuous time Markov chain associated with $\mathfrak{e}$. (Recall that after running Wilson's algorithm, the loops are obtained by dividing, at each vertex, the concatenation of the erased excursions according to a Poisson-Dirichlet distribution, then by forgetting the base points.) We denote by $\mathbb{P}_\mathcal{L}$ and $\mathbb{P}_\mathcal{T}$ their distributions.

We denote by $N_e(\mathcal{L})$ (by $N_{e^o}(\mathcal{L})$) the number of crossings of the edge e (of the oriented edge e^o) by the loops of $\mathcal{L}$ and by $\hat{\mathcal{L}}^x$ the total time spent by the loops of $\mathcal{L}$ at the vertex x.

Recall that for any complex function q, $|q| \leq 1$, defined on the set E^o of oriented edges, and $\chi \geq 0$ defined on X, denoting by $\circ$ the Hadamard product,

$$E\left(\prod_{e^o} q_{e^o}^{N_{e^o}(\mathcal{L})} e^{-\sum_x \chi_x \hat{\mathcal{L}}_x}\right) = \frac{\det(M_\lambda - C)}{\det(M_{\lambda+\chi} - C \circ q)}. \tag{1}$$

Definition 1 The real Gaussian free field is ϕ the real centered Gaussian process indexed by X whose covariance function is given by the Green function G. Let ϕ_1 and ϕ_2 be two independent copies of ϕ. The complex free field $\phi_1 + i\phi_2$ will be denoted by φ.

Remind also the following:

Theorem 1 *The fields* $\widehat{\mathcal{L}}$ *and* $\frac{1}{2}|\varphi|^2$ *have the same distribution.*

Note also that in the finite case, if φ is the complex Gaussian free field associated with the energy $\mathfrak{e}$ and $\mathfrak{e}'$ denotes a different energy, by Eq. 1,

$$\mathbb{E}\left(\prod_{(x,y)}\left[\frac{C'_{x,y}}{C_{x,y}}\right]^{N_{x,y}(\mathcal{L})} e^{-\langle \lambda'-\lambda, \widehat{\mathcal{L}}\rangle}\right) = \left[\frac{\mathcal{Z}_{\mathfrak{e}'}}{\mathcal{Z}_\mathfrak{e}}\right] = \mathbb{E}_\varphi(e^{-\frac{1}{2}[\mathfrak{e}'-\mathfrak{e}](\varphi)}) \tag{2}$$

If ω is a real one-form, i.e. if $\omega_{x,y} = -\omega_{y,x}$ we set:

$$\mathfrak{e}'^{(\omega)}(\varphi, \overline{\varphi}) = \frac{1}{2}\sum_{x,y} C_{x,y}(\varphi(x) - e^{i\omega_{x,y}}\varphi(y))(\overline{\varphi}(x) - e^{-i\omega_{x,y}}\overline{\varphi}(y)) + \sum_x \kappa_x \varphi(x)\overline{\varphi}(x).$$

Then, more generally

$$\mathbb{E}\left(\prod_{(x,y)}\left[\frac{C'_{x,y}e^{i\omega_{x,y}}}{C_{x,y}}\right]^{N_{x,y}(\mathcal{L})} e^{-\langle \lambda'-\lambda, \widehat{\mathcal{L}}\rangle}\right) = \left[\frac{\mathcal{Z}_{\mathfrak{e}',\omega}}{\mathcal{Z}_\mathfrak{e}}\right] = \mathbb{E}_\varphi(e^{-\frac{1}{2}[\mathfrak{e}',\omega-\mathfrak{e}](\varphi)}) \tag{3}$$

2 Interaction Between Tree and Loops

Given a parameter $0 < \beta < 1$ we can define an interacting pair $(\mathcal{T}, \mathcal{L})$ by the joint distribution:

$$\mathbb{P}^{(\beta)}_{\mathcal{T},\mathcal{L}}(T, dL) = \frac{1}{Z^{(\beta)}} \prod_{e \in T} \beta^{N_e(\mathcal{L})} \mathbb{P}_{\mathcal{L}_1}(dL)\, \mathbb{P}_{\mathcal{T}}(T),$$

$Z^{(\beta)}$ denoting the normalization constant $\sum_T [\int \prod_{e\in T} \beta^{N_e(\mathcal{L})} \mathbb{P}_{\mathcal{L}_1}(dL)]\, \mathbb{P}_{\mathcal{T}}(T)$,

As β tends to 0, the loops of $\mathcal{L}$ tend to avoid the tree. If $\beta = 1$, $\mathcal{T}$ and $\mathcal{L}$ are independent.

We can also define another interaction by the joint distribution:

$$\mathbb{P}^{(\beta-)}_{\mathcal{T},\mathcal{L}}(T, dL) = \frac{1}{Z^{(\beta-)}} \prod_{e \notin T} \beta^{N_e(\mathcal{L})} \mathbb{P}_{\mathcal{L}_1}(dL)\, \mathbb{P}_{\mathcal{T}}(T),$$

$Z^{(\beta-)}$ being a normalization constant.

As β tends to 0, the loops of $\mathcal{L}$ tend to be carried by $\mathcal{T}$. In particular, they tend to be contractible to a point.

3 Fock Spaces

In the beginning of this section, we allow the graph to be infinite, assuming only the transience hypothesis. We recall and complete the presentation of Fock spaces given in [2].

Let us first recall the construction of the Bosonic Fock space and its relation with the free field. Let $\mathbb{D}_n$ be the Hilbert space of complex functions v on X^n invariant under any permutation of the variables such that

$$\epsilon_n(v) = n! \sum_{x,y \in X^n} v(x)\bar{v}(y) Per(G_{x_i,y_j}, 1 \le i, j \le n) < \infty.$$

ϵ_n induces naturally a scalar product on $\mathbb{D}_n$. By definition, $\mathbb{D}_0 = \mathbb{C}$.

Note that If $v_1, \dots, v_n$ belong to $\mathbb{D}_1$, $\frac{1}{\sqrt{n!}} \sum_{\sigma \in \mathcal{S}_n} v_{\sigma(1)}(x_1) \dots v_{\sigma(n)}(x_n)$, denoted $v_1 \odot \dots \odot v_n(x_1, \dots, x_n)$, belongs to $\mathbb{D}_n$ with

$$\epsilon_n(v_1 \odot \dots \odot v_n) = Per\left(\sum_{x,y} G_{x,y} v_i(x)\bar{v}_j(y),\ 1 \le i, j \le n\right),$$

and linear combinations of such elements are dense in $\mathbb{D}_n$. In particular, If v belong to $\mathbb{D}_1$, $\epsilon_n(v^{\odot n}) = n!\epsilon_1(v)^n$. The symmetric Fock space $\mathbb{F}_B$ is defined as the space of sequences $u = (u_n, n \ge 0)$, $u_n \in \mathbb{D}_n$ such that the series $\epsilon(u) = \sum \epsilon_n(u_n)$ converges. In particular, If v belong to $\mathbb{D}_1$, $\exp^{\odot}(v)$ belongs to $\mathbb{F}_B$, with energy

$$\epsilon(\exp^{\odot}(v)) = \exp(\epsilon_1(v))$$

The construction of $\mathbb{F}_B$ is known as the bosonic second quantization.

It is well known that there exists a unique isomorphism of Hilbert spaces mapping $\mathbb{F}_B$ onto the space of square integrable functionals of the Gaussian free field, such that, for any real function u in $\mathbb{D}_1$,

$$\exp^{\odot}(u) \rightleftarrows e^{\sum_x \phi(x)\, u(x) - \frac{1}{2}\sum_{x,y} G_{x,y} u(x) u(y)}. \tag{4}$$

Elements of the non-completed Fock space $\oplus_0^{\infty} \mathbb{D}_n$, i.e. the space of finite sequences are mapped by the isomorphism onto square integrable polynomial functions of the free field which can be computed from identity 4.

For any $v \in \mathbb{D}_1$, the annihilation operator a_v and the creation operator a_v^* are defined on $\oplus_0^{\infty} \mathbb{D}_n$: For all v in $\mathbb{D}_1$, $a_v 1 = 0$ and $a_v^* 1 = v$. For any u_n in $\mathbb{D}_n$,

$$a_v u_n(x_1, \ldots, x_n - 1) = \sqrt{\frac{n!}{(n-1)!}} \sum_{y,z} G_{y,z} v(y) \sum_k u_n(x_1 \ldots x_k - 1, z, x_k, \ldots, x_{n-1})$$

$$a_v^* u_n(x_1, \ldots, x_n + 1) = \sqrt{\frac{(n-1)!}{n!}} \sum_k v(x_k) u_n(x_1 \ldots x_k - 1, x_{k+1}, \ldots, x_{n+1})$$

In particular

$$a_v(\mu_1 \odot \ldots \odot \mu_n) = \sum_k \sum_{x,y} G_{x,y} v(x) \mu_k(y) \mu_1 \odot \mu_{k-1} \ldots \odot \mu_{k+1} \ldots \odot \mu_n.$$

$$a_v^* \odot \mu_1 \odot \ldots \odot \mu_n = v \odot \mu_1 \odot \ldots \odot \mu_n.$$

These operator a_v and a_v^* are easily seen to be dual of each other. Set $a_x = a_{\delta_x}$ and $a_x^* =, a_{\delta_x}^*$. These operators verify the bosonic canonical commutation relations:

$$[a_x, a_y^*] = G_{x,y} \;;\; [a_x^*, a_y^*] = [a_x, a_y] = 0.$$

which determine the whole structure. The isomorphism allows to represent these operators on polynomials of the free field as follows:

$$a_x \rightleftarrows \sum_y G_{x,y} \frac{\partial}{\partial \phi(y)}$$
$$a_x^* \rightleftarrows \phi(x) - \sum_y G_{x,y} \frac{\partial}{\partial \phi(y)}$$

Here, $\phi(x)$ is identified to the operator of multiplication by $\phi(x)$. Hence,

$$\phi(x) = a_x + a_x^*$$

Therefore, the Fock space structure is entirely transported on the space of square integrable functionals of the free field. Conversely, expectation calculations of functionals of the Gaussian free field can be expressed in terms of the Fock space scalar product.

In the case of a complex field φ, the space of square integrable functionals of φ and $\overline{\varphi}$ is isomorphic to the tensor product of two copies of the symmetric Fock space $\mathbb{F}_B$, which is the closure of $\oplus_{n,m}\mathbb{D}_n \otimes \mathbb{D}_m$, and which we will denote by $\mathcal{F}_B$. We now get two commuting sets of creation and annihilation operators verifying the Bososnic canonical commutation relations. The Fock space structure is transported as before to the space of square integrable functionals of φ and $\overline{\varphi}$ using these two commuting sets of adjoint creation and annihilation operators defined on polynomials of the field:

$$a_x = \sqrt{2}\sum_y G_{x,y}\frac{\partial}{\partial\varphi(y)} \quad a_x^* = \frac{\varphi(x)}{\sqrt{2}} - \sqrt{2}\sum_y G_{x,y}\frac{\partial}{\partial\overline{\varphi}(y)}$$

$$b_x = \sqrt{2}\sum_y G_{x,y}\frac{\partial}{\partial\overline{\varphi}(y)} \quad b_x^* = \frac{\overline{\varphi}(x)}{\sqrt{2}} - \sqrt{2}\sum_y G_{x,y}\frac{\partial}{\partial\varphi(y)}$$

(Recall that if $z = x + iy$, $\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial \overline{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$).

We have

$$\varphi(x) = \sqrt{2}(b_x + a_x^*) \quad \text{and} \quad \overline{\varphi}(x) = \sqrt{2}(a_x + b_x^*)$$

We now present the skew symmetric counterpart of the free field, defined on the fermionic Fock space.

Let $\mathbb{I}_n$ be the Hilbert space of skew-symmetric complex functions w on X^n, i.e. such that for any permutation σ, $w(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = (-1)^{m(\sigma)}w(x_1, \ldots, x_n)$ and such that

$$\epsilon_n(w) = n! \sum_{x,y\in X^n} w(x)\bar{w}(y)\det(G_{x_i,y_j},\ 1 \le i, j \le n) < \infty.$$

ϵ_n induces naturally a scalar product on $\mathbb{I}_n$. By definition, $\mathbb{I}_0 = \mathbb{C}$.

If $v_1, \ldots v_n$ belong to $\mathbb{D}_1$, $\frac{1}{\sqrt{n!}}\sum_{\sigma\in\mathcal{S}_n}(-1)^{m(\sigma)}v_{\sigma(1)}(x_1)\ldots v_{\sigma(n)}(x_n)$ is denoted $v_1 \wedge \ldots \wedge v_n(x_1, \ldots, x_n)$ and belongs to $\mathbb{I}_n$ with

$$\epsilon_n(v_1 \wedge \ldots \wedge v_n) = \det\left(\sum_{x,y} G_{x,y}v_i(x)\bar{v}_j(y),\ 1 \le i, j \le n\right).$$

The skew-symmetric Fock space $\mathbb{F}_F$ is defined as the space of sequences $u = (w_n,\ n \ge 0),\ w_n \in \mathbb{I}_n$ such that the series $\epsilon(w, w) = \sum \epsilon_n(w_n, w_n)$ converges. Note that if X is finite, these series have at most $|X|+1$ non vanishing terms. The construction of $\mathbb{F}_F$ is known as the fermionic second quantization.

For any $v \in \mathbb{D}_1$, the annihilation operator c_v and the creation operator c_v^* are defined on the uncompleted Fock space $\bigoplus_0^\infty \mathbb{I}_n$ of finite sequences as follows: $c_v 1 = 0$ and $c_v^* 1 = v$, and for any u_n in $\mathbb{I}_n$,

$$c_v u_n(x_1, \ldots, x_{n-1}) \\ = \sqrt{\frac{n!}{(n-1)!}}(-1)^{k-1} \sum_{y,z} G_{y,z} v(y) \sum_k u_n(x_1 \ldots x_k - 1, z, x_k, \ldots, x_{n-1}),$$

$$c_v^* u_n(x_1, \ldots, x_n + 1) = \sqrt{\frac{(n-1)!}{n!}} \sum_k (-1)^{k-1} v(x_k) u_n(x_1 \ldots x_{k-1}, x_{k+1}, \ldots, x_{n+1}).$$

In particular

$$c_v(\mu_1 \wedge \ldots \wedge \mu_n) \\ = (-1)^{k-1} \sum_k \sum_{y,z} G_{y,z} v(y) \mu_k(y) \mu_1 \wedge \ldots \wedge \mu_{k-1} \wedge \mu_{k+1} \wedge \ldots \wedge \mu_n,$$

$$c_v^*(\mu_1 \wedge \ldots \wedge \mu_n) = v \wedge \mu_1 \wedge \ldots \wedge \mu_n.$$

We set $c_x = c_{\delta_x}$ and $c_x^* =, c_{\delta_x}^*$. It can be easily checked that c_y^* is the dual of c_y. The anticommutator $c_x c_y^* + c_y^* c_x$ denoted $[c_x, c_y^*]^+$ equals $G_{x,y}$ and all others anticommutators vanish.

We will work on the complex fermionic Fock space $\mathcal{F}_F$ defined as the tensor product of two copies of $\mathbb{F}_F$. The complex Fock space structure is defined by the two sets of creation and annihilation operators acting on the two copies of $\mathbb{F}_F$. $\mathcal{F}_F$ is generated by the vector $1 \otimes 1$ and creation/annihilation operators c_x, c_x^*, d_x, d_x^* with $[c_x, c_y^*]^+ = [d_x, d_y^*]^+ = G_{x,y}$ and all others anticommutators vanishing.

Anticommuting Grassmann operators $\psi(x), \overline{\psi}(x)$ are defined as operators on the Fermionic Fock space $\mathcal{F}_F$ by:

$$\psi(x) = \sqrt{2}(d_x + c_x^*) \quad \text{and} \quad \overline{\psi}(x) = \sqrt{2}(-c_x + d_x^*).$$

The following anticommutation relations hold

$$[\psi(x), \psi(y)]^+ = [\overline{\psi}(x), \psi(y)]^+ = [\overline{\psi}(x), \overline{\psi}(y)]^+ = 0$$

In particular, $(\sum_x \lambda_x \psi(x))^2 = (\sum_x \lambda_x \overline{\psi}(x))^2 = 0$. A simple calculation yields this property of the scalar product on $\mathcal{F}_F$:

$$\left\langle 1, \psi(x_m) \ldots \psi(x_1) \overline{\psi}(y_1) \ldots \overline{\psi}(y_n) 1 \right\rangle_{\mathcal{F}_F} = \delta_{nm} 2^n \det(G_{x_i, y_j}). \tag{5}$$

Therefore

$$\left\langle 1, \psi(x_n)\overline{\psi}(y_n)\ldots\psi(x_1)\overline{\psi}(y_1)1\right\rangle_{\mathcal{F}_F} = 2^n \det(G_{x_i,y_j}).$$

We now consider again a finite graph.

Proposition 1 *If $\mathfrak{e}'$ is a Dirichlet form defined by another set of conductances and another killing measure:*

$$\left\langle 1, \exp(-\frac{1}{2}\mathfrak{e}(\psi,\overline{\psi}) + \frac{1}{2}\mathfrak{e}'(\psi,\overline{\psi}))1\right\rangle_{\mathcal{F}_F} = \frac{\det(G)}{\det(G')} = \frac{\mathcal{Z}_e}{\mathcal{Z}_{e'}}. \tag{6}$$

Proof Indeed, if f_i is an orthonormal basis of the Dirichlet space in which $\mathfrak{e}'$ is diagonal with eigenvalues λ_i, the first side equals

$$\left\langle 1, \prod_i \exp(\frac{1}{2}(-1+\lambda_i)\langle\psi, f_i\rangle\langle\overline{\psi}, f_i\rangle)1\right\rangle_{\mathcal{F}_F}$$
$$= \left\langle 1, \prod_i (1+\frac{1}{2}(-1+\lambda_i)\langle\psi, f_i\rangle\langle\overline{\psi}, f_i\rangle)1\right\rangle_{\mathcal{F}_F}$$
$$= 1 + \sum_k \sum_{i_1<\ldots i_k} (-1+\lambda_{i_1})\ldots(-1+\lambda_i) = \prod \lambda_i.$$

In particular, for any positive measure χ on X,

$$\left\langle 1, \exp(\frac{1}{2}\sum_x \chi_x \psi(x)\overline{\psi}(x))1\right\rangle_{\mathcal{F}_F} = \frac{\det(G)}{\det(G_\chi)}.$$

More generally

$$\left\langle 1, \exp(\frac{1}{2}\mathfrak{e}'^{(\omega)}(\psi,\overline{\psi}) - \frac{1}{2}\mathfrak{e}(\psi,\overline{\psi}))1\right\rangle_{\mathcal{F}_F} = \frac{\mathcal{Z}_e}{\mathcal{Z}_{e',(\omega)}}. \tag{7}$$

More results on Fock spaces calculations and their relation to Grassmann integration (Cf [1]) and Wick products can be found in [3].

In the finite case, we can find a fermionic analogue of the elementary definition of the free field through the Gaussian density function and provide a slightly more elementary approach to fermionic calculations which is also closer from the formal path integrals calculations performed in theoretical physics.

Proposition 2 *Let u_x and $\bar{u}_x$ be the canonical basis of two copies of $\mathbb{R}^{|X|}$. Set $\eta = \bigwedge_x u_x \wedge \bar{u}_x$. $\sigma = \sum_x \lambda_x u_x \wedge \bar{u}_x - \sum_{x,y} C_{x,y} u_x \wedge \bar{u}_y$ and $\nu = 1 + \sum_1^{|X|} \frac{\sigma^{\wedge k}}{k!}$. Then, for any antisymmetric polynomial P:*

$$\langle 1, P(\psi,\bar{\psi})1)\rangle_{\mathcal{F}_F} = \det(G)\langle P^{\wedge}(\sqrt{2}\,u, \sqrt{2}\,\bar{u}) \wedge \nu, \eta\rangle_{\bigwedge \mathbb{R}^{|X|}\oplus\,\mathbb{R}^{|X|}}.$$

Here $P^\wedge$ denotes the polynomial in which the product is replaced by the wedge product.

Note that $1 + \sum_1^{|X|} \frac{\sigma^{\wedge k}}{k!!}$ is formally equal to $\exp^\wedge(e^\wedge(u, \bar{u}))$ and that η does not depend on an order on vertices. Moreover, note that σ and ν is formally obtained by replacing the product by the wedge product respectively in the Dirichlet form and in the Gaussian density function of the complex free field with respect to Lebesgue measure. This formulation is essentially equivalent to the use of Grassmann variables [1].

Proof It is clear that to get a non-vanishing result, the total degrees of P in u and $\bar{u}$ have to be equal.

For any ONB $\alpha^i = \sum_x a_x^i u_x$ of $\mathbb{R}^{|X|}$, setting $\bar{\alpha}^i = \sum_x a_x^i \bar{u}_x$, we have $\eta = \bigwedge_i \alpha_i \wedge \bar{\alpha}_i$. We choose the α^i to be the eigenvectors of $M(\lambda) - C$ and denote by r_i the associated eigenvalues, so that $G_{x,y} = \sum_i \frac{1}{r_i} a_x^i a_y^i$ and $\sigma = \sum_i r_i \, \alpha_i \wedge \bar{\alpha}_i$. Note that $u_x = \sum_i a_x^i \alpha^i$ and $\bar{u}_x = \sum_i a_x^i \bar{\alpha}^i$.

$\langle \bigwedge_{j=1}^n u_{x_j} \wedge \bar{u}_{y_j} \wedge \frac{1}{k!}\sigma^{\wedge k}, \eta\rangle$ vanishes if $k \neq |X| - n$, and for $k = |X| - n$:

$$\bigwedge_{j=1}^n u_{x_j} \wedge \bar{u}_{y_j} \wedge \frac{1}{k!}\sigma^{\wedge k} = \sum_{i_1 < \cdots < i_k} \prod_{l=1}^k r_{i_l} \bigwedge_j u_{x_j} \wedge \bar{u}_{y_j} \wedge \alpha_{i_1} \wedge \bar{\alpha}_{i_1} \wedge \ldots \alpha_{i_k} \wedge \bar{\alpha}_{i_k}.$$

Note that $\det(G) = \prod_1^{|X|} \frac{1}{r_i}$, $u_x = \sum_i a_x^i \alpha^i$ and $\bar{u}_x = \sum_i a_x^i \bar{\alpha}^i$. Then

$$\begin{aligned}
&\bigwedge_{j=1}^n u_{x_j} \wedge \bar{u}_{y_j} \wedge \frac{1}{k!}\sigma^{\wedge k} \\
&= \frac{1}{\det(G)} \sum_{h_1 < \cdots < h_n} \det(a_{x_l}^{h_l},\ 1 \le l \le n) \det(a_{y_l}^{h_l},\ 1 \le l \le n) \prod_{l=1}^n \frac{1}{r_{h_l}} \\
&= \frac{\det(G_{x_i,y_j}\ 1 \le i, j \le n)}{\det(G)}
\end{aligned}$$

The proof then follows directly from Eq. 5.

Recall finally the following result of [2] which provides a relation between the Grassmann operators and the random spanning tree.

Theorem 2 *For distinct edges $\pm\xi_1, \ldots \pm \xi_k$:*

$$\mathbb{P}_{ST}^e(\pm\xi_1, \ldots \pm \xi_k \in \mathcal{T}) = 2^{-k} \prod C_{x_i,y_i} \left\langle 1, (\prod(\psi(y_i) - \psi(x_i))(\overline{\psi}(y_i) - \overline{\psi}(x_i))1 \right\rangle_{\mathcal{F}_F}.$$

For distinct vertices $x_1, \ldots, x_k$:

$$\mathbb{P}_{ST}^e((x_i, \delta) \in \tau) = 2^{-k} \prod \kappa_{x_i} \left\langle 1, \prod(\psi(x_i)\overline{\psi}(x_i)\ 1 \right\rangle_{\mathcal{F}_F}.$$

4 Local Interaction in Supersymmetric Fock Space

The independent pair $(\mathcal{T}, \mathcal{L})$, associating a spanning tree $\mathcal{T}$ and a Poissonian loop ensemble $\mathcal{L}$ can be interpreted in the framework of these symmetric and skew symmetric Fock spaces. We now show how the natural interactions between the random spanning trees and the loop ensemble defined in Sect. 2 correspond to *local interactions* between these two Fock spaces. The partition function and more generally expectations of various functionals of the random pair $(\mathcal{T}, \mathcal{L})$ can be expressed in terms of the supersymmetric Fock space associated with G.

First note that it follows from (1) and from Fock space calculations that for any complex function q, $|q| \leq 1$, defined on the set E^o of oriented edges, and $\chi \geq 0$ defined on X,

$$E(\prod_{e^o} q_{e^o}^{N_{e^o}(\mathcal{L})} e^{-\sum_x \chi_x \hat{\mathcal{L}}_x}) = \langle 1,\ \exp(-\frac{1}{2}\sum_{x,y} C_{x,y}[q_{x,y}-1]\varphi_x\bar{\varphi}_y - \frac{1}{2}\sum_x \chi_x\varphi_x\bar{\varphi}_x)\ 1\rangle \tag{8}$$

Then it follows from Theorem 2 and an inclusion-exclusion argument that for any pair of functions b and c defined on edges, setting

$$|d\psi_{\{x,y\}}|^2 = \frac{1}{2}C_{x,y}(\psi(x)-\psi(y))(\overline{\psi}(x)-\overline{\psi}(y)),$$

$$\mathbb{E}(\prod_e (b_e 1_{e\notin\mathcal{T}} + c_e 1_{e\in\mathcal{T}})) = \langle 1,\ \prod_e (b_e(1-|d\psi_e|^2) + c_e|d\psi_e|^2)\ 1\rangle \tag{9}$$

Theorem 3 *For* $0 < \beta \leq 1$, *setting* $\varphi\overline{\varphi}(x,y) = \varphi(x)\overline{\varphi}(y) + \varphi(y)\overline{\varphi}(x)$*:*

$$Z^{(\beta)} = \langle 1, e^{\sum_{\{x,y\}} |d\psi_{\{x,y\}}|^2 (e^{\frac{1}{2}C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y))}-1)}\ 1\rangle$$

and for any positive functional F,

$$\sum_T \int F(\hat{L})\mathbb{P}^{(\beta+)}_{\mathcal{T},\mathcal{L}}(T, dL) = \frac{1}{Z^{(\beta)}}\langle 1,\ F(\frac{1}{2}\varphi\overline{\varphi})\ e^{\sum_{\{x,y\}}[|d\psi_{\{x,y\}}|^2(e^{\frac{1}{2}C_{x,y}[1-\beta](\varphi\overline{\varphi}(x,y))}-1)]}\ 1\rangle$$

Note that Proposition 2 allows to give another expression of $\sum_T \int F(\hat{L})\mathbb{P}^{(\beta+)}_{\mathcal{T},\mathcal{L}}(T, dL)$ in term of Lebesgue and Grassmann integrals, which is similar to the expressions obtained in theoretical physics for some interacting fields.

Proof It is enough to prove the result for Laplace transforms.

$$\sum_T \int e^{-\sum_x \chi_x \hat{L}_x} \prod_{e\in\mathcal{T}} \beta^{N_e(L)}\, \mathbb{P}_{\mathcal{L}}(dL)\, \mathbb{P}_{\mathcal{T}}(T) \text{ equals}$$

$$\sum_T \int \langle 1,\; e^{-\frac{1}{2}\sum_x \chi_x \varphi(x)\overline{\varphi}(x)} \prod_{\{x,y\}\in T} e^{\frac{1}{2}C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y))}\; 1\rangle\, \mathbb{P}_{\mathcal{T}}(T)$$

$$= \langle 1,\; e^{-\frac{1}{2}\sum_x \chi_x \varphi(x)\overline{\varphi}(x)} \prod_{\{x,y\}} [e^{\frac{1}{2}C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y))} |d\psi_{\{x,y\}}|^2 + 1 - |d\psi_{\{x,y\}}|^2]\; 1\rangle$$

$$= \langle 1,\; e^{-\frac{1}{2}\sum_x \chi_x \varphi(x)\overline{\varphi}(x)} \prod_{\{x,y\}} [1 - |d\psi_{\{x,y\}}|^2(1 - e^{\frac{1}{2}C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y))})]\; 1\rangle$$

$$= \langle 1,\; e^{-\frac{1}{2}\sum_x \chi_x \varphi(x)\overline{\varphi}(x)} \prod_{\{x,y\}} e^{|d\psi_{\{x,y\}}|^2 (e^{\frac{1}{2}C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y))}-1)}\; 1\rangle$$

$$= \langle 1,\; e^{-\frac{1}{2}\sum_x \chi_x \varphi(x)\overline{\varphi}(x)}\, e^{\sum_{\{x,y\}} |d\psi_{\{x,y\}}|^2 (e^{\frac{1}{2}C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y))}-1)}\; 1\rangle$$

Remarks

- Note that for β close to 1, the joint distribution $\mathbb{P}^{(\beta)}_{\mathcal{T},\mathcal{L}}$ is a perturbation of the product $\mathbb{P}_{\mathcal{L}} \otimes \mathbb{P}_{\mathcal{T}}$. The Fock space representation allows to expand the partition function and related expressions according to powers of $1-\beta$.
- More general formulas relating trees and loop observables expectations to Fock space expressions can be derived from Eqs. 3, 7, and Theorem 2.
- A similar representation (in terms of $\psi\overline{\psi}$) can be given with the random set of vertices connected to the root by the spanning tree.

Similar results hold for $\mathbb{P}^{(\beta-)}_{\mathcal{T},\mathcal{L}}$:

Proposition 3 *Assuming in addition $\kappa > 1-\beta$ so that $e^{\frac{1}{2}C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y))}$ is well defined in Fock space:*

$$Z^{(\beta-)} = \langle 1,\; e^{+\frac{1}{2}\sum_{\{x,y\}}[(C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y)) + |d\psi_{\{x,y\}}|^2(e^{+\frac{1}{2}C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y))}-1)]}\; 1\rangle.$$

Proof $\sum_T \int \prod_{e\notin\mathcal{T}} \beta^{N_e(L)}\, \mathbb{P}_{\mathcal{L}}(dL)\, \mathbb{P}_{\mathcal{T}}(T)$ equals

$$\sum_T \int \langle 1,\; \prod_{\{x,y\}\notin T} e^{\frac{1}{2}C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y))}\; 1\rangle\, \mathbb{P}_{\mathcal{L}}(T)$$

$$= \langle 1,\; \prod_{\{x,y\}} [e^{\frac{1}{2}C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y))}(1 - |d\psi_{\{x,y\}}|^2) + |d\psi_{\{x,y\}}|^2]\; 1\rangle$$

$$= \langle 1,\; \prod_{\{x,y\}} [e^{\frac{1}{2}C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y))} + |d\psi_{\{x,y\}}|^2(1 - e^{-\frac{1}{2}C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y))})]\; 1\rangle$$

$$= \langle 1,\; \prod_{\{x,y\}} e^{\frac{1}{2}C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y))}[1 + |d\psi_{\{x,y\}}|^2(e^{-\frac{1}{2}C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y))} - 1)]\; 1\rangle$$

$$= \langle 1,\; e^{+\frac{1}{2}\sum_{\{x,y\}}[(C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y)) + |d\psi_{\{x,y\}}|^2(e^{-\frac{1}{2}C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y))}-1)]}\; 1\rangle$$

More generally, $\sum_T \int F(\hat{L})\mathbb{P}^{(\beta-)}_{\mathcal{T},\mathcal{L}}(T, dL)$ equals

$$\frac{1}{Z^{(\beta-)}}\langle 1,\ F(\frac{1}{2}\varphi\overline{\varphi})\, e^{+\frac{1}{2}\sum_{\{x,y\}}[(C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y))+|d\psi_{\{x,y\}}|^2(e^{+\frac{1}{2}C_{x,y}[\beta-1](\varphi\overline{\varphi}(x,y))}-1)]}\ 1\rangle.$$

References

1. F. Berezin, *The Method of Second Quantization* (Academic, 1966)
2. Y. Le Jan, Markov paths, loops and fields. *École d'Été de Probabilités de Saint-Flour XXXVIII—2008*. Lecture Notes in Mathematics, vol. 2026 (Springer, Berlin-Heidelberg, 2011)
3. Y. Le Jan, On the Fock space representation of functionals of the occupation field and their renormalization. J.F.A. **80**, 88–108 (1988)

Remarks on Quasi-regular Dirichlet Subspaces

Liping Li

Abstract Some remarks on quasi-regular Dirichlet subspaces are given in this paper. We first introduce this concept by extending the notion of regular Dirichlet subspaces and then derive a basic type theorem for quasi-regular Dirichlet subspaces. Further remarks on quasi-regular Dirichlet subspaces of concrete Dirichlet forms, especially associated with Brownian motions, are also presented.

Keywords Dirichlet forms · Quasi-regular Dirichlet subspaces · Regular Dirichlet subspaces · Brownian motions.

1 Introduction

Let E be a Hausdorff space endowed with the Borel σ-algebra, m a fully supported σ-finite measure on E, and $(\mathscr{E}, \mathscr{F})$ a symmetric Dirichlet form on $L^2(E, m)$. It is well known that when $(\mathscr{E}, \mathscr{F})$ satisfies some condition (say regularity or quasi-regularity), it would correspond to an m-symmetric Markov process on E (see [1, 3, 7]). On the other hand, $\mathscr{F}$ is a Hilbert space with respect to the inner product $\mathscr{E}_1(\cdot, \cdot) := \mathscr{E}(\cdot, \cdot) + (\cdot, \cdot)_m$, where $(\cdot, \cdot)_m$ is the inner product of $L^2(E, m)$. Then it is a very basic problem to explore the closed subspaces of $\mathscr{F}$ with probabilistic meaning: The existence of proper ones and how to characterize them if exist.

The notion of Dirichlet subspaces was first raised in [2] for so-called regular Dirichlet forms. Precisely, further let E be a locally compact separable metric space, m a fully supported Radon measure on E, and $(\mathscr{E}, \mathscr{F})$ a regular Dirichlet form on $L^2(E, m)$. Another regular Dirichlet form $(\mathscr{E}', \mathscr{F}')$ on $L^2(E, m)$ is called a *regular Dirichlet subspace* of $(\mathscr{E}, \mathscr{F})$ if

$$\mathscr{F}' \subset \mathscr{F}, \quad \mathscr{E}(u, v) = \mathscr{E}'(u, v), \quad u, v \in \mathscr{F}'. \tag{1}$$

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We need to emphasize that $(\mathscr{E}', \mathscr{F}')$ is on the same L^2-space (strictly speaking, the same state space E and symmetric measure m) as $(\mathscr{E}, \mathscr{F})$. By this definition, $\mathscr{F}'$ is clearly a closed subspace of $\mathscr{F}$ under the inner product $\mathscr{E}_1$, and moreover, enjoys two additional properties:

(1) Dirichlet property: $u \in \mathscr{F}'$ implies $u^+ \wedge 1 \in \mathscr{F}'$, where $u^+ := u \vee 0$;
(2) Regularity: $\mathscr{F}' \cap C_c(E)$ is dense in $\mathscr{F}'$ with respect to $\mathscr{E}_1$-norm and dense in $C_c(E)$ with respect to the uniform norm, where $C_c(E)$ is the family of all continuous functions on E with compact supports.

As we know, these two properties bring the probabilistic significance of $\mathscr{F}'$: There exists another m-symmetric Markov process associated with $(\mathscr{E}', \mathscr{F}')$. Furthermore, the Dirichlet property (1) is equivalent to the Markovian property of this associated process and the regularity (2) is a sufficient condition for the existence of this Markov process. Further considerations on regular Dirichlet subspaces are included in [2, 4–6]. Particularly, not only for the existence and characterization of proper regular Dirichlet subspaces, it is also possible to analyse the structures of associated Markov processes through some special methods, see [6] for the discussions about the regular Dirichlet subspaces of one-dimensional Brownian motion.

Note that the regularity is not a necessary condition for a Dirichlet form associated with a Markov process. Quasi-regular Dirichlet forms are more general (especially on an infinite dimensional state space) and in one-to-one correspondence with good Markov processes due to [7]. This inspires us to replace the regularity (2) by the quasi-regularity (Clearly, the Dirichlet property (1) cannot be removed) and to consider another kind of Dirichlet subspaces. We name them the *quasi-regular Dirichlet subspaces*.

This paper is organized as follows. In Sect. 2, we shall introduce the concept of quasi-regular Dirichlet subspaces imposing an additional condition (S2) to ensure the inclusion relation in the sense of quasi-continuous equivalence classes. Particularly, a regular Dirichlet subspace is always a quasi-regular one. Then a basic type theorem for the quasi-regular Dirichlet subspaces is derived in Theorem 1. In Sect. 3, the quasi-regular Dirichlet subspaces of Brownian motions are considered. It is shown that for one-dimensional Brownian motion, its quasi-regular Dirichlet subspaces and regular Dirichlet subspaces coincide, but for multi-dimensional Brownian motion, we can raise some quasi-regular Dirichlet subspaces, which are not regular. Further remarks are presented in Sect. 3.2.2 for other attempts on multi-dimensional Brownian motion and in Sect. 4 for our future plans.

2 Quasi-regular Dirichlet Subspaces

As in Sect. 1, let E be a Hausdorff space and m a fully supported σ-finite measure on E. We are given a quasi-regular Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $L^2(E, m)$. For the definition of quasi-regularity and other concepts related to Dirichlet forms, we refer to [1, 3, 7]. The quasi-regular Dirichlet subspace is defined as follows.

Definition 1 Let $(\mathscr{E}, \mathscr{F})$ and $(\mathscr{E}', \mathscr{F}')$ be two quasi-regular symmetric Dirichlet forms on $L^2(E, m)$. We say $(\mathscr{E}', \mathscr{F}')$ is a *quasi-regular Dirichlet subspace* of $(\mathscr{E}, \mathscr{F})$, if

(S1) $\mathscr{F}' \subset \mathscr{F}, \quad \mathscr{E}(u, v) = \mathscr{E}'(u, v), \quad \forall u, v \in \mathscr{F}'$;
(S2) Any $\mathscr{E}'$-nest is also an $\mathscr{E}$-nest.

Remark 1 (S2) implies

(1) $\mathscr{E}'$-quasi-continuous function is also $\mathscr{E}$-quasi-continuous;
(2) $\mathscr{E}'$-exceptional set is also $\mathscr{E}$-exceptional.

In (S1) of the above definition, $\mathscr{F}' \subset \mathscr{F}$ is an inclusion relation in the sense of L^2-equivalence class. Precisely speaking, for any $u \in \mathscr{F}'$, it means that u is an L^2-equivalence class (not quasi-continuous equivalence class) and $u \in \mathscr{F}$. The quasi-continuous equivalence classes of u relative to $\mathscr{E}$ and $\mathscr{E}'$ may not satisfy the inclusion relation if without (S2), see the following example.

Example 1 Let $E = [0, 1]$ and m be the Lebesgue measure on $[0, 1]$. We first take $(\mathscr{E}', \mathscr{F}')$ to be the associated Dirichlet form of absorbing Brownian motion on $(0, 1)$, i.e.

$$\mathscr{F}' := \big\{u \in L^2((0, 1)) : u \text{ is absolutely continuous}, \\ u' \in L^2((0, 1)), u(0) = u(1) = 0\big\},$$

$$\mathscr{E}'(u, v) := \frac{1}{2}\int_0^1 u'(x)v'(x)dx, \quad u, v \in \mathscr{F}'.$$

Since $(\mathscr{E}', \mathscr{F}')$ is a regular Dirichlet form on $L^2((0, 1))$, it is quasi-regular on $L^2([0, 1])$. Note that $\{0, 1\}$ is an $\mathscr{E}'$-exceptional set.

Define another quadratic form on $L^2([0, 1])$ as follows:

$$\mathscr{F} := \{u \in L^2([0, 1]) : u \text{ is absolutely continuous}, \ u' \in L^2([0, 1])\big\},$$

$$\mathscr{E}(u, u) := \frac{1}{2}\int_0^1 u'(x)^2 dx + j(u(0) - u(1))^2 + k_0 u(0)^2 + k_1 u(1)^2, \quad u \in \mathscr{F},$$

where j, k_0 and k_1 are fixed non-negative constants. We assert $(\mathscr{E}, \mathscr{F})$ is a regular Dirichlet form on $L^2([0, 1])$. In fact, let $\mathbf{D}$ be the Dirichlet integral on $[0, 1]$, i.e. for any $u, v \in \mathscr{F}$, $\mathbf{D}(u, v) := \int_0^1 u'(x)v'(x)dx$. By [1, (2.2.31)], we have

$$\sup_{x\in[0,1]} u(x)^2 \leq c\mathbf{D}_1(u, u), \quad \forall u \in \mathscr{F},$$

where c is independent of u. Thus there exists another constant $C > 0$ such that for any $u \in \mathscr{F}$,

$$C^{-1}\mathbf{D}_1(u, u) \leq \mathscr{E}_1(u, u) \leq C\mathbf{D}_1(u, u).$$

From the regularity of $(\frac{1}{2}\mathbf{D}, \mathscr{F})$ (associated with the reflected Brownian motion on $[0, 1]$), we can conclude that $(\mathscr{E}, \mathscr{F})$ is a regular Dirichlet form on $L^2([0, 1])$. Furthermore, the $\mathscr{E}$-exceptional set must be the empty set.

Clearly, $(\mathscr{E}, \mathscr{F})$ and $(\mathscr{E}', \mathscr{F}')$ satisfy (S1). However, (S2) does not hold. For example, $\{[\frac{1}{n}, 1 - \frac{1}{n}] : n \geq 1\}$ is an $\mathscr{E}'$-nest, but not an $\mathscr{E}$-nest.

Finally, we show that the inclusion relation in the sense of quasi-continuous equivalence class does not hold for $\mathscr{E}$ and $\mathscr{E}'$. Take a function $\varphi \in C_c^\infty((0, 1)) \subset \mathscr{F}' \subset \mathscr{F}$. Denote the families of all $\mathscr{E}$ and $\mathscr{E}'$ quasi-continuous functions, which are $\mathscr{E}$-q.e. and $\mathscr{E}'$-q.e. equal to φ, by $[\varphi]_{\mathscr{E}}$ and $[\varphi]_{\mathscr{E}'}$ respectively. Then we have

$$[\varphi]_{\mathscr{E}'} = \left\{\phi \text{ on } [0, 1] : \phi|_{(0,1)} = \varphi\right\},$$
$$[\varphi]_{\mathscr{E}} = \left\{\phi \text{ on } [0, 1] : \phi|_{(0,1)} = \varphi, \phi(0) = \phi(1) = 0\right\}.$$

Notice that the function ϕ in $[\varphi]_{\mathscr{E}'}$ can be arbitrarily defined at $\{0, 1\}$ but however, $[\varphi]_{\mathscr{E}}$ has only one element. Therefore, $[\varphi]_{\mathscr{E}'}$ is not contained in $[\varphi]_{\mathscr{E}}$.

The following lemma indicates that in the context of regular Dirichlet forms, (S2) is not necessary for Dirichet subspaces. This is the reason why the regular Dirichlet subspace is defined as (1) in §1.

Lemma 1 *Assume that E is a locally compact separable metric space and m is a fully supported Radon measure on E. Let $(\mathscr{E}, \mathscr{F})$ and $(\mathscr{E}', \mathscr{F}')$ be two Dirichlet forms on $L^2(E, m)$. Assume further that*

(1) $(\mathscr{E}', \mathscr{F}')$ is regular;
(2) $(\mathscr{E}, \mathscr{F})$ has an $\mathscr{E}$-nest $\{K_n : n \geq 1\}$ of compact sets.

Then (S1) implies (S2). Particularly, a regular Dirichlet subspace is always a quasi-regular Dirichlet subspace.

Proof Take a positive function $\varphi \in L^2(E, m)$ with $0 < \varphi \leq 1$ and set $h := G_1\varphi$, where G_1 is the 1-resolvent of $(\mathscr{E}, \mathscr{F})$. Clearly, we have $0 < h \leq 1$. The h-capacity relative to $(\mathscr{E}, \mathscr{F})$ is denoted by Cap_h. Precisely, for any open set $D \subset E$, set

$$\mathscr{L}_{D,h} := \{u \in \mathscr{F} : u \geq h,\ m\text{-a.e. on } D\}.$$

Then

$$\text{Cap}_h(D) = \mathscr{E}_1(h_D, h_D),$$

where $h_D \in \mathscr{L}_{D,h}$ is the reduced function of h on D. Since $(\mathscr{E}', \mathscr{F}')$ is regular, we can define its 1-capacity, which is denoted by Cap'. Note that by (S1)

$$\text{Cap}'(D) = \inf\{\mathscr{E}_1'(u, u) : u \in \mathscr{L}_{D,1}\} = \inf\{\mathscr{E}_1(u, u) : u \in \mathscr{L}_{D,1}\},$$

where

$$\mathscr{L}_{D,1} = \{u \in \mathscr{F}' : u \geq 1,\ m\text{-a.e. on } D\}.$$

Since $\mathscr{F}' \subset \mathscr{F}$ and $h \leq 1$, it follows that $\mathscr{L}_{D,1} \subset \mathscr{L}_{D,h}$ and thus $\mathrm{Cap}'(D) \geq \mathrm{Cap}_h(D)$ for any open set D. This implies

$$\mathrm{Cap}'(A) \geq \mathrm{Cap}_h(A)$$

for any Borel subset A of E.

Now let $\{F_n\}$ be an $\mathscr{E}'$-nest. Equivalently, for any compact set $K \subset E$,

$$\lim_{n\to\infty} \mathrm{Cap}'(K \setminus F_n) = 0.$$

Fix a small constant $\varepsilon > 0$. Since $\{K_n\}$ is an $\mathscr{E}$-nest, we can take an integer m such that $\mathrm{Cap}_h(K_m^c) < \varepsilon/2$. Notice that K_m is compact, hence we can take another integer n_m such that $\mathrm{Cap}'(K_m \setminus F_{n_m}) < \varepsilon/2$. It follows that

$$\begin{aligned}
\mathrm{Cap}_h(F_{n_m}^c) &\leq \mathrm{Cap}_h\left(K_m^c \cup (K_m \setminus F_{n_m})\right) \\
&\leq \mathrm{Cap}_h(K_m^c) + \mathrm{Cap}_h(K_m \setminus F_{n_m}) \\
&\leq \mathrm{Cap}_h(K_m^c) + \mathrm{Cap}'(K_m \setminus F_{n_m}) \\
&< \varepsilon.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \mathrm{Cap}_h(F_n^c) = 0$ and thus $\{F_n\}$ is also an $\mathscr{E}$-nest. That completes the proof.

Next, we shall show a basic type theorem for quasi-regular Dirichlet subspaces. Every quasi-regular symmetric Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $L^2(E, m)$ satisfies the so-called Beurling-Deny decomposition: For any $u \in \mathscr{F}$,

$$\mathscr{E}(u, u) = \mathscr{E}^{(c)}(u, u) + \int_{E\times E\setminus d} (\tilde{u}(x) - \tilde{u}(y))^2\, J(dxdy) + \int_E \tilde{u}(x)^2 k(dx),$$

where $\tilde{u}$ is the quasi-continuous version of u, $\mathscr{E}^{(c)}$ is the strongly local part, J is a σ-finite symmetric measure on $E \times E \setminus d$ such that J does not charge any subset whose marginal projection is $\mathscr{E}$-polar and k is a σ-finite measure on E charging no $\mathscr{E}$-exceptional set. The following theorem is an extended result of [4, Theorem 2.1].

Theorem 1 *Let $(\mathscr{E}', \mathscr{F}')$ be a quasi-regular Dirichlet subspace of $(\mathscr{E}, \mathscr{F})$ on $L^2(E, m)$. Denote the jumping and killing measures of $(\mathscr{E}, \mathscr{F})$ and $(\mathscr{E}', \mathscr{F}')$ by J, J' and k, k' respectively. Then $J = J'$ and $k = k'$.*

Mimicking the proof of [7, VI, Theorem 1.2], we can assume, without loss of generality, E is a locally compact separable metric space, m is Radon and $(\mathscr{E}', \mathscr{F}')$ is regular on $L^2(E, m)$. Particularly, J' and k' are Radon measures. Then the rest of the proof is similar to that of [4, Theorem 2.1] and we omit it.

Remark 2 We have several facts relevant to this theorem:

(1) If without (S2) in Definition 1, then the conclusion of Theorem 1 may not hold. See Example 1.
(2) Both or neither of $(\mathscr{E}, \mathscr{F})$ and $(\mathscr{E}', \mathscr{F}')$ have jumping or killing parts.
(3) Clearly, $\mathscr{E}^{(c)} \equiv 0$ implies $\mathscr{E}'^{(c)} \equiv 0$, where $\mathscr{E}'^{(c)}$ is the strongly local part of $(\mathscr{E}', \mathscr{F}')$, but however not vice versa. There is an example in [6, Theorem 2.1] for 'not vice versa'.
(4) When $(\mathscr{E}, \mathscr{F})$ is of pure-jump type, i.e. $\mathscr{E}^{(c)} \equiv 0$ and $k = 0$, then a proper regular (hence quasi-regular) Dirichlet subspace for $(\mathscr{E}, \mathscr{F})$ may exist, see [5, Corollary 5.5].

3 Quasi-regular Dirichlet Subspaces of Concrete Dirichlet Forms

3.1 One-Dimensional Brownian Motion

The associated Dirichlet form of one-dimensional Brownian motion on $L^2(\mathbb{R})$ is $\left(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R})\right)$, where

$$H^1(\mathbb{R}) = \left\{u \in L^2(\mathbb{R}) : u \text{ is absolutely continuous and } u' \in L^2(\mathbb{R})\right\},$$
$$\frac{1}{2}\mathbf{D}(u, v) = \frac{1}{2}\int_{\mathbb{R}} u'(x)v'(x)dx, \quad u, v \in H^1(\mathbb{R}).$$

Clearly, $\left(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R})\right)$ is regular on $L^2(\mathbb{R})$. It is shown in [2] that each regular Dirichlet subspace of $\left(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R})\right)$ can be characterized by an absolutely continuous and strictly increasing function $\mathtt{s}$ on $\mathbb{R}$ such that

$$\mathtt{s}' = 0 \text{ or } 1, \quad a.e. \tag{2}$$

The function $\mathtt{s}$ is usually called the scale function of the associated diffusion process of regular Dirichlet subspace. Following [6], we can give an alternative expression of the regular Dirichlet subspaces. In fact, set a family of a.e. defined sets as follows:

$$\mathfrak{F} := \{F \subset \mathbb{R} : \lambda\left(F^c \cap (a, b)\right) > 0, \ \forall a < b\},$$

where λ is the Lebesgue measure on $\mathbb{R}$. Some typical examples for $F \in \mathfrak{F}$ are the generalized Cantor sets. Note that every set $F \in \mathfrak{F}$ corresponds to a scale function $\mathtt{s}$ satisfying (2): $F = \{x : \mathtt{s}'(x) = 0\}$ and on the contrary, $\mathtt{s}(x) = \mathtt{s}(0) + \int_0^x 1_{F^c}(y)\lambda(dy)$. The following lemma is taken from [6, Lemma 2.1].

Lemma 2 *$(\mathscr{E}', \mathscr{F}')$ is a regular Dirichlet subspace of $\left(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R})\right)$ on $L^2(\mathbb{R})$ if and only if there exists a set $F \in \mathfrak{F}$ such that*

$$\mathscr{F}' = \{u \in H^1(\mathbb{R}) : u' = 0,\ a.e.\ on\ F\},$$
$$\mathscr{E}'(u, v) = \frac{1}{2}\mathbf{D}(u, v), \quad u, v \in \mathscr{F}'.$$

Now we turn to consider the quasi-regular Dirichlet subspaces of $\left(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R})\right)$. The following theorem indicates that the quasi-regular Dirichlet subspaces and regular Dirichlet subspaces for one-dimensional Brownian motion coincide.

Theorem 2 *Let $(\mathscr{E}', \mathscr{F}')$ be a quasi-regular Dirichlet subspace of $\left(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R})\right)$ on $L^2(\mathbb{R})$. Then $(\mathscr{E}', \mathscr{F}')$ is regular on $L^2(\mathbb{R})$.*

Proof By Remark 1, the $\mathscr{E}'$-quasi-continuous function is continuous and the $\mathscr{E}'$-exceptional set must be the empty set. Particularly, $\mathscr{F}' \subset H^1(\mathbb{R}) \subset C_\infty(\mathbb{R})$, where $C_\infty(\mathbb{R})$ is the family of all continuous functions vanishing at infinity. Since $(\mathscr{E}', \mathscr{F}')$ is quasi-regular, it follows that $\mathscr{F}'$ is an algebra separating $\mathbb{R}$ and there exists a strictly positive continuous function in $\mathscr{F}'$. These imply $\mathscr{F}'$ is dense in $C_\infty(\mathbb{R})$ with respect to the uniform norm by Stone-Weierstrass theorem. Therefore, $(\mathscr{E}', \mathscr{F}')$ is regular on $L^2(\mathbb{R})$. That completes the proof.

3.2 Multi-dimensional Brownian Motion

3.2.1 Quasi-regular but Non-regular Dirichlet Subspaces

Given an integer $d \geq 2$, the associated Dirichlet form of d-dimensional Brownian motion on $L^2(\mathbb{R}^d)$ is $\left(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}^d)\right)$, where $H^1(\mathbb{R}^d)$ is the 1-Sobolev space on $\mathbb{R}^d$. Analogically to Theorem 2, one may conjecture that the quasi-regular Dirichlet subspaces of $\left(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}^d)\right)$ coincide with the regular ones. However, the case of d-dimensional Brownian motion with $d \geq 2$ is totally different to the one-dimensional case. Especially, any singleton set is exceptional and not all quasi-continuous functions in $H^1(\mathbb{R}^d)$ are continuous. In fact, we can find quasi-regular but non-regular Dirichlet subspaces of $\left(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}^d)\right)$ via the skew-product method, which was introduced in [5].

Example 2 Fix $d \geq 3$. The Brownian motion $(B_t)_{t\geq 0}$ on $\mathbb{R}^d$ may be written as

$$B_t = r_t \vartheta_{A_t}, \tag{3}$$

where (r_t) is the d-Bessel process on $[0, \infty)$, ϑ is the spherical Brownian motion on S^{d-1} and (A_t) is a positive continuous additive functional (PCAF in abbreviation) of (r_t). Note that the restriction of (r_t) to $(0, \infty)$ (0 is an entrance boundary of (r_t)) is a regular diffusion with no killing inside, the speed (also, symmetric) measure

$l_d(dx) := x^{d-1}dx$ and the scale function $\mathtt{s}(x) = (1/(2-d)) \cdot x^{2-d}$. The associated regular Dirichlet form of (r_t) on $L^2((0,\infty), l_d)$ is given by

$$\begin{aligned} \mathscr{F}^{(\mathtt{s})} &:= \left\{ u \in L^2\left((0,\infty),\ l_d\right) : u \ll \mathtt{s}, \int_0^\infty \left(\frac{du}{d\mathtt{s}}\right)^2 d\mathtt{s} < \infty \right\}, \\ \mathscr{E}^{(\mathtt{s})}(u,v) &:= \frac{1}{2}\int_0^\infty \frac{du}{d\mathtt{s}}\frac{dv}{d\mathtt{s}} d\mathtt{s}, \quad u, v \in \mathscr{F}^{(\mathtt{s})}, \end{aligned} \tag{4}$$

where '$u \ll \mathtt{s}$' is read as 'u is absolutely continuous with respect to $\mathtt{s}$'. In fact, one may easily check that $(\mathscr{E}^{(\mathtt{s})}, \mathscr{F}^{(\mathtt{s})})$ is also regular on $L^2([0,\infty), l_d)$. Note that $\{0\}$ is an l_d-polar set relative to (r_t) and the Revuz measure of (A_t) relative to (r_t) is $\mu(dx) = x^{d-3}dx$. Notice that μ is Radon on $[0,\infty)$ since $d \geq 3$. We refer the skew product expression of the associated Dirichlet form on $L^2([0,\infty) \times S^{d-1})$ or $L^2((0,\infty) \times S^{d-1})$ of (3) to [5] and [8]. Particularly, since $(\mathscr{E}^{(\mathtt{s})}, \mathscr{F}^{(\mathtt{s})})$ is regular both on $L^2((0,\infty), l_d)$ and $L^2([0,\infty), l_d)$, we can deduce from [8, Theorem 1.3] that $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}^d))$ is regular both on $L^2(\mathbb{R}^d \setminus \{0\})$ and $L^2(\mathbb{R}^d)$.

Now take another continuous and strictly increasing function $\hat{\mathtt{s}}$ on $[0,\infty)$ such that

$$d\hat{\mathtt{s}} \ll d\mathtt{s}, \quad \frac{d\hat{\mathtt{s}}}{d\mathtt{s}} = 0 \text{ or } 1, \text{ a.e.}, \quad \hat{\mathtt{s}}(0) = 0.$$

We refer the existence of $\hat{\mathtt{s}}$ to [2, (4.4)]. Let $(\mathscr{E}^{(\hat{\mathtt{s}})}, \mathscr{F}^{(\hat{\mathtt{s}})})$ be the Dirichlet form (4) replacing $\mathtt{s}$ by $\hat{\mathtt{s}}$. It follows from [1, Theorem 2.2.11] that $(\mathscr{E}^{(\hat{\mathtt{s}})}, \mathscr{F}^{(\hat{\mathtt{s}})})$ is regular on $L^2([0,\infty), l_d)$. Denote its associated diffusion process on $[0,\infty)$ by $(\hat{r}_t)$. Particularly, $\{0\}$ is not an l_d-polar set relative to $(\hat{r}_t)$ since $\hat{\mathtt{s}}(0) > -\infty$. Then by using [8, Theorem 1.3] again, the associated Dirichlet form $(\hat{\mathscr{E}}, \hat{\mathscr{F}})$ of

$$X_t := \hat{r}_t \vartheta_{\hat{A}_t}, \quad t \geq 0,$$

where $(\hat{A}_t)$ is the PCAF of $(\hat{r}_t)$ associated with the Radon smooth measure μ, is a regular Dirichlet form on $L^2(\mathbb{R}^d)$ and $\{0\}$ is of positive $\hat{\mathscr{E}}$-capacity. Furthermore, from [5, Theorem 3.1], we can conclude that $(\hat{\mathscr{E}}, \hat{\mathscr{F}})$ is a regular Dirichlet subspace of $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$. Let $(\mathscr{E}', \mathscr{F}')$ be the part Dirichlet form of $(\hat{\mathscr{E}}, \hat{\mathscr{F}})$ on $\mathbb{R}^d \setminus \{0\}$, i.e.

$$\mathscr{F}' = \{u \in \hat{\mathscr{F}} : \tilde{u}(0) = 0\}.$$

Clearly, $(\mathscr{E}', \mathscr{F}')$ is a regular Dirichlet form on $L^2(\mathbb{R}^d \setminus \{0\})$ and thus quasi-regular on $L^2(\mathbb{R}^d)$. Since any quasi-continuous function $\tilde{u}$ in $\mathscr{F}'$ satisfies $\tilde{u}(0) = 0$, we can obtain that $(\mathscr{E}', \mathscr{F}')$ is not regular on $L^2(\mathbb{R}^d)$.

Finally, we assert $(\mathscr{E}', \mathscr{F}')$ is a quasi-regular Dirichlet subspace of $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$. Clearly, (S1) holds for $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}^d))$ and $(\mathscr{E}', \mathscr{F}')$. We need only to prove that any $\mathscr{E}'$-nest is also a $\frac{1}{2}\mathbf{D}$-nest. To do this, fix an $\mathscr{E}'$-nest $\{F_n : n \geq 1\}$

relative to $(\mathscr{E}', \mathscr{F}')$ on $L^2(\mathbb{R}^d)$. Firstly, we assert that $\{F_n \setminus \{0\} : n \geq 1\}$ is an $\mathscr{E}'$-nest relative to $(\mathscr{E}', \mathscr{F}')$ on $L^2(\mathbb{R}^d \setminus \{0\})$. Note the $F_n \setminus \{0\}$ is closed in $\mathbb{R}^d \setminus \{0\}$ since F_n is closed in $\mathbb{R}^d$. By the definition of $\mathscr{E}'$-nest, we know that $\cup_{n\geq 1}\mathscr{F}'_{F_n}$ is $\mathscr{E}'_1$-dense in $\mathscr{F}'$ (on $L^2(\mathbb{R}^d)$), where

$$\begin{aligned}\mathscr{F}'_{F_n} &:= \{u \in \mathscr{F}' \subset L^2(\mathbb{R}^d) : u = 0 \text{ a.e. on } \mathbb{R}^d \setminus F_n\} \\ &= \{u \in \mathscr{F}' \subset L^2(\mathbb{R}^d \setminus \{0\}) : u = 0 \text{ a.e. on } (\mathbb{R}^d \setminus \{0\}) \setminus (F_n \setminus \{0\})\} \\ &=: \mathscr{F}'_{F_n \setminus \{0\}}.\end{aligned}$$

In this equality, $\mathscr{F}'_{F_n}$ (resp. $\mathscr{F}'_{F_n\setminus\{0\}}$) is defined with respect to $(\mathscr{E}', \mathscr{F}')$ on $L^2(\mathbb{R}^d)$ (resp. $L^2(\mathbb{R}^d \setminus \{0\})$. Hence $\cup_{n\geq 1}\mathscr{F}'_{F_n\setminus\{0\}}$ is also $\mathscr{E}'_1$-dense in $\mathscr{F}'$ (on $L^2(\mathbb{R}^d \setminus \{0\})$), and as a result, $\{F_n \setminus \{0\} : n \geq 1\}$ is an $\mathscr{E}'$-nest relative to $(\mathscr{E}', \mathscr{F}')$ on $L^2(\mathbb{R}^d \setminus \{0\})$. Secondly, on account of Lemma 1, $\{F_n \setminus \{0\} : n \geq 1\}$ is a $\frac{1}{2}\mathbf{D}$-nest relative to $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d \setminus \{0\})$, because $(\mathscr{E}', \mathscr{F}')$ is a regular Dirichlet subspace of $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d \setminus \{0\})$. Thirdly, since

$$\begin{aligned}H^1(\mathbb{R}^d)_{F_n} &:= \{u \in H^1(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) : u = 0 \text{ a.e. on } \mathbb{R}^d \setminus F_n\} \\ &= \{u \in H^1(\mathbb{R}^d) \subset L^2(\mathbb{R}^d \setminus \{0\}) : u = 0 \text{ a.e. on } (\mathbb{R}^d \setminus \{0\}) \setminus (F_n \setminus \{0\})\} \\ &=: H^1(\mathbb{R}^d)_{F_n\setminus\{0\}},\end{aligned}$$

an analogous argument of the first step leads to that $\{F_n : n \geq 1\}$ is a $\frac{1}{2}\mathbf{D}$-nest relative to $(\frac{1}{2}\mathbf{D}, H^1(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$. Therefore we eventually arrive at the desirable conclusion.

3.2.2 Other Attempts

The characterization of quasi-regular (or regular) Dirichlet subspaces for multi-dimensional Brownian motion is not complete yet. Except for the skew-product method in [5], another available method, i.e. the direct-product method, was introduced in [4] for the regular Dirichlet subspaces. Roughly speaking, the family of all regular Dirichlet subspaces, corresponding to the diffusion processes with independent components, of multi-dimensional Brownian motion can be completely characterized. We refer the details of direct-product method to [4, Theorem 3.2].

The result of Lemma 2 for one-dimensional Brownian motion inspires us to take the following problem into consideration: Is there any singular set $F \subset \mathbb{R}^d$ (like the set in $\mathfrak{F}$ for $d = 1$) such that

$$\mathscr{F}' := \{u \in H^1(\mathbb{R}^d) : \nabla u = 0 \text{ a.e. on } F\} \tag{5}$$

is a proper (quasi-regular) Dirichlet subspace of $H^1(\mathbb{R}^d)$ with the quadratic form $\frac{1}{2}\mathbf{D}$? We have no idea till now. The advantage of (5) is $\mathscr{F}'$ is that clearly closed and satisfies the Dirichlet property: $u \in \mathscr{F}'$ implies $u^+ \wedge 1 \in \mathscr{F}'$. However to check its quasi-regularity, we need quasi-continuous functions separating points in $\mathbb{R}^d$ (outside an exceptional set). This almost tells us that F must be very wired, and then it seems not easy to find suitable concrete functions (such to $\nabla u = 0$ a.e. on F) separating points in $\mathbb{R}^d$.

4 Further Remarks

Loosely speaking, the condition $\nabla u = 0$ a.e. on F in (5) implies u is harmonic on F. Thus generally, for a quasi-regular Dirichlet form $(\mathscr{E}, \mathscr{F})$ on $L^2(E, m)$ associated with an m-symmetric Markov process $(X_t)_{t\geq 0}$, we can introduce another family of harmonic functions on a nearly Borel set F with $m(F^c) > 0$ instead of (5):

$$\mathscr{F}' := \left\{u = H^1_{F^c} f : f \in \mathscr{F}\right\}, \tag{6}$$

where $H^1_{F^c} f(x) := \mathbf{E}_x \left(\mathrm{e}^{-\sigma_{F^c}} f(X_{\sigma_{F^c}})\right)$ and σ_{F^c} is the first hitting time of F^c relative to (X_t). Clearly, $\mathscr{F}'$ is a closed subspace of $\mathscr{F}$ under the inner product $\mathscr{E}_1$ and separating F^c outside an exceptional set. We are not clear at moment whether and how $\mathscr{F}'$ separates F, but it seems valuable to make an attempt since the functions in (6) enjoy rich properties. However, what disappoints us is that $\mathscr{F}'$ given by (6) usually does not satisfy the Dirichlet property. Note that $1 \in \mathscr{F}$ is not essential for the following equivalence but only simplifies the proof.

Lemma 3 *Let $\mathscr{F}'$ be given by (6) with a nearly Borel set F such that $m(F^c) > 0$. Assume that $(\mathscr{E}, \mathscr{F})$ is irreducible and $1 \in \mathscr{F}$. Then the following assertions are equivalent:*

(1) $\mathscr{F}'$ satisfies the Dirichlet property: $u \in \mathscr{F}'$ implies $u^+ \wedge 1 \in \mathscr{F}'$;
(2) $\mathscr{F}' = \mathscr{F}$.

Proof Clearly, (2) implies (1) and we need only to prove the contrary. Assume that $\mathscr{F}'$ satisfies the Dirichlet property. Let $G := F^c$ and G^r be the set of all regular points of G, i.e. $x \in G^r$ means $\mathbf{P}_x(\sigma_G = 0) = 1$. Take any $H^1_G f \in \mathscr{F}'$ for non-negative $f \in \mathscr{F}$. Then $H^1_G f \wedge 1 \in \mathscr{F}'$. That means, there exists another function $g \in \mathscr{F}$ such that $H^1_G f \wedge 1 = H^1_G g$. Particularly, $f \wedge 1 = g$ q.e. on G. So

$$H^1_G f \wedge 1 = H^1_G g = H^1_G (f \wedge 1).$$

Now let $f := 2 \in \mathscr{F}$ by $1 \in \mathscr{F}$. Then $2p^1_G \wedge 1 = p^1_G$, where $p^1_G = \mathbf{E}_x e^{-\sigma_G}$. This implies

$$p^1_G = 0 \text{ or } 1, \quad \text{q.e.}$$

Note that $\{p^1_G = 1\} = G^r$. Thus for q.e. $x \notin G \cup G^r$,

$$\mathbf{P}_x(\sigma_{G\cup G^r} < \infty) = \mathbf{P}_x(\sigma_G < \infty) = 0.$$

Therefore, $(G \cup G^r)^c$ is invariant and $m\left((G \cup G^r)^c\right) = 0$ since $(\mathscr{E}, \mathscr{F})$ is irreducible and $m(G) > 0$. Note that $G \cup G^r$ is quasi closed. It follows that $(G \cup G^r)^c$ is quasi-open and thus an $\mathscr{E}$-exceptional set by $m\left((G \cup G^r)^c\right) = 0$. On the other hand, for any quasi-continuous $f \in \mathscr{F}$ and q.e. $x \in G \cup G^r$ ($G \setminus G^r$ is semipolar, so $\mathscr{E}$-exceptional),

$$H^1_G f(x) = \mathbf{E}_x\left(e^{-\sigma_G} f(X_{\sigma_G})\right) = f(x).$$

This indicates $H^1_G f = f$ q.e. and $\mathscr{F}' = \mathscr{F}$. That completes the proof.

We end this section with some proposals for the topic of quasi-regular Dirichlet subspaces. On one hand, one of our main purposes is to obtain proper quasi-regular Dirichlet subspaces for concrete Dirichlet forms, for example, those associated with the multi-dimensional Brownian motions, the α-stable processes and (further) the O-U process on the abstract Wiener space. These tasks, as we showed above, are actually not complete. We even do not know whether the Dirichlet forms associated with α-stable processes on $\mathbb{R}^d$ or O-U process have a proper quasi-regular Dirichlet subspace. On the other hand, we are also interested in the applications of quasi-regular (also, regular) Dirichlet subspaces. Regular Dirichlet subspaces were already applied to SDEs and diffusion processes in several recent works by the author and his co-authors. We believe that the quasi-regular Dirichlet spaces would also play important roles in the relevant (and more) fields.

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References

1. Z.-Q. Chen, M. Fukushima, *Symmetric Markov Processes, Time Change, and Boundary Theory* (Princeton University Press, Princeton, 2012)
2. X. Fang, M. Fukushima, J. Ying, On regular Dirichlet subspaces of $H^1(I)$ and associated linear diffusions. Osaka J. Math. **42**, 27–41 (2005)
3. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes* (Walter de Gruyter & Co., Berlin, 2011)
4. L. Li, J. Ying, Regular subspaces of Dirichlet forms, in *Festschrift Masatoshi Fukushima* (World Scientific Publications, Hackensack, 2015), pp. 397–420
5. L. Li, J. Ying, Regular subspaces of skew product diffusions. Forum Math. **28**, 857–872 (2016)
6. L. Li, J. Ying, On structure of regular Dirichlet subspaces for one-dimensional Brownian motion. Ann. Probab. **45**, 2631–2654 (2017)

7. Z.M. Ma, M. Röckner, *Introduction to the Theory of (Nonsymmetric) Dirichlet Forms* (Springer, Berlin, 1992)
8. H. Ôkura, Recurrence criteria for skew products of symmetric Markov processes. Forum Math. **1**, 331–357 (1989)

Power-Law Dynamic Arising from Machine Learning

Wei Chen, Weitao Du, Zhi-Ming Ma, and Qi Meng

Abstract We study a kind of new SDE that was arisen from the research on optimization in machine learning, we call it power-law dynamic because its stationary distribution cannot have sub-Gaussian tail and obeys power-law. We prove that the power-law dynamic is ergodic with unique stationary distribution, provided the learning rate is small enough. We investigate its first exist time. In particular, we compare the exit times of the (continuous) power-law dynamic and its discretization. The comparison can help guide machine learning algorithm.

Keywords Machine learning · Stochastic gradient descent · Stochastic differential equation · Power-law dynamic

1 Introduction

In the past ten years, we have witnessed the rapid development of machine learning technology. We successfully train deep neural networks (DNN) and achieve big breakthroughs in AI tasks, such as computer vision [7, 8, 14], speech recognition [21, 23, 24] and natural language processing [5, 26, 27], etc.

Stochastic gradient descent (SGD) is a mainstream optimization algorithm in deep machine learning. Specifically, in each iteration, SGD randomly sample a minibatch of data and update the model by the stochastic gradient. For large DNN models,

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the gradient computation over each instance is costly. Thus, compared to gradient descent which updates the model by the gradient over the full batch data, SGD can train DNN much more efficiently. In addition, the gradient noise may help SGD to escape from local minima of the non-convex optimization landscape.

Researchers are investigating how the noise in SGD influences the optimization and generalization of deep learning. Recently, more and more work take SGD as the numerical discretization of the stochastic differential equations (SDE) and investigate the dynamic behaviors of SGD by analyzing the SDE, including the convergence rate [9, 15, 22], the first exit time [4, 17, 31, 32], the PAC-Bayes generalization bound [6, 19, 25] and the optimal hyper-parameters [6, 15]. Most of the results in this research line are derived from the dynamic with state-independent noise, assuming that the diffusion coefficient of SDE is a constant matrix independent of the state (i.e., model parameters in DNN). However, the covariance of the gradient noise in SGD does depend on the model parameters.

In our recent work [17, 18], we studied the dynamic behavior of SGD with state-dependent noise. We found that the covariance of the gradient noise of SGD in the local region of local minima can be well approximated by a quadratic function of the state. Then, we proposed to investigate the dynamic behavior of SGD by a stochastic differential equation (SDE) with a quadratic state-dependent diffusion coefficient. As shown in [17, 18], the new SDE with quadratic diffusion coefficient can better match the behavior of SGD compared with the SDE with constant diffusion coefficient.

In this paper, we study some mathematical properties of the new SDE with quadratic diffusion coefficient. After briefly introducing its machine learning background and investigating its preliminary properties (Sect. 2), we show in Sect. 3 that the stationary distribution of this new SDE is a power-law distribution (hence we call the corresponding dynamic a *power-law dynamic*), and the distribution possesses heavy-tailed property, which means that it cannot have sub-Gaussian tail. Employing coupling method, in Sect. 4 we prove that the power-law dynamic is ergodic with unique stationary distribution, provided the learning rate is small enough. In the last two sections we analyze the first exit time of the power-law dynamic. We obtain an asymptotic order of the first exit time in Sect. 5, we then in Sect. 6 compare the exit times of the (continuous) power-law dynamic and its discretization. The comparison can help guide machine learning algorithm.

2 Background and Preliminaries on Power-Law Dynamic

2.1 Background in Machine Learning

Suppose that we have training data $S_n = \{(x_1, y_1), \dots, (x_n, y_n)\}$ with inputs $\{x_j\}_{j=1}^n \in \mathbb{R}^{d_1 \times n}$ and outputs $\{y_j\}_{j=1}^n \in \mathbb{R}^{d_2 \times n}$. For a model $f_w(x) : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ with parameter (vector) $w \in \mathbb{R}^d$, its loss over the training instance (x_j, y_j) is $l(f_w(x_j), y_j)$, where $\ell(\cdot, \cdot)$

is the loss function. In machine learning, we are minimizing the empirical loss over the training data, i.e.,

$$\min_w L(w) := \frac{1}{n}\sum_{j=1}^{n} \ell(f_w(x_j), y_j). \tag{2.1}$$

Stochastic gradient descent (SGD) and its variants are the mainstream approaches to minimize $L(w)$. In SGD, the update rule at the kth iteration is

$$w_{k+1} = w_k - \eta \cdot \tilde{g}(w_k), \tag{2.2}$$

where η denotes the learning rate,

$$\tilde{g}(w) := \frac{1}{b}\sum_{j\in S_b} \nabla_w \ell(f_w(x_j, y_j)) \tag{2.3}$$

is the stochastic gradient, with S_b being a random sampled subset of S_n with size $b := |S_b|$. In the literature, S_b is called mini-batch.

We know that $\tilde{g}(w)$ is an unbiased estimator of the full gradient $\nabla L(w)$. The gap between the full gradient and the stochastic gradient, i.e.,

$$R(w) := \nabla L(w) - \tilde{g}(w), \tag{2.4}$$

is called the gradient noise in SGD. In the literature, e.g. [15, 17, 32], the gradient noise $R(w)$ is assumed to be drawn from Gaussian distribution,[1] that is, $R(w) \sim \mathcal{N}(0, \Sigma(R(w)))$, where $\Sigma(R(w))$ is the covariance matrix of $R(w)$. Denote $\Sigma(R(w))$ by $C(w)$, the update rule of SGD in Eq. (2.2) is then approximated by:

$$w_{k+1} = w_k - \eta\nabla L(w_k) + \eta\xi_k, \quad \xi_k \sim \mathcal{N}(0, C(w_k)). \tag{2.5}$$

Further, for small enough learning rate η, Eq. (2.5) can be viewed as the numerical discretization of the following stochastic differential equation (SDE) [9, 15, 17],

$$dw_t = -\nabla L(w_t)dt + \sqrt{\eta C(w_t)}dB_t, \tag{2.6}$$

where B_t is the standard Brownian Motion in $\mathbb{R}^d$. This viewpoint enable the researchers to investigate the dynamic properties of SGD by means of stochastic analysis. In this line, recent work studied the dynamic of SGD with the help of SDE. However, most of the quantitative results in this line of work were obtained for the dynamics with state-independent noise. More precisely, the authors assumed that the covariance $C(w_t)$ in Eq. (2.6) is a constant matrix independent with the state

[1] Under mild conditions the assumption is approximately satisfied by the Central Limit Theorem.

w_t. This assumption of constant diffusion coefficient simplifies the calculation and the corresponding analysis. But it is over simplified because the noise covariance in SGD does depend on the model parameters.

In our recent work [17, 18], we studied the dynamic behavior of SGD with state-dependent noise. The theoretical conduction and empirical observations of our research show that the covariance of the gradient noise of SGD in the local region of local minima can be well approximated by a quadratic function of the state w_t as briefly reviewed below.

Let w^* be a local minimum of the (empirical/training) loss function defined in (2.1). We assume that the loss function in the local region of w^* can be approximated by the second-order Taylor expansion as

$$L(w) = L(w^*) + \nabla_w L(w^*)(w - w^*) + \frac{1}{2}(w - w^*)^T H(w - w^*), \tag{2.7}$$

where H is the Hessian matrix of loss at w^*. Since $\nabla_w L(w^*) = 0$ at the local minimum w^*, (2.7) is reduced to

$$L(w) = L(w^*) + \frac{1}{2}(w - w^*)^T H(w - w^*), \tag{2.8}$$

Under the above setting, the full gradient of training loss is

$$\nabla L(w) = H(w - w^*), \tag{2.9}$$

and the stochastic gradient (2.3) is

$$\tilde{g}(w) := \tilde{g}(w^*) + \tilde{H}(w - w^*) \tag{2.10}$$

where $\tilde{g}(\cdot)$ and $\tilde{H}(\cdot)$ are the gradient and Hessian calculated by the minibatch. More explicitly, the ith component of $\tilde{g}(w)$ is

$$\tilde{g}_i(w) = \tilde{g}_i(w^*) + \sum_{a=1}^{d} \tilde{H}_{ia}(w_a - w_a^*). \tag{2.11}$$

Assuming that $Cov(\tilde{g}_i(w^*), \tilde{H}_{jk}) = 0$ for $i, j, k \in \{1, \ldots, d\}$,[2] we have

$$C(w)_{ij} = Cov(\tilde{g}_i(w), \tilde{g}_j(w)) = \Sigma_{ij} + (w - w^*)^T A^{ij}(w - w^*), \tag{2.12}$$

[2] This assumption holds for additive noise case and the squared loss [29]. Specifically, for $\ell(w) = (y - f_w(x))^2$, the gradient and Hessian are $g(w) = 2f_w'(x)(y - f_w(x))$ and $H(w) = 2(f_w'(x))^2 + 2f_w''(x)(y - f_w(w)) \approx 2(f_w'(x))^2$. With additive noise, we have $y = f_{w^*}(x) + \epsilon$ where ϵ is white noise independent with the input x. Then, $g(w^*) = 2f_w'(x)\epsilon$, $H(w^*) \approx 2(f_{w^*}'(x))^2$, and we have $Cov(\tilde{g}_i(w^*), \tilde{H}_{jk}) \approx 0$.

where $\Sigma_{ij} = Cov(\tilde{g}_i(w^*), \tilde{g}_j(w^*))$, A^{ij} is a $d \times d$ matrix with elements $A^{ij}_{ab} = Cov(\tilde{H}_{ia}, \tilde{H}_{jb})$.

Thus, we can convert $C(w)$ into an analytic tractable form as follows.

$$C(w) = \Sigma_g(I + (w - w^*)^T \Sigma_H (w - w^*)) \tag{2.13}$$

where Σ_g and Σ_H are positive definite matrix. The empirical observations in [17, 18] is consistent with the covariance structure (2.13). Thus the SDE (2.6) takes the form

$$dw_t = -H(w_t - w^*)dt + \sqrt{\eta C(w_t)}dB_t, \tag{2.14}$$

where $C(w)$ is given by (2.13). We call the dynamic driven by (2.14) a *power-law dynamic* because its stationary distribution obeys power-law (see Theorem 1 below). As shown in [17, 18], power-law dynamic can better match the behavior of SGD compared to the SDE with constant diffusion coefficient.

2.2 Preliminaries on Power-Law Dynamic

For the power-law dynamic (2.14), the infinitesimal generator exists and has the following form:

$$\mathcal{A} = \sum_i \sum_j \frac{\eta}{2} (\Sigma_g)_{ij} (1 + w^T \Sigma_H w) \frac{\partial}{\partial w^i} \frac{\partial}{\partial w^j} - \sum_i H_{ij} w^j \frac{\partial}{\partial w^i}.$$

We will use the infinitesimal generator to specify a coupling in the subsequent sections. Write $v_t = w_t - w^*$, then

$$dv_t = -Hv_t dt + \sqrt{\eta C(v_t)}dB_t, \tag{2.15}$$

where $C(v) = \Sigma_g(1 + v^T \Sigma_H v)$ (comparing with (2.14), here we slightly abused the notation C).

In machine learning, we often assume that the dynamic in (2.15) can be decoupled [18, 32, 33]. More explicitly, we assume that Σ_g, Σ_H and H are codiagonalizable by an orthogonal matrix Q, then under the affine transformation $v_t = Q(w_t - w_t^*)$, (2.15) is decoupled and can be written as

$$dv_t = -h_i v_t^i dt + \sqrt{\eta\sigma_i + \eta\rho_i (v_t^i)^2} dB_t^i, \quad i \in \{1, \ldots, d\}, \tag{2.16}$$

where σ_i, ρ_i are positive constants.[3]

[3] This decoupling property is empirically observed in machine learning, i.e., the directions of eigenvectors of the hessian matrix and the gradient covariance matrix are often nearly coincide at the minimum [32]. An explanation of this phenomenon is that under expectations the Hessian equals to Fisher information [11, 32, 35].

Following the convention of probabilistic literature, in what follows we shall write

$$\mu(v_t) = -Hv_t \tag{2.17}$$

$$\sigma^2(v_t) = \eta C(v_t). \tag{2.18}$$

Suppose $x, y \in \mathbb{R}^d$, by the mean value theorem, we have the following inequality,

$$|\sqrt{a + bx^2} - \sqrt{a + by^2}| \leq \sqrt{b}|x - y|. \tag{2.19}$$

Then, it is easy to check that both $\mu(\cdot)$ and $\sigma(\cdot)$ are local Lipschitz and have linear growth. Therefore, by standard theory of stochastic differential equations, the SDE (2.15) has a unique strong solution $v(t)$, which has continuous paths and possesses strong Markov property.

Consider the decoupled dynamic in (2.16), we use the fact that as $|x_i| \to \infty$,

$$\left|\frac{\mu_i(x_i)}{\sigma_i^2(x_i)}\right| = \left|\frac{h_i x_i}{\eta\sigma_i + \eta\rho_i x_i^2}\right| \sim O\left(\frac{1}{|x_i|}\right).$$

Then, for any fixed x_0, we have

$$\int_{-\infty}^{x_0} \exp\left(-\int_{x_0}^{x} \frac{2\mu_i(s)}{\sigma_i^2(s)} ds\right) \left(\int_{x}^{x_0} \frac{\exp(\int_{x_0}^{y} \frac{2\mu_i(s)}{\sigma_i^2(s)} ds)}{\sigma_i^2(y)} dy\right) dx = \infty,$$

and

$$\int_{x_0}^{\infty} \exp\left(-\int_{x_0}^{x} \frac{2\mu(s)}{\sigma^2(s)} ds\right) \left(\int_{x_0}^{x} \frac{\exp(\int_{x_0}^{y} \frac{2\mu(s)}{\sigma^2(s)} ds)}{\sigma^2(y)} dy\right) dx = \infty,$$

which implies that each component of v_t will not blow up in finite time.

To conclude, the stochastic differential equation (2.16) admits a unique strong solution $v(t)$, which has continuous paths and will not blow up in finite time. In subsequent sections we shall study more properties of the dynamic $v(t)$.

3 Property of the Stationary Distribution

In this section, we show that the stationary distribution of the SDE (2.15) possesses heavy-tailed property, and its decoupled form is a product of power-law distributions. The existence and uniqueness of the stationary distribution will be given in the next section.

Let Q be an orthogonal matrix such that $H' = QHQ^T$ is a diagonal matrix. Then

$$d(Qv_t) = -H'Qv_t dt + \sqrt{1 + (Qv_t)^T \tilde{\Sigma}_H Qv_t} \cdot \sqrt{\eta \tilde{\Sigma}_g} d\tilde{B}_t, \tag{3.1}$$

where $\tilde{\Sigma}_H = Q\Sigma_H Q^T$, $\tilde{\Sigma}_g = Q\Sigma_g Q^T$. Note that $\tilde{B}_t = QB_t$ is still a Brownian motion. (3.1) is just the power-law dynamic (2.15) under a new orthogonal coordinate system, so we will abuse the notation and denote the transformed dynamic by v_t as well. Since we care about the tail behavior of the power-law dynamic, we show first that v_t does not have finite higher moments as $t \to \infty$. This implies that v_t cannot have exponential decay on the tail.

Theorem 1 *(i) We can find $m \geq 2$ such that the moments of the power-law dynamic (3.1) of order greater than m will explode as the time $t \to \infty$.*

(ii) For the decoupled case in (2.16), the probabilistic density of the stationary distribution is a product of power-law distributions (the terminology follows from [34]) as below:

$$p(x) = \frac{1}{Z} \prod_{i=1}^{d} \left(1 + \frac{\rho_i}{\sigma_i} x_i^2\right)^{\kappa_i}, \tag{3.2}$$

where $\kappa_i = -\frac{\eta\rho_i + h_i}{\eta\rho_i}$ and Z is the normalization constant.

Proof (i) Denote the 2kth moment of v_t as $m_{2k}(t) := \sum_i \mathbb{E}(v_t^i)^{2k}$. Then $m_0(t) = \mathbb{E}[v_t^0] = 1$. By Ito's formula, we have

$$d \sum_i \mathbb{E}(v_t^i)^{2k} = \sum_i \{-H'_{ii}(v_t^i)^{2k} + k(2k-1)(1 + v^T \tilde{\Sigma}_H v)\tilde{\Sigma}_{g_{ii}} \mathbb{E}(v_t^i)^{2k-2}\}.$$

Let $h_{\max}$ be the maximal diagonal element of H' and $g_{\min}$ be the minimal diagonal element of $\tilde{\Sigma}_g$, then we get the recursion inequality (note that it may not hold for the odd degree moments):

$$dm_{2k}(t) \geq (-kh_{\max} + k(2k-1)g_{\min}H_{\min})m_{2k}(t) + k(2k-1)g_{\min}m_{2k-2}(t), \tag{3.3}$$

where $H_{\min}$ is the minimal eigenvalue of the positive definite matrix $\tilde{\Sigma}_H$. Let $a_k = -kh_{\max} + k(2k-1)g_{\min}H_{\min}$ and $b_k = k(2k-1)g_{\min}$, then

$$m_{2k}(t) \geq e^{a_k t}(x_0^{2k} + \int_0^t R_k(s)\exp(-a_k s)ds),$$

the remainder term is defined by $R_k(s) := k(2k-1)g_{\min}m_{2k-2}(s)$. From the above relation, we can prove the following inequality by induction:

$$dm_{2k}(t) \geq \sum_{i=0}^{k} c_k^i \exp(a_k^i t). \tag{3.4}$$

By tracking the related coefficients carefully, it is not difficult to find the recurrence relations for c_k^i. For example,

$$c_k^k = m_{2k}(0) - \sum_{i=0}^{k-1} c_k^i.$$

Since $g_{\min}H_{\min}$ is positive, a_i becomes positive when i is large. From this fact, we can always find a k such that

$$\lim_{t\to\infty} m_{2k}(t) = \infty,$$

which means the moment generating function of the stationary distribution blows up. Therefore, the stationary distribution of v_t cannot have sub-Gaussian tail.

(ii) Now we turn to the decoupled case of Eq. (2.16), since each coordinate is self-dependent, we know that the probabilistic density is of the product form. To investigate the probabilistic density $p_i(t,x)$ of one fixed coordinate v_t^i, we need to study the forward Kolmogolov equation satisfied by $p_i(t,x)$. Since $\mu(x)$ and $\sigma(x)$ have linear growth, we have

$$\frac{\partial p_i}{\partial t} = \frac{1}{2}\Delta(\sigma^2 p) - \nabla(\mu p) = \frac{1}{2}\Delta[(\eta\sigma_i + \eta\rho_i x_i^2)p] + \nabla[h_i x_i p].$$

We first transform the Kolmogorov forward equation into the Smoluchowski form:

$$\begin{aligned}\frac{1}{2}\Delta[(\eta\sigma_i + \eta\rho_i x_i^2)p] + \nabla[h_i x_i p] &= \nabla\cdot[\eta\rho_i x_i p + \frac{\sigma^2}{2}\nabla p + h_i x_i p] \\ &= \nabla[(h_i x_i + \eta\rho_i x_i)p] + \nabla[\frac{\sigma^2}{2}\nabla p].\end{aligned} \tag{3.5}$$

Define the fluctuation-dissipation relation κ_i by

$$\kappa_i = -\frac{\eta\rho_i + h_i}{\eta\rho_i}.$$

Then the stationary distribution p_s satisfies

$$0 = \nabla[(h_i x_i + \eta\rho_i x_i)p] + \nabla[\frac{\sigma_i^2}{2}\nabla p_i] \tag{3.6}$$

$$= \frac{1}{2}\nabla[\sigma^2(1 + \frac{\rho_i}{\sigma_i}x_i^2)^{\kappa_i}\nabla((1 + \frac{\rho_i}{\sigma_i}x_i^2)^{-\kappa_i}p_s)] \tag{3.7}$$

Therefore,

$$p_s(x) = \frac{1}{Z_i}\left(1 + \frac{\rho_i}{\sigma_i}x_i^2\right)^{\kappa_i}, \tag{3.8}$$

where Z_i is a constant. When $\kappa_i < -\frac{1}{2}$, Z_i can be chosen such that p_s is a probabilistic distribution. Since each component is decoupled, the stationary distribution is the product of (3.8), then the proof is completed.

Remark 1 (i) The above calculation is inspired by the idea from a statistical physics literature [34]. In [34], the tail index κ_i (depending on the hyper-parameter η) plays an important role in locating the large learning rate region.

(ii) Another way to view the power-law dynamic is to apply the results in the groundbreaking article [16]. Roughly speaking, the authors of [16] gave a complete classification in the Fourier space with a determined stationary distribution. Following the notation in [16], suppose we write the SDE in the following form:

$$dz = f(z)dt + \sqrt{2D(z)}dB(t),$$

where D(z) is a positive semi-definite diffusion matrix (a Riemannian metric). Suppose the stationary distribution $p_s(z) \propto \exp(-H(z))$, then the drift term $f(z)$ must satisfy:

$$f(z) = -[D(z) + Q(z)]\nabla H(z) + \Gamma(z),$$

where $Q(z)$ is an arbitrary skew-symmetric matrix (a symplectic form) and $\Gamma(z)$ is defined by

$$\Gamma_i(z) = \sum_{j=1}^{d} \frac{\partial}{\partial z_j}(D_{ij}(z) + Q_{ij}(z)).$$

When $d = 1$, due to the skew-symmetry, $Q(z) \equiv 0$. If the stationary distribution is given by (3.8), $H(z) = \kappa \ln(1 + \frac{\rho}{\sigma}z^2)$. Thus,

$$\nabla H(z) = \frac{-\kappa\rho z}{\sigma + \rho z^2}.$$

We get that

$$f(z) = 2(1 + \kappa)\eta\rho z.$$

In this way, we automatically obtain the fluctuation-dissipation relation.

4 Existence and Uniqueness of the Stationary Distribution

In this section, we shall prove that the power-law dynamic is ergodic with unique stationary distribution, provided the learning rate η is small enough (see Theorem 2 (ii) below). Note that unlike Langevin dynamics, we have a state-dependent diffusion term in the power-law dynamic and its stationary distribution does not have a sub-Gaussian tail, which makes the diffusion process break the log-Sobolev inequality condition. Instead of treating v_t as a gradient flow, we shall use coupling method to bound the convergence of v_t to its stationary distribution.

Let the drift vector $\mu(x) = -Hx$ and the diffusion matrix $\sigma^2(x) = \eta C(x)$, where $x \in \mathbb{R}^d$, be defined as in (2.17) and (2.18) respectively. We set

$$\theta := \inf_{x,y} \left\{ - < \mu(x) - \mu(y), x - y > / \|x - y\|^2 \right\}, \tag{4.1}$$

$$\lambda := \sup_{x,y} \left\{ \max_i \sum_{1 \le j \le d} (\sigma_{ij}(x) - \sigma_{ij}(y))^2 / \|x - y\|^2 \right\}. \tag{4.2}$$

Theorem 2 *(i) Let $p(t, x, \cdot)$ be the transition probability of the power-law dynamic driven by (2.15), we have*

$$\mathbb{W}_2(p(t, x, \cdot), p(t, y, \cdot)) \le \| x - y \| \, e^{(d \cdot \lambda - \theta)t}, \tag{4.3}$$

where $\mathbb{W}_2(\cdot, \cdot)$ is the Wasserstein distance between two probability distributions.

(ii) Employing the notations used in the previous section, we write $h_{\min}$ for the minimal diagonal element of the matrix H', $g_{\max}$ for the maximal element of $\sqrt{\tilde{\Sigma}_g}$, and H_{sum} for the sum of the eigenvalues of $\tilde{\Sigma}_H$. Suppose that

$$\eta < \frac{h_{\min}}{d^2 \cdot g_{\max}^2 H_{sum}}, \tag{4.4}$$

then the power-law dynamic in (2.15) is ergodic and its stationary distribution is unique.

Proof (i) We shall employ the coupling method of Markov processes in this proof and in the rest of this paper. The reader may refer to Chap. 2 of [2], especially page 24 and Example 2.16, for the relevant contents. Recall that every infinitesimal generator of an $\mathbb{R}^d$-valued diffusion process has the form $\mathcal{A}_s = \sum_x \alpha(x) \frac{\partial^2}{\partial x^2} + \sum_x \beta(x) \frac{\partial}{\partial x}$. To specify a coupling between two power-law dynamics starting from different points of $\mathbb{R}^d$, we define a coupling infinitesimal generator $\mathcal{A}_s(x, y)$, $(x, y) \in \mathbb{R}^{2d}$, as follows:

$$\alpha_s(x, y) = \begin{pmatrix} \sigma(x)\sigma(x)^T, \ \sigma(x)\sigma(y)^T \\ \sigma(y)\sigma(x)^T, \ \sigma(y)\sigma(y)^T \end{pmatrix}, \quad \beta_s(x, y) = \begin{pmatrix} \mu(x) \\ \mu(y) \end{pmatrix},$$

where $\sigma(\cdot)$ and $\mu(\cdot)$ are specified by (2.17) and (2.18) respectively, $\alpha_s(x, y)$ corresponds to the second order differentiation and $\beta_s(x, y)$ corresponds to the first order differentiation.
Let $r(x, y) = \|x - y\|^2$ and let $\mathcal{A}_s$ act on $r(x, y)$, we get

$$\begin{aligned}\mathcal{A}_s r(x, y) &= 2 < \mu(x) - \mu(y), x - y > + \sum_i \sum_j (\sigma_{ij}(x) - \sigma_{ij}(y))^2 \\ &\leq -2\theta \|x - y\|^2 + 2d\lambda \|x - y\|^2 \\ &\leq cr(x, y),\end{aligned}$$

where $c := 2d\lambda - 2\theta$. Denote by X_t the dynamic starting at x and Y_t the dynamic starting at y, by Ito's formula, we have

$$\frac{d\mathbb{E}r(X_t, Y_t)}{dt} \leq c\mathbb{E}r(X_t, Y_t).$$

Applying Gronwall's inequality, we get

$$\mathbb{E}r(X_t, Y_t) \leq r(x, y)e^{ct},$$

which implies that

$$\mathbb{W}_2(p(t, x, \cdot), p(t, y, \cdot)) \leq \sqrt{r(x, y)}e^{ct/2} = \|x - y\| e^{(d\cdot\lambda-\theta)t},$$

verifying (4.3).

(ii) In view of (4.3), we need only to check that if (4.4) holds, then $(d \cdot \lambda - \theta) < 0$. We have

$$< \mu(x) - \mu(y), x - y >= -\sum_i H_i'(x_i - y_i)^2 \leq -h_{\min} \|x - y\|^2,$$

therefore

$$\theta \geq h_{\min}. \tag{4.5}$$

On the other hand, let $g_{\max}$ be the maximal element of $\sqrt{\tilde{\Sigma}_g}$, then for all i,

$$\sum_{1\leq j\leq d} (\sigma_{ij}(x) - \sigma_{ij}(y))^2 \leq \eta \cdot g_{\max}^2 (\sqrt{1 + x^T \tilde{\Sigma}_H x} - \sqrt{1 + y^T \tilde{\Sigma}_H y})^2.$$

Since $\|x - y\|$ is preserved under orthogonal transformation, then by the mean value theorem and Cauchy inequality, we can find $(\theta_1, \ldots, \theta_d)$, such that

$$(\sigma_{ij}(x) - \sigma_{ij}(y))^2 \leq \eta \cdot g_{\max}^2 |(\frac{h_1\theta_1}{\sqrt{1 + \theta^T \tilde{\Sigma}_H \theta}}, \ldots, \frac{h_d\theta_d}{\sqrt{1 + \theta^T \tilde{\Sigma}_H \theta}})$$

$$\cdot (x_1 - y_1, \ldots, x_d - y_d)|^2 \le \eta \cdot g_{\max}^2 H_{sum} \|x - y\|^2 ,$$

where $\{h_i\}$ denote the eigenvalues of $\tilde{\Sigma_H}$, and H_{sum} denotes the sum of the eigenvalues. Thus,

$$\max_i \sum_{1 \le j \le d} (\sigma_{ij}(x) - \sigma_{ij}(y))^2 \le d \cdot \eta \cdot g_{\max}^2 H_{sum} \|x - y\|^2 .$$

Consequently,

$$\lambda \le d \cdot \eta \cdot g_{\max}^2 H_{sum}. \tag{4.6}$$

Combining (4.5) and (4.6), we see that (4.4) implies $(d \cdot \lambda - \theta) < 0$. Therefore Assertion (ii) holds by the virtue of (4.3). The proof is completed.

Remark 2 If we restrict ourselves in the decoupled case (2.16), we can get the exponential convergence to stationary distribution under much weaker condition of η. Notice that now $\sigma(x)$ is a diagonal matrix. Using short hand writing $(\sigma_{ii}(x) - \sigma_{ii}(y))^2$ for $\sum_{1 \le j \le d} (\sigma_{ij}(x) - \sigma_{ij}(y))^2$, we have

$$\begin{aligned}(\sigma_{ii}(x) - \sigma_{ii}(y))^2 &\le (\sqrt{\eta(\sigma_i + \rho_i(x^i)^2)} - \sqrt{\eta(\sigma_i + \rho_i(y^i)^2)})^2 \\ &\le \eta\rho_i(x_i - y_i)^2.\end{aligned}$$

Then, we have

$$\begin{aligned}L_s r(x, y) &= \sum_i (\sigma_{ii}(x) - \sigma_{ii}(y))^2 - \sqrt{\eta(\sigma_1 + \rho_i(y^i)^2)})^2 - h_i(x^i - y^i)^2] \\ &\le \sum_i (\eta\rho_i - h_i)(x^i - y^i)^2 \\ &\le c_s r(x, y),\end{aligned}$$

where $c_s := \max_i[\eta\rho_i - h_i]$, which does not involve the dimension d.

5 First Exit Time: Asymptotic Order

From now on, we investigate the first exit time from a ball of the power-law dynamic, which is an important issue in machine learning. By leveraging the transition rate results from the large deviation theory (see e.g. [13]), in this section we obtain an asymptotic order of the first exit time for the decoupled power-law dynamic.

Theorem 3 *Suppose* 0 *is the only local minimum of the loss function inside* $B(0, r)$. *Let* $\tau_r^x(\eta)$ *be the first exit time from* $B(0, r)$ *of the decoupled power-law dynamic in (2.16), with learning rate* η, *starting at* $x \in B(0, r)$, *then*

$$\lim_{\eta\to 0} \eta \log \mathbb{E}\tau_r^x(\eta) = C \cdot \inf_{\zeta=(\zeta^1,\ldots,\zeta^d)\in\partial B(0,r)} \sum_i -\frac{h_i}{\rho_i} \log[\sigma_i + \rho_i(\zeta^i)^2], \tag{5.1}$$

where C is a prefactor to be determined.

When $d = 1$, we have an explicit expression of the first exit time from an interval $[a, b]$ starting at $x \in (a, b)$:

$$\mathbb{E}\tau_x = g(x) := 2\int_x^b \frac{e^{\phi(y)}}{\sigma^2(y)} dy \int_a^y e^{-\phi(z)} dz, \tag{5.2}$$

where $\phi = 2 - \kappa \ln(1 + \frac{\rho}{\sigma} x^2)$ and $\kappa = -\frac{\eta\rho+h}{\eta\rho}$.

Proof Let τ be a stopping time with finite expectation and let $\mathcal{A}$ be the infinitesimal generator of v_t, then recall that Dynkin's formula tells us:

$$\mathbb{E}[f(v_\tau)] = f(x) + \mathbb{E}[\int_0^\tau \mathcal{A}f(v_s)ds], \quad f \in C_0^2(\mathbb{R}^d),$$

where $v_0 = x$. Suppose f solves the following boundary problem:

$$\begin{cases} \mathcal{A}f(x) = -1, & x \in B(0, r), \\ f(x) = 0, & x \in \partial B(0, r), \end{cases} \tag{5.3}$$

then $\mathbb{E}\tau_r^x = f(x)$, where τ_r^x denote the first exit time of v_t starting at x from the ball $B(0, r)$.

We consider first the situation of $d = 1$, let $\tau_{(a,b)}^x(\eta)$ be the first exit time of $v(t)$ from an interval $[a, b]$ starting at $x \in (a, b)$. Note that the diffusion coefficient function $\sigma(x) = \sqrt{\eta\sigma + \eta\rho x^2} > 0$, then by Dynkin's formula, $\mathbb{E}\tau_{(a,b)}^x(\eta) = g(x)$, where $g(x)$ solves the following second order ODE:

$$\begin{cases} \mathcal{A}_1 g(x) = -1, & x \in (a, b), \\ g(x) = 0, & x \in \{a, b\}, \end{cases} \tag{5.4}$$

where $\mathcal{A}_1 = -hx\frac{\partial}{\partial x} + (\eta\sigma + \eta\rho x^2)\frac{\partial^2}{\partial x^2}$ is the infinitesimal generator of the one dimensional power-law diffusion. Now we introduce the integration factor $\phi(x) := -\kappa \ln(1 + \frac{\rho}{\sigma} x^2)$, following (3.5),

$$\nabla[(hx + \eta\rho x)f(x)] + \nabla[\frac{\sigma^2}{2}\nabla f(x)] = \frac{1}{2}\nabla[\sigma^2 e^{-\phi}\nabla(e^{\phi} f(x))] = 0. \tag{5.5}$$

Then,

$$\begin{aligned}\mathcal{A}_1 f(x) &= \frac{1}{2}\nabla[\sigma^2 e^{-\phi}\nabla(e^{\phi}f(x))] - \nabla[\eta\rho x f(x)] + hf(x) - \eta\rho x\nabla f(x) - 2\nabla[hxf(x)]\\ &= \frac{1}{2}\nabla[\sigma^2 e^{-\phi}\nabla(e^{\phi}f(x))] - \frac{1}{2}\nabla[\dot{\phi}\sigma^2(x)f(x)] - \frac{1}{2}\dot{\phi}\sigma^2\nabla f(x)\\ &= \frac{1}{2}e^{\phi}\nabla[e^{-\phi}\sigma^2\nabla f(x)],\end{aligned}$$

where we denote $\dot{\phi} = 2(\eta\rho + h)x/\sigma^2$ to get the second line. Therefore (5.4) is equivalent to

$$\mathcal{A}_1 g(x) = \frac{1}{2}e^{\phi}\nabla[e^{-\phi}\sigma^2\nabla g(x)] = -1.$$

Integrating the above equation, we recover (13) of [3]:

$$\mathbb{E}\tau^x_{(a,b)}(\eta) = g(x) = 2\int_x^b dy\frac{e^{\phi(y)}}{\sigma^2(y)}\int_a^y e^{-\phi(z)}dz. \tag{5.6}$$

The reader can check Sect. 12.3 of [30] for the asymptotic analysis of (5.6) as $\eta \to 0$.

We now investigate the general situation of $d > 1$. We have only asymptotic estimates on the exit time as the learning rate $\eta \to 0$. For this purpose, it is convenient to introduce the geometric reformulation of (2.16). Suppose B_t is the standard brownian motion, recall that in local coordinates, the Riemannian Brownian motion W_t with metric $\{g_{ij}\}$ has the following form (cf. e.g. [10]):

$$dW^i_t = \sigma^i_j(W_t)dB^j_t + \frac{1}{2}b^i(W_t)dt, \tag{5.7}$$

where $b^i(x) = g^{jk}(x)\Gamma^i_{jk}(x)$ and $\sigma_{ij}(x) = \sqrt{g^{ij}(x)}$. Comparing (5.7) with the martingale part of (2.16), we define the inverse metric as $g^{ij} = (\sigma_i + \rho_i(x^i)^2)\delta_{ij}$. Then the metric g_{ij} is also a diagonal matrix. The Christoffel symbols can be calculated under the new metric:

$$\Gamma^i_{jk}(x) = \rho_i x^i \cdot (\rho_i + \rho_i(x^i)^2), \quad \text{when } i = j = k,$$

and $\Gamma^i_{jk}(x) = 0$, otherwise. Denote the gradient vector filed of a smooth function $f(x)$ by $\nabla f(x)$, then

$$\nabla f(x) = (\partial_i f)g^{ij}\partial_j(x),$$

where $\partial_i(x) := \frac{\partial}{\partial x_i}(x),\ 1 \le i \le d$, denotes the ith coordinate tangent vector at $x \in \mathbb{R}^d$. To emphasis the parameter η that appears in the diffusion term in the power-law dynamic, we denote the dynamic and its corresponding exit time by $v_t(\eta)$ and $\tau^x_r(\eta)$. Let $f_\eta(x) := -\sum_i \frac{h_i}{2\rho_i}\log(\sigma_i + \rho_i(x^i)^2) - \frac{\sqrt{\eta}}{2}\rho_i(\frac{\sigma_i}{2}(x^i)^2 + \frac{\rho_i}{3}(x^i)^3)$, then $v_t(\eta)$ in (2.16) can be seen as a diffusion process under the new metric:

$$dv_t(\eta) = \nabla f_\eta(v_t)dt + \sqrt{\eta}dW_t, \tag{5.8}$$

where W_t is the Riemannian Brownian motion by (5.7) and 0 is a local minima of the limit function: $f_{lim}(x) := \lim_{\eta\to 0} f_\eta(x) = -\sum_i \frac{h_i}{2\rho_i}\log(\sigma_i + \rho_i(x^i)^2)$. Note that both the drift term and the diffusion term are intrinsically defined with respect to the metric $\{g_{ij}\}_{1\le i,j\le d}$. By large deviation theory, the rate function $I_\eta(\phi)$ of a path $\phi : [0, T] \to \mathbb{R}^d$ is:

$$I_{lim}(\phi) = \frac{1}{2}\int_0^T \left\|\dot{\phi}(t) - \nabla f_{lim}(\phi(t))\right\|^2 dt + 2[f_{lim}(\phi(T)) - f_{lim}(\phi(0))],$$

where the norm is with respect to the Riemannian metric g_{ij}. It follows that the quasi-potential of the ball $B(0, r)$ is given by

$$\bar{f}_{lim} = 2[\inf_{\zeta=(\zeta^1,\ldots,\zeta^d)\in\partial B(0,r)} f_{lim}(\zeta) - f_{lim}(0)].$$

By Theorem 2.2 and Corollary 2.4 of [1], if 0 is the only local minima of f_{lim} in $B(0, r)$, then there exists a constant $C > 0$ such that

$$\begin{aligned}\lim_{\eta\to 0}\eta\log\mathbb{E}\tau_r^x(\eta) &= C\cdot 2[\inf_{\zeta=(\zeta^1,\ldots,\zeta^d)\in\partial B(0,r)} f_{lim}(\zeta) - f_{lim}(0)]\\ &= C\cdot\inf_{\zeta=(\zeta^1,\ldots,\zeta^d)\in\partial B(0,r)}\sum_i -\frac{h_i}{\rho_i}\log[1+\frac{\rho_i}{\sigma_i}(\zeta^i)^2].\end{aligned}$$

The proof is completed.

Remark 3 When the dimension $d = 1$, we can get similar results with a precise prefactor by applying semiclassical approximation to the integral (5.6), see [12]. Taking exponential of (5.1), it is obvious that the leading order of the average exit time is of the power law form with respect to the radius r.

6 First Exit Time: From Continuous to Discrete

In this section, we compare the exit times of the continuous power-law dynamic (2.15) and its discretization:

$$z_{k+1} = z_k + \epsilon\mu(z_k) + \epsilon\epsilon_k, \quad \epsilon_k \sim \mathbb{N}(0, \sigma^2(z_k)). \tag{6.1}$$

Note that the first exit time of the discretized dynamic (6.1) is an integer that measures how many steps it takes to escape from the ball, thus the K time steps correspond to $K\epsilon$ amount of time. In this view point, the comparison can help guide machine learning algorithm provided the time interval ϵ coincides with the learning rate η.

However, since in power law dynamic the covariance matrix $\sigma^2(w_k)$ contains η, for the convenience of theoretic discussion, we should temporarily distinguish ϵ from η before arriving at the conclusion.

To shorten the length of the article, we shall confine ourselves in the situation of $d = 1$. Assume the local minima is located at the origin 0. Denote the ball centered at 0 with radius $r > 0$ by $B(0, r)$, let τ_r^0 be the first exit time from the ball $B(0, r)$ of the one dimensional continuous power-law dynamic in (2.15) and let $\bar{\tau}_r^0$ be the corresponding first exit time of the discretized dynamic (6.1), both starting from 0.

Remark 4 From the definition, if $\bar{\tau}_r^0$ equals an integer k, then it takes k steps for the discretized dynamic to escape from the ball $B(0, r)$. Since the time interval is set to be ϵ, the corresponding exit time for the continuous dynamic to take is $k\epsilon$.

Given $|a| < r$, letτ_{r+a}^0 be the first exit time from the ball $B(0, r + a)$ of the one dimensional continuous power-law dynamic in (2.15), then we have the following comparison of $\mathbb{P}[\bar{\tau}_r^0 > K]$ with the corresponding quantities related to the first exit time of the continuous dynamic.

Theorem 4 *Suppose* $\delta, \bar{\delta} > 0$ *and satisfy* $\delta + \bar{\delta} < r$, *given a large integer* K, *we have*

$$\mathbb{P}[\tau_{r-\delta}^0 > K\epsilon] - \frac{4}{3\delta^2}\frac{E(\epsilon)K}{c\epsilon} \leq \mathbb{P}[\bar{\tau}_r^0 > K]$$
$$\leq \mathbb{P}[\tau_{r+\delta+\bar{\delta}}^0 > K\epsilon] + \frac{4}{3\delta^2}\frac{E(\epsilon)K}{c\epsilon} + 1 - \left(1 - \frac{C(\epsilon, \eta)}{\bar{\delta}^4}\right)^K,$$

where $E(\epsilon) \sim O(\epsilon^2)$ *and* $C(\epsilon, \eta) \sim O(\epsilon)$ *as* $\epsilon \to 0$.

Proof For our purpose we introduce an interpolation process z_t as follows:

$$dz_t = -hz_k dt + \sqrt{\eta\sigma + \eta\rho(z_k)^2} dB_t, \quad t \in [(k-1)\epsilon, k\epsilon), \tag{6.2}$$

where ϵ is the discretization step size. More precisely, the drift coefficient and the diffusion coefficient of (6.2) will remain unchanged when $t \in [(k-1)\epsilon, k\epsilon)$ for each $1 \leq k \leq K$. Note that if we rewrite $\sqrt{\eta\sigma + \eta\rho(z_k)^2}$ as $\sigma(z_k)$, (6.2) is expressed as

$$dz_t = -hz_k dt + \sigma(z_k) dB_t, \quad t \in [(k-1)\epsilon, k\epsilon), \tag{6.3}$$

which reduced to (6.1) when $t = k\epsilon$ for each k.

We shall adopt a similar strategy as in [20] to transfer from the average exit time of the power-law dynamic v_t to its discretization z_t. Below in the discussion we follow also the notations in the previous sections. Roughly speaking, the proof can be divided into two steps:

(i) Fix the number of iteration steps as K, prove that

$$\mathbb{P}((z_\epsilon, \ldots, z_{K\epsilon}) - (v_\epsilon, \ldots, v_{K\epsilon}) \notin B_\delta) \le \bar{\epsilon},$$

where $\bar{\epsilon}$ is a small positive constant to be determined, and $B_\delta = \underbrace{B(0,\delta) \times \cdots \times B(0,\delta)}_{K}$ is the hyper-cube of radius $\delta > 0$. This can be done by bounding the W_2-distance of $v_{k\epsilon}$ and $z_{k\epsilon}$ for $1 \le k \le K$. Let

$$A_\delta = \underbrace{B(0, r+\delta) \times \cdots \times B(0, r+\delta)}_{K},$$

where $r > |\delta| > 0$. For simplicity, denote the exit time of the power-law dynamic from A_δ by τ_δ. For the interpolation process z_t, we denote the corresponding exit time (an integer) with a bar above it: $\tau_\delta \to \bar{\tau_\delta}$. Then,

$$\begin{aligned}\mathbb{P}[(v_\epsilon, \ldots, v_{K\epsilon}) \in A_{-\delta}] - \bar{\delta} &\le \mathbb{P}[\bar{\tau_0} > K] \\ &\le \mathbb{P}[(v_\epsilon, \ldots, v_{K\epsilon}) \in A_\delta] + \bar{\epsilon}.\end{aligned} \tag{6.4}$$

Note that the event $\{\bar{\tau_\tau} > K\}$ indicates that the interpolation process z_t remains in A_δ when $t \le K\epsilon$.

(ii) Step 1 guarantees that if $v(t)$ is trapped in a ball with a different size when $t = \epsilon, 2\epsilon, \ldots, K\epsilon$, then the interpolation process $z(t)$ is also trapped in a ball. However,

$$\mathbb{P}[(v_\epsilon, \ldots, v_{k\epsilon}) \in A_\delta] > \mathbb{P}[\tau_\delta > K\epsilon],$$

since $v(t)$ may drift outside the ball when $t \in [(k-1)\epsilon, k\epsilon)$ for $1 \le k \le K$. We define this 'anomalous' random event by

$$R := \{\max_{0 \le k \le K-1} \sup_{t \in (k\epsilon, (k+1)\epsilon)} \|v_t - v_{k\epsilon}\| > \bar{\delta}\}.$$

Then obviously,

$$\begin{aligned}&\mathbb{P}[(v_\epsilon, \ldots, v_{k\epsilon}) \in A_\delta] \\ &\quad \le \mathbb{P}[\tau_{\delta+\bar{\delta}} > k\epsilon] + \mathbb{P}[(v_\epsilon, \ldots, v_{k\epsilon}) \in R^c],\end{aligned} \tag{6.5}$$

where R^c denotes the complement of the event R. We would expect that the probability of $\{(v(\epsilon), \ldots, v(k\epsilon)) \in R^c\}$ to be small if the diffusion coefficient of the dynamic is bounded, in which case we can apply Gaussian concentration results. However, the diffusion part of the power-law dynamic is not bounded, so additional technical issue should be taken care of.

Now we introduce the same form of coupling as in the previous sections between (v_t, z_t) for $t \in [(k-1)\epsilon, k\epsilon),\ \forall 1 \le k \le K$. Following the notations in the proof of Theorem 2, we set the $\alpha_s(v,z)$ and $\beta_s(v,z)$ of the infinitesimal generator $\mathcal{A}_s$ of the coupling as:

$$\alpha_s(v,z) = \begin{pmatrix} \sigma(v)\sigma^T(v), & \sigma(v)\sigma(z_{k\epsilon}) \\ \sigma(z_{k\epsilon})\sigma(z), & \sigma(z_{k\epsilon})\sigma^T(z_{k\epsilon}) \end{pmatrix}, \quad \beta_s(v,z) = \begin{pmatrix} -hv \\ -hz_{k\epsilon} \end{pmatrix}. \tag{6.6}$$

Suppose $(v_0, z_0) = (x, x)$, denote the marginal distribution of v_t and z_t at $t = k\epsilon$ by p_k^v and p_k^z respectively.

The remainder proof of the theorem will be accomplished by three lemmas. We first prove the following lemma for the one dimensional decoupled dynamic (2.16):

Lemma 1 *Suppose that the coefficients of* (2.16) *satisfy:*

$$\{\eta\rho, \rho\} \le h \le \frac{1}{\epsilon}, \tag{6.7}$$

(which is fulfilled in SGD algorithm for large batch size and small learning rate). Let $(v(t), z(t))$ *be the coupling process defined by* (6.6)*, and* $v(0) = z(0) = x$*. When* $t = K\epsilon$*, the Wasserstein distance between the marginal distribution* p_K^z *of* $z(t)$ *and the marginal distribution* p_K^v *of the power-law dynamic* $v(t)$ *is bounded by*

$$\mathcal{W}_2^2(p_K^v, p_K^z) \le \frac{4}{3}\frac{E(\epsilon)}{c\epsilon}, \tag{6.8}$$

where $c = 2h - \eta\rho > 0$*. Moreover,* $E(\epsilon) > 0$ *is independent of the number of steps* K *and is of order* $O(\epsilon^2)$ *when the time interval* $\epsilon \to 0$*.*

***Proof of Lemma* 1** Denote the transition probability of v_t and z_t from time $(k-1)\epsilon$ to $(k-1)\epsilon + t$ by $p^v_{(k-1)\epsilon+t}(v_{(k-1)\epsilon}, \cdot)$ and $p^z_{(k-1)\epsilon+t}(z_{(k-1)\epsilon}, \cdot)$ respectively. Let $\mathcal{A}_s$ act on the function $r(v,z) := \|v - z\|^2$ and use the Gronwall's inequality, we can deduce the following recursion inequality for $1 \le k \le K$:

$$\begin{aligned}
\mathcal{W}_2^2(p_k^v(\cdot), p_k^z(\cdot)) \le \int_{\mathbb{R}^d \times \mathbb{R}^d} & \mathcal{W}_2^2(p^v_{(k-1)\epsilon} + t(v_{(k-1)\epsilon}, \cdot), p^z_{(k-1)\epsilon}(z_{(k-1)\epsilon}, \cdot)) \\
& d\pi(p^v_{k-1}(v_{(k-1)\epsilon}), p^z_{k-1}(z_{(k-1)\epsilon})) \\
& \le e^{-c\epsilon}\mathcal{W}_2^2(p^v_{k-1}(\cdot), p^z_{k-1}(\cdot)) \\
& \quad + \int_{\mathbb{R}} \frac{1}{2}\epsilon^3 \rho(8\eta\rho + 4h)\, \|x\|^2\, dp^v_{k-1}(x)
\end{aligned}$$

$$+ \int_{\mathbb{R}} [\frac{1}{3}\epsilon^3 h(4\eta\rho + 2h)\rho + \frac{1}{2}\epsilon^2\sigma(8\eta\rho + 3h)] \|y\|^2 \, dp^z_{k-1}(y)$$

$$+ \epsilon^2\eta\rho(4\eta\rho + 2h) + \frac{1}{2}\epsilon^2\sigma(8\eta\rho + 3h),$$

where $c = 2h - \eta\rho > 0$ and $\pi(p^v_{k-1}(\cdot), p^z_{k-1}(\cdot))$ is the optimal coupling between $p^v_{k-1}(\cdot)$ and $p^z_{k-1}(\cdot)$. (Note that in our context the optimal coupling always exists, see e.g. Proposition 1.3.2 and Theorem 2.3.3 in [28].)

For the interpolation process z_t starting at y_0, by Ito's formula, we have the following estimate:

$$\int_{\mathbb{R}^d} \|y\|^2 \, dp^z_k(y) \le e^{-(2h-\eta\rho)k\epsilon}(y_0^2 - \frac{\eta\rho}{2h - \eta\rho}) + \frac{\eta\rho}{2h - \eta\rho}.$$

Similarly, the second moment of the continuous dynamic v_t starting at x_0 can be bounded by:

$$\mathbb{E}\,\|v_t\|^2 \le e^{-(2h-\eta\rho)t}(x_0^2 - \frac{\eta\rho}{2h - \eta\rho}) + \frac{\eta\rho}{2h - \eta\rho}. \tag{6.9}$$

Notice that the distance between v_t and z_t is zero at initialization, then by applying the recursion relation from $k = 1$ to $k = K$, we conclude that there exits $E(\epsilon) > 0$, such that

$$\mathcal{W}_2^2(p^v_K, p^x_K) \le \frac{4}{3}\frac{E(\epsilon)}{c\epsilon},$$

where $E(\epsilon) \sim O(\epsilon^2)$ is independent of K, which completes the proof of Lemma 1.

By the definition of the W_2-distance,

$$\begin{aligned}\mathbb{P}((z_\epsilon, \dots, z_{K\epsilon}) - (v_\epsilon, \dots, v_{K\epsilon}) \notin B_\delta) &\le \sum_{k=1}^{K} \frac{W_2^2(p^v_k, p^z_k)}{\delta^2} \\ &\le \frac{4}{3\delta^2} \cdot \frac{E(\epsilon)K}{c\epsilon}.\end{aligned} \tag{6.10}$$

Below we denote the above right hand side $\frac{4}{3\delta^2}\frac{E(\epsilon)K}{c\epsilon}$ by $\bar{\epsilon}$. For the second step, from (6.4) and (6.5), it follows that

$$\mathbb{P}[\tau_{-\delta} > K\epsilon] - \bar{\epsilon} \le \mathbb{P}[\bar{\tau}_0 > K] \le \mathbb{P}[\tau_{\delta+\bar{\delta}} > K\epsilon] + \mathbb{P}[(v_\epsilon, \dots, v_{K\epsilon}) \in R^c] + \bar{\epsilon}. \tag{6.11}$$

Therefore, we are left to estimate $\mathbb{P}[(v_\epsilon, \dots, v_{K\epsilon}) \in R^c]$. Under the condition $h > \frac{1}{2}\eta\rho$, we have the following lemma:

Lemma 2 *Let $\delta > 0$ be fixed. Conditioning on the event that v_t is inside $B(0, b + \delta)$ when $t = k\epsilon$ for all $1 \le k \le K$, we have*

$$\mathbb{E}\sup_{s\in[k\epsilon,(k+1)\epsilon)}(v_s)^4 \le D(\eta,\rho,\epsilon),$$

where $D(\eta,\rho,\epsilon) := [(2+2/\delta)\frac{(b+\delta)^2\eta\rho}{2h-\eta\rho} + (5+\frac{1}{\delta})(\eta\sigma)^2\epsilon]\exp\{12(1+\delta)\eta\rho\epsilon\}$.

Remark 5 The above lemma tells us that the fourth moment won't change too much if the time interval ϵ is small. Intuitively, since the martingale part of v_t is $\sqrt{\eta\sigma+\eta\rho v_t^2}dB_t$, if $|v_t|$ is bounded, then by the time change theorem, we know that the marginal distribution of the martingale part behaves like a scaled Gaussian.

Proof of Lemma 2 By Ito's formula,

$$d(v_t)^2 = \eta\sigma dt - (2h-\eta\rho)(v_t)^2dt + 2v_t\sqrt{\eta\sigma+\eta\rho_i(v_t)^2}dB_t.$$

Define a local martingale $M_t := 2\sum\int_0^t e^{(2h-\eta\rho)s}\sqrt{\eta\sigma+\eta\rho(v_s)^2}v_s dB_s$, then by Gronwall's inequality,

$$(v_t)^2 \le e^{-(2h-\eta\rho)t}(v_0)^2 + e^{-(2h-\eta\rho)t}M_t$$
$$+ e^{-(2h-\eta\rho)t}\int_0^t e^{(2h-\eta\rho)s}\eta\sigma ds.$$

Let

$$S_t := \mathbb{E}\sup_{s\in[k\epsilon,k\epsilon+t]}(v_s)^4,\quad t\in[0,\epsilon).$$

Then, for the fixed $\delta > 0$,

$$S_t \le (1+\delta)e^{-2(2h-\eta\rho)t}\mathbb{E}\sup_{s\in[0,t]}(M_s)^2 + (2+2/\delta)e^{-2(2h-\eta\rho)t}\mathbb{E}(v_{k\epsilon})^4$$
$$+ (2+2/\delta)e^{-2(2h-\eta\rho)t}\mathbb{E}(\int_0^t e^{(2h-\eta\rho)s}\eta\sigma ds)^2.$$

Applying Doob's inequality, we get

$$S_t \le 2(1+\delta)e^{-2(2h-\eta\rho)t}\mathbb{E}M_t^2 + (1+\frac{1}{\delta})\frac{e^{-2(2h-\eta\rho)t}-1}{(\eta\rho-2h)t}$$
$$\cdot\,\mathbb{E}\int_0^t(\eta\sigma)^2ds + (2+2/\delta)e^{-2(2h-\eta\rho)t}\mathbb{E}(v_{k\epsilon})^4.$$

Now, Ito's isometry implies that

$$S_t \le 12(1+\delta)\eta\rho \int_0^t S_s ds + (2+2/\delta)e^{-2(2h-\eta\rho)t}\mathbb{E}(v_{k\epsilon})^4$$
$$+ (5+\frac{1}{\delta})(\eta\sigma)^2 t$$
$$\le [(2+2/\delta)\mathbb{E}(v_{k\epsilon})^4 + (5+\frac{1}{\delta})(\eta\sigma)^2 t]\exp\{12(1+\delta)\eta\rho t\},$$

where we used Gronwall's inequality to derive the last line. For all $1 \le k \le K$, by taking $x_0 = 0$ in (6.9), we get

$$\mathbb{E}\{(v_{k\eta})^4 \mathbb{I}_{(v_\epsilon,\dots,v_{K\epsilon})\in A_\delta}\} \le \frac{(b+\delta)^2\eta\rho}{2h-\eta\rho}.$$

Then, conditioning on the event that $\{v_{k\epsilon} \in B(0, b+\delta)\}$, we have

$$\mathbb{E} \sup_{s\in[k\epsilon,(k+1)\epsilon)} (v_s)^4 \le [(2+2/\delta)\frac{(b+\delta)^2\eta\rho}{2h-\eta\rho} + (5+\frac{1}{\delta})(\eta\sigma)^2\epsilon]$$
$$\cdot \exp\{12(1+\delta)\eta\rho\epsilon\} := D(\eta,\rho,\epsilon),$$

which completes the proof of Lemma 2.

Lemma 3 *Let $\bar{\delta}$ be a positive constant, then for every $k \in [0,\dots,K-1]$,*

$$\mathbb{P}(\sup_{t\in[k\epsilon,(k+1)\epsilon)} \|v_t - v_{k\epsilon}\| > \bar{\delta}) \le \frac{C(\epsilon,\eta)}{\bar{\delta}^4},$$

where $C(\epsilon,\eta) \sim O(\epsilon)$ when $\epsilon \to 0$.

Proof of Lemma 3 Let $r(x) = \|x - v_{k\epsilon}\|^2$, then by Ito's lemma,

$$dr(v_t) \le -2hv_t(v_t - v_{k\epsilon})dt + 2\sqrt{\eta\sigma + \eta\rho(v_t)^2}(v_t - v_{k\epsilon})dB_t + (\eta\sigma + \eta\rho(v_t)^2)dt$$
$$\le -2h(v_t - v_{k\epsilon})^2 dt + 3h(v_{k\epsilon})^2 dt + \eta\sigma dt$$
$$+ (\eta\rho + h)(v_t)^2 dt + 2\sqrt{\eta\sigma + \eta\rho(v_t)^2}(v_t - v_{k\epsilon})dB_t,$$

and obviously we know that $r(v_{k\epsilon}) = 0$. Let

$$M_t := 2\int_{k\epsilon}^t e^{2hs}\sqrt{\eta\sigma + \eta\rho(v_s)^2}(v_s - v_{k\epsilon})dB_s,$$

we have

$$r(v_t) \le e^{-2ht} M_t + e^{-2ht} \int_{k\epsilon}^{t} e^{2hs}[3h(v_{k\epsilon})^2 + \eta\sigma + (\eta\rho + h)(v_s)^2]ds.$$

Let $S_t := \mathbb{E}\sup_{s\in[k\epsilon,t]} r^2(v_s)$, then

$$\begin{aligned}
S_t &\le 2e^{-4h(t-k\epsilon)}\mathbb{E}\sup_{s\in[k\epsilon,t]}(M_s)^2 + 2e^{-4h(t-k\epsilon)} \\
&\quad \cdot \mathbb{E}(\int_{k\epsilon}^{t} e^{2hs}[3h(v_{k\epsilon})^2 + \eta\sigma + (\eta\rho + h)(v_s)^2]ds)^2 \\
&\le 4e^{4h(t-k\epsilon)}\mathbb{E}(M_t)^2 + \frac{1 - e^{-4h(t-k\epsilon)}}{h}\mathbb{E}\int_{k\epsilon}^{t}[3h(v_{k\epsilon})^2 + \eta\sigma + (\eta\rho + h)(v_s)^2]^2 ds \\
&\le 16\mathbb{E}\int_{k\epsilon}^{t}(\eta\sigma + \eta\rho(v_s)^2)(v_s - v_{k\epsilon})^2 ds \\
&\quad + \frac{1 - e^{-4h(t-k\epsilon)}}{h}\mathbb{E}\int_{k\epsilon}^{t}[3h(v_{k\epsilon})^2 + \eta\sigma + (\eta\rho + h)(v_s)^2]^2 ds.
\end{aligned}$$

Let $t = (k+1)\epsilon$, and denote the above right hand side as $C(\epsilon, \eta)$. Since $\mathbb{E}|v_s|^2 \le \sqrt{\mathbb{E}|v_s|^4}$, by Lemma 2,

$$C(\epsilon, \eta) \sim O\left(\frac{\exp(\eta\epsilon) - 1}{\eta}\right) = O(\epsilon).$$

Therefore,

$$\mathbb{P}(\sup_{t\in[k\epsilon,(k+1)\epsilon)} \|v(t) - v(k\epsilon)\| > \bar{\delta}) \le \frac{\mathbb{E}S_{(k+1)\epsilon}}{\bar{\delta}^4} \le \frac{C(\epsilon, \eta)}{\bar{\delta}^4}.$$

The proof of Lemma 3 is completed.

From the previous lemma, we can easily deduce the following bound on $\mathbb{P}[(v_\epsilon, \dots, v_{K\epsilon}) \in R^c]$:

$$\mathbb{P}[(v_\epsilon, \dots, v_{K\epsilon}) \in R^c] \le 1 - (1 - \frac{C(\epsilon, \eta)}{\bar{\delta}^4})^K. \tag{6.12}$$

Combing (6.12) and (6.11), we have finished the proof of the theorem.

Remark 6 Theorem 4 discussed only the 1-dimensional case. For high dimensional case, there have been some intuitive discussion in machine learning literature [32]. Roughly speaking, the escaping path will concentrate on the critical paths, i.e., the paths on the direction of the eigenvector of the Hessian, when the noise is much smaller than the barrier height with high probability. If there are multiple parallel exit paths, the total exiting rate, i.e., the inverse of the expected exit time, equals to the sum of the first exiting rate for every path (cf. Rule 1 in [32]).

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References

1. N. Berglund, Kramers' law: validity, derivations and generalisations. *arXiv preprint* arXiv:1106.5799 (2013)
2. M.-F. Chen, *Eigenvalues, Inequalities, and Ergodic Theory* (Springer, London, 2006)
3. J.-L. Du, Power-law distributions and fluctuation-dissipation relation in the stochastic dynamics of two-variable Langevin equations. J. Stat. Mech.: Theory Exp. **2012**(02), P02006 (2012)
4. M. Gurbuzbalaban, U. Simsekli, L. Zhu, The heavy-tail phenomenon in SGD. *arXiv preprint* arXiv:2006.04740 (2020)
5. D. He, Y. Xia, T. Qin, L. Wang, N. Yu, T.-Y. Liu, W.-Y. Ma, Dual learning for machine translation, in *Advances in Neural Information Processing Systems* (2016), pp. 820–828
6. F. He, T. Liu, D. Tao, Control batch size and learning rate to generalize well: theoretical and empirical evidence, in *Advances in Neural Information Processing Systems* (2019), pp. 1141–1150
7. K. He, X. Zhang, S. Ren, J. Sun, Delving deep into rectifiers: surpassing human-level performance on ImageNet classification, in *Proceedings of the IEEE International Conference on Computer Vision* (2015), pp. 1026–1034
8. K. He, X. Zhang, S. Ren, J. Sun, Deep residual learning for image recognition, in *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition* (2016), pp. 770–778
9. L. He, Q. Meng, W. Chen, Z.-M. Ma, T.-Y. Liu, Differential equations for modeling asynchronous algorithms, in *Proceedings of the 27th International Joint Conference on Artificial Intelligence* (2018), pp. 2220–2226
10. E.P. Hsu, *Stochastic Analysis on Manifolds*, vol. 38 (American Mathematical Soc., Providence, RI, 2002)
11. S. Jastrzkebski, Z. Kenton, D. Arpit, N. Ballas, A. Fischer, Y. Bengio, A. Storkey, Three factors influencing minima in SGD. *arXiv preprint* arXiv:1711.04623 (2017)
12. V.N. Kolokoltsov, *Semiclassical Analysis for Diffusions and Stochastic Processes* (Springer, Berlin, 2007)
13. R.C. Kraaij, F. Redig, R. Versendaal, Classical large deviation theorems on complete Riemannian manifolds. Stoch. Processes Appl. **129**(11), 4294–4334 (2019)
14. A. Krizhevsky, I. Sutskever, G.E. Hinton, ImageNet classification with deep convolutional neural networks. Adv. Neural Inf. Process. Syst. **25**, 1097–1105 (2012)

15. Q. Li, C. Tai, et al., Stochastic modified equations and adaptive stochastic gradient algorithms, in *Proceedings of the 34th International Conference on Machine Learning*, vol. 70 (JMLR.org, 2017), pp. 2101–2110
16. Y.-A. Ma, T. Chen, E.B. Fox, A complete recipe for stochastic gradient MCMC. *arXiv preprint* arXiv:1506.04696 (2015)
17. Q. Meng, S. Gong, W. Chen, Z.-M. Ma, T.-Y. Liu, Dynamic of stochastic gradient descent with state-dependent noise. *arXiv preprint* arXiv:2006.13719v3 (2020)
18. Q. Meng, S. Gong, W. Du, W. Chen, Z.-M. Ma, T.-Y. Liu, A fine-grained study on the escaping behavior of stochastic gradient descent (2021, under review)
19. W. Mou, L. Wang, X. Zhai, K. Zheng, Generalization bounds of SGLD for non-convex learning: two theoretical viewpoints. *arXiv preprint* arXiv:1707.05947 (2017)
20. T.H. Nguyen, U. Simsekli, M. Gurbuzbalaban, G. Richard, First exit time analysis of stochastic gradient descent under heavy-tailed gradient noise, in *Advances in Neural Information Processing Systems* (2019), pp. 273–283
21. A. van den Oord, S. Dieleman, H. Zen, K. Simonyan, O. Vinyals, A. Graves, N. Kalchbrenner, A. Senior, K. Kavukcuoglu, Wavenet: a generative model for raw audio. *arXiv preprint* arXiv:1609.03499 (2016)
22. A. Rakhlin, O. Shamir, K. Sridharan, Making gradient descent optimal for strongly convex stochastic optimization, in *Proceedings of the 29th International Conference on International Conference on Machine Learning* (2012), pp. 1571–1578
23. Y. Ren, Y. Ruan, X. Tan, T. Qin, S. Zhao, Z. Zhao, T.-Y. Liu, Fastspeech: fast, robust and controllable text to speech. *arXiv preprint* arXiv:1905.09263 (2019)
24. J. Shen, R. Pang, R.J. Weiss, M. Schuster, N. Jaitly, Z. Yang, Z. Chen, Y. Zhang, Y. Wang, R. Skerrv-Ryan, et al., Natural TTs synthesis by conditioning Wavenet on MEL spectrogram predictions, in *2018 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)* (IEEE, 2018), pp. 4779–4783
25. Q.V. Le, S.L. Smith, A Bayesian perspective on generalization and stochastic gradient descent. *arXiv preprint* arXiv:1710.06451 (2017)
26. M. Sundermeyer, R. Schlüter, H. Ney, LSTM neural networks for language modeling, in *Thirteenth Annual Conference of the International Speech Communication Association* (2012)
27. A. Vaswani, N. Shazeer, N. Parmar, J. Uszkoreit, L. Jones, A.N. Gomez, Ł. Kaiser, I. Polosukhin, Attention is all you need, in *Advances in Neural Information Processing Systems* (2017), pp. 5998–6008
28. F.-Y. Wang, *Analysis for Diffusion Processes on Riemannian Manifolds*, vol. 18 (World Scientific, Singapore, 2014)
29. C. Wei, S. Kakade, T. Ma, The implicit and explicit regularization effects of dropout, in *International Conference on Machine Learning* (PMLR, 2020), pp. 10181–10192
30. E. Weinan, T. Li, E. Vanden-Eijnden, *Applied Stochastic Analysis*, vol. 199 (American Mathematical Soc., Providence, RI, 2019)
31. L. Wu, E. Weinan, C. Ma, How SGD selects the global minima in over-parameterized learning: a dynamical stability perspective. Adv. Neural Inf. Process. Syst. **31**, 8279–8288 (2018)
32. M. Sugiyama, Z. Xie, I. Sato, A diffusion theory for deep learning dynamics: stochastic gradient descent escapes from sharp minima exponentially fast. *arXiv preprint* arXiv:2002.03495 (2020)
33. G. Zhang, L. Li, Z. Nado, J. Martens, S. Sachdeva, G. Dahl, C. Shallue, R.B. Grosse, Which algorithmic choices matter at which batch sizes? Insights from a noisy quadratic model, in *Advances in Neural Information Processing Systems* (2019), pp. 8196–8207

34. Y. Zhou, J. Du, Kramers escape rate in overdamped systems with the power-law distribution. Physica A: Stat. Mech. Appl. **402**, 299–305 (2014)
35. Z. Zhu, J. Wu, B. Yu, L. Wu, J. Ma, The anisotropic noise in stochastic gradient descent: its behavior of escaping from sharp minima and regularization effects, in *Proceedings of International Conference on Machine Learning* (2019), pp. 7654–7663

Hölder Estimates for Resolvents of Time-Changed Brownian Motions

Kouhei Matsuura

Abstract In this paper, we study time changes of Brownian motions by positive continuous additive functionals. Under a certain regularity condition on the associated Revuz measures, we prove that the resolvents of the time-changed Brownian motions are locally Hölder continuous in the spatial components. We also obtain lower bounds for the indices of the Hölder continuity.

Keywords Brownian motion · Time change · Hölder continuity · Resolvent · Coupling

Mathematics Subject Classification 31C25 · 60J35 · 60J55 · 60J60 · 60J45

1 Introduction

Let $B = (\{B_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^d})$ be a Brownian motion on the d-dimensional Euclidean space $\mathbb{R}^d$. Let $A = \{A_t\}_{t\geq 0}$ be a positive continuous additive functional (PCAF in abbreviation) of B. Then, the time-changed Brownian motion $\check{B} = (\{\check{B}_t\}_{t\geq 0}, \{\check{P}_x\}_{x\in F})$ by the PCAF A is defined as

$$\check{B}_t = B_{\tau_t},\ \check{P}_x = P_x, \quad (t, x) \in [0, \infty) \times F.$$

Here, we denote by $\{\tau_t\}_{t\geq 0}$ the right continuous inverse of $\{A_t\}_{t\geq 0}$, and F stands for the support of A (see (3) below for the definition). From the Revuz correspondence (see (2)), the PCAF A induces a Borel measure μ on $\mathbb{R}^d$, which is called the Revuz measure of A. It is known that the time-changed Brownian motion $\check{B}$ becomes a μ-symmetric right process on F (see, e.g. [4, Theorem 5.2.1]). On account of this fact, in what follows, we use the symbol B^μ (A^μ and F^μ, respectively) to denote $\check{B}$ (A and F, respectively).

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A typical example of Revuz measures is of the form $f\,dm$. Here, $f : \mathbb{R}^d \to [0, \infty)$ is a locally bounded Borel measurable function, and m stands for the Lebesgue measure on $\mathbb{R}^d$. Then, we have $A_t^{f\,dm} = \int_0^t f(B_s)\,ds$, $t \geq 0$. However, a Revuz measure μ can be singular with respect to m. Then, the behavior of B^μ would be quite different from that of the standard Brownian motion. Nevertheless, if $F^\mu = \mathbb{R}^d$, we can simply describe the Dirichlet form $(\mathcal{E}, \mathcal{F}^\mu)$ of B^μ by using the extended Dirichlet space $H_e^1(\mathbb{R}^d)$ of B. If $d \in \{1, 2\}$, we see from [4, Theorem 2.2.13] that $H_e^1(\mathbb{R}^d)$ is identified with

$$\{f \in L^2_{\mathrm{loc}}(\mathbb{R}^d, m) \mid |\nabla f| \in L^2(\mathbb{R}^d, m)\}.$$

Here, $L^2_{\mathrm{loc}}(\mathbb{R}^d, m)$ is the space of locally square integrable functions on $\mathbb{R}^d$ with respect to m, and ∇f denotes the distributional gradient of f. Even if $d > 2$, the extended Dirichlet space is characterized with distributional derivatives ([4, Theorem 2.2.12]). From these facts and [4, (5.2.17)], we find that the Dirichlet form $(\mathcal{E}, \mathcal{F}^\mu)$ is identified with

$$\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}^d} (\nabla f(x), \nabla g(x))\,dm(x), \quad f, g \in \mathcal{F}^\mu, \tag{1}$$

$$\mathcal{F}^\mu = \{\widetilde{u} \in H_e^1(\mathbb{R}^d) \mid \widetilde{u} \in L^2(\mathbb{R}^d, \mu)\}.$$

Here, we denote by $(\cdot, \cdot)$ the standard inner product on $\mathbb{R}^d$, and $\widetilde{u}$ is the quasi-continuous version of $u \in H_e^1(\mathbb{R}^d)$. See [5, Lemma 2.1.4 and Theorem 2.1.7] for the existence and the uniqueness. However, even in this setting, it is generally difficult to write down other analytical objects associated with B^μ, such as the semigroup and the resolvent. Therefore, it is non-trivial to clarify how these objects depend on μ.

In this paper, we study the continuity of the resolvent of B^μ in the spatial component. Even though this kind of problem can be formulated for other Markov processes, the current setting allows us to quantitatively clarify how the continuity depends on μ. In the main theorem of this paper (Theorem 1), we prove that the resolvent of B^μ is Hölder continuous in the spatial component under a certain condition on μ. The condition is given in (5) below, and the index there represents a regularity of μ. This also describes a lower bound for the index of the Hölder continuity of the resolvent. In particular, we see that the resolvent is $(1 - \varepsilon)$-Hölder continuous if the index is sufficiently large. Condition (5) can be regarded as a generalized concept of the d-measure. We refer the reader to [6] for basic facts on time-changed Hunt processes by PCAFs associated with d-measures. We also note that the Liouville measure is one of examples which satisfies (5). The reader is referred to [7, 8] and references therein for more details and the time changed planar Brownian motion by the PCAF associated with the Liouvllle measure.

If $F^\mu = \mathbb{R}^d$, it is not very hard to see that the resolvent of B^μ is just Hölder continuous. In fact, we see from (1) that any bounded harmonic function h on $B(z, 2r)$ ($z \in \mathbb{R}^d$, $r > 0$) with respect to B^μ is also harmonic with respect to the standard

Brownian motion. Here, $B(z, r)$ denotes the open ball centered at z with radius $r > 0$. Then, from [1, Chapter II (1.3) Proposition], there exists a positive constant C independent of z and r such that

$$|h(x) - h(y)| \le C \sup_{z \in \mathbb{R}^d} |h(z)| \frac{|x - y|}{r}, \quad x, y \in B(z, r).$$

Furthermore, since A^μ is a homeomorphism on $[0, \infty)$ (here, we used the assumption that $F^\mu = \mathbb{R}^d$), $A^\mu_{\tau_{B(x,r)}}$ is identified with the exit time of B^μ from $B(x, r)$, where $\tau_{B(x,r)}$ denotes the first exit time of B from $B(x, r)$. This observation and the regularity condition (5) lead us to a mean exit time estimate for B^μ (see also Lemma 2). Then, the same argument as in [2, (5.6) Proposition, Section VII] shows that the resolvent of B^μ is Hölder continuous. We note that this kind of argument is applicable to several situations (see, e.g., [3, Proposition 3.3 and Theorem 3.5]); however, even if $\mu = m$, this only implies that the index of the Hölder continuity is greater than or equal to $2/3$. This estimate is not sharp because B^m is the standard Brownian motion and the resolvent is Lipschitz continuous. Thus, even if $F^\mu = \mathbb{R}^d$, our result does not directly follow from the method stated above, and implies rather sharp result.

For the proof of Theorem 1, we use the mirror coupling of d-dimensional Brownian motions. The key to our proof is an inductive argument based on the strong Markov property (of the coupling) and some estimates of the coupling time (Lemmas 3 and 5). Since mirror couplings of stochastic processes are universal concepts, our arguments may be useful for estimating the indices of Hölder continuity of resolvents for other time-changed Markov processes.

The remainder of this paper is organized as follows. In Sect. 2, we set up a framework and state the main theorem (Theorem 1). In Sect. 3, we provide some preliminary estimates for PCAFs of the d-dimensional Brownian motion. In Sect. 4, we introduce some lemmas on the mirror coupling of Brownian motions, and prove Theorem 1.

Notation. In the paper, we use the following symbols and conventions.

- $(\cdot, \cdot)$ and $|\cdot|$ denote the standard inner product and norm of $\mathbb{R}^d$, respectively.
- For $x \in \mathbb{R}^d$ and $r > 0$, $B(x, r)$ (resp. $\bar{B}(x, r)$) denotes the open (resp. closed) ball in $\mathbb{R}^d$ with center x and radius r.
- For a subset $S \subset \mathbb{R}^d$ and $f : S \to [-\infty, \infty]$, we set $\|f\|_\infty := \|f\|_{\infty,S} := \sup_{x \in S} |f(x)|$.
- For a topological space S, we write $\mathcal{B}_b(S)$ for the space of bounded Borel measurable functions on S.
- For $a, b \in [-\infty, \infty]$, we write $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.
- $\inf \emptyset = \infty$ by convention.

2 Main Results

Let $B = (\{B_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^d})$ be a Brownian motion on $\mathbb{R}^d$. The Dirichlet form is identified with

$$\mathcal{E}(f,g) = \frac{1}{2}\int_{\mathbb{R}^d} (\nabla f(x), \nabla g(x))\, dm(x), \quad f, g \in H^1(\mathbb{R}^d).$$

Here, $H^1(\mathbb{R}^d)(= H^1_{\mathrm{e}}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d, m))$ denotes the first-order Sobolev space on $\mathbb{R}^d$. For an open subset $U \subset \mathbb{R}^d$ and for a subset $A \subset \mathbb{R}^d$, we define

$$\mathrm{cap}(U) = \inf\left\{ \mathcal{E}(f,f) + \int_{\mathbb{R}^d} f^2\, dm \,\middle|\, f \in H^1(\mathbb{R}^d),\ f \geq 1,\ m\text{-a.e. on } U \right\},$$

$$\mathrm{Cap}(A) = \inf\left\{\mathrm{cap}(U) \mid A \subset U \text{ and } U \text{ is an open subset of } \mathbb{R}^d\right\}.$$

A non-negative Radon measure μ on $\mathbb{R}^d$ is said to be *smooth* if $\mu(A) = 0$ for any $A \subset \mathbb{R}^d$ with $\mathrm{Cap}(A) = 0$. For a smooth measure μ, by [4, Theorem 4.1.1], there exists a unique PCAF $A^\mu = \{A^\mu_t\}_{t\geq 0}$ of B such that for any non-negative functions $f, g \in \mathcal{B}_b(\mathbb{R}^d)$ and $\alpha > 0$,

$$\int_{\mathbb{R}^d} E_x\left[\int_0^\infty e^{-\alpha t} f(B_t)\, dA^\mu_t\right] g(x)\, dm(x) = \int_{\mathbb{R}^d} E_x\left[\int_0^\infty e^{-\alpha t} g(B_t)\, dt\right] f(x)\, d\mu(x), \quad (2)$$

where E_x denotes the expectation under P_x. See [4, Sect. 4] and [5, Sect. 5] for the definition and further details on PCAFs. We also note that the exceptional set of A^μ can be taken to be empty (see [4, Theorem 4.1.11]).

Let $\{\tau^\mu_t\}_{t\geq 0}$ be the right continuous inverse of A^μ. We define

$$B^\mu_t = B_{\tau^\mu_t},\ P^\mu_x = P_x, \quad (t,x) \in [0,\infty) \times F^\mu,$$

where F^μ denotes the support of A^μ:

$$F^\mu = \{x \in \mathbb{R}^d \mid \inf\{t > 0 \mid A^\mu_t > 0\} = 0,\ P_x\text{-a.s.}\}. \quad (3)$$

We note that F^μ is a nearly Borel subset with respect to B (see the paragraph after [4, (A.3.11)]). The support F^μ is also regarded as a topological subspace of $\mathbb{R}^d$. By [4, Theorems 5.2.1 and A.3.11], $B^\mu = (\{B^\mu_t\}_{t\geq 0}, \{P^\mu_x\}_{x\in F^\mu})$ is a μ-symmetric right process on F^μ. The resolvent $\{G^\mu_\alpha\}_{\alpha>0}$ is given by

$$G_\alpha^\mu f(x) = E_x\left[\int_0^\infty e^{-\alpha t} f(B_t^\mu)\,dt\right], \quad \alpha > 0,\ f \in \mathcal{B}_b(F^\mu),\ x \in F^\mu.$$

Let $\mathcal{B}_b^*(\mathbb{R}^d)$ denote the space of bounded universally measurable functions on $\mathbb{R}^d$. That is, any $f \in \mathcal{B}_b^*(\mathbb{R}^d)$ is bounded and measurable with respect to the σ-field $\mathcal{B}^*(\mathbb{R}^d)$; the family of universally measurable subsets of $\mathbb{R}^d$: $\mathcal{B}^*(\mathbb{R}^d) := \bigcap_{\mu\in\mathcal{P}(\mathbb{R}^d)} \mathcal{B}^\mu(\mathbb{R}^d)$. Here, $\mathcal{P}(\mathbb{R}^d)$ denotes the family of all probability measures on $\mathbb{R}^d$ and $\mathcal{B}^\mu(\mathbb{R}^d)$ is the completion of the Borel σ-field $\mathcal{B}(\mathbb{R}^d)$ on $\mathbb{R}^d$ with respect to $\mu \in \mathcal{P}(\mathbb{R}^d)$. For $\alpha > 0, f \in \mathcal{B}_b^*(\mathbb{R}^d)$, and $x \in \mathbb{R}^d$, we define

$$V_\alpha^\mu f(x) = E_x\left[\int_0^\infty e^{-\alpha A_t^\mu} f(B_t)\,dA_t^\mu\right].$$

We see from [4, Exercise A.1.29] that F^μ is a universally measurable subsets of $\mathbb{R}^d$. By noting this fact and using [4, Lemma A.3.10], we have

$$G_\alpha^\mu f(x) = V_\alpha^\mu f(x). \tag{4}$$

for any $\alpha > 0, f \in \mathcal{B}_b(F^\mu)$, and $x \in F^\mu$.

Now we are in a position to state our main theorem.

Theorem 1 *Let $p \in \mathbb{R}^d$ and assume that there exist $\kappa > d-2$, $R \in (0,1]$, and $K > 0$ such that*

$$\mu(B(x,r)) \le Kr^\kappa \tag{5}$$

for any $r \le R$ and $x \in \mathbb{R}^d$ with $|x-p| \le r$. Then, for any $\alpha > 0$ and $\varepsilon \in (0, (2-d+\kappa)/(3-d+\kappa))$, there exists $C > 0$ depending on d, p, κ, K, R, ε, and α such that

$$\left|V_\alpha^\mu f(x) - V_\alpha^\mu f(y)\right| \le C\|f\|_\infty |x-y|^{\{(2-d+\kappa)\wedge 1\}-\varepsilon} \tag{6}$$

for any $f \in \mathcal{B}_b^*(\mathbb{R}^d)$ and $x, y \in B(p, 2^{-C'}R)$, where C' is a positive number depending on d, ε and κ. In particular, we have

$$\left|G_\alpha^\mu f(x) - G_\alpha^\mu f(y)\right| \le C\|f\|_\infty |x-y|^{\{(2-d+\kappa)\wedge 1\}-\varepsilon} \tag{7}$$

for any $f \in \mathcal{B}_b(F^\mu)$ and $x, y \in F^\mu \cap B(p, 2^{-C'}R)$.

3 Preliminary Lemmas

For an open subset $U \subset \mathbb{R}^d$, we denote by $U \cup \{\partial_U\}$ the one-point compactification. We set $\tau_U = \inf\{t \in [0,\infty) \mid B_t \notin U\}$. Then, the absorbing Brownian motion $B^U = (\{B_t^U\}_{t\geq 0}, \{P_x\}_{x\in U})$ on U is defined as

$$B_t^U = \begin{cases} B_t, & t < \tau_U, \\ \partial_U, & t \geq \tau_U. \end{cases}$$

We write $p^U = p_t^U(x,y) \colon (0,\infty) \times U \times U \to [0,\infty)$ for the transition density of B^U. That is, p^U is the jointly continuous function such that

$$P_x(B_t^U \in dy) = p_t^U(x,y)\, dm(y), \quad t > 0,\ x \in U.$$

The Green function of B^U is defined by

$$g_U(x,y) = \int_0^\infty p_t^U(x,y)\, dt, \quad x, y \in U.$$

Lemma 1 *Let $U \subset \mathbb{R}^d$ be an open subset. Then, for any $x \in U$, $t > 0$, and non-negative $f \in \mathcal{B}_b(U)$,*

$$E_x\left[\int_0^{t\wedge\tau_U} f(B_s)\, dA_s^\mu\right] = \int_U \left(\int_0^t p_s^U(x,y)\, ds\right) f(y)\, d\mu(y).$$

In particular, we have

$$E_x\left[\int_0^{\tau_U} f(B_s)\, dA_s^\mu\right] = \int_U g_U(x,y) f(y)\, d\mu(y).$$

Proof We fix $t > 0$ and non-negative functions $f, g \in \mathcal{B}_b(U)$. We may assume that f is compactly supported. By [4, Proposition 4.1.10], we have

$$\int_U E_z\left[\int_0^{t\wedge\tau_U} f(B_s)\, dA_s^\mu\right] g(z)\, dm(z) = \int_0^t \left(\int_U (P_s^U g)(x) f(x)\, d\mu(x)\right) ds. \tag{8}$$

We use Fubini's theorem to obtain that

$$\int_0^t \left(\int_U (P_s^U g)(x) f(x)\, d\mu(x) \right) ds$$
$$= \int_U \left(\int_0^t \left(\int_U p_s^U(x,z) f(x)\, d\mu(x) \right) ds \right) g(z)\, dm(z). \tag{9}$$

Because μ is a Radon measure, by letting $g = \mathbf{1}_U$ in (8) and (9), we see that

$$E_{(\cdot)}\left[\int_0^{t\wedge\tau_U} f(B_s)\, dA_s^\mu \right] \quad \text{and} \quad \int_0^t \left(\int_U p_s^U(x,\cdot) f(x)\, d\mu(x) \right) ds$$

are integrable on U with respect to m. Moreover, because g is arbitrarily taken, (8) and (9) imply that for m-a.e. $z \in U$,

$$E_z\left[\int_0^{t\wedge\tau_U} f(B_s)\, dA_s^\mu \right] = \int_0^t \left(\int_U p_s^U(x,z) f(x)\, d\mu(x) \right) ds. \tag{10}$$

By following the convention that $f(\partial_U) = 0$, we see that the left-hand side of (10) is equal to $E_z[\int_0^t f(B_s^U)\, dA_{s\wedge\tau_U}^\mu]$. Hence, we have for m-a.e. $z \in U$,

$$E_z\left[\int_0^t f(B_s^U)\, dA_{s\wedge\tau_U}^\mu \right] = \int_0^t \left(\int_U p_s^U(x,z) f(x)\, d\mu(x) \right) ds. \tag{11}$$

We see from [4, Exercise 4.1.9 (iii)] that $\{A_{s\wedge\tau_U}^\mu\}_{s\ge 0}$ is the PCAF of B^U. By using the additivity, the Markov property of B^U, and (11), we obtain that for any $x \in U$,

$$E_x\left[\int_0^{t\wedge\tau_U} f(B_s)\, dA_s^\mu \right] = \lim_{u\downarrow 0} E_x\left[\int_u^t f(B_s^U)\, dA_{s\wedge\tau_U}^\mu \right]$$
$$= \lim_{u\downarrow 0} E_x\left[E_{B_u^U}\left[\int_0^{t-u} f(B_s^U)\, dA_{s\wedge\tau_U}^\mu \right] \right]$$
$$= \lim_{u\downarrow 0} \int_U p_u^U(x,z) \left(\int_0^{t-u} \left(\int_U p_s^U(y,z) f(y)\, d\mu(y) \right) ds \right) dm(z)$$
$$= \lim_{u\downarrow 0} \int_u^t \left(\int_U p_s^U(x,y) f(y)\, d\mu(y) \right) ds = \int_0^t \left(\int_U p_s^U(x,y) f(y)\, d\mu(y) \right) ds,$$

which completes the proof. "In particular" part immediately follows from the monotone convergence theorem. □

Let $d \geq 2$, $r \in (0, 1)$, and $x, y \in \mathbb{R}^d$. Then, by [5, Example 1.5.1],

$$g_{B(x,r)}(x, y) = \begin{cases} -\dfrac{1}{\pi} \log |x - y|, & d = 2, \\ \dfrac{\Gamma(d/2 - 1)}{2\pi^{d/2}} |x - y|^{2-d}, & d \geq 3. \end{cases} \tag{12}$$

Here, Γ denotes the gamma function. If $d = 1$, we see from [9, Lemma 20.10] that for any $a, b \in \mathbb{R}$ with $a < b$,

$$g_{(a,b)}(x, y) = \frac{2(x \wedge y - a)(b - x \vee y)}{b - a}, \quad x, y \in (a, b). \tag{13}$$

For $s \in (0, 1]$ and $t > 0$, we define

$$\zeta_d(s, t) = \begin{cases} s^{2-d+t}, & d \geq 3 \text{ or } d = 1, \\ -s^t \log s, & d = 2. \end{cases}$$

Lemma 2 *Let $p \in \mathbb{R}^d$ and take constants $\kappa > d - 2$, $R \in (0, 1]$, and $K > 0$ so that (5) holds. Then, there exists $C \in (0, \infty)$ depending on d, p, κ, R, and K such that for any $r \in (0, R/2]$ and $x \in B(p, r)$*

$$\int_{B(x,r)} g_{B(x,r)}(x, y)\, d\mu(y) \leq C\zeta_d(r, \kappa).$$

In particular, we have $E_x[A^{\mu}_{\tau_{B(x,r)}}] \leq C\zeta_d(r, \kappa)$.

Proof In view of (5), we have for any $r \in (0, R/2]$ and $x \in B(p, r)$,

$$\mu(B(x, r)) \leq Kr^{\kappa}. \tag{14}$$

Therefore, when $d = 1$, we use (13) to obtain that

$$\int_{B(x,r)} g_{B(x,r)}(x, y)\, d\mu(y) \leq 4Kr^{\kappa+1}.$$

Equation (12) implies that for any $k \in \mathbb{N}$,

$$\sup_{y \in \mathbb{R}^d \setminus B(x, r2^{-k})} g_{B(x,r)}(x, y) \le \begin{cases} -\dfrac{1}{\pi} \log(r2^{-k}), & d = 2, \\ \dfrac{\Gamma(d/2-1)}{2\pi^{d/2}} (r2^{-k})^{2-d}, & d \ge 3. \end{cases}$$

Thus, for $d = 2$, we obtain from (14) that

$$\begin{aligned} \int_{B(x,r)} g_{B(x,r)}(x, y)\, d\mu(y) &= \sum_{k=1}^{\infty} \int_{B(x, r2^{-(k-1)}) \setminus B(x, r2^{-k})} g_{B(x,r)}(x, y)\, d\mu(y) \\ &\le -\frac{K}{\pi} \sum_{k=1}^{\infty} (r2^{-(k-1)})^{\kappa} \log(r2^{-k}) \le -Cr^{\kappa} \log r. \end{aligned}$$

Here, C is a positive constant depending on κ and K. For $d \ge 3$, we similarly use (14) to obtain that

$$\begin{aligned} \int_{B(x,r)} g_{B(x,r)}(x, y)\, d\mu(y) &\le \frac{K\Gamma(d/2-1)}{2\pi^{d/2}} \sum_{k=1}^{\infty} (r2^{-(k-1)})^{\kappa} (r2^{-k})^{2-d} \\ &\le \left(\frac{K\Gamma(d/2-1)2^{\kappa}}{2\pi^{d/2}} \sum_{k=1}^{\infty} 2^{-\{(2-d)+\kappa\}k} \right) r^{2-d+\kappa}. \end{aligned}$$

Because $\kappa > d-2$, we have

$$\int_{B(x,r)} g_{B(x,r)}(x, y)\, d\mu(y) \le Cr^{2-d+\kappa}.$$

Here, C is a positive constant depending on d, K and κ. "In particular" part immediately follows from Lemma 1. □

4 Proof of Theorem 1

Let $x, y \in \mathbb{R}^d$ and $\{W_t\}_{t \ge 0}$ a d-dimensional Brownian motion starting at the origin. The mirror coupling $(Z^x, \widetilde{Z}^y) = (\{Z_t^x\}_{t \ge 0}, \{\widetilde{Z}_t^y\}_{t \ge 0})$ of d-dimensional Brownian motions is defined as follows:

- For any $t < \inf\{s > 0 \mid Z_s^x = \widetilde{Z}_s^y\}$,

$$\begin{aligned} Z_t^x &= x + W_t, \\ \widetilde{Z}_t^y &= y + W_t - 2\int_0^t \frac{Z_s^x - \widetilde{Z}_s^y}{|Z_s^x - \widetilde{Z}_s^y|^2}(Z_s^x - \widetilde{Z}_s^y,\, dW_s). \end{aligned} \tag{15}$$

- For any $t \geq \inf\{s > 0 \mid Z_s^x = \widetilde{Z}_s^y\}$, we have $Z_t^x = \widetilde{Z}_t^y$.

Remark 1 (1) The mirror coupling $(Z^x, \widetilde{Z}^y)$ is a special case of couplings for diffusion processes studied in [10, Sect. 3].
(2) For $x, y \in \mathbb{R}^d$ with $x \neq y$ and $t < \inf\{s > 0 \mid Z_s^x = \widetilde{Z}_s^y\}$, we have

$$Z_t^x - \widetilde{Z}_t^y = x - y + 2\int_0^t \frac{Z_s^x - \widetilde{Z}_s^y}{|Z_s^x - \widetilde{Z}_s^y|^2}(Z_s^x - \widetilde{Z}_s^y,\, dW_s).$$

This implies that the random vector $Z_t^x - Z_t^y$ is parallel to $x - y$. We then see from (15) that $\widetilde{Z}_t^y$ coincides with the mirror image of Z_t^x with respect to the hyperplane $H_{x,y} = \{z \in \mathbb{R}^d \mid (z - (x+y)/2, x - y) = 0\}$. Further, $\inf\{s > 0 \mid Z_s^x = \widetilde{Z}_s^y\} = \inf\{s > 0 \mid Z_s^x \in H_{x,y}\}$. Then, it is easy to see that $(Z^x, \widetilde{Z}^y)$ is a strong Markov process on $\mathbb{R}^d \times \mathbb{R}^d$.

For $x, y \in \mathbb{R}^d$, we define $\xi_{x,y} = \inf\{t > 0 \mid Z_t^x = Z_t^y\}$. We denote by $P_{x,y}$ the distribution of $(Z^x, \widetilde{Z}^y)$. For $t \geq 0$, we set

$$A_t^{\mu,x} = A_t^\mu(Z^x), \quad \widetilde{A}_t^{\mu,y} = A_t^\mu(\widetilde{Z}^y),$$

where we regard A_t^μ as $[0, \infty]$-valued functions on $C([0, \infty), \mathbb{R}^d)$, the space of $\mathbb{R}^d$-valued continuous functions on $[0, \infty)$. Then, $\{A_t^{\mu,x}\}_{t\geq 0}$ and $\{\widetilde{A}_t^{\mu,y}\}_{t\geq 0}$ become PCAFs of Z^x and $\widetilde{Z}^y$, respectively. Furthermore, $A^{\mu,x}$ and $\widetilde{A}^{\mu,y}$ can be regarded as PCAFs of the coupled process $(Z^x, \widetilde{Z}^y)$ in the natural way.

For $x, y \in \mathbb{R}^d$, we define

$$\mathcal{I}_{x,y} = E_{x,y}\big[A_{\xi_{x,y}}^{\mu,x} \wedge 1\big], \quad \widetilde{\mathcal{I}}_{x,y} = E_{x,y}\big[\widetilde{A}_{\xi_{x,y}}^{\mu,y} \wedge 1\big], \tag{16}$$

where $E_{x,y}$ denotes the expectation under $P_{x,y}$. At the end of this section (see (44) below), we will show that for any $f \in \mathcal{B}_b^*(\mathbb{R}^d)$ with $\|f\|_\infty \leq 1$,

$$|V_\alpha^\mu f(x) - V_\alpha^\mu f(y)| \leq 2(1 + \alpha^{-1})(\mathcal{I}_{x,y} + \widetilde{\mathcal{I}}_{x,y}), \quad \alpha > 0,\ x, y \in \mathbb{R}^d.$$

We now introduce some lemmas to estimate the expectations in (16).

Lemma 3 *Let $x, y \in \mathbb{R}^d$, and τ be a stopping time of $(Z^x, \widetilde{Z}^y)$. Then,*

$$E_{x,y}\left[\left|Z^x_{\xi_{x,y}\wedge\tau} - \widetilde{Z}^y_{\xi_{x,y}\wedge\tau}\right|^\theta\right] \le |x-y|^\theta$$

for any $\theta \in (0, 1]$.

Proof We fix $t \ge 0$ and $x, y \in \mathbb{R}^d$ with $x \ne y$. To simplify the notation, we write Z_t (resp. $\widetilde{Z}_t$, L_t, $\widetilde{L}_t$, ξ) for Z^x_t (resp. Z^y_t, L^x_t, $\widetilde{L}^y_t$, $\xi_{x,y}$). We also fix $n \in \mathbb{N}$ such that $|x-y| \ge 1/n$, and set $\xi_n = \inf\{s > 0 \mid |Z_s - \widetilde{Z}_s| \le 1/n\}$.

For $s < \xi_n \wedge \tau$, we define

$$\alpha_s = (Z_s - \widetilde{Z}_s)(Z_s - \widetilde{Z}_s)^T/|Z_s - \widetilde{Z}_s|^2.$$

Here, $(Z_s - \widetilde{Z}_s)^T$ denotes the transpose of $Z_s - \widetilde{Z}_s$. From Itô formula,

$$|Z_{t\wedge\xi_n\wedge\tau} - \widetilde{Z}_{t\wedge\xi_n\wedge\tau}|^2 - |x-y|^2 = 2\int_0^{t\wedge\xi_n\wedge\tau} (Z_s - \widetilde{Z}_s, \alpha_s\, dW_s) + t\wedge\xi_n\wedge\tau$$

and for any $\theta \in (0, 1]$,

$$\begin{aligned}
&|Z_{t\wedge\xi_n\wedge\tau} - \widetilde{Z}_{t\wedge\xi_n\wedge\tau}|^\theta - |x-y|^\theta \\
&= \theta \int_0^{t\wedge\xi_n\wedge\tau} |Z_s - \widetilde{Z}_s|^{\theta-2}(Z_s - \widetilde{Z}_s, \alpha_s\, dW_s) \\
&\quad + \{\theta/2 + \theta\,(\theta/2 - 1)\} \int_0^{t\wedge\xi_n\wedge\tau} |Z_s - \widetilde{Z}_s|^{\theta-2}\, ds. \qquad (17)
\end{aligned}$$

Since the first term above is a martingale and the second one is non-positive, by taking the expectations of both sides of (17), we arrive at

$$E_{x,y}\left[\left|Z_{t\wedge\xi_n\wedge\tau} - \widetilde{Z}_{t\wedge\xi_n\wedge\tau}\right|^\theta\right] \le |x-y|^\theta. \qquad (18)$$

Letting $n \to \infty$ in (18), we complete the proof. □

Lemma 4 *It holds that*

$$P_{x,y}(t < \xi_{x,y}) \le \frac{|x-y|}{\sqrt{2\pi t}}$$

for any $t > 0$ and $x, y \in \mathbb{R}^d$.

Proof Let $t \ge 0$ and $x, y \in \mathbb{R}^d$ with $x \ne y$. We take $n \in \mathbb{N}$ such that $|x-y| \ge 1/n$. Letting $\theta = 1$ in (17), we have

$$|Z_{t\wedge\xi_n} - \widetilde{Z}_{t\wedge\xi_n}| - |x-y| = \int_0^{t\wedge\xi_n} |Z_s - \widetilde{Z}_s|^{-1}(Z_s - \widetilde{Z}_s, \alpha_s\, dW_s). \tag{19}$$

The quadratic variation of the right-hand side of (19) equals to $t \wedge \xi_n$, $t \geq 0$. Hence, by the Dambis–Dubins–Schwartz theorem, there is a one-dimensional Brownian motion $\beta = \{\beta_s\}_{s\geq 0}$ such that

$$\beta_{t\wedge\xi_n} = \int_0^{t\wedge\xi_n} |Z_s - \widetilde{Z}_s|^{-1}(Z_s - \widetilde{Z}_s, \alpha_s\, dW_s). \tag{20}$$

By using (19), (20), and the reflection principle of the Brownian motion, we have

$$\begin{aligned} P_{x,y}(\xi_n > t) &\leq P_{x,y}\left(-|x-y| \leq \inf_{0\leq s\leq t} \beta_s\right) \\ &= 1 - 2\int_{-\infty}^{-|x-y|} \frac{1}{\sqrt{2\pi t}} \exp(-u^2/2t)\, du \\ &= \int_{-|x-y|}^{|x-y|} \frac{1}{\sqrt{2\pi t}} \exp(-u^2/2t)\, du \leq \frac{|x-y|}{\sqrt{2\pi t}}. \end{aligned} \tag{21}$$

Letting $n \to \infty$ in (21) completes the proof. □

For $x, y \in \mathbb{R}^d$ and an open subset $U \subset \mathbb{R}^d$, we define

$$\tau_U^x = \tau_U(Z^x), \quad \widetilde{\tau}_U^y = \tau_U(\widetilde{Z}^y)$$

where we regard τ_U as $[0, \infty]$-valued function on $C([0, \infty), \mathbb{R}^d)$. We note that $\tau_U^x = \inf\{t > 0 \mid (Z_t^x, \widetilde{Z}_t^y) \notin U \times \mathbb{R}^d\}$. Hence, τ_U^x and $\widetilde{\tau}_U^y$ are exit times of $(Z^x, \widetilde{Z}^y)$. We also see from [11, Lemma II.1.2] that there exists $C > 0$ depending on d such that

$$P_{x,y}(\tau_{B(x,r)}^x \leq t) \leq C \exp(-r^2/Ct) \tag{22}$$

for any $(x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times (0, \infty)$ and $r \in (0, \infty)$.

Lemma 5 *Let $\chi, \varepsilon \in (0, 1]$, $R > 0$ and $n \geq 1$ be positive numbers. Then, there is a positive constant C depending on ε, R, and n such that*

$$P_{x,y}(\tau_{B(x,2^{-n}R|x-y|^\chi)}^x \leq \xi_{x,y}) \leq C|x-y|^{1-\chi-\varepsilon}$$

for any $x, y \in \mathbb{R}^d$ with $|x-y| \in (0, 1]$.

Proof We fix $\chi, \varepsilon \in (0, 1]$, $R > 0$, $n \geq 1$. Let $x, y \in \mathbb{R}^d$ with $|x - y| \in (0, 1]$. By (22) and Lemma 4, there exists $C > 0$ such that for any $t > 0$,

$$\begin{aligned}
&P_{x,y}(\tau^x_{B(x,2^{-n}R|x-y|^\chi)} \leq \xi_{x,y}) \\
&\leq P_{x,y}(\tau^x_{B(x,2^{-n}R|x-y|^\chi)} \leq t) + P_{x,y}(\xi_{x,y} > t) \\
&\leq C \exp(-2^{-2n}R^2|x-y|^{2\chi}/Ct) + |x-y|/\sqrt{2\pi t}.
\end{aligned} \tag{23}$$

Then, letting $t = |x - y|^{2\chi+2\varepsilon} (\in (0, 1])$ in (23), we have

$$\begin{aligned}
&P_{x,y}(\tau^x_{B(x,2^{-n}R|x-y|^\chi)} \leq \xi_{x,y}) \\
&\leq C \exp(-2^{-2n}R^2|x-y|^{-2\varepsilon}/C) + |x-y|^{1-\chi-\varepsilon}.
\end{aligned} \tag{24}$$

For any $\delta \in (0, 1]$ and $c \in (0, \infty)$ there exists $c_\delta \in (0, \infty)$ depending on δ and c such that for any $r \in [0, \infty)$,

$$\exp(-cr^{-\delta}) \leq c_\delta r. \tag{25}$$

By using (24) and (25), we obtain the desired inequality. □

From now on, we fix $p \in \mathbb{R}^d$, and take constants $\kappa > d - 2$, $R \in (0, 1]$, and $K > 0$ so that (5) holds. Theorem 1 is proved by an inductive argument. The following lemma is the first step.

Lemma 6 *Let $\varepsilon \in (0, (2 - d + \kappa)/(3 - d + \kappa))$. There exists $C > 0$ depending on d, ε, p, κ, R, and K such that*

$$\mathcal{I}_{x,y} \leq C|x - y|^{(2-d+\kappa)/(3-d+\kappa)-\varepsilon}$$

for any $x, y \in B(p, R/2)$.

Proof Let $\varepsilon \in (0, (2 - d + \kappa)/(3 - d + \kappa))$. We fix $x, y \in B(p, R/2)$ and set

$$r = \frac{R|x-y|^\chi}{2},$$

where $\chi \in (0, 1]$ is a positive number which will be chosen later. Because $|x - y| \leq 1$, we have $r \leq R/2$. A straightforward calculation gives

$$\mathcal{I}_{x,y} \leq E_{x,y}\left[A^{\mu,x}_{\tau^x_{B(x,r)}}\right] + P_{x,y}(\tau^x_{B(x,r)} \leq \xi_{x,y}). \tag{26}$$

By applying Lemmas 2 and 5 to (26), we obtain that

$$\mathcal{I}_{x,y} \leq C\{\zeta_d(r, \kappa) + |x - y|^{1-\chi-\varepsilon}\}. \tag{27}$$

Here, $C > 0$ is a positive constant depending on d, ε, χ, p, κ, R, and K.

Next, we optimize the right-hand side of (27) in χ. Note that we have for any $a, b > 0$ with $b \le a$,

$$-s^a \log s \le (1/b)s^{a-b}, \quad s \in (0, 1]. \tag{28}$$

Thus, if $d = 2$, we have

$$\zeta_d(r, \kappa) \le \frac{\chi}{\varepsilon}\left(\frac{R}{2}\right)^{\kappa-(\varepsilon/\chi)} |x-y|^{\kappa\chi-\varepsilon}$$

provided that $\varepsilon/\chi \le \kappa$. Let χ be the solution to $\kappa\chi - \varepsilon = 1 - \chi - \varepsilon$. Then, $\chi = 1/(\kappa+1)$. Further, $\varepsilon/\chi \le \kappa$ and

$$\mathcal{I}_{x,y} \le C'|x-y|^{\frac{\kappa}{\kappa+1}-\varepsilon},$$

where C' is a positive constant depending on ε, χ, p, κ, R, and K.

If $d \ge 3$ or $d = 1$, we have

$$\begin{aligned}\zeta_d(r, \kappa) \le r^{2-d+\kappa-(\varepsilon/\chi)} &= \left(\frac{R|x-y|^\chi}{2}\right)^{2-d+\kappa-(\varepsilon/\chi)} \\ &= \left(\frac{R}{2}\right)^{2-d+\kappa-(\varepsilon/\chi)} |x-y|^{\chi(2-d+\kappa)-\varepsilon}.\end{aligned}$$

Let χ be the solution to $\chi(2-d+\kappa) - \varepsilon = 1 - \chi - \varepsilon$. Then, $\chi = 1/(3-d+\kappa) \in (0, 1]$ and

$$\mathcal{I}_{x,y} \le C''|x-y|^{(2-d+\kappa)/(3-d+\kappa)-\varepsilon}.$$

Here, C'' is a positive constant depending on d, ε, χ, p, κ, R, and K. □

For $\varepsilon \in (0, 1)$ and $n \in \mathbb{N}$, we set

$$q_{n,\kappa,\varepsilon} = \frac{r_n - \varepsilon r_{n-1}}{r_n + 1} - \varepsilon,$$

where r_n is a positive number defined by

$$r_n = (2-d+\kappa)(r_{n-1}+1), \quad r_0 = 0.$$

We then find that $r_n = \sum_{l=1}^{n}(2-d+\kappa)^l$ and $\{r_n\}_{n=1}^{\infty}$ is increasing. For any $n \in \mathbb{N}$,

$$
\begin{aligned}
q_{n,\kappa,\varepsilon} > 0 &\Longleftrightarrow \frac{r_n}{r_n + r_{n-1} + 1} > \varepsilon \\
&\Longleftrightarrow \frac{(2-d+\kappa)r_n}{(2-d+\kappa)r_n + (2-d+\kappa)(r_{n-1}+1)} > \varepsilon \\
&\Longleftrightarrow \frac{2-d+\kappa}{3-d+\kappa} > \varepsilon. \qquad (29)
\end{aligned}
$$

Lemma 7 *Let $n \in \mathbb{N}$ and $\varepsilon \in (0, (2-d+\kappa)/(3-d+\kappa))$. Then, there exists a positive constant C_1 depending on d, ε, p, κ, R, K, and n such that*

$$
\mathcal{I}_{x,y} \le C_1 |x-y|^{q_{n,\kappa,\varepsilon}} \qquad (30)
$$

for any $x, y \in B(p, 2^{-n}R)$.

Proof If $n = 1$, the conclusion follows from Lemma 6. In what follows, we suppose that (30) holds for some $n \in \mathbb{N}$. Then, for any $\varepsilon \in (0, (2-d+\kappa)/(3-d+\kappa))$, there exists $C_1 > 0$ depending on $d, \varepsilon, p, \kappa, R, K$, and n such that

$$
\mathcal{I}_{x,y} \le C_1 |x-y|^{q_{n,\kappa,\varepsilon}} \qquad (31)
$$

for any $x, y \in B(p, 2^{-n}R)$.

Let $\chi \in (0, 1]$ be a positive number which will be chosen later. We fix $\varepsilon \in (0, (2-d+\kappa)/(3-d+\kappa))$, and $x, y \in B(p, 2^{-n-1})$. To simplify the notation, we write

$$
\tau = \tau^x_{B(x, 2^{-n-1}R|x-y|^\chi)}, \quad \widetilde{\tau} = \widetilde{\tau}^y_{B(y, 2^{-n-1}R|x-y|^\chi)}, \quad \xi = \xi_{x,y}.
$$

In view of Remark 1 (2), we have $\tau = \widetilde{\tau}$. It is straightforward to show that

$$
\begin{aligned}
\mathcal{I}_{x,y} &\le E_{x,y}\big[A^{\mu,x}_\tau \wedge 1 : \xi \le \tau\big] + E_{x,y}\big[A^{\mu,x}_\tau \wedge 1 : \tau < \xi\big] \\
&\quad + E_{x,y}\big[(A^{\mu,x}_\xi - A^{\mu,x}_\tau) \wedge 1 : \tau < \xi\big] \\
&= E_{x,y}\big[A^{\mu,x}_\tau \wedge 1\big] + E_{x,y}\big[(A^{\mu,x}_\xi - A^{\mu,x}_\tau) \wedge 1 : \tau < \xi\big] \\
&=: I_1 + I_2. \qquad (32)
\end{aligned}
$$

On the event $\{\xi > \tau\}$, we have $A^{\mu,x}_\xi - A^{\mu,x}_\tau = A^{\mu,x}_{\xi-\tau} \circ \theta_\tau \le A^{\mu,x}_\xi \circ \theta_\tau$, where $\{\theta_t\}_{t \ge 0}$ denotes the shift operator of the coupled process $(Z^x, \widetilde{Z}^y)$. We know from Remark 1 (2) that $(Z^x, \widetilde{Z}^y)$ is a strong Markov process on $\mathbb{R}^d \times \mathbb{R}^d$. Therefore, we obtain that

$$
\begin{aligned}
I_2 &= E_{x,y}\left[\left(A^{\mu,x}_{\xi-\tau} \circ \theta_\tau\right) \wedge 1 : \tau < \xi\right] \\
&\le E_{x,y}\left[E_{Z^x_\tau, \widetilde{Z}^y_\tau}\left[A^{\mu,x}_\xi \wedge 1\right] : \tau < \xi\right] = E_{x,y}\left[\mathcal{I}_{Z^x_\tau, \widetilde{Z}^y_\tau} : \tau < \xi\right]. \qquad (33)
\end{aligned}
$$

Observe that $Z^x_\tau \in \overline{B}(x, 2^{-n-1}R)$ and $\widetilde{Z}^y_\tau = \widetilde{Z}^y_{\widetilde{\tau}} \in \overline{B}(y, 2^{-n-1}R)$. Furthermore, by noting that $x, y \in B(p, 2^{-n-1}R)$, we have $|p - Z^x_\tau| < 2^{-n}R$ and $|p - \widetilde{Z}^y_{\widetilde{\tau}}| < 2^{-n}R$. Then, we use (31) to obtain that

$$E_{x,y}\left[\mathcal{I}_{Z_\tau^x,\widetilde{Z}_\tau^y} : \tau < \xi\right] \leq C_1 E_{x,y}\left[\left|Z_\tau^x - \widetilde{Z}_\tau^y\right|^{q_{n,\kappa,\varepsilon}} : \tau < \xi\right]. \tag{34}$$

Let $a_n = (r_n + 1)/r_n$ and $b_n = r_n + 1$, Then, $a_n^{-1} + b_n^{-1} = 1$. Because $\varepsilon \in (0, (2 - d + \kappa)/(3 - d + \kappa))$, (29) implies that $0 < a_n q_{n,\kappa,\varepsilon}$ and

$$a_n q_{n,\kappa,\varepsilon} = \frac{r_n + 1}{r_n}\left(\frac{r_n - \varepsilon r_{n-1}}{r_n + 1} - \varepsilon\right) \leq 1.$$

By using Hölder's inequality, Lemmas 3 and 5, we obtain that

$$\begin{aligned}
&E_{x,y}\left[\left|Z_\tau^x - \widetilde{Z}_\tau^y\right|^{q_{n,\kappa,\varepsilon}} : \tau < \xi\right] \\
&\leq E_{x,y}\left[\left|Z_{\tau\wedge\xi} - \widetilde{Z}_{\tau\wedge\xi}\right|^{a_n q_{n,\kappa,\varepsilon}}\right]^{1/a_n} P_{x,y}(\tau < \xi)^{1/b_n} \\
&\leq |x - y|^{a_n q_{n,\kappa,\varepsilon}/a_n} P_{x,y}(\tau < \xi)^{1/b_n} \\
&\leq C_2 |x - y|^{q_{n,\kappa,\varepsilon} + (1-\chi-\varepsilon)/b_n},
\end{aligned} \tag{35}$$

where C_2 is a positive constant depending on ε, R, and n. Therefore, (33), (34), and (35) imply

$$I_2 \leq C_3 |x - y|^{q_{n,\kappa,\varepsilon} + (1-\chi-\varepsilon)/b_n}. \tag{36}$$

Here, C_3 is a positive constant depending on d, ε, p, κ, R, K, and n.

On the other hand, Lemma 2 yields

$$I_1 \leq E_{x,y}\left[A_\tau^{\mu,x}\right] \leq C_4 \zeta_d(2^{-n-1} R |x - y|^\chi, \kappa), \tag{37}$$

where C_4 is a positive constant depending on d, p, κ, R, and K. If $d = 2$, we use (28) to obtain that

$$\zeta_d(2^{-n-1} R |x - y|^\chi, \kappa) \leq (\chi/\varepsilon)(2^{-n-1} R)^{\kappa - (\varepsilon/\chi)} |x - y|^{\kappa\chi - \varepsilon}, \tag{38}$$

provided that $\varepsilon/\chi \leq \kappa$. If $d \geq 3$ or $d = 1$,

$$\begin{aligned}
\zeta_d(2^{-n-1} R |x - y|^\chi, \kappa) &= (2^{-n-1} R |x - y|^\chi)^{2-d+\kappa} \\
&\leq (2^{-n-1} R)^{2-d+\kappa} |x - y|^{\chi(2-d+\kappa)-\varepsilon}.
\end{aligned} \tag{39}$$

Therefore, if $\varepsilon \leq (2 - d + \kappa)\chi$, regardless of the value of d, we obtain from (37)–(39) that

$$I_1 \leq C_5 |x - y|^{(2-d+\kappa)\chi - \varepsilon}. \tag{40}$$

Here, C_5 is a positive constant depending on d, ε, p, κ, R, K, and n.

Let η be the solution to

$$(2-d+\kappa)\eta-\varepsilon=q_{n,\kappa,\varepsilon}+(1-\eta-\varepsilon)/b_n.$$

Then, a direct calculation and the definition of b_n imply that

$$\eta=\frac{b_n q_{n,\kappa,\varepsilon}+1+\varepsilon(b_n-1)}{b_n(2-d+\kappa)+1}=\frac{(r_n+1)q_{n,\kappa,\varepsilon}+1+\varepsilon r_n}{(2-d+\kappa)(r_n+1)+1}.$$

From the relation $r_{n+1}=(2-d+\kappa)(r_n+1)$ and the definition of $q_{n,\kappa,\varepsilon}$,

$$\begin{aligned}&(2-d+\kappa)\eta-\varepsilon\\&=\frac{(2-d+\kappa)(r_n+1)q_{n,\kappa,\varepsilon}+(2-d+\kappa)(1+\varepsilon r_n)}{r_{n+1}+1}-\varepsilon\\&=\frac{r_{n+1}-\varepsilon r_n}{r_{n+1}+1}-\varepsilon=q_{n+1,\kappa,\varepsilon}.\end{aligned}$$

Because $q_{n+1,\kappa,\varepsilon}>0$, we have $(2-d+\kappa)\eta>\varepsilon$. Noting the fact that $\{r_n\}_{n=1}^{\infty}$ is increasing, we have

$$0<\eta=\frac{r_{n+1}-\varepsilon r_n}{(2-d+\kappa)(r_{n+1}+1)}\le\frac{r_{n+1}}{(2-d+\kappa)(r_{n+1}+1)}=\frac{r_{n+1}}{r_{n+2}}\le 1.$$

Therefore, we can set $\chi=\eta$. By combining (32), (36), and (40), we see (30) holds for $n+1$. □

Because $\{r_n\}_{n=1}^{\infty}$ is increasing, we have for any $n\in\mathbb{N}$ and $\varepsilon\in(0,1)$,

$$q_{n,\kappa,\varepsilon}=\frac{r_n-\varepsilon r_{n-1}}{r_n+1}-\varepsilon\ge\frac{r_n}{r_n+1}-2\varepsilon.$$

If $2-d+\kappa\ge 1$, $\lim_{n\to\infty}r_n=\infty$. If $2-d+\kappa\in(0,1)$,

$$\lim_{n\to\infty}r_n=\frac{2-d+\kappa}{1-(2-d+\kappa)}.$$

Therefore, we obtain that $\underline{\lim}_{n\to\infty}q_{n,\kappa,\varepsilon}\ge(2-d+\kappa)\wedge 1-2\varepsilon$. Since the same estimate as Lemma 7 holds for $\widetilde{\mathcal{I}}_{x,y}$, we have the following corollary.

Corollary 1 *For any $\varepsilon\in(0,(2-d+\kappa)/(3-d+\kappa))$, there exists a positive constants C depending on d, ε, p, κ, R, and K such that*

$$\mathcal{I}_{x,y}+\widetilde{\mathcal{I}}_{x,y}\le C|x-y|^{(2-d+\kappa)\wedge 1-\varepsilon}$$

for any $x,y\in B(p,2^{-C'}R)$, where C' is a positive constant depending on d, ε and κ.

We now prove Theorem 1.

Proof of Theorem 1 Let $\alpha > 0$, $x, y \in \mathbb{R}^d$ and $f \in \mathcal{B}_b^*(\mathbb{R}^d)$. Without loss of generality, we may assume that $\|f\|_\infty \le 1$. We write $\xi = \xi_{x,y}$ for the simplicity. Then, we have

$$
\begin{aligned}
&V_\alpha^\mu f(x) - V_\alpha^\mu f(y) \\
&= E_{x,y}\left[\int_\xi^\infty \exp(-\alpha A_t^{\mu,x}) f(Z_t^x)\, dA_t^{\mu,x}\right] - E_{x,y}\left[\int_\xi^\infty \exp(-\alpha \widetilde{A}_t^{\mu,y}) f(\widetilde{Z}_t^y)\, d\widetilde{A}_t^{\mu,y}\right] \\
&\quad + E_{x,y}\left[\int_0^\xi \exp(-\alpha A_t^{\mu,x}) f(Z_t^x)\, dA_t^{\mu,x}\right] - E_{x,y}\left[\int_0^\xi \exp(-\alpha \widetilde{A}_t^{\mu,y}) f(\widetilde{Z}_t^y)\, d\widetilde{A}_t^{\mu,y}\right] \\
&=: J_1 - J_2 + J_3 - J_4. \qquad (41)
\end{aligned}
$$

Because $\{A_t^{\mu,x}\}_{t\ge 0}$ is a PCAF of $(Z^x, \widetilde{Z}^y)$, we have $A_{t+\xi}^{\mu,x} = A_\xi^{\mu,x} + A_t^{\mu,x} \circ \theta_\xi$ and $dA_{t+\xi}^{\mu,x} = dA_t^{\mu,x} \circ \theta_\xi$. By using these equations and the strong Markov property of $(Z^x, \widetilde{Z}^y)$, we obtain that

$$
\begin{aligned}
J_1 &= E_{x,y}\left[\int_0^\infty \exp(-\alpha A_{t+\xi}^{\mu,x}) f(Z_{t+\xi}^x)\, dA_{t+\xi}^{\mu,x}\right] = E_{x,y}\left[\exp(-\alpha A_\xi^{\mu,x}) V_\alpha^\mu f(Z_\xi^x)\right], \\
J_2 &= E_{x,y}\left[\exp(-\alpha \widetilde{A}_\xi^{\mu,y}) V_\alpha^\mu f(\widetilde{Z}_\xi^y)\right].
\end{aligned}
$$

Since $Z_\xi^x = \widetilde{Z}_\xi^y$, we have

$$
\begin{aligned}
J_1 - J_2 &= E_{x,y}\left[\exp(-\alpha A_\xi^{\mu,x}) V_\alpha^\mu f(Z_\xi^x)\right] - E_{x,y}\left[\exp(-\alpha A_\xi^{\mu,x}) V_\alpha^\mu f(\widetilde{Z}_\xi^y)\right] \\
&\quad + E_{x,y}\left[\exp(-\alpha A_\xi^{\mu,x}) V_\alpha^\mu f(\widetilde{Z}_\xi^y)\right] - E_{x,y}\left[\exp(-\alpha \widetilde{A}_\xi^{\mu,y}) V_\alpha^\mu f(\widetilde{Z}_\xi^y)\right] \\
&= 0 + E_{x,y}\left[\left\{\exp(-\alpha A_\xi^{\mu,x}) - \exp(-\alpha \widetilde{A}_\xi^{\mu,y})\right\} V_\alpha^\mu f(\widetilde{Z}_\xi^y)\right].
\end{aligned}
$$

Because $|\alpha V_\alpha^\mu f(\widetilde{Z}_\xi^y)| \le \|f\|_\infty = 1$ and the function $s \mapsto e^{-\alpha s}$ is α-Lipschitz continuous on $[0, \infty)$, we obtain that

$$
\begin{aligned}
|J_1 - J_2| &\le E_{x,y}\left[\left|A_\xi^x - \widetilde{A}_\xi^y\right| \wedge \alpha^{-1}\right] \\
&\le (1 + \alpha^{-1})(\mathcal{I}_{x,y} + \widetilde{\mathcal{I}}_{x,y}). \qquad (42)
\end{aligned}
$$

From Jensen's inequality,

$$
\begin{aligned}
|J_3 - J_4| &\le \alpha^{-1} E_{x,y}\left[1 - \exp(-\alpha A_\xi^x)\right] + \alpha^{-1} E_{x,y}\left[1 - \exp(-\alpha \widetilde{A}_\xi^y)\right] \\
&\le E_{x,y}\left[A_\xi^x \wedge \alpha^{-1}\right] + E_{x,y}\left[\widetilde{A}_\xi^y \wedge \alpha^{-1}\right] \\
&\le (1 + \alpha^{-1})(\mathcal{I}_{x,y} + \widetilde{\mathcal{I}}_{x,y}). \qquad (43)
\end{aligned}
$$

By using (41)–(43), we arrive at

$$\left|V_\alpha^\mu f(x) - V_\alpha^\mu f(y)\right| \leq 2(1+\alpha^{-1})\left(\mathcal{I}_{x,y} + \widetilde{\mathcal{I}}_{x,y}\right). \tag{44}$$

Corollary 1 and (44) yield the desired estimate. "In particular" part immediately follows from (4). □

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References

1. R.F. Bass, *Probabilistic Techniques in Analysis*, Probability and its Applications (New York) (Springer, New York, 1995)
2. R.F. Bass, *Diffusions and Elliptic Operators* Probability and Its Applications (New York). (Springer, New York, 1998)
3. R.F. Bass, M. Kassmann, T. Kumagai, Symmetric jump processes: localization, heat kernels and convergence. Ann. Inst. Henri Poincaré Probab. Stat. **46**, 59–71 (2010)
4. Z.-Q. Chen, M. Fukushima, *Symmetric Markov Processes, Time Change, and Boundary Theory*, London Mathematical Society Monographs Series, vol. 35. (Princeton University Press, Princeton, NJ, 2012)
5. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet forms and symmetric Markov processes*, De Gruyter Studies in Mathematics, vol. 19, second revised and extended edition (Walter de Gruyter & Co., Berlin, 2011)
6. M. Fukushima, T. Uemura, Capacitary bounds of measures and ultracontractivity of time changed processes. J. Math. Pures Appl. **82**(9), 553–572 (2003)
7. C. Garban, R. Rhodes, V. Vargas, Liouville Brownian motion. Ann. Probab. **44**, 3076–3110 (2016)
8. C. Garban, R. Rhodes, V. Vargas, On the heat kernel and the Dirichlet form of Liouville Brownian motion. Electron. J. Probab. **19**(96), 25 (2014)
9. O. Kallenberg, *Foundations of Modern Probability*, 2nd edn, Probability and Its Applications (Springer, New York, 2002)
10. T. Lindvall, L.C.G. Rogers, Coupling of multidimensional diffusions by reflection. Ann. Probab. **14**, 860–872 (1986)
11. D.W. Stroock, Diffusion semigroups corresponding to uniformly elliptic divergence form operators, *Séminaire de Probabilités, XXII*, Lecture Notes in Mathematics, vol. 1321 (Springer, Berlin, 1988), pp. 316–347

On the Continuity of Half-Plane Capacity with Respect to Carathéodory Convergence

Takuya Murayama

Abstract We study the continuity of half-plane capacity as a function of boundary hulls with respect to the Carathéodory convergence. In particular, our interest lies in the case that hulls are unbounded. Under the assumption that every hull is contained in a fixed hull with finite imaginary part and finite half-plane capacity, we show that the half-plane capacity is indeed continuous. We also discuss the extension of this result to the case that the underlying domain is finitely connected.

Keywords Half-plane capacity · Carathéodory convergence · Harmonic measure · Brownian motion with darning

Mathematics Subject Classification Primary: 60J45 · Secondary: 30C20 · 30C85

1 Introduction

The *half-plane capacity* is a "capacity" that measures $\mathbb{H}$-*hulls* (or *boundary hulls*) growing from the boundary of the complex upper half-plane $\mathbb{H}$. This capacity has several geometric meanings [5, 11, 17] and plays a role of time for chordal Loewner chains, that is, families of normalized conformal mappings defined in the complements of expanding (or shrinking) $\mathbb{H}$-hulls. In particular, the concept of Loewner chain is commonly utilized not just in complex analysis but also in probability theory because of the great importance of Schramm–Loewner evolution (SLE). Roughly speaking, the SLE hull is a random curve in the half-plane with all of its loops filled-in. Also, $\mathbb{H}$-hulls of more general form appear naturally in some applications; A recent example is the correspondence between monotone-independent increment processes

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and chordal Loewner chains established by Franz et al. [7] in non-commutative probability theory. Under such a background, we study the continuity of half-plane capacity, as a function of $\mathbb{H}$-hulls of broad class, with respect to the *Carathéodory kernel convergence* in this paper.

To be precise, concepts studied in this paper are defined as follows: A set $F \subset \mathbb{H}$ is called an $\mathbb{H}$-hull if F is relatively closed in $\mathbb{H}$ (i.e., $F = \overline{F} \cap \mathbb{H}$) and if $\mathbb{H} \setminus F$ is a simply connected domain. Given an appropriate sequence of $\mathbb{H}$-hulls, we can define its convergence in Carathéodory's sense (see Definition 2 in Sect. 2.1). Let $Z = ((Z_t)_{t \geq 0}, (\mathbb{P}_z)_{z \in \mathbb{H}})$ be an absorbed Brownian motion (ABM for brevity) in $\mathbb{H}$. The half-plane capacity of F is defined by[1]

$$\operatorname{hcap}(F) := \lim_{y \to \infty} y\, \mathbb{E}_{iy}\big[\operatorname{Im} Z_{\sigma_F}\big] = \lim_{y \to \infty} y\, \mathbb{E}_{iy}\big[\operatorname{Im} Z_{\tau_{\mathbb{H} \setminus F}}\big] \tag{1}$$

as long as this limit exists. Here, σ_F (resp. $\tau_{\mathbb{H} \setminus F}$) is the first hitting time (resp. exit time) of Z to F (resp. from $\mathbb{H} \setminus F$).

Originally, the half-plane capacity appears in a purely analytic way. Let F be a *bounded* $\mathbb{H}$-hull. Using Riemann's mapping theorem, we can show that there exists a unique conformal mapping $g_F \colon \mathbb{H} \setminus F \to \mathbb{H}$ with Laurent expansion

$$g_F(z) = z + \frac{a_F}{z} + o(z^{-1}), \quad z \to \infty, \tag{2}$$

around the point at infinity. (g_F is sometimes called the mapping-out function of F) The non-negative constant a_F is exactly the half-plane capacity of F. We note that

$$a_F = -\operatorname*{Res}_{z=\infty} g_F(z)\, dz = \operatorname*{Res}_{z=\infty} g_F^{-1}(z)\, dz.$$

If hulls are assumed to be uniformly bounded, then the continuity of half-plane capacity is proved quite easily. Let F_n, $n \in \mathbb{N} \cup \{\infty\}$, be $\mathbb{H}$-hulls that are contained in a disk $B(0, \rho)$ with center 0 and radius ρ. We suppose that F_n converges to F_∞ in Carathéodory's sense. Using a version of the Carathéodory kernel theorem [14, Theorem 3.8] and taking the Schwarz reflection of g_{F_n}'s across the real axis, we can show that $g_{F_n}(z)$ converges to $g_{F_\infty}(z)$ uniformly in $z \in \partial B(0, \rho)$. By (2) we have

$$\lim_{n \to \infty} a_{F_n} = \lim_{n \to \infty} \frac{1}{2\pi i} \int_{|z|=\rho} g_{F_n}(z)\, dz = \frac{1}{2\pi i} \int_{|z|=\rho} g_{F_\infty}(z)\, dz = a_{F_\infty}.$$

In contrast to the preceding case, the continuity of half-plane capacity fails if hulls are allowed to be unbounded. In this case, $\operatorname{hcap}(F)$ should be the *angular* residue of g_F^{-1} at infinity, but the angular residue is not a continuous functional of g_F^{-1} (see

[1] We adopt the usual convention that a function on $\mathbb{H}$ takes value zero at the cemetery of the ABM. The second equality in (1) is understood in this sense.

Goryainov and Ba [9, p. 1211]). In order to retrieve the continuity of half-plane capacity, some restriction must be imposed on $\mathbb{H}$-hulls.

In this paper, we assume that our boundary hulls are contained in a fixed hull with finite imaginary part and finite half-plane capacity. Under this assumption, our main result is stated as follows:

Theorem 1 *Let $\tilde{F}$ be an $\mathbb{H}$-hull with*

$$\operatorname{Im} \tilde{F} := \sup\{\operatorname{Im} z \,;\, z \in \tilde{F}\} < \infty \quad \textit{and} \quad \operatorname{hcap}(\tilde{F}) < \infty.$$

Suppose that $\mathbb{H}$-hulls F_n, $n \in \mathbb{N}$, and F_∞ are all contained in $\tilde{F}$ and that F_n converges to F_∞ as $n \to \infty$ (in the sense of Definition 2). Then

$$\lim_{n\to\infty} \operatorname{hcap}(F_n) = \operatorname{hcap}(F_\infty). \tag{3}$$

For the proof of Theorem 1, we make a full use of the probabilistic definition (1) of half-plane capacity instead of (angular) residue. The merit of our method is that the hitting probability of the ABM to a hull F is given by the harmonic measure of $\mathbb{H} \setminus F$. In fact, the Carathédory convergence implies the weak convergence of harmonic measures under a certain assumption, which is a key result to proving (3). Modifying the original argument of Binder et al. [1], we provide a proof of this result in Appendix 2.

After the proof of Theorem 1, we discuss its extension to *finitely connected domains*. Let D be a *parallel slit half-plane*, namely, the upper half-plane with some line segments parallel to the real axis removed. For an $\mathbb{H}$-hull F in D, its *half-plane capacity* $\operatorname{hcap}^D(F)$ *relative to D* is defined by replacing ABM with *Brownian motion with darning* (BMD for short) in the expression (1). Such replacement is naturally considered in the study of the Komatu–Loewner differential equation; See a series of recent papers [2–4]. We prove the continuity of hcap^D so defined under such assumptions as in Theorem 1 (see Theorem 2 in Sect. 3.1).

Remark 1 The question of the continuity of half-plane capacity originally arose from the joint work [10], which is in progress. In this work, a very similar result is proved by means of angular residues and an integral formula for conformal mappings. (We shall mention the relation of Theorem 1 to this result again in Remark 2 in Sect. 4.2.) Thus, our main contribution lies in the simple probabilistic method, which works for finitely connected domains as well, rather than the results themselves.

The rest of this paper is organized as follows: Sect. 2 is devoted to the case that the underlying domain is simply connected, namely, the upper half-plane. In Sect. 2.1, we provide the definition of Carathéodory's convergence and prove Theorem 1. In Sect. 2.2, we show that hcap is strictly monotone (with respect to the inclusion relation of $\mathbb{H}$-hulls) and give a partial converse of Theorem 1 for a monotone sequence of hulls. Section 3 is devoted to the case that the underlying domain is finitely connected, namely, a parallel slit half-plane. In Sect. 3.1, we generalize the definition of half-plane capacity in terms of BMD. The proof of Theorem 2 is done through Sects. 3.2

and 3.3. The final section, Sect. 4, is devoted to some remarks on the relation of our results to geometric function theory. (The reader can read Sect. 4 independently of Sect. 3.) There are two appendices. In Appendix 1, we prove a lemma on some hitting probability needed in the proof of Proposition 11. In Appendix 2, we prove the above-mentioned fact on the weak convergence of harmonic measures.

2 Study on the Upper Half-Plane

In this section, we study boundary hulls in the upper half-plane $\mathbb{H}$. Our main result, Theorem 1, is proved after some definitions are given. We also discuss the case that a sequence of hulls is monotone. Most of the results in this section will be carried over into the setting of Sect. 3.

2.1 Basic Definitions and Proof of Theorem 1

Definition 1 (*Convergence of Domains*) Let D_n, $n \in \mathbb{N}$, and D_∞ be domains in the complex plane $\mathbb{C}$ which have a point $z_0 \in \mathbb{C}$ in common. It is said that the sequence $(D_n)_{n\in\mathbb{N}}$ *converges* to D_∞ *in the kernel sense* or *in Carathéodory's sense* with respect to the reference point z_0 if the following hold:

- Each compact subset K of D_∞ is a subset of D_n for all but finitely many n;
- If a domain U contains z_0 and is a subset of D_n for infinitely many n, then $U \subset D_\infty$.

In Definition 1, we have skipped the definition of "kernel" and defined the kernel convergence directly. The equivalence of Definition 1 to the original definition (see for instance Roseblum and Rovnyak [18, §7.9]) is easy to check and left to the interested reader.

Definition 2 (*Convergence of* $\mathbb{H}$*-Hulls*) Suppose that $\mathbb{H}$-hulls F_n, $n \in \mathbb{N}$, and F_∞ are contained in another hull $\tilde{F}$. We say that the sequence $(F_n)_{n\in\mathbb{N}}$ *converges* to F_∞ (in Carathéodory's sense) if the complement $\mathbb{H} \setminus F_n$ converges to $\mathbb{H} \setminus F_\infty$ as $n \to \infty$ in the kernel sense with respect to some $z_0 \in \mathbb{H} \setminus \tilde{F}$. (This definition is independent of the choice of the reference point z_0.)

We use the following notation: $Z = ((Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in\mathbb{H}})$ is an ABM in $\mathbb{H}$. For a set $B \subset \mathbb{H}$, the symbol σ_B (resp. τ_B) denotes the first hitting time (resp. exit time) of Z to B (resp. from B). For a domain D, the expression

$$\mathrm{hm}_D(z, B) := \mathbb{P}_z(Z_{\tau_D -} \in B)$$

defines the *harmonic measure* of B in D seen from a point z. It is regarded as a Borel measure on $\mathbb{C}$. The value of any function at the cemetery of Z is set to be zero.

Let us begin the proof of Theorem 1. The following observation comes from the expression of half-plane capacity mentioned by Lalley et al. [11]: For an $\mathbb{H}$-hull F with $\operatorname{Im} F < \infty$, let $y > \eta > \operatorname{Im} F$ and $\mathbb{H}_\eta := \{ z \in \mathbb{C} \,;\, \operatorname{Im} z > \eta \}$. Then

$$\mathbb{E}_{iy}\left[\operatorname{Im} Z_{\sigma_F}\right] = \mathbb{E}_{iy}\left[\mathbb{E}_{Z_{\tau_{\mathbb{H}_\eta}}}\left[\operatorname{Im} Z_{\sigma_F}\right]\right] = \int_{\partial\mathbb{H}_\eta} \mathbb{E}_\zeta\left[\operatorname{Im} Z_{\sigma_F}\right] \operatorname{hm}_{\mathbb{H}_\eta}(iy, d\zeta)$$
$$= \frac{1}{\pi}\int_{\mathbb{R}} \mathbb{E}_{\xi+i\eta}\left[\operatorname{Im} Z_{\sigma_F}\right] \frac{y-\eta}{\xi^2+(y-\eta)^2}\, d\xi.$$

Hence we have

$$y\,\mathbb{E}_{iy}\left[\operatorname{Im} Z_{\sigma_F}\right] = \frac{1}{\pi}\int_{\mathbb{R}} \mathbb{E}_{\xi+i\eta}\left[\operatorname{Im} Z_{\sigma_F}\right] \frac{y(y-\eta)}{\xi^2+(y-\eta)^2}\, d\xi \quad \text{for } y > \eta > \operatorname{Im} F. \tag{4}$$

Proposition 1 (Expression of hcap) *Let F be an $\mathbb{H}$-hull with* $\operatorname{Im} F < \infty$ *and* $\operatorname{hcap}(F) < \infty$. *Then, for any $\eta > \operatorname{Im} F$,*

$$\operatorname{hcap}(F) = \frac{1}{\pi}\int_{\mathbb{R}} \mathbb{E}_{\xi+i\eta}\left[\operatorname{Im} Z_{\sigma_F}\right] d\xi. \tag{5}$$

Proof We apply Fatou's lemma to (4):

$$\frac{1}{\pi}\int_{\mathbb{R}} \mathbb{E}_{\xi+i\eta}\left[\operatorname{Im} Z_{\sigma_F}\right] d\xi \le \liminf_{y\to\infty} y\,\mathbb{E}_{iy}\left[\operatorname{Im} Z_{\sigma_F}\right] = \operatorname{hcap}(F) < \infty.$$

Hence the function $\xi \mapsto \mathbb{E}_{\xi+i\eta}\left[\operatorname{Im} Z_{\sigma_F}\right]$ is integrable. Moreover,

$$\frac{y(y-\eta)}{\xi^2+(y-\eta)^2} \le \frac{y}{y-\eta} < 2 \quad \text{for all } \xi \in \mathbb{R} \text{ and } y > 2\eta. \tag{6}$$

The dominated convergence theorem thus applies to (4), which yields (5).

Proposition 2 (Weak Monotonicity) *Let F and $\tilde{F}$ be $\mathbb{H}$-hulls with $F \subset \tilde{F}$. If $\tilde{F}$ satisfies* $\operatorname{Im} \tilde{F} < \infty$ *and* $\operatorname{hcap}(\tilde{F}) < \infty$, *then the limit* $\operatorname{hcap}(F)$ *exists and enjoys*

$$\operatorname{hcap}(F) \le \operatorname{hcap}(\tilde{F}). \tag{7}$$

Proof The process $(\operatorname{Im} Z_t)_{t\ge 0}$ is just a one-dimensional Brownian motion stopped when it hits the origin. Hence it is a non-negative martingale. Since $\lim_{t\to\infty} \operatorname{Im} Z_t = 0$ a.s., it is also a supermartingle with last element zero. Thus, the optional sampling theorem implies that

$$\mathbb{E}_{\xi+i\eta}\left[\operatorname{Im} Z_{\sigma_F}\right] \le \mathbb{E}_{\xi+i\eta}\left[\operatorname{Im} Z_{\sigma_{\tilde{F}}}\right] \quad \text{for } \xi \in \mathbb{R} \text{ and } \eta > \operatorname{Im} \tilde{F}. \tag{8}$$

The right-hand side of (8) is integrable as a function of ξ by Proposition 1. Thus, by virtue of (6), the dominated convergence theorem works in (4). This yields (7).

We now consider $\mathbb{H}$-hulls F_n, $n \in \mathbb{N} \cup \{\infty\}$, in Theorem 1. The next proposition then applies to the complementary domains $\mathbb{H} \setminus F_n$.

Proposition 3 (Uniform Regularity of Simply Connected Domains) *Simply connected domains none of which is $\mathbb{C}$ are uniformly regular. Namely, for any $\varepsilon > 0$ there exists $\delta > 0$ such that every simply connected domain $D \subsetneq \mathbb{C}$ satisfies*

$$\mathrm{hm}_D(z, B(z, \varepsilon)) > 1 - \varepsilon \quad \textit{for any } z \in D \textit{ with } dist(z, \partial D) < \delta.$$

Here, $B(z, \varepsilon)$ denotes the disk with center z and radius ε, and $dist(z, \partial D) := \inf\{\, |z - w| \,;\, w \in \partial D \,\}$.

There are several proofs of Proposition 3, and we refer the reader to Markowsky [13, Lemma 1] for a short and elegant probabilistic one. Using this proposition, we prove the following:

Proposition 4 *Under the assumption of Theorem 1, it holds that*

$$\lim_{n\to\infty} \mathbb{E}_\zeta \left[\mathrm{Im}\, Z_{\sigma_{F_n}} \right] = \mathbb{E}_\zeta \left[\mathrm{Im}\, Z_{\sigma_{F_\infty}} \right] \quad \textit{for every } \zeta \in \mathbb{H} \setminus \tilde{F}. \tag{9}$$

Proof The domains $\mathbb{H} \setminus F_n$, $n \in \mathbb{N} \cup \{\infty\}$, are uniformly regular by Proposition 3, and $\mathbb{H} \setminus F_n$ converges to $\mathbb{H} \setminus F_\infty$ in Carathéodory's sense by assumption. Thus, it follows from Theorem 4 in Appendix 2 that $\mathrm{hm}_{\mathbb{H}\setminus F_n}(\zeta, \cdot)$ converges weakly to $\mathrm{hm}_{\mathbb{H}\setminus F_\infty}(\zeta, \cdot)$. Putting $\psi(z) := \max\{0, \min\{\mathrm{Im}\, z, \mathrm{Im}\, \tilde{F}\}\}$, we have

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}_\zeta \left[\mathrm{Im}\, Z_{\sigma_{F_n}} \right] &= \lim_{n\to\infty} \int_{\mathbb{C}} \psi(z) \mathrm{hm}_{\mathbb{H}\setminus F_n}(\zeta, dz) \\ &= \int_{\mathbb{C}} \psi(z) \mathrm{hm}_{\mathbb{H}\setminus F_\infty}(\zeta, dz) = \mathbb{E}_\zeta \left[\mathrm{Im}\, Z_{\sigma_{F_\infty}} \right].\end{aligned}$$

Proposition 4 completes the proof of Theorem 1. Indeed, Proposition 2 ensures that $\mathrm{hcap}(F_n) < \infty$ for all n. Then by virtue of (8) and (9) with $F = F_n$, the dominated convergence theorem applies to (5) with $F = F_n$. This yields (3).

2.2 *Strict Monotonicity*

In Proposition 2, we have seen that the half-plane capacity is a weakly monotone function of $\mathbb{H}$-hulls. In fact, it is strictly monotone as follows:

Proposition 5 (Strict Monotonicity) *Let F and $\tilde{F}$ be $\mathbb{H}$-hulls with $F \subsetneq \tilde{F}$,* $\mathrm{Im}\, \tilde{F} < \infty$*, and* $\mathrm{hcap}(\tilde{F}) < \infty$*. Then*

$$\operatorname{hcap}(F) < \operatorname{hcap}(\tilde{F}).$$

Proof We assume

$$\operatorname{hcap}(F) = \operatorname{hcap}(\tilde{F}) \tag{10}$$

to the contrary and deduce a contradiction.

By (5), (8), and (10), we have, for each fixed $\eta > \operatorname{Im} \tilde{F}$,

$$\mathbb{E}_{\xi+i\eta}\left[\operatorname{Im} Z_{\sigma_F}\right] = \mathbb{E}_{\xi+i\eta}\left[\operatorname{Im} Z_{\sigma_{\tilde{F}}}\right] \quad \text{for a.e. } \xi \in \mathbb{R}.$$

Both sides of this equality are harmonic functions of variable $\zeta = \xi + i\eta$. Hence the identity theorem applies:

$$\mathbb{E}_{\zeta}\left[\operatorname{Im} Z_{\sigma_F}\right] = \mathbb{E}_{\zeta}\left[\operatorname{Im} Z_{\sigma_{\tilde{F}}}\right] \quad \text{for any } \zeta \in \mathbb{H} \setminus \tilde{F}.$$

In particular, if $z_0 \in (\mathbb{H} \cap \partial \tilde{F}) \setminus F$, then

$$\mathbb{E}_{z_0}\left[\operatorname{Im} Z_{\sigma_F}\right] = \mathbb{E}_{z_0}\left[\operatorname{Im} Z_{\sigma_{\tilde{F}}}\right] = \operatorname{Im} z_0 \tag{11}$$

because z_0 is a regular point of $\partial(\mathbb{H} \setminus \tilde{F})$ by Proposition 3. Here, the set $(\mathbb{H} \cap \partial \tilde{F}) \setminus F$ is not empty. Otherwise, $\tilde{F} \setminus F = \tilde{F}^{\circ} \setminus F$ would be open ($\tilde{F}^{\circ}$ stands for the interior of $\tilde{F}$), and the domain $\mathbb{H} \setminus F$ would be divided into disjoint open sets $\mathbb{H} \setminus \tilde{F}$ and $\tilde{F} \setminus F$.

Now, we define a harmonic function

$$u(z) := \mathbb{E}_{z}\left[\operatorname{Im} Z_{\sigma_F}\right] - \operatorname{Im} z, \quad z \in \mathbb{H} \setminus F.$$

Since $u(z)$ enjoys

$$\lim_{z \to \zeta} u(z) = 0 \text{ for any } \zeta \in \partial(\mathbb{H} \setminus F) \quad \text{and} \quad \limsup_{z \to \infty} \frac{u(z)}{\log|z|} \leq 0,$$

a corollary of the Phragmén–Lindelöf principle [16, Corollary 2.3.3] yields $u \leq 0$ in $\mathbb{H} \setminus F$. Hence (11) implies that u takes its maximum at z_0. By the maximum principle, we have $u \equiv 0$, namely,

$$\mathbb{E}_{z}\left[\operatorname{Im} Z_{\sigma_F}\right] = \operatorname{Im} z \quad \text{for all } z \in \mathbb{H} \setminus F.$$

This is absurd.

Using the strict monotonicity, we can show that $\operatorname{hcap}(F_n) \to \operatorname{hcap}(F_\infty)$ implies $F_n \to F_\infty$, the converse of Theorem 1, if $(F_n)_{n \in \mathbb{N}}$ is monotone. To this end, we recall the limit of monotone hulls.

Proposition 6 (Limit of Monotone Hulls) *Let* $(F_n)_{n \in \mathbb{N}}$ *be a sequence of* $\mathbb{H}$*-hulls.*

(a) *Suppose that* $(F_n)_{n\in\mathbb{N}}$ *is decreasing and set* $F := \bigcap_{n\in\mathbb{N}} F_n$. *Then* F *is an* $\mathbb{H}$*-hull, and* $F_n \to F$ *in Carathéodory's sense (with respect to any point* $z_0 \in \mathbb{H} \setminus F_0$*).*
(b) *Suppose that* $(F_n)_{n\in\mathbb{N}}$ *is increasing with* $G := \bigcup_{n\in\mathbb{N}} F_n \subset \tilde{F}$ *for some hull* $\tilde{F}$. *Let* F *be the union of* $\overline{G} \cap \mathbb{H}$ *and all the connected components of* $\mathbb{H} \setminus \overline{G}$ *that are disjoint from* $\mathbb{H} \setminus \tilde{F}$. *Then* F *is an* $\mathbb{H}$*-hull, and* $F_n \to F$ *in Carathéodory's sense (with respect to any point* $z_0 \in \mathbb{H} \setminus \tilde{F}$*).*

Proposition 6 is well known if $\mathbb{H}$-hulls are uniformly bounded. It is also not so difficult to prove on the basis of Definitions 1 and 2. We omit the detail here.

Proposition 7 *Let* F_n, $n \in \mathbb{N} \cup \{+\infty\}$, *be a monotone sequence of* $\mathbb{H}$*-hulls. Suppose that there exists a hull* $\tilde{F}$ *such that*

$$\bigcup_n F_n \subset \tilde{F}, \ \operatorname{Im} \tilde{F} < \infty, \ \textit{and}\ \operatorname{hcap}(\tilde{F}) < \infty.$$

Then $\lim_{n\to\infty} \operatorname{hcap}(F_n) = \operatorname{hcap}(F_\infty)$ *implies* $F_n \to F_\infty$ *in Carathéodory's sense.*

Proof We assume first that $F_n, n \in \mathbb{N} \cup \{+\infty\}$, are decreasing. Let $F := \bigcap_{n\in\mathbb{N}} F_n \supset F_\infty$. We have $F_n \to F$ by Proposition 6(a) and hence $\operatorname{hcap}(F_n) \to \operatorname{hcap}(F)$ by Theorem 1. Thus, we obtain

$$F_\infty \subset F \quad \text{and} \quad \operatorname{hcap}(F_\infty) = \operatorname{hcap}(F).$$

Proposition 5 yields $F_\infty = F$.

We assume next that $F_n, n \in \mathbb{N} \cup \{+\infty\}$, are increasing. Let F be the hull defined in Proposition 6(b). We have $F_n \to F$ and hence $\operatorname{hcap}(F_n) \to \operatorname{hcap}(F)$ by Theorem 1. Since $\mathbb{H} \setminus F_\infty \subset \bigcap_{n\in\mathbb{N}}(\mathbb{H} \setminus F_n)$, Definition 1 implies that $\mathbb{H} \setminus F_\infty \subset \mathbb{H} \setminus F$. Thus,

$$F_\infty \supset F \quad \text{and} \quad \operatorname{hcap}(F_\infty) = \operatorname{hcap}(F).$$

Proposition 5 yields $F_\infty = F$ again.

3 Study on Parallel Slit Half-Planes

In this section, we formulate and prove an extension of Theorem 1 in parallel slit half-planes, a standard type of finitely connected domains.

3.1 BMD Half-Plane Capacity

Let $N \in \mathbb{N} \setminus \{0\}$. For disjoint horizontal line segments C_j, $j = 1, 2, \dots, N$, in $\mathbb{H}$, we put $K := \bigcup_{j=1}^{N} C_j$ and $D := \mathbb{H} \setminus K$. Such a domain D is called a parallel slit half-plane. We also consider the quotient space D^* of $\mathbb{H}$ in which each C_j is identified with one point c_j^*. With $K^* := \{c_1^*, c_2^*, \dots, c_N^*\}$, it is written as $D^* := D \cup K^*$.

A BMD in D^*, which we denote by $Z^* = ((Z_t^*)_{t\geq 0}, (\mathbb{P}_z^*)_{z\in D^*})$, is a symmetric[2] diffusion process in D^* with the following two properties:

- The killed process of Z^* when it exits from D is an ABM in D;
- Z^* admits no killing in K^*.

See Chen et al. [3] for basic properties of BMD.

The (BMD) half-plane capacity of an $\mathbb{H}$-hull $F \subset D$ relative to D can be defined by

$$\operatorname{hcap}^D(F) := \lim_{y\to\infty} y\,\mathbb{E}_{iy}^*\left[\operatorname{Im} Z_{\sigma_F}^*\right] \tag{12}$$

as long as this limit exists. Here, σ_F denotes the first hitting time of Z^* to F. As before, the right-hand side of (12) typically coincides with the (angular) residue of some conformal mapping from or onto $D \setminus F$ (see Chen and Fukushima [2, p. 591]), which ensures the validity of this definition.

The main result of this section is the following:

Theorem 2 *Let $\tilde{F} \subset D$ be an $\mathbb{H}$-hull with* $\operatorname{Im} \tilde{F} < \infty$ *and* $\operatorname{hcap}^D(\tilde{F}) < \infty$. *Suppose that $\mathbb{H}$-hulls F_n, $n \in \mathbb{N}$, and F_∞ are all contained in $\tilde{F}$ and that F_n converges to F_∞ as $n \to \infty$ (in the sense of Definition 2). Then*

$$\lim_{n\to\infty} \operatorname{hcap}^D(F_n) = \operatorname{hcap}^D(F_\infty). \tag{13}$$

Notice that $D \setminus F_n$ converges to $D \setminus F_\infty$ in Carathéodory's sense if $F_n \to F_\infty$ in the sense of Definition 2. This is easily seen in view of Definition 1.

Propositions 1 and 2 hold for BMD and BMD half-plane capacity with obvious modifications. Thus, in order to prove Theorem 2, it suffices to show the following:

Proposition 8 *Under the assumption of Theorem 2, it holds that*

$$\lim_{n\to\infty} \mathbb{E}_\zeta^*\left[\operatorname{Im} Z_{\sigma_{F_n}}^*\right] = \mathbb{E}_\zeta^*\left[\operatorname{Im} Z_{\sigma_{F_\infty}}^*\right] \quad \text{for every } \zeta \in D \setminus \tilde{F}. \tag{14}$$

We take two steps to prove Proposition 8:

(a) Express $\mathbb{E}_\zeta^*\left[\operatorname{Im} Z_{\sigma_{F_n}}^*\right]$ in terms of ABM;
(b) Prove the uniform regularity (Definition 3 in Appendix 2) of $D \setminus F_n$'s to show that $\operatorname{hm}_{D\setminus F_n}(\zeta, \cdot) \to \operatorname{hm}_{D\setminus F_\infty}(\zeta, \cdot)$ weakly.

To carry out the step (a), we study Markov chains on K^* associated with BMD in Sect. 3.2. To implement the step (b), we make a comparison of $\operatorname{hm}_{D\setminus F_n}$ and $\operatorname{hm}_{\mathbb{H}\setminus F_n}$, the latter of which behaves well by Proposition 3, in Sect. 3.3.

[2] 'Symmetric' means "symmetric with respect to the Lebesgue measure on D^*". The measure of K^* is set to be zero.

3.2 Markov Chains Induced by BMD

Let F and $\tilde{F}$ be $\mathbb{H}$-hulls with $F \subset \tilde{F} \subset D$ and $\operatorname{Im} \tilde{F} < \infty$. Following Lawler [12, Sects. 5.2 and 5.3] and Chen et al. [3, Appendix 1], we observe that BMD naturally induces Markov chains in $K^* \cup \{c_0^*\}$ (c_0^* represents the cemetery of the chains). We shall use their transition probabilities to compute the expectation with respect to BMD.

For each $j = 1, 2, \dots, N$, let η_j be a smooth Jordan curve in $D \setminus \tilde{F}$ surrounding C_j. We define a measure on η_j by

$$\nu_j(dz) := \mathbb{P}^*_{c_j^*}\left(Z^*_{\sigma_{\eta_j}} \in dz\right).$$

We also put

$$\varphi^{(j)}(z) := \mathrm{hm}_{D\setminus F}(z, C_j) = \mathbb{P}^*_z(Z^*_{\sigma_{K^*}} = c_j^*, \sigma_{K^*} < \sigma_F)$$

for $z \in D \setminus F$. We consider a Markov chain $X = (X_n)_{n\in\mathbb{N}}$ whose transition probability from c_j^* to c_k^*, $j, k \in \{1, 2, \dots, N\}$, is given by

$$p_{jk} := \mathbb{E}^*_{c_j^*}\left[\mathbb{P}^*_{Z^*_{\sigma_{\eta_j}}}\left(Z^*_{\sigma_{K^*}} = c_k^*, \sigma_{K^*} < \sigma_F\right)\right] = \int_{\eta_j} \varphi^{(k)}(z)\, \nu_j(dz). \tag{15}$$

Hence the chain X moves from c_j^* to c_k^* when a BMD restricted in $D^* \setminus F$ moves from c_j^* to c_k^* after passing η_j. The probabilities p_{j0}, $0 \leq j \leq N$, are defined in an obvious way.

We condition the chain X defined above not to stay at the same state in one step. Then the corresponding transition probability is given by

$$q_{jk} = \frac{p_{jk}}{1 - p_{jj}}\ (j \neq 0), \quad q_{0k} = \delta_{0k}.$$

This conditioned chain satisfies $q_{j0} > 0$ for all j. Thus, the matrix $Q := (q_{jk})_{j,k=1}^N$ has eigenvalues which are all less than one, and the inverse $M = (M_{jk})_{j,k=1}^N := (I - Q)^{-1}$ exists.

Using the transition probabilities introduced above, let us compute

$$V^*(z) := \mathbb{E}^*_z\left[\operatorname{Im} Z^*_{\sigma_F}\right], \quad z \in D^* \setminus F.$$

We also define,[3] for the ABM Z in $\mathbb{H}$,

[3] The symbol $\sigma_\cdot$ is used for two meanings here: the hitting times of the ABM Z in $\mathbb{H}$ and of the BMD Z^* in D^*. Although this is abuse of notation, there will be no confusion.

$$V(z) := \mathbb{E}_z\left[\operatorname{Im} Z_{\sigma_F}; \sigma_F < \sigma_K\right], \quad z \in D \setminus F.$$

Proposition 9 *The function $V^*(z)$ satisfies*

$$V^*(z) = V(z) + \sum_{j=1}^{N} \varphi^{(j)}(z) \sum_{k=1}^{N} \frac{M_{jk}}{1 - p_{kk}} \int_{\eta_k} V(z)\, \nu_k(dz), \quad z \in D \setminus F. \tag{16}$$

Proof For $z \in D \setminus F$,

$$\begin{aligned} V^*(z) &= \mathbb{E}_z^*\left[\operatorname{Im} Z^*_{\sigma_F}; \sigma_F < \sigma_{K^*}\right] + \mathbb{E}_z^*\left[\operatorname{Im} Z^*_{\sigma_F}; \sigma_{K^*} < \sigma_F\right] \\ &= V(z) + \sum_{j=1}^{N} V^*(c_j^*)\, \mathbb{P}_z^*\left(Z^*_{\sigma_{K^*}} = c_j^*, \sigma_{K^*} < \sigma_F\right) \\ &= V(z) + \sum_{j=1}^{N} \varphi^{(j)}(z) V^*(c_j^*). \end{aligned} \tag{17}$$

Integrating the both side by ν_k and using the strong Markov property, we have

$$V(c_k^*) = \int_{\eta_k} V(z)\, \nu_k(dz) + \sum_{j=1}^{N} p_{kj} V^*(c_j^*).$$

This is equivalent to

$$\sum_{j=1}^{N} (\delta_{kj} - q_{kj}) V(c_j^*) = \frac{1}{1 - p_{kk}} \int_{\eta_k} V(z)\, \nu_k(dz).$$

Hence we finally get

$$V^*(c_j^*) = \sum_{k=1}^{N} \frac{M_{jk}}{1 - p_{kk}} \int_{\eta_k} V(z)\, \nu_k(dz). \tag{18}$$

Substituting (18) into (17) yields (16).

We now consider the case that F coincides with F_n in Theorem 2. In this case, we write the above functions V^*, V, and $\varphi^{(j)}$ as V_n^*, V_n, and $\varphi_n^{(j)}$, respectively. We also denote the above p_{jk} and M_{jk} by p_{jk}^n and M_{jk}^n, respectively.

Proposition 10 *If*

$$\operatorname{hm}_{D\setminus F_n}(\zeta, \cdot) \xrightarrow{w} \operatorname{hm}_{D\setminus F_\infty}(\zeta, \cdot) \textit{ as } n \to \infty \textit{ for every } \zeta \in D \setminus \tilde{F}, \tag{19}$$

then $V_n(\zeta)$, $\varphi_n^{(j)}(\zeta)$, p_{jk}^n, *and* M_{jk}^n *converge to* $V_\infty(\zeta)$, $\varphi_\infty^{(j)}(\zeta)$, p_{jk}^∞, *and* M_{jk}^∞, *respectively, as* $n \to \infty$.

Proof Suppose (19). Let $\psi_1(z)$ be a bounded continuous function which is equal to $\operatorname{Im} z$ on $\tilde{F}$ and zero on K. Then

$$\lim_{n\to\infty} V_n(\zeta) = \lim_{n\to\infty} \int_{\mathbb{C}} \psi_1(z) \mathrm{hm}_{D\setminus F_n}(\zeta, dz)$$

$$= \int_{\mathbb{C}} \psi_1(z) \mathrm{hm}_{D\setminus F_\infty}(\zeta, dz) = V_\infty(\zeta).$$

In order to show $\lim_{n\to\infty} \varphi_n^{(j)}(\zeta) = \varphi_\infty^{(j)}(\zeta)$, we just replace $\psi_1(z)$ by a bounded continuous function $\psi_2(z)$ which takes value one on C_j and zero on $\tilde{F} \cup (K \setminus C_j)$. Since $0 \le \varphi_n^{(j)}(\zeta) \le 1$, the dominated convergence theorem applies to (15), which yields $p_{jk}^n \to p_{jk}^\infty$. Finally, $M_{jk}^n \to M_{jk}^\infty$ follows from the above-mentioned construction.

From Propositions 9 and 10, we get the following:

Corollary 1 *The convergence of harmonic measures* (19) *implies that of expectations* (14) *in Proposition 8.*

3.3 Uniform Regularity of Slit Domains

In order to verify the convergence of harmonic measures (19), we prove the uniform regularity of domains $D \setminus F_n$.

Proposition 11 *Let* D *and* F_n, $n \in \mathbb{N} \cup \{\infty\}$, *be as in Theorem 2. Then the domains* $D \setminus F_n$ *are uniformly regular (in the sense of Definition 3 in Appendix 2).*

Proof In this proof, we consider

$$\tilde{A} := \tilde{F} \cup \partial\mathbb{H} \quad \text{and} \quad A_n := F_n \cup \partial\mathbb{H}, \ n \in \mathbb{N} \cup \{\infty\},$$

instead of $\tilde{F}$ and F_n. Needless to say, $D \setminus F_n = D \setminus A_n$.

Since K and $\tilde{A}$ are disjoint, we have $r := \operatorname{dist}(K, \tilde{A}) > 0$. If $\varepsilon \in (0, r/4)$, then $B(z, \varepsilon)$ intersects only one of the sets K and A_n. Therefore, we can divide the proof of the uniform regularity into two cases:

(a) $B(z, \varepsilon) \cap A_n \neq \emptyset$ and $B(z, \varepsilon) \cap K = \emptyset$;
(b) $B(z, \varepsilon) \cap A_n = \emptyset$ and $B(z, \varepsilon) \cap K \neq \emptyset$.

In what follows, let $Z = ((Z_t)_{t\ge 0}, (\mathbb{P}_z)_{z\in\mathbb{C}})$ be a complex Brownian motion.

(a) We consider the case (a). Since the domains $\mathbb{H} \setminus A_n$ are uniformly regular by Proposition 3, there exists $\delta \in (0, \varepsilon)$ such that

$$1 - \varepsilon < \mathrm{hm}_{\mathbb{H}\setminus A_n}(z, B(z, \varepsilon)) = \mathbb{P}_z \left(Z_{\sigma_{A_n}} \in B(z, \varepsilon) \right) \tag{20}$$

for $z \in D \setminus A_n$ with $\mathrm{dist}(z, A_n) < \delta$. We decompose the right-hand side as follows:

$$\begin{aligned}
&\mathbb{P}_z \left(Z_{\sigma_{A_n}} \in B(z, \varepsilon) \right) \\
&\quad = \mathbb{P}_z \left(Z_{\sigma_{A_n}} \in B(z, \varepsilon), \sigma_{A_n} < \sigma_K \right) + \mathbb{P}_z \left(Z_{\sigma_{A_n}} \in B(z, \varepsilon), \sigma_K < \sigma_{A_n} \right) \\
&\quad = \mathrm{hm}_{D\setminus A_n}(z, B(z, \varepsilon)) + \mathbb{E}_z \left[\mathbb{P}_{Z_{\sigma_K}} \left(Z_{\sigma_{A_n}} \in B(z, \varepsilon) \right) ; \sigma_K < \sigma_{A_n} \right] \\
&\quad \le \mathrm{hm}_{D\setminus A_n}(z, B(z, \varepsilon)) + \sup_{w \in K} \mathbb{P}_w \left(Z_{\sigma_{A_n}} \in B(z, \varepsilon) \right) \\
&\quad \le \mathrm{hm}_{D\setminus A_n}(z, B(z, \varepsilon)) + \sup_{w \in K} \mathbb{P}_w \left(\sigma_{B(z,\varepsilon)} < \tau_{\overline{\mathbb{H}}} \right).
\end{aligned} \tag{21}$$

In the last expression, the following uniform convergence is not difficult to see:

$$\lim_{\varepsilon \to 0} \sup \left\{ \mathbb{P}_w \left(\sigma_{B(z,\varepsilon)} < \tau_{\overline{\mathbb{H}}} \right) ; \ z \in \bar{N}_\delta(\tilde{A}), \ w \in K \right\} = 0. \tag{22}$$

Here, a closed subset

$$\bar{N}_\delta(\tilde{A}) := \{ z \in \overline{\mathbb{H}} \ ; \ \mathrm{dist}(z, \tilde{A}) \le \delta \}$$

of $\overline{\mathbb{H}}$ is disjoint from K by definition. We provide the proof of (22) for the sake of completeness in Appendix 1. Finally, (20), (21), and (22) ensure the condition for the uniform regularity in the case (a).

(b) We consider the case (b). In this case, we have

$$\begin{aligned}
\mathrm{hm}_{D\setminus A_n}(z, B(z, \varepsilon)) &= \mathbb{P}_z \left(Z_{\sigma_K} \in B(z, \varepsilon); \sigma_K < \sigma_{A_n} \right) \\
&\ge \mathbb{P}_z \left(Z_{\sigma_K} \in B(z, \varepsilon); \sigma_K < \sigma_{\tilde{A}} \right) \\
&\ge \mathbb{P}_z \left(\sigma_K \le \tau_{B(z,\varepsilon)} \right) = \mathrm{hm}_{B(z,\varepsilon)\setminus K}(z, K).
\end{aligned}$$

Hence it suffices to show that, for some fixed $\varepsilon_0 \in (0, r/4]$ and for every $\varepsilon \in (0, \varepsilon_0)$,

$$\lim_{\delta \to 0} \inf \left\{ \mathrm{hm}_{B(z,\varepsilon)\setminus K}(z, K) \ ; \ z \in \mathbb{C}, \ \mathrm{dist}(z, K) < \delta \right\} = 1. \tag{23}$$

There may be several ways to prove this, and one way is as follows: If ε_0 is small enough, then for every $\varepsilon \in (0, \varepsilon_0)$, the disk $\overline{B(z, \varepsilon)}$ intersects only one slit C_j of K, and the length of C_j is greater than 2ε. For such an ε, let $\delta \in (0, \varepsilon)$. For any point z with $\rho_z := \mathrm{dist}(z, K) < \delta$, we put $E := C_j \cap \overline{B(z, \varepsilon)}$. In the disk

$\overline{B(z, \varepsilon)}$, we consider the circular projection

$$\hat{E} := \{\, |w - z| + z \;;\; w \in E \,\}$$

of E onto the horizontal radius. Clearly, $\hat{E}$ is the line segment that connects $z + \rho_z$ and $z + \varepsilon$. By Beurling's projection theorem (see, e.g., Garnett and Marshall [8, Theorem 9.2, Chap. III]), we have

$$\begin{aligned} \mathrm{hm}_{B(z,\varepsilon)\setminus K}(z, K) \geq \mathrm{hm}_{B(z,\varepsilon)\setminus \hat{E}}(z, \hat{E}) &= \frac{2}{\pi} \arctan\left[\frac{1}{2}\left(\sqrt{\frac{\varepsilon}{\rho_z}} - \sqrt{\frac{\rho_z}{\varepsilon}}\right)\right] \\ &\geq \frac{2}{\pi} \arctan\left[\frac{1}{2}\left(\sqrt{\frac{\varepsilon}{\delta}} - \sqrt{\frac{\delta}{\varepsilon}}\right)\right]. \end{aligned}$$

The last expression goes to one as $\delta \to 0$. This proves (23).

Proposition 11 and Theorem 4 in Appendix 2 immediately yield the next corollary.

Corollary 2 *Let D and F_n, $n \in \mathbb{N} \cup \{\infty\}$, be as in Theorem 2. Then* (19) *holds, that is,* $\mathrm{hm}_{D\setminus F_n}(\zeta, \cdot)$ *converges weakly to* $\mathrm{hm}_{D\setminus F_\infty}(\zeta, \cdot)$ *for every* $\zeta \in D \setminus \tilde{F}$.

Corollaries 1 and 2 prove Proposition 8 and hence Theorem 2.

4 Relation to Geometric Function Theory

We give some remarks on Theorem 1 in the case that $\mathbb{H}$-hulls are unbounded. They are also the case with Theorem 2.

4.1 Half-Plane Capacity and Angular Residue at Infinity

In view of geometric function theory, a natural way to define the half-plane capacity of an unbounded $\mathbb{H}$-hull is to define it as the angular residue of the Riemann map at infinity. More precisely, suppose that there exists a (unique) conformal mapping $f_F : \mathbb{H} \to \mathbb{H} \setminus F$ with the following two properties [9, Lemma 1(b)]:

$$\lim_{\substack{z\to\infty \\ \operatorname{Im} z>\eta}} (f_F(z) - z) = 0 \quad \text{for any } \eta > 0, \tag{24}$$

and there exists $a_F \in \mathbb{C}$ such that

$$\lim_{\substack{z\to\infty \\ \arg z\in(\theta,\pi-\theta)}} z(z - f_F(z)) = a_F \quad \text{for any } \theta \in (0, \pi/2). \tag{25}$$

The constant a_F turns out to be non-negative and is called the angular residue at infinity.

It is known that the inverse mapping f_F^{-1} has angular residue $-a_F$ at infinity. Expressing the harmonic function $\mathrm{Im}(f_F^{-1}(z) - z)$ in terms of ABM, we can see that

(A) if there exists a conformal mapping $f_F \colon \mathbb{H} \to \mathbb{H} \setminus F$ which enjoys (24) and (25), then we have $\mathrm{Im}\, F < \infty$ and $\mathrm{hcap}(F) = a_F$.

In particular, the existence of the vertical limit (1) follows from that of the angular limit (25). On the other hand, it seems difficult to tell whether the following statement, which is the converse of (A), is true or not:

(B) If the limit (1) exists, then there exists a conformal mapping $f_F \colon \mathbb{H} \to \mathbb{H} \setminus F$ which enjoys (24) and (25).

In order to explain the difficulty of the above problem, suppose that the limit (1) exists. We can then take a conformal mapping $f \colon \mathbb{H} \to \mathbb{H} \setminus F$ with $f(\infty) = \infty$ in the sense of angular limit. Compared with (24) and (25), however, the last condition is too weak to relate the behavior of the inverse $f^{-1}(z)$ around $z = \infty$ to the quantity $\mathrm{hcap}(F)$. Typical tools concerning holomorphic self-mappings in $\mathbb{H}$ might be useful, such as the Pick–Nevanlinna integral representation, the Julia–Wolff–Carathéodory theorem, and so on, but at this moment they do not directly imply that $f^{-1}(z)$ behaves well near ∞.

We note that some probabilistic methods could be available for constructing the conformal mapping f_F above. In the case that the hull F is bounded, such a probabilistic construction of conformal mappings is studied by Lawler [12, Sect. 5.2] and by Chen et al. [3, Theorem 7.2]. It will be a natural question whether we can obtain such results in the present case as well.

4.2 Carathéodory Convergence and Locally Uniform Convergence

In the classical context, the Carathéodory convergence of domains (Definition 1) is associated with the locally uniform convergence of the Riemann maps by the following theorem (see for example Rosenblum and Rovnyak [18, p. 170]):

Theorem 3 (Carathéodory Kernel Theorem) *Let f_n be a conformal mapping from the unit disk $\mathbb{D}$ onto D_n with $f_n(0) = 0$ and $f_n'(0) > 0$ for each $n \in \mathbb{N}$. Then the following are equivalent:*

(a) *$(f_n)_{n\in\mathbb{N}}$ converges to a non-constant function locally uniformly in $\mathbb{D}$;*
(b) *$(D_n)_{n\in\mathbb{N}}$ converges to a proper subdomain of $\mathbb{C}$ in Carathéodory's sense with respect to the origin.*

Compared to Theorem 3, it is naturally expected that, for $\mathbb{H}$-hulls $(F_n)_{n\in\mathbb{N}}$ and the corresponding conformal mappings $(f_{F_n})_{n\in\mathbb{N}}$ with (24) and (25), the following are equivalent:

(a′) $(f_{F_n})_{n\in\mathbb{N}}$ converges locally uniformly in $\mathbb{H}$;
(b′) $(F_n)_{n\in\mathbb{N}}$ converges in the sense of Definition 2.

However, this equivalence is not obvious from the classical Carathéodory kernel theorem. The point is that, whereas the Riemann maps in Theorem 3 fix the origin, an interior point of $\mathbb{D}$, the mappings f_{F_n} fix the point at infinity, a boundary point of $\mathbb{H}$.

Although we omit their details, there are several variants of Carathéodory's kernel theorem known. We here notice that, in the case that the $\mathbb{H}$-hulls F_n are uniformly bounded, the author gave a proof [14, Theorem 3.8] of the equivalence of (a′) and (b′).

Remark 2 We have seen that, at this moment, the probabilistic definition (1) of half-plane capacity is not completely the same as the analytic one (25) for unbounded $\mathbb{H}$-hulls. In the joint work [10] referred to in Remark 1, it is proved in an analytic way that, under assumptions like (24) and (25), the locally uniform convergence $f_{F_n} \to f_{F_\infty}$ implies $a_{F_n} \to a_{F_\infty}$. This statement is thus different than Theorem 1 from a technical point of view. As it exceeds the scope of this article, we just point out this difference and do not go into the detail of such a technicality here.

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Appendix 1: A Result on Hitting Probability

Let $Z = ((Z_t)_{t\geq 0}, (\mathbb{P}_z)_{z\in\mathbb{C}})$ be a complex Brownian motion. We recall and prove (22):

Proposition 12 *Let S be a closed set in $\overline{\mathbb{H}}$ and K be a compact set in $\mathbb{H}$. Suppose $S \cap K = \emptyset$ and* $\operatorname{Im} S < \infty$. *Then*

$$\lim_{\varepsilon\to 0} \sup \left\{ \mathbb{P}_w \left(\sigma_{B(z,\varepsilon)} < \tau_{\overline{\mathbb{H}}}\right) ; z \in S,\ w \in K \right\} = 0. \tag{26}$$

Here, $\sigma_\cdot$ and $\tau_\cdot$ denote the first hitting and exit times of Z, respectively.

Proof We reduce the problem to the case in which S is compact. Let $a > 0$ be such that $K \subset \{ z ; |\operatorname{Re} z| < a \}$. We take $\varepsilon_0 > 0$ so small that a compact set

$$S' := (S \cap \{ z ; |\operatorname{Re} z| < a \}) \cup \{ z ; a \leq |\operatorname{Re} z| \leq 2a + \varepsilon_0,\ 0 \leq \operatorname{Im} z \leq \operatorname{Im} S \}.$$

satisfies $\operatorname{dist}(K, S') \geq \varepsilon_0$. It follows that

$$\sup_{\substack{z\in S\\ w\in K}} \mathbb{P}_w\left(\sigma_{B(z,\varepsilon)} < \tau_{\overline{\mathbb{H}}}\right) \le \sup_{\substack{z\in S'\\ w\in K}} \mathbb{P}_w\left(\sigma_{B(z,\varepsilon)} < \tau_{\overline{\mathbb{H}}}\right) \quad \text{for } \varepsilon \in (0, \varepsilon_0). \tag{27}$$

This inequality is seen as follows: Let $L := \{\, z \,;\, \operatorname{Re} z = 2a + \varepsilon_0 \,\}$. For $z \in S$ with $2a + \varepsilon_0 < \operatorname{Re} z \le 3a + \varepsilon_0$, we have

$$\begin{aligned}\mathbb{P}_w\left(\sigma_{B(z,\varepsilon)} < \tau_{\overline{\mathbb{H}}}\right) &= \mathbb{E}_w\left[\mathbb{P}_{Z_{\sigma_L}}\left(\sigma_{B(z,\varepsilon)} < \tau_{\overline{\mathbb{H}}}\right); \sigma_L < \tau_{\overline{\mathbb{H}}}\right]\\ &= \mathbb{E}_w\left[\mathbb{P}_{Z_{\sigma_L}}\left(\sigma_{B(2a+\varepsilon_0-\bar{z},\varepsilon)} < \tau_{\overline{\mathbb{H}}}\right); \sigma_L < \tau_{\overline{\mathbb{H}}}\right]\\ &= \mathbb{P}_w\left(\sigma_L < \sigma_{B(2a+\varepsilon_0-\bar{z},\varepsilon)} < \tau_{\overline{\mathbb{H}}}\right) \le \mathbb{P}_w\left(\sigma_{B(2a+\varepsilon_0-\bar{z},\varepsilon)} < \tau_{\overline{\mathbb{H}}}\right).\end{aligned}$$

Hence

$$\sup_{\substack{z\in S,\ 2a+\varepsilon_0<\operatorname{Re} z\le 3a+\varepsilon_0\\ w\in K}} \mathbb{P}_w\left(\sigma_{B(z,\varepsilon)} < \tau_{\overline{\mathbb{H}}}\right) \le \sup_{\substack{z\in S'\\ w\in K}} \mathbb{P}_w\left(\sigma_{B(z,\varepsilon)} < \tau_{\overline{\mathbb{H}}}\right).$$

Repeating such a reflection argument, we can conclude (27).

By virtue of (27), it suffices to prove (26) with S replaced by S'. We define half-planes $\overline{\mathbb{H}} - z := \{\, w - z \,;\, w \in \overline{\mathbb{H}} \,\}$, $z \in \mathbb{C}$, and $H := \overline{\mathbb{H}} - i \operatorname{Im} S$. Then

$$\mathbb{P}_w\left(\sigma_{B(z,\varepsilon)} < \tau_{\overline{\mathbb{H}}}\right) = \mathbb{P}_{w-z}\left(\sigma_{B(0,\varepsilon)} < \tau_{\overline{\mathbb{H}}-z}\right) \le \mathbb{P}_{w-z}\left(\sigma_{B(0,\varepsilon)} < \tau_H\right).$$

We now put $K - S' := \{\, w - z \,;\, w \in K,\ z \in S' \,\}$ and $f_\varepsilon(\zeta) := \mathbb{P}_\zeta\left(\sigma_{B(0,\varepsilon)} < \tau_H\right)$ for $\zeta \in K - S'$. The function $f_\varepsilon(\zeta)$ is continuous in ζ and decreases to zero as $\varepsilon \to 0$ for each ζ. By Dini's theorem, we have $\lim_{\varepsilon\to 0} \sup_{\zeta\in K-S'} f_\varepsilon(\zeta) = 0$, which gives (26).

Appendix 2: Weak Convergence of Harmonic Measures

Following Binder et al. [1, (3.1)], we define the uniform regularity of domains as follows:

Definition 3 (Uniform Regularity) A collection $\mathcal{D}$ of proper subdomains of $\mathbb{C}$ is said to be *uniformly regular* if, for any $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ such that every $D \in \mathcal{D}$ satisfies

$$\operatorname{hm}_D(z, B(z, \varepsilon)) > 1 - \varepsilon \quad \text{for any } z \in D \text{ with } \operatorname{dist}(z, \partial D) < \delta. \tag{28}$$

Only is the Euclidean distance used in Definition 3. This definition is slightly different from the original one, which involves the spherical distance. In the subsequent argument, we use only the Euclidean distance and do not consider the spherical one.

The aim of this appendix is to provide a self-contained proof of the following theorem, which is part of Binder et al. [1, Theorems 2.3 and 3.1]:

Theorem 4 *Let D_n, $n \in \mathbb{N}$, and D_∞ be proper subdomains of $\mathbb{C}$ which have a point z_0 in common. Suppose that these domains are uniformly regular. If D_n converges to D_∞ as $n \to \infty$ in Carathéodory's sense with respect to z_0, then* $\mathrm{hm}_{D_n}(z_0, \cdot)$ *converges weakly to* $\mathrm{hm}_{D_\infty}(z_0, \cdot)$.

Remark 3 The weak convergence of harmonic measures (in simply connected domains) was also proved under a different (but closely related) assumption in the earlier paper of Snipes and Ward [19, Remark 1] in the context of harmonic measure distribution functions.

In what follows, we keep the assumption of Theorem 4. Namely, D_n, $n \in \mathbb{N}$, and D_∞ are uniformly regular domains, and D_n converges to D_∞ in Carathéodory's sense with respect to a point z_0.

We construct a "common interior approximation" of D_n for sufficiently large n (cf. [1, Definition 2.2]). Let $\delta_0 := \mathrm{dist}(z_0, \partial D_\infty) > 0$. For $\delta \in (0, \delta_0)$ and $r > 0$, we define $U_{\delta,r}$ as the connected component of an open set

$$\left\{ z \in D_\infty \ ; \ \mathrm{dist}(z, \partial D_\infty) > \frac{\delta}{4}, \ |z - z_0| < r \right\}$$

that contains z_0. We also set

$$\Gamma_\delta := \left\{ z \in \mathbb{C} \ ; \ \mathrm{dist}(z, \partial D_\infty) = \frac{\delta}{4} \right\}.$$

Lemma 1 *For every $\delta \in (0, \delta_0)$ and $r > 0$, there exists $N \in \mathbb{N}$ such that, for any $N < n \leq \infty$, the domain D_n enjoys*

$$U_{\delta,r} \subset D_n \quad \textit{and} \quad \textit{dist}(z, \partial D_n) < \delta \quad \textit{for all } z \in \Gamma_\delta \cap \partial U_{\delta,r}. \tag{29}$$

Proof By definition, we have

$$\overline{U_{\delta,r}} \Subset D_\infty \quad \text{and} \quad \mathrm{dist}(z, \partial D_\infty) = \frac{\delta}{4} \quad \text{for all } z \in \Gamma_\delta \cap \partial U_{\delta,r}. \tag{30}$$

Since $D_n \to D_\infty$, there exists $N' \in \mathbb{N}$ such that $\overline{U_{\delta,r}} \subset \bigcap_{N' \leq n < \infty} D_n$. We here see that, for a fixed $z \in \Gamma_\delta \cap \partial U_{\delta,r}$, there are only finitely many n such that $\mathrm{dist}(z, \partial D_n) \geq \delta/2$; Otherwise, $U_{\delta,r} \cup B(z, \delta/2)$ would be a subdomain of D_n for infinitely many n, and hence this domain would be a subset of D_∞ by Definition 1. This contradicts (30). In this way, we can define

$$n(z) := \max\{ n \geq N' \ ; \ \mathrm{dist}(z, \partial D_n) \geq \delta/2 \}$$

with the maximum set to be N' if this set is empty.

By the compactness, we can choose finitely many points $z_k \in \Gamma_\delta \cap \partial U_{\delta,r}$ so that $\Gamma_\delta \cap \partial U_{\delta,r} \subset \bigcup_k B(z_k, \delta/2)$. Set $N := \max_k n(z_k)$. For any $z \in \Gamma_\delta \cap \partial U_{\delta,r}$, we can find z_k with $z \in B(z_k, \delta/2)$, and hence

$$\operatorname{dist}(z, \partial D_n) \leq |z - z_k| + \operatorname{dist}(z_k, \partial D_n) < \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

for all $n > N$.

Let $Z = ((Z_t)_{t \geq 0}, (\mathbb{P}_z)_{z \in \mathbb{C}})$ be a planar Brownian motion. We write the exit time of Z from $U_{\delta,r}$ as $\tau_{\delta,r} := \tau_{U_{\delta,r}}$.

Lemma 2 *For every $\varepsilon > 0$, there exists $R > 0$ such that*

$$\mathbb{P}_{z_0}\left(Z_{\tau_{\delta,r}} \in \Gamma_\delta\right) \geq 1 - \varepsilon \tag{31}$$

for all $\delta \in (0, \delta_0)$ and $r > R$.

Proof Since D_∞ is a regular domain by assumption, its complement is non-polar. Hence we can take a compact non-polar subset K of D_∞^c (see for instance Port and Stone [15, Proposition 2.4, Chap. 2]). In two dimension, a non-polar set is recurrent for Brownian motion [15, Propositions 2.9 and 2.10, Chap. 2], which means

$$\lim_{r \to \infty} \mathbb{P}_{z_0}\left(\sigma_K < \sigma_{\partial B(z_0, r)}\right) = \mathbb{P}_{z_0}\left(\sigma_K < \infty\right) = 1.$$

Here, the Brownian motion starting at z_0 has to hit Γ_δ before it hits $K \subset D_\infty^c$. Therefore, for any $\varepsilon > 0$, there exists $R > 0$ such that, if $r > R$, then

$$1 - \varepsilon \leq \mathbb{P}_{z_0}\left(\sigma_K < \sigma_{\partial B(z_0, r)}\right) \leq \mathbb{P}_{z_0}\left(Z_{\tau_{\delta,r}} \in \Gamma_\delta\right).$$

Let $\mathrm{Lip}_b(\mathbb{C})$ be the space of bounded Lipschitz functions in $\mathbb{C}$ with norm

$$\|f\|_{\mathrm{Lip}_b(\mathbb{C})} := \sup_{z \in \mathbb{C}} |f(z)| + \sup_{\substack{z, w \in \mathbb{C} \\ z \neq w}} \frac{|f(z) - f(w)|}{|z - w|}.$$

A distance

$$d(\mu, \nu) := \sup_{\substack{f \in \mathrm{Lip}_b(\mathbb{C}) \\ \|f\|_{\mathrm{Lip}_b(\mathbb{C})} \leq 1}} \left| \int_{\mathbb{C}} f \, d\mu - \int_{\mathbb{C}} f \, d\nu \right|, \quad \mu, \nu \in \mathcal{P}(\mathbb{C}),$$

is known to metrize the weak topology of the space $\mathcal{P}(\mathbb{C})$ of Borel probability measures on $\mathbb{C}$ (see for instance Dudley [6, Theorem 11.3.3]). Using this distance, we now prove Theorem 4.

***Proof of Theorem* 4** Fix $\varepsilon > 0$. By Lemma 2 and the assumption of the uniform regularity, we can take $\delta \in (0, \delta_0)$ and $r > 0$ so that (31) and (28) with $D = D_n$ for all $n \in \mathbb{N} \cup \{\infty\}$ hold. For these δ and r, there exists $N \in \mathbb{N}$ such that (29) holds for all $N < n \le \infty$ by Lemma 1.

For $N < n \le \infty$ and $f \in \mathrm{Lip}_b(\mathbb{C})$ with $\|f\|_{\mathrm{Lip}_b(\mathbb{C})} \le 1$, let us consider the difference

$$\left| \int_{\mathbb{C}} f(z) \mathrm{hm}_{U_{\delta,r}}(z_0, dz) - \int_{\mathbb{C}} f(z) \mathrm{hm}_{D_n}(z_0, dz) \right| \le \mathbb{E}_{z_0}\left[|f(Z_{\tau_{\delta,r}}) - f(Z_{\tau_{D_n}})| \right].$$

Note that the random variable $\varDelta_n := |f(Z_{\tau_{\delta,r}}) - f(Z_{\tau_{D_n}})|$ satisfies

$$\varDelta_n \le \|f\|_{\mathrm{Lip}_b(\mathbb{C})} |Z_{\tau_{\delta,r}} - Z_{\tau_{D_n}}| \le |Z_{\tau_{\delta,r}} - Z_{\tau_{D_n}}|$$

and

$$\varDelta_n \le 2\|f\|_{\mathrm{Lip}_b(\mathbb{C})} \le 2.$$

We take the expectation of $\varDelta_n$ on three disjoint events. First,

$$\mathbb{E}_{z_0}\left[\varDelta_n; Z_{\tau_{\delta,r}} \in \varGamma_\delta,\ |Z_{\tau_{\delta,r}} - Z_{\tau_{D_n}}| < \varepsilon\right] \le \varepsilon. \tag{32}$$

Second,

$$\begin{aligned} &\mathbb{E}_{z_0}\left[\varDelta_n; Z_{\tau_{\delta,r}} \in \varGamma_\delta,\ |Z_{\tau_{\delta,r}} - Z_{\tau_{D_n}}| \ge \varepsilon\right] \\ &\le 2\,\mathbb{P}_{z_0}\left(Z_{\tau_{\delta,r}} \in \varGamma_\delta,\ |Z_{\tau_{\delta,r}} - Z_{\tau_{D_n}}| \ge \varepsilon\right) \\ &= 2\,\mathbb{E}_{z_0}\left[\mathrm{hm}_{D_n}(Z_{\tau_{\delta,r}}, B(Z_{\tau_{\delta,r}}, \varepsilon)^c); Z_{\tau_{\delta,r}} \in \varGamma_\delta\right] \le 2\varepsilon. \end{aligned} \tag{33}$$

Third,

$$\mathbb{E}_{z_0}\left[\varDelta_n; Z_{\tau_{\delta,r}} \notin \varGamma_\delta\right] \le 2\,\mathbb{P}_{z_0}(Z_{\tau_{\delta,r}} \notin \varGamma_\delta) \le 2\varepsilon. \tag{34}$$

Combining (32)–(34) yields $\mathbb{E}_{z_0}[\varDelta_n] \le 5\varepsilon$. Since f was arbitrary, we have

$$d(\mathrm{hm}_{U_{\delta,r}}(z_0, \cdot), \mathrm{hm}_{D_n}(z_0, \cdot)) \le 5\varepsilon$$

for all $N < n \le \infty$. Finally, it follows from the triangle inequality that

$$d(\mathrm{hm}_{D_n}(z_0, \cdot), \mathrm{hm}_{D_\infty}(z_0, \cdot)) \le 10\varepsilon$$

for all $n > N$, which proves Theorem 4.

References

1. I. Binder, C. Rojas, M. Yampolsky, Carathéodory convergence and harmonic measure. Potential Anal. **51**, 499–509 (2019)
2. Z.-Q. Chen, M. Fukushima, Stochastic Komatu-Loewner evolutions and BMD domain constant. Stoch. Process. Appl. **128**, 545–594 (2018)
3. Z.-Q. Chen, M. Fukushima, S. Rohde, Chordal Komatu-Loewner equation and Brownian motion with darning in multiply connected domains. Trans. Amer. Math. Soc. **368**, 4065–4114 (2016)
4. Z.-Q. Chen, M. Fukushima, H. Suzuki, Stochastic Komatu-Loewner evolutions and SLEs. Stoch. Process. Appl. **127**, 2068–2087 (2017)
5. V.N. Dubinin, M. Vuorinen, Ahlfors-Beurling conformal invariant and relative capacity of compact sets. Proc. Amer. Math. Soc. **142**, 3865–3879 (2014)
6. R.M. Dudley, in *Real Analysis and Probability*. Cambridge Studies in Advanced Mathematics, vol. 74 (Cambridge University Press, Cambridge, 2002)
7. U. Franz, T. Hasebe, S. Schleißinger, Monotone increment processes, classical Markov processes, and Loewner chains. Diss. Math. **552**, 119 (2020)
8. J.B. Garnett, D.E. Marshall, *Harmonic Measure*. New Mathematical Monographs, vol. 2 (Cambridge University Press, 2005)
9. V. V. Goryainov, I. Ba, Semigroups of conformal mappings of the upper half-plane into itself with hydrodynamic normalization at infinity. Ukr. Math. J. **44**, 1209–1217. Transl. Ukr. Math. Zh. **44**(1992), 1320–1329 (1992)
10. T. Hasebe, I. Hotta, T. Murayama, Additive processes on the real line and Loewner chains (in preparation)
11. S. Lalley, G. Lawler, H. Narayanan, Geometric interpretation of half-plane capacity. Electron. Commun. Probab. **14**, 566–571 (2009)
12. G.F. Lawler, The Laplacian-b random walk and the Schramm-Loewner evolution. Illinois J. Math. **50**, 701–746 (2006)
13. G. Markowsky, A remark on the probabilistic solution of the Dirichlet problem for simply connected domains in the plane. J. Math. Anal. Appl. **464**, 1143–1146 (2018)
14. T. Murayama, Chordal Komatu-Loewner equation for a family of continuously growing hulls. Stoch. Process. Appl. **129**, 2968–2990 (2019)
15. S.C. Port, C.J. Stone, *Brownian Motion and Classical Potential Theory*. Probability and Mathematical Statistics (Academic Press, New York, London, 1978)
16. T. Ransford, *Potential Theory in the Complex Plane*. London Mathematical Society Student Texts, vol. 28 (Cambridge University Press, Cambridge, 1995)
17. S. Rohde, C. Wong, Half-plane capacity and conformal radius. Proc. Amer. Math. Soc. **142**, 931–938 (2014)
18. M. Rosenblum, J. Rovnyak, *Topics in Hardy Classes and Univalent Functions*. Birkhäuser Advanced Texts: Basler Lehrbücher (Birkhäuser Verlag, Basel, 1994)
19. M.A. Snipes, L.A. Ward, Convergence properties of harmonic measure distributions for planar domains. Complex Var. Elliptic Equ. **53**, 897–913 (2008)

Dyson's Model in Infinite Dimensions Is Irreducible

Hirofumi Osada and Ryosuke Tsuboi

Abstract Dyson's model in infinite dimensions is a system of Brownian particles interacting via a logarithmic potential with an inverse temperature of $\beta = 2$. The stochastic process is given as a solution to an infinite-dimensional stochastic differential equation. Additionally, a Dirichlet form with the sine_2 point process as a reference measure constructs the stochastic process as a functional of the associated configuration-valued diffusion process. In this paper, we prove that Dyson's model in infinite dimensions is irreducible.

Keywords Dyson's model · Random matrices · Irreducibility · Diffusion process · Interacting Brownian motion · Infinite-dimensional stochastic differential equations · Logarithmic potential · Gaussian unitary ensembles

Mathematics Subject Classification 60B20 · 60H10 · 60J40 · 60J60 · 60K35

1 Introduction

This paper considers an infinite-dimensional stochastic differential equation (ISDE) of the form

$$X_t^i - X_0^i = B_t^i + \frac{\beta}{2} \int_0^t \lim_{r\to\infty} \sum_{|X_u^i - X_u^j|<r,\ j\neq i}^{\infty} \frac{1}{X_u^i - X_u^j} du \quad (i \in \mathbb{Z}). \tag{1.1}$$

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For $\beta = 2$, the ISDE was introduced by Spohn [27], who called it Dyson's model. Spohn derived (1.1) for $\beta = 2$ as an informal limit of Dyson's Brownian motion in finite dimensions. Here, Dyson's Brownian motion is a solution of a finite-dimensional stochastic differential equation (SDE) such that

$$X_t^{N,i} - X_0^{N,i} = B_t^i + \frac{\beta}{2}\int_0^t \sum_{j\neq i}^N \frac{1}{X_u^{N,i} - X_u^{N,j}} du - \frac{\beta}{2N}\int_0^t \frac{1}{X_u^{N,i}} du. \tag{1.2}$$

If $\beta = 2$, then SDE (1.2) describes the dynamics of the eigenvalues of Gaussian unitary ensembles of order $N \in \mathbb{N}$ [3, 16]. Spohn [27] constructed the limit dynamics as the L^2-Markovian semi-group given by the Dirichlet form

$$\mathcal{E}(f, g) = \int_{\mathfrak{S}} \mathbb{D}[f, g] d\mu \tag{1.3}$$

on $L^2(\mathfrak{S}, \mu)$, where $\mathfrak{S}$ is the configuration space over $\mathbb{R}$, $\mathbb{D}$ is the standard carré du champ on $\mathfrak{S}$, and μ is the sine_2 random point field. Furthermore, the domain of the Dirichlet form is taken to be the closure of the polynomials on $\mathfrak{S}$.

Let μ be the sine_β-random point field. If $\beta = 2$, then μ becomes a determinantal random point field whose m-point correlation function ρ^m with respect to the Lebesgue measure is given by

$$\rho^m(\mathbf{x}) = \det[\mathcal{K}_{\sin,2}(x^i, x^j)]_{i,j=1}^m.$$

Here, $\mathcal{K}$ is the sine kernel given by

$$\mathcal{K}(x, y) = \frac{\sin\{\theta\sqrt{2}(x-y)\}}{\pi(x-y)}.$$

Spohn [27] proved the closability of $\mathcal{E}$ on $L^2(\mathfrak{S}, \mu)$ with a predomain consisting of polynomials on $\mathfrak{S}$ for $\beta = 2$.

In [17], the first author proved that $(\mathcal{E}, \mathcal{D}_\circ^\mu)$ is closable on $L^2(\mathfrak{S}, \mu)$, and that its closure is a quasi-regular Dirichlet form. Here, $\mathcal{D}_\circ$ is the set consisting of local and smooth functions on $\mathfrak{S}$ and $\mathcal{D}_\circ^\mu$ is given by

$$\mathcal{D}_\circ^\mu = \{f \in \mathcal{D}_\circ\, ;\, \mathcal{E}_1(f, f) < \infty\}.$$

Thus, Osada constructed the L^2-Markovian semi-group as well as the diffusion

$$\mathfrak{X}(t) = \sum_{i\in\mathbb{Z}} \delta_{X^i(t)}$$

associated with the Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(\mathfrak{S}, \mu)$. We call $\mathfrak{X}$ the unlabeled dynamics or unlabeled diffusion because the state space of the process is $\mathfrak{S}$. The unlabeled diffusion can be constructed for $\beta = 1, 4$ [21], and the associated labeled process $\mathbf{X} = (X^i)_{i\in\mathbb{N}}$ satisfies ISDE (1.1) for $\beta = 1, 2, 4$ [20]. These cases have been proved as examples of the general theory developed in various papers [19–22]. In [20], the meaning of a solution to an ISDE is a weak solution; the uniqueness of a weak solution of an ISDE and the Dirichlet form is left open in [20, 21]. (See [8] for the concept of strong and weak solutions of stochastic differential equations).

Tsai [29] solved ISDE (1.1) for all $\beta \in [1, \infty)$. He proved the existence of a strong solution and the path-wise uniqueness of this solution. The method used by Tsai depends on an artistic coupling specific to Dyson's model. A non-equilibrium solution is obtained in the sense that the ISDE is solved by starting at each point in an explicitly given subset $\mathfrak{S}_0 \subset \mathfrak{S}$ such that $\mu(\mathfrak{S}_0) = 1$.

The μ-reversibility of the associated unlabeled diffusion is left open in [29]. Combining [20] and [29], we find that the unlabeled process given by the solution of (1.2) obtained in [29] is reversible with respect to μ for $\beta = 1, 4$. For a general $\beta > 0$, note that the reversible probability measure of the unlabeled diffusion given by the solution to ISDE (1.1) is expected to be a sine$_\beta$-random point field. This remains an open problem, except for $\beta = 1, 2, 4$ [21].

One of the authors and Tanemura [24] also proved the existence of a strong solution and the path-wise uniqueness of this solution for $\beta = 1, 2, 4$. Their method can be applied to quite a wide range of examples. Using the result in [24], Kawamoto *et al.* proved the uniqueness of Dirichlet forms [11]. They checked the infinite system of finite-dimensional SDEs with consistency (IFC) condition in [12], which plays an important role in the theory developed in [24]. Kawamoto and the second author of [12] derive a solution to the ISDE from N-particle systems [9, 10].

The goal of this paper is to prove that the solution of (1.3) for $\beta = 2$ is irreducible (Theorem 1). In the remainder of this paper, we consider the case $\beta = 2$. Hence, we take μ to be the sine$_2$ random point field.

By definition, the configuration $\mathfrak{S}$ over $\mathbb{R}$ is given by

$$\mathfrak{S} = \Big\{ \mathfrak{s} = \sum_i \delta_{s^i} \ ; \ \mathfrak{s}(K) < \infty \ \text{ for any compact } K \Big\}.$$

We endow $\mathfrak{S}$ with the vague topology. Under the vague topology, $\mathfrak{S}$ is a Polish space. A probability measure on $(\mathfrak{S}, \mathscr{B}(\mathfrak{S}))$ is called a random point field (also called a point process). Let

$$\mathfrak{S}_{\mathrm{s,i}} = \Big\{ \mathfrak{s} \in \mathfrak{S} \ ; \ \mathfrak{s}(\{s\}) \le 1 \text{ for all } s \in \mathbb{R}, \ \mathfrak{s}(\mathbb{R}) = \infty \Big\}.$$

In [18, 21], we proved that the sine$_2$ random point field μ satisfies

$$\mathrm{Cap}((\mathfrak{S}_{\mathrm{s,i}})^c) = 0. \tag{1.4}$$

Furthermore, μ is translation invariant and tail trivial [15, 23]. Hence, by the individual ergodic theorem, we have that, for μ-a.s. $\mathfrak{s}$,

$$\lim_{R\to\infty} \frac{\mathfrak{s}([-R,R])}{R} = \int_{\mathfrak{S}} \mathfrak{s}([-1,1]) d\mu.$$

Then, we set

$$\mathfrak{S}_n = \Big\{ \mathfrak{s} \in \mathfrak{S}\, ;\ \frac{1}{n} \le \frac{\mathfrak{s}([-R,R])}{R} \le n \text{ for all } R \in \mathbb{N} \Big\}. \tag{1.5}$$

Using the argument in the proof of Theorem 1 in [17, p.127], we see that

$$\mathrm{Cap}\Big(\big(\bigcup_{n=1}^{\infty} \mathfrak{S}_n \big)^c \Big) = 0. \tag{1.6}$$

We write $\mathbf{s} = (s^i)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$, and we set

$$\mathbb{R}^{\mathbb{Z}}_{<} = \{ \mathbf{s} = (s^i)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}\, ;\ s^i < s^{i+1} \ \text{ for all } \ i\, \}.$$

Let $\mathfrak{u}$ be a map on $\mathbb{R}^{\mathbb{Z}}_{<}$ such that $\mathfrak{u}(\mathbf{s}) = \sum_{i\in\mathbb{Z}} \delta_{s^i}$. Let $\mathfrak{l} : \mathfrak{S}_{\mathrm{s,i}} \to \mathbb{R}^{\mathbb{Z}}_{<}$ be a function such that $\mathfrak{u} \circ \mathfrak{l} = \mathrm{id}.$. We call $\mathfrak{u}$ an unlabeling map and $\mathfrak{l}$ a labeling map. There exist many labeling maps. We can take $\mathfrak{l}$ in such a way that $\mathfrak{l}^0(\mathfrak{s}) = \min\{s^i; s^i \ge 0,\ \mathfrak{s} = \sum_{i\in\mathbb{Z}} \delta_{s^i}\}$ and $\mathfrak{l}^i(\mathfrak{s}) < \mathfrak{l}^{i+1}(\mathfrak{s})$ for all $i \in \mathbb{Z}$, where $\mathfrak{l}(\mathfrak{s}) = (\mathfrak{l}^i(\mathfrak{s}))_{i\in\mathbb{Z}}$. This choice of $\mathfrak{l}$ is just for convenience and has no specific meaning. Let

$$\mathrm{W} = C([0,\infty); \mathbb{R}^{\mathbb{Z}}_{<}). \tag{1.7}$$

Let $\mathfrak{l}_{\mathrm{path}} = \{\mathfrak{l}_{\mathrm{path}}(\mathfrak{w})_t\}_{t\in[0,\infty)}$ be the label path map generated by $\mathfrak{l}$ (see [24, pp. 1148–1149] and (2.6) in [12]). By definition, $\mathfrak{l}_{\mathrm{path}}$ is the map from $C([0,\infty); \mathfrak{S}_{\mathrm{s,i}})$ to W such that $\mathfrak{l}_{\mathrm{path}}(\mathfrak{w})_0 = \mathfrak{l}(\mathfrak{w}_0)$, where $\mathfrak{w} = \{\mathfrak{w}_t\}_{t\in[0,\infty)} \in C([0,\infty); \mathfrak{S}_{\mathrm{s,i}})$.

Let $\mathbf{X} = (X^i)_{i\in\mathbb{Z}}$ be a solution to ISDE (1.1) with $\beta = 2$ defined on a filtered space $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t\})$. We set

$$\mu^{\infty} = \mu \circ \mathfrak{l}^{-1} \tag{1.8}$$

and assume that

$$\mu^{\infty} = P(\mathbf{X}_0 \in \cdot). \tag{1.9}$$

The associated unlabeled process

$$\mathfrak{X} = \sum_{i\in\mathbb{Z}} \delta_{X^i}$$

is a μ-reversible diffusion given by the Dirichlet form $(\mathcal{E}, \mathcal{D})$ in (1.3) [24]. Note also that the labeled process $\mathbf{X} = \mathfrak{l}_{\text{path}}(\mathfrak{X})$ obtained by the Dirichlet form in [20, 24] coincides with the solution obtained by Tsai [29].

From (1.4) and $\mathbf{X} = \mathfrak{l}_{\text{path}}(\mathfrak{X})$ we find that

$$P(\mathbf{X} \in \mathrm{W}) = 1.$$

We set $\mathbf{w} = (w^i)_{i\in\mathbb{Z}} \in \mathrm{W}$ and

$$P^{\infty} = P \circ \mathbf{X}^{-1}, \quad P^{\infty}_{\mathbf{x}} = P^{\infty}(\cdot|\mathbf{w}_0 = \mathbf{x}). \tag{1.10}$$

Theorem 1 *$\{P^{\infty}_{\mathbf{x}}\}$ is irreducible. That is, if* $\mathbf{A}$ *and* $\mathbf{B} \in \mathscr{B}(\mathbb{R}^{\mathbb{Z}}_{<})$ *satisfy*

$$P^{\infty}(\mathbf{w}_0 \in \mathbf{A},\ \mathbf{w}_t \in \mathbf{B}) = 0, \tag{1.11}$$

then $P^{\infty}(\mathbf{w}_0 \in \mathbf{A}) = 0$ *or* $P^{\infty}(\mathbf{w}_t \in \mathbf{B}) = 0$.

We do not know whether $\mathbf{X}$ has an invariant probability measure that is absolutely continuous with respect to μ^{∞}. If this is the case, then Theorem 1 implies that $\mathbf{X}$ is irreducible in the usual sense.

From Theorem 1, (1.8), and (1.10), we immediately have the following.

Corollary 1 *The solution of* (1.1) *with* $\beta = 2$ *is irreducible in the sense that, if* $\mathbf{A}$ *and* $\mathbf{B} \in \mathscr{B}(\mathbb{R}^{\mathbb{Z}}_{<})$ *satisfy*

$$P(\mathbf{X}_0 \in \mathbf{A},\ \mathbf{X}_t \in \mathbf{B}) = 0,$$

then $P(\mathbf{X}_0 \in \mathbf{A}) = 0$ *or* $P(\mathbf{X}_t \in \mathbf{B}) = 0$.

To prove Theorem 1, we prepare two results, Theorems 2 and 3.

For $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}}$, we set $\mathbf{x}^m = (x^i)_{|i|<m}$ and $\mathbf{x}^{m*} = (x^i)_{|i|\geq m}$. We set

$$\mathbb{R}^{m}_{<} = \{\mathbf{x}^m = (x^i)_{|i|<m};\, x^i < x^{i+1} \quad \text{for all } -m < i < m-1\},$$
$$\mathbb{R}^{m*}_{<} = \{\mathbf{x}^{m*} = (x^i)_{|i|\geq m};\ x^i < x^{i+1} \quad \text{for all } i < -m,\ m \leq i\}.$$

Let $\mathbf{X}^m = (X^i)_{|i|<m}$ and $\mathbf{X}^{m*} = (X^i)_{|i|\geq m}$. Let $\mathbf{w}^{m*} = (w^i)_{|i|\geq m}$ for $\mathbf{w} = (w^i)_{i\in\mathbb{Z}}$. We introduce the regular conditional probabilities such that

$$\begin{aligned} P^{m}_{\mathbf{w}} &= P(\mathbf{X}^m \in \cdot\, |\mathbf{X}^{m*} = \mathbf{w}^{m*}), \\ P^{m}_{\mathbf{x},\mathbf{w}} &= P(\mathbf{X}^m \in \cdot|\mathbf{X}^m_0 = \mathbf{x}^m, \mathbf{X}^{m*} = \mathbf{w}^{m*}). \end{aligned} \tag{1.12}$$

By construction, $\mathbf{X}^m$ under $\{P^{m}_{\mathbf{x},\mathbf{w}}\}$ is a time-inhomogeneous diffusion. The heat equations describing the transition probability density are given by (3.20) and (3.21).

Theorem 2 *Let $P^{m*} = P \circ (\mathbf{X}^{m*})^{-1}$. For each $m \in \mathbb{N}$, $\{P^m_{\mathbf{x},\mathbf{w}}\}$ is irreducible for P^{m*}-a.s. $\mathbf{w}$. That is, if $\mathbf{A}$ and $\mathbf{B} \in \mathscr{B}(\mathbb{R}^m_<)$ satisfy*

$$P^m_{\mathbf{w}}(\mathbf{w}^m_0 \in \mathbf{A},\ \mathbf{w}^m_t \in \mathbf{B}) = 0 \quad \text{for } P^{m*}\text{-a.s. } \mathbf{w}, \tag{1.13}$$

then $P^m_{\mathbf{w}}(\mathbf{w}^m_0 \in \mathbf{A}) = 0$ for P^{m*}-a.s. $\mathbf{w}$ or $P^m_{\mathbf{w}}(\mathbf{w}^m_t \in \mathbf{B}) = 0$ for P^{m*}-a.s. $\mathbf{w}$.

Theorem 3 *Let $P^m = P \circ (\mathbf{X}^m)^{-1}$. For each $m \in \mathbb{N}$, the process P^m is irreducible. That is, if $\mathbf{A}$ and $\mathbf{B} \in \mathscr{B}(\mathbb{R}^m_<)$ satisfy*

$$P^m(\mathbf{w}^m_0 \in \mathbf{A},\ \mathbf{w}^m_t \in \mathbf{B}) = 0, \tag{1.14}$$

then $P^m(\mathbf{w}^m_0 \in \mathbf{A}) = 0$ or $P^m(\mathbf{w}^m_t \in \mathbf{B}) = 0$.

Let $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ and $\Psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ be measurable functions. A stochastic process given by a solution $\mathbf{X} = (X^i)_i$ of the ISDE

$$X^i_t - X^i_0 = B^i_t + \frac{1}{2}\int_0^t \nabla\Phi(X^i_u)du + \frac{1}{2}\int_0^t \sum_{j \neq 1} \nabla\Psi(X^i_u, X^j_u)du$$

is called an interacting Brownian motion (in infinite dimensions) with potential (Φ, Ψ). Here, $(\nabla\Psi)(x, y) = \nabla_x\Psi(x, y)$. The study of interacting Brownian motions was initiated by Lang [13, 14], who solved the ISDE for $(0, \Psi)$ with $\Psi \in C^3_0(\mathbb{R}^d)$ such that Ψ is of Ruelle's class in the sense that it is super stable and regular. Fritz [5] constructed non-equilibrium solutions for the same potentials as in [13, 14] with a further restriction that the dimension $d \le 4$. Tanemura solved the ISDE for the hard-core potential [28], while Fradon–Roelly–Tanemura solved the ISDE for the hard-core potential with long range interactions, but still of Ruelle's class [4]. Various ISDEs with logarithmic interaction potentials have also been solved [7, 11, 20, 22, 24, 25, 29].

There are fewer results for the irreducibility and ergodicity of solutions of interacting Brownian motions. Albeverio–Kondratiev–Röckner [1] proved the equivalence of the ergodicity of Dirichlet forms and the extremal property of the associated (grand canonical or canonical) Gibbs measures with potentials of Ruelle's class [26]. Corwin-Sun [2] proved the ergodicity of the Airy line ensembles, for which the dynamics are related to the Airy_2 random point field. A general result concerning the ergodicity of Dirichlet forms can be found in [6].

The remainder of this paper is organized as follows. In Sect. 2, we recall the concept of the m-labeled process and the Lyons–Zheng decomposition for interacting Brownian motions. In Sect. 3, we prove Theorem 2 and Theorem 3. Finally, in Sect. 4, we prove Theorem 1.

2 The *m*-Labeled Process and the Lyons–Zheng Decomposition

We introduce the m-labeled process $\mathbf{X}^{[m]} = (\mathbf{X}^m, \mathfrak{X}^{m*})$, where

$$\mathfrak{X}_t^{m*} = \sum_{|j|\ge m} \delta_{X_t^j}.$$

The process $\mathbf{X}^{[m]}$ is given by the Dirichlet form $(\mathcal{E}^{[m]}, \mathcal{D}^{[m]})$ on $L^2(\mathbb{R}_<^m \times \mathfrak{S}, \mu^{[m]})$ such that

$$\mathcal{E}^{[m]}(f, g) = \int_{\mathbb{R}_<^m \times \mathfrak{S}} \mathbb{D}^{[m]}[f, g]\, d\mu^{[m]}.$$

Here, $\mathbb{D}^{[m]}$ is the standard carré du champ on $\mathbb{R}_<^m \times \mathfrak{S}$ [24], $\mathcal{D}^{[m]}$ is the closure of

$$\{f_1 \otimes f_2 \in C_0^\infty \otimes \mathcal{D}_\circ;\ \mathcal{E}_1^{[m]}(f_1 \otimes f_2, f_1 \otimes f_2) < \infty\},$$

and $\mu^{[m]}$ is the m-reduced Campbell measure such that

$$\mu^{[m]}(A \times B) = \int_A \rho^m(\mathbf{x}^m)\mu_{\mathbf{x}^m}(B) d\mathbf{x}^m.$$

Moreover, ρ^m is the m-point correlation function of μ, and $\mu_{\mathbf{x}^m}$ is a reduced Palm measure conditioned at $\mathbf{x}^m = (x^i)_{|i|<m}$ given by

$$\mu_{\mathbf{x}^m} = \mu(\cdot - \sum_{|i|<m} \delta_{x^i} | \mathfrak{l}^i(\mathfrak{s}) = x^i, \quad \text{for all } |i| < m). \tag{2.1}$$

The standard definition of the reduced Palm measure $\mu_{\mathbf{x}^m}$ is

$$\mu_{\mathbf{x}^m} = \mu(\cdot - \sum_{|i|<m} \delta_{x^i} | \mathfrak{s}(\{x^i\}) \ge 1 \quad \text{for all } |i| < m).$$

Because the state space of process $\mathbf{X}$ is $\mathbb{R}_<^{\mathbb{Z}}$, we take $\mu_{\mathbf{x}^m}$ given by (2.1). We set

$$P_{\mathbf{x},\mathfrak{s}}^{[m]} = P(\mathbf{X}^{[m]} \in \cdot | \mathbf{X}_0^{[m]} = (\mathbf{x}, \mathfrak{s})).$$

From [19, 24], we have that $\{P_{\mathbf{x},\mathfrak{s}}^{[m]}\}$ is a diffusion associated with the Dirichlet form $(\mathcal{E}^{[m]}, \mathcal{D}^{[m]})$ on $L^2(\mathbb{R}_<^m \times \mathfrak{S}, \mu^{[m]})$. By construction, $\{P_{\mathbf{x},\mathfrak{s}}^{[m]}\}$ is $\mu^{[m]}$-symmetric and $\mu^{[m]}$ is an invariant measure of $\{P_{\mathbf{x},\mathfrak{s}}^{[m]}\}$. One of the most critical properties of $\{P_{\mathbf{x},\mathfrak{s}}^{[m]}\}$ is its consistency. To explain the consistency, we prepare some notations.

Let $\mathfrak{l}_{\mathrm{path}}$ be the path label introduced in Sect. 1. We write $\mathfrak{l}_{\mathrm{path}} = (\mathfrak{l}^i_{\mathrm{path}})_{i\in\mathbb{Z}}$. We set $\mathfrak{l}^{[m]}_{\mathrm{path}}$ from $\mathfrak{l}_{\mathrm{path}}$ as follows:

$$\mathfrak{l}^{[m]}_{\mathrm{path}}(\mathfrak{w})_t = ((\mathfrak{l}^i_{\mathrm{path}}(\mathfrak{w})_t)_{|i|<m}, \sum_{|j|\geq m} \delta_{\mathfrak{l}^j_{\mathrm{path}}(\mathfrak{w})_t}).$$

For an $\mathbb{R}^{[m]}$-valued path $\mathbf{w}^{[m]}$ such that $\mathbf{w}^{[m]}_t = ((w^i_t)_{|i|<m}, \sum_{|j|\geq m} \delta_{w^j_t})$, we set

$$\mathfrak{u}^{[m]}_{\mathrm{path}}(\mathbf{w}^{[m]})_t = \sum_{i\in\mathbb{Z}} \delta_{w^i_t}.$$

Clearly, $\mathfrak{u}^{[m]}_{\mathrm{path}}(\mathbf{w}^{[m]})_t = \mathfrak{u}(\mathbf{w}_t)$, where $\mathfrak{u}$ is the unlabeling map defined in Sect. 1. Additionally, $\mathfrak{u}(\mathbf{x}, \mathfrak{s}) = \sum_i \delta_{x^i} + \mathfrak{s}$ for $\mathbf{x} = (x^i)$.

We have the following consistency.

Lemma 1 *For each* $m \in \{0\} \cup \mathbb{N}$,

$$P^{[m]}_{\mathbf{x},\mathfrak{s}} \circ (\mathfrak{u}^{[m]}_{\mathrm{path}})^{-1} = P^{[0]}_{\mathfrak{u}(\mathbf{x},\mathfrak{s})},$$
$$P^{[0]}_{\mathfrak{s}} \circ (\mathfrak{l}^{[m]}_{\mathrm{path}})^{-1} = P^{[m]}_{\mathfrak{l}_m(\mathfrak{s})}.$$

Proof Applying Theorem 2.4 in [19] to Dyson's model, we obtain Lemma 1. □

Let $\mathbf{x}^{[m]} = (\mathbf{x}^m, \mathfrak{x}^{m*})$ for $\mathbf{x} = (x^i)_{i\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}_{<}$, where $\mathfrak{x}^{m*} = \sum_{|i|\geq m} \delta_{x^i}$.

Lemma 2 *For each* $m \in \{0\} \cup \mathbb{N}$

$$P^{\infty}_{\mathbf{x}} \circ (\mathfrak{u}^{[m]}_{\mathrm{path}})^{-1} = P^{[m]}_{\mathbf{x}^{[m]}}.$$

Proof Lemma 2 follows from (1.10) and Lemma 1. □

Note that w^j under $P^{[m]}$ is a solution to SDE (1.1) for $|j| < m$. Thus, the martingale term of the Fukushima decomposition of w^j describes Brownian motion. Hence, applying the Lyons–Zheng decomposition to w^j under $P^{[m]}$, we obtain

$$w^j_t - w^j_u = \frac{1}{2}\{B^j_t - B^j_u + \hat{B}^j_t - \hat{B}^j_u\} \quad \text{for } 0 \leq t, u \leq T, \tag{2.2}$$

where $\hat{B}^j_t = B^j_{T-t}$. Because $P^{[m]}$ is a symmetric diffusion, $\hat{B}^j_t$ describes Brownian motion. Furthermore, $\{B^j_t\}_{|j|<m}$ is a sequence of independent Brownian motions under $P^{[m]}$. Because $\hat{B}^j_t$ is a time reversal of B^j_t, $\{\hat{B}^j_t\}_{|j|<m}$ is a sequence of independent Brownian motions under $P^{[m]}$. We refer to Sect. 9 in [12] for the proof of the Lyons–Zheng decomposition of this form. Because of the consistency in Lemma 2, we have that (2.2) holds for all $j \in \mathbb{Z}$ under P^{∞}. Furthermore, $\{B^j_t\}_{j\in\mathbb{Z}}$ and $\{\hat{B}^j_t\}_{j\in\mathbb{Z}}$ are sequences of independent Brownian motions under P^{∞}. Thus, $\{B^j_t - B^j_u\}_{j\in\mathbb{Z}}$

and $\{\hat{B}_t^j - \hat{B}_u^j\}_{j\in\mathbb{Z}}$ are sequences of increments of independent Brownian motions. Collecting these statements together, we obtain the following.

Lemma 3 *(1) For each $j \in \mathbb{Z}$, we have (2.2) for P^∞-a.s.*
(2) $\{B_t^j - B_u^j\}_{j\in\mathbb{Z}}$ and $\{\hat{B}_t^j - \hat{B}_u^j\}_{j\in\mathbb{Z}}$ are sequences of increments of independent Brownian motions under P^∞.

3 Proof of Theorems 2 and 3

Let d^μ be the logarithmic derivative of μ. By definition, d^μ is a function defined on $\mathbb{R} \times \mathfrak{S}$ such that $\mathsf{d}^\mu \in L^1_{\text{loc}}(\mu^{[1]})$ and

$$\int_{\mathbb{R}\times\mathfrak{S}} \mathsf{d}^\mu(s, \mathfrak{s})\varphi(s, \mathfrak{s})d\mu^{[1]} = -\int_{\mathbb{R}\times\mathfrak{S}} \nabla\varphi(s, \mathfrak{s})d\mu^{[1]}$$

for all $\varphi \in C_0^\infty(\mathbb{R}) \otimes \mathcal{D}_\circ^b$, where $\mathcal{D}_\circ^b$ is the set consisting of bounded, local, and smooth functions on $\mathfrak{S}$ [24]. We write $\mathfrak{s} = \sum_i \delta_{s^i}$. In [20], it is proved that μ has a logarithmic derivative such that

$$\begin{aligned} \mathsf{d}^\mu(s, \mathfrak{s}) &= 2 \lim_{R\to\infty} \sum_{s^i\in S_R} \frac{1}{s - s^i} \quad \text{in } L^2_{\text{loc}}(\mu^{[1]}) \\ &= 2 \lim_{R\to\infty} \sum_{|s-s^i|<R} \frac{1}{s - s^i} \quad \text{in } L^2_{\text{loc}}(\mu^{[1]}). \end{aligned} \tag{3.1}$$

The sums in (3.1) converge because μ is translation invariant, $d = 1$, and the variance of $\mathfrak{s}([-R, R])$ under μ increases logarithmically as $R \to \infty$. The second equality in (3.1) comes from $d = 1$.

Note that the Ginibre random point field μ_{gin} satisfies the following [20]:

$$\begin{aligned} \mathsf{d}^{\mu_{\text{gin}}}(s, \mathfrak{s}) &= -2s + 2 \lim_{R\to\infty} R \sum_{s^i\in S_R} \frac{s - s^i}{|s - s^i|^2} \quad \text{in } L^2_{\text{loc}}(\mu^{[1]}_{\text{gin}}) \\ &= 2 \lim_{R\to\infty} R \sum_{|s-s^i|<R} \frac{s - s^i}{|s - s^i|^2} \quad \text{in } L^2_{\text{loc}}(\mu^{[1]}_{\text{gin}}). \end{aligned} \tag{3.2}$$

The Ginibre random point field μ_{gin} is the counterpart of μ in $\mathbb{R}^2$, because μ_{gin} is rotation- and translation-invariant, and the interaction potential of μ_{gin} is the logarithmic potential with an inverse temperature of $\beta = 2$. Compare (3.1) and (3.2). The first equalities in (3.1) and (3.2) have different expressions according to the dimension d.

Recall that $\beta = 2$. Then, the ISDE in question is given by

$$X_t^i - X_0^i = B_t^i + \int_0^t \lim_{r\to\infty} \sum_{|X_u^i - X_u^j| < r,\ j\neq i} \frac{1}{X_u^i - X_u^j} du \quad (i \in \mathbb{Z}). \tag{3.3}$$

Using (3.1) and (3.3), we have

$$X_t^i - X_0^i = B_t^i + \frac{1}{2}\int_0^t \mathsf{d}^{\mu}(X_u^i, \sum_{j\neq i} \delta_{X_u^j}) du \quad (i \in \mathbb{Z}). \tag{3.4}$$

From (3.1), (3.3), and (3.4), it is easy to see that, $i \in \mathbb{Z}$,

$$\begin{aligned} X_t^i - X_0^i &= B_t^i + \int_0^t \sum_{|j|<m,\ j\neq i}^{m} \frac{1}{X_u^i - X_u^j} du + \int_0^t \lim_{r\to\infty} \sum_{\substack{|X_u^i - X_u^j| < r \\ m\le |j|,\ j\neq i}} \frac{1}{X_u^i - X_u^j} du \\ &= B_t^i + \int_0^t \sum_{|j|<m,\ j\neq i}^{m} \frac{1}{X_u^i - X_u^j} du + \int_0^t \lim_{n\to\infty} \sum_{m\le |j|\le n,\ j\neq i} \frac{1}{X_u^i - X_u^j} du. \end{aligned}$$

Taking this equation into account, we set $b_{\mathbf{w}}^m = (b_{\mathbf{w}}^{m,i})_{|i|<m}$ such that

$$b_{\mathbf{w}}^{m,i}(\mathbf{x}^m, t) = \sum_{\substack{j\neq i \\ |j|<m}} \frac{1}{x^i - x^j} + \lim_{n\to\infty} \sum_{m\le |j|\le n} \frac{1}{x^i - w_t^j}. \tag{3.5}$$

Let $\mathbf{x}^m = (x^i)_{|i|<m}$, $\mathbf{x} = (x^i)_{i\in\mathbb{Z}}$, and $\mathbf{y} = (y^i)_{i\in\mathbb{Z}}$. For $\mathbf{y} \in \mathbb{R}_<^{\mathbb{Z}}$, we set

$$\mathbb{R}_<^m(\mathbf{y}) = \{\mathbf{x}^m \in \mathbb{R}_<^m\,;\ y^{-m} < x^{-m+1},\ x^{m-1} < y^m\}.$$

Let $\mathscr{O}_{T,\mathbf{w}}^m$ be a time-dependent open set in $\mathbb{R}_<^m$ such that

$$\mathscr{O}_{T,\mathbf{w}}^m = \{(\mathbf{x}^m, t) \in \mathbb{R}_<^m \times [0, T]\,;\ \mathbf{x}^m \in \mathbb{R}_<^m(\mathbf{w}_t)\}.$$

For $(\mathbf{x}^m, t)$, we set $\mathbf{x}_t^m = (x_t^i)_{|i|<m}$ such that $(\mathbf{x}_t^m, t) = (\mathbf{x}^m, t)$. For $\epsilon \geq 0$, $m, T \in \mathbb{N}$, and $\mathbf{w} \in \mathrm{W}$, we set

$$\begin{aligned} \mathscr{O}_{T,\mathbf{w}}^{m,\epsilon} = \{(\mathbf{x}^m, t) \in \mathscr{O}_{T,\mathbf{w}}^m\,; |x_t^i - x_t^{i+1}| > \epsilon,\ -m < i < m-1 \\ |x_t^{-m+1} - w_t^{-m}| > \epsilon,\ |x_t^{m-1} - w_t^m| > \epsilon\}. \end{aligned} \tag{3.6}$$

Suppose that $\epsilon > 0$ and that $\mathscr{O}_{T,\mathbf{w}}^{m,\epsilon}$ is nonempty and connected. For P^{∞}-a.s. $\mathbf{w}$, we find a connected open set $\mathscr{Q}_{T,\mathbf{w}}^{m,\epsilon}$ in $\mathbb{R}_<^m \times [0, T]$ with smooth boundary such that

$$\mathscr{O}_{T,\mathbf{w}}^{m,\epsilon} \subset \mathscr{Q}_{T,\mathbf{w}}^{m,\epsilon} \subset \mathscr{O}_{T,\mathbf{w}}^{m,\epsilon/2}. \tag{3.7}$$

Lemma 4 *For each $T, m \in \mathbb{N}$, and P^{∞}-a.s. $\mathbf{w}$, the following hold.*

(1) $b_{\mathbf{w}}^{m}(\mathbf{x}^m, t)$ *is Hölder continuous in t in $\mathscr{Q}_{T,\mathbf{w}}^{m,\epsilon}$ for each $\mathbf{x}^m$.*
(2) $b_{\mathbf{w}}^{m}(\mathbf{x}^m, t)$ *is Lipschitz continuous in $\mathbf{x}^m$ in $\mathscr{Q}_{T,\mathbf{w}}^{m,\epsilon}$.*

Proof Let $(\mathbf{x}^m, t), (\mathbf{x}^m, u) \in \mathscr{Q}_{T,\mathbf{w}}^{m,\epsilon}$ and fix i such that $|i| < m$. Then from (3.5)

$$\begin{aligned} b_{\mathbf{w}}^{m,i}(\mathbf{x}^m, t) - b_{\mathbf{w}}^{m,i}(\mathbf{x}^m, u) &= \sum_{|j|\geq m} \frac{1}{x^i - w_t^j} - \sum_{|j|\geq m} \frac{1}{x^i - w_u^j} \\ &= \sum_{|j|\geq m} \frac{w_t^j - w_u^j}{(x^i - w_t^j)(x^i - w_u^j)}. \end{aligned} \tag{3.8}$$

From Lemma 3, we can deduce for P^{∞}-a.s. that

$$w_t^j - w_u^j = \frac{1}{2}\{B_t^j - B_u^j + \hat{B}_t^j - \hat{B}_u^j\}, \tag{3.9}$$

where $\{B_t^j - B_u^j\}_{j\in\mathbb{Z}}$ and $\{\hat{B}_t^j - \hat{B}_u^j\}_{j\in\mathbb{Z}}$ are sequences of increments of independent Brownian motions under P^{∞}. From (3.8) and (3.9), we have that

$$\begin{aligned} b_{\mathbf{w}}^{m,i}(\mathbf{x}^m, t) - b_{\mathbf{w}}^{m,i}(\mathbf{x}^m, u) &= \frac{1}{2} \sum_{|j|\geq m} \frac{B_t^j - B_u^j + \hat{B}_t^j - \hat{B}_u^j}{(x^i - w_t^j)(x^i - w_u^j)} \\ &= \frac{1}{2} \sum_{|j|\geq m} \frac{B_t^j - B_u^j}{(x^i - w_t^j)(x^i - w_u^j)} + \frac{1}{2} \sum_{|j|\geq m} \frac{\hat{B}_t^j - \hat{B}_u^j}{(x^i - w_t^j)(x^i - w_u^j)}. \end{aligned} \tag{3.10}$$

Let W be as in (1.7). To control the denominator in (3.10), we set

$$A_n = \Big\{\mathbf{w} \in \mathrm{W}\,;\ \big\{\min_{t\in[a,b]} |x^i - w_t^j|\big\} \geq \frac{|j|}{n} \text{ for all } |j| \geq m\Big\}. \tag{3.11}$$

Using (3.11), we deduce that, for P^{∞}-a.s. $\mathbf{w} \in A_n$,

$$\begin{aligned} \sup_{t\in[a,b]} \Big\{ \sum_{|j|\geq m} \frac{|j|}{|x^i - w_t^j|^3} \Big\} &\leq \Big\{ \sum_{|j|\geq m} \frac{|j|}{\min_{t\in[a,b]} |x^i - w_t^j|^3} \Big\} \\ &\leq \Big\{ \sum_{|j|\geq m} \frac{|j|}{(\frac{|j|}{n})^3} \Big\} \quad \text{by (3.11)} \\ &= n^3 \Big\{ \sum_{|j|\geq m} \frac{1}{|j|^2} \Big\} < \infty. \end{aligned} \tag{3.12}$$

We set $Q(\mathbf{x}^m) = \{\mathbf{w} \in \mathrm{W}\,;\ \mathscr{O}_{T,\mathbf{w}}^{m,\epsilon/2} \cap (\{\mathbf{x}^m\} \times [0,T]) \neq \emptyset\}$. Then using (1.6), (3.6), and (3.7), we deduce

$$P^{\infty}(\{\bigcup_{n\in\mathbb{N}} A_n\}^c\,;\ Q(\mathbf{x}^m)) = 0. \tag{3.13}$$

Hence, from (3.12) and (3.13), we obtain, for P^{∞}-a.s. $\mathbf{w} \in Q(\mathbf{x}^m)$,

$$c_1(\mathbf{w}) := \sup_{t\in[a,b]} \Big\{ \sum_{|j|\geq m} \frac{|j|}{|x^i - w_t^j|^3} \Big\} < \infty. \tag{3.14}$$

Using Young's inequality and (3.14), we have

$$\begin{aligned} &\sum_{|j|\geq m} \Big| \frac{B_t^j - B_u^j}{(x^i - w_t^j)(x^i - w_u^j)} \Big| \qquad (3.15)\\ \leq&\Big(\sup_{t\in[a,b]} \sum_{|j|\geq m} \frac{|j|}{|x^i - w_t^j|^3} \Big)^{1/3} \Big(\sup_{u\in[a,b]} \sum_{|j|\geq m} \frac{|j|}{|x^i - w_u^j|^3} \Big)^{1/3} \Big(\sum_{|j|\geq m} \frac{|B_t^j - B_u^j|^3}{|j|^2} \Big)^{1/3} \\ =&c_1(\mathbf{w})^{2/3} \Big(\sum_{|j|\geq m} \frac{|B_t^j - B_u^j|^3}{|j|^2} \Big)^{1/3}. \end{aligned}$$

Similarly, for P^{∞}-a.s. $\mathbf{w} \in Q(\mathbf{x}^m)$, we have that

$$\sum_{|j|\geq m} \Big| \frac{\hat{B}_t^j - \hat{B}_u^j}{(x^i - w_t^j)(x^i - w_u^j)} \Big| \leq c_1(\mathbf{w})^{2/3} \Big(\sum_{|j|\geq m} \frac{|\hat{B}_t^j - \hat{B}_u^j|^3}{|j|^2} \Big)^{1/3}. \tag{3.16}$$

Recall that $c_t(\mathbf{w}) < \infty$ for P^{∞}-a.s. $\mathbf{w} \in Q(\mathbf{x}^m)$ from (3.14). Note that $\{B_t^j\}_{j\in\mathbb{Z}}$ and $\{\hat{B}_t^j\}_{j\in\mathbb{Z}}$ are sequences of independent Brownian motions. Then, we obtain Lemma 4 (1) from (3.10), (3.15), and (3.16).

Let $(\mathbf{x}^m, t)$ and $(\mathbf{y}^m, t) \in \mathscr{Q}_{T,\mathbf{w}}^{m,\epsilon}$. From (3.5), we have that

$$\begin{aligned} &b_{\mathbf{w}}^{m,i}(\mathbf{x}^m, t) - b_{\mathbf{w}}^{m,i}(\mathbf{y}^m, t) \qquad (3.17)\\ =&\sum_{\substack{j\neq i\\ |j|<m}} \frac{1}{x^i - x^j} - \sum_{\substack{j\neq i\\ |j|<m}} \frac{1}{y^i - y^j} + \sum_{|j|\geq m} \frac{1}{x^i - w_t^j} - \sum_{|j|\geq m} \frac{1}{y^i - w_t^j} \\ =&\sum_{\substack{j\neq i\\ |j|<m}} \frac{1}{x^i - x^j} - \sum_{\substack{j\neq i\\ |j|<m}} \frac{1}{y^i - y^j} + \sum_{|j|\geq m} \frac{y^i - x^i}{(x^i - w_t^j)(y^i - w_t^j)}. \end{aligned}$$

Then, using (1.6) and (3.17), we obtain (2). □

We define the probability measure on $\mathbb{R}^m_<$ by

$$P^m_{\mathbf{w}}(t) = P^\infty(\mathbf{w}^m_t \in \cdot \,|\mathbf{w}^{m*}_0). \tag{3.18}$$

Let $\mathscr{O}^{m,\epsilon}_{T,\mathbf{w}}$ be as in (3.6). Let $\mathscr{O}^{m,\epsilon}_{T,\mathbf{w}}(t)$ be the cross section of $\mathscr{O}^{m,\epsilon}_{T,\mathbf{w}}$ such that

$$\mathscr{O}^{m,\epsilon}_{T,\mathbf{w}}(t) = \{\mathbf{x}^m \in \mathbb{R}^m_<;\ (\mathbf{x}^m, t) \in \mathscr{O}^{m,\epsilon}_{T,\mathbf{w}}\}. \tag{3.19}$$

Proof of Theorem 2 We consider the time-inhomogeneous heat equation on $\mathscr{Q}^{m,\epsilon}_{T,\mathbf{w}}$ such that the associated backward equation is given by

$$\begin{aligned}\Big\{\frac{\partial}{\partial t} + \frac{1}{2}\sum_{|i|<m}(\frac{\partial}{\partial x^i})^2 + \sum_{|i|,|j|<m,i\neq j}\frac{1}{x^i - x^j}\frac{\partial}{\partial x^i} \\ + \sum_{|i|<m\leq|j|}\frac{1}{x^i - w^j_t}\frac{\partial}{\partial x^i}\Big\}p(t,x,u,y) = 0\end{aligned} \tag{3.20}$$

and the forward equation is given by

$$\begin{aligned}\Big\{\frac{\partial}{\partial u} - \frac{1}{2}\sum_{|i|<m}(\frac{\partial}{\partial x^i})^2 - \sum_{|i|,|j|<m,i\neq j}\frac{1}{x^i - x^j}\frac{\partial}{\partial x^i} \\ - \sum_{|i|<m\leq|j|}\frac{1}{x^i - w^j_u}\frac{\partial}{\partial x^i}\Big\}p(t,x,u,y) = 0.\end{aligned} \tag{3.21}$$

From Lemma 4, we have constants c_2 and α such that $0 < \alpha < 1$ and

$$|b^{m,i}_{\mathbf{w}}(\mathbf{x}^m, t) - b^{m,i}_{\mathbf{w}}(\mathbf{y}^m, u)| \leq c_2\{|\mathbf{x}^m - \mathbf{y}^m| + |t-u|^\alpha\}. \tag{3.22}$$

From (3.22), we can apply a general theorem of heat equations to determine that the fundamental solution (the transition probability density) of (3.20) and (3.21) on $\mathscr{O}^{m,\epsilon}_{T,\mathbf{w}}$ under a Dirichlet boundary condition on the boundary is positive and continuous. Taking $\epsilon \to 0$ and using the obvious inequality such that the heat kernel dominates that with the Dirichlet boundary condition, we find that the heat kernel $p(t,x,u,y) = p^{m,0}_{T,\mathbf{w}}(t,x,u,y)$ on $\mathscr{O}^{m,0}_{T,\mathbf{w}}(t) \times \mathscr{O}^{m,0}_{T,\mathbf{w}}(u)$ is a positive density of the transition probability with respect to the Lebesgue measure. Using (1.13), we find

$$\int_{\mathbf{A}\times\mathbf{B}} p(0,x,t,y)dxdy = 0. \tag{3.23}$$

From (3.23) and positivity of p, we deduce that $\mathbf{A}$ or $\mathbf{B}$ have Lebesgue measure zero. Hence, either of the following hold:

$$P^m_{\mathbf{w}}(\mathbf{w}^m_0 \in \mathbf{A}) = \int_{\mathbf{A}\times\mathscr{O}^{m,0}_{T,\mathbf{w}}(t)} p(0,x,t,y)dxdy = 0 \tag{3.24}$$

or

$$P^m_{\mathbf{w}}(\mathbf{w}^m_t \in \mathbf{B}) = \int_{\mathscr{O}^{m,0}_{T,\mathbf{w}}(0)\times\mathbf{B}} p(0,x,t,y)dxdy = 0. \tag{3.25}$$

We thus obtain Theorem 2. □

Proof of Theorem 3 Using (1.14) and Fubini's theorem, we deduce (1.13). Then, applying Theorem 2, we have that

$$P^m_{\mathbf{w}}(\mathbf{w}^m_0 \in \mathbf{A}) = 0 \quad \text{for } P^{m*}\text{-a.s.}\mathbf{w}$$

or

$$P^m_{\mathbf{w}}(\mathbf{w}^m_t \in \mathbf{B}) = 0 \quad \text{for } P^{m*}\text{-a.s. w}.$$

Integrating these with respect to P^{m*}, we conclude that Theorem 3 holds from (1.12). □

4 Proof of Theorem 1

Let $\mathrm{W}^{m*} = C([0,\infty); \mathbb{R}^{m*}_{<})$. Let $\varpi^{m*} : \mathrm{W} \to \mathrm{W}^{m*}$ be the projection such that $\mathbf{w} = (w^i)_{i\in\mathbb{Z}} \mapsto \mathbf{w}^{m*} = (w^i)_{|i|\ge m}$. Let

$$\mathbf{T} = \{\mathbf{t} = (t_1,\ldots,t_l)\,;\ 0 < t_k < t_{k+1}\ (1 \le k < l),\ \ l \in \mathbb{N}\}.$$

We set $\varpi^{m*}_{\mathbf{t}}(\mathbf{w}) = \mathbf{w}^{m*}_{\mathbf{t}} = (w^i_{\mathbf{t}})_{|i|\ge m}$, where $w^i_{\mathbf{t}} = (w^i_{t_1},\ldots,w^i_{t_l})$, and

$$\mathscr{C}^{\infty*}_{\text{path}} = \bigvee_{\mathbf{t}\in\mathbf{T}} \bigcap_{m=1}^{\infty} \sigma[\varpi^{m*}_{\mathbf{t}}].$$

We know that μ is tail trivial [15, 23]. That is, $\mu(A) \in \{0,1\}$ for each $A \in \mathcal{T}(\mathfrak{S})$, where

$$\mathcal{T}(\mathfrak{S}) = \bigcap_{R=1}^{\infty} \sigma[\pi^c_R].$$

The tail triviality of μ can be refined to the triviality of $\mathscr{C}_{\mathrm{path}}^{\infty *}$ with respect to P^{∞} using Lemma 5. The triviality of $\mathscr{C}_{\mathrm{path}}^{\infty *}$ with respect to P^{∞} is one of the critical properties in the proof of the uniqueness of solutions to ISDEs in [24]. We require a rather difficult argument for the proof of this fact.

Lemma 5 *$\mathscr{C}_{\mathrm{path}}^{\infty *}$ is trivial with respect to P^{∞}. That is,*

$$P^{\infty}(\mathscr{A}) \in \{0, 1\} \textit{ for each } \mathscr{A} \in \mathscr{C}_{\mathrm{path}}^{\infty *}.$$

Proof Lemma 5 follows directly from Theorem 5.3 in [24]. □

Proof of Theorem 1 Recall that $\varpi_t^{m*}(\mathbf{w}) = (w_t^i)_{m \le |i|}$ for $\mathbf{w} = (w^i)_{i \in \mathbb{Z}}$. We set

$$\mathscr{F}_{0,t}^{m*} = \sigma[\varpi_0^{m*}, \varpi_t^{m*}].$$

Let $\varpi^m : \mathrm{W} \to \mathrm{W}^m$ be the projection such that $\mathbf{w} = (w^i)_{i \in \mathbb{Z}} \mapsto \mathbf{w}^m = (w^i)_{|i|<m}$, where $\mathrm{W}^m = C([0, \infty); \mathbb{R}_<^m)$. We set $\varpi_u(\mathbf{w}) = \mathbf{w}_u$, $\varpi_u^m(\mathbf{w}) = \mathbf{w}_u^m$, and $\varpi_u^{m*}(\mathbf{w}) = \mathbf{w}_u^{m*}$. For $\mathscr{A}$ and $\mathscr{B} \subset \mathrm{W}$, we set

$$\begin{aligned}
\mathbf{A}_0 &= \varpi_0(\mathscr{A}), & \mathbf{A}_0^m &= \varpi_0^m(\mathscr{A}), & \mathbf{A}_0^{m*} &= \varpi_0^{m*}(\mathscr{A}),\\
\mathbf{B}_t &= \varpi_t(\mathscr{B}), & \mathbf{B}_t^m &= \varpi_t^m(\mathscr{B}), & \mathbf{B}_t^{m*} &= \varpi_t^{m*}(\mathscr{B}).
\end{aligned}$$

Let $\mathbf{A}$ and $\mathbf{B}$ be as in the statement of Theorem 1. We take $\mathscr{A} = \varpi_0^{-1}(\mathbf{A})$ and $\mathscr{B} = \varpi_t^{-1}(\mathbf{B})$. Then we find $\mathbf{A} = \mathbf{A}_0$ and $\mathbf{B} = \mathbf{B}_t$. Noting $\mathbf{A}_0^{m*}, \mathbf{B}_t^{m*} \in \mathscr{F}_{0,t}^{m*}$ and using (1.12), we deduce that, for P^{m*}-a.s. $\mathbf{w}$,

$$\begin{aligned}
&P_{\mathbf{w}}^m(\mathbf{w}_0^m \in \mathbf{A}_0^m, \mathbf{w}_t^m \in \mathbf{B}_t^m | \mathscr{F}_{0,t}^{m*}) 1_{\mathbf{A}_0^{m*}}(\mathbf{w}_0^{m*}) 1_{\mathbf{B}_t^{m*}}(\mathbf{w}_t^{m*}) \\
=&P_{\mathbf{w}}^m(\mathbf{w}_0^m \in \mathbf{A}_0^m, \mathbf{w}_t^m \in \mathbf{B}_t^m, \mathbf{w}_0^{m*} \in \mathbf{A}_0^{m*}, \mathbf{w}_t^{m*} \in \mathbf{B}_t^{m*} | \mathscr{F}_{0,t}^{m*}) \\
=&P_{\mathbf{w}}^m(\mathbf{w}_0 \in \mathbf{A}_0, \mathbf{w}_t \in \mathbf{B}_t | \mathscr{F}_{0,t}^{m*}) \\
=&P^{\infty}(\mathbf{w}_0 \in \mathbf{A}_0, \mathbf{w}_t \in \mathbf{B}_t | \mathscr{F}_{0,t}^{m*}) \quad \text{by (1.12)}.
\end{aligned} \tag{4.1}$$

From (4.1) and (1.11), we have

$$\begin{aligned}
&\int_{\mathrm{W}} P_{\mathbf{w}}^m(\mathbf{w}_0^m \in \mathbf{A}_0^m, \mathbf{w}_t^m \in \mathbf{B}_t^m | \mathscr{F}_{0,t}^{m*}) 1_{\mathbf{A}_0^{m*}}(\mathbf{w}_0^{m*}) 1_{\mathbf{B}_t^{m*}}(\mathbf{w}_t^{m*}) P^{m*}(d\mathbf{w}) \\
=&\int_{\mathrm{W}} P^{\infty}(\mathbf{w}_0 \in \mathbf{A}_0, \mathbf{w}_t \in \mathbf{B}_t | \mathscr{F}_{0,t}^{m*}) P^{m*}(d\mathbf{w}) \quad \text{by (4.1)} \\
=&P^{\infty}(\mathbf{w}_0 \in \mathbf{A}_0, \mathbf{w}_t \in \mathbf{B}_t) = 0 \quad \text{by (1.11)}.
\end{aligned}$$

Using this, we obtain, for P^{m*}-a.s. $\mathbf{w}$,

$$P_{\mathbf{w}}^m(\mathbf{w}_0^m \in \mathbf{A}_0^m,\ \mathbf{w}_t^m \in \mathbf{B}_t^m | \mathscr{F}_{0,t}^{m*}) 1_{\mathbf{A}_0^{m*}}(\mathbf{w}_0^{m*}) 1_{\mathbf{B}_t^{m*}}(\mathbf{w}_t^{m*}) = 0.$$

From this, we easily deduce, for P^{m*}-a.s. $\mathbf{w}$,

$$P_{\mathbf{w}}^m(\mathbf{w}_0^m \in \mathbf{A}_0^m,\ \mathbf{w}_t^m \in \mathbf{B}_t^m)1_{\mathbf{A}_0^{m*}}(\mathbf{w}_0^{m*})1_{\mathbf{B}_t^{m*}}(\mathbf{w}_t^{m*}) = 0. \tag{4.2}$$

Using (4.2) and Theorem 2, we deduce

$$P_{\mathbf{w}}^m(\mathbf{w}_0^m \in \mathbf{A}_0^m)1_{\mathbf{A}_0^{m*}}(\mathbf{w}_0^{m*})1_{\mathbf{B}_t^{m*}}(\mathbf{w}_t^{m*}) = 0 \quad \text{for } P^{m*} - \text{a.s. } \mathbf{w} \tag{4.3}$$

or

$$P_{\mathbf{w}}^m(\mathbf{w}_t^m \in \mathbf{B}_t^m)1_{\mathbf{A}_0^{m*}}(\mathbf{w}_0^{m*})1_{\mathbf{B}_t^{m*}}(\mathbf{w}_t^{m*}) = 0 \quad \text{for } P^{m*} - \text{a.s. } \mathbf{w}. \tag{4.4}$$

Suppose (4.3). Then, using $P^{m*} = P^{\infty} \circ (\varpi^{m*})^{-1}$, we obtain

$$P_{\mathbf{w}}^m(\mathbf{w}_0^m \in \mathbf{A}_0^m)1_{\mathbf{A}_0^{m*}}(\mathbf{w}_0^{m*})1_{\mathbf{B}_t^{m*}}(\mathbf{w}_t^{m*}) = 0 \quad \text{for } P^{\infty} - \text{a.s. } \mathbf{w}. \tag{4.5}$$

Taking $\mathscr{B} = \mathrm{W}$ in (4.1) and using $P^{m*} = P^{\infty} \circ (\varpi^{m*})^{-1}$, we obtain

$$P_{\mathbf{w}}^m(\mathbf{w}_0^m \in \mathbf{A}_0^m)1_{\mathbf{A}_0^{m*}}(\mathbf{w}_0^{m*}) = P^{\infty}(\mathbf{w}_0 \in \mathbf{A}_0|\mathscr{F}_{0,t}^{m*}) \quad \text{for } P^{\infty} - \text{a.s. } \mathbf{w}.$$

Hence, we deduce

$$\begin{aligned} &P_{\mathbf{w}}^m(\mathbf{w}_0^m \in \mathbf{A}_0^m)1_{\mathbf{A}_0^{m*}}(\mathbf{w}_0^{m*})1_{\mathbf{B}_t^{m*}}(\mathbf{w}_t^{m*}) \\ =&P^{\infty}(\mathbf{w}_0 \in \mathbf{A}_0|\mathscr{F}_{0,t}^{m*})1_{\mathbf{B}_t^{m*}}(\mathbf{w}_t^{m*}) \quad \text{for } P^{\infty} - \text{a.s. } \mathbf{w}. \end{aligned} \tag{4.6}$$

From (4.5) and (4.6), we obtain

$$P^{\infty}(\mathbf{w}_0 \in \mathbf{A}_0|\mathscr{F}_{0,t}^{m*})1_{\mathbf{B}_t^{m*}}(\mathbf{w}_t^{m*}) = 0 \quad \text{for } P^{\infty}\text{-a.s. } \mathbf{w} \tag{4.7}$$

Integrating (4.7) with respect to P^{∞}, we obtain

$$\int_{\mathrm{W}} P^{\infty}(\mathbf{w}_0 \in \mathbf{A}_0|\mathscr{F}_{0,t}^{m*})1_{\mathbf{B}_t^{m*}}(\mathbf{w}_t^{m*})P^{\infty}(d\mathbf{w}) = 0. \tag{4.8}$$

Next, suppose (4.4). Then, similarly as (4.8), we obtain

$$\int_{\mathrm{W}} P^{\infty}(\mathbf{w}_t \in \mathbf{B}_t|\mathscr{F}_{0,t}^{m*})1_{\mathbf{A}_0^{m*}}(\mathbf{w}_0^{m*})P^{\infty}(d\mathbf{w}) = 0. \tag{4.9}$$

Thus, we see either (4.8) or (4.9) holds for each $m \in \mathbb{N}$. Hence, (4.8) holds for infinitely many $m \in \mathbb{N}$ or (4.9) holds for infinitely many $m \in \mathbb{N}$.

Note that the sequence of σ-fields $\{\mathscr{F}_{0,t}^{m*}\}_{m\in\mathbb{N}}$ is decreasing. Furthermore, the sequences of sets

$$\{(\varpi_0^{m*})^{-1}(\mathbf{A}_0^{m*})\}_{m\in\mathbb{N}} \quad \text{and} \quad \{(\varpi_t^{m*})^{-1}(\mathbf{B}_t^{m*})\}_{m\in\mathbb{N}}$$

are increasing and the limits

$$\widetilde{\mathscr{A}} := \bigcup_{m=1}^{\infty}(\varpi_0^{m*})^{-1}(\mathbf{A}_0^{m*}) \quad \text{and} \quad \widetilde{\mathscr{B}} := \bigcup_{m=1}^{\infty}(\varpi_t^{m*})^{-1}(\mathbf{B}_t^{m*})$$

are $\mathscr{C}_{\mathrm{path}}^{\infty *}$-measurable. The sets $\widetilde{\mathscr{A}}$ and $\widetilde{\mathscr{B}}$ contain $\mathscr{A}$ and $\mathscr{B}$, respectively. Hence, using the martingale convergence theorem and the Lebesgue convergence theorem, we find that, P^{∞}-a.s. and in $L^1(\mathrm{W}, P^{\infty})$,

$$\begin{aligned} &\lim_{m\to\infty} P^{\infty}(\mathbf{w}_0 \in \mathbf{A}_0 | \mathscr{F}_{0,t}^{m*}) 1_{\mathbf{B}_t^{m*}}(\mathbf{w}_t^{m*}) \\ &= P^{\infty}(\mathbf{w}_0 \in \mathbf{A}_0 | \bigcap_{m=1}^{\infty} \mathscr{F}_{0,t}^{m*}) 1_{\widetilde{\mathscr{B}}}(\mathbf{w}), \end{aligned} \tag{4.10}$$

$$\begin{aligned} &\lim_{m\to\infty} P^{\infty}(\mathbf{w}_t \in \mathbf{B}_t | \mathscr{F}_{0,t}^{m*}) 1_{\mathbf{A}_0^{m*}}(\mathbf{w}_0^{m*}) \\ &= P^{\infty}(\mathbf{w}_t \in \mathbf{B}_t | \bigcap_{m=1}^{\infty} \mathscr{F}_{0,t}^{m*}) 1_{\widetilde{\mathscr{A}}}(\mathbf{w}). \end{aligned} \tag{4.11}$$

From Lemma 5 and $\bigcap_{m=1}^{\infty} \mathscr{F}_{0,t}^{m*} \subset \mathscr{C}_{\mathrm{path}}^{\infty *}$, we deduce $P^{\infty}(\widetilde{\mathscr{A}}) \in \{0, 1\}$. Furthermore, $\{\mathbf{w}; \mathbf{w}_0 \in \mathbf{A}_0\} \subset \widetilde{\mathscr{A}}$ by construction. Hence,

$$\int_{\mathscr{A}} P^{\infty}(\mathbf{w}_0 \in \mathbf{A}_0 | \bigcap_{m=1}^{\infty} \mathscr{F}_{0,t}^{m*}) dP^{\infty} = P^{\infty}(\mathscr{A}) P^{\infty}(\mathbf{w}_0 \in \mathbf{A}_0). \tag{4.12}$$

Similarly, we have

$$\int_{\mathscr{B}} P^{\infty}(\mathbf{w}_t \in \mathbf{B}_t | \bigcap_{m=1}^{\infty} \mathscr{F}_{0,t}^{m*}) dP^{\infty} = P^{\infty}(\mathscr{B}) P^{\infty}(\mathbf{w}_t \in \mathbf{B}_t). \tag{4.13}$$

Suppose $P^{\infty}(\widetilde{\mathscr{A}}) = 0$. Then $P^{\infty}(\mathbf{w}_0 \in \mathbf{A}_0) = 0$ because $\{\mathbf{w}; \mathbf{w}_0 \in \mathbf{A}_0\} \subset \widetilde{\mathscr{A}}$. Suppose $P^{\infty}(\widetilde{\mathscr{A}}) = 1$. If, in addition, (4.8) holds for infinitely many $m \in \mathbb{N}$, then from (4.8), (4.10), and (4.12), we deduce $P^{\infty}(\mathbf{w}_0 \in \mathbf{A}_0) = 0$.

Similarly, $P^{\infty}(\widetilde{\mathscr{B}}) = 0$ implies $P^{\infty}(\mathbf{w}_t \in \mathbf{B}_t) = 0$. If $P^{\infty}(\widetilde{\mathscr{B}}) = 1$ and (4.9) holds for infinitely many $m \in \mathbb{N}$, then $P^{\infty}(\mathbf{w}_t \in \mathbf{B}_t) = 0$ from (4.9), (4.11), and (4.13).

Combining these and recalling $\mathbf{A} = \mathbf{A}_0$ and $\mathbf{B} = \mathbf{B}_t$ complete the proof. □

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References

1. S. Albeverio, Yu.G. Kondratiev, M. Röckner, Analysis and geometry on configuration spaces: the Gibbsian case. J. Funct. Anal. **157**(1), 242–291 (1998)
2. I. Corwin, X. Sun, Ergodicity of the airy line ensemble. Electric. Commun. Probab. **19**(49), 1–11 (2014)
3. F.J. Dyson, A Brownian-motion model for the eigenvalues of a random matrix. J. Math. Phys. **3**, 1191–1198 (1962)
4. M. Fradon, S. Roelly, H. Tanemura, An infinite system of Brownian balls with infinite range interaction. Stochastic Process Their Appl. **90**(1), 43–66 (2000)
5. J. Fritz, Gradient dynamics of infinite point systems. Ann. Probab. **15**, 478–514 (1987)
6. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, 2nd ed. (Walter de Gruyter, 2011)
7. R. Honda, H. Osada, Infinite-dimensional stochastic differential equations related to Bessel random point fields. Stochastic Processes Their Appl. **125**(10), 3801–3822 (2015)
8. N. Ikeda, S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, 2nd ed (North-Holland, 1989)
9. Y. Kawamoto, H. Osada, Finite particle approximations of interacting Brownian particles with logarithmic potentials. J. Math. Soc. Japan **70**(3), 921–952 (2018). https://doi.org/10.2969/jmsj/75717571
10. Y. Kawamoto, H. Osada, Dynamical universality for random matrices. Partial Differ. Equ. Appl. **3**, 27 (2022). https://doi.org/10.1007/s42985-022-00154-7
11. Y. Kawamoto, H. Osada, H. Tanemura, Uniqueness of Dirichlet forms related to infinite systems of interacting Brownian motions. Potential Anal. **55**, 639–676 (2021). https://doi.org/10.1007/s11118-020-09872-2
12. Y. Kawamoto, H. Osada, H. Tanemura, Infinite-dimensional stochastic differential equations and tail σ-fields II: the IFC condition. J. Math. Soc. Japan **74**(1), 79–128 (2022)
13. R. Lang, Unendlich-dimensionale Wienerprocesse mit Wechselwirkung I Z. Wahrschverw. Gebiete **38**, 55–72 (1977)
14. R. Lang, Unendlich-dimensionale Wienerprocesse mit Wechselwirkung II Z. Wahrschverw. Gebiete **39**, 277–299 (1978)
15. R. Lyons, A note on tail triviality for determinantal point processes. Electron. Commun. Probab. **23**, 1–3, paper no. 72 (2018). ISSN: 1083-589X
16. M.L. Mehta, *Random Matrices*, 3rd edn. (Elsevier, Amsterdam, 2004)
17. H. Osada, Dirichlet form approach to infinite-dimensional Wiener processes with singular interactions. Commun. Math. Phys. **176**, 117–131 (1996)
18. H. Osada, Non-collision and collision properties of Dyson's model in infinite dimensions and other stochastic dynamics whose equilibrium states are determinantal random point fields, in *Stochastic Analysis on Large Scale Interacting Systems*. Advanced Studies in Pure Mathematics, Vol. 39, ed. by T. Funaki, H. Osada (2004), pp. 325–343
19. H. Osada, Tagged particle processes and their non-explosion criteria. J. Math. Soc. Japan **62**(3), 867–894 (2010)
20. H. Osada, Infinite-dimensional stochastic differential equations related to random matrices. Probab. Theory Related Fields **153**, 471–509 (2012)
21. H. Osada, Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials. Ann. Probab. **41**, 1–49 (2013)
22. H. Osada, Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials II: airy random point field. Stochastic Process. Their Appl. **123**, 813–838 (2013)
23. H. Osada, S. Osada, Discrete approximations of determinantal point processes on continuous spaces: tree representations and tail triviality. J. Stat. Phys. **170**, 421 (2018). https://doi.org/10.1007/s10955-017-1928-2
24. H. Osada, H. Tanemura, Infinite-dimensional stochastic differential equations and tail σ-fields. Probab. Theory Relat. Fields **177**, 1137–1242 (2020). https://doi.org/10.1007/s00440-020-00981-y

25. H. Osada, H. Tanemura, *Infinite-dimensional stochastic differential equations arising from Airy random point fields* (preprint). arXiv:1408.0632 [math.PR] (ver. 8)
26. D. Ruelle, Superstable interactions in classical statistical mechanics. Commun. Math. Phys. **18**, 127–159 (1970)
27. H. Spohn, Interacting Brownian particles:a study of Dyson's model, in *Hydrodynamic Behavior and Interacting Particle Systems*. IMA Volumes in Mathematics and its Applications, vol. 9, ed. by G. Papanicolaou (Springer, Berlin, 1987), pp. 151–179
28. H. Tanemura, A system of infinitely many mutually reflecting Brownian balls in $\mathbb{R}^d$. Probab. Theory Relat. Fields **104**, 399–426 (1996)
29. L.-C. Tsai, Infinite dimensional stochastic differential equations for Dyson's model. Probab. Theory Relat. Fields **166**, 801–850 (2016)

(Weak) Hardy and Poincaré Inequalities and Criticality Theory

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Abstract In this paper we study Hardy and Poincaré inequalities and their weak versions for quadratic forms satisfying the first Beurling-Deny criterion. We employ these inequalities to establish a criticality theory for such forms.

Keywords Criticality theory · Hardy inequality · Poincaré inequality

1 Introduction

This paper deals with (weak) Hardy/Poincaré inequalities for certain quadratic forms. First we recall the classical inequalities for nonempty open subsets $\Omega \subseteq \mathbb{R}^n$. We let

$$\mathcal{E}_\Omega : L^2(\mathbb{R}^n) \to [0, \infty], \quad \mathcal{E}_\Omega(f) = \begin{cases} \int_\Omega |\nabla f|^2 dx & \text{if } f \in H^1(\Omega) \\ \infty & \text{else} \end{cases}$$

denote the Dirichlet-integral. With this notation, for $n \geq 3$ and $f \in L^2(\mathbb{R}^n)$ the classical Hardy inequality reads

$$\frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^2} dx \leq \mathcal{E}_{\mathbb{R}^n}(f).$$

If Ω is bounded and connected, the classical Poincaré inequality states that there exists $C > 0$ such that for all $f \in L^2(\Omega)$ with $f \perp 1$ we have

$$C \int_\Omega |f|^2 dx \leq \mathcal{E}_\Omega(f).$$

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In this paper we consider similar inequalities with $\mathcal{E}_\Omega$ replaced by a more general quadratic form and the left side of these inequalities replaced by a weighted L^2-norm. Furthermore, we discuss weak versions that incorporate perturbations of the quadratic form on the right side.

Hardy's inequality yields $\ker \mathcal{E}_{\mathbb{R}^n} = \{0\}$. For bounded and connected Ω Poincaré's inequality implies $\ker \mathcal{E}_\Omega = \mathbb{R} \cdot 1$, so that $f \perp 1$ is equivalent to $f \perp \ker \mathcal{E}_\Omega$. Hence, in both of the cases $\Omega = \mathbb{R}^n$ with $n \geq 3$ or Ω bounded and connected, the classical Hardy respectively Poincaré inequality is an inequality between a weighted L^2-norm and the quadratic form $\mathcal{E}_\Omega$ that holds for all functions orthogonal to $\ker \mathcal{E}_\Omega$. This is the starting point for the following more abstract discussion.

Let $q\colon L^2(X, \mu) \to [0, \infty]$ be a closed quadratic form satisfying the first Beurling-Deny criterion, i.e., $q(|f|) \leq q(f)$ for all $f \in L^2(X, \mu)$. A typical example is the Dirichlet integral $\mathcal{E}_\Omega$ considered above or, more generally, a quadratic form of a (discrete) Schrödinger operator, see e.g. [12, 20, 30] or a Dirichlet form, see e.g. [5]. We deal with inequalities of the form

$$(\Diamond) \qquad \int_X f^2 w d\mu \leq \alpha(r) q(f) + r\Phi(f), \quad f \perp_w \ker q,\ r > 0.$$

Here, $w\colon X \to (0, \infty)$ is measurable, $\alpha\colon (0, \infty) \to (0, \infty)$ is decreasing, $\Phi\colon L^2(X, \mu) \to [0, \infty]$ is homogenous (i.e. $\Phi(\lambda f) = |\lambda|^2 \Phi(f)$ for any $\lambda \in \mathbb{R}$, $f \in L^2(X, \mu)$) and the symbol $\perp_w$ indicates that orthogonality is considered in $L^2(X, w\mu)$. In view of our discussion of the Dirichlet integral, we call Inequality $\Diamond$ *weak Hardy inequality* if $\ker q = \{0\}$ and *weak Poincaré inequality* if $\ker q \neq \{0\}$. In the case $\Phi = 0$, the function α becomes a constant and Inequality $\Diamond$ is referred to as *Hardy inequality* if $\ker q = \{0\}$, respectively *Poincaré inequality* if $\ker q \neq \{0\}$.

The goal of this paper is to provide abstract criteria for (weak) Hardy/Poincaré inequalities with respect to the homogeneous functionals $\Phi = 0$ and $\Phi(f) = \|f/h\|_\infty^2$, where $h\colon X \to (0, \infty)$. We study which w are eligible but do not aim at giving explicit bounds for α if $\Phi \neq 0$. Moreover, we apply these inequalities to establish a criticality theory for general forms satisfying the first Beurling-Deny criterion.

It turns out that if one replaces $\int f^2 w d\mu$ on the left side of Inequality $\Diamond$ by $(\int f w d\mu)^2$, then for generic homogeneous Φ there exists a function α such this inequality holds, see Theorem 4.2. For forms with the first Beurling-Deny criterion and $\ker q = \{0\}$, this always leads to a weak Hardy inequality with respect to $\Phi(f) = \|f/h\|_\infty^2$ as long as $h \in L^2(X, \mu)$ and $w \in L^2(X, h^2\mu)$, see Theorem 4.3. Thus, one can say that weak Hardy inequalities hold generically for forms with the first Beurling-Deny criterion and trivial kernel.

Forms satisfying a Hardy inequality are called *subcritical* and there is a large amount of literature on subcriticality, see e.g. (and references therein) [16, 19, 21] for elliptic operators with real coefficients, [4, 30, 32] for generalized Schrödinger forms (Dirichlet form plus potential term) and [12] for discrete Schrödinger operators. It turns out that weak Hardy inequalities can be employed to study subcriticality. We give a comprehensive characterization in Theorem 5.2, which should cover most previous results (for quadratic forms), simplifies/unifies proofs and also gives some new insights.

If q is irreducible, in the situations where subcriticality is well understood (see the mentioned references on subcriticality), there is a dichotomy. Either q is subcritical or there exists a sequence (φ_n) with $q(\varphi_n) \to 0$ that converges pointwise to a strictly positive function h. In this second case, q is called *critical* and the function h is unique up to multiplication by a constant, the so-called *Agmon ground state* of q. Our discussion in Theorem 5.2 shows that this dichotomy may fail in general. It becomes a trichotomy and the third case (besides criticality and subcriticality) happens if and only if the form q does not possess an excessive function, see Corollary 5.3. So far we do not have a concrete example for this case. The reason is that in the situations where subcriticality was studied previously, the corresponding semigroups are semigroups of kernel operators and we show in Appendix 6 that irreducible semigroups of kernel operators always admit excessive functions. Along the way we give a partial answer to a question of Schep [24] on more general positive operators on L^p-spaces.

Weak Poincaré inequalities were introduced in [23] for conservative Dirichlet forms on finite measure spaces to study the rate of convergence of their semigroups to equilibrium when there is no spectral gap, i.e., when they do not satisfy a Poincaré inequality with $w = 1$. Similarly, the weak Hardy inequalities mentioned above can be employed to establish the rate of convergence to 0 of semigroups coming from forms with trivial kernel, see the remark after Theorem 4.3.

A closed quadratic form q on $L^2(X, \mu)$ with the first Beurling-Deny criterion can be extended to a lower semicontinuous quadratic form q_e on $L^0(X, \mu)$, the so-called extended form. With this at hand the Agmon ground state (if it exists) is an element of the kernel of q_e and criteria for subcriticality can be formulated conveniently in terms of q_e. One of the observations is that subcriticality is equivalent to the domain of q_e being a Hilbert space, a fact which is well-known for Dirichlet forms (where subcriticality is called transience and the domain of the extended form is the extended Dirichlet space), see e.g. [5, Section 1.6]. Moreover, it is known for critical Dirichlet forms (usually called recurrent Dirichlet forms) that the quotient of the extended Dirichlet space modulo constants is a Hilbert space if a Poincaré inequality holds, see e.g. [5, Section 4.8]. We show in Theorem 6.1 that some sort of converse holds in our setting: In the critical case completeness of the domain of q_e modulo the kernel of q_e is equivalent to a weak Poincaré inequality. This observation seems to be new. An example from [23] shows that there are irreducible conservative Dirichlet forms without weak Poincaré inequality and as a consequence we obtain that their extended Dirichlet space modulo constants (i.e. the domain of the extended form modulo its kernel) is not complete, see Corollary 6.3. To the best of our knowledge it is a new observation that such forms exist.

At the heart of the considerations regarding the completeness of the domain of the extended form lies the following observation: If q has an excessive function, weak Hardy/Poincaré inequalities are equivalent to the continuity of the embedding of (a quotient) of the domain of the extended form equipped with the norm coming from q_e into (a quotient) of $L^0(X, \mu)$ equipped with (the quotient topology of) the topology of local convergence in measure. That this continuity is equivalent to completeness of the the domain of the extended form is a consequence of the abstract Theorem 2.3, which is taken from [25].

For technical reasons and in order to understand the third case besides criticality and subcriticality in the mentioned trichotomy better, we introduce another functional q^+ on $L^+(X, \mu)$ (the cone of $[0, \infty]$-valued μ-a.e. defined functions), which is a lower semicontinuous extension of q_e. If q is irreducible, the mentioned trichotomy then reads either $\ker q^+ = \{0\}$ (subcriticality) or $\ker q^+ = \{\lambda h \mid \lambda \in [0, \infty]\}$ (criticality, with Agmon ground state h) or $\ker q^+ = \{0, \infty\}$.

As discussed above, besides new insights, our approach to criticality theory also yields simplified/unified proofs for facts, which are known for Dirichlet forms and some more general situations. We lay out the material such that our proofs do not depend on known aspects of criticality theory, for otherwise they would not really simplify/unify arguments. To this end, we discuss properties of forms with the first Beurling-Deny criterion in Sect. 3 in some detail. Among other things we give a short independent proof for the existence of the extended forms q_e, see Proposition 3.6, which otherwise could be deduced from [27].

The paper is structured as follows: Sect. 2 discusses closedness and closability of quadratic forms on the topologcial vector space $L^0(X, \mu)$, which is similar to situation for forms on $L^2(X, \mu)$. Moreover, it recalls properties of positivity preserving operators. In Sect. 3 we review the Beurling-Deny criteria and their consequences. Particular emphasis is laid on the smallest closed extensions to $L^0(X, \mu)$ (this is the so-called extended form) and to $L^+(X, \mu)$. Section 4 discusses abstract weak Hardy/Poincaré inequalities and why certain weak Hardy inequalities hold generically for forms with the first Beurling Deny criterion and trivial kernel. These results are applied in Sect. 5 to obtain a comprehensive treatement of criticality theory for forms with the first Beurling-Deny criterion. Section 6 provides a characterization for weak Poincaré inequalites and provides an example for a Dirichlet form whose extended Dirichlet space is not complete. Existence of excessive functions is essential for applying criticality theory. Appendix A shows the existence of excessive functions for irreducible semigroups of kernel operators (and hence for the associated forms). The given proof is somewhat more general and provides a partial answer to a question of Schep [24] on positive operators on L^p-spaces.

2 Preliminaries

2.1 Closed Quadratic Forms on $L^2(X, \mu)$ and $L^0(X, \mu)$

Throughout this text $(X, \mathfrak{A}, \mu)$ is a σ-finite measure space. By $L^+(X, \mu)$ we denote the quotient of the space of measurable functions $f : X \to [0, \infty]$ with respect to equivalence μ-a.e. and by $L^0(X, \mu)$ we denote the quotient of the space of measurable functions $f : X \to \mathbb{R}$ with respect to equivalence μ-a.e. For $f \in L^+(X, \mu)$ we write $f > 0$ if $f(x) > 0$ for μ-a.e. $x \in X$. For any sequence (f_n) in $L^+(X, \mu)$ the pointwise a.e. defined functions $\liminf_{n\to\infty} f_n$ and $\limsup_{n\to\infty} f_n$ exist in $L^+(X, \mu)$.

We equip $L^0(X,\mu)$ and $L^+(m)$ with the topology of *local convergence in measure*. More precisely, we choose an integrable $\varphi\colon X \to (0,\infty)$ and define for $f \in L^0(X,\mu)$ the F-norm (which is not a norm)

$$\|f\|_0 = \int_X (|f| \wedge 1)\varphi d\mu.$$

It induces a metric d on $L^0(X,\mu)$ by $d(f,g) = \|f-g\|_0$ and a metric ρ on $L^+(X,\mu)$ by $\rho(f,g) = d(\arctan(f), \arctan(g))$ (with the convention $\arctan(\infty) = \pi/2$). A sequence (f_n) in $L^0(X,\mu)$ or in $L^+(X,\mu)$ converges with respect to these metrics to a function f if and only if any subsequence of (f_n) has a subsequence converging to f μ-a.e. In particular, the topologies generated by d and ρ coincide on $L^0_+(X,\mu) = L^0(X,\mu) \cap L^+(X,\mu)$. Both $L^0(X,\mu)$ and $L^+(X,\mu)$ are complete and $L^0(X,\mu)$ is a topological vector space, i.e., addition and scalar multiplication are continuous.

As usual, for $p \in [1,\infty]$ we denote by $L^p(X,\mu)$ the Lebesgue spaces of p-integrable real-valued functions with corresponding norm $\|\cdot\|_p$. The scalar product on $L^2(X,\mu)$ is denoted by $\langle\cdot,\cdot\rangle$. By $L^p_+(X,\mu)$ we denote the cone of nonnegative functions in $L^p(X,\mu)$, $p \in \{0\} \cup [1,\infty]$. If $h \in L^0_+(X,\mu)$ is strictly positive, we let

$$L^\infty_h(X,\mu) = \{f \in L^0(X,\mu) \mid f/h \in L^\infty(X,\mu)\}$$

and equip it with the norm $\|f\|_{h,\infty} = \|f/h\|_\infty$.

In this text we consider quadratic forms on the Hilbert space $L^2(X,\mu)$ and on the topological vector space $L^0(X,\mu)$, and certain homogenous functionals on the cone $L^+(X,\mu)$.

Let $p \in \{0,2\}$ and let $q\colon L^p(X,\mu) \to [0,\infty]$ be a quadratic form with *domain*

$$D(q) = \{f \in L^p(X,\mu) \mid q(f) < \infty\}.$$

By polarization it induces a bilinear form on its domain, which we also denote by q. In this sense we have $q(f) = q(f,f)$ for $f \in D(q)$.

We call the quadratic form q *closed* if $D(q)$ is complete with respect to the metric $d_q(f,g) = q(f-g)^{1/2} + \|f-g\|_p$. The topology induced by this metric on $D(q)$ is called the *form topology*. As is well known in the case $p=2$, closedness of a quadratic form is equivalent to lower semicontinuity. According to Lemma [26, Lemma A.4], the same is true for quadratic forms on any complete metrizable topological vector space. Since this is important for us we formulate it as a lemma.

Lemma 2.1 *Let $p \in \{0,2\}$ and let $q\colon L^p(X,\mu) \to [0,\infty]$ be a quadratic form. The following assertions are equivalent.*

(i) *q is closed.*
(ii) *q is lower semicontinuous, i.e., for all (f_n) in $L^p(X,\mu)$ the convergence $f_n \to f$ in $L^p(X,\mu)$ implies*

$$q(f) \leq \liminf_{n\to\infty} q(f_n).$$

If $p = 2$, both are equivalent to:

(iii) *q is weakly lower semicontinuous, i.e., for all (f_n) in $L^2(X, \mu)$ the convergence $f_n \to f$ weakly in $L^2(X, \mu)$ implies*

$$q(f) \leq \liminf_{n\to\infty} q(f_n).$$

Remark What makes this lemma nontrivial is that in general $L^0(X, \mu)$ is not a locally convex vector space. Hence, one can not just use a modified version of the Hilbert space proof.

A quadratic form q' on $L^p(X, \mu)$ is called an extension of q if $D(q) \subseteq D(q')$ and $q' = q$ on $D(q)$. We call q *closable* if it possesses a closed extension. As for closedness the standard characterization of closability extends from the Hilbert space case to $L^0(X, \mu)$, see [26, Lemma A.3]. Again there is some difficulty because $L^0(X, \mu)$ is not locally convex.

Lemma 2.2 *Let $p \in \{0, 2\}$ and let $q\colon L^p(X, \mu) \to [0, \infty]$ be a quadratic form. The following assertions are equivalent.*

(i) *q is closable.*
(ii) *q is lower semicontinuous on its domain, i.e., for all (f_n) in $D(q)$ and $f \in D(q)$ the convergence $f_n \to f$ in $L^p(X, \mu)$ implies*

$$q(f) \leq \liminf_{n\to\infty} q(f_n).$$

In this case, q possesses a smallest closed extension $\bar{q}\colon L^p(X, \mu) \to [0, \infty]$, which is given by

$$\bar{q}(f) = \begin{cases} \lim\limits_{n\to\infty} q(f_n) & \text{if there ex. } q\text{-Cauchy sequence } (f_n) \text{ with } f_n \to f \text{ in } L^p(X, \mu) \\ \infty & \text{else} \end{cases}.$$

Sometimes closedness of a quadratic form on $L^p(X, \mu)$ is equivalent to $(D(q)/\ker q, q)$ being a Hilbert space (i.e. to ensure completeness the convergence with respect to $\|\cdot\|_p$ can be omitted). The following theorem shows that this is precisely the case if $(D(q), q)$ continuously embeds into a quotient of $L^p(X, \mu)$. It is a special case of [25, Theorem 1.38]. Since here we only deal with metrizable topological vector spaces, the proof simplifies and we include it for the convenience of the reader.

Theorem 2.3 *Let $p \in \{0, 2\}$ and let $q\colon L^p(X, \mu) \to [0, \infty]$ be a quadratic form. Each two of the following statements imply the third.*

(i) *q is closed.*
(ii) $\ker q$ *is closed in* $L^p(X, \mu)$ *and the embedding*

$$\iota\colon (D(q)/\ker q, q) \to L^p(X, \mu)/\ker q, \quad f \mapsto f$$

is continuous. Here, $L^p(X, \mu)/\ker q$ *is equipped with the quotient topology.*
(iii) $(D(q)/\ker q, q)$ *is a Hilbert space.*

Proof In both cases $p = 0, 2$, we equipped $L^p(X, \mu)$ with the translation invariant metric $d_p(f, g) = \|f - g\|_p$. If $F \subseteq L^p(X, \mu)$ is a closed subspace, then

$$d_F(f + F, g + F) = \inf\{\|f - g + h\|_p \mid h \in F\}$$

is a translation invariant metric inducing the quotient topology on $L^p(X, \mu)/F$. With respect to this metric the quotient space is complete.

(i) and (ii) $\Rightarrow$ (iii): Let $(f_n + \ker q)$ be q-Cauchy. Since the metrics are translation invariant, (ii) implies that $(f_n + \ker q)$ is Cauchy in $L^p(X, \mu)/\ker q$. By completeness of the quotient we obtain $f \in L^p(X, \mu)$ such that $f_n + \ker q \to f + \ker q$ in $L^p(X, \mu)/\ker q$ and hence $f_n + k_n \to f$ in $L^p(X, \mu)$ for some $k_n \in \ker q$. Using lower semicontinuity guaranteed by (i), we obtain

$$q(f - f_n) \leq \liminf_{m\to\infty} q(f_m + k_m - f_n) = \liminf_{m\to\infty} q(f_m - f_n).$$

This yields $f_n + \ker q \to f + \ker q$ with respect to q.

(i) and (iii) $\Rightarrow$ (ii): Lower semicontinuity implies that $\ker q$ is closed. Since $(D(q)/\ker q, q)$ is complete and by the closedness of $\ker q$ the space $L^p(X, \mu)/\ker q$ is complete, we can use the closed graph theorem (which holds for maps between complete metrizable topological vector spaces, see e.g. [8]). Hence, it suffices to show that the map ι is closed. To this end, consider a sequence $(f_n + \ker q)$ with $f_n + \ker q \to f + \ker q$ with respect to q and $f_n + \ker q \to g + \ker q$ in $L^p(X, \mu)/\ker q$. Using lower semicontinuity guaranteed by (i) (which passes to the quotient space), we obtain

$$q(f - g) \leq \liminf_{n\to\infty} q(f - f_n) = 0.$$

This yields $f + \ker q = g + \ker q$ and closedness of ι is established.

(ii) and (iii) $\Rightarrow$ (i): Let (f_n) be $L^p(X, \mu)$-Cauchy and q-Cauchy. We need to show that (f_n) converges with respect to the form topology. By completeness of $L^p(X, \mu)$ there exists $f \in L^p(X, \mu)$ with $f_n \to f$. Since also $(D(q)/\ker q, q)$ is a Hilbert space, we have $q(g - f_n) \to 0$ for some $g \in D(q)$. Then (ii) implies $f_n + k_n \to g$ in $L^p(X, \mu)$ for some $k_n \in \ker q$. Altogether, (k_n) converges in $L^p(X, \mu)$ to $k = g - f$. Since $\ker q$ is closed and $k_n \in \ker q$, we obtain $g - f \in \ker q$. This implies $q(f - f_n) \to 0$ and we obtain $f_n \to f$ with respect to the form topology. □

2.2 Extensions of Positivity Preserving Operators

Let $T: L^p(X, \mu) \to L^p(X, \mu)$ be a positivity preserving operator, i.e., $f \geq 0$ implies $Tf \geq 0$. We abuse notation and extend it to an operator $T: L^+(X, \mu) \to L^+(X, \mu)$ by letting

$$Tf = \lim_{n\to\infty} Tf_n$$

for a sequence (f_n) in $L^p_+(X, \mu)$ with $f_n \nearrow f$ μ-a.s. The limit always exists μ-a.s. and does not depend on the choice of the sequence (f_n) (here we use the continuity of positivity preserving operators, see e.g. [15, Proposition 1.3.5]). This extension is a linear map on the cone $L^+(X, \mu)$. It follows from the definition that T satisfies Fatou's lemma, i.e., for any sequence (f_n) in $L^+(X, \mu)$ we have

$$T\left(\liminf_{n\to\infty} f_n\right) \leq \liminf_{n\to\infty} Tf_n.$$

If $f \in L^0(X, \mu)$ such that $T|f| \in L^0(X, \mu)$, then we set $Tf = Tf_+ - Tf_-$. This yields a linear operator on $L^0(X, \mu)$ with domain $\{f \in L^0(X, \mu) \mid T|f| \in L^0(X, \mu)\}$, which extends the given operator T. Moreover, there is also a version of Lebesgue's dominated convergence theorem for the extension. If $g \in L^0_+(X, \mu)$ such that $Tg \in L^0_+(X, \mu)$, then $f_n \to f$ in $L^0(X, \mu)$ and $|f_n| \leq g$ imply $T|f| \in L^0(X, \mu)$ and $Tf = \lim_{n\to\infty} Tf_n$ in $L^0(X, \mu)$.

3 The Beurling-Deny Criteria, Excessive Functions and Extended Forms

In this section we review properties of forms satisfying the first Beurling-Deny criterion. Most of the results here are known but we include proofs for two reasons: (1) Some of the proofs in the literature use parts of the theory we develop in subsequent sections. This can lead to intransparent cross-references, which we try to avoid. (2) Making consequent use of lower semicontinuity and excessive functions leads to shorter proofs of some known results.

3.1 Basics and Excessive Functions

Let $p \in \{0, 2\}$ and let $q: L^p(X, \mu) \to [0, \infty]$ be a quadratic form. We say that q satisfies the *first Beurling-Deny criterion* if $q(|f|) \leq q(f)$ for all $f \in L^p(X, \mu)$. A function $h \in L^0(X, \mu)$ (with $h_- \in L^2(X, \mu)$ in the case $p = 2$) is called *q-excessive* if $q(f \wedge h) \leq q(f)$ for all $f \in L^p(X, \mu)$, where $f \wedge h = \min\{f, h\}$. If the constant function 1 is q-excessive, then q is said to satisfy the *second Beurling-Deny criterion*.

A form on $L^2(X, \mu)$, which satisfies the second Beurling-Deny criterion and has dense domain in $L^2(X, \mu)$, is called *Dirichlet form*. As is common in the literature, below $\mathcal{E}$ will denote a Dirichlet form on $L^2(X, \mu)$.

Remark (a) For closed forms the existence of a nonnegative excessive function implies the first Beurling-Deny criterion. In particular, closed forms with the second Beurling-Deny criterion satisfy the first. The question which forms satisfying the first Beurling-Deny criterion admit an excessive function is of interest and will be discussed in detail below.

(b) If $p = 2$, we could also look at lower semibounded quadratic forms. There are two reasons why if one assumes one of the Beurling-Deny criteria, it suffices to consider nonnegative forms: 1) A lower semibounded quadratic form satisfying the second Beurling-Deny is always nonnegative. 2) A quadratic form q satisfies the first Beurling-Deny criterion if and only if for one/all $\alpha \in \mathbb{R}$ the quadratic form $q + \alpha \|\cdot\|_2^2$ satisfies the first Beurling-Deny criterion. Hence, q is either nonnegative anyway or one can add a multiple of the square of the L^2-norm to obtain a nonnegative form without loosing Beurling-Deny criteria.

Lemma 3.1 *Let $p \in \{0, 2\}$ and let $q\colon L^p(X, \mu) \to [0, \infty]$ be a quadratic form satisfying the first Beurling-Deny criterion. For all $f, g \in L^p(X, \mu)$*

$$q(f \wedge g) + q(f \vee g) \leq q(f) + q(g).$$

In particular, $D(q)$ is a lattice and functions in $\ker q$ are q-excessive.

Proof The parallelogram identity and the first Beurling-Deny criterion yield

$$\begin{aligned} 2q(f \wedge g) + 2q(f \vee g) &= q(f \wedge g + f \vee g) + q(f \wedge g - f \vee g) \\ &= q(f + g) + q(|f - g|) \\ &\leq q(f + g) + q(f - g) \\ &= 2q(f) + 2q(g). \quad \square \end{aligned}$$

The following lemma shows that the set of nonnegative excessive functions is closed under local convergence in measure.

Lemma 3.2 *Let $p \in \{0, 2\}$ and let $q\colon L^p(X, \mu) \to [0, \infty]$ be a closed quadratic form. Let (h_n) be a sequence of nonnegative q-excessive functions converging in $L^0(X, \mu)$ to h. Then h is q-excessive.*

Proof Since h_n and h are nonnegative, we have $f \wedge h_n = f_+ \wedge h_n - f_- \to f_+ \wedge h - f_- = f \wedge h$ in $L^p(X, \mu)$ (this is clear for $p = 0$, for $p = 2$ it follows from Lebesgue's dominated convergence theorem). Hence, lower semicontinuity of q on $L^p(X, \mu)$ implies

$$q(f \wedge h) \leq \liminf_{n \to \infty} q(f \wedge h_n) \leq q(f).$$

This shows that h is q-excessive. $\square$

The following lemma is due to Ancona, see [2, Proposition 4]. It shows that for forms satisfying the first Beurling-Deny criterion taking the absolute value is continuous with respect to the form topology (which is induced by the metric $d_q(f, g) = q(f - g)^{1/2} + \|f - g\|_p$ on $D(q)$). The setting of [2] is a bit different and therefore we explain why Ancona's proof can be applied in our situation.

Lemma 3.3 *For $p \in \{0, 2\}$ let q be a closed quadratic form on $L^p(X, \mu)$ satisfying the first Beurling-Deny criterion. Then*

$$|\cdot|\colon D(q) \to D(q), \quad f \mapsto |f|$$

is continuous with respect to the form topology.

Proof The equations necessary for proving that $f_n \to f$ with respect to the form topology implies $|f_n| \to |f|$ with respect to q are contained in the proof of [2, Proposition 4]. In contrast to our assumptions, the setting of [2] requires that $(D(q), q)$ is a Hilbert space and that $(D(q), q)$ continuously embeds into $L^0(X, \mu)$. However, what Ancona really uses in the proof of [2, Proposition 4] is that if (f_n) in $D(q)$ is q-bounded and $f_n \to f$ in $L^p(X, \mu)$, then $f \in D(q)$ and $f_n \to f$ q-weakly. This is a standard result if $p = 2$ and proved in [26, Lemma A.5] for $p = 0$. □

Assumption From now on we assume that all quadratic forms on $L^p(X, \mu)$, $p \in \{0, 2\}$, are *densely defined*, i.e., $D(q)$ is dense in $L^p(X, \mu)$.

Remark For forms q on $L^p(X, \mu)$, $p \in \{0, 2\}$, satisfying the first Beurling-Deny criterion, this assumption is not a restriction. In this case, $D(q)$ is a lattice and hence the $L^p(X, \mu)$-closure of $D(q)$ is a closed vector lattice in $L^p(X, \mu)$. Such lattices are given by $L^p(Y, \mathfrak{B}, \mu|_{\mathfrak{B}})$, where $Y \subseteq X$ and $\mathfrak{B}$ is a σ-algebra on Y contained in the original σ-algebra. Hence, q can always be considered to be densely defined on some L^p-space.

Let q be a closed form on $L^2(X, \mu)$. For $\alpha > 0$ we define the quadratic form

$$q_\alpha\colon L^2(X, \mu) \to [0, \infty], \quad q_\alpha(f) = q(f) + \alpha \|f\|_2^2 .$$

By the Riesz representation theorem for $\alpha > 0$ and $f \in L^2(X, \mu)$ there exists a unique element $G_\alpha f \in D(q)$ such that

$$q(G_\alpha f, g) + \alpha \langle G_\alpha f, g\rangle = \langle f, g\rangle$$

for all $g \in D(q)$. The induced family of self-adjoint operators $(G_\alpha)_{\alpha>0}$ on $L^2(X, \mu)$ is called *resolvent family* of q. Since q is densely defined, it corresponds to a unique nonnegative self-adjoint operator L, the so-called *generator of q* through $G_\alpha = (L + \alpha)^{-1}$, $\alpha > 0$. Moreover, it is strongly continuous, i.e., for $f \in L^2(X, \mu)$ we have $\alpha G_\alpha f \to f$, as $\alpha \to \infty$. Using spectral calculus of L, we obtain a self-adjoint C_0-semigroup $(T_t)_{t>0} = (e^{-tL})_{t>0}$ on $L^2(X, \mu)$, which we call *semigroup associated with q*.

It is well-known that G_α, $\alpha > 0$, and T_t, $t > 0$, are positivity preserving if and only if q satisfies the first Beurling-Deny criterion. Moreover, $(G_\alpha)_{\alpha>0}$ and $(T_t)_{t>0}$ are Markovian (i.e. $0 \le f \le 1$ implies $0 \le \alpha G_\alpha f \le 1$ respectively $0 \le T_t f \le 1$) if and only if q satisfies the second Beurling-Deny criterion. See e.g. [22, Theorem XIII.50 and Theorem XIII.51]. We need the following extension of this result, which is basically taken from [9, Proposition 4].

Lemma 3.4 *Let $q\colon L^2(X,\mu) \to [0,\infty]$ be a quadratic form satisfying the first Beurling-Deny criterion. For $h \in L^0_+(X,\mu)$ the following assertions are equivalent.*

(i) *h is q-excessive.*
(ii) *$\alpha G_\alpha h \le h$ for all $\alpha > 0$.*
(iii) *$T_t h \le h$ for all $t > 0$.*

If, additionally, $h \in D(q)$, this is equivalent to $q(h,g) \ge 0$ for all nonnegative $g \in D(q)$.

Proof This follows directly from Ouhabaz' invariance criterion for closed convex sets under the resolvent respectively semigroup, see [18]. See also the proof of [9, Proposition 4] for more details. □

Remark There are two immediate consequences of this lemma, which we will use below:

(a) Assertion (ii) yields that if $h \in L^0_+(X,\mu)$ is excessive and strictly positive, then αG_α and T_t extend to contractions on $L^\infty_h(X,\mu)$. By duality we obtain that αG_α and T_t extend to a contraction on $L^1(X,h\mu)$.
(b) The characterization of excessive functions in the form domain shows that for nonnegative $f \in L^2(X,\mu)$ and $\beta \ge \alpha$, the resolvent $G_\alpha f$ is q_β-excessive.

If q satisfies the first Beurling-Deny criterion, for $\alpha \le \beta$ the resolvent identity $G_\alpha = G_\beta + (\beta-\alpha)G_\alpha G_\beta$ extends from $L^2_+(X,\mu)$ to the extended operators on $L^+(X,\mu)$. This implies that for $f \in L^+(X,\mu)$ the map $(0,\infty) \to L^+(X,\mu)$, $\alpha \mapsto G_\alpha f$ is monotone decreasing. Hence, for each nonnegative $f \in L^+(X,\mu)$ the limit

$$Gf = \lim_{\alpha \to 0+} G_\alpha f$$

exists in $L^+(X,\mu)$. The map $G\colon L^+(X,\mu) \to L^+(X,\mu)$ is called the *Green operator* of q.

Lemma 3.5 *Let $q\colon L^2(X,\mu) \to [0,\infty]$ be a quadratic form satisfying the first Beurling-Deny criterion.*

(a) *Let $g \in L^+(X,\mu)$ and suppose that $Gg \in L^0(X,\mu)$. Then Gg is q-excessive.*
(b) *Suppose $g \in L^2_+(X,\mu)$ is strictly positive such that for all $f \in D(q)$*

$$\int_X |f| g \, d\mu \le q(f)^{1/2}.$$

Then Gg is strictly positive and $Gg \in L^1_+(X, g\mu)$.

Proof (a): According to the previous lemma, for $f \in L^2_+(X,\mu)$ and $\beta \leq \alpha$ the resolvent $G_\beta f$ is q_α-excessive. Now Gg is the limit in $L^0(X,\mu)$ of functions of the form $G_\beta f$ with $\beta \leq \alpha$ and $f \leq g$. Hence, Gg is also q_α-excessive by Lemma 3.2. We obtain

$$q(f \wedge Gg) + \alpha \, \|f \wedge Gg\|_2^2 = q_\alpha(f \wedge Gg) \leq q_\alpha(f) = q(f) + \alpha \, \|f\|_2^2 .$$

Letting $\alpha \to 0+$ yields the claim.

(b): Strict positivity follows from $\alpha G_\alpha g \to g$, as $\alpha \to \infty$, and the fact that $\alpha \to G_\alpha g$ is decreasing. Using the definition of G_α we obtain

$$\left(\int_X g G_\alpha g d\mu \right)^2 \leq q(G_\alpha g) = \langle g, G_\alpha g \rangle - \alpha \, \|G_\alpha g\|_2^2 .$$

This implies $\int_X g G_\alpha g d\mu \leq 1$ and we obtain the statement from the monotone convergence theorem. □

3.2 The Extensions q_e and q^+

Every quadratic form q on $L^2(X,\mu)$ can be interpreted to be a quadratic form on $L^0(X,\mu)$ by letting $q(f) = \infty$ for $f \in L^0(X,\mu) \setminus L^2(X,\mu)$. The following lemma shows that forms on $L^2(X,\mu)$ satisfying the first Beurling-Deny criterion are closable when viewed in this sense as forms on $L^0(X,\mu)$.

Proposition 3.6 *(Existence of the extended form) Let q be a closed quadratic form on $L^2(X,\mu)$ satisfying the first Beurling-Deny criterion. Then q is closable on $L^0(X,\mu)$ and its closure q_e is a quadratic form satisfying the first Beurling-Deny criterion. Moreover, $D(q_e) \cap L^2(X,\mu) = D(q)$ and a function $h \in L^0_+(X,\mu)$ is q-excessive if and only if it is q_e-excessive.*

Proof According to Lemma 2.2 we need to show that q is lower semicontinuous with respect to $L^0(X,\mu)$-convergence on its domain. Thus, let (f_n) and $f \in D(q)$ such that $f_n \to f$ in $L^0(X,\mu)$. Without loss of generality we can assume $\liminf_{n\to\infty} q(f_n) < \infty$ and $f_n \to f$ μ-a.e. (else choose a subsequence).

Case 1: There exists a q-excessive function $h > 0$.

First we assume there exists $C > 0$ such that $|f_n| \leq Ch$ for all $n \in \mathbb{N}$. For $\varphi \in L^1(X, h\mu) \cap L^2(X,\mu)$ and $\alpha > 0$ we obtain

$$q(f, G_\alpha \varphi) = \langle f, \varphi - \alpha G_\alpha \varphi \rangle = \lim_{n\to\infty} \langle f_n, \varphi - \alpha G_\alpha \varphi \rangle = q(f_n, G_\alpha \varphi).$$

For the second equality we used Lebesgue's dominated convergence theorem, which can be applied since $\alpha G_\alpha \varphi \in L^1(X, h\mu)$ by h being excessive. Elements of the form $G_\alpha \varphi$ as above are dense in $D(q)$ with respect to q. Since (f_n) is q-bounded, this implies $f_n \to f$ weakly with respect to q. Using the Cauchy-Schwarz inequality, weak convergence yields

$$q(f) \leq \liminf_{n\to\infty} q(f_n).$$

This shows lower semicontinuity for sequences, which are uniformly bounded by an excessive function.

With this at hand we can treat the general case. Since h is excessive, for any $C > 0$ the function Ch is excessive, and since $h > 0$, we have $f = \lim_{C\to\infty}(f \wedge (Ch)) \vee (-Ch)$ in $L^2(X, \mu)$. The L^2-lower semicontinuity of q, Ch being excessive and the lower semicontinuity for sequences which are uniformly bounded by an excessive function yield

$$\begin{aligned} q(f) &\leq \liminf_{C\to\infty} q((f \wedge (Ch)) \vee (-Ch)) \\ &\leq \liminf_{C\to\infty} \liminf_{n\to\infty} q((f_n \wedge (Ch)) \vee (-Ch)) \\ &\leq \liminf_{n\to\infty} q(f_n). \end{aligned}$$

This is the claim we wanted to prove.

Case 2: q does not have a strictly positive excessive function. By pointwise convergence the pointwise supremum $F = \sup_n |f_n|$ exists and is finite μ-a.e. We let $g = \varphi/(F \vee 1)$ for some $0 < \varphi \in L^1(X, \mu) \cap L^\infty(X, \mu)$. Then by construction (f_n) is bounded in $L^2(X, g\mu)$.

Next, for $\varepsilon > 0$ we consider the quadratic form $\tilde{q}\colon L^2(X, \mu) \to [0, \infty]$ defined by

$$\tilde{q}(f) = q(f) + \varepsilon \int_X f^2 g d\mu.$$

By Lemma 3.5 the Green operator of the form $\tilde{q}$ applied to g exists and is strictly positive. Therefore, $\tilde{q}$ has an excessive function $h > 0$. Moreover, $\liminf_{n\to\infty} \tilde{q}(f_n) < \infty$ by our choice of g. Hence, Case 1 shows

$$\tilde{q}(f) \leq \liminf_{n\to\infty} \tilde{q}(f_n).$$

Since (f_n) is bounded in $L^2(X, g\mu)$, we can take the limit $\varepsilon \to 0+$ and obtain the desired lower semicontinuity.

q_e satisfies the first Beurling Deny criterion: For $f \in D(q_e)$ we can choose $f_n \in D(q)$ with $f_n \to f$ in $L^0(X, \mu)$ and $q(f_n) \to q_e(f)$. The lower semicontinuity of q_e and $q_e = q$ on $D(q)$ yield

$$q_e(|f|) \leq \liminf_{n\to\infty} q_e(|f_n|) = \liminf_{n\to\infty} q(|f_n|) \leq \lim_{n\to\infty} q(f_n) = q_e(f).$$

It remains to prove $D(q_e) \cap L^2(X, \mu) = D(q)$ and the statement on excessive functions. The inclusion $D(q) \subseteq D(q_e) \cap L^2(X, \mu)$ is trivial. Let $f \in D(q_e) \cap L^2(X, \mu)$ and let (f_n) be a q-Cauchy sequence with $f_n \to f$ in $L^0(X, \mu)$. Without loss of generality we can assume $f_n \to f$ μ-a.e. (else choose a subsequence). Lebesgue's dominated convergence theorem implies

$$f = \lim_{n\to\infty} (f \wedge |f_n|) \vee (-|f_n|) = \lim_{n\to\infty} \lim_{m\to\infty} (f_m \wedge |f_n|) \vee (-|f_n|)$$

in $L^2(X, \mu)$. The L^2-lower semicontinuity of q and the subadditivity of q with respect to taking suprema yield

$$q(f) \le \liminf_{n\to\infty} \liminf_{m\to\infty} q((f_m \wedge |f_n|) \vee (-|f_n|)) \le 3 \lim_{n\to\infty} q(f_n) < \infty.$$

This shows $f \in D(q)$.

Since $q = q_e$ on $L^2(X, \mu)$ (here we use $D(q_e) \cap L^2(X, \mu) = D(q)$), nonnegative q_e-excessive functions are q-excessive. Now suppose $h \in L^0_+(X, \mu)$ is q-excessive. For $f \in D(q_e)$ we choose (f_n) in $D(q)$ with $f_n \to f$ in $L^0(X, \mu)$ and $q(f_n) \to q_e(f)$. The lower semicontinuity of q_e and $q = q_e$ on $D(q)$ yield

$$q_e(f \wedge h) \le \liminf_{n\to\infty} q_e(f_n \wedge h) = \liminf_{n\to\infty} q(f_n \wedge h) \le \liminf_{n\to\infty} q(f_n) = q_e(f).$$

This shows that h is also q_e-excessive. □

Definition 3.7 (*Extended form*) The closed quadratic form q_e on $L^0(X, \mu)$ introduced in the previous proposition is called the *extended form of q*.

Remark (a) Using the formula for q_e from Lemma 2.2 and the characterization of convergence in $L^0(X, \mu)$ in terms of μ-a.e. convergent subsequences, it is easy to see that if $\mathcal{E}$ is a Dirichlet form, then $D(\mathcal{E}_e)$ is the extended Dirichlet space of $\mathcal{E}$ and $\mathcal{E}_e$ is the extension of $\mathcal{E}$ to the extended Dirichlet space, see e.g. [3, Definition 1.1.4] for the definition of the extended Dirichlet space.

(b) The first assertion of this proposition (namely closability of q on $L^0(X, \mu)$) is known. For Dirichlet forms it is equivalent to the fact that the extension of the Dirichlet form to the extended Dirichlet space is well-defined, see [5, Theorem 1.5.2], which has its origin in [28]. In the generality we use here, the lower semicontinuity of q on its domain with respect to a.e.-convergence is the main result of [27]. As mentioned in the introduction, the proof given in [27] uses parts of the criticality theory we develop below. Therefore, we gave an independent proof. The idea of first proving lower semicontinuity for forms with excessive functions and then extending it to the general case is taken from [27] but our proof of Case 1, which is based on properties of excessive functions, is much shorter than the one in [27]. For Dirichlet forms a version of the argument is contained in the proof of [25, Theorem 1.59].

(c) To the best of our knowledge the literature only contains that the extension of q to the extended space $D(q_e)$ is lower semicontinuous on its domain, i.e.,

$f_n \to f$ in $L^0(X, \mu)$ and $f \in D(q_e)$ implies $q_e(f) \le \liminf_{n\to\infty} q_e(f_n)$, see e.g. [3, Corollary 1.1.9]. The statement of our proposition is stronger. Since our approach automatically gives closedness of q_e, we have lower semicontinuity of q_e on the whole space $L^0(X, \mu)$. This is not a mere technicality, but crucial for our considerations below.

Lemma 3.8 *Let q be a closed quadratic form on $L^2(X, \mu)$ satisfying the first Beurling-Deny criterion. For $\alpha > 0$ the operator $\alpha G_\alpha \colon D(q) \to D(q)$ extends by continuity with respect to q to a contraction $\alpha G_\alpha \colon D(q_e) \to D(q_e)$ with respect to q_e. This extension is compatible with the extension of αG_α to $L^0(X, \mu)$, which was discussed in Sect. 2.2. Moreover, for any $f \in D(q_e)$ we have $f - \alpha G_\alpha f \in L^2(X, \mu)$ and*

$$\lim_{\alpha\to 0+} q_e(\alpha G_\alpha f) = 0.$$

Proof By the spectral theorem for the (nonnegative) generator of q we have

$$q(\alpha G_\alpha f) = \int_{[0,\infty)} \frac{\lambda\alpha^2}{(\lambda+\alpha)^2} d\sigma_f(\lambda) \le \int_{[0,\infty)} \lambda d\sigma_f(\lambda) = q(f)$$

and

$$\alpha \|f - \alpha G_\alpha f\|^2 = \int_{[0,\infty)} \frac{\alpha\lambda^2}{(\lambda+\alpha)^2} d\sigma_f(\lambda) \le \int_{[0,\infty)} \lambda d\sigma_f(\lambda) = q(f),$$

where σ_f is the spectral measure of the generator at $f \in L^2(X, \mu)$. Using Lebesgue's dominated convergence theorem, this shows $\lim_{\alpha\to 0+} q(\alpha G_\alpha f) = 0$ for $f \in D(q)$.

Now let $f \in D(q_e)$ and let (f_n) be q-Cauchy with $f_n \to f$ in $L^0(X, \mu)$. By the lower semicontinuity of q_e we have $f_n \to f$ in the q_e-form topology. The inequalities above show that $(\alpha G_\alpha f_n)$ is q-Cauchy and that $(f_n - \alpha G_\alpha f_n)$ is $L^2(X, \mu)$-Cauchy. Hence, there exists $R_\alpha f \in L^0(X, \mu)$ such that $\alpha G_\alpha f_n \to R_\alpha f$ in $L^0(X, \mu)$ with $f - R_\alpha f \in L^2(X, \mu)$. This implies $R_\alpha f \in D(q_e)$ and

$$q_e(R_\alpha f) \le \liminf_{n\to\infty} q(\alpha G_\alpha f_n) \le \liminf_{n\to\infty} q(f_n) = q_e(f).$$

We proved that $\alpha G_\alpha \colon D(q) \to D(q)$ extends to a contraction $R_\alpha \colon D(q_e) \to D(q_e)$ with the continuity property that (f_n) in $D(q)$ with $f_n \to f$ in the form topology yields $\alpha G_\alpha f_n \to R_\alpha f$ in $L^0(X, \mu)$.

It remains to show $R_\alpha f = \alpha G_\alpha f$, where the right side of this equation denotes the extension of αG_α discussed in Sect. 2.2. Lemma 3.3 yields $|f_n| \to |f|$ in the q_e-form topology. Fatou's lemma for positivity preserving operators and our definition of R_α show

$$\alpha G_\alpha |f| \le \lim_{n\to\infty} \alpha G_\alpha |f_n| = R_\alpha |f| \in D(q_e).$$

This implies that $\alpha G_\alpha f$ exists. Now consider $g_n = (f_n \wedge |f|) \vee (-|f|)$. Since $D(q_e) \cap L^2(X, \mu) = D(q)$, we have $g_n \in D(q)$. Moreover, by Lemma 3.3 (g_n) converges to f in the form topology. Using $|g_n| \leq f$ we can apply Lebesgue's theorem for positivity preserving operators and obtain

$$\alpha G_\alpha f = \lim_{n \to \infty} \alpha G_\alpha g_n = R_\alpha f.$$

For the last equality we used our definition of R_α.

The convergence $\lim_{\alpha \to 0+} q_e(\alpha G_\alpha f) = 0$ follows from the fact that $D(q)$ is dense in $D(q_e)$ with respect to the form topology and that αG_α is a contraction. □

Remark The previous lemma is a resolvent version of [32, Lemma 2.8], which treats the semigroup. We included a short proof because in [32] the relation of the extension of G_α to $D(q_e)$ by continuity and by positivity are not explained.

There is another lower semicontinuous extension of q_e to $L^+(X, \mu)$, which we discuss next. Since the topologies of $L^+(X, \mu)$ and $L^0(X, \mu)$ agree on $L^0_+(X, \mu)$, the restriction of q_e to $D(q_e) \cap L^+(X, \mu)$ is lower semicontinuous with respect to $L^+(X, \mu)$-convergence. This implies that $q^+ : L^+(X, \mu) \to [0, \infty]$ given by

$$q^+(f) = \begin{cases} \lim\limits_{n \to \infty} q_e(f_n) & \text{if there ex. } q_e\text{-Cauchy sequence } (f_n) \text{ with } f_n \to f \text{ in } L^+(X, \mu) \\ \infty & \text{else} \end{cases}$$

is well-defined. The same arguments used in the proof showing that q_e is lower semicontinuous also yield that q^+ is lower semicontinuous, see the proof of [26, Lemma A.3]. The functional $q^+ : L^+(X, \mu) \to [0, \infty]$ is not a quadratic form as $L^+(X, \mu)$ is not even a vector space. However, it is readily verified that it is homogeneous, i.e., $\lambda^2 q^+(f) = q^+(\lambda f)$ for all $\lambda \geq 0$ and $f \in L^+(X, \mu)$.

The lower semicontinuity of q^+ and the corresponding inequality for q_e, Lemma 3.1, show

$$q^+(f \wedge g) + q^+(f \vee g) \leq q^+(f) + q^+(g).$$

With this at hand, if we let $D(q^+) = \{f \in L^+(X, \mu) \mid q^+(f) < \infty\}$, the same proof as the one we gave for the identity $D(q_e) \cap L^2(X, \mu) = D(q)$ yields

$$D(q^+) \cap L^0_+(X, \mu) = D(q_e) \cap L^0_+(X, \mu).$$

Moreover, if h is q-excessive (or equivalently q_e-excessive), then also

$$q^+(f \wedge h) \leq q^+(f), \quad f \in L^+(X, \mu).$$

The functional q^+ can assign finite values to functions taking the value ∞ on a set of positive measure. However, the following lemma shows that this is only the case if $\ker q^+$ is nontrivial.

Lemma 3.9 *Let q be a closed quadratic form on $L^2(X, \mu)$ satisfying the first Beurling-Deny criterion. For $f \in L^+(X, \mu)$ with $q^+(f) < \infty$ we have $\infty \cdot 1_{\{f=\infty\}} \in \ker q^+$.*

Proof Let $h = \infty \cdot 1_{\{f=\infty\}} = \lim_{\lambda \to 0+} \lambda f$. The lower semicontinuity and homogeneity of q^+ yield

$$q^+(h) \leq \liminf_{\lambda \to 0+} q^+(\lambda f) = \liminf_{\lambda \to 0+} \lambda^2 q^+(f) = 0. \quad \square$$

Lemma 3.10 *Let q be a closed quadratic form satisfying the first Beurling-Deny criterion. Let (f_n) be a sequence in $L^+(X, \mu)$ with $\sum_{k=1}^{\infty} q^+(f_k) < \infty$. Then*

$$q^+(\liminf_{n\to\infty} f_n) = q^+(\limsup_{n\to\infty} f_n) = 0.$$

In particular, if $\ker q^+ = \{0\}$, then $f_n \to 0$ μ-a.e.

Proof We only treat $\limsup f_n$, the statement on $\liminf f_n$ follows along the same lines. We have

$$\limsup_{n\to\infty} f_n = \lim_{n\to\infty} \sup_{l\geq n} f_l = \lim_{n\to\infty} \lim_{N\to\infty} \sup_{N\geq l\geq n} f_l$$

pointwise μ-a.e. and hence in $L^+(X, \mu)$. Since q^+ is lower semicontinuous on $L^+(X, \mu)$, we obtain

$$q^+(\limsup_{n\to\infty} f_n) \leq \liminf_{n\to\infty} \liminf_{N\to\infty} q^+(\sup_{N\geq l\geq n} f_l) \leq \liminf_{n\to\infty} \liminf_{N\to\infty} \sum_{l=n}^{N} q^+(f_l) = 0.$$

For the second inequality we used the subadditivity of q^+ with respect to taking suprema. The 'in particular'-statement follows immediately. $\square$

Remark The previous lemma is a continuity property for the functional q^+. If $\ker q^+ = \{0\}$, it yields that $(D(q_e), q_e)$ continuously embeds into $L^0(X, \mu)$. This observation is exploited further in Sect. 5.

3.3 *Invariant Sets and Irreducibilty*

Let q be a quadratic form on $L^p(X, \mu)$, $p \in \{0, 2\}$. We say that a measurable set $A \subseteq X$ is *q-invariant* if $q(1_A f) \leq q(f)$ for all $f \in L^p(X, \mu)$. Indeed, this definition of irreducibilty equals the usual one, see e.g. [25, Lemma 2.32]. The quadratic form q is called *irreducible* or *ergodic* if every q-invariant set A satisfies $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. Using lower semicontinuity it is easy to see that for a closed form satisfying the first Beurling-Deny criterion a set A is q-invariant if and only if it is q_e-invariant. In particular, q is irreducible if and only if q_e is irreducible.

Proposition 3.11 *For $p \in \{0, 2\}$ let q be a closed quadratic form on $L^p(X, \mu)$ satisfying the first Beurling-Deny criterion. Let $h \in L^0(X, \mu)$ be q-excessive.*

(a) $h_- \in \ker q$. *In particular, if* $\ker q = \{0\}$, *then every excessive function is nonnegative.*
(b) *If h is nonnegative, then $\{h > 0\}$ is q-invariant. In particular, if q is irreducible, every nontrivial nonnegative excessive function is strictly positive.*

Proof (a): $h_- = (-h) \vee 0 = -(h \wedge 0)$. We obtain

$$q(h_-) = q(-h_-) = q(h \wedge 0) \leq q(0) = 0.$$

This shows $h_- \in \ker q$.

(b): For $f \in L^p(X, \mu)$ the identity

$$1_{\{h>0\}} f = \lim_{n \to \infty} (f \wedge (nh)) \vee (-nh)$$

holds in $L^p(X, \mu)$. Since nh is also excessive, we obtain using lower semicontinuity

$$q(1_{\{h>0\}} f) \leq \liminf_{n \to \infty} q((f \wedge (nh)) \vee (-nh)) \leq q(f).$$

This shows the invariance of $\{h > 0\}$. □

Corollary 3.12 *For $p \in \{0, 2\}$ let q be an irreducible closed quadratic form on $L^p(X, \mu)$ satisfying the first Beurling-Deny criterion. Then* $\ker q$ *is at most one-dimensional and if* $\ker q \neq \{0\}$, *there exists a strictly positive $h \in L^p(X, \mu)$ such that* $\ker q = \mathbb{R}h$.

Proof Assume there exists $0 \neq h \in \ker q$ (else there is nothing to show). Since q satisfies the first Beurling-Deny criterion, we can assume $h \geq 0$ (else consider $|h|$ instead of h). Part (b) of the previous lemma yields $h > 0$ (use that functions in $\ker q$ are excessive).

Now let $h' \in \ker q_e$ and for $\alpha \in \mathbb{R}$ consider the function $g_\alpha = h' - \alpha h \in \ker q$. Using part (b) of the previous lemma again, g_α has fixed sign, i.e., either $g_\alpha = 0$, $g_\alpha > 0$ or $g_\alpha < 0$. Let $\alpha_0 = \sup\{\alpha \mid g_\alpha > 0\}$. Then, obviously, $0 = g_{\alpha_0} = h' - \alpha_0 h$. □

4 (Very) Weak and Abstract Poincaré and Hardy inequalities

In this section we discuss very weak Poincaré and Hardy inequalities, which are valid for most quadratic forms on Hilbert spaces. We then show how they yield weak Hardy inequalities for forms satisfying the first Beurling-Deny criterion with trivial kernel.

Moreover, we prove abstract Hardy inequalites, which hold for all forms satisfying the first Beurling-Deny criterion.

Our very weak Poincaré and Hardy inequalities are based on the following rather elementary observation.

Lemma 4.1 *(Completeness of weakly compact sets and continuity) Let $q\colon H \to [0,\infty]$ be a closed quadratic form on the Hilbert space H and let C be a weakly compact set in H. Then $D(q) \cap C$ equipped with the pseudometric induced by the seminorm $q^{1/2}$ is complete and for any $w \in (\ker q)^{\perp}$ the map*

$$D(q) \cap C \to \mathbb{R}, \quad f \mapsto \langle w, f \rangle$$

is continuous with respect to q.

Proof Let (f_n) be q-Cauchy in $D(q) \cap C$. By the Eberlein-Smulian theorem (f_n) has a subsequence (f_{n_k}) that converges weakly to some $f \in C$. The weak lower semicontinuity of q yields

$$q(f - f_n) \leq \liminf_{k \to \infty} q(f_{n_k} - f_n).$$

This shows $f \in D(q)$ and $f_n \to f$ with respect to q.

Let $w \in H$ with $w \perp \ker q$. Assume that the map $f \mapsto \langle w, f \rangle$ is not continuous. Then there exists $\varepsilon > 0$ and $f_n, f \in D(q) \cap C$ with $|\langle w, f - f_n \rangle| \geq \varepsilon$ for all $n \geq 1$ and $q(f - f_n) \to 0$, as $n \to \infty$. Employing the Eberlein-Smulian theorem again, we can assume without loss of generality $f - f_n \to h$ weakly in H. The weak lower semicontinuity of q yields

$$q(h) \leq \liminf_{n \to \infty} q(f - f_n) = 0,$$

so that $h \in \ker q$. But then

$$0 = |\langle w, h \rangle| = \lim_{n \to \infty} |\langle w, f_n - f \rangle| \geq \varepsilon,$$

a contradicition. □

We recall that we call a functional $\Phi\colon H \to [0,\infty]$ homogeneous if $\Phi(\lambda f) = |\lambda|^2 \Phi(f)$ for all $\lambda \in \mathbb{R}$ and $f \in H$.

Theorem 4.2 *(Very weak Poincaré/Hardy inequality) Let $q\colon H \to [0,\infty]$ be a closed quadratic form on the Hilbert space H and let $\Phi\colon H \to [0,\infty]$ be a homogeneous functional such that for every $R \geq 0$ the sublevel set $B_R^{\Phi} = \{f \in H \mid \Phi(f) \leq R\}$ is closed and bounded in H. Then for every $w \in (\ker q)^{\perp}$ there exists a decreasing function $\alpha = \alpha_w\colon (0,\infty) \to (0,\infty)$ such that for any $r > 0$ and all $f \in H$ we have*

$$|\langle w, f \rangle|^2 \leq \alpha(r) q(f) + r\Phi(f).$$

Proof Suppose that the statement does not hold. Then there exist $r > 0$ and a sequence (f_n) in $D(q)$ with

$$|\langle w, f_n\rangle|^2 > nq(f_n) + r\Phi(f_n).$$

In particular, $|\langle w, f_n\rangle| > 0$. Using that q is a quadratic form and that Φ is homogeneous, we can assume $|\langle w, f_n\rangle| = 1$. This implies $q(f_n) \to 0$ and $f_n \in B^{\Phi}_{1/r}$. Since in Hilbert spaces closed bounded convex sets are weakly compact, the previous lemma applied to $C = B^{\Phi}_{1/r}$ yields $\langle w, f_n\rangle \to 0$, a contradiction. □

Remark (a) We call these inequalities very weak Hardy/Poincaré inequalities because in the L^2-case they have $|\int_X fwd\mu|^2$ on their left side, while for weak Hardy/Poincaré inequalities we would like to have $\int_X f^2 wd\mu$ for some nonnegative w on the left side.

(b) The statement can be strengthened a bit. Since $w \in (\ker q)^{\perp}$, the functional Φ can be replaced by the homogeneous functional

$$\widetilde{\Phi}(f) = \inf\{\Phi(f + h) \mid h \in \ker q\}.$$

Theorem 4.3 *(*Weak Hardy inequality) Let q be a closed quadratic form on $L^2(X, \mu)$ satisfying the first Beurling-Deny criterion with $\ker q = \{0\}$. For every strictly positive $h \in L^2(X, \mu)$ and every nonnegative $w \in L^2(X, \mu) \cap L^1(X, h^2\mu)$, there exists a decreasing function $\alpha = \alpha_{w,h}\colon (0, \infty) \to (0, \infty)$ such that

$$\int_X |f|^2 wd\mu \le \alpha(r)q(f) + r\,\|f/h\|^2_\infty, \quad f \in D(q).$$

Proof Consider the homogeneous functional $\Phi\colon L^2(X, \mu) \to [0, \infty]$, $\Phi(f) = \|f/h\|^2_\infty$. It satisfies $\Phi(f) \le C^2$ if and only if $|f| \le Ch$. Since $h \in L^2(X, \mu)$, this implies that the sublevel sets of Φ are bounded and closed. Since $\ker q = \{0\}$, Theorem 4.2 applied to w and Φ yields a decreasing function $\alpha\colon (0, \infty) \to (0, \infty)$ with

$$|\langle f, w\rangle|^2 \le \alpha(r)q(f) + r\,\|f/h\|^2_\infty$$

for all $r > 0$ and $f \in L^2(X, \mu)$. Now suppose that the claimed inequality does not hold. Then there exists $r > 0$ and (f_n) with $\|f_n/h\|_\infty = 1$ and

$$\int_X |f_n|^2 wd\mu > nq(f_n) + r.$$

Since $|f_n| \le h$ and $wh^2 \in L^1(X, \mu)$, this implies $q(f_n) \to 0$. Moreover, by the first Beurling-Deny criterion, we can assume without loss of generality $f_n \ge 0$ (else consider $|f_n|$). Inequality ♡ then yields $f_n w \to 0$ in $L^1(X, \mu)$. We can assume without loss of generality that this convergence also holds μ-a.e. (else pass to a suitable subsequence). With this at hand and $|f_n|^2 w \le h^2 w \in L^1(X, \mu)$, we deduce with Lebesgue's dominated convergence theorem

$$0 = \lim_{n\to\infty} \int_X |f_n|^2 w d\mu > \liminf_{n\to\infty} nq(f_n) + r > r > 0,$$

a contradiction. □

Remark (a) This statement shows that for forms with the first Beurling-Deny criterion, the condition $\ker q = \{0\}$ implies a weak Hardy inequality. So, naturally, one may wonder whether the same is true for forms with $\ker q \neq \{0\}$. Of course, in this case, functions in $D(q)$ should be replaced by functions in $(\ker q)^\perp$ (with respect to the $L^2(X, w\mu)$-inner product) and hence one is looking for a weak Poincaré inequality. However, examples from [23] show that weak Poincaré inequalities need not hold in general and in Sect. 6 we discuss which additional assumptions are needed.

(b) Weak Poincaré inequalities were systematically introduced in [23] to characterize the rate of convergence to equilibrium for conservative Markovian semigroups without spectral gap. In our case $\ker q = \{0\}$, the associated semigroup (T_t) does not converge to an equilibrium (or projection to a ground state) but strongly to 0, as $t \to \infty$, see e.g. [10, Theorem 1.1]. Still one can ask for the corresponding rate of convergence.
If q has a strictly positive excessive function h, then (T_t) is a contraction on $L_h^\infty(X, \mu)$. With the same arguments as in the proof of [23, Theorem 2.1], one can then show that the inequality

$$(\diamond) \qquad \int_X |f|^2 d\mu \le \alpha(r)q(f) + r\,\|f/h\|_\infty^2, \quad r > 0, f \in D(q),$$

with some decreasing function $\alpha\colon (0, \infty) \to (0, \infty)$ implies

$$\|T_t f\|_2^2 \le \xi(t)(\|f\|_2^2 + \|f/h\|_\infty^2), \quad f \in L^2(X, \mu),$$

with $\xi(t) = \inf\{r > 0 \mid -\frac{1}{2}\alpha(r)\log r \le t\}$. Since $\xi(t) \to 0$ as $t \to \infty$, this is a uniform rate of convergence for the semigroup.
If $\mathcal{E}$ is a Dirichlet form on $L^2(X, \mu)$ with $\mu(X) < \infty$ and $\ker \mathcal{E} = \{0\}$, our theorem above applied to $w = 1$ and the excessive function $h = 1$ yields Inequality $\diamond$ for an appropriate function α. Note however, that our result only yields the existence of α but does not give an estimate for α (and ξ).

We can extend this weak Hardy inequality to all strictly positive h by passing to extended spaces. In this case, the condition on the kernel is a condition on the kernel of the extended form.

Corollary 4.4 *Let q be a closed quadratic form on $L^2(X, \mu)$ satisfying the first Beurling-Deny criterion with* $\ker q_e = \{0\}$. *Let $h \in L^0(X, \mu)$ be strictly positive. For all strictly positive $w \in L^1(X, (1 + h^2)\mu)$ there exists a decreasing function $\alpha = \alpha_{w,h} : (0, \infty) \to (0, \infty)$ such that for all $r > 0$*

$$\int_X |f|^2 w d\mu \leq \alpha(r)q(f) + r \|f/h\|_\infty^2, \quad f \in D(q_e).$$

Proof We let $\tilde{\mu} = w\mu$. The restriction of q_e to $L^2(X, \tilde{\mu})$ is a closed quadratic form satisfying the first Beurling-Deny criterion and it has trivial kernel. Our assumptions on w yield $1 \in L^2(X, \tilde{\mu}) \cap L^1(X, h^2\tilde{\mu})$. The previous theorem yields a decreasing function α sucht that

$$\int_X |f|^2 d\tilde{\mu} \leq \alpha(r)q_e(f) + r \|f/h\|_\infty^2,$$

for all $f \in D(q_e) \cap L^2(X, \tilde{\mu})$. By our choice of w the inclusion $f/h \in L^\infty(X, \mu)$ implies $f \in L^2(X, \tilde{\mu})$. Hence, the above inequality is indeed true for all $f \in D(q_e)$. Since $\tilde{\mu} = w\mu$, the claim follows. □

We finish this section with an abstract Hardy inequality that is valid for all forms satisfying the first Beurling-Deny criterion. It compares a transform of the given form with a quadratic form, which is not necessarily positive.

Proposition 4.5 (Abstract Hardy inequality) Let q be a closed quadratic form on $L^2(X, \mu)$ satisfying the first Beurling-Deny criterion. Let $h \in D(q_e)$ be nonnegative and let $f \in D(q_e)$ such that $hf, hf^2 \in D(q_e)$. Then

$$q_e(hf) \geq q_e(hf^2, h).$$

Proof Without loss of generality we can assume $hf, hf^2 \in D(q)$ (else change the measure μ to μ' to make sure that $hf, hf^2 \in D(q_e) \cap L^2(X, \mu') = D(q')$, where q' is the restriction of q_e to $L^2(X, \mu')$). The spectral theorem implies

$$\begin{aligned} q(hf) - q(hf^2, h) &= \lim_{\alpha \to 0+} \alpha \langle (I - \alpha G_\alpha)hf, hf \rangle - \lim_{\alpha \to 0+} \alpha \langle (I - \alpha G_\alpha)hf^2, h \rangle \\ &= \lim_{\alpha \to 0+} \alpha^2 \left(\langle G_\alpha hf^2, h \rangle - \langle G_\alpha hf, hf \rangle \right). \end{aligned}$$

Hence, it suffices to show that $\langle \alpha G_\alpha hf^2, h \rangle - \langle \alpha G_\alpha hf, hf \rangle$ is positive. Since $hf, hf^2 \in L^2(X, \mu)$ and since the resolvents are continuous, it suffices to verify positivity for simple functions $f = \sum_i \alpha_i 1_{A_i}$ with pairwise disjoint sets A_i with $h1_{A_i} \in L^2(X, \mu)$. Using the symmetry of G_α, for such a simple function we obtain

$$\begin{aligned}\langle G_\alpha hf^2, h\rangle - \langle G_\alpha hf, hf\rangle &= \sum_i \alpha_i^2 \langle G_\alpha h1_{A_i}, h\rangle - \sum_{i,j} \alpha_i\alpha_j \langle G_\alpha h1_{A_i}, h1_{A_j}\rangle \\ &= \frac{1}{2}\sum_{i,j} \langle G_\alpha h1_{A_i}, h1_{A_j}\rangle (\alpha_i - \alpha_j)^2 \\ &\quad + \sum_i \alpha_i^2 \langle G_\alpha h1_{A_i}, h\rangle - \sum_i \alpha_i^2 \langle G_\alpha h1_{A_i}, h1_{\cup_j A_j}\rangle.\end{aligned}$$

Since G_α is positivity preserving, the right side of this equation is nonnegative. □

Corollary 4.6 (Hardy inequality for perturbed forms) Let q be a closed quadratic form on $L^2(X, \mu)$ satisfying the first Beurling-Deny criterion. For all strictly positive $g \in L^2(X, \mu)$ and all $f \in D(q)$ and $\alpha > 0$ we have

$$q(f) + \alpha \|f\|_2^2 \geq \int_X f^2 \frac{g}{G_\alpha g} d\mu.$$

Proof We apply the previous proposition to the closed form $q_\alpha = q + \alpha \|\cdot\|^2$ with $D(q_\alpha) = D(q)$ and $h = G_\alpha g$. For $f \in D(q)$ we have $(f/h)h = f \in D(q)$. Hence, if also $(f/h)^2 h \in D(q)$, the previous proposition yields

$$\begin{aligned}q_\alpha(f) &= q_\alpha((f/h)h) \geq q_\alpha((f/h)^2 h, h) \\ &= q_\alpha((f/G_\alpha g)^2 G_\alpha g, G_\alpha g) = \int_X f^2 \frac{g}{G_\alpha g} d\mu.\end{aligned}$$

Next we show $(f/h)^2 h \in D(q)$ for all $f \in D(q) \cap L_h^\infty(X, \mu)$. As discussed in Lemma 3.5, the function h is q_α excessive and strictly positive (here we use that G_α is the Green operator of q_α). The form $q_{\alpha,h}\colon L^2(X, \mu) \to [0, \infty]$ defined by $q_{\alpha,h}(f) = q_\alpha(hf)$ is a Dirichlet form. Indeed, $h > 0$ implies that $q_{\alpha,h}$ is densely defined. Moreover, h is q_α-excessive (see Remark after Lemma 3.4) so that

$$q_{\alpha,h}(f \wedge 1) = q_\alpha(h(f \wedge 1)) = q_\alpha(hf \wedge h) \leq q_\alpha(hf) = q_{\alpha,h}(f)$$

shows that the constant function 1 is $q_{\alpha,h}$-excessive. Then $D(q_{\alpha,h}) \cap L^\infty(X, \mu)$ is an algebra, see e.g. [5, Theorem 1.4.2]. Since $D(q_{\alpha,h}) = \{f \in L^2(X, \mu) \mid fh \in D(q)\}$, we obtain for $f \in D(q) \cap L_h^\infty(X, \mu)$ that $f/h \in D(q_{\alpha,h}) \cap L^\infty(X, \mu)$. The algebra property yields $(f/h)^2 \in D(q_{\alpha,h}) \cap L^\infty(X, \mu)$, so that $(f/h)^2 h \in D(q)$.

Now let $f \in D(q)$. What we have shown so far yields the desired inequality for the functions $f_n = (f \wedge nh) \vee (-nh) \in D(q) \cap L_h^\infty(X, \mu)$. Using that nh is strictly positive and q_α-excessive shows

$$\int_X f^2 \frac{g}{G_\alpha g} d\mu = \lim_{n\to\infty} \int_X f_n^2 \frac{g}{G_\alpha g} d\mu \leq \limsup_{n\to\infty} q_\alpha(f_n) \leq q_\alpha(f). \quad \square$$

5 From Weak Hardy Inequalities to Hardy Inequalities—Subcriticality

In this section we discuss under which conditions weak Hardy inequalities lead to Hardy inequalities, i.e., when the function α in the weak hardy inequality is bounded. Forms that satisfy a Hardy inequality are called subcritical in the literature. Hence, the content of this section is devoted to characterizing subcriticality for quadratic forms satisfying the first Beurling-Deny criterion. Our arguments will rely on our weak and abstract Hardy inequalities and their corollaries.

Definition 5.1 (*Subcriticality*) A quadratic form q on $L^2(X, \mu)$ is called *subcritical* if there exists a strictly positive $w \in L^0(X, \mu)$ such that the following *Hardy inequality* holds

$$\int_X |f|^2 w d\mu \le q(f), \quad f \in D(q).$$

In this case, w is called *Hardy weight for q*.

Remark The definition of q^+ and q_e and Fatou's lemma yield that Hardy inequalities as in the last definition extend to $D(q^+)$ and $D(q_e)$. In particular, for subcritical q we have $D(q^+) = D(q_e) \cap L^0_+(X, \mu)$.

The aim of this section is to prove the following theorem.

Theorem 5.2 (Characterization subcriticality) Let q be a closed quadratic form on $L^2(X, \mu)$ satisfying the first Beurling-Deny criterion. The following assertions are equivalent.

(i) q is subcritical.
(ii) There exists a strictly positive $g \in L^0(X, \mu)$ such that for all $f \in D(q)$

$$\int_X |f| g d\mu \le q(f)^{1/2}.$$

(iii) $\ker q^+ = \{0\}$.
(iv) $\ker q_e = \{0\}$ and there exists a strictly positive q-excessive function $h \in D(q_e)$.
(v) For one strictly positive $f \in L^0(X, \mu)$ the limit $Gf = \lim_{\alpha \to 0+} G_\alpha f$ exists in $L^0(X, \mu)$.
(vi) For all strictly positive q-excessive functions $h \in L^0(X, \mu)$ and all $f \in L^1_+(X, h\mu)$ the limit $Gf = \lim_{\alpha \to 0+} G_\alpha f$ exists in $L^0(X, \mu)$.
(vii) $\ker q_e = \{0\}$ and the map

$$D(q_e) \to D(q_e), \quad f \mapsto |f|$$

is continuous with respect to q_e.

(viii) The embedding $(D(q_e), q_e) \to L^0(X, \mu)$, $f \mapsto f$ is continuous.
(ix) $(D(q_e), q_e)$ is a Hilbert space.

Proof *Step 1*: We first discuss the equivalence of (iii), (iv), (vii), (viii) and (ix).

(viii) $\Leftrightarrow$ (ix): This is the content of Theorem 2.3.

(viii) and (ix) $\Rightarrow$ (iv): Let $g \in D(q_e)$ with $g > 0$ be given. By (viii) the set $\{f \in D(q_e) \mid f \geq g\}$ is a closed nonempty convex set in the Hilbert space $(D(q_e), q_e)$. Hence, by the approximation theorem in Hilbert spaces, there exists $h \in D(q_e)$ with

$$q_e(h) = \inf\{q_e(f) \mid f \geq g\}.$$

For nonnegative $\varphi \in D(q_e)$ and $\varepsilon > 0$ we have $h + \varepsilon\varphi \geq g$ such that

$$q_e(h) \leq q_e(h + \varepsilon\varphi) = q_e(h) + 2\varepsilon q_e(h, \varphi) + \varepsilon^2 q_e(\varphi).$$

Letting $\varepsilon \to 0+$ yields $q_e(h, \varphi) \geq 0$, so that h is q_e-excessive by Lemma 3.4 (choose a measure $\mu' = \varphi\mu$ such that $h \in L^2(X, \mu')$ and the result follows from Lemma 3.4 applied to the restriction of q_e to $L^2(X, \mu')$).

Since $L^0(X, \mu)$ is Hausdorff, the continuity of the embedding $D(q_e) \to L^0(X, \mu)$ implies that $(D(q_e), q_e)$ is Hausdorff as well. This means $\ker q_e = \{0\}$.

(vii) $\Rightarrow$ (viii): Let $h \in D(q_e)$ be strictly positive. Since $\ker q_e = \{0\}$, Corollary 4.4 yields a function $\alpha\colon (0, \infty) \to (0, \infty)$ and $w > 0$ such that

$$\int_X (|f| \wedge h)^2 w \, d\mu \leq \alpha(r) q_e(|f| \wedge h) + r$$

for all $r > 0$ and $f \in D(q_e)$. Let (f_n) in $D(q_e)$ with $q_e(f_n) \to 0$ be given. The continuity of $|\cdot|$ with respect to q_e implies that the map $D(q_e) \to D(q_e)$, $f \mapsto |f| \wedge h$ is also continuous with respect to q_e. We obtain $q_e(|f_n| \wedge h) \to 0$ and our weak Hardy inequality implies

$$\int_X (|f_n| \wedge h)^2 w \, d\mu \to 0.$$

This yields $f_n \to 0$ in $L^0(X, \mu)$ and we arrive at (viii).

(viii) $\Rightarrow$ (vii): Assertion (viii) implies that the form topology is induced by the norm $q_e^{1/2}$. With this at hand, the assertion follows from Lemma 3.3.

(iv) $\Rightarrow$ (iii): Let $f \in \ker q^+$ and let $f_n \in D(q_e)$ with $f_n \to f$ and $q_e(f_n) \to 0$. Let $h \in D(q_e)$ be excessive and strictly positive. Using $f_n \wedge h \to f \wedge h$ in $L^0(X, \mu)$ and the lower semicontinuity of q_e we obtain

$$q_e(f \wedge h) \leq \liminf_{n\to\infty} q_e(f_n \wedge h) \leq \liminf_{n\to\infty} q_e(f_n) = 0.$$

This implies $f \wedge h \in \ker q_e$, so that $f \wedge h = 0$ by our assumption. The strict positivity of h yields $f = 0$.

(iii) $\Rightarrow$ (viii): Let (f_n) be in $D(q_e)$ with $q_e(f_n) \to 0$. Since (f_n) is an arbitrary sequence with this property, it suffices to show that (f_n) has a subsequence converging to 0 μ-a.e. Using the first Beurling-Deny criterion, we choose a subsequence (f_{n_k}) such that

$$\sum_{k=1}^{\infty} q^+(|f_{n_k}|) = \sum_{k=1}^{\infty} q_e(|f_{n_k}|) < \infty.$$

With this and $\ker q^+ = \{0\}$ at hand, $|f_{n_k}| \to 0$ μ-a.e. follows from Lemma 3.10.

Step 2: Now that we established the equivalence of (iii), (iv), (vii), (viii) and (ix), we show that all of these are equivalent to (ii), (v) and (vi).

(vi) $\Rightarrow$ (v): This is trivial.

(v) $\Rightarrow$ (ii): Without loss of generality we can assume that f is strictly positive and in $L^1(X, \mu) \cap L^2(X, \mu)$. Consider the function $F = f/(Gf \vee 1)$. Then $F \leq f$ and $F \leq f/Gf$, and so the monotonicity of G yields $\int_X FGFd\mu \leq \|f\|_1$. This implies

$$q_\alpha(G_\alpha F) = \langle F, G_\alpha F\rangle \leq \|f\|_1 .$$

For $g \in D(q)$ we obtain

$$\langle |g|, F\rangle = q_\alpha(|g|, G_\alpha F) \leq q_\alpha(|g|)^{1/2} q_\alpha(G_\alpha F)^{1/2} \leq \|f\|_1^{1/2}\, q_\alpha(g)^{1/2}.$$

Letting $\alpha \to 0+$ yields (ii).

(ii) $\Rightarrow$ (iv): If (ii) holds we have $\ker q_e = \{0\}$ (since the inequality extends to the extended space, see the remark before this theorem). The existence of a strictly positive excessive function und (ii) is the content of Lemma 3.5 (b).

(iv) $\Rightarrow$ (vi): Let $h \in L^0(X, \mu)$ be striclty positive and q-excessive and let $f \in L^1_+(X, h\mu)$. Without loss of generality we can assume $h \in D(q_e)$ (else consider $h \wedge h'$ for a strictly positive excessive function $h' \in D(q_e)$). Lemma 3.8 and (viii) imply $\alpha G_\alpha h \to 0$ in $L^0(X, \mu)$ as $\alpha \to 0+$. The resolvent identity and $\alpha G_\alpha h \leq h$, see Lemma 3.4, show that this convergence is monotone and that

$$G_\beta(h - \alpha G_\alpha h) = G_\alpha h - \beta G_\alpha G_\beta h \leq G_\alpha h \leq \alpha^{-1} h.$$

Let $A_\alpha = \{\alpha G_\alpha h \leq h/2\}$ and note that $A_\alpha \nearrow X$ as $\alpha \to 0+$ by our previous considerations. Hence, it suffices to show that Gf is a.s. finite on A_α. Using symmetry of the extended resolvents and the inequality above, for $\beta > 0$ we estimate

$$\int_{A_\alpha} G_\beta fh d\mu \leq 2 \int_X f G_\beta(h - \alpha G_\alpha h) d\mu = \frac{1}{\alpha} \int_X fh d\mu.$$

Hence, we obtain $1_{A_\alpha} Gf \in L^1(X, h\mu)$ and arrive at (ii).

Step 3: Assertions (ii) to (ix) are equivalent to (i):

(i) $\Rightarrow$ (ii): Without loss of generality we can assume that the Hardy weight w is in $L^1(X, \mu)$. For $f \in D(q)$ we obtain

$$\int_X |f| w d\mu \le \left(\int_X |f|^2 w d\mu \right)^{1/2} \left(\int_X w d\mu \right)^{1/2} \le C q(f)^{1/2}.$$

This yields the claim.

(v) $\Rightarrow$ (i): Without loss of generality we can assume $Gg \in L^0(X, \mu)$ for some strictly positive $g \in L^2(X, \mu)$. Then Corollary 4.6 yields the claim for the Hardy weight $w = g/Gg$ after letting $\alpha \to 0+$. □

Remark (*Hardy weights*) The proof of the theorem shows that for all strictly positive $g \in L^0(X, \mu)$ with $Gg \in L^0(X, \mu)$ the function $w = g/Gg$ is a Hardy weight for q. A criterion for the existence of Gg is given by assertion (iv).

Remark (*State of the art*)

(a) Let $\mathcal{E}$ be a Dirichlet form. In this case, the constant function 1 is $\mathcal{E}$-excessive and assertion (iii) reduces to $\ker \mathcal{E}_e = \{0\}$. This is one characterization of transience of the Dirichlet form $\mathcal{E}$ and the equivalence of transience to (ii), (iii) (iv), (v), (vi) and (ix) is well-known, see the discussion in [5, Section 1.6 and Notes]. The equivalence of transience to subcriticality with Hardy weight $w = g/Gg$ is more or less contained in [4] for regular Dirichlet forms. The additional regularity assumption in [4] allows that the measure $w\mu$ on one side of the Hardy inequality can even be replaced by a smooth measure. The equivalence of transience to (viii) is based on Theorem 2.3 and taken from [25].

(b) For second-order linear elliptic operators the relation of Hardy inequalities and the behavior of the resolvent at the infimum of the spectrum (which in our case can always be taken to equal 0), i.e., the equivalence of (i) and (v), (vi) is well known, see e.g. [16, 19, 21] and references therein.
The connection of subcriticality to transience of h-transformed Schrödinger type forms (Dirichlet form + form induced by a potential) is studied in the recent [32], with previous results in [30, 31]. For Schrödinger type forms on discrete spaces corresponding results were obtained in [12].

(c) The implication (viii) and (ix) $\Rightarrow$ (iv) uses a standard argument showing that certain minimizers of the 'energy' q are 'superharmonic' functions. The idea to use the function $f/(Gf \vee 1)$ in the proof of implication (v) $\Rightarrow$ (ii) is taken from the proof of [5, Theorem 1.5.1].

(d) The equivalence of subcriticality to (iii) and (vii) seems to be a new observation.

With this at hand we obtain that there are three types of irreducible forms satisfying the first Beurling-Deny criterion.

Corollary 5.3 *Let q be an irreducible quadratic form on $L^2(X, \mu)$ satisfying the first Beurling-Deny criterion. Then precisely one the following assertions holds.*

(i) *q is subcritical.*
(ii) $\ker q_e = \mathbb{R}h$ *for some strictly positive $h \in L^0(X, \mu)$.*
(iii) $\ker q^+ = \{0, \infty\}$.

Moreover, (iii) holds if and only if q does not possess a nontrivial nonnegative excessive function.

Proof Suppose (i) does not hold. According to our theorem $\ker q^+ \neq \{0\}$. Now there are two cases:

Case 1: $\ker q^+ \cap L^0(X, \mu) \neq \{0\}$. Since $D(q_e) \cap L^0_+(X, \mu) = D(q^+) \cap L^0_+(X, \mu)$, this implies $\ker q_e \neq \emptyset$. Now (ii) follows from Corollary 3.12. Since $\ker q_e \cap L^0_+(X, \mu) \subseteq \ker q^+$, this implies that (iii) does not hold.

Case 2: $\ker q^+ \cap L^0(X, \mu) = \{0\}$. Let $0 \neq h \in \ker q^+$. Since $h \notin L^0(X, \mu)$, the set $A = \{h = \infty\}$ has positive measure. We show that it is q_e-invariant as this implies (iii) by the irreducibilty of q.

According to Lemma 3.9, we have $1_A \cdot \infty \in \ker q^+$. For nonnegative $f \in D(q_e)$ we obtain

$$q_e(1_A f) = q_e(f \wedge (1_A \cdot \infty)) = q^+(f \wedge (1_A \cdot \infty)) \leq q^+(f) + q^+(1_A \cdot \infty) = q_e(f).$$

Since q_e satisfies the first Beurling-Deny criterion, this implies $f 1_A \in D(q_e)$ for all $f \in D(q_e)$ and

$$\begin{aligned} q_e(1_A f) &= q_e(1_A f_+) + q_e(1_A f_-) - 2q_e(1_A f_+, 1_A f_-) \\ &\leq q_e(f_+) + q_e(f_-) - 2q_e(1_A f_+, 1_A f_-). \end{aligned}$$

It remains to prove $q_e(1_A f_+, 1_A f_-) \geq q_e(f_+, f_-)$ to establish the invariance of A. Nonnegative $h, h' \in D(q_e)$ with $h \wedge h' = 0$ satisfy $q_e(h, h') \leq 0$ (this is a consequence of Lemma 3.1). Hence,

$$\begin{aligned} q_e(f_+, f_-) &= q_e(1_A f_+, 1_A f_-) + q_e(1_A f_+, 1_{X\setminus A} f_-) \\ &\quad + q_e(1_{X\setminus A} f_+, 1_A f_-) + q_e(1_A f_+, 1_{X\setminus A} f_-) \\ &\leq q_e(1_A f_+, 1_A f_-). \end{aligned}$$

It remains to prove the 'Moreover' statement. If (iii) does not hold, either (i) or (ii) are satisfied. If (i) holds, then q has an excessive function by the previous theorem and if (ii) holds, h is an excessive function.

Now suppose q has a nontrivial excessive function $h \in L^0_+(X, \mu)$. If ∞ were in $\ker q^+$, we would obtain

$$q^+(h) = q^+(\infty \wedge h) \leq q^+(\infty) = 0.$$

This implies $\ker q_+ \neq \{0, \infty\}$, so that (iii) does not hold. □

Definition 5.4 (*Criticality*) An irreducible form satisfying (ii) in the previous corollary is called *critical* and the strictly positive function h is an *Agmon ground state* of q.

Remark For irreducible Dirichlet forms the dichotomy between subcriticality (transience) and criticality (recurrence) is well-known, see [5, Section 1.6]. It is also known for classical Schrödinger operators [20], certain generalized Schrödinger forms [30] and discrete Schrödinger operators [12].

For general irreducible forms however, it may happen happen that they are neither critical nor subcritical. According to the corollary, this is precisely the case if they do not possess excessive functions.

In concrete models the existence of an excessive function is often known. Indeed, we do not have a counterexample of a closed form satisfiying the first Beurling-Deny criterion without excessive function. Excessive functions usually correspond to superharmonic functions with respect to an associated 'weakly defined operator'. Existence results for such functions are sometimes referred to as Allegretto-Piepenbrink type theorems, see e.g. [6, 14, 29] and references therein. Below in Appendix 6 we prove existence of excessive functions for irreducible forms for which the semigroup admits a heat kernel, an assumption that is satisfied for the models considered in the mentioned [12, 20, 30]. Our existence result relies on a weak Harnack principle, which is shown to hold for kernel operators with strictly positive kernel.

6 Weak Poincaré Inequalities and Completeness of Extended form Domains

In this section we discuss when weak Poincaré inequalities hold. For conservative Dirichlet forms on $L^2(X, \mu)$ with finite μ, they have been introduced and extensively studied in [23], which also contains an abundance of examples and further references. Therefore, here we restrict ourselves to two additional abstract criteria for the validity of such an inequality.

For simplicity we assume irreducibility of the form even though this is not necessary. Moreover, we write $f \perp_w h$ to state that the functions f and h are orthogonal in the Hilbert space $L^2(X, w\mu)$.

Theorem 6.1 *Let q be an irreducible closed quadratic form on $L^2(X, m)$ satisfying the first Beurling-Deny criterion. Assume further that q is critical such that* $\ker q_e = \mathbb{R}h$ *for some strictly positive $h \in L^0(X, \mu)$. The following assertions are equivalent.*

(i) *$(D(q_e)/\mathbb{R}h, q_e)$ is a Hilbert space.*
(ii) *The map*

$$(D(q_e)/\mathbb{R}h, q_e) \to L^0(X, \mu)/\mathbb{R}h, \quad f + \mathbb{R}h \mapsto f + \mathbb{R}h$$

is continuous. Here, $L^0(X, \mu)/\mathbb{R}h$ is equipped with the quotient topology.

(iii) *For one/all strictly positive $w \in L^1(X, h^2\mu)$ there exists a decreasing $\alpha : (0, \infty) \to (0, \infty)$ such that for all $f \in L^2(X, w\mu)$ with $f \perp_w h$ and all $r > 0$ we have*

$$\int_X f^2 w d\mu \leq \alpha(r) q_e(f) + r \|f/h\|_\infty^2 .$$

Proof The equivalence of (i) and (ii) is the content of Theorem 2.3.

(iii) $\Rightarrow$ (ii): It suffices to show that if (f_n) is a sequence in $D(q_e)$ with $q_e(f_n) \to 0$, then there exist $C_n \in \mathbb{R}$ such that $f_n - C_n h \to 0$ in $L^0(X, \mu)$. Let $w \in L^1(X, h^2\mu)$ be strictly positive such that the weak Poincaré inequality holds and consider

$$T : L^0(X, \mu) \to L^2(X, w\mu), \quad Tf = (f \wedge h) \vee (-h).$$

For each $n \in \mathbb{N}$ there exists $C_n \in \mathbb{R}$ such that $T(f_n - C_n h) \perp_w h$. Since also $|Tf| = |f| \wedge h$, the weak Poincaré inequality applied to $T(f_n - C_n h)$ and $h \in \ker q_e$ imply

$$\int_X (|f_n - C_n h| \wedge h)^2 w d\mu \leq \alpha(r) q_e(T(f_n - C_n h)) + r \leq \alpha(r) q_e(f_n) + r, \quad r > 0.$$

Since h, w are strictly positive, we infer $f_n - C_n h \to 0$ in $L^0(X, \mu)$ from $q_e(f_n) \to 0$.

(ii) $\Rightarrow$ (iii): Suppose that the weak Poincaré inequality does not hold for a given strictly positive $w \in L^1(X, h^2\mu)$. Then there exist $r > 0$ and a sequence (f_n) in $D(q_e) \cap L_h^\infty(X, \mu)$ with $f_n \perp_w h$ such that $\int f_n^2 w d\mu = 1$ and

$$1 > n q_e(f_n) + r^2 \|f_n/h\|_\infty^2 .$$

This implies $|f_n| \leq h/r$ and $q_e(f_n) \to 0$. Assertion (ii) yields $C_n \in \mathbb{R}$ such that $f_n - C_n h \to 0$ in $L^0(X, \mu)$. Without loss of generality we assume the convergence holds μ-a.e. (else pass to a suitable subsequence). It suffices to show $C_n \to 0$ and hence $f_n \to 0$ μ-a.e. Indeed, using $|f_n| \leq h/r$ and $h \in L^2(X, w\mu)$, Lebesgue's dominated convergence theorem and $f_n \to 0$ μ-a.e. imply

$$0 = \lim_{n\to\infty} \int_X f_n^2 w d\mu = 1,$$

a contradiction.

From $h > 0$ and

$$|C_n h| \leq |C_n h - f_n| + |f_n| \leq |C_n h - f_n| + h/r \to h/r, \quad n \to \infty,$$

it follows that C_n is bounded. Hence, $f_n - C_n h$ is bounded by some constant times h. Using this and $f_n \perp_w h$, we conclude with the help of Lebesgue's dominated convergence theorem

$$-C_n \int_X h^2 w d\mu = \int_X (f_n - C_n h) h w d\mu \to 0, \quad n \to \infty.$$

This shows the required $C_n \to 0$. □

Remark (a) Instead of working with the norm $\|\cdot/h\|_\infty$ on the right side of the inequality in (iii), we could have also used $\delta_h(f) = \operatorname{ess\,sup} f/h - \operatorname{ess\,inf} f/h$. Note that δ_h is the quotient norm on $L^\infty_h(X,\mu)/\mathbb{R}h$.

(b) It is known for recurrent (critical) irreducible Dirichlet forms $\mathcal{E}$ that Poincaré type inequalites yield that $(D(\mathcal{E}_e)/\mathbb{R}1, \mathcal{E}_e)$ is a Hilbert space, see e.g. the discussion in [5, Section 4.8], which is based on [17] and treats Harris recurrent Dirichlet forms. The observation that the converse can be characterized by a weak Poincaré inequality seems to be new.
Recall Theorem 5.2, which states that if $(D(q_e), q_e)$ is a Hilbert space (and in particular $\ker q_e = \{0\}$), not only weak Hardy inequalites but Hardy inequalites hold with respect to certain Hardy weights. It would be interesting to know whether Poincaré inequalites with respect to certain weights hold under the condition that $(D(q_e)/\ker q_e, q_e)$ is a Hilbert space or what else has to be assumed. As mentioned above, for Dirichlet forms Harris recurrence is sufficient for a Poincaré inequality to hold.
In some sense a Poincaré inequality for q can be interpreted as subcriticality of the form q considered on the quotient space $L^2(X,\mu)/\mathbb{R}h \simeq (\mathbb{R}h)^\perp$ (or q_e on the quotient $L^0(X,\mu)/\mathbb{R}h$). These quotients do not carry a good order structure. Hence, the methods used in the proof of Theorem 5.2, which heavily rely on the order structure of the function spaces, are not available.

For an irreducible conservative Dirichlet form $\mathcal{E}$ and finite μ, in [23] it is noted that the validity of a weak Poincaré inequality (with $h = w = 1$) is equivalent to Kusuoka-Aida's weak spectral gap property discussed in [1, 13]. The *weak spectral gap property* is said to hold if sequences (f_n) in $D(\mathcal{E})$ with $\|f_n\|_2 \le 1, f_n \perp 1$ and $\mathcal{E}(f_n) \to 0$ converge to 0 in measure with respect to μ. In this sense, our main observation is that the weak spectral gap property is the same as the continuity of the embedding in assertion (ii). As shown in [1, Lemma 2.6], the weak spectral gap property holds if the semigroup is a semigroup of kernel operators. Hence, we obtain the following corollary.

Corollary 6.2 *Let $\mathcal{E}$ be an irreducible conservative Dirichlet form on $L^2(X,\mu)$ with finite μ. If the associated semigroup admits an integral kernel, then $(D(\mathcal{E}_e)/\mathbb{R}1, \mathcal{E}_e)$ is a Hilbert space.*

Reference [23, Theorem 7.1] gives a sharp criterion for the weak Poincaré inequality for a conservative irreducible Dirichlet form on a configuration space over a non-compact manifold M (with a finite measure μ on the configuration space, $h = w = 1$ and with respect to the quotient norm $\delta(f) = \operatorname{ess\,sup} f - \operatorname{ess\,inf} f$ on the right side of the weak Poincaré inequality). In particular, it shows that if

$$\lambda(r) = \inf\{\|\nabla f\|_{\mathbf{L}^2(M)} \mid f \in C_c^\infty(M) \text{ with } \|f\|_{L^2(M)} = 1 \text{ and } \|f\|_\infty^2 \le r\} = 0$$

for some $r > 0$, then the considered Dirichlet form on the configuration space over M does not satisfy a weak Poincaré inequality. But for $M = \mathbb{R}$ it is readily verified that $\lambda(r) = 0$ for any $r > 0$ (for $n \in \mathbb{N}$ consider smoothed versions of $f_n\colon \mathbb{R} \to \mathbb{R}$, $f_n(x) = (-|x|/n^2 + 1/n)_+$). Hence, on $M = \mathbb{R}$ the Dirichlet forms on configuration space over $\mathbb{R}$ described in [23, Section 7] do not satisfy weak Poincaré inequalities and according to our theorem their extended Dirichlet space is not complete. The existence of such examples seems to be a new observation and therefore we state it as a corollary to our theorem.

Corollary 6.3 *There exists an irreducible Dirichlet form $\mathcal{E}$ with* $\ker \mathcal{E}_e = \mathbb{R}1$*, such that* $(D(\mathcal{E}_e)/\mathbb{R}1, \mathcal{E}_e)$ *is not a Hilbert space.*

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Appendix: Existence of Excessive Functions and a Question of Schep

In this appendix we show that an irreducible form with the first Beurling-Deny criterion has a strictly positive excessive function if its semigroup is a semigroup of kernel operators. This result is based on a more general observation for positivity preserving kernel operators on L^p-spaces, which answers a question of Schep for this particular class of operators.

In this section (Y, ν) denotes another σ-finite measure space. We assume $1 < p < \infty$ and consider positivity preserving operators $T\colon L^p(X, \mu) \to L^p(Y, \nu)$, which are automatically continuous, see e.g. [15, Proposition 1.3.5]. Their adjoint is denoted by $T^*\colon L^q(Y, \nu) \to L^q(X, \mu)$ with $1/q + 1/p = 1$. Note that if $f \in L^p_+(X, \mu)$, then $f^{p-1} \in L^q(X, \mu)$. In particular, if T is positivity preserving, $T^*(Tf)^{p-1}$ is well-defined for $f \in L^p_+(X, \mu)$ and belongs to $L^q(X, \mu)$.

We consider the quantity

$$\lambda(T) = \inf\{\lambda \geq 0 \mid \text{ ex. strictly positive } f \in L^p_+(X, \mu) \text{ with } T^*(Tf)^{p-1} \leq \lambda f^{p-1}\}.$$

It turns out that $\lambda(T) = \|T\|^p$, see [24, Theorem 4 and Theorem 8], but we shall not use this fact.

By definition, for $\lambda > \lambda(T)$ there exists a strictly positive $f \in L^p_+(X, \mu)$ with $T^*(Tf)^{p-1} \leq \lambda f^{p-1}$. Schep asked in [24, Section 8] what happens at $\lambda = \lambda(T)$. In this case, one cannot expect to find a corresponding $f \in L^p_+(X, \mu)$, but can still hope for a strictly positive $f \in L^0_+(X, \mu)$. The following theorem shows that this is indeed true if T is a kernel operator with strictly positive integral kernel.

Theorem A.1 *Let $1 < p < \infty$ and let $T : L^p(X, \mu) \to L^p(Y, \nu)$ be a kernel operator with strictly positive integral kernel. Then there exists a strictly positive $h \in L^0(X, \mu)$ such that*

$$T^*(Th)^{p-1} \le \lambda(T)h^{p-1}.$$

This theorem and our existence result for excessive functions are a consequence of the following lemma, whose proof we give at the end of this section.

Lemma A.2 *Let $1 < p < \infty$ and let $T : L^p(X, \mu) \to L^p(Y, \nu)$ be a kernel operator with strictly positive integral kernel. Let $\lambda \ge 0$ and let $f_n \in L^p(X, \mu)$ be strictly positive with $T^*(Tf_n)^{p-1} \le \lambda f_n^{p-1}$. Then there exist $C_n > 0$ such that $h = \liminf_{n\to\infty} C_n f_n$ satisfies $0 < h < \infty$ μ-a.e. and*

$$T^*(Th)^{p-1} \le \lambda h^{p-1}.$$

Proof of Theorem A.1 Choose a sequence $\lambda_n \searrow \lambda(T)$ and let $f_n \in L^p(X, \mu)$ be strictly positive with $T^*(Tf_n)^{p-1} \le \lambda_n f_n^{p-1}$. Choose C_n according to Lemma A.2 and set $h = \liminf_{n\to\infty} C_n f_n$. Then h is striclty positive and in $L^0(X, \mu)$. Fatou's lemma for positivity preserving operators yields

$$T^*(Th)^{p-1} \le \liminf_{n\to\infty} T^*(T(C_n f_n))^{p-1} \le \liminf_{n\to\infty} \lambda_n (C_n f_n)^{p-1} = \lambda(T)h^{p-1}.\square$$

Theorem A.3 *Let q be an irreducible quadratic form on $L^2(X, \mu)$ satisfying the first Beurling-Deny criterion. Assume that the associated semigroup is a semigroup of kernel operators. Then there exists a strictly positive q-excessive function.*

Proof Irreducibility yields that the integral kernel of the associated semigroup (T_t) is strictly positive. We choose a strictly positive function $g > 0$. Then $f_\alpha = G_\alpha g$ is strictly positive and $T_t f_\alpha \le e^{t\alpha} f_\alpha$ (use that $G_\alpha g$ is q_α-excessive and that $(e^{-t\alpha}T_t)_{t>0}$ is the semigroup of q_α, then apply Lemma 3.4). According to Lemma A.2 applied to $p = 2$ and $T_t = (T_{t/2})^* T_{t/2}$, there exist $C_\alpha > 0$ such that $h = \liminf_{\alpha\to 0+} C_\alpha f_\alpha$ is strictly positive and in $L^0(X, \mu)$. Using Fatou's lemma for positivity preserving operators we obtain

$$T_t h \le \liminf_{\alpha\to 0+} T_t C_\alpha f_\alpha \le \liminf_{\alpha\to 0+} e^{t\alpha} C_\alpha f_\alpha = h.$$

Since this is true for any $t > 0$, Lemma 3.4 yields that h is excessive. □

Remark The existence of an integral kernel for the semigroup (the heat kernel) is e.g. guaranteed if X is a separable metric space, μ is a Borel measure of full support on X and $T_t L^2(X, \mu) \subseteq C(X)$, $t > 0$, see e.g. [11]. This property is a question of local regularity for solutions to the corresponding heat equation and satisfied in the situations discussed in [12, 20, 30]. Another criterion ensuring the existence of heat kernels is L^1-L^∞ ultracontractivity in the case of Dirichlet forms.

We now prove Lemma A.2 for operators T which are *positivity improving* ($f \geq 0$ and $f \neq 0$ implies that Tf is strictly positive) and satisfy a weak Harnack principle. Then we show that the weak Harnack principle holds for kernel operators with strictly positive kernel. We start with two ergodicity properties for postivity improving operators.

Lemma A.4 *(Ergodicity) Let $1 < p < \infty$ and let $T : L^p(X, \mu) \to L^p(Y, \nu)$ be a positivity improving operator. Every measurable $A \subseteq X$ with*

$$T^*(T 1_A f)^{p-1} \leq 1_A T^*(Tf)^{p-1}$$

for all $f \in L^p_+(X, \mu)$ satisfies $\mu(A) = 0$ or $\mu(X \setminus A) = 0$.

Proof If T is positivity improving, then T^* is also positivity improving. If $\mu(A) \neq 0$, we can choose $f \in L^p_+(X, \mu)$ with $f \neq 0$ on A. The positivity improving property for T and T^* and the inequality for A yields

$$0 < T^*(T 1_A f)^{p-1} \leq 1_A T^*(Tf)^{p-1}.$$

Hence, $\mu(X \setminus A) = 0$. □

Lemma A.5 *Let $1 < p < \infty$ and let $T : L^p(X, \mu) \to L^p(Y, \nu)$ be positivity improving. Then $\lambda(T) > 0$. Let $h \in L^+(X, \mu)$ satisfy*

$$T^*(Th)^{p-1} \leq K h^{p-1}$$

for some $K \geq \lambda(T) > 0$. Then either $h = 0$, $0 < h < \infty$ or $h = \infty$ holds μ-a.s.

Proof $\lambda(T) > 0$ directly follows from T (and hence T^*) being positivity improving.

We show that the sets $A_1 = \{h > 0\}$ and $A_2 = \{h = \infty\}$ satisfy

$$T^*(T 1_{A_i} f)^{p-1} \leq 1_{A_i} T^*(Tf)^{p-1}$$

for all $f \in L^p_+(X, \mu)$, $i = 1, 2$. With this at hand, the statement follows from Lemma A.4.

For any $f \in L^p_+(X, \mu)$ we have

$$1_{\{h>0\}} f = \lim_{n\to\infty} f \wedge (nh) \text{ and } 1_{\{h=\infty\}} f = \lim_{n\to\infty} f \wedge (n^{-1}h),$$

where the limits hold in $L^p(X, \mu)$ due to Lebesgue's dominated convergence theorem. Moreover, since T and T^* are positivity preserving, we obtain

$$T^*(T(f \wedge (nh)))^{p-1} \leq T^*(Tf)^{p-1}$$

and

$$T^*(T(f \wedge (nh)))^{p-1} \leq T^*(T(nh))^{p-1} \leq K(nh)^{p-1}.$$

These observations yield

$$\begin{aligned} T^*(T1_{\{h>0\}}f)^{p-1} &= \lim_{n\to\infty} T^*(Tf \wedge (nh))^{p-1} \\ &\leq \lim_{n\to\infty} (K(nh)^{p-1}) \wedge T^*(Tf)^{p-1} \\ &= 1_{\{h>0\}} T^*(Tf)^{p-1}. \end{aligned}$$

A similar computation yields the statement for the set $\{h = \infty\}$. □

Definition A.6 (Weak Harnack principle) Let $1 < p < \infty$ and let $T: L^p(X, \mu) \to L^p(Y, \nu)$ be a positivity preserving linear operator. We say that T satisfies the *weak Harnack principle at* $\lambda \geq 0$ if there exist $D > 0$ and measurable sets $A, B \subseteq X$ with $\mu(A), \mu(B) > 0$, such that for every $f \in L^p_+(X, \mu)$ with

$$T^*(Tf)^{p-1} \leq \lambda f^{p-1}$$

we have

$$\int_A f d\mu \leq D \operatorname*{ess\,inf}_{x\in B} f.$$

We can now prove a version of Lemma A.2 for operators satisfying a weak Harnack principle.

Lemma A.7 *Let* $1 < p < \infty$ *and let* $T: L^p(X, \mu) \to L^p(Y, \nu)$ *be a positivity improving operator that satisfies the weak Harnack principle at some* $\lambda \geq \lambda(T)$. *Let* $f_n \in L^p(X, \mu)$ *be strictly positive with* $T^*(Tf_n)^{p-1} \leq \lambda f_n^{p-1}$. *Then there exist* $C_n > 0$ *such that* $h = \liminf_{n\to\infty} C_n f_n$ *satisfies* $0 < h < \infty$ μ*-a.e. and*

$$T^*(Th)^{p-1} \leq \lambda h^{p-1}.$$

Proof Let $A, B \subseteq X$ and let $D > 0$ as in the weak Harnack principle, which we apply to f_n. From $\int_A f_n d\mu > 0$ we obtain $\operatorname{ess\,inf}_{x\in B} f_n > 0$. Hence, we can choose $C_n > 0$ such that $\operatorname{ess\,inf}_{x\in B} C_n f_n = 1$. We set

$$h = \liminf_{n\to\infty} C_n f_n \in L^+(M).$$

Fatou's lemma for positivity preserving operators yields

$$T^*(Th)^{p-1} \leq \liminf_{n\to\infty} T^*(TC_n f_n)^{p-1} \leq \liminf_{n\to\infty} \lambda (C_n f_n)^{p-1} = \lambda h^{p-1}.$$

According to Lemma A.5, it suffices to exclude the cases $h = 0$ and $h = \infty$ to obtain the desired statement. From $\operatorname{ess\,inf}_{x\in B} C_n f_n = 1$ we infer $h \geq 1$ on B. Therefore, $h \neq 0$. Moreover, Fatou's Lemma yields

$$\int_A h d\mu \leq \liminf_{n\to\infty} \int_A C_n f_n d\mu \leq \liminf_{n\to\infty} D \operatorname*{ess\,inf}_{x\in B} C_n f_n = D.$$

In particular, h is μ-a.s. finite on A and so we obtain $h \neq \infty$. □

Lemma A.8 *Let $k\colon Y \times X \to (0,\infty)$ be bimeasurable. Then the function*

$$\tilde{k}\colon X \times X \to [0,\infty), \quad \tilde{k}(x,y) = \int_Y k(z,x)k(z,y)d\nu(z)$$

is bimeasurable and there exists $c > 0$ and a measurable set $A \subseteq X$ with $\mu(A) > 0$ such that

$$\tilde{k} \geq c \text{ on } A \times A.$$

Proof The measurability of $\tilde{k}$ follows from Fubini's theorem. For the estimate we can assume that μ and ν are probability measures (else restrict everything to sets of finite measure and rescale). For $n \in \mathbb{N}$ we consider the sets $D_n = \{(y,x) \in Y \times X \mid k(y,x) \geq 1/n\}$ and for $x \in X$ we consider the measurable sections $(D_n)^x = \{y \in Y \mid (y,x) \in D_n\}$. By definition we have

$$\tilde{k}(x,y) = \int_Y k(z,y)k(z,x)d\nu(z) \geq \frac{1}{n^2}\nu((D_n)^x \cap (D_n)^y).$$

Hence, we need to prove that for some $n \in \mathbb{N}$ there exists $C > 0$ and a set A with $\mu(A) > 0$ and $\nu((D_n)^x \cap (D_n)^y) \geq C > 0$ for all $x, y \in A$.

Our assumption $k > 0$ yields $D_n \nearrow Y \times X$, as $n \to \infty$. Using the monotone convergence theorem, for $\varepsilon > 0$ we find $n_\varepsilon \in \mathbb{N}$ with

$$(\nu \otimes \mu)(D_{n_\varepsilon}) \geq 1 - \varepsilon.$$

We claim that $A_\varepsilon = \{x \in X \mid \nu((D_{n_\varepsilon})^x) > 1 - 2\varepsilon\}$ satisfies $\mu(A_\varepsilon) > 0$ (here we use Fubini's theorem for the measurability of A_ε). If this were not the case, we would have $\nu((D_{n_\varepsilon})^x) \leq 1 - 2\varepsilon$ for μ-a.e. $x \in X$. But then Fubini's theorem leads to the contradiction

$$(\nu \otimes \mu)(D_{n_\varepsilon}) = \int_Y \nu((D_{n_\varepsilon})^x)d\mu(x) \leq 1 - 2\varepsilon.$$

Hence, for $x, y \in A_\varepsilon$ we have

$$\nu((D_{n_\varepsilon})^x \cap (D_{n_\varepsilon})^y) = \nu((D_{n_\varepsilon})^x) + \nu((D_{n_\varepsilon})^y) - \nu((D_{n_\varepsilon})^x \cup (D_{n_\varepsilon})^y) > 1 - 4\varepsilon,$$

so that $A = A_\varepsilon$ for some $\varepsilon < 1/4$ is a set as required. □

Remark If there exists $c > 0$ and not neglibile sets $A \subseteq X$, $B \subseteq Y$ such that $k \geq c$ on $B \times A$, then the statement of the previous lemma trivially follows from the definition of $\tilde{k}$. Note however that $k > 0$ does not imply the existence of such sets, see e.g. [7, Theorem B.1].

Lemma A.9 *(Weak Harnack principle for kernel operators) Let* $1 < p < \infty$ *and let* $T: L^p(X, \mu) \to L^p(Y, \nu)$ *be a kernel operator with strictly positive kernel. Then* T *satisfies the weak harnack principle at every* $\lambda > 0$.

Proof Let $\lambda > 0$ and let $f \in L^p_+(X, \mu)$ with $T^*(Tf)^{p-1} \leq \lambda f^{p-1}$. Using the previous lemma we choose $c > 0$ and a set A with $\mu(A) > 0$ such that

$$\tilde{k}(x, y) = \int_Y k(z, x)k(z, y)d\nu(z) \geq c$$

for all $x, y \in A$. Let q such that $1/q + q/p = 1$ and let $\varphi \in L^q_+(X, \mu)$. Using Minkowski's inequality for integrals and Fubini's theorem we obtain

$$\begin{aligned}
\lambda \int_X f(x)\varphi(x)d\mu(x) &\geq \int_X \varphi(x) \left(\int_Y k(z, x) \left(\int_X k(z, y)f(y)d\mu(y) \right)^{p-1} d\nu(z) \right)^{1/(p-1)} d\mu(x) \\
&\geq \left(\int_X \varphi(x) \left(\int_Y k(z, x) \int_X k(z, y)f(y)d\mu(y)d\nu(z) \right)^{p-1} d\mu(x) \right)^{1/(p-1)} \\
&= \left(\int_X \varphi(x) \left(\int_X \tilde{k}(x, y)f(y)d\mu(y) \right)^{p-1} d\mu(x) \right)^{1/(p-1)} \\
&\geq c \int_A \varphi(x)d\mu(x) \int_A f(y)d\mu(y).
\end{aligned}$$

Now we let $\varphi = 1_B$, where $B \subseteq A$ is a set wit $0 < m(B) < \infty$ such that $f \leq \operatorname{ess\,inf}_{x\in A} f + \varepsilon$ on B. We obtain

$$\lambda(\operatorname*{ess\,inf}_{x\in A} f + \varepsilon)m(B) \geq cm(B) \int_A f(y)d\mu(y).$$

Since $\varepsilon > 0$ is arbitrary, this proves the claim with constant $D = \lambda/c > 0$. □

Remark We proved that in principle the weak Harnack principle holds at all $\lambda > 0$. However, nontrivial functions with $T^*(Tf)^{p-1} \leq \lambda f^{p-1}$ only exist for $\lambda \geq \lambda(T)$.

Proof of Lemma A.2 This follows directly from the previous two lemmas. □

References

1. S. Aida, Uniform positivity improving property, Sobolev inequalities, and spectral gaps. J. Funct. Anal. **158**(1), 152–185 (1998)
2. A. Ancona, Continuité des contractions dans les espaces de Dirichlet. In *Séminaire de Théorie du Potentiel de Paris, No. 2. Lecture Notes in Mathematics, vol. 563 (University of Paris, Paris, 1975–1976)*, pp. 1–26
3. Z.-Q. Chen, M. Fukushima, *Symmetric Markov processes, time change, and boundary theory*. London Mathematical Society Monographs Series, vol. 35 (Princeton University Press, Princeton, NJ, 2012)
4. P.J. Fitzsimmons, Hardy's inequality for Dirichlet forms. J. Math. Anal. Appl. **250**(2), 548–560 (2000)
5. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet forms and symmetric Markov processes*, vol. 19. de Gruyter Studies in Mathematics (Walter de Gruyter & Co., Berlin, extended edition, 2011)
6. S. Haeseler, M. Keller, Generalized solutions and spectrum for Dirichlet forms on graphs, in *Random walks, boundaries and spectra*, vol. 64 of. Progress in Probability (Birkhäuser/Springer Basel AG, Basel, 2011), pp. 181–199
7. P.R. Halmos, V.S. Sunder, Bounded integral operators on L^2 spaces, in *Ergebnisse der Mathematik und ihrer Grenzgebiete* [*Results in Mathematics and Related Areas*], vol. 96. (Springer, Berlin, 1978)
8. T. Husain, *The Open Mapping and Closed Graph Theorems in Topological Vector Spaces* (Clarendon Press, Oxford, 1965)
9. N. Kajino, Equivalence of recurrence and Liouville property for symmetric Dirichlet forms. Mat. Fiz. Komp yut. Model.**3**(40), 89–98 (2017)
10. M. Keller, D. Lenz, H. Vogt, R. Wojciechowski, Note on basic features of large time behaviour of heat kernels. J. Reine Angew. Math. **708**, 73–95 (2015)
11. M. Keller, D. Lenz, H. Vogt, R. Wojciechowski, Note on basic features of large time behaviour of heat kernels. J. Reine Angew. Math. **708**, 73–95 (2015)
12. M. Keller, Y. Pinchover, F. Pogorzelski, Criticality theory for Schrödinger operators on graphs. J. Spectr. Theory **10**(1), 73–114 (2020)
13. Shigeo Kusuoka, Analysis on Wiener spaces. II. Differential forms. J. Funct. Anal. **103**(2), 229–274 (1992)
14. D. Lenz, P. Stollmann, I. Veselić, The Allegretto-Piepenbrink theorem for strongly local Dirichlet forms. Doc. Math. **14**, 167–189 (2009)
15. P. Meyer-Nieberg, *Banach lattices* (Springer, Berlin, 1991)
16. M. Murata, Structure of positive solutions to $(-\Delta + V)u = 0$ in $\mathbf{R}^n$, in *Proceedings of the Conference on Spectral and Scattering Theory for Differential Operators*(Fujisakura-so,Seizô Itô, Tokyo, 1986), pp.64–108
17. Y. Ōshima, Potential of recurrent symmetric Markov processes and its associated Dirichlet spaces, in *Functional Analysis in Markov Processes (Katata/Kyoto, 1981)*, vol. 923. Lecture Notes in Mathematics (Springer, Berlin, 1982), pp. 260–275
18. E..-M.. Ouhabaz, Invariance of closed convex sets and domination criteria for semigroups. Potential Anal. **5**(6), 611–625 (1996)
19. Y. Pinchover, Topics in the theory of positive solutions of second-order elliptic and parabolic partial differential equations, in *Spectral Theory and Mathematical Physics: A Festschrift in Honor of Barry Simon's 60th Birthday*, vol. 76. *Proceedings of Symposium on Pure Mathematics* (American Mathematical Society, Providence, RI, 2007), pp. 329–355
20. Y. Pinchover, K. Tintarev, A ground state alternative for singular Schrödinger operators. J. Funct. Anal. **230**(1), 65–77 (2006)
21. R.G. Pinsky, *Positive harmonic functions and diffusion*. Cambridge Studies in Advanced Mathematics, vol. 45 (Cambridge University Press, Cambridge, 1995)
22. M. Reed, B. Simon. *Methods of Modern Mathematical Physics. IV. Analysis of Operators* (Academic [Harcourt Brace Jovanovich, Publishers], New York, 1978)

23. M. Röckner, F.-Y. Wang, Weak Poincaré inequalities and L^2-convergence rates of Markov semigroups. J. Funct. Anal. **185**(2), 564–603 (2001)
24. A.R. Schep, Positive operators on L^p-spaces. In *Positivity, Trends in Mathematics* (Birkhäuser, Basel, 2007), pp. 229–254
25. M. Schmidt. Energy forms. Disseration (2017) arXiv:1703.04883
26. M. Schmidt, A note on reflected Dirichlet forms. Potential Anal. **52**(2), 245–279 (2020)
27. B. Schmuland, Positivity preserving forms have the Fatou property. Potential Anal. **10**(4), 373–378 (1999)
28. M.L. Silverstein, *Symmetric Markov processes*, Lecture Notes in Mathematics, vol. 426 (Springer, Berlin, 1974)
29. B. Simon, Schrödinger semigroups. Bull. Amer. Math. Soc. (N.S.) **7**(3), 447–526 (1982)
30. M. Takeda, Criticality and subcriticality of generalized Schrödinger forms. Illinois J. Math. **58**(1), 251–277 (2014)
31. M. Takeda, Criticality for Schrödinger type operators based on recurrent symmetric stable processes. Trans. Amer. Math. Soc. **368**(1), 149–167 (2016)
32. M. Takeda, T. Uemura, Criticality of Schrödinger forms and recurrence of Dirichlet forms (2021). Preprint, arXiv:2102.06869

Maximal Displacement of Branching Symmetric Stable Processes

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Abstract We determine the limiting distribution and the explicit tail behavior for the maximal displacement of a branching symmetric stable process with spatially inhomogeneous branching structure. Here the branching rate is a Kato class measure with compact support and can be singular with respect to the Lebesgue measure.

Keywords Branching symmetric stable process · Symmetric stable process · Kato class measure · Additive functional

Mathematics Subject Classification 60J80 · 60J75 · 60F05 · 60J55 · 60J46

1 Introduction

We studied in [26] the limiting distributions for the maximal displacement of a branching Brownian motion with spatially inhomogeneous branching structure. In this paper, we show the corresponding results for a branching symmetric stable process. Our results clarify how the tail behavior of the underlying process would affect the long time asymptotic properties of the maximal displacement.

There has been great interest in the maximal displacement of branching Brownian motions and branching random walks with light tails for which the associated branching structures are spatially homogeneous. We refer to [9] as a pioneering work, and to [8, Sects. 5–7] and references therein, and [22, 32] for recent developments on this research subject. On the other hand, Durrett [17] proved the weak convergence of a properly normalized maximal displacement of a branching random walk on $\mathbb{R}$ with regularly varying tails. This result in particular shows that the maximal displacement grows exponentially in contrast with the light tailed model. Bhattacharya-Hazra-Roy [3, Theorem 2.1] further showed the weak convergence of point processes associated with the scaled particle positions. We refer to [20, 24] for related studies on

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the maximal displacement of branching stable processes or branching random walks with regularly varying tails.

Here we are interested in how the spatial inhomogeneity of the branching structure would affect the behavior of the maximal displacement. For a branching Brownian motion on $\mathbb{R}$, Erickson [18] determined the linear growth rate of the maximal displacement in terms of the principal eigenvalue of the Schrödinger type operator. Lalley-Sellke [23] then showed that a properly shifted maximal displacement is weakly convergent to a random shift of the Gumbel distribution with respect to the limiting martingale. Bocharov and Harris [6, 7] proved the results corresponding to [18, 23] for a catalytic branching Brownian motion on $\mathbb{R}$ in which reproduction occurs only at the origin. We showed in [29, 30] that the results in [6, 18] are valid for the branching Brownian motion motion on $\mathbb{R}^d$ $(d \geq 1)$ in which the branching rate is a Kato class measure with compact support. Under the same setting, Nishimori and the author [26] further determined the limiting distribution of the shifted maximal displacement as in [7, 23]. This result reveals that, even though reproduction occurs only on a compact set unlike [23], the spatial dimension d appears in the lower order term of the maximal displacement. We refer to [5, 25] for further developments.

Carmona-Hu [14] and Bulinskaya [11, 12] obtained the linear growth rate and weak convergence for the maximal displacement of a branching random walk on $\mathbb{Z}^d$ in which reproduction occurs only on finite points and the underlying random walk has light tails. We note that the underlying random walk in [11, 12, 14] is irreducible and allowed to be nonsymmetric, and the so-called $L \log L$ condition is sufficient for the validity of these results as proved by [11, 12]. Recently, Bulinskaya [13] showed the weak convergence of a properly normalized maximal displacement of a branching random walk on $\mathbb{Z}$ with regularly varying tails as in [3, 17] and reproduction occurs only on finite points. As for the spatially homogeneous model, the growth rate of the maximal displacement is exponential in contrast with the light tailed model.

In this paper, we prove the weak convergence and the long time tail behavior for the maximal displacement of a branching symmetric stable process on $\mathbb{R}^d$ with spatially inhomogeneous branching structure (Theorems 17 and 18). We will then see that the maximal displacement grows exponentially and the growth rate is determined by the principal eigenvalue of the Schrödinger type operator associated with the fractional Laplacian. The spatial dimension d also affects the limiting distribution and tail behavior of the maximal displacement. Our results are applicable to a branching symmetric stable process in which reproduction occurs only on singular sets.

Our results can be regarded as a continuous state space and multidimensional analogue of [13]. In particular, we provide an explicit form of the limiting distribution for the maximal displacement. On the other hand, since our approach is based on the second moment method as for [26], we need the second moment condition on the offspring distribution, which is stronger than the $L \log L$ condition as imposed in [3, 13, 17].

We note that the functional analytic approach works well for the continuous space model. Our model of branching symmetric stable processes is closely related to the Schrödinger type operator associated with the fractional Laplacian and Kato class measure through the first moment formula on the expected population (Lemma 15).

It is possible to calculate or estimate the principal eigenvalue of the Schrödinger type operator as in Sect. 3.3.

The rest of this paper is organized as follows: In Sect. 2, we first prove the resolvent asymptotic behaviors of a symmetric stable process. We then discuss the invariance of the essential spectra, and the asymptotics of the integral associated with the ground state, for the Schrödinger type operator. We finally determine asymptotic behaviors of the Feynman-Kac functional. In Sect. 3, we first introduce a model of branching symmetric stable processes. We then present our results in this paper with examples. In Sect. 4, we prove the weak convergence result (Theorem 17) by following the approach of [26, Theorem 2.4].

2 Symmetric Stable Processes

For $\alpha \in (0, 2)$, let $\mathbf{M} = (\{X_t\}_{t\geq 0}, \{P_x\}_{x\in\mathbb{R}^d})$ be a symmetric α-stable process on $\mathbb{R}^d$ generated by $-(-\Delta)^{\alpha/2}/2$. Let $p_t(x, y)$ be the transition density function of $\mathbf{M}$,

$$P_x(X_t \in A) = \int_A p_t(x, y)\,\mathrm{d}y, \quad t > 0, x \in \mathbb{R}^d, A \in \mathcal{B}(\mathbb{R}^d).$$

Here $\mathcal{B}(\mathbb{R}^d)$ is the family of Borel measurable sets on $\mathbb{R}^d$. By [4, Theorem 2.1] and [37, Lemmas 2.1 and 2.2], we have

Lemma 1 *There exists a positive continuous function g on $[0, \infty)$ such that*

$$p_t(x, y) = \frac{1}{t^{d/\alpha}} g\left(\frac{|x - y|}{t^{1/\alpha}}\right). \tag{1}$$

Moreover, the function g satisfies the following.

(i) *The value $g(0)$ is given by*

$$g(0) = \frac{2^{d/\alpha}\Gamma(d/\alpha)}{\alpha 2^{d-1}\pi^{d/2}\Gamma(d/2)}.$$

(ii) *There exists $c > 0$ such that for any $r \geq 0$,*

$$0 \leq g(0) - g(r) \leq cr^2. \tag{2}$$

(iii) *The next equality holds.*

$$\lim_{r\to\infty} r^{d+\alpha} g(r) = \frac{\alpha \sin(\alpha\pi/2)\Gamma((d+\alpha)/2)\Gamma(\alpha/2)}{2^{2-\alpha}\pi^{1+d/2}} (=: C_{d,\alpha}). \tag{3}$$

2.1 Resolvent Asymptotics

In this subsection, we prove the asymptotic behaviors and the so-called 3G-type inequality for the resolvent density of **M**. For $\beta > 0$, let $G_\beta(x, y)$ denote the β-resolvent density of **M**,

$$G_\beta(x, y) = \int_0^\infty e^{-\beta t} p_t(x, y)\,\mathrm{d}t.$$

Define

$$w_\beta(r) = \int_0^\infty e^{-\beta t} \frac{1}{t^{d/\alpha}} g\left(\frac{r}{t^{1/\alpha}}\right)\,\mathrm{d}t \ (r \geq 0)$$

so that $G_\beta(x, y) = w_\beta(|x - y|)$ by (1).

Lemma 2 *Let* $\beta > 0$.

(i) *The function* w_β *satisfies*

$$\lim_{r\to\infty} r^{d+\alpha} w_\beta(r) = \beta^{-2} C_{d,\alpha} \tag{4}$$

and

$$w_\beta(r) \sim \begin{cases} \beta^{(d-\alpha)/\alpha} \Gamma((\alpha - d)/d) g(0) & (d < \alpha), \\ \alpha g(0) \log r^{-1} & (d = \alpha), \\ \alpha r^{\alpha-d} \int_0^\infty s^{d-\alpha-1} g(s)\,\mathrm{d}s & (d > \alpha), \end{cases} \quad (r \to +0). \tag{5}$$

(ii) *There exists* $C > 0$ *such that for any* $x, y, z \in \mathbb{R}^d$,

$$G_\beta(x, y) G_\beta(y, z) \leq C G_\beta(x, z) \left(G_\beta(x, y) + G_\beta(y, z)\right).$$

Proof (ii) follows by (i) and direct calculation. We now show (i). The relation (4) follows by (3) and the dominated convergence theorem:

$$r^{d+\alpha} w_\beta(r) = \int_0^\infty e^{-\beta t} \left(\frac{r}{t^{1/\alpha}}\right)^{d+\alpha} g\left(\frac{r}{t^{1/\alpha}}\right) t\,\mathrm{d}t \to \beta^{-2} C_{d,\alpha} \quad (r \to \infty).$$

We next show (5). If $d < \alpha$, then as $r \to +0$,

$$w_\beta(r) \to \int_0^\infty e^{-\beta t} t^{-d/\alpha} \, \mathrm{d}t \, g(0) = \beta^{(d-\alpha)/\alpha} \Gamma\left(\frac{\alpha - d}{\alpha}\right) g(0).$$

If $d > \alpha$, then (3) implies that as $r \to +0$,

$$r^{d-\alpha} w_\beta(r) = \alpha \int_0^\infty e^{-\beta (r/s)^\alpha} s^{d-\alpha-1} g(s) \, \mathrm{d}s \to \alpha \int_0^\infty s^{d-\alpha-1} g(s) \, \mathrm{d}s.$$

We now assume that $d = \alpha (= 1)$. Let

$$w_\beta(r) = \int_0^{r^\alpha} e^{-\beta t} t^{-1} g\left(\frac{r}{t^{1/\alpha}}\right) \mathrm{d}t + \int_{r^\alpha}^\infty e^{-\beta t} t^{-1} g\left(\frac{r}{t^{1/\alpha}}\right) \mathrm{d}t = \text{(I)} + \text{(II)}.$$

Then by (3),

$$\text{(I)} \le \frac{c_1}{r^{2\alpha}} \int_0^{r^\alpha} e^{-\beta t} t \, \mathrm{d}t \le c_2.$$

Let

$$\text{(II)} = \int_{r^\alpha}^\infty e^{-\beta t} t^{-1} \left(g\left(\frac{r}{t^{1/\alpha}}\right) - g(0) \right) \mathrm{d}t + g(0) \int_{r^\alpha}^\infty e^{-\beta t} t^{-1} \, \mathrm{d}t.$$

Then by (2),

$$\left| \int_{r^\alpha}^\infty e^{-\beta t} t^{-1} \left(g\left(\frac{r}{t^{1/\alpha}}\right) - g(0) \right) \mathrm{d}t \right| \le c_3 r^2 \int_{r^\alpha}^\infty e^{-\beta t} t^{-1-2/\alpha} \, \mathrm{d}t \le c_4.$$

Since

$$\int_{r^\alpha}^\infty e^{-\beta t} t^{-1} \, \mathrm{d}t \sim \alpha \log\left(\frac{1}{r}\right) \quad (r \to +0),$$

we obtain the desired assertion for $d = \alpha$. □

If $d > \alpha$, then $\mathbf{M}$ is transient and the Green function $G(x, y) = \int_0^\infty p_t(x, y) \, \mathrm{d}t$ is given by

$$G(x, y) = \frac{2^{1-\alpha} \Gamma((d-\alpha)/2)}{\pi^{d/2} \Gamma(\alpha/2)} |x - y|^{\alpha - d}. \tag{6}$$

We use the notation $G_0(x, y) = G(x, y)$ for $d > \alpha$.

2.2 Spectral Properties of Schrödinger Type Operators with the Fractional Laplacian

In this subsection, we study spectral properties of a Schrödinger type operator associated with the fractional Laplacian and Green tight Kato class measure. We first introduce Kato class and Green tight measures, and the associated bilinear forms.

Definition 3 (i) Let μ be a positive Radon measure on $\mathbb{R}^d$. Then μ belongs to the Kato class ($\mu \in \mathcal{K}$ in notation) if

$$\lim_{\beta\to\infty} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} G_\beta(x, y)\, \mu(\mathrm{d}y) = 0.$$

(ii) A measure $\mu \in \mathcal{K}$ is (1-)Green tight ($\mu \in \mathcal{K}_\infty(1)$ in notation) if

$$\lim_{R\to\infty} \sup_{x\in\mathbb{R}^d} \int_{|y|>R} G_1(x, y)\, \mu(\mathrm{d}y) = 0.$$

Any Kato class measure with compact support in $\mathbb{R}^d$ belongs to $\mathcal{K}_\infty(1)$ by definition.

Let $(\mathcal{E}, \mathcal{F})$ be a regular Dirichlet form on $L^2(\mathbb{R}^d)$ associated with $\mathbf{M}$,

$$\mathcal{F} = \left\{ u \in L^2(\mathbb{R}^d) \mid \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} \,\mathrm{d}x\mathrm{d}y < \infty \right\},$$

$$\mathcal{E}(u, u) = \frac{1}{2}\mathcal{A}(d, \alpha) \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} \,\mathrm{d}x\mathrm{d}y,\ u \in \mathcal{F}$$

with

$$\mathcal{A}(d, \alpha) = \frac{\alpha 2^{d-1}\Gamma((d + \alpha)/2)}{\pi^{d/2}\Gamma(1 - \alpha/2)}$$

([19, Example 1.4.1]). For $\beta > 0$, we let $\mathcal{E}_\beta(u, u) = \mathcal{E}(u, u) + \beta \int_{\mathbb{R}^d} u^2\,\mathrm{d}x$. Since any function in $\mathcal{F}$ admits a quasi continuous version ([19, Theorem 2.1.3]), we may and do assume that if we write $u \in \mathcal{F}$, then u denotes its quasi continuous version.

For $\mu \in \mathcal{K}$ and $\beta > 0$, let $G_\beta\mu(x) = \int_{\mathbb{R}^d} G_\beta(x, y)\, \mu(\mathrm{d}y)$. Then $\|G_\beta\mu\|_\infty < \infty$ by the definition of the Kato class measure. Moreover, the Stollmann-Voigt inequality ([33, Theorem 3.1] and [19, Exercise 6.4.4]) holds:

$$\int_{\mathbb{R}^d} u^2\,\mathrm{d}\mu \le \|G_\beta\mu\|_\infty \mathcal{E}_\beta(u, u),\ u \in \mathcal{F}. \tag{7}$$

In particular, any function $u \in \mathcal{F}$ belongs to $L^2(\mu)$. We also know by (7) that the embedding

$$I_\mu : (\mathcal{F}, \sqrt{\mathcal{E}_\beta}) \to L^2(\mu), \quad I_\mu f = f, \ \mu\text{-a.e.}$$

is continuous. Even if μ is a signed measure, we can define the continuous embedding I_μ as above by replacing μ with $|\mu|$.

Let ν be a signed Borel measure on $\mathbb{R}^d$ such that the measures ν^+ and ν^- in the Jordan decomposition $\nu = \nu^+ - \nu^-$ belong to $\mathcal{K}$. Let $(\mathcal{E}^\nu, \mathcal{F})$ be the quadratic form on $L^2(\mathbb{R}^d)$ defined by

$$\mathcal{E}^\nu(u, u) = \mathcal{E}(u, u) - \int_{\mathbb{R}^d} u^2 \, \mathrm{d}\nu, \ u \in \mathcal{F}.$$

Since ν charges no set of zero capacity ([1, Theorem 3.3]), $(\mathcal{E}^\nu, \mathcal{F})$ is well-defined. Furthermore, it is a lower bounded closed symmetric bilinear form on $L^2(\mathbb{R}^d)$ ([1, Theorem 4.1]) so that the associated self-adjoint operator $\mathcal{H}^\nu$ on $L^2(\mathbb{R}^d)$ is formally written as $\mathcal{H}^\nu = (-\Delta)^{\alpha/2}/2 - \nu$.

Let $\{p_t^\nu\}_{t>0}$ be the strongly continuous symmetric semigroup on $L^2(\mathbb{R}^d)$ generated by $(\mathcal{E}^\nu, \mathcal{F})$. Let $A_t^{\nu^+}$ and $A_t^{\nu^-}$ be the positive continuous additive functionals in the Revuz correspondence to ν^+ and ν^-, respectively (see [19, p.401] for details). If we define $A_t^\nu = A_t^{\nu^+} - A_t^{\nu^-}$, then

$$p_t^\nu f(x) = E_x\left[e^{A_t^\nu} f(X_t)\right], \quad f \in L^2(\mathbb{R}^d) \cap \mathcal{B}_b(\mathbb{R}^d).$$

Here $\mathcal{B}_b(\mathbb{R}^d)$ is the family of bounded Borel measurable functions on $\mathbb{R}^d$. Moreover, there exists a jointly continuous integral kernel $p_t^\nu(x, y)$ on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ such that $p_t^\nu f(x) = \int_{\mathbb{R}^d} p_t^\nu(x, y) f(y) \, \mathrm{d}y$ ([1, Theorem 7.1]).

For $\beta \geq 0$, we define $G_\beta^\nu f(x) = \int_0^\infty e^{-\beta t} p_t^\nu f(x) \, \mathrm{d}t$ provided that the right hand side above makes sense. Then

$$G_\beta^\nu f(x) = E_x\left[\int_0^\infty e^{-\beta t + A_t^\nu} f(X_t) \, \mathrm{d}t\right].$$

If we define $G_\beta^\nu(x, y) = \int_0^\infty e^{-\beta t} p_t^\nu(x, y) \, \mathrm{d}t$, then $G_\beta^\nu f(x) = \int_{\mathbb{R}^d} G_\beta^\nu(x, y) f(y) \, \mathrm{d}y$.

We next discuss the invariance of the essential spectra of $(-\Delta)^{\alpha/2}/2$ under the perturbation with respect to the finite Kato class measure. For a self-adjoint operator $\mathcal{L}$ on $L^2(\mathbb{R}^d)$, let $\sigma_{\mathrm{ess}}(\mathcal{L})$ denote its essential spectrum.

Proposition 4 *If ν^+ and ν^- are finite Kato class measures, then* $\sigma_{\mathrm{ess}}(\mathcal{H}^\nu) = \sigma_{\mathrm{ess}}((-\Delta)^{\alpha/2}/2) = [0, \infty)$.

We refer to [2, 10] for $\alpha = 2$. To prove Proposition 4, we follow the argument of [10, Theorem 3.1]. More precisely, we show three lemmas below for the proof of Proposition 4.

Suppose that ν is a signed Borel measure on $\mathbb{R}^d$ such that ν^+ and ν^- are positive Radon measures on $\mathbb{R}^d$ charging no set of zero capacity. For $\beta > 0$, let $G_\beta \nu(x) = \int_{\mathbb{R}^d} G_\beta(x, y)\,\nu(\mathrm{d}y)$. Then for any $f \in L^2(|\nu|)$, $(f\nu)^+$ and $(f\nu)^-$ are positive Radon measures on $\mathbb{R}^d$ charging no set of zero capacity and

$$G_\beta(f\nu)(x) = E_x\left[\int_0^\infty e^{-\beta t} f(X_t)\,\mathrm{d}A_t^\nu\right] = E_x^\beta\left[A_\zeta^{f\nu}\right].$$

Here P_x^β is the law of the $e^{-\beta t}$-subprocess of $\mathbf{M}$ and ζ is the lifetime of this subprocess. By [16, Theorem 6.7.4], we have $G_\beta(f\nu) \in \mathcal{F}$ and $\mathcal{E}_\beta(G_\beta(f\nu), v) = \int_{\mathbb{R}^d} f \cdot I_\nu v\,\mathrm{d}\nu$ for any $v \in \mathcal{F}$. Hence if we define $K_\beta f(x) = G_\beta(f\nu)(x)$ for $f \in L^2(|\nu|)$, then $K_\beta f \in \mathcal{F}$.

Lemma 5 *If ν^+ and ν^- belong to the Kato class, then for any $\beta > 0$, $I_\nu K_\beta$ is a bounded linear operator on $L^2(|\nu|)$. Moreover, there exists $\beta_0 > 0$ such that for any $\beta > \beta_0$, the associated operator norm $\|I_\nu K_\beta\|$ satisfies $\|I_\nu K_\beta\| < 1$.*

Proof We prove this lemma only for $\nu^- = 0$ because a similar calculation applies to the general case. By (7), we have for any $f \in L^2(\nu)$,

$$\int_{\mathbb{R}^d} (I_\nu K_\beta f)^2\,\mathrm{d}\nu = \int_{\mathbb{R}^d} (K_\beta f)^2\,\mathrm{d}\nu \le \|G_\beta \nu\|_\infty \mathcal{E}_\beta(K_\beta f, K_\beta f) < \infty$$

so that $I_\nu K_\beta f \in L^2(\nu)$. Combining this with the relation

$$\mathcal{E}_\beta(K_\beta f, K_\beta f) = \int_{\mathbb{R}^d} (I_\nu K_\beta f) f\,\mathrm{d}\nu \le \sqrt{\int_{\mathbb{R}^d} (I_\nu K_\beta f)^2\,\mathrm{d}\nu}\sqrt{\int_{\mathbb{R}^d} f^2\,\mathrm{d}\nu},$$

we get $\|I_\nu K_\beta\| \le \|G_\beta \nu\|_\infty$. Since ν is a Kato class measure, we have $\|G_\beta \nu\|_\infty \to 0$ as $\beta \to \infty$ so that the desired assertion holds. □

Lemma 5 implies that for any $\beta > \beta_0$, we can define $(1 - I_\nu K_\beta)^{-1}$ as a bounded linear operator on $L^2(|\nu|)$.

Lemma 6 *Let β_0 and $\nu^\pm$ be as in Lemma 5. Then for any $\beta > \beta_0$,*

$$G_\beta^\nu f - G_\beta f = K_\beta((1 - I_\nu K_\beta)^{-1} I_\nu G_\beta f), \quad f \in L^2(\mathbb{R}^d).$$

Proof As in Lemma 5, we may assume that $\nu^- = 0$. Fix $\beta > \beta_0$ and $f \in L^2(\mathbb{R}^d)$. Then by Lemma 5, we can define the bounded linear operator $(1 - I_\nu K_\beta)^{-1}$ on $L^2(\nu)$ and $u = (1 - I_\nu K_\beta)^{-1} I_\nu G_\beta f \in L^2(\nu)$. For any $v \in \mathcal{F}$,

$$\mathcal{E}^{\nu}_{\beta}(G_\beta f + K_\beta u, v) = \mathcal{E}_\beta(G_\beta f + K_\beta u, v) - \int_{\mathbb{R}^d} I_\nu(G_\beta f + K_\beta u)\cdot I_\nu v \, \mathrm{d}\nu$$
$$= \int_{\mathbb{R}^d} f v \, \mathrm{d}x + \int_{\mathbb{R}^d} u \cdot I_\nu v \, \mathrm{d}\nu - \int_{\mathbb{R}^d} I_\nu(G_\beta f + K_\beta u)\cdot I_\nu v \, \mathrm{d}\nu.$$

Since $I_\nu(G_\beta f + K_\beta u) = u$, we have $\mathcal{E}^{\nu}_{\beta}(G_\beta f + K_\beta u, v) = \int_{\mathbb{R}^d} f v \, \mathrm{d}x$ so that the proof is complete. □

Lemma 7 *Let ν^+ and ν^- be finite Kato class measures, and let β_0 be as in Lemma 5.*

(i) *For any $\beta > 0$, K_β is a compact operator from $L^2(|\nu|)$ to $L^2(\mathbb{R}^d)$.*
(ii) *For any $\beta > \beta_0$, $G^{\nu}_{\beta} - G_\beta$ is a compact operator on $L^2(\mathbb{R}^d)$.*

Proof As in the previous lemmas, we may assume that $\nu^- = 0$. We first prove (i). For $n = 1, 2, 3, \ldots$, we define

$$K^{(n)}_{\beta} f(x) = \int_{|x-y|\geq 1/n} G_\beta(x, y) f(y)\, \nu(\mathrm{d}y), \quad f \in L^2(\nu).$$

Since (4) yields

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \left(G_\beta(x, y)\mathbf{1}_{|x-y|\geq 1/n}\right)^2 \nu(\mathrm{d}y)\mathrm{d}x \leq c_1\nu(\mathbb{R}^d) \int_{|x|\geq 1/n} \frac{\mathrm{d}x}{|x|^{2(d+\alpha)}} < \infty,$$

$K^{(n)}_{\beta}$ is a Hilbert-Schmidt operator so that it is compact from $L^2(\nu)$ to $L^2(\mathbb{R}^d)$ (see, e.g., [21, Corollary 4.6]).

We now assume that $d > \alpha$. Let $\varepsilon \in (0, \alpha/2)$ and

$$k^{(n)}_1(x, y) = \frac{\mathbf{1}_{\{|x-y|<1/n\}}}{|x-y|^{d/2-\varepsilon}}, \quad k^{(n)}_2(x, y) = \frac{\mathbf{1}_{\{|x-y|<1/n\}}}{|x-y|^{d/2-\alpha+\varepsilon}}.$$

By (5), there exists $c_1 > 0$ such that for any $n = 1, 2, 3, \ldots$,

$$G_\beta(x, y)\mathbf{1}_{\{|x-y|<1/n\}} \leq c_1 k^{(n)}_1(x, y) k^{(n)}_2(x, y), \ x, y \in \mathbb{R}^d$$

and

$$K_\beta f(x) - K^{(n)}_{\beta} f(x) = \int_{|x-y|<1/n} G_\beta(x, y) f(y)\, \nu(\mathrm{d}y), \ x \in \mathbb{R}^d.$$

Therefore,

$$\|K_\beta f - K_\beta^{(n)} f\|_{L^2(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \left(\int_{|x-y|<1/n} G_\beta(x,y) f(y)\, \nu(\mathrm{d}y) \right)^2 \mathrm{d}x$$
$$\le c_2 \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} k_1^{(n)}(x,y)^2 f(y)^2\, \nu(\mathrm{d}y) \right) \left(\int_{\mathbb{R}^d} k_2^{(n)}(x,y)^2\, \nu(\mathrm{d}y) \right) \mathrm{d}x$$
$$\le c_2 \left\{ \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} k_1^{(n)}(x,y)^2 f(y)^2\, \nu(\mathrm{d}y) \right) \mathrm{d}x \right\} \sup_{x \in \mathbb{R}^d} \left(\int_{\mathbb{R}^d} k_2^{(n)}(x,y)^2\, \nu(\mathrm{d}y) \right). \quad (8)$$

Let $\omega_d = 2\pi^{d/2}\Gamma(d/2)^{-1}$ be the surface area of the unit ball in $\mathbb{R}^d$. Then by the Fubini theorem,

$$\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} k_1^{(n)}(x,y)^2 f(y)^2\, \nu(\mathrm{d}y) \right) \mathrm{d}x$$
$$= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} k_1^{(n)}(x,y)^2\, \mathrm{d}x \right) f(y)^2 \nu(\mathrm{d}y) = \frac{\omega_d \|f\|_{L^2(\nu)}^2}{2\varepsilon n^{2\varepsilon}}. \quad (9)$$

Since $\alpha > 2\varepsilon$, we have by (5),

$$\int_{\mathbb{R}^d} k_2^{(n)}(x,y)^2\, \nu(\mathrm{d}y) = \int_{|x-y|<1/n} |x-y|^{-d+2\alpha-2\varepsilon}\, \nu(\mathrm{d}y)$$
$$\le \frac{1}{n^{\alpha-2\varepsilon}} \int_{|x-y|<1/n} |x-y|^{-d+\alpha}\, \nu(\mathrm{d}y) \le \frac{c_3}{n^{\alpha-2\varepsilon}} \int_{\mathbb{R}^d} G_1(x,y)\, \nu(\mathrm{d}y).$$

Hence

$$\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} k_2^{(n)}(x,y)^2\, \nu(\mathrm{d}y) \le \frac{c_3}{n^{\alpha-2\varepsilon}} \|G_1 \nu\|_\infty.$$

Combining this inequality with (9), we see by (8) that

$$\|K_\beta - K_\beta^{(n)}\| := \sup_{f \in L^2(\nu), f\neq 0} \frac{\|K_\beta f - K_\beta^{(n)} f\|_{L^2(\mathbb{R}^d)}}{\|f\|_{L^2(\nu)}} \le \frac{c_4}{\varepsilon n^{\alpha/2}} \to 0 \quad (n \to \infty).$$

For $d \le \alpha$, we also have $\|K_\beta - K_\beta^{(n)}\| \to 0$ as $n \to \infty$ by (5) and direct calculation. Since $K_\beta^{(n)}$ is compact, so is K_β by [27, Theorem VI.12]. This completes the proof of (i).

Since G_β is a bounded linear operator from $L^2(\mathbb{R}^d)$ to $(\mathcal{F}, \sqrt{\mathcal{E}_\beta})$ and $(1 - I_\nu K)^{-1} I_\nu$ is a bounded linear operator from $(\mathcal{F}, \sqrt{\mathcal{E}_\beta})$ to $L^2(\nu)$, Lemma 6 and (i) imply (ii). □

Proof of Proposition 4. Since $\sigma_{\mathrm{ess}}((-\Delta)^{\alpha/2}/2) = [0, \infty)$, the assertion follows by Lemma 7 and [27, Theorem VIII.14]. □

Remark 8 Let ν^+ and ν^- be Kato class measures such that $\nu = \nu^+ - \nu^-$ forms a signed Borel measure on $\mathbb{R}^d$. If $\tilde{\nu}^+ - \tilde{\nu}^-$ is the Jordan decomposition of ν, then $\tilde{\nu}^+$ and $\tilde{\nu}^-$ are also Kato class measures and $A_t^{\nu^+} - A_t^{\nu^-} = A_t^{\tilde{\nu}^+} - A_t^{\tilde{\nu}^-}$ by the uniqueness of the Revuz correspondence ([19, Theorem 5.13]). In particular, Proposition 4 is true as it is even if $\nu = \nu^+ - \nu^-$ is not the Jordan decomposition of ν.

We next discuss the asymptotic behavior of an integral associated with the ground state of $\mathcal{H}^\nu$. In what follows, we may and do assume that ν can be decomposed as $\nu = \nu^+ - \nu^-$ for some $\nu^+, \nu^- \in \mathcal{K}_\infty(1)$. Let $\lambda(\nu)$ be the bottom of the L^2-spectrum of $\mathcal{H}^\nu$. Then

$$\lambda(\nu) = \inf \left\{ \mathcal{E}(u,u) - \int_{\mathbb{R}^d} u^2 \, \mathrm{d}\nu \mid u \in C_0^\infty(\mathbb{R}^d), \int_{\mathbb{R}^d} u^2 \, \mathrm{d}x = 1 \right\},$$

where $C_0^\infty(\mathbb{R}^d)$ is the totality of smooth functions with compact support in $\mathbb{R}^d$. Moreover, if $\lambda(\nu) < 0$, then $\lambda(\nu)$ is the eigenvalue and the corresponding eigenfunction, which is called the ground state, has a bounded and strictly positive continuous version ([35, Theorem 2.8 and Sect. 4]). We write h for this version with L^2-normalization $\|h\|_{L^2(\mathbb{R}^d)} = 1$.

By the same proof as for [35, Lemma 4.1] and [30, Lemma A.1], we see that for any positive constants p and p' with $p' < 1 < p$, there exist positive constants c and C such that

$$\frac{c}{|x|^{(d+\alpha)p}} \le h(x) \le \frac{C}{|x|^{(d+\alpha)p'}} \quad (|x| \ge 1). \tag{10}$$

If ν^+ and ν^- are in addition compactly supported in $\mathbb{R}^d$, then (10) is valid for $p = p' = 1$.

Let $\lambda := \lambda(\nu)$ and

$$c_\star = \frac{C_{d,\alpha}\omega_d}{\alpha(-\lambda)^2} = \frac{\sin(\pi\alpha/2)\Gamma((d+\alpha)/2)\Gamma(\alpha/2)}{(-\lambda)^2 2^{1-\alpha}\pi\Gamma(d/2)}. \tag{11}$$

The next lemma determines the asymptotic behavior of the ground state h integrated outside the ball.

Lemma 9 *Suppose that* $\lambda < 0$.

(i) *For any* $x \in \mathbb{R}^d$,

$$h(x) = \int_{\mathbb{R}^d} G_{-\lambda}(x, y)h(y)\,\nu(\mathrm{d}y).$$

(ii) *If* ν^+ *and* ν^- *are compactly supported in* $\mathbb{R}^d$, *then* $\int_{\mathbb{R}^d} h(y)\,\nu(\mathrm{d}y) > 0$ *and*

$$R^\alpha \int_{|y|>R} h(y)\,\mathrm{d}y \to c_\star \int_{\mathbb{R}^d} h(y)\,\nu(\mathrm{d}y) \quad (R \to \infty). \tag{12}$$

Proof Let ν^+ and ν^- belong to $\mathcal{K}_\infty(1)$ and $\lambda < 0$. Since [26, Lemma 3.1 (i)] and its proof remain valid under the current setting, we have (i) in the same way as for the proof of [26, Lemma 3.1 (iii)].

We assume in addition that ν^+ and ν^- are compactly supported in $\mathbb{R}^d$. Then by (10) with $p = p' = 1$, $\int_{|y|>R} h(y)\,\mathrm{d}y$ is convergent for any $R > 0$. Since (i) yields

$$\begin{aligned}\int_{|y|>R} h(y)\,\mathrm{d}y = &\int_{|y|>R} \left(\int_{\mathbb{R}^d} G_{-\lambda}(y, z)h(z)\,\nu^+(\mathrm{d}z) \right) \mathrm{d}y \\ &- \int_{|y|>R} \left(\int_{\mathbb{R}^d} G_{-\lambda}(y, z)h(z)\,\nu^-(\mathrm{d}z) \right) \mathrm{d}y,\end{aligned}$$

we have by (4),

$$\begin{aligned}\int_{|y|>R} \left(\int_{\mathbb{R}^d} G_{-\lambda}(y, z)h(z)\,\nu^\pm(\mathrm{d}z) \right) \mathrm{d}y &\sim \frac{C_{d,\alpha}}{(-\lambda)^2} \int_{|y|>R} \frac{\mathrm{d}y}{|y|^{d+\alpha}} \int_{\mathbb{R}^d} h(z)\,\nu^\pm(\mathrm{d}z) \\ &= \frac{c_\star}{R^\alpha} \int_{\mathbb{R}^d} h(z)\,\nu^\pm(\mathrm{d}z),\end{aligned}$$

whence (12) holds. Moreover, since there exist $c_1 > 0$ and $c_2 > 0$ by (10) such that $c_1 \leq R^\alpha \int_{|y|>R} h(y)\,\mathrm{d}y \leq c_2$ for any $R \geq 1$, we obtain $\int_{\mathbb{R}^d} h(y)\,\nu(\mathrm{d}y) > 0$. □

For $\alpha = 2$, we proved in [26, Lemma 3.1 (iv)] the assertion corresponding to Lemma 9 (ii), but the scaling order there is exponential in contrast with the polynomial order in (12).

Suppose that ν^+ and ν^- are Kato class measures with compact support in $\mathbb{R}^d$ and $\lambda < 0$. Then for any $\beta > -\lambda$, we have

$$\inf\left\{\mathcal{E}_\beta(u,u)-\int_{\mathbb{R}^d}u^2\,\mathrm{d}\nu \mid u\in C_0^\infty(\mathbb{R}^d),\int_{\mathbb{R}^d}u^2\,\mathrm{d}x=1\right\}=\beta+\lambda>0$$

so that by [34, Lemma 3.5],

$$\inf\left\{\mathcal{E}_\beta(u,u)+\int_{\mathbb{R}^d}u^2\,\mathrm{d}\nu^- \mid u\in C_0^\infty(\mathbb{R}^d),\int_{\mathbb{R}^d}u^2\,\mathrm{d}\nu^+=1\right\}>1.$$

Hence by [15, Lemma 3.5 (1), Theorems 3.6 and 5.2] with Lemma 2 (ii), there exist positive constants c and C for any $\beta>-\lambda$ such that

$$cG_\beta(x,y)\le G_\beta^\nu(x,y)\le CG_\beta(x,y). \tag{13}$$

Let $\lambda_2(\nu)$ be the second bottom of the spectrum for $\mathcal{H}^\nu$,

$$\lambda_2(\nu)=\inf\left\{\mathcal{E}(u,u)-\int_{\mathbb{R}^d}u^2\,\mathrm{d}\nu \mid u\in C_0^\infty(\mathbb{R}^d),\int_{\mathbb{R}^d}u^2\,\mathrm{d}x=1,\int_{\mathbb{R}^d}uh\,\mathrm{d}x=0\right\}.$$

If $\lambda<0$, then Proposition 4 implies that $(\lambda<)\lambda_2(\nu)\le 0$. Then as in [26, Lemma 3.1 (ii)], there exists $C>0$ such that

$$|p_t^\nu(x,y)-e^{-\lambda t}h(x)h(y)|\le Ce^{-\lambda_2 t},\quad t\ge 1. \tag{14}$$

2.3 Asymptotic Behaviors of Feynman-Kac Functionals

In this subsection, we prove the asymptotic properties of the Feynman-Kac functionals for a symmetric stable process. Even though our approach is similar to that of [26, Proposition 3.2], we need calculations by taking into account the polynomial decay property of the tail distribution of the symmetric stable process in (3). Throughout this subsection, we assume that ν^+ and ν^- are the Kato class measures with compact support in $\mathbb{R}^d$ and that $\lambda<0$.

Let

$$q_t(x,y)=p_t^\nu(x,y)-p_t(x,y)-e^{-\lambda t}h(x)h(y) \tag{15}$$

so that for $R>0$,

$$E_x[e^{A_t^\nu};|X_t|>R]=P_x(|X_t|>R)+e^{-\lambda t}h(x)\int_{|y|>R}h(y)\,\mathrm{d}y+\int_{|y|>R}q_t(x,y)\,\mathrm{d}y. \tag{16}$$

For $r > 0$, let $B(r) = \{y \in \mathbb{R}^d \mid |y| < r\}$ be an open ball with radius r centered at the origin. Fix $M > 0$ so that the support of $|\nu|$ is included in $B(M)$. For $c > 0$, define

$$I_c(t, R) = \begin{cases} e^{ct}/R^{\alpha} & (\lambda_2 < 0), \\ (t P_0(|X_t| > R - M)) \wedge (e^{ct}/R^{\alpha}) & (\lambda_2 = 0) \end{cases}$$

and

$$J(t, R) = e^{-\lambda t}(R - M)^d \int_{t^{1/\alpha}}^{\infty} e^{\lambda u^{\alpha}} g\left(\frac{R - M}{u}\right) \frac{\mathrm{d}u}{u^{d+1}},$$

where g is the same function as in (1).

Proposition 10 *For any $c > 0$ with $c > -\lambda_2$, there exists $C > 0$ such that for any $x \in \mathbb{R}^d$, $t \geq 1$ and $R > 2M$,*

$$\left| \int_{|y|>R} q_t(x, y)\,\mathrm{d}y \right| \leq C\left(h(x) P_0(|X_t| > R - M) + I_c(t, R) + h(x) J(t, R)\right).$$

Proof As for [26, (3.19)], we have

$$\begin{aligned} \int_{|y|>R} q_t(x, y)\,\mathrm{d}y &= \int_0^1 \left(\int_{\mathbb{R}^d} p_s^{\nu}(x, z) P_z(|X_{t-s}| > R)\,\nu(\mathrm{d}z) \right) \mathrm{d}s \\ &\quad + \int_1^t \left(\int_{\mathbb{R}^d} (p_s^{\nu}(x, z) - e^{-\lambda s} h(x) h(z)) P_z(|X_{t-s}| > R)\,\nu(\mathrm{d}z) \right) \mathrm{d}s \\ &\quad - e^{-\lambda t} h(x) \int_{t-1}^{\infty} e^{\lambda s} \left(\int_{\mathbb{R}^d} h(z) P_z(|X_s| > R)\,\nu(\mathrm{d}z) \right) \mathrm{d}s \\ &= \text{(I)} + \text{(II)} - \text{(III)}. \end{aligned} \tag{17}$$

For any $s \in [0, t]$ and $z \in \mathbb{R}^d$,

$$P_z(|X_{t-s}| > R) \leq P_0(|X_{t-s}| > R - |z|) \leq P_0(|X_t| > R - |z|) \tag{18}$$

by the spatial uniformity and scaling property of the symmetric stable process. Then for any $\varepsilon > 0$, we see by (4), (10) (with $p = p' = 1$) and (13) (with $\beta = -\lambda + \varepsilon$) that

$$\int_0^1 \left(\int_{\mathbb{R}^d} p_s^\nu(x,z)\,|\nu|(\mathrm{d}z) \right) \mathrm{d}s \le e^{-\lambda+\varepsilon} \int_{\mathbb{R}^d} \left(\int_0^1 e^{(\lambda-\varepsilon)s} p_s^\nu(x,z)\,\mathrm{d}s \right) |\nu|(\mathrm{d}z)$$
$$\le e^{-\lambda+\varepsilon} \int_{\mathbb{R}^d} G^\nu_{-\lambda+\varepsilon}(x,z)\,|\nu|(\mathrm{d}z) \le c_1 \int_{\mathbb{R}^d} G_{-\lambda+\varepsilon}(x,z)\,|\nu|(\mathrm{d}z) \le c_2 h(x).$$

Hence by (18),

$$\text{(I)} \le P_0(|X_t| > R - M) \int_0^1 \left(\int_{\mathbb{R}^d} p_s^\nu(x,z)\,|\nu|(\mathrm{d}z) \right) \mathrm{d}s$$
$$\le c_3 P_0(|X_t| > R - M) h(x).$$

Fix $c > 0$ with $c \ge -\lambda_2$. Then by (18) and Lemma 2 (i),

$$\int_1^t \left(\int_{\mathbb{R}^d} e^{-\lambda_2 s} P_z(|X_{t-s}| > R)\,|\nu|(\mathrm{d}z) \right) \mathrm{d}s$$
$$\le |\nu|(\mathbb{R}^d) \int_1^t e^{-\lambda_2 s} P_0(|X_{t-s}| > R - M)\,\mathrm{d}s$$
$$\le |\nu|(\mathbb{R}^d) e^{ct} \int_{|y|>R-M} G_c(0,y)\,\mathrm{d}y \le c_4 e^{ct} \int_{|y|>R-M} \frac{\mathrm{d}y}{|y|^{d+\alpha}} \le \frac{c_5 e^{ct}}{R^\alpha}.$$

If $\lambda_2 = 0$, then by (18) again,

$$\int_1^t \left(\int_{\mathbb{R}^d} P_z(|X_{t-s}| > R)\,|\nu|(\mathrm{d}z) \right) \mathrm{d}s \le c_6 t P_0(|X_t| > R - M).$$

Hence by (14),

$$|\text{(II)}| \le \int_1^t \left(\int_{\mathbb{R}^d} |p_s^\nu(x,z) - e^{-\lambda s} h(x) h(z)| P_z(|X_{t-s}| > R)\,|\nu|(\mathrm{d}z) \right) \mathrm{d}s$$
$$\le c_7 \int_1^t \left(\int_{\mathbb{R}^d} e^{-\lambda_2 s} P_z(|X_{t-s}| > R)\,|\nu|(\mathrm{d}z) \right) \mathrm{d}s \le c_8 I_c(t,R).$$

Since (1) yields

$$P_0(|X_s| > R) = \omega_d \int_{R/s^{1/\alpha}}^{\infty} g(r) r^{d-1}\, \mathrm{d}r = \omega_d R^d \int_0^{s^{1/\alpha}} g\left(\frac{R}{u}\right) \frac{\mathrm{d}u}{u^{d+1}}, \tag{19}$$

we have by (19) and integration by parts formula,

$$\int_t^{\infty} e^{\lambda s} P_0(|X_s| > R)\, \mathrm{d}s = \frac{e^{\lambda t}}{-\lambda} P_0(|X_t| > R) + \frac{\omega_d}{-\lambda} R^d \int_{t^{1/\alpha}}^{\infty} e^{\lambda u^\alpha} g\left(\frac{R}{u}\right) \frac{\mathrm{d}u}{u^{d+1}}.$$

We also see by (18) that

$$\begin{aligned}\int_{t-1}^{\infty} e^{\lambda s} P_0(|X_s| > R - M)\, \mathrm{d}s &= \int_t^{\infty} e^{\lambda(s-1)} P_0(|X_{s-1}| > R - M)\, \mathrm{d}s \\ &\le e^{-\lambda} \int_t^{\infty} e^{\lambda s} P_0(|X_s| > R - M)\, \mathrm{d}s.\end{aligned}$$

Therefore,

$$\begin{aligned}|(\mathrm{III})| &\le e^{-\lambda t} h(x) \int_{t-1}^{\infty} e^{\lambda s} P_0(|X_s| > R - M)\, \mathrm{d}s \left(\int_{\mathbb{R}^d} h(z)\, |\nu|(\mathrm{d}z) \right) \\ &\le c_9 e^{-\lambda t} h(x) \int_t^{\infty} e^{\lambda s} P_0(|X_s| > R - M)\, \mathrm{d}s \\ &\le c_{10} h(x) \left(P_0(|X_t| > R - M) + J(t, R) \right),\end{aligned}$$

which completes the proof. □

Remark 11 Let us take $R = 0$ in (17). Then by (14) and (15), there exists $c > 0$ such that for any $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $t \ge 1$,

$$\sup_{x \in \mathbb{R}^d} \left| e^{\lambda t} E_x \left[e^{A_t^\nu} f(X_t) \right] - h(x) \int_{\mathbb{R}^d} f(y) h(y)\, \mathrm{d}y \right| \le c \|f\|_\infty e^{\lambda t} \left(t \vee e^{-\lambda_2(\nu) t} \right).$$

The right hand side above goes to 0 as $t \to \infty$ because $\lambda < \lambda_2(\nu) \le 0$. This result extends the assertion in [26, Remark 3.4] for the Brownian motion to the symmetric stable process, and provides a convergence rate bound in [35, (1.3)].

Let $R(t)$ be a positive measurable function on $(0, \infty)$ such that $R(t) \to \infty$ as $t \to \infty$. Then by Lemma 9 (ii),

$$\eta(t) := e^{-\lambda t} \int_{|y|>R(t)} h(y)\,\mathrm{d}y \sim c_* \frac{e^{-\lambda t}}{R(t)^\alpha} \tag{20}$$

with

$$c_* = c_\star \int_{\mathbb{R}^d} h(y)\,\nu(\mathrm{d}y). \tag{21}$$

The next lemma reveals the exact asymptotic behavior of the Feynman-Kac semi-group conditioned that the particle at time t sits outside the ball with radius $R(t)$.

Lemma 12 *Let K be a compact set in $\mathbb{R}^d$. If $R(t)/t^{1/\alpha} \to \infty$ as $t \to \infty$, then there exist positive constants c_1, c_2 and T such that for any $x \in K$, $t \geq T$ and $s \in [0, t-1]$,*

$$E_x\left[e^{A^\nu_{t-s}}; |X_{t-s}| > R(t)\right] = e^{\lambda s} h(x)\eta(t)(1+\theta_{s,x}(t))$$

with

$$|\theta_{s,x}(t)| \leq c_1 e^{-c_2(t-s)}.$$

Here c_1 and c_2 can be independent of the choice of the function $R(t)$. In particular,

$$\lim_{t\to\infty} \sup_{x\in K} \left| \frac{1}{h(x)\eta(t)} E_x\left[e^{A^\nu_t}; |X_t| > R(t)\right] - 1 \right| = 0.$$

Proof Take $M > 0$ so that $B(M)$ includes both K and the support of $|\nu|$. Then for any $s \in [0, t-1]$,

$$(R(t) - M)/(t-s)^{1/\alpha} \geq (R(t) - M)/t^{1/\alpha}$$

and the right hand above goes to ∞ as $t \to \infty$. Hence by (1) and (3), there exist $c_1 > 0$, $c_2 > 0$ and $T_1 > 1$ such that for any $x \in K$ and $t \geq T_1$ and $s \in [0, t-1]$,

$$P_x(|X_{t-s}| > R(t)) \leq P_0(|X_{t-s}| > R(t) - M) = \omega_d \int_{\frac{R(t)-M}{(t-s)^{1/\alpha}}}^{\infty} g(u) u^{d-1}\,\mathrm{d}u$$

$$\leq c_1 \int_{\frac{R(t)-M}{(t-s)^{1/\alpha}}}^{\infty} \frac{\mathrm{d}u}{u^{\alpha+1}} \leq c_2 \frac{t-s}{R(t)^\alpha} = c_2 e^{\lambda(t-s)}(t-s)\frac{e^{-\lambda(t-s)}}{R(t)^\alpha}. \tag{22}$$

For any $c > 0$,

$$I_c(t-s, R(t)) \leq \frac{e^{c(t-s)}}{R(t)^\alpha} = e^{(c+\lambda)(t-s)} \frac{e^{-\lambda(t-s)}}{R(t)^\alpha}. \tag{23}$$

By (3), there exists $T_2 > 1$ such that for all $t \geq T_2$,

$$\int_{t^{1/\alpha}}^{R(t)} e^{\lambda u^\alpha} g\left(\frac{R(t)-M}{u}\right) \frac{\mathrm{d}u}{u^{d+1}} \leq \frac{c_3}{R(t)^{d+\alpha}} \int_{t^{1/\alpha}}^{\infty} e^{\lambda u^\alpha} u^{\alpha-1}\,\mathrm{d}u \leq \frac{c_4 e^{\lambda t}}{R(t)^{d+\alpha}}$$

and

$$\int_{R(t)}^{\infty} e^{\lambda u^\alpha} g\left(\frac{R(t)-M}{u}\right) \frac{\mathrm{d}u}{u^{d+1}} \leq c_5 \int_{R(t)}^{\infty} e^{\lambda u^\alpha} \frac{\mathrm{d}u}{u^{d+1}} \leq \frac{c_6 e^{\lambda R(t)^\alpha}}{R(t)^{d+\alpha}} \leq \frac{c_7 e^{\lambda t}}{R(t)^{d+\alpha}}.$$

Hence

$$\begin{aligned} J(t, R(t)) &= e^{-\lambda t}(R(t)-M)^d \int_{t^{1/\alpha}}^{\infty} e^{\lambda u^\alpha} g\left(\frac{R(t)-M}{u}\right) \frac{\mathrm{d}u}{u^{d+1}} \\ &\leq \frac{c_8}{R(t)^\alpha} = c_8 e^{\lambda(t-s)} \frac{e^{-\lambda(t-s)}}{R(t)^\alpha}. \end{aligned} \tag{24}$$

Note that all the constants c_i can be independent of the choice of the function $R(t)$.

Fix $c \in (-\lambda_2(\nu), -\lambda)$. Then by combining (16) and Proposition 10 with (22)–(24), there exist positive constants c_9, c_{10} and c_{11}, and $T \geq 1$ such that for any $x \in K$, $t \geq T$ and $s \in [0, t-1]$,

$$\begin{aligned} &\left|E_x\left[e^{A^\nu_{t-s}}; |X_{t-s}| > R(t)\right] - e^{\lambda s}\eta(t)h(x)\right| \\ &\leq P_x(|X_{t-s}| > R(t)) + \left|\int_{|y|>R(t)} q_{t-s}(x,y)\,\mathrm{d}y\right| \\ &\leq \frac{c_9 e^{-\lambda(t-s)}}{R(t)^\alpha}\left(e^{\lambda(t-s)}(t-s) + e^{(c+\lambda)(t-s)} + e^{\lambda(t-s)}\right) \leq c_{10} e^{\lambda s} e^{-c_{11}(t-s)} \frac{e^{-\lambda t}}{R(t)^\alpha}. \end{aligned}$$

Then by (20), the proof is complete. □

Recall that by [28, Lemma 3.4], we have for any $\mu \in \mathcal{K}_\infty(1)$,

$$\sup_{x \in \mathbb{R}^d} E_x\left[\int_0^\infty e^{2\lambda s + A^\nu_s}\,\mathrm{d}A^\mu_s\right] < \infty. \tag{25}$$

The next two lemmas will be used later for the second moment estimates of the expected population for a branching symmetric stable process.

Lemma 13 *Let K be a compact set in $\mathbb{R}^d$ and μ a Kato class measure with compact support in $\mathbb{R}^d$. If $R(t)/t^{1/\alpha} \to \infty$ as $t \to \infty$, then there exist $C > 0$ and $T > 0$ such* *that for any* $t \ge T$,

$$\sup_{x\in K} E_x\left[\int_0^t e^{A_s^\nu} E_{X_s}\left[e^{A_{t-s}^\nu}; |X_{t-s}| > R(t)\right]^2 \, \mathrm{d}A_s^\mu\right] \le C\eta(t)^2.$$

Proof Fix $x \in K$. For $t \ge 1$,

$$\begin{aligned}
&E_x\left[\int_0^t e^{A_s^\nu} E_{X_s}\left[e^{A_{t-s}^\nu}; |X_{t-s}| > R(t)\right]^2 \, \mathrm{d}A_s^\mu\right] \\
&= E_x\left[\int_0^{t-1} e^{A_s^\nu} E_{X_s}\left[e^{A_{t-s}^\nu}; |X_{t-s}| > R(t)\right]^2 \, \mathrm{d}A_s^\mu\right] \\
&+ E_x\left[\int_{t-1}^t e^{A_s^\nu} E_{X_s}\left[e^{A_{t-s}^\nu}; |X_{t-s}| > R(t)\right]^2 \, \mathrm{d}A_s^\mu\right] = \text{(IV)} + \text{(V)}.
\end{aligned} \tag{26}$$

If $0 \le s \le t-1$, then Lemma 12 yields for any $z \in \operatorname{supp}[\mu]$

$$E_z\left[e^{A_{t-s}^\nu}; |X_{t-s}| > R(t)\right] \le c_1 e^{\lambda s}\eta(t)$$

so that by (25),

$$\text{(IV)} \le c_2\eta(t)^2 \sup_{x\in\mathbb{R}^d} E_x\left[\int_0^\infty e^{2\lambda s + A_s^\nu} \, \mathrm{d}A_s^\mu\right] \le c_3\eta(t)^2. \tag{27}$$

By [1, Theorem 6.1 (i)] and (18), there exists $c_4 > 0$ such that for any $M > 0$, $R > M$, $t \in [0, 1]$ and $x \in \mathbb{R}^d$ with $|x| \le M$,

$$E_x\left[e^{A_t^\nu}; |X_t| > R\right] \le c_4 P_0(|X_t| > R - M) \le c_4 P_0(|X_1| > R - M).$$

Hence (3) implies that for any $z \in \operatorname{supp}[\mu]$, all sufficiently large $t \ge 1$ and any $s \in [t-1, t]$,

$$E_z\left[e^{A_{t-s}^\nu}; |X_{t-s}| > R(t)\right] \le c_5 P_0(|X_1| > R(t) - M) \le \frac{c_6}{R(t)^\alpha}. \tag{28}$$

Since (25) yields

$$E_x\left[\int_{t-1}^{t} e^{A_s^\nu}\,\mathrm{d}A_s^\mu\right] \le e^{-2\lambda t}\sup_{x\in\mathbb{R}^d} E_x\left[\int_0^\infty e^{2\lambda s + A_s^\nu}\,\mathrm{d}A_s^\mu\right] \le c_7 e^{-2\lambda t},$$

we have by (20),

$$(\mathrm{V}) \le \frac{c_8}{R(t)^{2\alpha}} E_x\left[\int_{t-1}^{t} e^{A_s^\nu}\,\mathrm{d}A_s^\mu\right] \le \frac{c_9 e^{-2\lambda t}}{R(t)^{2\alpha}} \le c_{10}\eta(t)^2.$$

Combining this with (26) and (27), we complete the proof. □

For $\kappa > 0$, let $R^\kappa(t) = (e^{-\lambda t}\kappa)^{1/\alpha}$. Letting $R(t) = R^\kappa(t)$ in (20), we get

$$\eta(t) \to c_*\kappa^{-1}\ (t\to\infty). \tag{29}$$

Lemma 14 *Let $K \subset \mathbb{R}^d$ be a compact set.*

(i) *For any $\kappa > 0$,*

$$\lim_{t\to\infty}\sup_{x\in K}\left|\frac{\kappa}{h(x)} E_x\left[e^{A_t^\nu}; |X_t| > R^\kappa(t)\right] - c_*\right| = 0.$$

(ii) *Let μ be a Kato class measure with compact support in $\mathbb{R}^d$. Then*

$$\lim_{\kappa\to\infty}\limsup_{t\to\infty}\sup_{x\in K}\kappa E_x\left[\int_0^t e^{A_s^\nu} E_{X_s}\left[e^{A_{t-s}^\nu}; |X_{t-s}| > R^\kappa(t)\right]^2 \mathrm{d}A_s^\mu\right] = 0.$$

Proof (i) follows by Lemma 12 and (29). We now show (ii). By Lemma 12 and (28), there exist $c_1 > 0$ and $T = T(\kappa) > 1$ for any $\kappa > 0$ such that, for any $z \in \mathrm{supp}[\mu]$, $t \ge T$ and $s \in [0, t]$,

$$E_z\left[e^{A_{t-s}^\nu}; |X_{t-s}| > R^\kappa(t)\right] \le c_1 e^{\lambda s}\eta(t).$$

Hence by (25), there exists $c_2 > 0$ such that for all $t \ge T$,

$$\sup_{x\in K}\kappa E_x\left[\int_0^t e^{A_s^\nu} E_{X_s}\left[e^{A_{t-s}^\nu}; |X_{t-s}| > R^\kappa(t)\right]^2 \mathrm{d}A_s^\mu\right] \le \frac{c_2}{\kappa}(\kappa\eta(t))^2.$$

Then by (29),

$$\limsup_{t\to\infty}\sup_{x\in K}\kappa E_x\left[\int_0^t e^{A_s^\nu}E_{X_s}\left[e^{A_{t-s}^\nu};|X_{t-s}|>R^\kappa(t)\right]^2 \mathrm{d}A_s^\mu\right]\le\frac{c_2c_*^2}{\kappa}.$$

The right hand side above goes to 0 as $\kappa\to\infty$. □

3 Maximal Displacement of Branching Symmetric Stable Processes

In this section, we first introduce a model of branching symmetric stable processes. We then present our main results with examples.

3.1 Branching Symmetric Stable Processes

For $\alpha\in(0,2)$, let $\mathbf{M}=(\Omega,\mathcal{F},\{X_t\}_{t\ge0},\{P_x\}_{x\in\mathbb{R}^d},\{\mathcal{F}_t\}_{t\ge0})$ be a symmetric α-stable process on $\mathbb{R}^d$, where $\{\mathcal{F}_t\}_{t\ge0}$ is the minimal augmented admissible filtration.

Let us formulate the model of a branching symmetric α-stable process on $\mathbb{R}^d$ by following [28] and references therein. We first define the set $\mathbf{X}$ as follows: let $(\mathbb{R}^d)^{(0)}=\{\Delta\}$ and $(\mathbb{R}^d)^{(1)}=\mathbb{R}^d$. Let $n\ge2$. For $\mathbf{x}=(x^1,\dots,x^n)$ and $\mathbf{y}=(y^1,\dots,y^n)$ in $(\mathbb{R}^d)^n$, we write $\mathbf{x}\sim\mathbf{y}$ if there exists a permutation σ of $\{1,2,\dots,n\}$ such that $y^i=x^{\sigma(i)}$ for any $i=1,\dots,n$. Using this equivalence relation, we define $(\mathbb{R}^d)^{(n)}=(\mathbb{R}^d)^n/\sim$ for $n\ge2$ and $\mathbf{X}=\cup_{n=0}^\infty(\mathbb{R}^d)^{(n)}$.

Let $\mathbf{p}=\{p_n(x)\}_{n=0}^\infty$ be a probability function on $\mathbb{R}^d$, $0\le p_n(x)\le1$ and $\sum_{n=0}^\infty p_n(x)=1$ for any $x\in\mathbb{R}^d$. We assume that $p_0(x)+p_1(x)\not\equiv1$ to avoid the triviality. Fix $\mu\in\mathcal{K}$ and $\mathbf{p}$. We next introduce a particle system as follows: a particle starts from $x\in\mathbb{R}^d$ at time $t=0$ and moves by following the distribution P_x until the random time U. Here the distribution of U is given by

$$P_x(U>t\mid\mathcal{F}_\infty)=e^{-A_t^\mu}\ (t>0).$$

At time U, this particle dies leaving no offspring with probability $p_0(X_{U-})$, or splits into n particles with probability $p_n(X_{U-})$ for $n\ge1$. For the latter case, these n particles then move by following the distribution $P_{X_{U-}}$ and repeat the same procedure independently. If there exist n particles alive at time t, then the positions of these particles determine a point in $(\mathbb{R}^d)^{(n)}$. Let $\mathbf{X}_t$ denote such a point,

$$\mathbf{X}_t=(\mathbf{X}_t^{(1)},\dots,\mathbf{X}_t^{(n)})\in(\mathbb{R}^d)^{(n)}.$$

In this way, we can define the model of a branching symmetric α-stable process $\overline{\mathbf{M}} = (\{\mathbf{X}_t\}_{t\geq 0}, \{\mathbf{P}_\mathbf{x}\}_{\mathbf{x}\in\mathbf{X}})$ on $\mathbf{X}$ (or simply on $\mathbb{R}^d$) with branching rate μ and branching mechanism $\mathbf{p}$. Note that for $x \in \mathbb{R}^d$, $\mathbf{P}_x$ denotes the law of the process such that the initial state is a single particle at x.

Let S be the first splitting time of $\overline{\mathbf{M}}$ given by

$$\mathbf{P}_x(S > t \mid \sigma(X)) = P_x(U > t \mid \mathcal{F}_\infty) = e^{-A_t^\mu} \ (t > 0).$$

Let Z_t be the population at time t and $e_0 := \inf\{t > 0 \mid Z_t = 0\}$ the extinction time of $\overline{\mathbf{M}}$. Note that $Z_t = 0$ for all $t \geq e_0$. For $f \in \mathcal{B}_b(\mathbb{R}^d)$, we define

$$Z_t(f) = \begin{cases} \sum_{k=1}^{Z_t} f(\mathbf{X}_t^{(k)}) & (t < e_0), \\ 0 & (t \geq e_0). \end{cases}$$

For $A \in \mathcal{B}(\mathbb{R}^d)$, let $Z_t(A) := Z_t(\mathbf{1}_A)$ denote the population on A at time t.

Let $Q(x) = \sum_{n=0}^\infty np_n(x)$ and $\nu_Q(\mathrm{d}x) = Q(x)\mu(\mathrm{d}x)$. Let $R(x) = \sum_{n=1}^\infty n(n-1)p_n(x)$ and $\nu_R(\mathrm{d}x) = R(x)\mu(\mathrm{d}x)$. We here recall the next lemma on the first and second moments of $Z_t(f)$:

Lemma 15 ([26, Lemma 2.2] and [28, Lemma 3.3]) *Let $\mu \in \mathcal{K}$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$.*

(i) *If $\nu_Q \in \mathcal{K}$, then*

$$\mathbf{E}_x\left[Z_t(f)\right] = E_x\left[e^{A_t^{(Q-1)\mu}} f(X_t)\right].$$

(ii) *If $\nu_R \in \mathcal{K}$, then*

$$\mathbf{E}_x\left[Z_t(f)^2\right] = E_x\left[e^{A_t^{(Q-1)\mu}} f(X_t)^2\right] + E_x\left[\int_0^t e^{A_s^{(Q-1)\mu}} E_{X_s}\left[e^{A_{t-s}^{(Q-1)\mu}} f(X_{t-s})\right]^2 \mathrm{d}A_s^{\nu_R}\right].$$

3.2 Weak Convergence and Tail Asymptotics

Let $\overline{\mathbf{M}} = (\{\mathbf{X}_t\}_{t\geq 0}, \{\mathbf{P}_\mathbf{x}\}_{\mathbf{x}\in\mathbf{X}})$ be a branching symmetric α-stable process on $\mathbb{R}^d$ with branching rate $\mu \in \mathcal{K}$ and branching mechanism $\mathbf{p}$. We impose the next assumption on μ and $\mathbf{p}$:

Assumption 16 (i) The support of μ is compact in $\mathbb{R}^d$.
(ii) $\nu_R \in \mathcal{K}$.
(iii) $\lambda((Q-1)\mu) < 0$.

Let $\lambda = \lambda((Q-1)\mu)$. Under Assumption 16, λ is the principal eigenvalue of the operator $\mathcal{H}^{(Q-1)\mu}$ on $L^2(\mathbb{R}^d)$ as mentioned in Sect. 2.2. Let h denote the bounded and strictly positive continuous version of the corresponding ground state with L^2-normalization. We define $M_t = e^{\lambda t} Z_t(h)$. Then by [28, Lemma 3.4], $\{M_t\}_{t\geq 0}$ is a nonnegative square integrable $\mathbf{P}_x$-martingale so that $\mathbf{E}_x[M_t] = h(x)$ and $M_\infty = \lim_{t\to\infty} M_t$ exists $\mathbf{P}_x$-a.s. with $\mathbf{P}_x(M_\infty > 0) > 0$.

Let L_t denote the maximal Euclidean norm of particles alive at time t:

$$L_t = \begin{cases} \max_{1\leq k\leq Z_t} |\mathbf{X}_t^{(k)}| & (t < e_0), \\ 0 & (t \geq e_0). \end{cases}$$

Since each particle follows the law of the symmetric stable process and $\mathbf{P}_x(Z_t < \infty) = 1$ for any $t > 0$, L_t is well-defined and $\mathbf{P}_x(L_t < \infty) = 1$ for any $t \geq 0$.

In what follows, let c_* denote the positive constant given by (21) with $\nu = (Q-1)\mu$. For $\kappa > 0$, let $R^\kappa(t) = (e^{-\lambda t}\kappa)^{1/\alpha}$. We then have

Theorem 17 *For any $\kappa > 0$,*

$$\lim_{t\to\infty} \mathbf{P}_x(L_t > R^\kappa(t)) = \mathbf{E}_x\left[1 - \exp\left(-\kappa^{-1} c_* M_\infty\right)\right].$$

Theorem 17 extends [26, Theorem 2.4] for the branching Brownian motion to that for the branching symmetric stable process. Theorem 17 implies that L_t grows exponentially fast in contrast with the linear growth for the branching Brownian motion.

Since

$$\mathbf{P}_x(L_t > R^\kappa(t), e_0 < \infty) \leq \mathbf{P}_x(t < e_0 < \infty) \to 0 \ (t \to \infty)$$

and $\{e_0 < \infty\} \subset \{M_\infty = 0\}$, we obtain

$$\lim_{t\to\infty} \mathbf{P}_x(L_t > R^\kappa(t) \mid e_0 = \infty) = \mathbf{E}_x\left[1 - \exp\left(-\kappa^{-1} c_* M_\infty\right) \mid e_0 = \infty\right].$$

If we let $Y_t = e^{\lambda t/\alpha} L_t$, then the equality above reads

$$\lim_{t\to\infty} \mathbf{P}_x(Y_t \leq \kappa \mid e_0 = \infty) = \mathbf{E}_x\left[\exp\left(-\kappa^{-\alpha} c_* M_\infty\right) \mid e_0 = \infty\right]. \quad (30)$$

Moreover, if $d = 1$ and $1 < \alpha < 2$, then [28, Remark 3.14] yields $\{e_0 = \infty\} = \{M_\infty > 0\}$, $\mathbf{P}_x$-a.s. so that

$$\lim_{t\to\infty} \mathbf{P}_x(Y_t \leq \kappa \mid M_\infty > 0) = \mathbf{E}_x\left[\exp\left(-\kappa^{-\alpha} c_* M_\infty\right) \mid M_\infty > 0\right]. \quad (31)$$

Hence the distribution of Y_t under $\mathbf{P}_x(\cdot \mid M_\infty > 0)$ is weakly convergent to the average over the Fréchet distributions with parameter α scaled by $c_* M_\infty$ (see, e.g., [8, Theorem 1.12] and references therein for the terminologies about external distributions). On the other hand, if $d > \alpha$, then $\mathbf{M}$ is transient so that $\mathbf{P}_x(\{e_0 = \infty\} \cap \{M_\infty = 0\}) > 0$. In particular, we do not know the validity of (31).

For $R > 0$, let $Z_t^R = Z_t(\overline{B(R)}^c)$. The next theorem determines the long time asymptotic behavior of the tail distribution of L_t:

Theorem 18 *Let a be a positive measurable function on $(0, \infty)$ such that $a(t) \to \infty$ as $t \to \infty$, and let $R(t) = (e^{-\lambda t} a(t))^{1/\alpha}$.*

(i) *The next equality holds locally uniformly in $x \in \mathbb{R}^d$.*

$$\lim_{t\to\infty} \frac{\mathbf{P}_x(L_t > R(t))}{\mathbf{E}_x\left[Z_t^{R(t)}\right]} = 1.$$

(ii) *For each $k \in \mathbb{N}$, the next equality holds locally uniformly in $x \in \mathbb{R}^d$.*

$$\lim_{t\to\infty} \mathbf{P}_x(Z_t^{R(t)} = k \mid L_t > R(t)) = \begin{cases} 1 & (k = 1), \\ 0 & (k \geq 2). \end{cases}$$

The statement of this theorem is similar to those of [26, Theorem 2.5 and Corollary 2.6]; however, the tail distribution of the maximal displacement for the branching symmetric stable process is completely different from that for the branching Brownian motion (see [26, (2.16), (2,17)]). In fact, combining Theorem 18 with Lemmas 12 and 15, and (20), we have as $t \to \infty$,

$$\mathbf{P}_x(L_t > R(t)) \sim \mathbf{E}_x\left[Z_t^{R(t)}\right] \sim \frac{c_* h(x)}{a(t)}. \tag{32}$$

We omit the proof of Theorem 18 because it is identical with those of [26, Theorem 2.5 and Corollary 2.6], respectively.

3.3 Examples

In this subsection, we present three examples to which the results in the previous subsection are applicable.

Example 19 Let $d = 1$ and $\alpha \in (1, 2)$. Then δ_0, the Dirac measure at the origin, belongs to the Kato class. Let $\overline{\mathbf{M}}$ be a branching symmetric α-stable process on $\mathbb{R}$ with branching rate $\mu = c\delta_0$ $(c > 0)$ and branching mechanism $\mathbf{p} = \{p_n(x)\}_{n=0}^\infty$. We assume that $p_0(0) + p_2(0) = 1$ for simplicity. Then $\mathbf{P}_x(e_0 = \infty) > 0$ if and only if $p_2(0) > 1/2$ ([28, Example 4.4]). In particular, if $m = 2p_2(0) > 1$, then

$$\lambda := \lambda((Q-1)\mu) = -\left\{\frac{c(m-1)2^{1/\alpha}}{\alpha \sin(\pi/\alpha)}\right\}^{\alpha/(\alpha-1)}$$

and

$$c_* = \frac{C_{1,\alpha}\omega_1}{\alpha(-\lambda)^2} \times c(m-1) \int_{\mathbb{R}} h(y)\, \delta_0(\mathrm{d}y) = \frac{2c(m-1)C_{1,\alpha}}{\alpha(-\lambda)^2} h(0).$$

With these λ and c_*, (31) and (32) hold.

Example 20 Let $1 < \alpha < 2$ and $d > \alpha$. For $r > 0$, let δ_r be the surface measure on $\partial B(r) = \{y \in \mathbb{R}^d \mid |y| = r\}$. Let $\overline{\mathbf{M}}$ be a branching symmetric α-stable process on $\mathbb{R}^d$ with branching rate $\mu = c\delta_r$ $(c > 0)$ and branching mechanism $\mathbf{p} = \{p_n(x)\}_{n=0}^{\infty}$. We assume that $p_0 \equiv p_0(x)$, $p_2 \equiv p_2(x)$ and $p_0 + p_2 = 1$. Then $\mathbf{P}_x(e_0 = \infty) > 0$ holds irrelevantly of the value of p_2 because $\mathbf{M}$ is transient. If we let $m = 2p_2$ and $\lambda := \lambda((Q-1)\mu)$, then $\lambda < 0$ if and only if $p_2 > 1/2$ and

$$r > \left\{ \frac{\sqrt{\pi}\,\Gamma((d+\alpha-2)/2)\Gamma(\alpha/2)}{c(m-1)\Gamma((d-\alpha)/2)\Gamma((\alpha-1)/2)} \right\}^{1/(\alpha-1)}$$

(see [28, Example 4.7] and references therein). Under this condition, (30) and (32) hold.

Assume that $1 < \alpha < 2$ and $d > \alpha$. Let $r > 0$ and $\mu_r(\mathrm{d}x) = \mathbf{1}_{B(r)}(x)\,\mathrm{d}x$. To present the last example, we estimate

$$\check{\lambda}_\beta = \inf \left\{ \mathcal{E}(u,u) \mid u \in \mathcal{F},\, \beta \int_{B(r)} u^2\,\mathrm{d}x = 1 \right\} \quad (\beta > 0),$$

which is the bottom of the spectrum for the time changed Dirichlet form of $(\mathcal{E}, \mathcal{F})$ with respect to the measure $\beta\mu_r$ (see, e.g., [31, Section 3] for details).

Let $\check{\lambda} = \check{\lambda}_1$, and let $v(x) = \int_{B(r)} G(x,y)\,\mathrm{d}y$ be the 0-potential of the measure μ_r. Then

$$\check{\lambda} \le \frac{1}{\|v\|^2_{L^2(B(r))}} \mathcal{E}(v,v) = \frac{1}{\|v\|^2_{L^2(B(r))}} \int_{B(r)} v\,\mathrm{d}x \le \frac{1}{\inf_{y \in B(r)} v(y)}. \tag{33}$$

Let

$$I_{d,\alpha} = \alpha \int_0^1 u^{d-1}(1+u)^{\alpha-d}\,\mathrm{d}u, \quad \kappa_{d,\alpha} = \frac{\alpha\Gamma(d/2)\Gamma(\alpha/2)}{2^{2-\alpha}\Gamma((d-\alpha)/2)}.$$

Recall that $\omega_d = 2\pi^{d/2}\Gamma(d/2)^{-1}$ is the surface area of the unit ball in $\mathbb{R}^d$. Then for any $y \in B(r)$, $|y-z| \le |y| + |z| \le r + |z|$ and thus

$$\int_{B(r)} \frac{\mathrm{d}z}{|y-z|^{d-\alpha}} \ge \int_{B(r)} \frac{\mathrm{d}z}{(r+|z|)^{d-\alpha}} = \omega_d \int_0^r \frac{s^{d-1}}{(r+s)^{d-\alpha}}\,\mathrm{d}s = \frac{\omega_d I_{d,\alpha}}{\alpha} r^\alpha.$$

Since this inequality and (6) yield

$$\inf_{y\in B(r)} v(y) \geq \frac{I_{d,\alpha} r^{\alpha}}{\kappa_{d,\alpha}},$$

we get by (33),

$$\check{\lambda} \leq \frac{\kappa_{d,\alpha}}{I_{d,\alpha} r^{\alpha}}.$$

We also know by [31, Example 3.10] that

$$\check{\lambda} \geq \frac{\kappa_{d,\alpha}}{r^{\alpha}}.$$

Noting that $\check{\lambda}_{\beta} = \check{\lambda}/\beta$, we further obtain

$$\frac{\kappa_{d,\alpha}}{\beta r^{\alpha}} \leq \check{\lambda}_{\beta} \leq \frac{\kappa_{d,\alpha}}{\beta I_{d,\alpha} r^{\alpha}}. \tag{34}$$

We here note that the lower bound of $\check{\lambda}$ in [31, Example 3.10] is incorrect because of the computation error.

Let $\lambda_{\beta} = \lambda(\beta\mu_r)$. Then by [36, Lemma 2.2], $\lambda_{\beta} < 0$ if and only if $\check{\lambda}_{\beta} < 1$. Hence by (34), we obtain

$$r > \left(\frac{\kappa_{d,\alpha}}{\beta I_{d,\alpha}}\right)^{1/\alpha} \Rightarrow \lambda_{\beta} < 0, \quad r \leq \left(\frac{\kappa_{d,\alpha}}{\beta}\right)^{1/\alpha} \Rightarrow \lambda_{\beta} \geq 0. \tag{35}$$

We do not know if λ_{β} is negative or not for $(\kappa_{d,\alpha}/\beta)^{1/\alpha} < r \leq \left\{\kappa_{d,\alpha}/(\beta I_{d,\alpha})\right\}^{1/\alpha}$.

Example 21 Let $1 < \alpha < 2$ and $d > \alpha$. For $r > 0$, let $\mu_r(\mathrm{d}x) = \mathbf{1}_{B(r)}(x)\,\mathrm{d}x$. Let $\overline{\mathbf{M}}$ be a branching symmetric α-stable process on $\mathbb{R}^d$ with branching rate $\mu = c\mu_r$ ($c > 0$) and branching mechanism $\mathbf{p} = \{p_n(x)\}_{n=0}^{\infty}$. We assume that $p_0 \equiv p_0(x)$, $p_2 \equiv p_2(x)$ and $p_0 + p_2 = 1$. Then $\mathbf{P}_x(e_0 = \infty) > 0$ holds irrelevantly of the value of p_2.

Let $m = 2p_2$ and $\lambda := \lambda((Q-1)\mu)$. Assume that $p_2 > 1/2$. Then by (35), we have the following: if $r > \left\{\kappa_{d,\alpha}/(c(m-1)I_{d,\alpha})\right\}^{1/\alpha}$, then $\lambda < 0$ so that (30) and (32) hold. On the other hand, if $r \leq \{\kappa_{d,\alpha}/(c(m-1))\}^{1/\alpha}$, then we have $\lambda = 0$ so that Assumption 16 fails.

4 Proof of Theorem 17

Once we obtain the asymptotic behaviors of the Feynman-Kac functionals as in Sect. 2.3, we can establish Theorem 17 along the way as for the proof of [26, Theorem 2.4].

Let $\overline{\mathbf{M}} = (\{\mathbf{X}_t\}_{t\geq 0}, \{\mathbf{P}_{\mathbf{x}}\}_{\mathbf{x}\in\mathbf{X}})$ be a branching symmetric α-stable process on $\mathbb{R}^d$ with branching rate μ and branching mechanism $\mathbf{p}$ so that Assumption 16 is fulfilled. Let $\nu = (Q-1)\mu$ and $\lambda = \lambda((Q-1)\mu)$, and let c_* be the corresponding value in (21). Recall that for $\kappa > 0$, $R^\kappa(t) = (e^{-\lambda t}\kappa)^{1/\alpha}$.

Lemma 22 *Let K be a compact set in $\mathbb{R}^d$. Then*

$$\lim_{\kappa\to\infty}\limsup_{t\to\infty}\sup_{x\in K}\left|\frac{\kappa}{h(x)}\mathbf{P}_x(L_t > R^\kappa(t)) - c_*\right| = 0$$

and for any $c > 0$,

$$\lim_{\gamma\to+0}\limsup_{t\to\infty}\sup_{x\in K}\left|\frac{1}{\gamma h(x)}\mathbf{E}_x\left[1-e^{-\gamma c M_t}\right] - c\right| = 0.$$

We omit the proof of Lemma 22; by using Lemma 14, we can show Lemma 22 in the same way as for [26, Lemma 4.1].

Let $\mathcal{L}$ be the totality of compact sets in $\mathbb{R}^d$.

Lemma 23 *The following equalities hold*:

$$\lim_{\kappa\to\infty}\sup_{L\in\mathcal{L}}\limsup_{t\to\infty}\sup_{x\in L}\left|\frac{\kappa}{h(x)}\mathbf{P}_x(L_t > R^\kappa(t)) - c_*\right| = 0 \tag{36}$$

and for any $c > 0$,

$$\lim_{\gamma\to+0}\sup_{L\in\mathcal{L}}\limsup_{t\to\infty}\sup_{x\in L}\left|\frac{1}{\gamma h(x)}\mathbf{E}_x\left[1-e^{-\gamma c M_t}\right] - c\right| = 0.$$

Proof We can prove the assertion in the same way as for [26, Proposition 4.2]. We here prove (36) only. By the Chebyshev inequality and Lemma 15,

$$\begin{aligned}\mathbf{P}_x(L_t > R^\kappa(t)) &= \mathbf{P}_x\left(Z_t^{R^\kappa(t)} \geq 1\right)\\ &\leq \mathbf{E}_x\left[Z_t^{R^\kappa(t)}\right] = E_x\left[e^{A_t^{(Q-1)\mu}}; |X_t| > R^\kappa(t)\right].\end{aligned}$$

Hence by Lemma 14,

$$\limsup_{\kappa\to\infty}\sup_{L\in\mathcal{L}}\limsup_{t\to\infty}\sup_{x\in L}\frac{\kappa}{h(x)}\mathbf{P}_x(L_t > R^\kappa(t)) \leq c_*.$$

Then for the proof of (36), it suffices to show that

$$\liminf_{\kappa\to\infty}\inf_{L\in\mathcal{L}}\liminf_{t\to\infty}\inf_{x\in L}\frac{\kappa}{h(x)}\mathbf{P}_x(L_t > R^\kappa(t)) \geq c_*. \tag{37}$$

In what follows, we give a proof of (37). Lemma 22 says that for any $\varepsilon > 0$ and $K \in \mathcal{L}$, there exists $\kappa_0 = \kappa_0(\varepsilon, K) > 0$ such that for any $\kappa \geq \kappa_0$, there exists $T_0 = T_0(\varepsilon, K, \kappa) > 0$ such that for any $t \geq T_0$ and $x \in K$,

$$|\kappa \mathbf{P}_x(L_t > R^\kappa(t)) - c_* h(x)| < \varepsilon h(x).$$

Let $t > 0$ and $0 \leq s < t$. Then

$$R^\kappa(t) = (e^{-\lambda(t-s)})^{1/\alpha}(\kappa e^{-\lambda s})^{1/\alpha} = R^{\kappa e^{-\lambda s}}(t-s)$$

and $\kappa e^{-\lambda s} \geq \kappa \geq \kappa_0$. Hence for any $T > 0$, $t \geq T + T_0$ and $s \in [0, T]$,

$$|\kappa \mathbf{P}_x(L_{t-s} > R^\kappa(t)) - c_* h(x)| < \varepsilon h(x), \ x \in K. \tag{38}$$

Fix $K \in \mathcal{L}$ which includes the support of μ. Let σ be the first hitting time to K of some particle. Then σ is relevant to the initial particle only because particles can not branch outside K. We use the same notation σ to denote the first hitting time to K of a symmetric α-stable process $\mathbf{M} = (\{X_t\}_{t \geq 0}, \{P_x\}_{x \in \mathbb{R}^d})$. Since $X_\sigma \in K$, there exists $\kappa_1 = \kappa_1(\varepsilon) > 0$ for any $\varepsilon > 0$ such that for any $\kappa \geq \kappa_1$, there exists $T_1 = T_1(\varepsilon, \kappa) > 0$ such that for any $T > 0$ and $t \geq T + T_1$, we have by the strong Markov property and (38),

$$\begin{aligned} \kappa \mathbf{P}_x(L_t > R^\kappa(t)) &\geq E_x\left[\kappa \mathbf{P}_{X_\sigma}(L_{t-s} > R^\kappa(t))|_{s=\sigma}; \sigma \leq T\right] \\ &\geq (c_* - \varepsilon) E_x\left[h(X_\sigma); \sigma \leq T\right], \ x \in \mathbb{R}^d. \end{aligned} \tag{39}$$

Since $e^{\lambda t + A_t^{(Q-1)\mu}} h(X_t)$ is a P_x-martingale and $P_x(A_{t \wedge \sigma}^{(Q-1)\mu} = 0) = 1$ for any $t \geq 0$, the optional stopping theorem yields

$$E_x\left[e^{\lambda(T \wedge \sigma)} h(X_{T \wedge \sigma})\right] = E_x\left[e^{\lambda(T \wedge \sigma) + A_{T \wedge \sigma}^{(Q-1)\mu}} h(X_{T \wedge \sigma})\right] = h(x)$$

and thus

$$\begin{aligned} E_x\left[h(X_\sigma); \sigma \leq T\right] &\geq E_x\left[e^{\lambda \sigma} h(X_\sigma); \sigma \leq T\right] \\ &= E_x\left[e^{\lambda(T \wedge \sigma)} h(X_{T \wedge \sigma})\right] - E_x\left[e^{\lambda T} h(X_T); T < \sigma\right] \\ &\geq h(x) - e^{\lambda T}\|h\|_\infty. \end{aligned}$$

Then by (39), we have for any $t \geq T + T_1$,

$$\kappa \mathbf{P}_x(L_t > R^\kappa(t)) \geq (c_* - \varepsilon)(h(x) - e^{\lambda T}\|h\|_\infty), \ x \in \mathbb{R}^d.$$

In particular, for any $L \in \mathcal{L}$ and $t \geq T + T_1$,

$$\inf_{x \in L} \frac{\kappa}{h(x)} \mathbf{P}_x(L_t > R^\kappa(t)) \geq (c_* - \varepsilon)\left(1 - \frac{e^{\lambda T}\|h\|_\infty}{\inf_{x \in L} h(x)}\right).$$

Letting $t \to \infty$ and then $T \to \infty$, we have

$$\liminf_{t\to\infty} \inf_{x\in L} \frac{\kappa}{h(x)} \mathbf{P}_x(L_t > R^\kappa(t)) \geq c_* - \varepsilon.$$

Furthermore, since the right hand side above is independent of the choice of $L \in \mathcal{L}$, we obtain (37) by letting $\kappa \to \infty$ and then $\varepsilon \to 0$. □

Proof of Theorem 17. We follow the argument of [26, Proof of Theorem 2.4]. In what follows, we write $R(t) = R^\kappa(t)$ for simplicity. Since $\mathbf{P}_x(L_t < \infty) = 1$ for any $t \geq 0$, there exists $r_1 = r_1(\varepsilon, T_1)$ for any $\varepsilon > 0$ and $T_1 > 0$ such that $\mathbf{P}_x(L_{T_1} > r_1) \leq \varepsilon$. Hence for any $t \geq T_1$,

$$\mathbf{P}_x(L_t \leq R(t)) \leq \mathbf{P}_x(L_t \leq R(t), L_{T_1} \leq r_1) + \varepsilon,$$

which yields

$$\begin{aligned} &\mathbf{P}_x(L_t > R(t)) - \mathbf{E}_x\left[1 - \exp\left(-\kappa^{-1} c_* M_t\right)\right] \\ &\geq \mathbf{E}_x\left[\exp\left(-\kappa^{-1} c_* M_t\right) - \mathbf{1}_{\{L_t \leq R(t)\}}; L_{T_1} \leq r_1\right] - \varepsilon. \end{aligned} \tag{40}$$

Recall that $e_0 = \inf\{t > 0 \mid Z_t = 0\}$ is the extinction time of $\overline{\mathbf{M}}$. Then for any $t \geq e_0$, $M_t = 0$ and $L_t = 0$ by definition. Therefore, by the Markov property,

$$\begin{aligned} &\mathbf{E}_x\left[\exp\left(-\kappa^{-1} c_* M_t\right) - \mathbf{1}_{\{L_t \leq R(t)\}}; L_{T_1} \leq r_1\right] \\ &= \mathbf{E}_x\Big[\left\{\mathbf{E}_{\mathbf{X}_{T_1}}\left[\exp\left(-\kappa^{-1} c_* e^{\lambda T_1} M_{t-T_1}\right)\right] - \mathbf{P}_{\mathbf{X}_{T_1}}(L_{t-T_1} \leq R(t))\right\}; \\ &\qquad T_1 < e_0, L_{T_1} \leq r_1\Big] \\ &= (\mathrm{VI}). \end{aligned}$$

By Lemma 23, there exists $\kappa_0 = \kappa_0(\delta) > 0$ for any $\delta \in (0, c_*)$ such that if $T > 0$ satisfies $\kappa e^{-\lambda T} \geq \kappa_0$, then

$$\sup_{L\in\mathcal{L}} \limsup_{t\to\infty} \sup_{x\in L} \left| \frac{\kappa e^{-\lambda T}}{h(x)} \mathbf{P}_x(L_{t-T} > R(t)) - c_* \right| < \delta.$$

Let $T_1 = T_1(\kappa_0)$ satisfy $\kappa e^{-\lambda T_1} \geq \kappa_0$. Then there exists $T_2 = T_2(\varepsilon, \delta, T_1) > 0$ such that for any $y \in \overline{B(r_1)}$ and $t \geq T_1 + T_2$,

$$\kappa^{-1}(c_* - \delta) e^{\lambda T_1} h(y) \leq \mathbf{P}_y(L_{t-T_1} > R(t)) \leq \kappa^{-1}(c_* + \delta) e^{\lambda T_1} h(y). \tag{41}$$

Note that $1 - x \leq e^{-x}$ for any $x \in \mathbb{R}$ and there exists $r_0(\delta) > 0$ for any $\delta > 0$ such that $1 - x \geq e^{-(1+\delta)x}$ for any $x \in [0, r_0(\delta)]$. Hence if we take T_1 so large that $\kappa^{-1}(c_* + \delta) e^{\lambda T_1} \|h\|_\infty \leq r_0(\delta)$, then by (41),

$$\begin{aligned} \exp\left(-(1+\delta)\kappa^{-1}(c_* + \delta) e^{\lambda T_1} h(y)\right) &\leq \mathbf{P}_y(L_{t-T_1} \leq R(t)) \\ &\leq \exp\left(-\kappa^{-1}(c_* - \delta) e^{\lambda T_1} h(y)\right). \end{aligned}$$

By Lemma 23, we also obtain

$$\begin{aligned}\exp\left(-(1+\delta)\kappa^{-1}(c_*+\delta)e^{\lambda T_1}h(y)\right) &\leq \mathbf{E}_y\left[\exp\left(-\kappa^{-1}c_*e^{\lambda T_1}M_{t-T_1}\right)\right] \\ &\leq \exp\left(-\kappa^{-1}(c_*-\delta)e^{\lambda T_1}h(y)\right)\end{aligned}$$

so that for any $t \geq T_1 + T_2$,

$$(\mathrm{VI}) \geq \mathbf{E}_x\left[\exp\left(-(1+\delta)\kappa^{-1}(c_*+\delta)M_{T_1}\right)\right] - \mathbf{E}_x\left[\exp\left(-\kappa^{-1}(c_*-\delta)M_{T_1}\right)\right].$$

Since the right hand side goes to 0 as $t \to \infty$, $T_1 \to \infty$ and $\delta \to +0$, we have by (40),

$$\liminf_{t\to\infty}\left(\mathbf{P}_x(L_t > R(t)) - \mathbf{E}_x\left[1-\exp\left(-\kappa^{-1}c_*M_t\right)\right]\right) \geq 0.$$

In the same way, we also have

$$\limsup_{t\to\infty}\left(\mathbf{P}_x(L_t > R(t)) - \mathbf{E}_x\left[1-\exp\left(-\kappa^{-1}c_*M_t\right)\right]\right) \leq 0$$

so that the proof is complete. □

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References

1. S. Albeverio, P. Blanchard, Z.M. Ma, Feynman-Kac semigroups in terms of signed smooth measures, in *Random Partial Differential Equations*, ed. by U. Hornung et al. (Birkhäuser, Basel, 1991), pp. 1–31
2. A. Ben Amor, Invariance of essential spectra for generalized Schrödinger operators. Math. Phys. Electron. J. **10**, 18 (2004) (Paper 7)
3. A. Bhattacharya, R.S. Hazra, P. Roy, Point process convergence for branching random walks with regularly varying steps. Ann. Inst. Henri Poincaré Probab. Stat. **53**, 802–818 (2017)
4. R.M. Blumenthal, R.K. Getoor, Some theorems on stable processes. Trans. Amer. Math. Soc. **95**, 263–273 (1960)
5. S. Bocharov, Limiting distribution of particles near the frontier in the catalytic branching Brownian motion. Acta Appl. Math. **169**, 433–453 (2020)
6. S. Bocharov, S.C. Harris, Branching Brownian motion with catalytic branching at the origin. Acta Appl. Math. **134**, 201–228 (2014)
7. S. Bocharov, S.C. Harris: Limiting distribution of the rightmost particle in catalytic branching Brownian motion. Electron. Commun. Probab. **21**, 12 (2016)
8. A. Bovier, *Gaussian Processes on Trees. From Spin Glasses to Branching Brownian* (Cambridge University Press, Cambridge, 2017)
9. M.D. Bramson, Maximal displacement of branching Brownian motion. Comm. Pure Appl. Math. **31**, 531–581 (1978)
10. J.F. Brasche, P. Exner, Yu. A. Kuperin, P. Šeba, Schrödinger operators with singular interactions. J. Math. Anal. Appl. **184**, 112–139 (1994)

11. E. Vl, Bulinskaya: spread of a catalytic branching random walk on a multidimensional lattice. Stochastic Process. Appl. **28**, 2325–2340 (2018)
12. E. Vl, Bulinskaya: fluctuations of the propagation front of a catalytic branching walk. Theory Probab. Appl. **64**, 513–534 (2020)
13. E. VI. Bulinskaya, Maximum of catalytic branching random walk with regularly varying tails. J. Theoret. Probab. **34**, 141–161 (2021)
14. P. Carmona, Y. Hu, The spread of a catalytic branching random walk. Ann. Inst. H. Poincaré Probab. Statist. **50**, 327–351 (2014)
15. Z.-Q. Chen, Gaugeability and conditional gaugeability. Trans. Amer. Math. Soc. **354**, 4639–4679 (2002)
16. Z.-Q. Chen, M. Fukushima, *Symmetric Markov Processes, Time Change, and Boundary Theory* (Pinceton University Press, Princeton, 2012)
17. R. Durrett, Maxima of branching random walks. Z. Wahrsch. Verw. Gebiete **62**, 165–170 (1983)
18. K.B. Erickson, Rate of expansion of an inhomogeneous branching process of Brownian particles. Z. Wahrsch. Verw. Gebiete **66**, 129–140 (1984)
19. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, 2nd rev. and ext. ed. (Walter de Gruyter, 2011)
20. A. Getan, S. Molchanov, B. Vainberg, Intermittency for branching walks with heavy tails. Stoch. Dyn. **17**, 14 (2017) (1750044)
21. P.R. Halmos, V.S. Sunder, *Bounded Integral Operators on L^2 Spaces* (Springer, Berlin, 1978)
22. Y.H. Kim, E. Lubetzky, O. Zeitouni, The maximum of branching Brownian motion in $\mathbb{R}^d$, preprint. Ann. Appl. Probab. Available at arXiv:2104.07698
23. S. Lalley, T. Sellke, Traveling waves in inhomogeneous branching Brownian motions. I. Ann. Probab. **16**, 1051–1062 (1988)
24. S.P. Lalley, Y. Shao, Maximal displacement of critical branching symmetric stable processes. Ann. Inst. Henri Poincaré Probab. Stat. **52**, 1161–1177 (2016)
25. Y. Nishimori, in *Limiting distributions for particles near the frontier of spatially inhomogeneous branching Brownian motions*. Preprint, available at arXiv:2104.13063
26. Y. Nishimori, Y. Shiozawa, Limiting distributions for the maximal displacement of branching Brownian motions. J. Math. Soc. Japan. **74**, 177–216 (2022). https://doi.org/10.2969/jmsj/85158515
27. M. Reed, B. Simon, *Methods of Modern Mathematical Physics. IV. Analysis of Operators* (Academic, New York, 1978)
28. Y. Shiozawa, Exponential growth of the numbers of particles for branching symmetric α-stable processes. J. Math. Soc. Japan **60**, 75–116 (2008)
29. Y. Shiozawa, Spread rate of branching Brownian motions. Acta Appl. Math. **155**, 113–150 (2018)
30. Y. Shiozawa, Maximal displacement and population growth for branching Brownian motions. Illinois J. Math. **63**, 353–402 (2019)
31. Y. Shiozawa, M. Takeda, Variational formula for Dirichlet forms and estimates of principal eigenvalues for symmetric α-stable processes. Potential Anal. **23**, 135–151 (2005)
32. R. Stasiński, J. Berestycki, B. Mallein, Derivative martingale of the branching Brownian motion in dimension $d \geq 1$. Ann. Inst. Henri Poincaré Probab. Stat. **57**, 1786–1810 (2021)
33. P. Stollmann, J. Voigt, Perturbation of Dirichlet forms by measures. Potential Anal. **5**, 109–138 (1996)
34. M. Takeda, Conditional gaugeability and subcriticality of generalized Schrödinger operators. J. Funct. Anal. **191**, 343–376 (2002)
35. M. Takeda, Large deviations for additive functionals of symmetric stable processes. J. Theoret. Probab. **21**, 336–355 (2008)
36. M. Takeda, K. Tsuchida, Differentiability of spectral functions for symmetric α-stable processes processes. Trans. Amer. Math. Soc. **359**, 4031–4054 (2007)
37. M. Wada, Asymptotic expansion of resolvent kernels and behavior of spectral functions for symmetric stable processes. J. Math. Soc. Japan **69**, 673–692 (2017)

Random Riemannian Geometry in 4 Dimensions

Karl-Theodor Sturm

Abstract We construct and analyze conformally invariant random fields on 4-dimensional Riemannian manifolds (M, g). These centered Gaussian fields h, called *co-biharmonic Gaussian fields*, are characterized by their covariance kernels k defined as the integral kernel for the inverse of the *Paneitz operator*

$$\mathsf{p} = \frac{1}{8\pi^2}\Big[\Delta^2 + \mathsf{div}\Big(2\mathsf{Ric} - \frac{2}{3}\mathsf{scal}\Big)\nabla\Big].$$

The kernel k is invariant (modulo additive corrections) under conformal transformations, and it exhibits a precise logarithmic divergence

$$\Big|k(x, y) - \log\frac{1}{d(x, y)}\Big| \le C.$$

In terms of the co-biharmonic Gaussian field h, we define the *quantum Liouville measure*, a random measure on M, heuristically given as

$$d\mu(x) := e^{\gamma h(x) - \frac{\gamma^2}{2}k(x,x)}\, d\mathsf{vol}_g(x)\,,$$

and rigorously obtained a.s. for $|\gamma| < \sqrt{8}$ as weak limit of the RHS with h replaced by suitable regular approximations $(h_\ell)_{\ell\in\mathbb{N}}$. For the flat torus $M = \mathbb{T}^4$, we provide discrete approximations of the Gaussian field and of the Liouville measures in terms of semi-discrete random objects, based on Gaussian random variables on the discrete torus and piecewise constant functions in the isotropic Haar system.

Keywords Random Riemannian geometry · Gaussian field · Conformally invariant · Paneitz operator · Bi-Laplacian · Biharmonic · Membrane model · Quantum Liouville measure

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1 Random Riemannian Geometries and Conformal Invariance

The basic ingredients of any *Random Riemannian Geometry* are a family $\mathfrak{M}$ of Riemannian manifolds (M, g) and a probability measure $\mathbf{P}_{\mathfrak{M}}$ on $\mathfrak{M}$. Typically, $\mathfrak{M} = \{(M, g') : \ g' = e^{2h} g, \ \ h \in \mathcal{C}^\infty(M)\}$ for some given (M, g), and $\mathbf{P}_{\mathfrak{M}}$ is the push forward of a probability measure $\mathbf{P}_g$ on $\mathcal{C}^\infty(M)$ under the map $h \mapsto (M, e^{2h} g)$.

Of major interest are Random Riemannian Geometries which are *conformally invariant*. In the previous setting this means that

- $\mathbf{P}_{g'} = \mathbf{P}_g$ if $g' = e^{2\varphi} g$ for some $\varphi \in \mathcal{C}^\infty(M)$
- $h \overset{(d)}{=} h' \circ \Phi$ if $\Phi : (M, g) \to (M', g')$ is an isometry and h and h' are distributed according to $\mathbf{P}_g$ and $\mathbf{P}_{g'}$, resp.

In this respect, of course, the 2-dimensional case plays a particular role thanks to the powerful Riemannian Mapping Theorem—but the concept of conformally invariant random geometries is by no means restricted to this case.

Mostly, such probability measures $\mathbf{P}_g$ are Gaussian fields, informally given as

$$d\mathbf{P}_g(h) = \frac{1}{Z_g} \exp\Big(- \mathfrak{e}_g(h, h)\Big)\, dh \tag{1}$$

with some (non-existing) uniform distribution dh on $\mathcal{C}^\infty(M)$, a normalizing constant Z_g, and some bilinear form $\mathfrak{e}_g$. The rigorous definition of such probability measures $\mathbf{P}_g$ often requires to pass to spaces of distributions (rather than smooth functions). It is based on the Bochner–Minlos Theorem and the unique characterization of $\mathbf{P}_g$ as

$$\int e^{i\langle u,h\rangle}\, d\mathbf{P}_g(h) = \exp\Big(- \frac{1}{2}\mathfrak{k}_g(u, u)\Big) \quad \forall u \in \mathcal{C}^\infty(M) \tag{2}$$

where $\mathfrak{k}_g(u, u)^{1/2} := \sup_h \frac{\langle u,h\rangle}{\mathfrak{e}_g(h,h)^{1/2}}$ denotes the norm dual to $\mathfrak{e}_g$.

The conformal invariance requirement for the random geometry then amounts to the requirement

$$\mathfrak{e}_g(u, u) = \mathfrak{e}_{e^{2\varphi} g}(u, u) \quad \forall \varphi, \ \forall u. \tag{3}$$

In the two-dimensional case, this is a well-known property of the *Dirichlet energy*, cf. [6],

$$\mathcal{E}_g(u, u) := \int_M \big|\nabla_g u\big|^2 \, d\mathsf{vol}_g.$$

The conformally invariant random field defined and constructed in this way is the celebrated *Gaussian Free Field* [23]. It is a particular (and the most prominent) case of a log-correlated random field [5] and of a fractional Gaussian field [14]. It naturally arises as the scaling limit of various discrete models of random surfaces, for instance discrete Gaussian Free Fields or harmonic crystals [23]. It is also deeply related to another planar conformally invariant random object of fundamental importance, the *Schramm–Loewner evolution* [10, 11, 19]. For instance, level curves of the Discrete Gaussian Free Field converge to SLE_4 [20], and zero contour lines of the Gaussian Free Field are well-defined random curves distributed according to SLE_4 [21]. The work [15], and subsequent works in its series, thoroughly study the relation between the Schramm-Loewner evolution and Gaussian free field on the plane. The *Liouville Quantum Gravity* is a random measure, informally obtained as the Riemannian volume measure when the metric tensor is conformally transformed with the Gaussian Free Field as conformal weight. Since the Gaussian Free Field is only a distribution, the rigorous construction of the random measure requires a renormalization procedure due to Kahane [9]. This renormalization depends on a roughness parameter γ and works only for $|\gamma| < 2$. In [16] and subsequent work in its series, Miller and Sheffield prove that for the value $\gamma = \sqrt{8/3}$ the Liouville Quantum Gravity coincides with the Brownian map, that is a random metric measure space arising as a universal scaling limit of random trees and random planar graphs (see [12, 13] and the references therein). More recently, [4, 8] establish the existence of the Liouville Quantum Gravity metric for $\gamma \in (0, 2)$.

All these approaches to conformally invariant random objects so far, with exception of the recent contribution [1], are limited to the two-dimensional case. The main reason is not the lack of a Riemannian Mapping Theorem but the fact that the Dirichlet energy is no longer conformally invariant in dimension $n \neq 2$. One rather obtains

$$\mathcal{E}_{e^{2\varphi}g}(u, u) = \int_M \left|\nabla_g u\right|^2 e^{(n-2)\varphi} d\mathsf{vol}_g.$$

In the four-dimensional case, a more promising candidate appears to be the bi-Laplacian energy

$$\tilde{\mathfrak{e}}_g(u, u) := \int_M \left(\Delta_g u\right)^2 d\mathsf{vol}_g.$$

This energy functional is still not conformally invariant but it is close to:

$$\tilde{\mathfrak{e}}_{e^{2\varphi}g}(u, u) = \int_M \left(\Delta_g u + 2\nabla_g\varphi\, \nabla_g u\right)^2 d\mathsf{vol}_g = \tilde{\mathfrak{e}}_g(u, u) + \text{low order terms}.$$

Our search for a conformally invariant energy functional in dimensions $n = 4$ finally will lead us to considering

$$\mathfrak{e}_g(u, u) = c \int_M \left(\Delta_g u\right)^2 d\mathsf{vol}_g + \text{low order terms}.$$

Paneitz [18] found the precise formula for the conformally invariant energy functional in dimension 4. Subsequently, Graham et al. [7] showed the existence of a conformally invariant energy functional of the form

$$\mathfrak{e}_g(u,u) = c\int_M \big(-\Delta_g u\big)^{n/2}\, d\mathsf{vol}_g + \text{low order terms}$$

on Riemannian manifolds of even dimension n.

Based on these results, jointly with Dello Schiavo et al. [2], we constructed and analyzed conformally invariant random fields on Riemannian manifolds (M, g) of arbitrary even dimensions. In the subsequent Sects. 2 and 3, we will summarize these results in the particular case $n = 4$. In Sect. 4, we will present a detailed study of approximations of the random field and the random measure on the 4-dimensional flat torus in terms of corresponding random objects on the discrete tori $\mathbb{T}^4_\ell$, $\ell \in \mathbb{N}$.

2 Paneitz Energy on 4-Dimensional Manifolds

From now, we will be more specific. (M, g) will always be a 4-dimensional smooth, compact, connected Riemannian manifold without boundary. Integrable functions (or distributions) u on M will be called *grounded* if $\int_M u\, d\mathsf{vol}_g = 0$ (or $\langle u, \mathbf{1}\rangle = \mathbf{0}$, resp.). Let $(\varphi_j)_{j\in\mathbb{N}_0}$ denote the complete ON-basis of $L^2(M, \mathsf{vol}_g)$ consisting of eigenfunctions of $-\Delta_g$ with corresponding eigenvalues $(\lambda_j)_{j\in\mathbb{N}_0}$. Then the *grounded Sobolev spaces* $\mathring{H}^s(M, g) = (-\Delta_g)^{-s/2}\mathring{L}^2(M, \mathsf{vol}_g)$ for $s \in \mathbb{R}$ are given by

$$\mathring{H}^s(M,g) = \Big\{ u = \sum_{j\in\mathbb{N}} \alpha_j\varphi_j : \sum_{j\in\mathbb{N}} \lambda_j^s\, |\alpha_j|^2 < \infty \Big\},$$

whereas the usual Sobolev spaces are $H^s(M, g) = (1 - \Delta_g)^{-s/2}L^2(M, \mathsf{vol}_g) = \mathring{H}^s(M, g) \oplus \mathbb{R}\cdot\mathbf{1}$. Extending the scalar product in $L^2(M, \mathsf{vol}_g)$, the pairing between $u = \sum_{j\in\mathbb{N}_0} \alpha_j\varphi_j \in H^s$ and $v = \sum_{j\in\mathbb{N}_0} \beta_j\varphi_j \in H^{-s}$ is given by

$$\langle u, v\rangle := \langle u, v\rangle_{H^s,H^{-s}} := \sum_{j\in\mathbb{N}_0} \alpha_j\, \beta_j .$$

The Laplacian acts on these grounded spaces by $-\Delta_g : \mathring{H}^s \to \mathring{H}^{s-2}$, $\sum_{j\in\mathbb{N}} \alpha_j\varphi_j \mapsto \sum_{j\in\mathbb{N}} \lambda_j\alpha_j\varphi_j$. The operator inverse to it is the *grounded Green operator*

$$\mathring{\mathsf{G}}_g : \mathring{H}^s \to \mathring{H}^{s+2},\ \sum_{j\in\mathbb{N}} \alpha_j\varphi_j \mapsto \sum_{j\in\mathbb{N}} \frac{\alpha_j}{\lambda_j}\varphi_j .$$

On $\mathring{H}^0 = \mathring{L}^2$, it is given as an integral operator $\mathring{\mathsf{G}}_g u(x) = \int_M \mathring{G}_g(x, y)\, u(y)\, d\mathsf{vol}_g(y)$ in terms of the grounded Green kernel $\mathring{G}_g(x, y)$ on M. The latter is symmetric in x

and y, it is grounded (i.e. $\int_M \mathring{G}_g(x,y)\, d\mathsf{vol}_g(y) = 0$ for all x) and

$$\left|\mathring{G}_g(x,y) - \frac{1}{4\pi^2 \cdot d_g(x,y)^2}\right| \le C.$$

Definition 1 The *Paneitz energy* is defined as the bilinear form on $L^2(M, \mathsf{vol}_g)$ with domain $H^2(M)$ by

$$\mathfrak{e}_g(u,u) = \frac{1}{8\pi^2}\int_M \left[(\Delta_g u)^2 - 2\mathrm{Ric}_g(\nabla_g u, \nabla_g u) + \frac{2}{3}\mathsf{scal}_g \cdot |\nabla_g u|^2\right] d\mathsf{vol}_g. \quad (4)$$

In particular, for every 4-dimensional Einstein manifold with $\mathrm{Ric}_g = k\, g$ for $k \in \mathbb{R}$ (which implies $\mathsf{scal}_g = 4k$),

$$\mathfrak{e}_g(u,u) = \frac{1}{8\pi^2}\int_M \left[(\Delta_g u)^2 + \frac{2}{3}k\,|\nabla_g u|^2\right] d\mathsf{vol}_g. \quad (5)$$

Example 1 (a) For the 4-sphere $M = \mathbb{S}^4$,

$$\mathfrak{e}_g(u,u) = \frac{1}{8\pi^2}\int_M \left[(\Delta_g u)^2 + 2|\nabla_g u|^2\right] d\mathsf{vol}_g. \quad (6)$$

(b) For the 4-torus $M = \mathbb{T}^4$,

$$\mathfrak{e}_g(u,u) = \frac{1}{8\pi^2}\int_M (\Delta_g u)^2\, d\mathsf{vol}_g. \quad (7)$$

Theorem 1 ([18]) *The Paneitz energy is conformally invariant:*

$$\mathfrak{e}_g(u,u) = \mathfrak{e}_{e^{2\varphi}g}(u,u) \quad \forall \varphi \in \mathcal{C}^\infty(M),\ \forall u \in H^2(M).$$

Definition 2 The 4-manifold (M, g) is called *admissible* if $\mathfrak{e}_g > 0$ on $\mathring{H}^2(M)$.

As an immediate consequence of Theorem 1, we observe that admissibility is a conformal invariance. Large classes of 4-manifolds are admissible.

Proposition 1 ([2, Prop. 2.4, 2.5]) (a) *All compact Einstein 4-manifolds with nonnegative Ricci curvature are admissible.*
(b) *All compact hyperbolic 4-manifolds with spectral gap $\lambda_1 > 2$ are admissible.*

However, not every compact four-dimensional Riemannian manifold is admissible.

Example 2 ([2, Prop. 2.7]) Let M_1, M_2 be compact hyperbolic Riemannian surfaces such that $\lambda_1(M_1) \le \frac{2}{3}$. Then the Einstein 4-manifold $M = M_1 \times M_2$ is not admissible.

If (M, g) is admissible, then the *Paneitz operator* (or *co-bilaplacian*)

$$\mathsf{p}_g = \frac{1}{8\pi^2}\Big[\Delta_g^2 + \mathsf{div}\Big(2\mathsf{Ric}_g - \frac{2}{3}\mathsf{scal}_g\Big)\nabla\Big] \tag{8}$$

is a self-adjoint positive operator on $L^2(M, \mathsf{vol}_g)$ with domain $H^4(M)$. Here the curvature term $2\mathsf{Ric}_g - \frac{2}{3}\mathsf{scal}_g$ should be viewed as an endomorphism of the tangent bundle, acting on the gradient of a function. In coordinates:

$$\mathsf{p}_g u = \frac{1}{8\pi^2}\sum_{i,j}\nabla_i\Big[\nabla^i\nabla^j + 2\mathsf{Ric}_g^{ij} - \frac{2}{3}\mathsf{scal}_g \cdot g^{ij}\Big]\nabla_j u, \qquad \forall u \in \mathcal{C}^\infty(M).$$

Let $(\psi_j)_{j\in\mathbb{N}_0}$ denote a complete orthonormal basis of $L^2(M, \mathsf{vol}_g)$ consisting of eigenfunctions for p_g, and let $(\nu_j)_{j\in\mathbb{N}_0}$ denote the corresponding sequence of eigenvalues. Then the operator k_g, inverse to p_g on $\mathring{L}^2$, is given on $H^{-4}(M)$ by

$$\mathsf{k}_g : u \mapsto \mathsf{k}_g u := \sum_{j\in\mathbb{N}} \frac{1}{\nu_j}\langle u, \psi_j\rangle\, \psi_j,$$

and the associated bilinear form with domain $H^{-2}(M)$ is given by

$$\mathfrak{k}_g(u, v) := \langle u, \mathsf{k}_g v\rangle_{L^2} = \sum_{j\in\mathbb{N}} \frac{1}{\nu_j}\langle u, \psi_j\rangle\,\langle v, \psi_j\rangle.$$

The crucial properties of the kernel for the *co-biharmonic Green operator* k_g are its logarithmic divergence and its conformal invariance.

Theorem 2 ([2, Thm. 2.18]) *If (M, g) is admissible, then k_g is an integral operator with an integral kernel k_g which satisfies*

$$\Big|k_g(x, y) + \log d_g(x, y)\Big| \le C. \tag{9}$$

Furthermore, the kernel $k_g(x, y)$ is symmetric in x, y and grounded.

Theorem 3 ([2, Prop. 2.19]) *Assume that (M, g) is admissible and that $g' := e^{2\varphi} g$ for some $\varphi \in \mathcal{C}^\infty(M)$. Then the co-biharmonic Green kernel $k_{g'}$ for the metric g' is given by*

$$k_{g'}(x, y) = k_g(x, y) - \frac{1}{2}\bar{\phi}(x) - \frac{1}{2}\bar{\phi}(y) \tag{10}$$

with $\bar{\phi} \in \mathcal{C}^\infty(M)$ defined by

$$\bar{\phi} := \frac{2}{\mathsf{vol}_{g'}(M)} \int k_g(., z)\, d\mathsf{vol}_{g'}(z) - \frac{1}{\mathsf{vol}_{g'}(M)^2} \iint k_g(z, w)\, d\mathsf{vol}_{g'}(z)\, d\mathsf{vol}_{g'}(w)\ .$$

Example 3 Assume that (M, g) is Ricci flat. Then

$$k_g(x, y) = 8\pi^2\, \mathring{G}_g^{(2)}(x, y) := 8\pi^2 \int_M \mathring{G}_g(x, z)\, \mathring{G}_g(z, y)\, d\mathsf{vol}_g(z)$$

where $\mathring{G}_g$ denotes the grounded Green kernel on (M, g).

3 Co-biharmonic Gaussian Field and Quantum Liouville Measure

Throughout the sequel, assume that (M, g) is an admissible 4-manifold (compact, smooth, without boundary—as always).

3.1 Conformally Invariant Gaussian Field

Definition 3 A co-biharmonic Gaussian field h on (M, g) is a linear family

$$\big(\langle h, u\rangle\big)_{u\in H^{-2}}$$

of centered Gaussian random variables (defined on some probability space) with

$$\mathbf{E}\big[\langle h, u\rangle^2\big] = \mathfrak{k}_g(u, u) \quad \forall u \in H^{-2}(M).$$

Theorem 4 ([2, Prop. 3.9, Rem. 3.3]) *Let a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ be given and an i.i.d. sequence $(\xi_j)_{j\in\mathbb{N}}$ of $\mathcal{N}(0, 1)$ random variables. Furthermore, let $(\psi_j)_{j\in\mathbb{N}_0}$ and $(\nu_j)_{j\in\mathbb{N}_0}$ denote the sequences of eigenfunctions and eigenvalues for p_g (counted with multiplicities). Then a co-biharmonic field is given by*

$$h := \sum_{j\in\mathbb{N}} \nu_j^{-1/2}\, \xi_j\, \psi_j. \tag{11}$$

More precisely,

(a) *For each $\ell \in \mathbb{N}$, a centered Gaussian random variable h_ℓ with values in $\mathcal{C}^\infty(M)$ is given by*

$$h_\ell := \sum_{j=1}^{\ell} \nu_j^{-1/2}\, \xi_j\, \psi_j. \tag{12}$$

(b) *The convergence* $h_\ell \to h$ *holds in* $L^2(\mathbf{P}) \times H^{-\epsilon}(M)$ *for every* $\epsilon > 0$. *In particular, for a.e.* ω *and every* $\epsilon > 0$,

$$h^\omega \in H^{-\epsilon}(M),$$

(c) *For every* $u \in H^{-2}(M)$, *the family* $(\langle u, h_\ell\rangle)_{\ell\in\mathbb{N}}$ *is a centered* $L^2(\mathbf{P})$*-bounded martingale and*

$$\langle u, h_\ell\rangle \to \langle u, h\rangle \quad \text{in } L^2(\mathbf{P}) \text{ as } \ell \to \infty.$$

Remark 1 (a) A co-biharmonic Gaussian field on (M, g) can be regarded as a random variable with values in $\mathring{H}^{-\epsilon}(M)$ for any $\epsilon > 0$.
(b) Given any 'grounded' white noise Ξ on (M, g), then $h := \sqrt{\mathsf{k}_g}\,\Xi$ is a co-biharmonic Gaussian field on (M, g).

Theorem 5 ([2, Thm. 3.11]) *Let* $h : \Omega \to H^{-\epsilon}(M)$ *denote a co-biharmonic Gaussian field for* (M, g) *and let* $g' = e^{2\varphi} g$ *with* $\varphi \in \mathcal{C}^\infty(M)$. *Then*

$$h' := h - \frac{1}{\mathsf{vol}_{g'}(M)}\big\langle h, \mathbf{1}\big\rangle_{H^{-\epsilon}(M,g'),H^{\epsilon}(M,g')}$$

is a co-biharmonic Gaussian field for (M, g').

Besides the previous eigenfunction approximation, there are numerous other ways to approximate a given co-biharmonic Gaussian field h by 'smooth' Gaussian fields h_ℓ, $\ell \in \mathbb{N}$.

Proposition 2 *Let* ρ_ℓ *for* $\ell \in \mathbb{N}$ *be a family of bounded functions on* $M \times M$ *such that* $\rho_\ell(x, .)\mathsf{vol}_g$ *for each* $x \in M$ *is a family of probability measures on* M *which for* $\ell \to \infty$ *weakly converges to* δ_x. *Define centered Gaussian fields* h_ℓ *for* $\ell \in \mathbb{N}$ *by*

$$h_\ell(y) := \langle h \,|\, \rho_\ell(., y)\rangle\,. \tag{13}$$

Then, for every $u \in \mathcal{C}(M)$, *as* $\ell \to \infty$

$$\langle h_\ell \,|\, u\rangle \longmapsto \langle h \,|\, u\rangle \quad \mathbf{P}\text{-a.s. and in } L^2(\mathbf{P})\,.$$

The associated covariance kernels are given by

$$k_\ell(x, y) := \iint k(x', y')\rho_\ell(x', x)\rho_\ell(y', y)\, d\mathsf{vol}_g(x')\, d\mathsf{vol}_g(y')$$

for all $\ell \in \mathbb{N}$, *and* $k_\ell \to k$ *as* $\ell \to \infty$ *on locally uniformly on* $M \times M$ *off the diagonal.*

Proof Obviously, $\langle h_\ell \,|\, u\rangle = \langle h \,|\, \rho_\ell * u\rangle$ with $(\rho_\ell * u)(x) = \int \rho_\ell(x, y)u(y)\, d\mathsf{vol}_g(y)$, and $\rho_\ell * u \to u$ in L^2 as $\ell \to \infty$. Moreover,

$$\mathbf{E}\Big[\big|\langle h|u\rangle - \langle h_\ell u\rangle\big|^2\Big] = \mathbf{E}\Big[\big|\langle h|u - \rho_\ell * u\rangle\big|^2\Big]$$
$$= \sum_{j=1}^\infty \frac{1}{\nu_j}\big|\langle\psi_j|u - \rho_\ell * u\rangle\big|^2 \le C\,\big\|u - \rho_\ell * u\big\|^2_{H^{-2}}.$$

A particular case of such approximations through convolution kernels will be considered now.

Proposition 3 *Let $(\mathfrak{Q}_\ell)_{\ell\in\mathbb{N}}$ be a family of partitions of M with $\forall\ell, \forall Q \in \mathfrak{Q}_\ell : \exists m \in \mathbb{N}, \exists Q_1, \ldots, Q_m \in \mathfrak{Q}_{\ell+1} : Q = \bigcup_{i=1}^m Q_i$ and with $\sup\{\mathrm{diam}(Q) : Q \in \mathfrak{Q}_\ell\} \to 0$ as $\ell \to \infty$. For $\ell \in \mathbb{N}$ put*

$$\rho_\ell(x, y) := \sum_{Q\in\mathfrak{Q}_\ell} \frac{1}{\mathsf{vol}_g(Q)}\mathbf{1}_Q(x)\mathbf{1}_Q(y). \tag{14}$$

In other words, for given $x \in M$ we have $\rho_\ell(x, .) = \frac{1}{\mathsf{vol}_g(Q)}\mathbf{1}_Q$ with the unique $Q \in \mathfrak{Q}_\ell$ which contains x. Defining h_ℓ as before then yields

$$h_\ell(x) = \frac{1}{\mathsf{vol}_g(Q)}\big\langle h\,\big|\,\mathbf{1}_Q\big\rangle \quad \forall x \in Q, \forall Q \in \mathfrak{Q}_\ell. \tag{15}$$

For $\ell \in \mathbb{N}$, let $\mathfrak{F}_\ell$ denote the σ-field in $(\Omega, \mathfrak{F}, \mathbf{Q})$ generated by the random functions on M that are piecewise constant on each of the sets $Q \in \mathfrak{Q}_\ell$. Then $(h_\ell)_{\ell\in\mathbb{N}}$ is a $(\Omega, \mathfrak{F}, (\mathfrak{F}_\ell)_{\ell\in\mathbb{N}}, \mathbf{P})$-martingale and

$$h_\ell = \mathbf{E}\big[h\big|\mathfrak{F}_\ell\big] \quad \forall\ell \in \mathbb{N}. \tag{16}$$

3.2 Quantum Liouville Measure

Let an admissible 4-manifold (M, g) be given as well as a co-biharmonic Gaussian field h on it. Furthermore, let smooth approximations $(h_\ell)_{\ell\in\mathbb{N}}$ of it be given—informally defined as $h_\ell := \rho_\ell * h$ and formally by (13)—in terms of a sequence $(\rho_\ell)_{\ell\in\mathbb{N}_0}$ of bounded convolution densities on M. Fix $\gamma \in \mathbb{R}$.

For $\ell \in \mathbb{N}$ define a random measure $\mu_\ell = \rho_\ell\,\mathsf{vol}_g$ on M with density

$$\rho_\ell(x) := \exp\Big(\gamma h_\ell(x) - \frac{\gamma^2}{2}k_\ell(x, x)\Big)$$

with $k_\ell(x, y) := \iint k(x', y')\rho_\ell(x', x)\rho_\ell(y', y)\, d\mathsf{vol}_g(x')\, d\mathsf{vol}_g(y')$ as before.

Theorem 6 ([2, Thm. 4.1]) *If $|\gamma| < \sqrt{8}$, then there exists a random measure μ on M with $\mu_\ell \to \mu$. More precisely, for every $u \in \mathcal{C}(M)$,*

$$\int_M u\, d\mu_\ell \longrightarrow \int_M u\, d\mu \quad \textit{in } L^1(\mathbf{P}) \textit{ and } \mathbf{P}\textit{-a.s. as } \ell \to \infty.$$

The random measure μ is independent of the choice of the convolution densities $(\rho_\ell)_{\in\mathbb{N}}$.

If the $(\rho_\ell)_{\in\mathbb{N}}$ are chosen according to (14) then for each $u \in \mathcal{C}(M)$ the family $Y_\ell := \int_M u\, d\mathsf{vol}_g$, $\ell \in \mathbb{N}$, is a uniformly integrable martingale. If in addition $|\gamma| < 2$, then this martingale is even L^2-bounded.

The latter claim, indeed, can be seen directly:

$$\begin{aligned}
\sup_\ell \mathbf{E}\Big[{Y_\ell}^2\Big] &= \sup_\ell \mathbf{E} \iint e^{\gamma(h_\ell(x)+h_\ell(y) - \frac{\gamma^2}{2}(\mathbf{E}[h_\ell^2(x)+h_\ell^2(y)]} u(x)u(y)\, d\mathrm{vol}_g(x) d\mathrm{vol}_g(y) \\
&= \sup_\ell \iint e^{\gamma^2 k_\ell(x,y)} u(x)u(y)\, d\mathrm{vol}_g(x)\, d\mathrm{vol}_g(y) \\
&\le \|u\|_\infty^2 \cdot \sup_\ell \iint \Big[\iint \rho_\ell(x',x)\rho_\ell(y',y) e^{\gamma^2 k(x',y')}\, d\mathrm{vol}_g(x')\, d\mathrm{vol}_g(y') \Big] \\
&\qquad\qquad d\mathrm{vol}_g(x)\, d\mathrm{vol}_g(y) \\
&= \|u\|_\infty^2 \cdot \iint e^{\gamma^2 k(x',y')}\, d\mathrm{vol}_g(x')\, d\mathrm{vol}_g(y') \\
&\le \|u\|_\infty^2 \cdot \iint \frac{1}{d(x,y)^{\gamma^2}}\, d\mathrm{vol}_g(x)\, d\mathrm{vol}_g(y) + C'
\end{aligned}$$

by means of Jensen's inequality and the kernel estimate (9). Obviously, the final integral is finite if and only if $\gamma^2 < 4$.

Definition 4 The random measure $\mu := \lim_{\ell\to\infty} \mu_\ell$ is called *quantum Liouville measure*.

Remark 2 ([2, Cor 4.10, Prop. 4.14]) Assume $|\gamma| < \sqrt{8}$ and let $\omega \mapsto \mu^\omega$ denote the random measure constructed above. Then for **P**-a.e. ω, the measure μ^ω on M

- does not charge sets of vanishing H^2-capacity;
- does not charge sets of vanishing H^1-capacity provided $|\gamma| < 2$;
- is singular w.r.t. the volume measure on M whenever $\gamma \neq 0$.

Moreover, the random measure μ has finite moments of any negative order, i.e. for any $p > 0$,

$$\mathbf{E}\big[\mu(M)^{-p}\big] < \infty.$$

A key property of the quantum Liouville measure is its quasi-invariance under conformal transformations.

Theorem 7 ([2, Thm. 4.4]) *Let μ be the quantum Liouville measure for (M,g), and μ' be the quantum Liouville measure for (M,g') where $g' = e^{2\varphi} g$ for some $\varphi \in \mathcal{C}^\infty(M)$. Then*

$$\mu' \overset{(\mathrm{d})}{=} e^{-\gamma\xi + \frac{\gamma^2}{2}\bar{\varphi} + 4\varphi}\, \mu \tag{17}$$

where $\xi := \frac{1}{v'}\langle h, e^{4\varphi}\rangle$ and $\bar{\varphi} := \frac{2}{v'}\, \mathsf{k}_g(e^{4\varphi}) - \frac{1}{v'^2}\, \mathfrak{k}_g(e^{4\varphi}, e^{4\varphi})$ with $v' := \mathsf{vol}_{g'}(M)$.

4 Approximation by Random Fields and Liouville Measures on the Discrete 4-Torus

For the remaining part, we now focus on the 4-dimensional torus $\mathbb{T}^4 := \mathbb{R}^4/\mathbb{Z}^4$, equipped with the flat metric. With this choice of (M, g), we will drop the g from the notations: $k = k_g$, $G = G_g$ etc.

For the 4-torus, we will study approximations of the co-biharmonic field—now briefly called biharmonic field (since the underlying Paneitz operator or co-bilaplacian is now simply the bilaplacian)—and of the quantum Liouville measure by (semi-) discrete versions of such fields and measures, defined on the discrete tori $\mathbb{T}^4_\ell$ as $\ell \to \infty$.

4.1 The Isotropic Haar System

To begin with, for $\ell \in \mathbb{N}_0$ define the parameter sets

$$A_\ell := \left\{0, 1, \ldots, 2^\ell - 1\right\}^4, \quad B_\ell := \{0, 1\}^4 \setminus \{(0, 0, 0, 0)\}, \quad I_\ell := A_\ell \times B_\ell$$

and the discrete 4-torus

$$\mathbb{T}^4_\ell := 2^{-\ell} \cdot A_\ell = (2^{-\ell}\mathbb{Z}^4)/\mathbb{Z}^4.$$

Moreover, let $\mathfrak{Q}_\ell := \left\{Q_{\ell,\alpha} : \alpha \in A_\ell\right\}$ denote the set of all dyadic cubes

$$Q_{\ell,\alpha} := 2^{-\ell} \cdot \Big([\alpha_1, \alpha_1 + 1) \times [\alpha_2, \alpha_2 + 1) \times [\alpha_3, \alpha_3 + 1) \times [\alpha_4, \alpha_4 + 1)\Big) \subset \mathbb{T}^4$$

of edge length $2^{-\ell}$, and let $\mathcal{S}_\ell$ denote the set of all grounded functions $u : \mathbb{T}^4 \to \mathbb{R}$ which are constant on each of the cubes $Q \in \mathfrak{Q}_\ell$. With each $Q_{\ell,\alpha} \in \mathbb{T}^4_\ell$ we associate a set $\{\eta_{\ell,\alpha,\beta} : \beta \in B_\ell\} \subset \mathcal{S}_{\ell+1}$ of 15 multivariate Haar functions with support $Q_{\ell,\alpha}$ given by all possible tensor products

$$\eta_{\ell,\alpha,\beta}(x) := \tilde{\eta}_{\ell,\alpha_1,\beta_1}(x_1) \cdot \tilde{\eta}_{\ell,\alpha_2,\beta_2}(x_2) \cdot \tilde{\eta}_{\ell,\alpha_3,\beta_3}(x_3) \cdot \tilde{\eta}_{\ell,\alpha_4,\beta_4}(x_4)$$

where

$$\tilde{\eta}_{\ell,\alpha_k,\beta_k}(x_k) := \begin{cases} 2^{\ell/2} \cdot 1_{[\alpha_k,\alpha_k+1)}(2^\ell x_k), & \text{if } \beta_k = 0 \\ 2^{\ell/2} \cdot \Big(1_{[\alpha_k,\alpha_k+\frac{1}{2})} - 1_{[\alpha_k+\frac{1}{2},\alpha_k+1)}\Big)(2^\ell x_k), & \text{if } \beta_k = 1 \end{cases}$$

for $k = 1, 2, 3, 4$.

For $\ell \in \mathbb{N}_0$, the block

$$\mathcal{H}_\ell := \{\eta_{\ell,\alpha,\beta} : \alpha \in A_\ell,\ \beta \in B_\ell\}$$

consists of $15 \cdot 2^{4\ell}$ Haar functions which we call Haar functions of level ℓ. The union of all of them,

$$\mathcal{H} = \bigcup_{\ell=0}^{\infty} \mathcal{H}_\ell,$$

is a complete orthonormal system in $\mathring{L}^2(\mathbb{T}^4)$, called *isotropic 4-dimensional Haar system*, cf. [17]. Moreover,

$$\mathcal{S}_\ell = \mathrm{span}\Big(\bigcup_{\kappa=0}^{\ell-1} \mathcal{H}_\kappa\Big). \tag{18}$$

For $x \in \mathbb{T}^4$ and $\ell \in \mathbb{N}$, the unique cube $Q \in \mathfrak{Q}_\ell$ with $x \in Q$ will be denoted by $Q_\ell(x)$. Given a function $u \in \mathring{L}^1(\mathbb{T}^4)$, we define the function $u_\ell \in \mathcal{S}_\ell$ by

$$u_\ell(x) := 2^{4\ell} \int_{Q_\ell(x)} u \, d\mathcal{L}^4. \tag{19}$$

Restricted to $\mathring{L}^2(\mathbb{T}^4)$, the map $\pi_{\mathfrak{Q}_\ell} : u \mapsto u_\ell$ is the L^2-projection onto the linear subspace $\mathcal{S}_\ell$. Moreover,

$$u_\ell = \sum_{\kappa=0}^{\ell-1} \sum_{\iota \in I_\kappa} \langle u, \eta_{\kappa,\iota}\rangle \, \eta_{\kappa,\iota}. \tag{20}$$

4.2 The Semi-discrete Gaussian Field

Let an i.i.d. family of $\mathcal{N}(0,1)$ random variables $(\xi_{\ell,\iota})_{\ell \in \mathbb{N}_0, \iota \in I_\ell}$ be given with $I_\ell = A_\ell \times B_\ell$ as before. For $\ell \in \mathbb{N}$ put

$$\hat{h}^\omega_\ell(x) := \sqrt{8}\,\pi \sum_{\kappa=0}^{\ell-1} \sum_{\iota \in I_\kappa} \xi^\omega_{\kappa,\iota} \cdot \mathring{\mathsf{G}}\, \eta_{\kappa,\iota}(x).$$

Here $\mathring{\mathsf{G}}$ denotes the grounded Green operator on the 4-torus, given as an integral operator $\mathring{\mathsf{G}}u(x) = \int_{\mathbb{T}^4} \mathring{G}(x,y)\, u(y)\, d\mathcal{L}^4(y)$ in terms of the grounded Green kernel $\mathring{G}(x,y)$ on $\mathbb{T}^4$. (For related results with the Green kernel of the torus replaced by the Green kernel of the discrete torus, see Sect. 4.3.) Moreover, define the non-symmetric kernel $\mathring{G}_\ell(x,z) := \big(\pi_{\mathfrak{Q}_\ell} \mathring{G}(x,.)\big)(z) = 2^{4\ell} \int_{Q_\ell(z)} \mathring{G}(x,v)\, d\mathcal{L}^4(v)$ and put

$$\hat{k}_\ell(x,y) := 8\pi^2 \int_{\mathbb{T}^4} \mathring{G}_\ell(x,z) \mathring{G}_\ell(y,z)\, d\mathcal{L}^4(z).$$

As $\ell \to \infty$, this converges pointwise to $k(x,y) := 8\pi^2 \int_{\mathbb{T}^4} \mathring{G}(x,z)\, \mathring{G}(z,y)\, d\mathcal{L}^4(z)$, which—up to the pre-factor—is the Green kernel for the bi-Laplacian Δ^2.

Proposition 4 *For every $\ell \in \mathbb{N}$,*

(a) *for every ω, the function $\hat{h}^\omega_\ell$*

- *is in* $\mathcal{C}^1(\mathbb{T}^4)$ *and grounded (i.e.* $\int_{\mathbb{T}^4} \hat{h}^\omega_\ell \, d\mathcal{L}^4 = 0$*);*
- *is smooth off the boundaries of dyadic cubes* $Q \in \mathfrak{Q}_\ell$*;*
- *has constant Laplacian on the interior of each dyadic cube* $Q \in \mathfrak{Q}_\ell$*;*
- *is the sum* $\sum_{\kappa=0}^{\ell-1} \sum_{\iota \in I_\kappa} \hat{h}^\omega_{\kappa,\iota}$ *of functions* $\hat{h}^\omega_{\kappa,\iota} = \sqrt{8}\,\pi\, \xi^\omega_{\kappa,\iota} \cdot \mathring{\mathsf{G}}\, \eta_{\kappa,\iota}$ *each of which is harmonic on* $\mathbb{T}^4 \setminus \bar{Q}_{\kappa,\iota}$ *for the dyadic cube* $Q_{\kappa,\iota} \in \mathfrak{Q}_\ell$*;*

(b) *for every* $x \in \mathbb{T}^4$*, the random variable* $\hat{h}_\ell(x)$ *is centered and Gaussian with variance* $\hat{k}_\ell(x,x)$*, the latter being independent of* x*;*
(c) $\hat{h}_\ell$ *is a centered Gaussian field with covariance function* $\hat{k}_\ell(x,y)$*.*

Proof We show (c), the rest is straightforward. By the very definition of $\hat{h}_\ell$, the i.i.d. property of the $\xi_{\kappa,\iota}$ and the projection properties (19) and (20),

$$\begin{aligned}
\mathbf{E}\Big[\hat{h}_\ell(x)\cdot \hat{h}_\ell(y)\Big] &= 8\pi^2 \cdot \sum_{\kappa=0}^{\ell-1} \sum_{\iota\in I_\kappa} \big\langle \mathring{G}(x,.), \eta_{\kappa,\iota}\big\rangle \cdot \big\langle \mathring{G}(y,.), \eta_{\kappa,\iota}\big\rangle \\
&= 8\pi^2 \cdot \Big\langle \pi_{\mathfrak{Q}_\ell} \mathring{G}(x,.), \pi_{\mathfrak{Q}_\ell} \mathring{G}(y,.)\Big\rangle \\
&= 8\pi^2 \cdot 2^{8\ell} \int\limits_{\mathbb{T}^4} \Bigg(\int\limits_{Q_\ell(z)} \mathring{G}(x,v)\, d\mathcal{L}^4(v) \cdot \int\limits_{Q_\ell(z)} \mathring{G}(y,w)\, d\mathcal{L}^4(w)\Bigg)\, d\mathcal{L}^4(z) \\
&= \hat{k}_\ell(x,y).
\end{aligned}$$

Theorem 8 (a) *The centered Gaussian random field* h *with covariance function* k *(as introduced and studied in Sect. 3.1) is given in the case of the 4-torus by*

$$h := \sqrt{8}\,\pi \sum_{\kappa=0}^{\infty} \sum_{\iota\in I_\kappa} \xi_{\kappa,\iota} \cdot \mathring{\mathsf{G}}\eta_{\kappa,\iota}$$

and called biharmonic Gaussian field.
(b) *The convergence* $\hat{h}_\ell \to h$ *holds in* $L^2(\mathbf{P}) \times H^{-\epsilon}(\mathbb{T}^4)$ *for every* $\epsilon > 0$*. In particular, for a.e.* ω *and every* $\epsilon > 0$*,*

$$h^\omega \in H^{-\epsilon}(\mathbb{T}^4).$$

(c) *For every* $u \in H^{-2}(\mathbb{T}^4)$*, the family* $(\langle u, \hat{h}_\ell\rangle)_{\ell\in\mathbb{N}}$ *is a centered* $L^2(\mathbf{P})$*-bounded martingale and*

$$\langle u, \hat{h}_\ell\rangle \to \langle u, h\rangle \quad \text{in } L^2(\mathbf{P}) \text{ as } \ell \to \infty.$$

Proof (a) For convergence (and well-definedness) of the infinite sum, see (b) and/or (c) below. To identify the covariance, observe that

$$\mathbf{E}\Big[\langle u, h\rangle^2\Big] = 8\pi^2 \cdot \sum_{\kappa=0}^{\infty} \sum_{\iota\in I_\kappa} \big\langle \mathring{G}u, \eta_{\kappa,\iota}\big\rangle^2 = 8\pi^2 \cdot \big\|\mathring{G}u\big\|^2 = \big\langle u, ku\big\rangle.$$

(b) For $s > 0$, let $\mathring{\mathsf{G}}^s$ denote the s-th power of the operator $\mathring{\mathsf{G}}$ and let $\mathring{G}^{(s)}$ denote its kernel which is given by the formula

$$\mathring{G}^{(s)}(x,y) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \mathring{p}_t(x,y)dt$$

in terms of the grounded heat kernel $\mathring{p}_t(x,y) = p_t(x,y) - 1$. Then for $\epsilon > 0$,

$$\begin{aligned}\frac{1}{8\pi^2}\mathbf{E}\Big[\|h\|^2_{H^{-\epsilon}}\Big] &= \frac{1}{8\pi^2}\mathbf{E}\Big[\big\|\mathring{\mathsf{G}}^{\epsilon} h\big\|^2_{L^2}\Big] = \sum_{\kappa=0}^{\infty}\sum_{\iota\in I_\kappa} \|\mathring{\mathsf{G}}^{1+\epsilon}\eta_{\kappa,\iota}\|^2_{L^2} \\ &= \int_{\mathbb{T}^4} \Big\|\mathring{G}^{(1+\epsilon)}(.,z)\Big\|^2 d\mathcal{L}^4(z) = \int_{\mathbb{T}^4} \mathring{G}^{(2+2\epsilon)}(z,z)\,d\mathcal{L}^4(z) \\ &= \mathring{G}^{(2+2\epsilon)}(0,0) < \infty\end{aligned}$$

since $\mathring{G}^{(s)}$ for $s > n/2 = 2$ is a bounded function, see [2]. This proves that $\|h\|_{H^{-\epsilon}} < \infty$ for a.e. ω. The convergence $\hat{h}_\ell \to h$ in $H^{-\epsilon}(\mathbb{T}^4)$ follows similarly.

(c) By construction, for every $x \in \mathbb{T}^4$, the family $(\hat{h}_\ell(x))_{\ell\in\mathbb{N}}$ is a centered martingale. The martingale property immediately carries over to the family $(\langle u, \hat{h}_\ell\rangle)_{\ell\in\mathbb{N}}$ for any function or distribution u. The L^2-boundedness follows from

$$\frac{1}{8\pi^2}\cdot\sup_\ell \mathbf{E}\Big[\langle u, \hat{h}_\ell\rangle^2\Big] = \sup_\ell \|\pi_{\mathfrak{Q}_\ell}\mathring{\mathsf{G}}u\|^2_{L^2} = \|\mathring{\mathsf{G}}u\|^2_{L^2} < \infty.$$

Remark 3 (a) With $\eta_{\ell,\iota}$ for $\iota = (\alpha,\beta)$ in I_ℓ as above and $\psi_{\ell,\iota} := \mathring{\mathsf{G}}\eta_{\ell,\iota}$, the family $(\psi_{\ell,\iota})_{\ell\in\mathbb{N}_0,\iota\in I_\ell}$ is a complete orthonormal system in the Hilbert space $H^2(\mathbb{T}^4)$, equipped with the scalar product

$$\langle u, v\rangle_{H^2} := \langle \Delta u, \Delta v\rangle_{L^2}.$$

(b) Given any complete orthonormal system $(\psi_k)_{k\in\mathbb{N}}$ in the Hilbert space $\mathring{H}^2(\mathbb{T}^4)$ and any i.i.d. sequence of $\mathcal{N}(0,1)$ random variables $(\xi_k)_{k\in\mathbb{N}}$, with the same arguments as for Theorem 8 one can prove that the Gaussian random field

$$h_\ell := \sqrt{8}\pi \sum_{k=1}^{\ell} \xi_k \cdot \psi_k$$

converges in $L^2(\mathbf{P}) \times H^{-\epsilon}(\mathbb{T}^4)$ for every $\epsilon > 0$ as $\ell \to \infty$ to the biharmonic Gaussian field h on the 4-torus. Moreover, $\langle u, h_\ell\rangle \to \langle u, h\rangle$ in $L^2(\mathbf{P})$ for every $u \in H^{-2}(\mathbb{T}^4)$.

4.3 The Semi-discrete Liouville Measure

Given a Gaussian field h as considered in Theorem 8, then following Proposition 3 a semi-discrete approximation of it is defined by

$$h_\ell := \pi_{\mathfrak{Q}_\ell} h \quad (\forall \ell \in \mathbb{N}).$$

For each ℓ, this is a centered Gaussian field with covariance given by

$$k_\ell(x,y) := 8\pi^2\, 2^{8\ell} \int\limits_{Q_\ell(x)} \int\limits_{Q_\ell(y)} \left(\int\limits_{\mathbb{T}^4} \mathring{G}(z,v)\mathring{G}(z,w)\, d\mathcal{L}^4(z) \right) d\mathcal{L}^4(w)\, d\mathcal{L}^4(v)$$

$$= 8\pi^2\, 2^{8\ell} \int\limits_{Q_\ell(x)} \int\limits_{Q_\ell(y)} k(v,w)\, d\mathcal{L}^4(w)\, d\mathcal{L}^4(v).$$

For any $\gamma \in \mathbb{R}$ and $\ell \in \mathbb{N}$, we define the *semi-discrete quantum Liouville measure* $\mu_\ell = \rho_\ell\, \mathcal{L}^4$ as the random measure on $\mathbb{T}^4$ with density w.r.t. $\mathcal{L}^4$ given by

$$\rho_\ell^\omega(x) := e^{\gamma h_\ell^\omega(x) - \frac{\gamma^2}{2} k_\ell(x,x)}.$$

Corollary 1 *For $|\gamma| < \sqrt{8}$ and a.e. ω, the measures μ_ℓ^ω as $\ell \to \infty$ weakly converge to the Borel measure μ^ω introduced and studied in Sect. 3.2,*

$$\mu_\ell \to \mu \quad \mathbf{P}\text{-a.s.}$$

For $|\gamma| < 2$, the convergence also holds in $L^2(\mathbf{P})$.

4.4 Discrete Random Objects

To end up with fully discrete random objects, we have to replace the Green function on the continuous torus by the Green function on the discrete torus. To do so, for $\ell \in \mathbb{N}$ we define the *discrete Laplacian* Δ_ℓ acting on functions $u \in L^2(\mathbb{T}^4_\ell)$ by

$$-\Delta_\ell u = 2^{2\ell+3} \cdot \big(u - p_\ell u\big), \qquad p_\ell u(i) := \frac{1}{8} \sum_{j \in J_\ell} u(i+j)$$

with $\quad J_\ell := \Big\{(k,0,0,0),(0,k,0,0),(0,0,k,0),(0,0,0,k) : \quad k \in \{-2^{-\ell}, 2^{-\ell}\}\Big\}.$
Note that the associated discrete Dirichlet form $\mathcal{E}_\ell(u,u) := -\langle u, \Delta_\ell u\rangle_{L^2}$ on $L^2(\mathbb{T}^4_\ell)$ has a positive spectral gap $\lambda_\ell^1 := \inf\left\{\frac{\mathcal{E}_\ell(u,u)}{\|u\|^2_{L^2}} : u \in \mathring{L}^2(\mathbb{T}^4_\ell)\right\}$. Furthermore, we define the grounded transition kernel by $\mathring{p}_\ell(i,j) := \frac{1}{8}\mathbf{1}_{J_\ell}(i-j) - 2^{-4\ell}$.

The *discrete Green operator* acting on grounded functions $u \in \mathring{L}^2(\mathbb{T}^4_\ell)$ is defined by

$$\dot{\mathsf{G}}_\ell u(i) := 2^{-2\ell-3} \sum_{k=0}^{\infty} p_\ell^k u(i) = 2^{-2\ell-3} \sum_{k=0}^{\infty} \dot{p}_\ell^k u(i) = \sum_{j\in\mathbb{T}^4_\ell} \dot{G}_\ell(i,j)u(j) \tag{21}$$

where $\dot{G}_\ell(i,j) := 2^{-2\ell-3} \sum_{k=0}^{\infty} \dot{p}_\ell^k(i,j)$ denotes the grounded Green kernel. The convergence of the operator sum is granted by the positivity of λ_ℓ^1:

$$\|p_\ell\|_{\mathring{L}^2,\mathring{L}^2} \le \|p_\ell^{1/2}\|^2_{\mathring{L}^2,\mathring{L}^2} = \sup_{u\in\mathring{L}^2} \frac{\langle u, p_\ell u\rangle}{\|u\|^2_{\mathring{L}^2}} = 1 - \inf_{u\in\mathring{L}^2} \frac{\mathcal{E}_\ell(u,u)}{\|u\|^2_{\mathring{L}^2}} = 1 - \lambda_\ell^1 < 1.$$

This operator is then extended to an operator acting on functions $u \in S_\ell$ by

$$\bar{\mathsf{G}}_\ell u(x) := \dot{\mathsf{G}}_\ell\Big(u|_{\mathbb{T}^4_\ell}\Big)(2^{-\ell}\alpha) \quad \forall x \in Q_{\ell,\alpha}, \alpha \in A_\ell.$$

In terms of the (extended) discrete Green operator we define the *discrete Gaussian field*

$$\dot{h}_\ell^\omega(i) := \sqrt{8}\,\pi \sum_{\kappa=0}^{\ell-1} \sum_{\iota\in I_\kappa} \xi^\omega_{\kappa,\iota} \cdot \dot{\mathsf{G}}_\ell\left(\eta_{\kappa,\iota}|_{\mathbb{T}^4_\ell}\right)(i)$$

on $\mathbb{T}^4_\ell$ and its piecewise constant extension

$$\bar{h}_\ell^\omega(x) := \sqrt{8}\,\pi \sum_{\kappa=0}^{\ell-1} \sum_{\iota\in I_\kappa} \xi^\omega_{\kappa,\iota} \cdot \bar{\mathsf{G}}_\ell\, \eta_{\kappa,\iota}(x)$$

on $\mathbb{T}^4$. For $\gamma \in \mathbb{R}$, the *discrete quantum Liouville measure* is given by

$$\dot{\mu}_\ell^\omega := 2^{-4\ell} \exp\Big(-\frac{\gamma^2}{2}\dot{k}_\ell\Big) \sum_{i\in\mathbb{T}^4_\ell} \exp\Big(\gamma \dot{h}_\ell^\omega(i)\Big)\,\delta_i$$

with

$$\dot{k}_\ell := \mathbf{E}\big[\dot{h}_\ell(i)^2\big] = 8\pi^2 \cdot 2^{-4\ell} \sum_{j\in\mathbb{T}^4_\ell} \dot{\mathsf{G}}_\ell(i,j)^2 = \pi^2 \cdot 2^{-8\ell-3} \sum_{k=0}^{\infty} k\, \dot{p}_\ell^k(i,i),$$

independent of $i \in \mathbb{T}^4_\ell$.

Alternatively—and equivalent in distribution according to (18)—we can define the discrete Gaussian field by

$$\dot{h}_\ell^\omega(i) := \sqrt{8}\,\pi\,\big(\dot{\mathsf{G}}_\ell\,\dot{\xi}^\omega\big)(i) = 2^{-2\ell-3/2}\,\pi \sum_{k=0}^{\infty} \sum_{j\in\mathbb{T}^4_\ell} \dot{p}_\ell^k\,(i,j)\;\dot{\xi}_j^\omega \tag{22}$$

on $\mathbb{T}^4_\ell$ with a sequence of $\mathcal{N}(0,1)$-i.i.d. random variables $(\xi_j)_{j\in\mathbb{T}^4_\ell}$ and $\dot{\xi}_i := \xi_i - 2^{-4\ell}\sum_{j\in\mathbb{T}^4}\xi_j$. In other words,

$$-\Delta_\ell \dot{h}^\omega_\ell(i) = \sqrt{8}\pi\,\dot{\xi}^\omega_i.$$

Moreover,

$$\mathbf{E}\Big[\langle u, \dot{h}_\ell\rangle^2_{L^2}\Big] = 8\pi^2\,\big\|\dot{\mathsf{G}}_\ell u\big\|^2_{L^2} \quad \forall u \in L^2(\mathbb{T}^4_\ell)$$

and thus the distribution of the Gaussian field $\dot{h}_\ell$ is explicitly given by the probability measure

$$d\mathbf{P}_\ell(\zeta) := \frac{1}{Z_\ell}\exp\bigg(-\frac{1}{16\pi^2}\Big\|\Delta_\ell\zeta\Big\|^2_{L^2(\mathbb{T}^4_\ell)}\bigg)\prod_{j\in\mathbb{T}^4_\ell}\mathcal{L}^1(d\zeta_j), \tag{23}$$

conditioned to the hyperplane $\big\{\sum_i \zeta_i = 0\big\}$ in $\mathbb{R}^{\mathbb{T}^4_\ell}$. Here $\zeta = (\zeta_i)_{i\in\mathbb{T}^4_\ell}$,

$$\Big\|\Delta_\ell\zeta\Big\|^2_{L^2(\mathbb{T}^4_\ell)} = 2^{-4\ell}\sum_{i\in\mathbb{T}^4_\ell}\Big|\zeta_i - \frac{1}{8}\sum_{j\in J_\ell}\zeta_{i+j}\Big|^2,$$

and $Z_\ell \in (0,\infty)$ denotes a suitable normalization constant.

The convergence $\bar{h}_\ell \to h$ and $\dot{\mu}_\ell \to \mu$ as $\ell \to \infty$ will be analyzed in detail in the forthcoming paper [3]. For related convergence questions concerning biharmonic Gaussian random fields on the cube $[0,1]^4$ with Dirichlet boundary conditions, see [22].

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References

1. B. Cerclé, Liouville conformal field theory on the higher-dimensional sphere (2019). arXiv:1912.09219
2. L. Dello Schiavo, R. Herry, E. Kopfer, K.-T. Sturm, Conformally invariant random fields, quantum Liouville measures, and random Paneitz operators on Riemannian manifolds of even dimension (2021). Arxiv 2105.13925
3. L. Dello Schiavo, R. Herry, E. Kopfer, K.-T. Sturm, Polyharmonic fields and Liouville geometry in arbitrary dimension: from discrete to continuous (in preparation) (2021)
4. J. Ding, J. Dubédat, A. Dunlap, H. Falconet, Tightness of Liouville first passage percolation for $\gamma \in (0,2)$. Publ. Math. Inst. Hautes Études Sci. **132**, 353–403 (2020)
5. B. Duplantier, R. Rhodes, S. Sheffield, V. Vargas, Log-correlated Gaussian fields: an overview, in *Geometry, Analysis and Probability* (2017), pp. 191–216
6. M. Fukushima, Y. Oshima, M. Takeda, in *Dirichlet Forms and Symmetric Markov Processes. De Gruyter Studies in Mathematics*. vol. 19, extended edn. (de Gruyter, 2011)
7. C.R. Graham, R. Jenne, L.J. Mason, G.A.J. Sparling, Conformally invariant powers of the Laplacian. I. Existence. J. Lond. Math. Soc. (2) **46**(3), 557–565 (1992)

8. E. Gwynne, J. Miller, Existence and uniqueness of the Liouville quantum gravity metric for $\gamma \in (0, 2)$. Invent. Math. **223**(1), 213–333 (2021)
9. J.-P. Kahane, Sur le Chaos Multiplicatif. Ann. Sci. Math. Québec **9**(2), 105–150 (1985)
10. G.F. Lawler, in *Conformally Invariant Processes in the Plane, Mathematical Surveys and Monographs*, vol. 114 (American Mathematical Society, Providence, RI, 2005)
11. G.F. Lawler, Conformally invariant loop measures, in *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary Lectures* (World Scientific Publishing, Hackensack, NJ, 2018), pp. 669–703
12. J.-F. Le Gall, Brownian geometry. Jpn. J. Math. **14**(2), 135–174 (2019)
13. J.-F. Le Gall, G. Miermont, Scaling limits of random trees and planar maps, in *Probability and Statistical Physics in Two and More Dimensions. Clay Mathematics Proceedings*, vol. 15 (American Mathematical Society, Providence, RI, 2012), pp. 155–211
14. A. Lodhia, S. Sheffield, X. Sun, S.S. Watson, Fractional Gaussian fields: a survey. Probab. Surv. **13**, 1–56 (2016)
15. J. Miller, S. Sheffield, Imaginary geometry I: interacting SLEs. Probab. Theory Relat. Fields **164**(3–4), 553–705 (2016)
16. J. Miller, S. Sheffield, Liouville quantum gravity and the Brownian map I: the QLE$(8/3, 0)$ metric. Invent. Math. **219**(1), 75–152 (2020)
17. P. Oswald, Haar system as Schauder basis in Besov spaces: the limiting cases for $0<p \leq 1$. INS-preprint no. 1810. Bonn University
18. S.M. Paneitz, A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds (summary). SIGMA Symmetry Integr. Geom. Methods Appl., **4**, Paper 036, 3 (1983). Published in 2008
19. O. Schramm, Conformally invariant scaling limits: an overview and a collection of problems, in *International Congress of Mathematicians*, vol. I (European Mathematical Society, Zürich, 2007), pp. 513–543
20. O. Schramm, S. Sheffield, Contour lines of the two-dimensional discrete Gaussian free field. Acta Math. **202**(1), 21–137 (2009)
21. O. Schramm, S. Sheffield, A contour line of the continuum Gaussian free field. Probab. Theory Relat. Fields **157**(1–2), 47–80 (2013)
22. F. Schweiger, On the membrane model and the discrete Bilaplacian. Ph.D. thesis. Bonn University (2021)
23. S. Sheffield, Gaussian free fields for mathematicians. Probab. Theory Relat. Fields **139**(3–4), 521–541 (2007)

Infinite Particle Systems with Hard-Core and Long-Range Interaction

Hideki Tanemura

Abstract A system of Brownian hard balls is regarded as a reflecting Brownian motion in the configuration space and can be represented by a solution to a Skorohod-type equation. In this article, we consider the case that there are an infinite number of balls, and the interaction between balls is given by the long-range pair interaction. We discuss the existence and uniqueness of strong solutions to the infinite-dimensional Skorohod equation.

Keywords Stochastic differential equations · Infinite particle systems · Skorohod equations

Mathematics Subject Classification 60K35 · 60J46 · 60J60

1 Introduction

In this article, we study systems of interacting Brownian motions on $\mathbb{R}^d$, $d \geq 2$. Let $\Phi : \mathbb{R}^d \to (-\infty, \infty]$ be a self (free) potential and $\Psi : \mathbb{R}^d \times \mathbb{R}^d \to (-\infty, \infty)$ be a symmetric pair-interaction potential. In the case that these potentials are smooth, the system is described by the stochastic differential equation (SDE)

$$dX_t^j = dB_t^j - \frac{1}{2}\nabla\Phi(X_t^j)dt - \frac{1}{2}\sum_{k\in\Lambda, k\neq j}\nabla\Psi(X_t^j, X_t^k)dt, \quad j \in \mathbb{I},$$

where B_t^j, $j \in \Lambda$ are independent Brownian motions and $\mathbb{I}$ is a countable index set. We consider in this article the case that Φ is smooth and Ψ is a pair potential with a hard core of radius $r > 0$ (i.e., $\Psi = \Psi_{\text{hard}} + \Psi_{\text{sm}}$):

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$$\Psi_{\text{hard}}(x,y) = \begin{cases} 0 & \text{if } |x-y| \geq r, \\ \infty & \text{if } |x-y| < r, \end{cases} \quad \text{the hard-core pair potential,}$$

$$\Psi_{\text{sm}}(x,y) = \Psi_{\text{sm}}(x-y) : \text{a translation invariant smooth potential.}$$

The system can be regarded as that of balls with a radius $r > 0$.

When $\mathbb{I}$ is a finite subset of $\mathbb{N}$, Saisho [16] and Saisho and Tanaka [17] showed that a system of interacting Brownian balls with potential Φ and Ψ_{sm} can be represented by the unique solution of the Skorohod-type equation

$$\begin{aligned} dX_t^j &= dB_t^j - \frac{1}{2}\nabla\Phi(X_t^j)dt - \frac{1}{2}\sum_{k\in\mathbb{I},k\neq j} \nabla\Psi_{\text{sm}}(X_t^j - X_t^k)dt \\ &\quad + \sum_{k\in\mathbb{I},k\neq j} (X_t^j - X_t^k)dL_t^{jk}, \quad j \in \mathbb{I}, \qquad \text{(SKE-}\mathbb{I}) \\ |X_t^j &- X_t^k| \geq r, \quad j,k \in \mathbb{I}, \end{aligned}$$

where L_t^{jk} $j,k \in \mathbb{I}$ are non-decreasing functions satisfying

$$L_t^{jk} = L_t^{kj} = \int_0^t \mathbf{1}(|X_s^j - X_s^k| = r)dL_s^{jk}, \quad j,k \in \mathbb{I}.$$

For $\mathbb{I} = \mathbb{N}$, the existence and uniqueness of solutions of (SKE-$\mathbb{N}$) have been solved in the cases that $\Phi = \Psi_{\text{sm}} = 0$ [18] and $\Phi = 0$ and Ψ_{sm} has stretched exponential decay [5]. In these cases, the interaction among particles has a short range. The purpose of this article is to generalize the results for long-range interaction including the case that Ψ_{sm} has polynomial decay.

Let $\mathfrak{X}$ be the configuration space of unlabeled balls with diameter $r > 0$. The space $\mathfrak{X}$ is a compact Polish space with the vague topology. Using Dirichlet form theory, we can construct an $\mathfrak{X}$-valued process Ξ describing an interacting system with an infinite number of unlabeled particles [8, 11] including the case with a hard-core interaction. See, for example, [2, 3] for the relation between Dirichlet forms and reflecting Brownian motions. For a system with a finite number of balls, the existence of the solution to the Skorohod equation is derived through Fukushima decomposition from the process constructed using a Dirichlet form (see, for instance, [1]) For a system with an infinite number of balls, we need to label the balls because the coordinate function is not locally in the domain of the Dirichlet form. To label the balls in the system, we use a sequence of tagged particle processes $\{(\boldsymbol{X}^m, \Xi^m)\}_{M\in\mathbb{N}}$ with consistency, as introduced by Osada [9]. Additionally, we can apply the argument in [10] to the case with a hard-core interaction. We can then apply the Fukushima decomposition to our case and show the existence of a solution $\boldsymbol{X}$ for SKE-$\mathbb{N}$.

The existence and uniqueness of strong solutions have been discussed [14]. In the cited paper, we introduced an infinite system of finite-dimensional SDEs associated with a solution X of (SKE-$\mathbb{N}$). We showed that under the condition that each finite-dimensional SDE has a unique strong solution, referenced as (**IFC**), there exists a strong solution and a unique strong solution of (SKE-$\mathbb{N}$) under some constraints. However, we are not sure if the IFC holds in our model with a hard-core interaction. In the present paper, we introduce two conditions, namely $\mathfrak{I}$-IFC and the finite cluster property (**FCP**). We present a result for the existence and uniqueness of strong solutions of (SKE-$\mathbb{N}$) under ($\mathfrak{I}$-**IFC**) and (**FCP**) and verify these conditions in our setting.

Section 2 prepares notations and cites results on the construction of the unlabeled process Ξ and labeled process X and the Skorohod equation. Section 3 presents results, Proposition 3.1 and Theorem 3.3. Section 4 presents the proof of Theorem 3.3.

2 Preliminaries

2.1 Systems of Unlabeled Hard Balls

We denote the configuration space of unlabeled balls with radius $r > 0$ in $\mathbb{R}^d$ by

$$\mathfrak{X} = \{\xi = \{x^j\}_{j\in\mathbb{I}} : |x^j - x^k| \geq r \ j \neq k,\ \mathbb{I}\text{ is countable}\}.$$

We can regard an element $\xi = \{x^j\}_{j\in\mathbb{I}} \in \mathfrak{X}$ as a Radon measure $\sum_{j\in\mathbb{I}} \delta_{x^j}$ and $\mathfrak{X}$ as a subset of the set $\mathfrak{M}$ of non-negative Radon measures:

$$\mathfrak{M} = \mathfrak{M}(\mathbb{R}^d) = \Big\{\xi(\cdot) = \sum_{j\in\mathbb{I}} \delta_{x^j}(\cdot) : \xi(K) < \infty, \forall K \subset \mathbb{R}^d \text{ compact}\Big\}.$$

where δ_x is the delta measure at x. We remark that $\mathfrak{X}$ is compact with the vague topology. We denote by $\pi_A(\xi)$ the restriction of ξ on $A \in \mathbb{R}^d$.

We cite the definition of the quasi-Gibbs measure on $\mathfrak{M}$ [12, Definition 2.1]. For $\zeta \in \mathfrak{M}$, the Hamiltonian of Φ, Ψ on $U_\ell = \{x \in \mathbb{R}^d : |x| \leq \ell\}$ is given by

$$H_\ell(\zeta) = \sum_{x\in\mathrm{supp}\zeta\cap U_\ell} \Phi(x) + \sum_{x,y\in\mathrm{supp}\zeta\cap U_\ell, x\neq y} \Psi(x, y).$$

Let Λ be the Poisson random field on $\mathbb{R}^d$ with intensity measure dx.

Definition 2.1 (*Quasi-Gibbs State*) A probability measure μ is called a (Φ, Ψ)-quasi-Gibbs state if

$$\mu^m_{\ell,\xi}(d\zeta) = \mu(\pi_{U_\ell}(\zeta) \in d\zeta | \pi_{U^c_\ell}(\xi) = \pi_{U^c_\ell}(\zeta), \zeta(U_\ell) = m)$$

satisfies that for $\ell, m, k \in \mathbb{N}$, μ-a.s. ξ,

$$c^{-1}e^{-H_\ell(\zeta)}\Lambda_\ell^m(d\zeta) \le \mu_{\ell,\xi}^m(d\zeta) \le ce^{-H_\ell(\zeta)}\Lambda_\ell^m(d\zeta),$$

where $c = c(\ell, m, \xi) > 0$ is a constant depending on ℓ, m, ξ and

$$\Lambda_\ell^m(\cdot) = \Lambda(\pi_{U_\ell} \in \cdot|\mathfrak{M}_\ell^m) \quad \text{with}\, \mathfrak{M}_\ell^m = \{\xi(U_\ell) = m\}.$$

A function f on $\mathfrak{X}$ is called a polynomial function if it can be expressed as

$$f(\xi) = Q\left(\langle\varphi_1, \xi\rangle, \langle\varphi_2, \xi\rangle, \dots, \langle\varphi_\ell, \xi\rangle\right)$$

with a polynomial function Q on $\mathbb{R}^\ell$ and smooth functions φ_j, $1 \le j \le \ell$, on $\mathbb{R}^d$ with compact support, where

$$\langle\varphi, \xi\rangle = \int_{\mathbb{R}^d} \varphi(x)\xi(dx).$$

We denote by $\mathcal{P}$ the set of all polynomial functions on $\mathfrak{M}$. A polynomial function is a local and smooth function; i.e., there is a compact set K such that

$$f(\xi) = f(\pi_K(\xi))$$

and symmetric smooth functions $\hat{f}_n$, $n \in \mathbb{N}$ such that

$$f(\xi) = \hat{f}(x_1, \dots, x_n), \quad \text{if}\, \xi \cap K = \sum_{j=1}^n \delta_{x^j}.$$

The sequence of the functions $\hat{f}_n$, $n \in \mathbb{N}$ is called a K-representation of f.

For $f \in \mathcal{P}$, we introduce the square field on $\mathfrak{M}$ defined by

$$\mathbb{D}(f, g)(\xi) = \frac{1}{2}\int_{\mathbb{R}^d} \xi(dx)\nabla_x f(\xi) \cdot \nabla_x g(\xi). \tag{2.1}$$

For a probability measure μ on $\mathfrak{X}$, we introduce the bilinear form on $L^2(\mu)$ defined by

$$\mathcal{E}^\mu(f, g) = \int_{\mathfrak{M}} \mathbb{D}(f, g)(\xi)\mu(d\xi), \quad f, g \in \mathcal{D}_\circ^\mu,$$
$$\mathcal{D}_\circ^\mu = \{f \in \mathcal{P} : \| f \|_1 < \infty\},$$

where

$$\| f \|_1^2 = \| f \|_{L^2(\mu)}^2 + \mathcal{E}^\mu(f, f).$$

We make the following assumptions.

(**A0**) The pair potential Ψ has a hard-core interaction with radius $r > 0$; i.e., $\Psi(x, y) = \infty$ if $|x - y| \leq r$.
(**A1**) μ is a (Φ, Ψ)-quasi-Gibbs state, and $\Phi : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ and $\Psi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ satisfy

$$c^{-1}\Phi_0(x) \leq \Phi(x) \leq c\ \Phi_0(x),$$
$$c^{-1}\Psi_0(x - y) \leq \Psi(x, y) \leq c\ \Psi_0(x - y)$$

for some $c > 1$ and are locally bounded from below and upper semi-continuous functions Φ_0, Ψ_0 with $\{x \in \mathbb{R}^d : \Psi_0(x) = \infty\}$ being compact.

From (**A0**), μ satisfies $\mu(\mathfrak{X}) = 1$. We cite the result in [11, Lemma 2.1] with [13].

Lemma 2.2 ([11, 13]) *Assume (**A0**) and (**A1**).*

(i) $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu}_{\circ})$ *is closable on* $L^2(\mathfrak{X}, \mu)$*, and the closure* $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu})$ *is a regular Dirichlet form.*
(ii) *There is a diffusion process* $(\Xi_t, \mathbb{P}_{\xi})$ *associated with* $(\mathcal{E}^{\mu}, \mathcal{D}^{\mu})$ *on* $L^2(\mu)$.
(iii) $(\Xi_t, \mathbb{P}_{\mu})$ *is a reversible process, where* $\mathbb{P}_{\mu} = \int_{\mathfrak{X}} \mu(d\xi)\mathbb{P}_{\xi}$.

2.2 Systems of Labeled Balls

We denote the unlabeled configuration space of an infinite number of balls by

$$\mathfrak{X}_{\infty} = \{\xi = \{x^j\}_{j\in\mathbb{N}} : |x^j - x^k| \geq r\ j \neq k\}.$$

We make the following assumption because we are studying a system of infinite particles.

(**A2**) $\mu(\mathfrak{X}_{\infty}) = 1$.

We denote the configuration space of labeled balls by

$$\boldsymbol{S}_{\text{hard}} = \{\boldsymbol{x} = (x^j)_{j\in\mathbb{N}} \in (\mathbb{R}^d)^{\mathbb{N}} : |x^j - x^k| \geq r, \quad j \neq k\}.$$

We introduce the unlabel map $\mathfrak{u} : \boldsymbol{S}_{\text{hard}} \to \mathfrak{X}_{\infty}$ defined by

$$\mathfrak{u}((x^j)_{j\in\mathbb{N}}) = \{x^j\}_{j\in\mathbb{N}} \tag{2.2}$$

and a label map $\mathfrak{l} : \mathfrak{X}_{\infty} \to \mathbf{S}_{\text{hard}}$ such that

$$\mathfrak{l}(\xi) = (x^j)_{j\in\mathbb{N}}, \quad \text{if}\ \xi = \{x^j\}_{j\in\mathbb{N}} \in \mathfrak{X}_{\infty}. \tag{2.3}$$

From the hard-core condition of the configuration space $\mathfrak{X}$ and the continuity of the trajectory of the process Ξ, we can lift the unlabeled dynamics $\Xi = \{X^j\}_{j\in\mathbb{N}}$ to labeled dynamics $\boldsymbol{X} = (X^j)_{j\in\mathbb{N}}$. Here, X^j is an $\mathbb{R}^d$-valued continuous process on one of the intervals of the form $[0, b)$ and (a, b), $0 < a < b \le \infty$. We refer to X^j as a tagged particle. If $b < \infty$, we say the tagged particle explodes. If $a > 0$, we say that the tagged particle enters. We make the following assumption.

(**NEE**) Ξ_t is an $\mathfrak{X}$-valued diffusion process in which no tagged particle explodes or enters.

For a topological space S, $W(S)$ denotes the set of continuous paths from $\mathbb{R}_+ := [0, \infty)$ to S and we put $W_x(S) = \{w \in W(S) : w(0) = x\}$ for $x \in S$. Under **(NEE)**, we can construct a labeled map $\mathfrak{l}_{\mathrm{path}}$ from $W(\mathfrak{X}_\infty)$ to $W(\mathbf{S}_{\mathrm{hard}})$ such that for $\Xi = \{X^j\}_{j\in\mathbb{N}} \in W(\mathfrak{X}_\infty)$ we have

$$\mathfrak{l}_{\mathrm{path}}(\Xi) = (X^j)_{j\in\mathbb{N}} \equiv \boldsymbol{X}.$$

We remark that $\mathfrak{l}_{\mathrm{path}}(\Xi)_t \neq \mathfrak{l}(\Xi_t)$. We also put for $m \in \mathbb{N}$

$$\mathfrak{l}_{\mathrm{path}}^{[m]}(\Xi) = ((X^j)_{j=1}^m, \{X_j\}_{j=m+1}^\infty) =: (\boldsymbol{X}^m, \Xi^m).$$

We quote the results in [9, Theorem 2.5]. See also [14, Lemma 1.2].

Lemma 2.3 ([9]) *Assume* (***A0***)–(***A2***). $(\Xi_t, \mathbb{P}_\mu)$ *satisfies* (***NEE***).

From the above lemma we can lift the unlabeled process $(\Xi, \mathbb{P}_\xi)$ to a labeled process $(\boldsymbol{X}, \mathbf{P}_{\boldsymbol{x}})$ such that

$$\boldsymbol{X} = \mathfrak{l}_{\mathrm{path}}(\Xi), \quad \mathbf{P}_{\boldsymbol{x}} = \mathbb{P}_{\mathfrak{u}(\boldsymbol{x})} \quad \text{and} \quad \boldsymbol{x} = \mathfrak{l}(\xi).$$

$\mathfrak{l}_{\mathrm{path}}(\Xi)_t$ depends on not only Ξ_t but also the trajectory of Ξ, and $\boldsymbol{X}^m = (X^1, X^2, \ldots, X^m)$, $m \in \mathbb{N}$, is thus not a Dirichlet process for Ξ. Then, using the argument in [9], we introduce the m-labelled processes $(\boldsymbol{X}^m, \Xi^m)$, $m \in \mathbb{N} \cup \{0\}$, for which $\boldsymbol{X}^m$ is a Dirichlet process.

We shall present the Dirichlet form associated with the m-labeled process. Let $\mu^{[m]}$ be the reduced m-Campbell measure on $(\mathbb{R}^d)^m \times \mathfrak{X}$ for μ defined as

$$\mu^{[m]}(d\boldsymbol{x}^m d\eta) = \rho^m(\boldsymbol{x}^m)\mu_{\boldsymbol{x}^m}(d\eta)d\boldsymbol{x}^m,$$

where ρ^m is the m-point correlation function of μ with respect to the Lebesgue measure $d\boldsymbol{x}^m$ and $\mu_{\boldsymbol{x}^m}$ is the reduced Palm measure conditioned at $\boldsymbol{x}^m \in (\mathbb{R}^d)^m$. See, for instance, [7] for these definitions. We introduce the bilinear form $(\mathcal{E}^{\mu^{[m]}}, \mathcal{D}_\circ^{\mu^{[m]}})$ defined by

$$\mathcal{E}^{\mu^{[m]}}(f,g) = \int_{(\mathbb{R}^d)^m \times \mathfrak{X}} \Big\{ \frac{1}{2} \sum_{i=1}^{m} \frac{df}{dx^i} \frac{dg}{dx^i} + \mathbb{D}(f,g) \Big\} \mu^{[m]}(d\boldsymbol{x}^m d\eta),$$

$$\mathcal{D}_{\circ}^{\mu^{[m]}} = \Big\{ f \in C_0^{\infty}((\mathbb{R}^d)^m) \otimes \mathcal{D}_{\circ}; \mathcal{E}^{\mu^{[m]}}(f,f) < \infty, f \in L^2((\mathbb{R}^d)^m \times \mathfrak{X}, \mu^{[m]}) \Big\},$$

where $\dfrac{\partial}{\partial x^j}$ is the nabla in $\mathbb{R}^d$, and $\mathbb{D}$ is defined by (2.1).

We quote the following.

Lemma 2.4 ([9]) *Assume (**A0**)–(**A2**). Let $m \in \mathbb{N}$.*

(i) *The bilinear form $(\mathcal{E}^{\mu^{[m]}}, \mathcal{D}_{\circ}^{\mu^{[m]}})$ is closable. Its closure, denoted by $(\mathcal{E}^{\mu^{[m]}}, \mathcal{D}^{\mu^{[m]}})$, is associated with the diffusion process $((\boldsymbol{X}_t^m, \Xi_t^m), \mathbb{P}^{[m]}_{(\boldsymbol{x}^m,\eta)})$.*

(ii) *The sequence $\{((\boldsymbol{X}_t^m, \Xi_t^m), \mathbb{P}^{[m]}_{(\boldsymbol{x}^m,\eta)})\}_{m\in\mathbb{N}}$ satisfies the consistency condition*

$$\mathbb{P}^{[m]}_{(\boldsymbol{x}^m,\eta)} = \mathbb{P}_{\mathfrak{u}(\boldsymbol{x}^m,\eta)} \circ (\mathfrak{l}^{[m]}_{\text{path}})^{-1}, \quad \mathbb{P}^{[m]}_{(\boldsymbol{x}^m,\eta)} \circ \mathfrak{u}^{-1} = \mathbb{P}_{\mathfrak{u}(\boldsymbol{x}^m,\eta)},$$

where $\mathfrak{u}(\boldsymbol{x}^m,\eta) = \{x^j\}_{j=1}^m \cup \eta \in \mathfrak{X}$, if $\{x^j\}_{j=1}^m \cap \eta = \emptyset$ and $\{x^j\}_{j=1}^m \cup \eta \in \mathfrak{X}$.

We can construct from this lemma the labeled process $\boldsymbol{X} = (X^1, X^2, \dots)$ satisfying

$$\boldsymbol{X}^m = (X^1, \dots, X^m), \quad \Xi^m = \{X^j\}_{j=m+1}^{\infty}, \quad m \in \mathbb{N}. \tag{2.4}$$

In particular, we can regard the process X^j, $j \le m$ as a Dirichlet process of the diffusion $(\boldsymbol{X}^m, \Xi^m)$ associated with the Dirichlet form $(\mathcal{E}^{\mu^{[m]}}, \mathcal{D}^{\mu^{[m]}})$.

2.3 Skorohod Equation

Let D be an open domain in $\mathbb{R}^N$, $N \in \mathbb{N}$. Let $\mathcal{N}_x = \mathcal{N}_x(D)$ be the set of inward normal unit vectors at $x \in \partial D$,

$$\mathcal{N}_x = \bigcup_{\ell > 0} \mathcal{N}_{x,\ell} \quad \mathcal{N}_{x,\ell} = \{\boldsymbol{n} \in \mathbb{R}^N : |\boldsymbol{n}| = 1, U_\ell(x - \ell\boldsymbol{n}) \cap D = \emptyset\}.$$

For $x \in \overline{D}$ and $w \in W_0(\mathbb{R}^N)$, we consider what is called the *Skorohod equation*:

$$\zeta(t) = x + w(t) + \varphi(t), \quad t \ge 0. \tag{SK}$$

A solution to (SK) is a pair (ζ, φ) satisfying (SK) with the following two conditions.

(1) $\zeta \in W(\overline{D})$.

(2) φ is an $\mathbb{R}^N$-valued continuous function with bounded variation on each finite time interval satisfying $\varphi(0) = 0$ and

$$\varphi(t) = \int_0^t \boldsymbol{n}(s) d\|\varphi\|_s, \quad \|\varphi\|_t = \int_0^t \mathbf{1}_{\partial D}(\zeta(s)) d\|\varphi\|_s,$$

where $\boldsymbol{n}(s) \in \mathcal{N}_{\zeta(s)}$ if $\zeta(s) \in \partial D$ and $\|\varphi\|_t$ denotes the total variation of φ on $[0, t]$.

We introduce the following conditions for D.

(A) (Uniform exterior sphere condition) There exists a constant $\alpha_0 > 0$ such that

$$\forall x \in \partial D, \quad \mathcal{N}_x = \mathcal{N}_{x,\alpha_0} \neq \emptyset.$$

(B) There exists constants $\delta_0 > 0$ and $\beta_0 \in [1, \infty)$ such that for any $x \in \partial D$ there exists a unit vector $\boldsymbol{l}_x$ verifying

$$\forall \boldsymbol{n} \in \bigcup_{y \in U_{\delta_0}(x) \cap \partial D} \mathcal{N}_y, \quad \langle \boldsymbol{l}_x, \boldsymbol{n} \rangle \geq \frac{1}{\beta_0}.$$

We quote the results in [16, 17] and [5, Lemmas 3.1 and 3.2].

Lemma 2.5 ([5, 16, 17]) (i) *Suppose D satisfies conditions (A) and (B). There is then a unique solution of (SK).*
(ii) *Suppose that D satisfies conditions (A) and (B). Let $(\zeta^{(i)}, \phi^{(i)})$ be the solution of (SK) for $x^{(i)}$ and $w^{(i)}$, $i = 1, 2$. Then, for each $T > 0$, there exists a constant $C = C(\alpha_0, \beta_0, \delta_0)$ such that*

$$|\zeta^{(1)}(t) - \zeta^{(2)}(t)| \leq (\|w^{(1)} - w^{(2)}\|_t + |x^{(1)} - x^{(2)}|) e^{C(\|\varphi^{(1)}\|_t + \|\varphi^{(2)}\|_t)} \tag{2.5}$$

and

$$\|\varphi^{(i)}\|_t \leq f\left(\Delta_{0,T,\cdot}(w^{(i)}), \sup_{0 \leq s \leq t} |w^{(i)}|\right), \quad 0 \leq t \leq T, \ i = 1, 2, \tag{2.6}$$

where f is a function on $W_0(\mathbb{R}_+) \times \mathbb{R}_+$ which depending on $\alpha_0, \beta_0, \delta_0$, and $\Delta_{0,T,\delta}(w)$ which denotes the modulus of continuity of w in $[0, T]$.
(iii) *The configuration space of n balls with diameter $r > 0$,*

$$D_n = \{\boldsymbol{x} = (x^1, x^2, \ldots, x^n) \in (\mathbb{R}^d)^n : |x^j - x^k| > r, \ j \neq k\},$$

satisfies conditions (A) and (B).

As a corollary of this lemma, the existence of a unique strong solution of Skorohod SDEs has been proved [16, Theorem 5.1]. We then see the existence of a unique strong solution of (SKE-$\mathbb{I}$) if $\mathbb{I}$ is a finite subset of $\mathbb{N}$. An approximation Skorohod-type equation was introduced in the proof of [16, Theorem 5.1]. In our setting, the equation is written as

$$dX_n^j(t) = dB_t^j - \frac{1}{2}\nabla\Phi(X_n^j(h_n(t))dt - \frac{1}{2}\sum_{k\in\mathbb{I},k\neq j}\nabla\Psi_{\mathrm{sm}}(X_n^j(h_n(t)) - X_n^k(h_n(t))dt$$
$$+ \sum_{k\in\mathbb{I},k\neq j}(X_n^j(t) - X_n^k(t))dL_n^{jk}(t), \quad j\in\mathbb{I}, \qquad \text{(SKE(n)-}\mathbb{I}\text{)}$$

with the initial condition $(X_n^j(0))_{j\in\mathbb{I}} = (X^j(0))_{j\in\mathbb{I}}$, where

$$h_n(0) = 0,\ h_n(t) = (k-1)2^{-n},\ (k-1)2^{-n} < t \le k2^{-n},\ k\in\mathbb{N},\ n\in\mathbb{N}.$$

By Lemma 2.5, (SKE(n)-$\mathbb{I}$) has a unique strong solution, and the limit of the sequence $\{((X_n^j)_{j\in\mathbb{I}}, (L_n^{jk})_{j,k\in\mathbb{I}})\}_{n\in\mathbb{N}}$ as $n\to\infty$ coincides with $((X^j)_{j\in\mathbb{I}}, (L^{jk})_{j,k\in\mathbb{I}})$.

3 Results

3.1 Existence of a Weak Solution

We make the following assumption.

(**A3**) A probability measure μ on $\mathfrak{X}$ has the log derivative $\mathsf{d}_\mu(x,\eta)\in L^1_{loc}(\mathbb{R}^d\times\mathfrak{X},\mu^{[1]})$, i.e., for any $f\in C_0^\infty(\mathbb{R}^d)\times\mathcal{P}$,

$$-\int_{\mathbb{R}^d\times\mathfrak{X}}\nabla_x f(x,\eta)\mu^{[1]}(dxd\eta) = \int_{\mathbb{R}^d\times\mathfrak{X}}\mathsf{d}_\mu(x,\eta)f(x,\eta)\mu^{[1]}(dxd\eta)$$
$$+ \int_{\{(x,\eta):\eta\in\mathfrak{X},x\in S_\eta\}}\boldsymbol{n}_\eta(x)f(x,\eta)\mathcal{S}_\eta(dx)\rho(x)\mu_x(d\eta),$$

where $\mathcal{S}_\eta$ is the surface measure on S_η,

$$S_\eta = \{x\in\mathbb{R}^d : |x-y| = r \text{ for some } y\in\eta\},$$

and $\boldsymbol{n}_\eta(x)$ is the inward normal vector of S_η at x.

We can extend the notion of the log derivative in distribution and write

$$\overline{\mathsf{d}}_\mu(x,\eta) = \mathbf{1}_{S_\eta}(x)\mathsf{d}_\mu(x,\eta) + \mathbf{1}_{\partial S_\eta}(x)\boldsymbol{n}_\eta(x)\delta_x.$$

If the log derivative exists, we put $b(x,\eta) = \frac{1}{2}\mathsf{d}_\mu(x,\eta)$. The following result is a modification of [10, Theorem 26].

Proposition 3.1 *Assume the conditions* (***A0***)–(***A3***). *There then exists* $\mathfrak{H} \subset \mathfrak{X}$ *with* $\mu(\mathfrak{H}) = 1$ *such that* $(\boldsymbol{X} = \mathfrak{l}(\Xi), \mathbb{P}_\xi)$, $\xi \in \mathfrak{H}$ *satisfies the infinite-dimensional stochastic differential equation of Skorohod type (ISKE)*

$$dX_t^j = dB_t^j + b\Big(X_t^j, \{X_t^k\}_{k\neq j}\Big)dt + \sum_{k\neq j}\Big(X_t^j - X_t^k\Big)dL_t^{jk}, \qquad \text{(ISKE)}$$
$$|X_t^j - X_t^k| \geq r, \quad j,k \in \mathbb{N},\ j \neq k,\ t \geq 0,$$

where L_t^{jk}, $k, j \in \mathbb{N}$, *is a non-decreasing function satisfying*

$$L_t^{jk} = L_t^{kj} = \int_0^t \mathbf{1}(|X_s^j - X_s^k| = r)dL_s^{jk}.$$

Proof Let $m \in \mathbb{N}$. The coordinate function x^j is locally in the domain of $\mathcal{D}^{\mu^{[m]}}$, and X^j is thus a Dirichlet process of $(\boldsymbol{X}^m, \Xi^m)$. Applying Fukushima decomposition ([4, Theorem 5.5.1] to x^j yields

$$X_t^{m,j} - X_0^{m,j} = M_t^{[x^j]} + N_t^{[x^j]}, \quad \text{under } \mathbb{P}^{[m]}_{(\boldsymbol{x}^m,\eta)}. \tag{3.7}$$

Here, $M^{[x^j]}$ is a martingale additive functional locally of finite energy and $N^{[x^j]}$ is a continuous additive functional locally of zero energy. Through a straightforward calculation using (**A2**), we have for $f \in C_0^\infty(\mathbb{R}^d) \otimes \mathcal{D}_\circ$ that

$$-\mathcal{E}^{\mu^{[m]}}(x^i, f) = \int_{\mathbb{R}^d\times\mathfrak{X}} \mathsf{d}_\mu(x^j, \{x^k\}_{k\neq j}^m \cup \eta) f(x,\eta)\mu^{[m]}(dxd\eta)$$
$$+ \int_{\{(x,\eta):\eta\in\mathfrak{X}, x\in S_\eta\}} f(x,\eta)\boldsymbol{n}_\eta(x^j)\mathcal{S}_\eta(dx^j)\mu_{\boldsymbol{x}^m}(d\eta)d\boldsymbol{x}^{m\diamond j},$$

where $\boldsymbol{x}^{m\diamond j} = (x^k)_{k\neq j}^m$. Hence, by [4, Theorem 5.2.4], we deduce that

$$N_t^{[x^j]} = \int_0^t b(X_s^{m,j}, \{X_s^k\}_{k\neq j}^m \cup \Xi_s^{[m]})ds + \sum_{\substack{1\leq k\leq m\\ k\neq j}} \int_0^t (X_s^{m,j} - X_s^k)dL_s^{m,jk}$$
$$+ \sum_{k=m+1}^\infty \int_0^t (X_s^{m,j} - X_s^k)dL_s^{m,jk}, \tag{3.8}$$

where $L_t^{m,jk}$, $1 \le j \le m$, $k \in \mathbb{N}$ are increasing functions satisfying

$$L_t^{m,jk} = \int_0^t \mathbf{1}(|X_s^{m,j} - X_s^{m,k}| = r) dL_s^{m,jk}, \quad j,k = 1,2,\dots,m,$$

$$L_t^{m,jk} = \int_0^t \mathbf{1}(|X_s^{m,j} - X_s^{k}| = r) dL_s^{m,jk}, \quad j = 1,2,\dots,m,\ k = m+1,\dots.$$

Here, we used the relation $\Xi^m = \{X^j\}_{j=m+1}^{\infty}$ from the consistency condition (2.4). We put

$$\mathbb{D}^m[f,g] = \frac{1}{2}\sum_{i=1}^{m} \frac{df}{dx_i}\frac{dg}{dx_i} + \mathbb{D}(f,g), \quad f,g \in C_0^{\infty}((\mathbb{R}^d)^m) \otimes \mathcal{D}_\circ.$$

For $1 \le j, \ell \le m$

$$2\mathbb{D}^m[x^j f, x^j] - \mathbb{D}^m[(x^j)^2, f] = 2\mathbb{D}^m[x^j, x^j]f = f,$$
$$2\mathbb{D}^m[(x^j \pm x^\ell)f, (x^j \pm x^\ell)] - \mathbb{D}^m[(x^j \pm x^\ell)^2, f]$$
$$= 2\mathbb{D}^m[x^j \pm x^\ell, x^j \pm x^\ell]f = \begin{cases} 0, & (j = \ell), \\ 2f & (j \ne \ell). \end{cases}$$

Then, from [4, Theorem5.2.3],

$$\langle M^{[x^j]}, M^{[x^\ell]}\rangle_t = \begin{cases} 0, & (j \ne \ell), \\ t & (j = \ell). \end{cases} \tag{3.9}$$

Combining (3.7), (3.8) and (3.9) with the consistency (2.4), we obtain the proposition. □

Let $\mathfrak{H}$ and $\mathfrak{X}_{\mathrm{sde}}$ be Borel subsets of $\mathfrak{X}$ such that

$$\mu(\mathfrak{H}) = \mu(\mathfrak{X}_{\mathrm{sde}}) = 1, \quad \mathfrak{H} \subset \mathfrak{X}_{\mathrm{sde}} \subset \mathfrak{X}_\infty.$$

Let b be a measurable function on $\mathbb{R}^d \times \mathfrak{X}$ that has a finite value on

$$\mathfrak{X}_{\mathrm{sde}}^{[1]} = \{x, \eta) \in \mathbb{R}^d \times \mathfrak{X} : \{x\} \cup \eta \in \mathfrak{X}_{\mathrm{sde}}\}.$$

Let $\mathfrak{l}$ be the label defined by (2.3). We put $\boldsymbol{H} = \mathfrak{l}(\mathfrak{H})$ and $\boldsymbol{S}_{\mathrm{sde}} = \mathfrak{l}(\mathfrak{X}_{\mathrm{sde}})$. Here, $\boldsymbol{H}$ is the set of initial starting points of solutions and $\boldsymbol{S}_{\mathrm{sde}}$ is the set in which the coefficient of (ISKE) is well defined. We consider the following ISKE with (3.10) and (3.11):

$$dX_t^j = dB_t^j + b\Big(X_t^j, \{X_t^k\}_{k\neq j}\Big)dt + \sum_{k\neq j}\Big(X_t^j - X_t^k\Big)dL_t^{jk}, \quad j\in\mathbb{N}, \quad \text{(ISKE)}$$

$$\boldsymbol{X}_0 \in \boldsymbol{H}, \quad \boldsymbol{X} \in W(\boldsymbol{S}_{\text{sde}}), \tag{3.10}$$

$$L_t^{jk} = \int_0^t \mathbf{1}(|X_s^j - X_s^k| = r)dL_s^{jk} \quad j,k\in\mathbb{N}, \tag{3.11}$$

where L_t^{jk}, $j,k\in\mathbb{N}$ is a non-decreasing real-valued function starting from zero.

Definition 3.2 (*Weak Solution*) By a weak solution of (ISKE) with (3.10) and (3.11), we mean an $(\mathbb{R}^d)^N \times \mathbb{R}_+^{\mathbb{N}\times\mathbb{N}} \times (\mathbb{R}^d)^{\mathbb{N}}$-valued stochastic process $(\boldsymbol{X}, \boldsymbol{L}, \boldsymbol{B})$ defined on a probability space $(\Omega, \mathcal{F}, P)$ with a reference family $\{\mathcal{F}_t\}_{t\geq 0}$ such that

(i) $(\boldsymbol{X}, \boldsymbol{L})$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted $\boldsymbol{S}_{\text{sde}} \times \mathbb{R}_+^{\mathbb{N}\times\mathbb{N}}$-valued process satisfying (3.10) and (3.11);
(ii) $\boldsymbol{B} = (B^j)_{j\in\mathbb{N}}$ is an $\mathbb{R}^{\mathbb{N}}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion with $\boldsymbol{B}_0 = \mathbf{0}$;
(iii) $\{b(X_t^j, \{X_t^k\}_{k\neq j})\}_{j\in\mathbb{N}}$ is a family of $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes with

$$E\Big[\int_0^T |b(X_t^j, \{X_t^k\}_{k\neq j})|dt\Big] < \infty \quad \text{for all } T; \text{ and}$$

(iv) with probability one, $(\boldsymbol{X}, \boldsymbol{L}, \boldsymbol{B})$ satisfies for all $t \geq 0$ that

$$X_t^j = X_0^j + B_t^j + \int_0^t b\Big(X_u^j, \{X_u^k\}_{k\neq j}\Big)du + \sum_{k\neq j}\int_0^t \big(X_u^j - X_u^k\big)dL_u^{jk}, \quad j\in\mathbb{N}.$$

Remark 1 Let μ be a canonical Gibbs state with the potentials $(\Phi, \Psi = \Psi_{\text{hard}} + \Psi_{\text{sm}})$ such that Φ and Ψ_{sm} are smooth. From the same argument as [14, Lemma 13.5] the log derivative of μ exists and is represented as

$$\mathsf{d}_\mu(x,\eta) = -\nabla\Phi(x) - \sum_{y\in\eta}\nabla\Psi_{\text{sm}}(x-y).$$

It follows from applying Proposition 3.1 that $(\boldsymbol{X}_t, \mathbb{P}_{\boldsymbol{x}})$ is a weak solution (SDE-$\mathbb{N}$).

3.2 Statement of the Results

We study the existence and uniqueness of strong solutions of (ISKE) with (3.10) and (3.11), whose definition are given in Definitions A.1 and A.3. In [14], we developed a general theory of the existence of a strong solution and the pathwise uniqueness of solutions for ISDEs concerning interacting Brownian motions. We apply the argument made in the cited paper.

Let $(\boldsymbol{X}, \boldsymbol{B})$ be an $(\mathbb{R}^d)^{\mathbb{N}} \times (\mathbb{R}^d)^{\mathbb{N}}$-valued continuous process defined on a standard filtered space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$. The regular conditional probability $P_{\boldsymbol{x}} = P(\cdot|\boldsymbol{X}_0 = \boldsymbol{x})$ then exists for $P \circ \boldsymbol{X}_0^{-1}$-a.s. $\boldsymbol{x}$.

We cite the conditions for a weak solution of $(\boldsymbol{X}, \boldsymbol{B}, P)$ given in [14, Section 3.2].

(**SIN**) $P(\boldsymbol{X} \in W(\boldsymbol{S}_{hard})) = 1$.
(μ-**AC**) (μ-absolutely continuity condition)

$$P(\mathfrak{u}(\boldsymbol{X}_t) \in \cdot) \prec \mu \quad \text{for all } t > 0,$$

where for the two Radon measures m_1 and m_2, we write $m_1 \prec m_2$ if m_1 is absolutely continuous with respect to m_2.
(**NBJ**) (No big jump condition) $\forall \ell, \forall T \in \mathbb{N}$

$$P(\mathsf{m}_{r,T}(\boldsymbol{X}) < \infty) = 1,$$

where

$$\mathsf{m}_{\ell,T}(\boldsymbol{X}) = \inf\{m \in \mathbb{N};\ |X^n(t)| > \ell, \forall n > m, \forall t \in [0, T]\}.$$

For a topological space S, we denote by $\mathcal{B}(S)$ the topological Borel field of S. We say a family of strong solutions $\boldsymbol{X} = F_{\boldsymbol{x}}(\boldsymbol{B})$ starting at $\boldsymbol{x}$ for $P \circ \boldsymbol{X}_0^{-1}$-a.s. $\boldsymbol{x}$ satisfies the measurable family condition if

(**MF**) $P(F_{\boldsymbol{x}}(\boldsymbol{B}) \in A)$ is $\overline{\mathcal{B}((\mathbb{R}^d)^{\mathbb{N}})}^{\boldsymbol{X}_0}$-measurable for any $A \in \mathcal{B}(W(\mathbb{R}^d))$,

where $\overline{\mathcal{B}((\mathbb{R}^d)^{\mathbb{N}})}^{\boldsymbol{X}_0}$ is the completion of $\boldsymbol{B}(W((\mathbb{R}^d)^{\mathbb{N}}))$ with respect to $P \circ \boldsymbol{X}_0^{-1}$.

The tail σ-field on $\mathfrak{X}$ is defined as

$$\mathcal{T}(\mathfrak{X}) = \bigcap_{r=1}^{\infty} \sigma(\pi_{S_r^c}).$$

We introduce the following condition on a probability measure μ.

(**TT**) (tail trivial) $\quad \mu(\mathsf{A}) \in \{0, 1\} \quad$ for any $\mathsf{A} \in \mathcal{T}(\mathfrak{X})$.

We make the following assumption.

(**R**) $\Phi \equiv 0$ and Ψ_{sm} is a Ruelle's class potential such that

$$\sup_{\xi\in\mathfrak{X}} \sum_{x\in\xi} |\nabla\Psi_{\mathrm{sm}}(x)| < \infty, \quad \sup_{\xi\in\mathfrak{X}} \sum_{x\in\xi} |\nabla^2\Psi_{\mathrm{sm}}(x)| < \infty.$$

For a subset $\mathbb{I}$ of $\mathbb{N}$, we put $\mathbb{I}^c = \mathbb{N}\setminus\mathbb{I}$. We introduce another condition ($\mathcal{I}$-**IFC**) weaker than (**IFC**) given in Sect. 4.4. Let $T \in \mathbb{N}$ and $\mathbb{M} = \{(\mathbb{I}_i, t_i)\}_{i=0}^{M}$ be sequences of pairs of finite index sets and times such that

$$\mathbb{I}_0 \supset \mathbb{I}_1 \supset \cdots \supset \mathbb{I}_{M-1} =: \mathbb{I}_*, \quad t_i = \frac{iT}{M}, \quad i = 0, 1, \ldots, M-1.$$

We introduce the sequence of SDEs

$$\begin{aligned} dY_t^{\mathbb{I}_i,j} &= dB_t^j + b_{\boldsymbol{X}}^{\mathbb{I}_i,j}(Y_t^{\mathbb{I}_i,j}, \mathbf{Y}_t^{\mathbb{I}_i})dt + \sum_{k\in\mathbb{I}_i\setminus\{j\}} (Y_t^{\mathbb{I}_i,j} - Y_t^{\mathbb{I}_i,k})dL_t^{\mathbb{I}_i,jk} \\ &\quad + \sum_{k\in\mathbb{I}_i^c}^{\infty} (Y_t^{\mathbb{I}_i,j} - X_t^k)dL_t^{\mathbb{I}_i,jk}, \quad j \in \mathbb{I}_i, \quad t \in [t_i, t_{i+1}], \qquad (\mathrm{SKE}_{\boldsymbol{X}}(\mathbb{M})) \\ Y_{t_0}^{\mathbb{I}_0,j} &= x^j,\ j \in \mathbb{I}_0, \quad Y_{t_i}^{\mathbb{I}_i,j} = Y_{t_i}^{\mathbb{I}_{i-1}},\ j \in \mathbb{I}_i,\ i = 1, 2, \ldots, M-1, \end{aligned}$$

where $i = 0, 1, \ldots, M-1$, $b_{\boldsymbol{X}}^{\mathbb{I}_i,j}(t, \boldsymbol{y}) = b\Big(y^j, \{y^k\}_{k\in\mathbb{I}_i\setminus\{j\}} + \{X_t^k\}_{k\in\mathbb{I}_i^c}\Big)$, and $L_t^{\mathbb{I}_i,jk}$, $j \in \mathbb{I}_i$, $k \in \mathbb{N}$ are increasing functions satisfying

$$L_t^{\mathbb{I}_i,jk} = \int_{t_i}^{t_{i+1}} \mathbf{1}(|Y_s^{\mathbb{I}_i,j} - Y_s^{\mathbb{I}_i,k}| = r)dL_s^{\mathbb{I}_i,jk}, \quad j, k \in \mathbb{I}_i,$$

$$L_t^{\mathbb{I}_i,jk} = \int_{t_i}^{t_{i+1}} \mathbf{1}(|Y_s^{\mathbb{I}_i,j} - X_s^k| = r)dL_s^{\mathbb{I}_i,jk}, \quad j \in \mathbb{I}_i,\ k \in \mathbb{I}_i^c.$$

We introduce the sequence $\{\Lambda(\mathbb{I}_i, [t_i, t_{i+1}])\}_{i=0}^{M-1}$ of the events defined by

$$\Lambda(\mathbb{I}_i, [t_i, t_{i+1}]) = \{|X_u^k - Y_u^{\mathbb{I}_i,j}| > r, \quad u \in [t_i, t_{i+1}],\ j \in \mathbb{I}_i,\ k \in \mathbb{I}_i^c\}$$

and put $\Lambda^{\mathbb{M}} = \bigcap_{i=0}^{M-1} \Lambda(\mathbb{I}_i, [t_i, t_{i+1}])$.

We denote by $\mathcal{C}^{\mathbb{I}_0,\mathbb{I}_*^c}$ the completion of $\mathcal{B}(W_{\mathbf{0}}((\mathbb{R}^d)^{\mathbb{I}_0}) \times W((\mathbb{R}^d)^{\mathbb{I}_*^c}))$ with respect to $P_{\boldsymbol{x}} \circ (\boldsymbol{B}^{\mathbb{I}_0}, \boldsymbol{X}^{\mathbb{I}_*^c})^{-1}$. Let $\boldsymbol{u} \in W((\mathbb{R}^d)^{\mathbb{N}})$ and $(\mathbf{v}, \boldsymbol{w}) \in W_{\mathbf{0}}((\mathbb{R}^d)^{\mathbb{I}_0}) \times W((\mathbb{R}^d)^{\mathbb{I}_*^c})$. We put $\mathcal{B}_t(W((\mathbb{R}^d)^{\mathbb{N}})) = \sigma[\boldsymbol{u}_s : 0 \le s \le t]$ and denote by $\mathcal{C}_t^{\mathbb{I}_0,\mathbb{I}_*^c}$ the completion of $\sigma[(\mathbf{v}_s, \boldsymbol{w}_s) : 0 \le s \le t]$ with respect to $P_{\boldsymbol{x}} \circ (\boldsymbol{B}^{\mathbb{I}_0}, \boldsymbol{X}^{\mathbb{I}_*^c})^{-1}$. We then make the following assumption.

($\mathfrak{I}$-**IFC**) For each $\mathbb{M} = \{(\mathbb{I}_i, t_i)\}_{i=0}^{M}$, the pathwise uniqueness of solutions of $(\mathrm{SKE}_{\boldsymbol{X}}(\mathbb{M}))$ on $\Lambda^{\mathbb{M}}$ holds under $P_{\boldsymbol{x}}$ for $P \circ \boldsymbol{X}_0^{-1}$-a.s $\boldsymbol{x}$. Moreover, there exists a $\mathcal{C}^{\mathbb{I}_0,\mathbb{I}_*}$-measurable function

$$F_{\boldsymbol{x}}^{\mathbb{M}} : W_0((\mathbb{R}^d)^{\mathbb{I}_0}) \times W((\mathbb{R}^d)^{\mathbb{I}_*^c}) \to W((\mathbb{R}^d)^{\mathbb{N}})$$

such that $F_{\boldsymbol{x}}^{\mathbb{M}}$ is $\mathcal{C}_t^{\mathbb{I}_0,\mathbb{I}_*}/\mathcal{B}_t(W((\mathbb{R}^d)^{\mathbb{N}}))$-measurable for each t and satisfies

$$F_{\boldsymbol{x}}^{\mathbb{M}}(\boldsymbol{B}^{\mathbb{I}_0}, \boldsymbol{X}^{\mathbb{I}_*^c})_s = (Y_s^{\mathbb{I}_i}, \boldsymbol{X}_s^{\mathbb{I}_i^c}),\ s \in [t_i, t_{i+1}],\ i = 0, \ldots, M-1,$$

on $\Lambda^{\mathbb{M}}$ under $P_{\boldsymbol{x}}$ for $P \circ \boldsymbol{X}_0^{-1}$-a.s $\boldsymbol{x}$.

We also introduce the other condition (**FCP**). We first introduce measurable subsets of $W(\boldsymbol{S}_{\mathrm{hard}})$. For $\varepsilon > 0$, $0 \le s < t < \infty$ and a bounded connected open set O of $\mathbb{R}^d$, we denote by $\mathfrak{C}(\varepsilon, [s, t], O)$ the set of all elements $\boldsymbol{X} = (X^1, X^2, \ldots)$ of $W(\boldsymbol{S}_{\mathrm{hard}})$ such that

$$U_{(r+\varepsilon)/2}(X_u^j) \in O,\ u \in [s, t], \quad \text{if } X_s^j \in O,$$
$$U_{(r+\varepsilon)/2}(X_u^j) \in \mathbb{R}^d \setminus O,\ u \in [s, t], \quad \text{if } X_s^j \in \mathbb{R}^d \setminus O.$$

For $\varepsilon > 0$, $p, T, a, M \in \mathbb{N}$, we denote by $\mathfrak{C}(\varepsilon, p, T, a, M)$ the set of elements $\boldsymbol{X}$ of $W(\boldsymbol{S}_{\mathrm{hard}})$ such that $\boldsymbol{X} \in \bigcap_{i=0}^{M-1} \mathfrak{C}(\varepsilon, [t_i, t_{i+1}], O_i)$, for $t_i = \frac{iT}{M}, i = 0, 1, \ldots, M-1$, and some decreasing sequence $\mathbb{O} = \{O_i\}_{i=0}^{M-1}$ of open subsets of $\mathbb{R}^d$ with

$$O_0 \subset U_{a+M+M^p}(0),\ U_\varepsilon(O_{i+1}) \subset O_i,\ 0 \le i \le M-2 \\ \text{and}, U_{a+M}(0) \subset O_{M-1}. \tag{3.12}$$

We denote a measurable subset $\mathfrak{C}$ of $W(\boldsymbol{S}_{\mathrm{hard}})$ by

$$\mathfrak{C} = \bigcup_{\varepsilon>0} \bigcup_{p=1}^{\infty} \bigcap_{T=1}^{\infty} \bigcap_{a=1}^{\infty} \bigcap_{M_0=1}^{\infty} \bigcup_{M=M_0}^{\infty} \mathfrak{C}(\varepsilon, p, T, a, M).$$

Note that $\boldsymbol{X} \in \mathfrak{C}$ implies $\theta_t \boldsymbol{X} \in \mathfrak{C}$ for any $t > 0$, where $\theta_t \boldsymbol{X}(u) = \boldsymbol{X}(u + t)$. We remark that we can define $\mathfrak{C}$ by a countable collection of bounded open subsets of $\mathbb{R}^d$, because if $\boldsymbol{X} \in \mathfrak{C}(\varepsilon, [s, t], O)$, there exists $\varepsilon' > 0$ and a polyhedron O' with vertices in $\varepsilon'\mathbb{Z}^d$ such that $\boldsymbol{X} \in \mathfrak{C}(\varepsilon', [s, t], O')$. We then assume $\mathbb{O}$ is chosen from a countable family $\mathcal{A} = \{\mathbb{O}(\ell)\}_{\ell \in \mathbb{N}}$.

We make the following assumption, called the finite cluster property (**FCP**).

(**FCP**) $P(\boldsymbol{X} \in \mathfrak{C}) = 1$.

The main theorem of this article is the following.

Theorem 3.3 *Assume (**TT**).*

(i) *Assume (**A0**), (**A2**), (**A3**), and (**R**). Put* $(\boldsymbol{X}, P) = (\mathfrak{l}_{\mathrm{path}}(\Xi), \mathbb{P}_{\mu})$. *Then, for* $\mu \circ \mathfrak{l}^{-1}$*-a.s.* $\boldsymbol{x}$, $(\boldsymbol{X}, P_{\boldsymbol{x}})$ *is a strong solution of (ISKE) with* (3.10) *and* (3.11) *starting at* $\boldsymbol{x}$. *Moreover,* $(\boldsymbol{X}, P)$ *satisfies (**MF**), (**ℑ-IFC**), (**FCP**), (*μ*-**AC**), (**SIN**), and (**NBJ**).*

(ii) *(ISKE) with* (3.10) *and* (3.11) *has a family of unique strong solutions* $\{F_{\boldsymbol{x}}\}$ *starting at* $\boldsymbol{x}$ *for* $P \circ \boldsymbol{X}_0^{-1}$*-a.s.* $\boldsymbol{x}$ *under the constraints of (**MF**), (**ℑ-IFC**), (**FCP**), (*μ*-**AC**), (**SIN**), and (**NBJ**).*

Remark 2 If (**A1**) is satisfied, we can decompose μ as

$$\mu = \int_{\mathfrak{X}} \mu(d\eta)\mu_{\mathrm{Tail}}^{\eta},$$

where $\mu_{\mathrm{Tail}}^{\eta} = \mu(\cdot|\mathcal{T}(\mathfrak{X}))(\eta)$ is the regular conditional distribution with respect to the tail σ-field. Note that (**TT**) for $\mu_{\mathrm{Tail}}^{\eta}$ holds. In the case that μ satisfies (**R**), (**MF**), (**ℑ-IFC**), (**FCP**), (**SIN**), and (**NBJ**), $\mu_{\mathrm{Tail}}^{\eta}$ also does for μ-a.s. η. Hence, assuming (μ_{Tail}-**AC**) for μ-a.s. η instead of (μ-**AC**), the counterpart of Theorem 3.3 is derived. The constraint of (μ_{Tail}-**AC**) means that there is no $A \in \mathcal{T}(\mathfrak{M})$ such that for $\mu_{\mathrm{Tail}}^{\eta}$-a.s. ξ,

$$P(\mathfrak{u}(\boldsymbol{X}_s) \in A|\mathfrak{u}(\boldsymbol{X}_0) = \xi) \neq P(\mathfrak{u}(\boldsymbol{X}_t) \in A|\mathfrak{u}(\boldsymbol{X}_0) = \xi)$$

for some $0 \leq s < t$. It is then possible that another solution $\boldsymbol{X}'$ that changes the tail exists. See [14, Section 3.3].

Example 3.4 Recalling Remark 1, we present two examples for Theorem 3.3.

(i) Lennard-Jones 6-12 potential ($d = 3$, $\Psi_{\mathrm{sm}}(x) = \Psi_{6,12}(x) = |x|^{-12} - |x|^{-6}$)

$$b(x, \{y^k\}) = \frac{\beta}{2} \sum_k \left\{ \frac{12(x - y^k)}{|x - y^k|^{14}} - \frac{6(x - y^k)}{|x - y^k|^8} \right\}.$$

(ii) Riesz potentials ($d < a \in \mathbb{N}$ and $\Psi_{\mathrm{sm}}(x) = \Psi_a(x) = (\beta/a)|x|^{-a}$)

$$b(x, \{y^k\}) = \frac{\beta}{2} \sum_k \frac{x - y^k}{|x - y^k|^{a+2}}.$$

4 Proof of the Main Theorem

We prepare lemmas for proving the main theorem.

4.1 Finite Cluster Property

Lemma 4.1 (Lemma 2.6 in [5]) *Assume (**A0**), (**A2**), and (**R**). Then, (**FCP**) for $\mathbf{X} = \mathfrak{l}_{\text{path}}(\Xi)$ holds.*

Remark 3 (i) In the proof of Lemma 2.6 in [5], we used an estimate of the modulus of continuity of each ball derived through Lyons–Zhen decomposition [4, Section 5.7]. We again use the estimate here.

(ii) In the proof, we used a property of the continuum percolation model associated with μ. See [5, Lemma 2.4]. It is seen that this property holds under (**R**) but not under (**A1**), which is an obstacle to generalizing Theorem 3.3 for quasi-Gibbs states. It is an interesting problem to study percolation theory for quasi-Gibbs states. See, for instance, Ghosh [6].

4.2 On the Lipschitz Continuity of $b_{\mathbf{X}}^{\mathbb{I}}$

In this subsection, we examine the Lipschitz continuity of $b_{\mathbf{X}}^{\mathbb{I}}$ for a finite subset $\mathbb{I}$ of $\mathbb{N}$. We assume (**A0**) and (**R**). We recall that $b(x, \eta) = -\frac{1}{2}\sum_{y\in\eta} \nabla\Psi_{\text{sm}}(x - y)$ in Remark 1. Let $\mathbb{I}$ be a finite subset of $\mathbb{N}$. We introduce the domain of the configurations of balls indexed by $\mathbb{I}$ given by

$$D^{\mathbb{I}} = \{\mathbf{x}^{\mathbb{I}} \in (\mathbb{R}^d)^{\mathbb{I}} : |x^j - x^k| > r, \quad j \neq k,\ j, k \in \mathbb{I}\}.$$

For $\mathbf{x} = (x^k)_{k\in\mathbb{I}} \in D^{\mathbb{I}}$ and $\eta \in \mathfrak{X}$ with $\{x^j\}_{j\in\mathbb{I}} \cap \eta = \emptyset,\ \{x^j\}_{j\in\mathbb{I}} \cup \eta \in \mathfrak{X}$, we set

$$b^{\mathbb{I}}(\mathbf{x}, \eta) = (b(x^j, \{x^k\}_{k\in\mathbb{I}\setminus\{j\}} + \eta))_{j\in\mathbb{I}}.$$

Let $K(\eta) \in [0, \infty]$ be a function defined by

$$K(\eta) = \sup\left\{\frac{b^{\mathbb{I}}(\mathbf{x}, \eta) - b^{\mathbb{I}}(\mathbf{y}, \eta)}{|\mathbf{x} - \mathbf{y}|} : \mathbf{x} \neq \mathbf{y}, \mathbf{x}, \mathbf{y} \in D^{\mathbb{I}}\right.$$
$$\left.\{x^j\}_{j\in\mathbb{I}} \cap \eta = \{y^j\}_{j\in\mathbb{I}} \cap \eta = \emptyset,\ \{x^j\}_{j\in\mathbb{I}} \cup \eta, \{y^j\}_{j\in\mathbb{I}} \cup \eta \in \mathfrak{X}\right\}.$$

It follows from (**R**) that

$$K := \sup_{\eta\in\mathfrak{X}} K(\eta) < \infty \tag{4.13}$$

and it follows from

$$b^{\mathbb{I}}_{\boldsymbol{X}}(t,\boldsymbol{x}) = b^{\mathbb{I}}\big(\boldsymbol{x}, \mathfrak{u}(\boldsymbol{X}^{\mathbb{I}^c}_t)\big)$$

that

$$|b^{\mathbb{I}}_{\boldsymbol{X}}(t,\boldsymbol{x}) - b^{\mathbb{I}}_{\boldsymbol{X}}(t,\boldsymbol{y})| \le K|\boldsymbol{x}-\boldsymbol{y}|. \tag{4.14}$$

Remark 4 In [14, Section 11], the Lipschitz continuity of $b^{\mathbb{I}}_{\boldsymbol{X}}$ was discussed in the case that μ is a quasi-Gibbs state. The method presented in that section is applicable to the case with a hard core.

4.3 $\mathfrak{I}$-IFC

Lemma 4.2 *Assume (**A0**), (**A2**), (**A3**), and (**R**). Let* $(\boldsymbol{X}, P_{\boldsymbol{x}}) = (\mathfrak{l}_{\text{path}}(\Xi), \mathbb{P}_{\mathfrak{u}(\boldsymbol{x})})$. *($\mathfrak{I}$-**IFC**) then holds for* $\boldsymbol{X}$.

Proof Let $\mathbb{M} = \{(\mathbb{I}_i, t_i)\}_{i=0}^{M-1}$. Suppose that $\omega \in \Lambda^{\mathbb{M}}$. Then, $\{\boldsymbol{Y}^{\mathbb{I}_i}\}$ satisfies

$$dY^{\mathbb{I}_i,j}_t = dB^j_t + b^{\mathbb{I}_i,j}_{\boldsymbol{X}}(t,\mathbf{Y}^{\mathbb{I}_i}_t)dt + \sum_{k\in\mathbb{I}_i\setminus\{j\}}(Y^{\mathbb{I}_i,j}_t - Y^{\mathbb{I}_i,k}_t)dL^{\mathbb{I}_i,jk}_t,$$
$$j\in\mathbb{I}_i,\quad t\in[t_i,t_{i+1}],\quad i=0,1,\dots,M-1. \tag{4.15}$$

Let $\boldsymbol{Y}^{\mathbb{I}_0}$ and $\tilde{\boldsymbol{Y}}^{\mathbb{I}_0}$ be solutions of (4.15) with $i=0$. Put

$$w_t = \boldsymbol{B}^{\mathbb{I}}_t + \int_0^t b^{\mathbb{I}_0,j}_{\boldsymbol{X}}(s,\mathbf{Y}^{\mathbb{I}_0}_s)ds,\quad t\in[0,t_1],$$
$$\tilde{w}_t = \boldsymbol{B}^{\mathbb{I}}_t + \int_0^t b^{\mathbb{I}_0,j}_{\boldsymbol{X}}(s,\tilde{\mathbf{Y}}^{\mathbb{I}_0}_s)ds,\quad t\in[0,t_1].$$

It then follows from the Lipschitz continuity (4.14) and (4.13) that

$$\|w-\tilde{w}\|_t \le \int_0^t |b^{\mathbb{I}_0,j}_{\boldsymbol{X}}(s,\mathbf{Y}^{\mathbb{I}_0}_s) - b^{\mathbb{I}_0,j}_{\boldsymbol{X}}(s,\tilde{\mathbf{Y}}^{\mathbb{I}_0}_s)|ds \le K\int_0^t |\mathbf{Y}^{\mathbb{I}_0}_s - \tilde{\mathbf{Y}}^{\mathbb{I}_0}_s|ds.$$

$\triangle_{0,t_1,\delta}(w) < \infty$, $\triangle_{0,t_1,\delta}(\tilde{w}) < \infty$, $\sup_{0\le s\le t_1} |w| < \infty$, $\sup_{0\le s\le t_1} |\tilde{w}| < \infty$, and we thus have the pathwise uniqueness in the case that $i = 0$ from (2.5) and (2.6). Repeating this procedure, we obtain the pathwise uniqueness for $i = 1, 2, \ldots, M-1$. Using an approximation process as in (SKE(n)-$\mathbb{I}$) and the above estimate, we can show the existence of a strong solution $F_{\boldsymbol{x}}^{\mathbb{M}}$. □

4.4 Proof of Theorem 3.3

We use the following lemma, which is a modification of [14, Theorem 3.1], with the condition (**IFC**) replaced by the pair of conditions ($\mathfrak{I}$-**IFC**) and (**FCP**).

Lemma 4.3 *Assume (**TT**) for μ. Assume that (ISKE) with* (3.10) *and* (3.11) *has a weak solution* $(\boldsymbol{X}, \boldsymbol{B})$ *under P satisfying ($\mathfrak{I}$-**IFC**), (**FCP**), (μ-**AC**), (**SIN**), and (**NBJ**). Then, (ISKE) with* (3.10) *and* (3.11) *has a family of unique strong solutions* $\{F_{\boldsymbol{x}}\}$ *starting at $\boldsymbol{x}$ for $P \circ \boldsymbol{X}_0^{-1}$-a.s. $\boldsymbol{x}$ under the constraints of (**MF**), ($\mathfrak{I}$-**IFC**), (**FCP**), (**AC**) for μ, (**SIN**), and (**NBJ**).*

From this lemma, Theorem 3.3 is shown, if we check that the weak solution $(\mathfrak{l}_{\text{path}}(\Xi), \mathbb{P}_\mu)$ of (ISKE) satisfies (**MF**), ($\mathfrak{I}$-**IFC**), (**FCP**), (μ-**AC**), (**SIN**), and (**NBJ**). (**MF**) is obvious. (**AC**) is derived from the reversibility of the process Ξ with respect to μ. (**SIN**) and (**NBJ**) follows from Ψ_{sm} being of Ruelle's class with a hard core. See Lemmas 10.2 and 10.3 in [14]. ($\mathfrak{I}$-**IFC**) is derived from Lemma 4.2. Hence, it is enough to show Lemma 4.3 to prove Theorem 3.3.

In the proof [14, Theorem 3.1], (**IFC**) is used in [14, Lemma 4.2]. That is to say the existence of a function $F_{\boldsymbol{x}}^{\infty} : W_0((\mathbb{R}^d)^{\mathbb{N}}) \times W((\mathbb{R}^d)^{\mathbb{N}}) \to W((\mathbb{R}^d)^{\mathbb{N}})$ satisfying

(i) $F_{\boldsymbol{x}}^{\infty}(\boldsymbol{B}, \boldsymbol{X}) = \boldsymbol{X}$ $P_{\boldsymbol{x}}$-a.s. and
(ii) $F_{\boldsymbol{x}}^{\infty}(\boldsymbol{b}, \cdot)$ is $\mathcal{T}_{\text{path}}((\mathbb{R}^d)^{\mathbb{N}})_{\boldsymbol{x},\boldsymbol{b}}$-measurable for $P_{Br}^{\infty} := P \circ B^{-1}$- a.s $\boldsymbol{b}$,

where $\mathcal{T}_{\text{path}}((\mathbb{R}^d)^{\mathbb{N}})_{\boldsymbol{x},\boldsymbol{b}}$ is the completion of

$$\mathcal{T}_{\text{path}}((\mathbb{R}^d)^{\mathbb{N}}) := \bigcap_{\mathbb{I}\subset\mathbb{N}, \sharp\mathbb{I}<\infty} \sigma(\boldsymbol{X}^{\mathbb{I}^c})$$

with respect to $P_{\boldsymbol{x}} \circ (\boldsymbol{X}, \boldsymbol{B})^{-1}(\cdot|\boldsymbol{B} = \boldsymbol{b})$. We construct $F_{\boldsymbol{x}}^{\infty}$ under the conditions (**SIN**), ($\mathfrak{I}$-**IFC**), and (**FCP**). Let $\mathbb{O} = \{O_i\}_{i=0}^{M-1}$ with (3.12). Put

$$\mathbb{I}(O_i) = \{j \in \mathbb{N} : X_{t_i}^j \in O_i\}, \quad \mathbb{M}(\mathbb{O}) = \{(\mathbb{I}(O_i), t_i)\}_{i=0}^{M-1}$$

and

$$F_{\boldsymbol{x}}^{[\mathbb{O}]} = F_{\boldsymbol{x}}^{\mathbb{M}(\mathbb{O})} := \sum_{\mathbb{M}} F_{\boldsymbol{x}}^{\mathbb{M}} \mathbf{1}(\mathbb{M}(\mathbb{O}) = \mathbb{M}) \mathbf{1}_{\Lambda^{\mathbb{M}}} \text{ on } \bigcup_{\mathbb{M}} \Lambda^{\mathbb{M}}.$$

Let $a \in \mathbb{N}$. From (**FCP**) and ($\mathfrak{I}$-**IFC**), for P_{Br}^{∞}-a.s. $\boldsymbol{b}$ and $P(\cdot|B = b)$-a.s $\boldsymbol{X}$, there exists $\mathbb{O} \in \mathcal{A} := \{\mathbb{O}(\ell)\}$ such that $U_{a+M} \subset O_{M-1}$ and

$$F_{\boldsymbol{x}}^{[\mathbb{O}]}(\boldsymbol{b}, \boldsymbol{X}) \in \Lambda^{\mathbb{M}(\mathbb{O})}.$$

We put $\ell(\boldsymbol{b}, \boldsymbol{X}) = \min\{\ell \in \mathbb{N} : F_{\boldsymbol{x}}^{\mathbb{O}(\ell)}(\boldsymbol{b}, \boldsymbol{X}) \in \Lambda^{\mathbb{M}(\mathbb{O}(\ell))}\}$ and

$$F_{\boldsymbol{x}}^{(a)}(\boldsymbol{b}, \boldsymbol{X}) = \sum_{\ell \in N} \mathbf{1}(\ell = \ell(\boldsymbol{b}, \boldsymbol{X})) F_{\boldsymbol{x}}^{[\mathbb{O}(\ell)]}(\boldsymbol{b}, \boldsymbol{X}).$$

Let $\mathbb{I}$ be a finite subset of $\mathbb{N}$. Then, $F_{\boldsymbol{x}}^{(a)}(\boldsymbol{b}, \cdot)\mathbf{1}(\mathbb{I}(O_{M-1}) \supset \mathbb{I})$ is $\overline{\sigma(\boldsymbol{X}^{\mathbb{I}^c})}$-measurable, where $\overline{\sigma(\boldsymbol{X}^{\mathbb{I}^c})}$ is the completion of $\sigma(\boldsymbol{X}^{\mathbb{I}^c})$ with respect to $P_{\boldsymbol{x}} \circ (\boldsymbol{X}, \boldsymbol{B})^{-1}(\cdot|\boldsymbol{B} = \boldsymbol{b})$. From (**SIN**), we see that $\lim_{a\to\infty} P_{\boldsymbol{x}}(\mathbb{I}(O_{M-1}) \supset \mathbb{I}|\boldsymbol{B} = \boldsymbol{b}) = 1$. Putting

$$F_{\boldsymbol{x}}^{\infty} = \lim_{a\to\infty} F_{\boldsymbol{x}}^{(a)},$$

we see that $F_{\boldsymbol{x}}^{\infty}(\boldsymbol{b}, \cdot)$ satisfies (ii). The claim (i) is derived from ($\mathfrak{I}$-**IFC**). Therefore, Lemma 4.3 is proved adopting the same procedure used in [14, Theorem 3.1]. □

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Appendix

A.1 Solutions of (ISKE)

We give precise definitions of solutions of the (ISKE).

Let $\overline{\mathcal{B}}$ and $\overline{\mathcal{B}}_t$ be the completions of $\mathcal{B}(W((\mathbb{R}^d)^{\mathbb{N}})$ and $\mathcal{B}_t(W((\mathbb{R}^d)^{\mathbb{N}})$ with respect to $P_{Br}^{\infty} = P(\boldsymbol{B} \in \cdot)$, respectively.

Definition A.1 (*Strong Solutions Starting at* $\boldsymbol{x}$) A weak solution $\boldsymbol{X}$ of (ISKE) with (3.10) and (3.11) and an $(\mathbb{R}^d)^{\mathbb{N}}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion $\boldsymbol{B}$ on $(\Omega, \mathcal{F}, P, \{\mathcal{F}\}_{t\geq 0})$ is called a strong solution starting at $\boldsymbol{x}$ if $\boldsymbol{X}_0 = \boldsymbol{x}$ a.s. and if there exists a function $F_{\boldsymbol{x}} : W_0((\mathbb{R}^d)^{\mathbb{N}}) \to W((\mathbb{R}^d)^{\mathbb{N}})$ such that $F_{\boldsymbol{x}}$ is $\overline{\mathcal{B}}/\mathcal{B}(W((\mathbb{R}^d)^{\mathbb{N}})$-measurable, and $\overline{\mathcal{B}}_t/\mathcal{B}_t(W((\mathbb{R}^d)^{\mathbb{N}})$-measurable for each t and that $F_{\boldsymbol{x}}$ satisfies $\boldsymbol{X} = F_{\boldsymbol{x}}(\boldsymbol{B})$ a.s. We also call $\boldsymbol{X} = F_{\boldsymbol{x}}(\boldsymbol{B})$ a strong solution starting at $\boldsymbol{x}$. Additionally, we call $F_{\boldsymbol{x}}$ itself a strong solution starting at $\boldsymbol{x}$.

Definition A.2 (*Unique Strong Solution Starting at* $\boldsymbol{x}$) We say (ISKE) with (3.10) and (3.11) has a unique strong solution starting at $\boldsymbol{x}$ if there exists a function $F_{\boldsymbol{x}}$: $W_0((\mathbb{R}^d)^{\mathbb{N}}) \to W((\mathbb{R}^d)^{\mathbb{N}})$ such that, for any weak solution $(\hat{\boldsymbol{X}}, \hat{\boldsymbol{L}}, \hat{\boldsymbol{B}})$ of (ISKE) with

(3.10) and (3.11) $\hat{\boldsymbol{X}} = F_{\boldsymbol{x}}(\hat{\boldsymbol{B}})$ a.s. and if, for any $(\mathbb{R}^d)^{\mathbb{N}}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion $\boldsymbol{B}$ defined on $(\Omega, \mathcal{F}, P, \{\mathcal{F}\}_{t\geq 0})$ with $\boldsymbol{B}_0 = 0$, the process $\boldsymbol{X} = F_{\boldsymbol{x}}(\boldsymbol{B})$ is a strong solution of (ISKE) with (3.10) and (3.11) starting at $\boldsymbol{x}$. We also call $F_{\boldsymbol{x}}$ a unique strong solution starting at $\boldsymbol{x}$.

We next present a variant of the notion of a unique strong solution.

Definition A.3 (*A Unique Strong Solution Under a Constraint*) For a condition (**Cond**), we say (ISKE) with (3.10) and (3.11) has a unique strong solution starting at $\boldsymbol{x}$ under the constraint of (**Cond**) if there exists a function $F_{\boldsymbol{x}} : W_0((\mathbb{R}^d)^{\mathbb{N}}) \to W((\mathbb{R}^d)^{\mathbb{N}})$ such that for any weak solution $(\hat{\boldsymbol{X}}, \hat{\boldsymbol{B}})$ of (ISKE) with (3.10) and (3.11) starting at $\boldsymbol{x}$ satisfying (**Cond**), it holds that $\hat{\boldsymbol{X}} = F_{\boldsymbol{x}}(\hat{\boldsymbol{B}})$ a.s. and if, for any $(\mathbb{R}^d)^{\mathbb{N}}$-valued $\{\mathcal{F}_t\}_{t\geq 0}$-Brownian motion $\boldsymbol{B}$ on $(\Omega, \mathcal{F}, P, \{\mathcal{F}\}_{t\geq 0})$ with $\boldsymbol{B}_0 = 0$, the process $\boldsymbol{X} = F_{\boldsymbol{x}}(\boldsymbol{B})$ is a strong solution of (ISKE) with (3.10) and (3.11) starting at $\boldsymbol{x}$ satisfying (**Cond**). We also call $F_{\boldsymbol{x}}$ a strong solution starting at $\boldsymbol{x}$ under the constraint of (**Cond**).

For a family of strong solutions $\{F_{\boldsymbol{x}}\}$ satisfying (**MF**), we put

$$P_{\{F_{\boldsymbol{x}}\}} = \int P(F_{\boldsymbol{x}}(\boldsymbol{B}) \in \cdot) P \circ \boldsymbol{X}_0^{-1}(d\boldsymbol{x}).$$

Let $(\boldsymbol{X}, \boldsymbol{L}, \boldsymbol{B})$ be a solution of (ISKE) with (3.10) and (3.11) under P. Suppose that $(\boldsymbol{X}, \boldsymbol{B})$ is a unique strong solution under $P_{\boldsymbol{x}}$ for $P \circ \boldsymbol{X}_0^{-1}$-a.s. $\boldsymbol{x}$. Let $\{F_{\boldsymbol{x}}\}$ be a family of the unique strong solution given by $(\boldsymbol{X}, \boldsymbol{B})$ under $P_{\boldsymbol{x}}$. Then, (**MF**) is automatically satisfied and $P_{\{F_{\boldsymbol{x}}\}} = P \circ \boldsymbol{X}^{-1}$.

Definition A.4 (*A Family of Unique Strong Solutions Under Constraints*) For a condition (**Cond**), we say (ISKE) with (3.10) and (3.11) has a family of unique strong solutions $\{F_{\boldsymbol{x}}\}$ starting at $\boldsymbol{x}$ for $P \circ \boldsymbol{X}_0^{-1}$-a.s. $\boldsymbol{x}$ under the constraints of (**MF**) and (**Cond**) if $\{F_{\boldsymbol{x}}\}$ satisfies (**MF**) and $P_{\{F_{\boldsymbol{x}}\}}$ satisfies (**Cond**). Furthermore, (i) and (ii) are satisfied.

(i) For any weak solution $(\hat{\boldsymbol{X}}, \hat{\boldsymbol{B}})$ under $\hat{P}$ of (ISKE) with (3.10) and (3.11) with $\hat{P} \circ \boldsymbol{X}_0^{-1} \prec P \circ \boldsymbol{X}_0^{-1}$ satisfying (**Cond**), it holds that, for $\hat{P} \circ \boldsymbol{X}_0^{-1}$-a.s. $\boldsymbol{x}$, $\hat{\boldsymbol{X}} = F_{\boldsymbol{x}}(\hat{\boldsymbol{B}})$ $\hat{P}_{\boldsymbol{x}}$-a.s., where $\hat{P}_{\boldsymbol{x}} = \hat{P}(\cdot|\hat{\boldsymbol{X}}_0 = \boldsymbol{x})$.
(ii) For an arbitrary $(\mathbb{R}^d)^{\mathbb{N}}$-valued $\{\mathcal{F}_t\}$-Brownian motion $\boldsymbol{B}$ on $(\Omega, \mathcal{F}, P, \{\mathcal{F}\}_{t\geq 0})$ with $\boldsymbol{B}_0 = \boldsymbol{0}$, $F_{\boldsymbol{x}}(\boldsymbol{B})$ is a strong solution of (ISKE) with (3.10) and (3.11) starting at $\boldsymbol{x}$ for $P \circ \boldsymbol{X}_0^{-1}$-a.s. $\boldsymbol{x}$.

A.2 Definition of (IFC)

In this subsection, we introduce (**IFC**) for our situation. Let $\mathbb{I}$ be a finite subset of $\mathbb{N}$. Put $\mathbb{I}^c = \mathbb{N} \setminus \mathbb{I}$. For $\boldsymbol{y} = (y^1, y^2, \dots) \in \boldsymbol{S}_{\text{hard}}$, we put $\boldsymbol{y}^{\mathbb{I}} = (y^j)_{j\in\mathbb{I}}$ and $\boldsymbol{y}^{\mathbb{I}^c} = (y^j)_{j\in\mathbb{I}^c}$ Let $(\boldsymbol{X}, \boldsymbol{B}) = ((X^j)_{j\in\mathbb{N}}, (B^j)_{j\in\mathbb{N}})$ be a weak solution of (ISKE) starting at $\boldsymbol{x} = (x^j)_{j\in\mathbb{N}}$ defined on $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$. We consider the SDE

$$
\begin{aligned}
dY_t^{\mathbb{I},j} &= dB_t^j + b_{\boldsymbol{X}}^{\mathbb{I},j}(t,\mathbf{Y}_t^{\mathbb{I}})dt + \sum_{k\in\mathbb{I}\setminus\{j\}}(Y_t^{\mathbb{I},j} - Y_t^{\mathbb{I},k})dL_t^{\mathbb{I},jk} \\
&\quad + \sum_{k\in\mathbb{I}^c}^{\infty}(Y_t^{\mathbb{I},j} - X_t^k)dL_t^{\mathbb{I},jk}, \quad j\in\mathbb{I}, \qquad (\mathrm{SKE}_{\boldsymbol{X}}(\mathbb{I})) \\
Y_0^{\mathbb{I},j} &= X_0^j = x^j, \quad j\in\mathbb{I},
\end{aligned}
$$

where $L_t^{\mathbb{I},jk}$, $j\in\mathbb{I}$, $k\in\mathbb{N}$ are increasing functions satisfying

$$
L_t^{\mathbb{I},jk} = \int_0^t \mathbf{1}(|Y_s^{\mathbb{I},j} - Y_s^{\mathbb{I},k}| = r)dL_s^{\mathbb{I},jk}, \quad j,k\in\mathbb{I},
$$

$$
L_t^{\mathbb{I},jk} = \int_0^t \mathbf{1}(|Y_s^{\mathbb{I},j} - X_s^k| = r)dL_s^{\mathbb{I},jk}, \quad j\in\mathbb{I},\ k\in\mathbb{I}^c.
$$

We denote by $\mathcal{C}^{\mathbb{I}}$ the completion of $\mathcal{B}(W_{\mathbf{0}}((\mathbb{R}^d)^{\mathbb{I}})\times W((\mathbb{R}^d)^{\mathbb{N}}))$ with respect to $P_{\boldsymbol{x}}\circ(\boldsymbol{B}^{\mathbb{I}},\boldsymbol{X}^{\mathbb{I}^c})^{-1}$. Let $(\mathbf{v},\boldsymbol{w})\in W_{\mathbf{0}}((\mathbb{R}^d)^{\mathbb{I}})\times W((\mathbb{R}^d)^{\mathbb{N}})$. We denote by $\mathcal{C}_t^{\mathbb{I}}$ the completion of $\sigma[(\mathbf{v}_s,\boldsymbol{w}_s):0\le s\le t]$ with respect to $P_{\boldsymbol{x}}\circ(\boldsymbol{B}^{\mathbb{I}},\boldsymbol{X}^{\mathbb{I}^c})^{-1}$.

Definition A.5 (*Strong Solution for* $(\boldsymbol{X},\boldsymbol{B})$ *Starting at* $\boldsymbol{x}^{\mathbb{I}}$) $\mathbf{Y}^{\mathbb{I}}$ is called a strong solution of $(\mathrm{SKE}_{\boldsymbol{X}}(\mathbb{I}))$ for $(\boldsymbol{X},\boldsymbol{B})$ under $P_{\boldsymbol{x}}$ if $(\mathbf{Y}^{\mathbb{I}},\boldsymbol{B}^{\mathbb{I}},\boldsymbol{X}^{\mathbb{I}^c})$ satisfies $(\mathrm{SKE}_{\boldsymbol{X}}(\mathbb{I}))$ and there exists a $\mathcal{C}^{\mathbb{I}}$-measurable function

$$
F_{\boldsymbol{x}}^{\mathbb{I}} : W_{\mathbf{0}}((\mathbb{R}^d)^{\mathbb{I}})\times W((\mathbb{R}^d)^{\mathbb{I}^c}) \to W((\mathbb{R}^d)^{\mathbb{I}})
$$

such that $F_{\mathbf{x}}^{\mathbb{I}}$ is $\mathcal{C}_t^{\mathbb{I}}/\mathcal{B}_t(W(\mathbb{R}^d)^{\mathbb{I}}$-measurable for each t, and $F_{\boldsymbol{x}}^{\mathbb{I}}$ satisfies $\mathbf{Y}^I = F_{\boldsymbol{x}}^{\mathbb{I}}(\boldsymbol{B}^{\mathbb{I}},\boldsymbol{X}^{\mathbb{I}^c})$, $P_{\boldsymbol{x}}$-a.s.

Definition A.6 (*A Unique Strong Solution for* $(\boldsymbol{X},\boldsymbol{B})$ Starting at $\boldsymbol{x}_m$) The SDE $(\mathrm{SKE}_{\boldsymbol{X}}(\mathbb{I}))$ is said to have a unique strong solution for $(\boldsymbol{X},\boldsymbol{B})$ under $P_{\boldsymbol{x}}$ if there exists a strong solution $F_{\boldsymbol{x}}^{\mathbb{I}}$ such that for any solution $(\hat{\mathbf{Y}}^{\mathbb{I}},\boldsymbol{B}^{\mathbb{I}},\boldsymbol{X}^{\mathbb{I}^c})$ of $(\mathrm{SKE}_{\boldsymbol{X}}(\mathbb{I}))$ under $P_{\boldsymbol{x}}$, $\hat{\mathbf{Y}}^{\mathbb{I}} = F_{\boldsymbol{x}}^{\mathbb{I}}(\boldsymbol{B}^{\mathbb{I}},\boldsymbol{X}^{\mathbb{I}^c})$ for $P_{\boldsymbol{x}}$-a.s..

We can then give the definition of (**IFC**).

(**IFC**) For each finite subset $\mathbb{I}\subset\mathbb{N}$, $(\mathrm{SKE}_{\boldsymbol{X}}(\mathbb{I}))$ has a unique strong solution under $P_{\boldsymbol{x}} := P(\cdot|\boldsymbol{X}_0=\boldsymbol{x})$ for $P\circ\boldsymbol{X}_0^{-1}$-a.s. $\boldsymbol{x}$.

References

1. Z.Q. Chen, On reflecting diffusion processes and Skorohod decomposition. Probab. Theory Relat. Fields **94**, 281–315 (1993)
2. M. Fukushima, A construction of reflecting barrier Brownian motions for bounded domains. Osaka J. Math. **94**, 183–215 (1968)
3. M. Fukushima, Regular representations of Dirichlet spaces. Trans. Amer. Math. Soc. **155**, 455–743 (1971)
4. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet forms and symmetric Markov processes*, 2nd edn. (Walter de Gruyter, Berlin, 2010)
5. M. Fradon, S. Roelly, H. Tanemura, An infinite system of Brownian balls with infinite range interaction. Stoch. Process. Appl. **90**, 43–66 (2000)
6. S. Gohsh, Continuum percolation for Gaussian zeroes and Ginibre eigenvalues. Ann. Probab. **44**, 3357–3384 (2016)
7. O. Kallenberg, in *Random Measures, Theory and Applications*. Probability Theory and Stochastic Modelling, vol. 77 (Springer, Cham, 2017)
8. H. Osada, Dirichlet form approach to infinite-dimensional Wiener processes with singular interactions. Commun. Math. Phys. **176**, 117–131 (1996)
9. H. Osada, Tagged particle processes and their non-explosion criteria. J. Math. Soc. Jpn. **62**, 867–894 (2010)
10. H. Osada, Infinite-dimensional stochastic differential equations related to random matrices. Probab. Theory Relat. Fields **153**, 471–509 (2012)
11. H. Osada, Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials. Ann. Probab. **41**, 1–49 (2013)
12. H. Osada, Interacting Brownian motions in infinite dimensions with logarithmic interaction potentials II?: Airy random point field. Stoch. Process. Appl. **123**, 813–838 (2013)
13. H. Osada, H. Tanemura, Cores of Dirichlet forms related to random matrix theory. Proc. Japan Acad. Ser. A Math. Sci. **90**, 145–150 (2014)
14. H. Osada, H. Tanemura, Infinite-dimensional stochastic differential equations and tail σ-fields. Probab. Theory Relat. Fields **177**, 1137–1242 (2020)
15. D. Ruelle, Super-stable interactions in classical statistical mechanics. Commun. Math. Phys. **18**, 127–159 (1970)
16. Y. Saisho, Stochastic differential equations for multidimensional domain with reflecting boundary. Probab. Theory Relat. Fields **104**, 455–477 (1987)
17. Y. Saisho, H. Tanaka, Stochastic differential equations for mutually reflecting Brownian balls. Osaka J. Math. **23**, 725–740 (1986)
18. H. Tanemura, A system of infinitely many mutually reflecting Brownian balls in $\mathbb{R}^d$. Probab. Theory Relat. Fields **104**, 399–426 (1996)
19. H. Tanemura, Uniqueness of Dirichlet forms associated with systems of infinitely many Brownian balls in $\mathbb{R}^d$. Probab. Theory Relat. Fields **109**, 275–299 (1997)

On Universality in Penalisation Problems with Multiplicative Weights

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Abstract We give a general framework for the universality classes of σ-finite measures in penalisation problems with multiplicative weights. We discuss penalisation problems for Brownian motions, Lévy processes and Langevin processes in our framework.

Keywords Markov process · Martingale · Limit theorem · Penalisation · Conditioning

Mathematics Subject Classification 60F05 · 60G44 · 60J57

1 Introduction

For a measure μ and a non-negative measurable function f, we write $\mu[f]$ for the integral $\int f \mathrm{d}\mu$.

For a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $(\mathcal{F}_s)_{s \geq 0}$, and for a non-negative process $\Gamma = (\Gamma_t)_{t \geq 0}$ called a *weight*, we mean by a *penalisation* a problem of finding a limit probability P^Γ on $(\Omega, \mathcal{F})$ called the *penalised probability* such that

$$\frac{P[F_s \Gamma_t]}{P[\Gamma_t]} \underset{t \to \infty}{\longrightarrow} P^\Gamma[F_s] \tag{1.1}$$

is satisfied for all $s \geq 0$ and all bounded $\mathcal{F}_s$-measurable functional F_s. Under the penalised probability P^Γ, the process $(\Gamma_t)_{t \geq 0}$ is prevented from taking small values; this is why Roynette et al. [14] (see also [15]) called this problem the penalisation. Conditioning a process to stay in a domain D may be regarded as a special case of the penalisation, as we take the weight $\Gamma_t = 1_{\{\tau_D > t\}}$ where τ_D denotes the exit time of D.

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Although the penalised probability P^{Γ} depends upon the weight Γ, we can often find a σ-finite measure $\mathscr{P}$ on $(\Omega, \mathcal{F})$ independent of a particular weight such that

$$P^{\Gamma}(A) = \frac{\mathscr{P}[\Gamma_\infty; A]}{\mathscr{P}[\Gamma_\infty]}, \quad A \in \mathcal{F} \tag{1.2}$$

holds with a suitable limit Γ_∞ of Γ_t in a certain class of weights Γ. In this case we say that Γ belongs to the *universality class* of $\mathscr{P}$. The aim of this paper is to gain a clear insight into the universality classes in penalisation problems. For this purpose, we confine ourselves to multiplicative weights.

Let $\{B = (B_t)_{t \ge 0}, W_x\}$ denote the canonical representation of the one-dimensional Brownian motion with $W_x(B_0 = x) = 1$ and let $\mathcal{F}_t^B = \sigma(B_s : s \le t)$ denote the natural filtration of the coordinate process B. Let $\tau_D = \inf\{t \ge 0 : B_t = 0\}$ denote the exit time of B from the non-zero real $D = \mathbb{R} \setminus \{0\}$. Let $x \in D$ be fixed. It is then well-known that

$$W_x[F_s | \tau_D > t] \underset{t \to \infty}{\longrightarrow} W_x^{\pm 3\mathrm{B}}[F_s] = \frac{1}{|x|} W_x\left[F_s |B_s| 1_{\{\tau_D > s\}}\right] \tag{1.3}$$

for all bounded $\mathcal{F}_s^B$-measurable functional F_s, where $W_x^{\pm 3\mathrm{B}}$ denotes the law of $\pm$ times 3-dimensional Bessel process starting from x. This conditioning to avoid zero may be regarded as a special case of the penalisation with the weight being given by $\Gamma_t = 1_{\{\tau_D > t\}}$. Note that $W_x^{\pm 3\mathrm{B}}$ is locally absolutely continuous with respect to W_x, i.e. $W_x^{\pm 3\mathrm{B}}|_{\mathcal{F}_s^B}$ is absolutely continuous with respect to $W_x|_{\mathcal{F}_s^B}$ for all $s \ge 0$. But $W_x^{\pm 3\mathrm{B}}$ and W_x are mutually singular on $\mathcal{F}_\infty^B := \sigma(B)$, because $W_x^{\pm 3\mathrm{B}}(\tau_D = \infty) = W_x(\tau_D < \infty) = 1$. While the original process $\{B, W_x\}$ is recurrent, the *penalised process* $\{B, W_x^{\pm 3\mathrm{B}}\}$ is transient.

Roynette et al. [12, 13] have studied the penalisation problems for the one-dimensional Brownian motion. They determined the penalised probabilities for $\Gamma_t = f(\overline{X}_t)$, a function of a supremum, $\Gamma_t = f(L_t)$, a function of a local time at 0, and $\Gamma_t = \exp(-\int_0^t v(B_s)\mathrm{d}s)$, a Kac killing weight. For the special case $\Gamma_t = \mathrm{e}^{-L_t}$, we have

$$\frac{W_0[F_s \mathrm{e}^{-L_t}]}{W_0[\mathrm{e}^{-L_t}]} \underset{t \to \infty}{\longrightarrow} W_0^{\Gamma}[F_s] = \frac{1}{1+|x|} W_0\left[F_s(1+|B_s|)\mathrm{e}^{-L_s}\right] \tag{1.4}$$

for all $s \ge 0$ and all bounded $\mathcal{F}_s^B$-measurable functional F_s. Although W_0^{Γ} is locally absolutely continuous with respect to W_0, the two measures W_0^{Γ} and W_0 are mutually singular on $\mathcal{F}_\infty^B$, because $W_0^{\Gamma}(L_\infty < \infty) = W_0(L_\infty = \infty) = 1$. While the original process $\{B, W_0\}$ is recurrent, the penalised process $\{B, W_0^{\Gamma}\}$ is transient.

Najnudel et al. [8] have introduced the σ-finite measure $\mathscr{W}_0$ defined by

$$\mathscr{W}_0 = \int_0^\infty \frac{\mathrm{d}u}{\sqrt{2\pi u}} \Pi^{(u)} \bullet W_0^{\mathrm{s3B}}, \tag{1.5}$$

where $\Pi^{(u)}$ stands for the law of the Brownian bridge from 0 to 0 of length u, W_0^{s3B} for the law of the symmetrised Bessel process, and $\bullet$ for the law of the concatenated path of two independent paths. They proved that the penalised probability W_0^{Γ} for any weight Γ in the previous paragraph is absolutely continuous on $\mathcal{F}_\infty^B$ with respect to $\mathscr{W}_0$:

$$W_0^{\Gamma}[F] = \frac{\mathscr{W}_0[F\Gamma_\infty]}{\mathscr{W}_0[\Gamma_\infty]} \tag{1.6}$$

for all bounded $\mathcal{F}_\infty^B$-measurable functional F. Moreover, if we define $\mathscr{W}_x(\cdot) = \mathscr{W}_0(x + B \in \cdot)$, we have

$$W_x^{\pm 3\mathrm{B}}[F] = \frac{\mathscr{W}_x[F; \tau_D = \infty]}{\mathscr{W}_x(\tau_D = \infty)} \tag{1.7}$$

for all $x > 0$ and all bounded $\mathcal{F}_\infty^B$-measurable functional F. In other words, all the weights belong to the universality class of $\mathscr{W}_x$.

Yano et al. [20, 21], Yano [22] and recently Takeda and Yano [16] studied the penalisation problems for one-dimensional stable Lévy processes and found out that there are two different universality classes. In this paper, we would like to give a general framework to characterise universality classes, where we will give some new results.

Groeneboom et al. [6] studied the conditioning to stay positive for the Langevin process. Profeta [10] studied penalisation problems with several kinds of weights. In this paper, we shall also discuss universality classes for those penalisation problems.

This paper is organized as follows. In Sect. 2 we develop a general study on penalised probabilities with multiplicative weights. In Sect. 3 we define the unweighted measures and discuss the subsequent Markov property of them. In Sect. 4 we state and prove our main theorems on universality classes. In Sect. 5 we give a general discussion on penalisation problems with multiplicative weights. In Sects. 6, 7 and 8, we look at some known results of penalisation problems for Brownian motions, Lévy processes and Langevin processes in our framework. In last section as an Appendix, we discuss extension of the transformed probability measures given by local absolute continuity.

2 Penalised Probability

For a measure μ and a non-negative measurable function f, we write $f \cdot \mu$ for the transformed measure defined by $(f \cdot \mu)(A) = \int_A f \mathrm{d}\mu$ for all measurable set A. Let $(\mathcal{F}_s)_{s \geq 0}$ be a filtration. For two measures μ and ν, we say that μ is *locally absolutely continuous* with respect to ν if $\mu|_{\mathcal{F}_s}$ is absolutely continuous with respect to $\nu|_{\mathcal{F}_s}$ for all $s \geq 0$. We say the two measures are *locally equivalent* if they are locally absolutely continuous with respect to each other. For a parameterised family $(\mu_\lambda)_\lambda$

of finite measures and a finite measure μ, we say that

$$\lim_\lambda \mu_\lambda = \mu \textit{ along } (\mathcal{F}_s)_{s\ge 0} \tag{2.1}$$

if

$$\lim_\lambda \mu_\lambda[F_s] = \mu[F_s] \tag{2.2}$$

holds for all $s \ge 0$ and all bounded measurable functional F_s.

Let S be a locally compact separable metric space and let $\mathbb{D}$ denote the space of càdlàg paths from $[0,\infty)$ to S. Let $X = (X_t)_{t\ge 0}$ denote the coordinate process: $X_t(\omega) = \omega(t)$ for $t \ge 0$ and $\omega \in \mathbb{D}$. Let $\mathcal{F}_t^X = \sigma(X_s : s \le t)$ denote the natural filtration of X and set $\mathcal{F}_t = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}^X$ so that $(\mathcal{F}_t)_{t\ge 0}$ is a right-continuous filtration. We write $\mathcal{F}_\infty = \sigma(\bigcup_{t\ge 0} \mathcal{F}_t) = \sigma(X)$. For $t \ge 0$, let θ_t denote the shift operator of $\mathbb{D}$: $\theta_t\omega(s) = \omega(t+s)$ for $s \ge 0$.

Let $\{X, \mathcal{F}_\infty, (P_x)_{x\in S}\}$ denote the canonical representation of a strong Markov process taking values in S with respect to the augmented filtration $(\mathcal{G}_t)_{t\ge 0}$ of $(\mathcal{F}_t)_{t\ge 0}$. A process $\Gamma = (\Gamma_t)_{t\ge 0}$ is called a *weight* if it is a non-negative càdlàg process. A weight Γ is called *multiplicative* if Γ is adapted to $(\mathcal{F}_t)_{t\ge 0}$ and

$$\Gamma_t = \Gamma_s \cdot (\Gamma_{t-s} \circ \theta_s),\ P_x\text{-a.s. for all } 0 \le s \le t < \infty \text{ and all } x \in S. \tag{2.3}$$

Let Γ be a multiplicative weight. Since $\Gamma_0 = \Gamma_0 \cdot (\Gamma_0 \circ \theta_0) = \Gamma_0^2$, we note that

$$\text{for any } x \in S \text{ we have either } P_x(\Gamma_0 = 1) = 1 \text{ or } P_x(\Gamma_0 = 0) = 1. \tag{2.4}$$

We set

$$S^\Gamma = \{x \in S : P_x(\Gamma_0 = 1) = 1\}. \tag{2.5}$$

It is easy to see that

$$\tau^\Gamma := \inf\{t \ge 0 : X_t \notin S^\Gamma\} = \inf\{t \ge 0 : \Gamma_t = 0\}\ P_x\text{-a.s. for all } x \in S, \tag{2.6}$$

since $[\Gamma_{t_0} = 0$ implies $\Gamma_t = 0$ for all $t \ge t_0]$ because of the multiplicativity.

We introduce the following assumptions:

(A1) There is a Borel function φ^Γ on S such that $\varphi^\Gamma > 0$ on S^Γ and

$$P_x[\Gamma_t \varphi^\Gamma(X_t)] = \varphi^\Gamma(x) \quad \text{for all } x \in S \quad \text{and} \quad t \ge 0. \tag{2.7}$$

(A2) It holds that

$$P_x[\Gamma_{\boldsymbol{e}(q)}] \to 0 \quad \text{as } q \downarrow 0 \quad \text{for all } x \in S^\Gamma, \tag{2.8}$$

where we abuse P_x for the extended probability measure of P_x supporting a standard exponential variable $\boldsymbol{e}$ independent of $\mathcal{F}_\infty$ and we set $\boldsymbol{e}(q) = \boldsymbol{e}/q$ for $q > 0$.

Note that, by the dominated convergence theorem, the condition (A2) follows from the following condition:

(A2′) It holds that

$$P_x[\Gamma_t] \to 0 \quad \text{as } t \to \infty \quad \text{for all } x \in S^\Gamma. \tag{2.9}$$

By the multiplicativity, the condition (2.7) is equivalent to the condition that

$$(\Gamma_t \varphi^\Gamma(X_t))_{t \geq 0} \text{ is a right-continuous } ((\mathcal{G}_t)_{t\geq 0}, P_x)\text{-martingale for all } x \in S \tag{2.10}$$

(for right-continuity, see, e.g., [5, Theorem 5.8]). Under (A1), for $x \in S^\Gamma$, we may define a probability measure P_x^Γ on $(\mathbb{D}, \mathcal{F}_\infty)$, which we call the *penalised probability* of P_x for Γ, by the following (see Appendix):

$$P_x^\Gamma|_{\mathcal{F}_t} = \frac{\Gamma_t \varphi^\Gamma(X_t)}{\varphi^\Gamma(x)} \cdot P_x|_{\mathcal{F}_t} \quad \text{for all } t \geq 0. \tag{2.11}$$

It is then immediate that the *penalised process* $\{X, \mathcal{F}_\infty, (P_x^\Gamma)_{x\in S}\}$ is a Markov process with respect to $(\mathcal{F}_t)_{t\geq 0}$.

We write $\xrightarrow{P}$ for convergence in probability. In addition to (A1) and (A2), we also introduce the following assumption:

(A3) There is a non-negative finite $\mathcal{F}_\infty$-measurable functional Γ_∞ such that

$$P_x^\Gamma\left(\Gamma_t \underset{t\to\infty}{\longrightarrow} \Gamma_\infty > 0\right) = 1 \quad \text{for all } x \in S^\Gamma. \tag{2.12}$$

Note that in many examples we have (A3) and $P_x(\liminf_{t\to\infty} \Gamma_t = 0) = 1$, which implies that the two measures P_x^Γ and P_x are mutually singular on $\mathcal{F}_\infty$.

The following is a routine argument.

Proposition 2.1 *Let Γ be a multiplicative weight. Then the following hold.*

(i) *Under* (A1)*, it holds that*

$$P_x^\Gamma(\tau^\Gamma = \infty) = 1 \quad \textit{for all } x \in S^\Gamma. \tag{2.13}$$

(ii) *Under* (A1), (A2) *and* (A3)*, it holds that*

$$P_x^\Gamma\left(\varphi^\Gamma(X_t) \underset{t\to\infty}{\longrightarrow} \infty\right) = 1 \quad \textit{for all } x \in S^\Gamma. \tag{2.14}$$

Proof (i) We apply the optional stopping theorem to the $((\mathcal{G}_t)_{t\geq 0}, P_x)$-martingale $M_t := \Gamma_t \varphi^\Gamma(X_t)/\varphi^\Gamma(x)$ [by (A1)] to see that

$$P_x^\Gamma(\tau^\Gamma > t) = P_x\left[M_t; \tau^\Gamma > t\right] \tag{2.15}$$
$$= P_x[M_{t\wedge\tau^\Gamma}] - P_x\left[M_{t\wedge\tau^\Gamma}; \tau^\Gamma \leq t\right] \tag{2.16}$$
$$= P_x[M_0] - P_x\left[M_{\tau^\Gamma}; \tau^\Gamma \leq t\right] = 1, \tag{2.17}$$

which implies that $P_x^\Gamma(\tau^\Gamma = \infty) = 1$.

(ii) Let $0 \leq s \leq t < \infty$ and $A_s \in \mathcal{F}_s$. We then have

$$\begin{aligned} P_x^\Gamma\left[\frac{1}{\Gamma_t\varphi^\Gamma(X_t)}; A_s\right] &= \frac{1}{\varphi^\Gamma(x)} P_x(A_s,\ \tau^\Gamma > t) \\ &\leq \frac{1}{\varphi^\Gamma(x)} P_x(A_s,\ \tau^\Gamma > s) \\ &= P_x^\Gamma\left[\frac{1}{\Gamma_s\varphi^\Gamma(X_s)}; A_s\right]. \end{aligned} \tag{2.18}$$

This shows that $N_t := 1/\{\Gamma_t\varphi^\Gamma(X_t)\}$ is a non-negative P_x^Γ-supermartingale with respect to the completed filtration $(\overline{\mathcal{F}}_t^{P_x^\Gamma})_{t\geq 0}$ of $(\mathcal{F}_t)_{t\geq 0}$, and consequently it converges P_x^Γ-a.s. as $t \to \infty$ to some random variable N_∞. By (A3), we see that

$$\frac{1}{\varphi^\Gamma(X_t)} = \Gamma_t N_t \underset{t\to\infty}{\longrightarrow} \Gamma_\infty N_\infty \quad P_x^\Gamma\text{-a.s.}, \tag{2.19}$$

which implies $1/\varphi^\Gamma(X_{\boldsymbol{e}(q)}) \overset{P_x^\Gamma}{\underset{q\downarrow 0}{\longrightarrow}} \Gamma_\infty N_\infty$. Using Fatou's lemma, we obtain

$$P_x^\Gamma[\Gamma_\infty N_\infty] \leq \liminf_{q\downarrow 0} P_x^\Gamma\left[\frac{1}{\varphi^\Gamma(X_{\boldsymbol{e}(q)})}\right] = \frac{1}{\varphi^\Gamma(x)} \lim_{q\downarrow 0} P_x\left[\Gamma_{\boldsymbol{e}(q)}\right] = 0 \tag{2.20}$$

by (A2). Hence we obtain (2.14). □

3 Subsequent Markov Property

Let Γ be a multiplicative weight satisfying (A1), (A2) and (A3). For $x \in S^\Gamma$, we may define a measure $\mathscr{P}_x^\Gamma$ on $(\mathbb{D}, \mathcal{F}_\infty)$, which we call the *unweighted measure* of P_x^Γ, by

$$\mathscr{P}_x^\Gamma = \varphi^\Gamma(x)\Gamma_\infty^{-1} \cdot P_x^\Gamma \quad \text{on } \mathcal{F}_\infty. \tag{3.1}$$

Note that $\mathscr{P}_x^\Gamma$ is σ-finite on $\mathcal{F}_\infty$, because $\mathbb{D} = \bigcup_{n\in\mathbb{N}}\{\Gamma_\infty > 1/n\}$, $\mathscr{P}_x^\Gamma$-a.e. and

$$\mathscr{P}_x^\Gamma(\Gamma_\infty > 1/n) \le n\varphi^\Gamma(x) < \infty \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

The family of the unweighted measures satisfies the following property.

Theorem 3.1 *Let Γ be a multiplicative weight satisfying* (A1)–(A3). *Then, for any $x \in S^\Gamma$, any non-negative $\mathcal{F}_t$-measurable functional F_t and any non-negative $\mathcal{F}_\infty$-measurable functional G, it holds that*

$$\mathscr{P}_x^\Gamma[F_t(G\circ\theta_t)] = P_x\big[F_t\mathscr{P}_{X_t}^\Gamma[G];\, \tau^\Gamma > t\big]. \tag{3.3}$$

Proof By definition of $\mathscr{P}_x^\Gamma$, we have

$$\begin{aligned}\mathscr{P}_x^\Gamma[(F_t\Gamma_t)((G\Gamma_\infty)\circ\theta_t)] &= \mathscr{P}_x^\Gamma[F_t(G\circ\theta_t)\Gamma_\infty] &(3.4)\\ &= \varphi^\Gamma(x)P_x^\Gamma[F_t(G\circ\theta_t)]. &(3.5)\end{aligned}$$

By the Markov property for X under P_x^Γ, by the local equivalence between P_x^Γ and P_x, and by the global equivalence between P_x^Γ and $\mathscr{P}_x^\Gamma$, we obtain

$$\begin{aligned}(3.5) &= \varphi^\Gamma(x)P_x^\Gamma\big[F_tP_{X_t}^\Gamma[G]\big] &(3.6)\\ &= P_x\big[F_t\varphi^\Gamma(X_t)\Gamma_tP_{X_t}^\Gamma[G]\big] &(3.7)\\ &= P_x\big[F_t\Gamma_t\mathscr{P}_{X_t}[G\Gamma_\infty]\big], &(3.8)\end{aligned}$$

where we used the fact obtained from Proposition 2.1 that $X_t \in S^\Gamma$, P_x-a.s. on $\{\Gamma_t > 0\}$. Thus we obtain

$$\mathscr{P}_x^\Gamma[F_t\Gamma_t(G\Gamma_\infty)\circ\theta_t] = P_x\big[F_t\Gamma_t\mathscr{P}_{X_t}^\Gamma[G\Gamma_\infty]\big]. \tag{3.9}$$

Replacing F_t by $F_t\Gamma_t^{-1}1_{\{\tau^\Gamma>t\}}$ and G by $G\Gamma_\infty^{-1}1_{\{\Gamma_\infty>0\}}$, we obtain the desired identity, since $\tau^\Gamma = \infty$ and $\Gamma_\infty > 0$, $\mathscr{P}_x^\Gamma$-a.e. The proof is now complete. □

Theorem 3.1 asserts that, the process under $\mathscr{P}_x^\Gamma$ behaves until a fixed time t as the process under P_x killed upon leaving S^Γ, and it starts afresh at time t to behave as the process under $\mathscr{P}_{X_t}^\Gamma$. In this sense, we may call this property (3.3) the *subsequent Markov property*.

4 Universality Class

Let $\mathcal{E}$ be a particular multiplicative weight satisfying (A1)–(A3). We would like to give a sufficient condition for existence of a positive function $c(x)$ such that

$$S^{\Gamma} \subset S^{\mathcal{E}} \quad \text{and} \quad \mathscr{P}_x^{\Gamma} = c(x) 1_{\{\Gamma_\infty > 0\}} \cdot \mathscr{P}_x^{\mathcal{E}} \quad \text{for all } x \in S^{\Gamma}. \tag{4.1}$$

We note that $[\mathscr{P}_x^{\Gamma} = c(x) 1_{\{\Gamma_\infty > 0\}} \cdot \mathscr{P}_x^{\mathcal{E}}]$ yields $[\Gamma$ belongs to the universality class of $\mathscr{P}_x^{\mathcal{E}}]$ in the sense we mentioned in Introduction.

Theorem 4.1 (Universality Theorem) *Let $\mathcal{E}$ and Γ be two multiplicative weights satisfying* (A1)–(A3). *Suppose there exists a positive function $c(x)$ such that*

$$P_x^{\mathcal{E}}\left(\Gamma_t \underset{t\to\infty}{\longrightarrow} \Gamma_\infty\right) = 1, \quad \frac{\varphi^{\Gamma}(X_t)}{\varphi^{\mathcal{E}}(X_t)} \xrightarrow[t\to\infty]{P_x^{\mathcal{E}}} c(x) \quad \textit{for all } x \in S^{\Gamma} \tag{4.2}$$

and

$$P_x^{\Gamma}\left(\mathcal{E}_t \underset{t\to\infty}{\longrightarrow} \mathcal{E}_\infty > 0\right) = 1, \quad \frac{\varphi^{\Gamma}(X_t)}{\varphi^{\mathcal{E}}(X_t)} \xrightarrow[t\to\infty]{P_x^{\Gamma}} c(x) \quad \textit{for all } x \in S^{\Gamma}. \tag{4.3}$$

(Notice that these assumptions do not follow from (A3).) Then (4.1) *holds.*

Proof Let $x \in S^{\Gamma}$ be fixed. Since $P_x = P_x^{\Gamma}$ on $\mathcal{F}_0$, we have

$$P_x(\mathcal{E}_0 = 1) = P_x^{\Gamma}(\mathcal{E}_0 = 1) \geq P_x^{\Gamma}(\mathcal{E}_\infty > 0) = 1, \tag{4.4}$$

which shows $x \in S^{\mathcal{E}}$. By the assumptions, we have

$$R_t \xrightarrow[t\to\infty]{P_x^{\mathcal{E}}} R_\infty \quad \text{and} \quad R_t \xrightarrow[t\to\infty]{P_x^{\Gamma}} R_\infty \quad \text{with } R_t = \frac{\Gamma_t \varphi^{\Gamma}(X_t)}{\varphi^{\mathcal{E}}(X_t)} \quad \text{and} \quad R_\infty = c(x)\Gamma_\infty. \tag{4.5}$$

Let $s > 0$ and let F_s be a non-negative $\mathcal{F}_s$-measurable functional. For $t > s$, we have

$$P_x^{\mathcal{E}}\left[F_s \cdot \frac{R_t}{1 + R_t + \mathcal{E}_t}\right] = \frac{1}{\varphi^{\mathcal{E}}(x)} P_x\left[F_s \cdot \frac{R_t}{1 + R_t + \mathcal{E}_t} \cdot \mathcal{E}_t \varphi^{\mathcal{E}}(X_t)\right] \tag{4.6}$$

$$= \frac{\varphi^{\Gamma}(x)}{\varphi^{\mathcal{E}}(x)} P_x^{\Gamma}\left[F_s \cdot \frac{R_t}{1 + R_t + \mathcal{E}_t} \cdot \frac{\mathcal{E}_t \varphi^{\mathcal{E}}(X_t)}{\Gamma_t \varphi^{\Gamma}(X_t)}\right] \tag{4.7}$$

$$= \frac{\varphi^{\Gamma}(x)}{\varphi^{\mathcal{E}}(x)} P_x^{\Gamma}\left[F_s \cdot \frac{\mathcal{E}_t}{1 + R_t + \mathcal{E}_t}\right]. \tag{4.8}$$

Letting $t \to \infty$ and applying the dominated convergence theorem, we obtain

$$P_x^{\mathcal{E}}\left[F_s \cdot \frac{R_\infty}{1 + R_\infty + \mathcal{E}_\infty}\right] = \frac{\varphi^{\Gamma}(x)}{\varphi^{\mathcal{E}}(x)} P_x^{\Gamma}\left[F_s \cdot \frac{\mathcal{E}_\infty}{1 + R_\infty + \mathcal{E}_\infty}\right]. \tag{4.9}$$

Since $s > 0$ and F_s are arbitrary, we obtain

$$c(x)\varphi^{\mathcal{E}}(x)\Gamma_\infty \cdot P_x^{\mathcal{E}} = \varphi^{\Gamma}(x)\mathcal{E}_\infty \cdot P_x^{\Gamma}, \tag{4.10}$$

which yields

$$c(x)1_{\{\Gamma_\infty>0\}} \cdot \mathscr{P}_x^{\mathcal{E}} = 1_{\{\mathcal{E}_\infty>0\}} \cdot \mathscr{P}_x^{\Gamma} = \mathscr{P}_x^{\Gamma}, \tag{4.11}$$

since $P_x^\Gamma(\mathcal{E}_\infty > 0) = 1$. We thus obtain the desired result. □

5 Penalisation Problems

We give two systematic methods of ensuring the conditions (A1) and (A2) in penalisation problems.

5.1 Constant Clock

We give a general framework for penalisation problems with constant clock.

Proposition 5.1 *Let Γ be a multiplicative weight. Let $\rho(t)$ be a function such that*

$$\rho(t) \underset{t\to\infty}{\longrightarrow} \infty \quad \textit{and} \quad \frac{\rho(t)}{\rho(t-s)} \underset{t\to\infty}{\longrightarrow} 1 \quad \textit{for all } s > 0, \tag{5.1}$$

or in other words, $\rho(\log t)$ is divergent and slowly varying at $t = \infty$. Suppose there exists a process $(M_s)_{s\ge0}$ such that $P_x(M_0 > 0) = P_x(\Gamma_0 = 1)$ for all $x \in S$ and

$$\rho(t)P_x[\Gamma_t|\mathcal{F}_s] \underset{t\to\infty}{\longrightarrow} M_s \quad \textit{in } L^1(P_x) \textit{ for all } x \in S \textit{ and all } s \ge 0. \tag{5.2}$$

Then the weight Γ satisfies (A1) *and* (A2′) *with*

$$\varphi^\Gamma(x) = \lim_{t\to\infty} \rho(t)P_x[\Gamma_t], \tag{5.3}$$

and the following penalisation limit with constant clock holds:

$$\frac{\Gamma_t \cdot P_x}{P_x[\Gamma_t]} \underset{t\to\infty}{\longrightarrow} P_x^\Gamma \quad \textit{along } (\mathcal{F}_s)_{s\ge0} \quad \textit{for all } x \in S^\Gamma. \tag{5.4}$$

Proof The convergence (5.2) for $s = 0$ becomes (5.3). By the multiplicativity $\Gamma_t = \Gamma_s \cdot (\Gamma_{t-s} \circ \theta_s)$ and by the Markov property, we have

$$\rho(t)P_x[\Gamma_t|\mathcal{F}_s] = \frac{\rho(t)}{\rho(t-s)}\Gamma_s \cdot \rho(t-s)P_{X_s}[\Gamma_{t-s}] \underset{t\to\infty}{\longrightarrow} \Gamma_s\varphi^\Gamma(X_s) \quad \text{in } P_x\text{-a.s.}. \tag{5.5}$$

which yields $M_s = \Gamma_s \varphi^\Gamma(X_s)$. Hence we have

$$P_x[\Gamma_t \varphi^\Gamma(X_t)] = \lim_{u\to\infty} \rho(u) P_x[P_x[\Gamma_u|\mathcal{F}_t]] = \lim_{u\to\infty} \rho(u) P_x[\Gamma_u] = \varphi^\Gamma(x), \quad (5.6)$$

which shows that (A1) is satisfied. As $\rho(t) \to \infty$, we obtain (A2′). For $s > 0$ and for a bounded $\mathcal{F}_s$-measurable functional F_s, we obtain

$$\rho(t) P_x[F_s \Gamma_t] = P_x[F_s \rho(t) P_x[\Gamma_t|\mathcal{F}_s]] \underset{t\to\infty}{\longrightarrow} P_x[F_s M_s] = \varphi^\Gamma(x) P_x^\Gamma[F_s]. \quad (5.7)$$

This shows (5.4). □

5.2 Exponential Clock

Conditioning and penalisation problems with exponential clock have been widely studied; see [3, 4, 9, 11, 19]. We give a general framework for them.

Proposition 5.2 *Let $r(q)$ be a function defined for small $q > 0$ such that $r(q) \to \infty$ as $q \downarrow 0$. We abuse P_x for the extended probability measure of P_x supporting a standard exponential variable $\boldsymbol{e}$ independent of $(\mathcal{F}_t)_{t\ge 0}$ and set $\boldsymbol{e}(q) = \boldsymbol{e}/q$ for $q > 0$. Suppose there exists a process $(M_s)_{s\ge 0}$ such that $P_x(M_0 > 0) = P_x(\Gamma_0 = 1)$ for all $x \in S$ and*

$$\lim_{q\downarrow 0} r(q) P_x[\Gamma_{\boldsymbol{e}(q)}|\mathcal{F}_s] = \lim_{q\downarrow 0} r(q) P_x[\Gamma_{\boldsymbol{e}(q)} 1_{\{\boldsymbol{e}(q)>s\}}|\mathcal{F}_s] = M_s \quad \text{in } L^1(P_x)$$
$$\text{for all } x \in S \text{ and all } s \ge 0. \quad (5.8)$$

Then the weight Γ satisfies (A1) and (A2) with

$$\varphi^\Gamma(x) = \lim_{q\downarrow 0} r(q) P_x[\Gamma_{\boldsymbol{e}(q)}], \quad (5.9)$$

and the following penalisation limit with exponential clock holds:

$$\lim_{q\downarrow 0} \frac{\Gamma_{\boldsymbol{e}(q)} \cdot P_x}{P_x[\Gamma_{\boldsymbol{e}(q)}]} = \lim_{q\downarrow 0} \frac{\Gamma_{\boldsymbol{e}(q)} 1_{\{\boldsymbol{e}(q)>s\}} \cdot P_x}{P_x[\Gamma_{\boldsymbol{e}(q)}; \boldsymbol{e}(q) > s]} = P_x^\Gamma \quad \text{along } (\mathcal{F}_s)_{s\ge 0} \quad \text{for all } x \in S^\Gamma. \quad (5.10)$$

Proof The convergence (5.8) for $s = 0$ becomes (5.9). By the multiplicativity $\Gamma_t = \Gamma_s \cdot (\Gamma_{t-s} \circ \theta_s)$, by the Markov property and by the memoryless property

$$\boldsymbol{e}(q) - s \text{ given } \{\boldsymbol{e}(q) > s\} \overset{\text{law}}{=} \boldsymbol{e}(q), \quad (5.11)$$

we have

$$r(q)P_x[\Gamma_{\boldsymbol{e}(q)}1_{\{\boldsymbol{e}(q)>s\}}|\mathcal{F}_s] = \mathrm{e}^{-qs}r(q)P_x[\Gamma_{\boldsymbol{e}(q)+s}|\mathcal{F}_s] \tag{5.12}$$

$$= \mathrm{e}^{-qs}\Gamma_s r(q)P_{X_s}[\Gamma_{\boldsymbol{e}(q)}] \underset{q\downarrow 0}{\longrightarrow} \Gamma_s\varphi^{\Gamma}(X_s) \quad P_x\text{-a.s.,} \tag{5.13}$$

which yields $M_s = \Gamma_s\varphi^{\Gamma}(X_s)$. Hence we obtain

$$P_x[\Gamma_t\varphi^{\Gamma}(X_t)] = P_x[M_t] = \lim_{q\downarrow 0} r(q)P_x[P_x[\Gamma_{\boldsymbol{e}(q)}|\mathcal{F}_t]]$$

$$= \lim_{q\downarrow 0} r(q)P_x[\Gamma_{\boldsymbol{e}(q)}] = \varphi^{\Gamma}(x), \tag{5.14}$$

which shows that (A1) is satisfied. As $r(q) \to \infty$, we obtain (A2). For $s > 0$ and for a bounded $\mathcal{F}_s$-measurable functional F_s, we obtain

$$r(q)P_x[F_s\Gamma_{\boldsymbol{e}(q)}] = P_x[F_s r(q)P_x[\Gamma_{\boldsymbol{e}(q)}|\mathcal{F}_s]] \underset{q\downarrow 0}{\longrightarrow} P_x[F_sM_s] = \varphi^{\Gamma}(x)P_x^{\Gamma}[F_s]. \tag{5.15}$$

This shows (5.10). □

6 Brownian Penalisation Revisited

Let us look at some results of Roynette et al. [12, 13] and Najnudel et al. [8] in our framework.

Let $\{B = (B_t)_{t\geq 0}, (W_x)_{x\in\mathbb{R}}\}$ denote the canonical representation of the one-dimensional Brownian motion with $W_x(B_0 = x) = 1$. Set $\overline{B}_t = \sup_{s\leq t} B_s$ and let L_t denote the local time of B at 0. For the shift operator on the path space, we have

$$B_{t+s} = B_t \circ \theta_s, \quad \overline{B}_{t+s} = \overline{B}_s \vee (\overline{B}_t \circ \theta_s), \quad L_{t+s} = L_s + (L_t \circ \theta_s). \tag{6.1}$$

For a technical reason, we set as the state space

$$S = \{(x, y, l) \in \mathbb{R}^3 : y \geq x,\ l \geq 0\} \tag{6.2}$$

and consider the coordinate process $X = (X_t)_{t\geq 0} = (X_t^B, X_t^{\mathrm{sup}}, X_t^{\mathrm{lt}})_{t\geq 0}$ on the space of càdlàg paths from $[0, \infty)$ to S. Writing $a \vee b = \max\{a, b\}$, we define $P_{(x,y,l)}$ by the law on $\mathbb{D}$ of $(B, y \vee \overline{B}, l + L)$ under W_x, and adopt the notation of Sect. 2. By the identities (6.1), we see that the process $\{X, \mathcal{F}_\infty, (P_{(x,y,l)})_{(x,y,l)\in S}\}$ is a strong Markov process with respect to the augmented filtration.

(1) Supremum penalisation. For an integrable function $f : \mathbb{R} \to [0, \infty)$ such that for some $-\infty < y_0 \leq \infty$ we have $f(y) > 0$ for $y \leq y_0$ and $f(y) = 0$ for $y > y_0$,

we set

$$\Gamma_t^{\sup,f} = \frac{f(X_t^{\sup})}{f(X_0^{\sup})} 1_{\{X_t^{\sup} \le y_0\}}, \quad S^{\sup,f} = \{(x,y,l) \in S : y \le y_0\}. \tag{6.3}$$

Then we see that $\Gamma^{\sup,f}$ is a multiplicative weight with $S^{\Gamma^{\sup,f}} = S^{\sup,f}$ (in what follows we will omit similar remarks). By Roynette–Vallois–Yor [12, Theorem 3.6], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = \sqrt{\pi t/2}$ and

$$\varphi^{\sup,f}(x,y,l) = y - x + \frac{1}{f(y)} \int_y^{y_0} f(u)\mathrm{d}u, \quad (x,y,l) \in S^{\sup,f}, \tag{6.4}$$

so that (A1) and (A2′) are satisfied. By the discussion of Roynette et al. [12, Sect. 1.4], we can derive that

$$P_{(x,y,l)}^{\sup,f}(X_\infty^{\sup} > a) = \frac{\int_a^{y_0} f(u)\mathrm{d}u}{(y-x)f(y) + \int_y^{y_0} f(u)\mathrm{d}u}, \quad y \le a < \infty, \tag{6.5}$$

and hence that $[X_t^{\sup} = X_\infty^{\sup}$ for large $t]$ and $[\Gamma_t^{\sup,f} \to \Gamma_\infty^{\sup,f} > 0]$ $P_{(x,y,l)}^{\sup,f}$-a.s., which shows (A3). By (ii) of Proposition 2.1, we obtain the following known results:

$$P_{(x,y,l)}^{\sup,f}\left(X_t^B \to -\infty, \ \frac{\varphi^{\sup,f}(X_t)}{|X_t^B|} \to 1 \right) = 1. \tag{6.6}$$

(2) Local time penalisation. For an integrable function $f : [0,\infty) \to [0,\infty)$ such that for some $0 \le l_0 \le \infty$ we have $f(l) > 0$ for $l \le l_0$ and $f(l) = 0$ for $l > l_0$, we set

$$\Gamma_t^{\mathrm{lt},f} = \frac{f(X_t^{\mathrm{lt}})}{f(X_0^{\mathrm{lt}})} 1_{\{X_t^{\mathrm{lt}} \le l_0\}}, \quad S^{\mathrm{lt},f} = \{(x,y,l) \in S : l \le l_0\}. \tag{6.7}$$

By Roynette et al. [12, Theorem 3.13 and Lemma 3.15], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = \sqrt{\pi t/2}$ and

$$\varphi^{\mathrm{lt},f}(x,y,l) = |x| + \frac{1}{f(l)} \int_l^{l_0} f(u)\mathrm{d}u, \quad (x,y,l) \in S^{\mathrm{lt},f}, \tag{6.8}$$

so that (A1) and (A2′) are satisfied. Moreover, (A3) is also satisfied and

$$P^{\mathrm{lt},f}_{(x,y,l)}\big(X^B_t \to \pm\infty\big) = \frac{x^\pm f(l) + \frac{1}{2}\int_l^{l_0} f(u)\mathrm{d}u}{|x| f(l) + \int_l^{l_0} f(u)\mathrm{d}u} \tag{6.9}$$

with $x^\pm = \max\{\pm x, 0\}$. It is then obvious that

$$P^{\mathrm{lt},f}_{(x,y,l)}\left(|X^B_t| \to \infty,\ \frac{\varphi^{\mathrm{lt},f}(X_t)}{|X^B_t|} \to 1\right) = 1. \tag{6.10}$$

Note that the conditioning to avoid zero, which we have mentioned in Introduction, can be regarded as a special case of the local time penalisation with the weight $1_{\{X^{\mathrm{lt}}_t=0\}} = \Gamma^{\mathrm{lt},f}_t$ for $f(l) = 1_{\{l=0\}}$.

(3) Kac killing penalisation with integrable potential. For an integrable function $v : \mathbb{R} \to [0,\infty)$ satisfying

$$0 < \int_{\mathbb{R}} (1+|x|)v(x)\mathrm{d}x < \infty, \tag{6.11}$$

we set

$$\Gamma^{\mathrm{Kac},v}_t = \exp\left(-\int_0^t v(X^B_s)\mathrm{d}s\right), \quad S^{\mathrm{Kac},v} = S. \tag{6.12}$$

By Roynette et al. [13, Theorem 4.1], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = \sqrt{\pi t/2}$ and $\varphi^{\mathrm{Kac},v}(x,y,l) = \varphi_v(x)$ where φ_v is the unique solution to the Sturm–Liouville equation

$$\frac{1}{2}\frac{\mathrm{d}^2\varphi_v}{\mathrm{d}x^2}(x) = v(x)\varphi_v(x), \quad \lim_{x\to\pm\infty}\frac{\mathrm{d}\varphi_v}{\mathrm{d}x}(x) = \pm 1. \tag{6.13}$$

so that (A1) and (A2′) are satisfied. Moreover, (A3) is also satisfied and

$$P^{\mathrm{Kac},v}_{(x,y,l)}\big(X^B_t \to -\infty\big) = \frac{1}{C_v}\int_x^\infty \frac{\mathrm{d}y}{\varphi_v(y)^2}, \quad P^{\mathrm{Kac},v}_{(x,y,l)}\big(X^B_t \to \infty\big) = \frac{1}{C_v}\int_{-\infty}^x \frac{\mathrm{d}y}{\varphi_v(y)^2} \tag{6.14}$$

with $C_v = \int_{\mathbb{R}} \frac{\mathrm{d}y}{\varphi_v(y)^2}$. By (6.13) it is obvious that

$$P^{\mathrm{Kac},v}_{(x,y,l)}\left(|X^B_t| \to \infty,\ \frac{\varphi^{\mathrm{Kac},v}(X_t)}{|X^B_t|} \to 1\right) = 1. \tag{6.15}$$

(4) Kac killing penalisation with Heaviside potential. For $\lambda > 0$, we set

$$\Gamma_t^{\mathrm{Hev},\lambda} = \exp\left(-\lambda \int_0^t 1_{\{X_s^B > 0\}} \mathrm{d}s \right), \quad S^{\mathrm{Kac},v} = S. \tag{6.16}$$

By Roynette et al. [13, Theorem 5.1 and Example 5.4], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = \sqrt{\pi t/2}$ and

$$\varphi^{\mathrm{Hev},\lambda}(x, y, l) = \begin{cases} \frac{1}{\sqrt{2\lambda}} \mathrm{e}^{-\sqrt{2\lambda}x} & (x \geq 0), \\ \frac{1}{\sqrt{2\lambda}} - x & (x < 0), \end{cases} \tag{6.17}$$

so that (A1) and (A2′) are satisfied. Moreover, (A3) is also satisfied and

$$P_{(x,y,l)}^{\mathrm{Hev},\lambda}\left(X_t^B \to -\infty, \ \frac{\varphi^{\mathrm{Hev},\lambda}(X_t)}{|X_t^B|} \to 1 \right) = 1. \tag{6.18}$$

($*$) The universality class of Brownian penalisation. Take $\mathcal{E}_t = \exp(-X_t^{\mathrm{lt}})$ as a special case of (2) with $f(l) = \mathrm{e}^{-l}$. (Note that, by Najnudel et al. [8, Theorem 1.1.2], the corresponding unweighted measure $\mathscr{P}_x^{\mathcal{E}}$ coincides with $\mathscr{W}_x$ given in Introduction.) By the above argument, we see that all the assumptions of Theorem 4.1 are satisfied with $\mathcal{E}$ and $\Gamma = \Gamma^{\sup,f}$, $\Gamma^{\mathrm{lt},f}$, $\Gamma^{\mathrm{Kac},v}$ or $\Gamma^{\mathrm{Hev},\lambda}$, so that we obtain the following known result:

$$\mathscr{P}_{(x,y,l)}^{\Gamma} = 1_{\{\Gamma_\infty > 0\}} \cdot \mathscr{P}_{(x,y,l)}^{\mathcal{E}} \quad \text{for all } (x, y, l) \in S^{\Gamma}. \tag{6.19}$$

We remark the following obvious facts: It holds up to $\mathscr{P}_{(x,y,l)}^{\mathcal{E}}$-null sets that

$$\mathbb{D} = \{X_t^B \to \infty \text{ or } X_t^B \to -\infty\}, \tag{6.20}$$

and that the event $\{\Gamma_\infty > 0\}$ becomes

$$\{\Gamma_\infty^{\sup,f} > 0\} = \{X_t^B \to -\infty \text{ and } X_\infty^{\sup} \leq y_0\}, \tag{6.21}$$

$$\{\Gamma_\infty^{\mathrm{lt},f} > 0\} = \{[X_t^B \to \infty \text{ or } X_t^B \to -\infty] \text{ and } X_\infty^{\mathrm{lt}} \leq l_0\}, \tag{6.22}$$

$$\{\Gamma_\infty^{\mathrm{Kac},v} > 0\} = \{X_t^B \to \infty \text{ or } X_t^B \to -\infty\}, \tag{6.23}$$

$$\{\Gamma_\infty^{\mathrm{Hev},\lambda} > 0\} = \{X_t^B \to -\infty\}. \tag{6.24}$$

7 Lévy Penalisation Revisited

Let us look at some results of Yano et al. [20, 21], Yano [22] and Takeda and Yano [16] in our framework.

Let $\{Z=(Z_t)_{t\geq 0}, (P_x^Z)_{x\in\mathbb{R}}\}$ denote the canonical representation of one-dimensional strictly α-stable process of index $1<\alpha<2$, skewness $-1\leq\beta\leq 1$ and scaling parameter $c_\theta>0$:

$$P_0^Z[\mathrm{e}^{i\lambda Z_t}] = \exp\Big(-c_\theta|\lambda|^\alpha\Big(1-i\beta\mathrm{sgn}(\lambda)\tan\frac{\pi\alpha}{2}\Big)\Big), \quad \lambda\in\mathbb{R}. \tag{7.1}$$

(For the facts in this paragraph, see e.g. [2, Sect. VIII].) We assume that $1<\alpha<2$ so as to exclude the Brownian case and to assure that zero is regular for itself: Writing $T_0=\inf\{t>0 : Z_t=0\}$ for the hitting time of zero, we have

$$P_0^Z(T_0>0)=1. \tag{7.2}$$

Set $\overline{Z}_t=\sup_{s\leq t} Z_s$ and let L_t denote the local time of Z at 0. Let

$$\rho := P_0^Z(Z_1>0) = \frac{1}{2}+\frac{1}{\pi\alpha}\arctan\Big(\beta\tan\frac{\pi\alpha}{2}\Big)\in[1-1/\alpha, 1/\alpha] \tag{7.3}$$

and let k denote the positive constant such that

$$\lim_{y\to\infty} y^\alpha P_0^Z(\overline{Z}>y)=k. \tag{7.4}$$

We set as the state space

$$S=\{(x,y,l)\in\mathbb{R}^3 : y\geq x,\ l\geq 0\} \tag{7.5}$$

and consider the coordinate process $X=(X_t)_{t\geq0}=(X_t^Z, X_t^{\mathrm{sup}}, X_t^{\mathrm{lt}})_{t\geq 0}$ on the space of càdlàg paths from $[0,\infty)$ to S. We define $P_{(x,y,l)}$ by the law on $\mathbb{D}$ of $(Z, y\vee\overline{Z}, l+L)$ under P_x^Z, and adopt the notation of Sect. 2.

(1) Supremum penalisation. For a non-increasing function $f:\mathbb{R}\to[0,\infty)$ such that for some $-\infty<y_0\leq\infty$ we have $f(y)>0$ for $y\leq y_0$ and $f(y)=0$ for $y>y_0$, and

$$\int_0^{y_0} x^{\alpha\rho-1} f(y)\mathrm{d}y<\infty, \tag{7.6}$$

we set

$$\Gamma_t^{\mathrm{sup},f} = \frac{f(X_t^{\mathrm{sup}})}{f(X_0^{\mathrm{sup}})} 1_{\{X_t^{\mathrm{sup}} \le y_0\}}, \quad S^{\mathrm{sup},f} = \{(x,y,l) \in S : y \le y_0\}. \tag{7.7}$$

By Yano et al. [21, Theorem 5.1], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = t^\rho / k$ and

$$\varphi^{\mathrm{sup},f}(x,y,l) = (y-x)^{\alpha\rho} + \frac{\alpha\rho}{f(y)} \int_y^{y_0} f(u)(u-x)^{\alpha\rho-1} \mathrm{d}u, \quad (x,y,l) \in S^{\mathrm{sup},f}, \tag{7.8}$$

so that (A1) and (A2′) are satisfied. In the same way as that of deducing (6.5), we see that $[X_t^{\mathrm{sup}} = X_\infty^{\mathrm{sup}}$ for large $t]$ and $[\Gamma_t^{\mathrm{sup},f} \to \Gamma_\infty^{\mathrm{sup},f} > 0]$ $P_{(x,y,l)}^{\mathrm{sup},f}$-a.s., which shows (A3). By (ii) of Proposition 2.1 and by the dominated convergence theorem, we obtain the following known results:

$$P_{(x,y,l)}^{\mathrm{sup},f}\left(X_t^Z \to -\infty, \ \frac{\varphi^{\mathrm{sup},f}(X_t)}{(-X_t^Z)^{\alpha\rho}} \to 1 \right) = 1. \tag{7.9}$$

Note that the special case of the supremum penalisation with the weight $1_{\{X_t^{\mathrm{sup}} = 0\}} = \Gamma_t^{\mathrm{sup},f}$ for $f(l) = 1_{\{y=0\}}$ corresponds to the conditioning to stay negative.

(2) Local time penalisation. For an integrable function $f : [0,\infty) \to [0,\infty)$ such that for some $0 \le l_0 \le \infty$ we have $f(l) > 0$ for $l \le l_0$ and $f(l) = 0$ for $l > l_0$, we set

$$\Gamma_t^{\mathrm{lt},f} = \frac{f(X_t^{\mathrm{lt}})}{f(X_0^{\mathrm{lt}})} 1_{\{X_t^{\mathrm{lt}} \le l_0\}}, \quad S^{\mathrm{lt},f} = \{(x,y,l) \in S : l \le l_0\}. \tag{7.10}$$

By Takeda and Yano [16] and by certain computation in [18, Sect. 5], we see that all the assumptions of Proposition 5.2 are satisfied with $r(q) = c_r q^{1/\alpha - 1}$ for a certain constant $c_r > 0$ and

$$\varphi^{\mathrm{lt},f}(x,y,l) = C_{\alpha,\beta}(1 - \beta \operatorname{sgn}(x))|x|^{\alpha-1} + \frac{1}{f(l)} \int_l^{l_0} f(u)\mathrm{d}u, \quad (x,y,l) \in S^{\mathrm{lt},f} \tag{7.11}$$

with a certain constant $C_{\alpha,\beta} > 0$, so that (A1) and (A2) are satisfied. In the same way as that of deducing (6.5), we see that $[X_t^{\mathrm{lt}} = X_\infty^{\mathrm{lt}}$ for large $t]$ and $[\Gamma_t^{\mathrm{lt},f} \to \Gamma_\infty^{\mathrm{lt},f} > 0]$ $P_{(x,y,l)}^{\mathrm{lt},f}$-a.s., which shows (A3). By (ii) of Proposition 2.1, we obtain

$$P^{\mathrm{lt},f}_{(x,y,l)}\left((1-\beta\mathrm{sgn}(X^Z_t))|X^Z_t|^{\alpha-1}\to\infty,\ \frac{\varphi^{\mathrm{lt},f}(X_t)}{C_{\alpha,\beta}(1-\beta\mathrm{sgn}(X^Z_t))|X^Z_t|^{\alpha-1}}\to 1\right)=1; \tag{7.12}$$

in particular,

$$P^{\mathrm{lt},f}_{(x,y,l)}\left(X^Z_t\to-\infty,\ \frac{\varphi^{\mathrm{lt},f}(X_t)}{(-X^Z_t)^{\alpha-1}}\to 2C_{\alpha,1}\right)=1 \quad (\text{if } \beta=1), \tag{7.13}$$

$$P^{\mathrm{lt},f}_{(x,y,l)}\left(X^Z_t\to\infty,\ \frac{\varphi^{\mathrm{lt},f}(X_t)}{(X^Z_t)^{\alpha-1}}\to 2C_{\alpha,-1}\right)=1 \quad (\text{if } \beta=-1). \tag{7.14}$$

In the case of $-1<\beta<1$, we have a stronger convergence result in Takeda and Yano [16]:

$$P^{\mathrm{lt},f}_{(x,y,l)}\big(\lim X^Z_t=\limsup X^Z_t=\limsup(-X^Z_t)=\infty\big)=1 \quad \text{if } -1<\beta<1. \tag{7.15}$$

Note that the special case of the local time penalisation with the weight $1_{\{X^{\mathrm{lt}}_t=0\}}=\Gamma^{\mathrm{lt},f}_t$ for $f(l)=1_{\{l=0\}}$ corresponds to the conditioning to avoid zero. See [17] for comparison of two types of conditionings for Lévy processes.
($*$) The universality classes of Lévy penalisation. By (7.9), it holds that

$$\{\Gamma^{\sup,f}_\infty>0\}=\{X^Z_t\to-\infty \text{ and } X^{\sup}_\infty\le y_0\} \quad \text{up to } \mathscr{P}^{\sup,f}_{(x,y,l)}\text{-null sets} \tag{7.16}$$

in any case of $-1\le\beta\le 1$.
($*$1) Consider the case of $-1<\beta<1$. By (7.15), it holds that

$$\{\Gamma^{\mathrm{lt},g}_\infty>0\}=\{\lim X^Z_t=\limsup X^Z_t=\limsup(-X^Z_t)=\infty \quad \text{and} \quad X^{\mathrm{lt}}_\infty\le y_0\}$$
$$\text{up to } \mathscr{P}^{\mathrm{lt},g}_{(x,y,l)}\text{-null sets}. \tag{7.17}$$

This shows that the two σ-finite measures $\mathscr{P}^{\sup,f}_{(x,y,l)}$ and $\mathscr{P}^{\mathrm{lt},g}_{(x,y,l)}$ are singular to each other. Note that (7.9) and (7.15) imply

$$P^{\sup,f}_{(x,y,l)}\left(\frac{\varphi^{\mathrm{lt},g}(X_t)}{\varphi^{\sup,f}(X_t)}\to 0\right)=1 \tag{7.18}$$

because $\alpha\rho>\alpha-1$, so that the assumption of Theorem 4.1 is not satisfied.
($*$2) Consider the case of $\beta=1$, the spectrally positive case. Take $\mathcal{E}_t=\exp(X^{\sup}_0-X^{\sup}_t)$ as a special case of (1) with $f(y)=\mathrm{e}^{-y}$. Then, since $\alpha\rho=\alpha-1$, all the assumptions of Theorem 4.1 are satisfied with $\mathcal{E}$ and $\Gamma=\Gamma^{\sup,f}$ or $\Gamma^{\mathrm{lt},g}$, so that we conclude as a new result that

$$\mathscr{P}^{\Gamma}_{(x,y,l)} = 1_{\{\Gamma_\infty > 0\}} \cdot \mathscr{P}^{\mathcal{E}}_{(x,y,l)} \quad \text{for all } (x, y, l) \in S^{\Gamma}. \tag{7.19}$$

It holds up to $\mathscr{P}^{\mathcal{E}}_{(x,y,l)}$-null sets that

$$\mathbb{D} = \{X^Z_t \to -\infty\}, \tag{7.20}$$

and that the event $\{\Gamma_\infty > 0\}$ becomes

$$\{\Gamma^{\mathrm{lt},g}_\infty > 0\} = \{X^Z_t \to -\infty \text{ and } X^{\mathrm{lt}}_\infty \le l_0\}. \tag{7.21}$$

(∗3) Consider the case of $\beta = -1$, the spectrally negative case. Then

$$\{\Gamma^{\mathrm{lt},g}_\infty > 0\} = \{X^Z_t \to \infty \text{ and } X^{\mathrm{lt}}_\infty \le l_0\} \quad \text{up to } \mathscr{P}^{\mathrm{lt},g}_{(x,y,l)}\text{-null sets}, \tag{7.22}$$

which shows that $\mathscr{P}^{\sup,f}_{(x,y,l)}$ and $\mathscr{P}^{\mathrm{lt},g}_{(x,y,l)}$ are singular to each other.

8 Langevin Penalisation Revisited

Let us look at some results of Profeta [10] in our framework.

Let $\{(B, A), (W_{(b,a)})_{(b,a)\in\mathbb{R}^2}\}$ denote the canonical representation of the two-dimensional diffusion $(B, A) = (B_t, A_t)_{t\ge 0}$ where B is a Brownian motion starting from b and

$$A_t = a + \int_0^t B_u \mathrm{d}u. \tag{8.1}$$

This two-dimensional diffusion is a special case of the *Langevin process* and the process A is called the *integrated Brownian motion*. Set $\overline{A}_t := \sup_{s\le t} A_s$.

We set

$$S = \{(b, a, y) \in \mathbb{R}^3 : y \ge a\} \tag{8.2}$$

as the state space and consider the coordinate process

$$X = (X_t)_{t\ge 0} = (X^B_t, X^A_t, X^{\sup}_t)_{t\ge 0} \tag{8.3}$$

on the space of càdlàg paths from $[0, \infty)$ to S. We define $P_{(b,a,y)}$ by the law on $\mathbb{D}$ of $(B, A, y \vee \overline{A})$ under $W_{(b,a)}$, and adopt the notation of Sect. 2.

We recall the confluent hypergeometric function (see [1, Chap. 13]):

$$U(\alpha,\beta,z)=\frac{1}{\Gamma(\alpha)}\int_0^\infty \mathrm{e}^{-zu}u^{\alpha-1}(1+u)^{\beta-\alpha-1}\mathrm{d}u,\quad \alpha>0,\ \beta\in\mathbb{R},\ z>0. \tag{8.4}$$

It is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}z}(z^\alpha U(\alpha,\beta,z))=-\alpha(\beta-\alpha-1)z^{\alpha-1}U(\alpha+1,\beta,z). \tag{8.5}$$

The following asymptotics are taken from [1, Formulae 13.5.2 and 13.5.8]:

$$\lim_{z\to\infty} z^\alpha U(\alpha,\beta,z)=1\ (\beta\in\mathbb{R}),\quad \lim_{z\downarrow 0} z^{\beta-1}U(\alpha,\beta,z)=\frac{\Gamma(\beta-1)}{\Gamma(\alpha)}\ (1<\beta<2). \tag{8.6}$$

(1) Conditioning to stay negative. We write $\tau^A=\inf\{t>0 : X_t^A\geq 0\}$ for the exit time from $(-\infty,0)$ for the process X^A and set

$$\Gamma_t^A=1_{\{\tau^A>t\}},\quad S^A=\{(b,a,y)\in S : y<0\}=\{(b,a,y)\in\mathbb{R}^3 : a\leq y<0\}. \tag{8.7}$$

By modifying Profeta [10, Theorem 5], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t)=c_1t^{1/4}$ for a certain constant $c_1>0$ and

$$\varphi^A(b,a,y)=h(-a,-b),\quad (b,a,y)\in S^A, \tag{8.8}$$

with a continuous function $h:(0,\infty)\times\mathbb{R}\to(0,\infty)$ given as

$$h(x,y)=\begin{cases}(\frac{9}{2}x)^{1/6}z^{1/3}U(\frac{1}{6},\frac{4}{3},z)=y^{1/2}z^{1/6}U(\frac{1}{6},\frac{4}{3},z) & (y>0),\\ \frac{1}{6}(\frac{9}{2}x)^{1/6}z^{1/3}U(\frac{7}{6},\frac{4}{3},z)\mathrm{e}^{-z}=\frac{1}{6}|y|^{1/2}z^{1/6}U(\frac{7}{6},\frac{4}{3},z)\mathrm{e}^{-z} & (y<0),\end{cases} \tag{8.9}$$

for $x>0$ and $z=\frac{2}{9}\frac{|y|^3}{x}$, so that (A1) and (A2′) are satisfied. Moreover, (A3) is also satisfied and

$$P_{(b,a,y)}^A\big(X_t^B\to-\infty \text{ and } X_t^A\to-\infty\big)=1. \tag{8.10}$$

Let us prove this fact, as the part $[X_t^B\to-\infty]$ was not mentioned in [10]. By the formulae (8.6), we see that both $z^{1/6}U(\frac{1}{6},\frac{4}{3},z)$ and $z^{1/6}U(\frac{7}{6},\frac{4}{3},z)\mathrm{e}^{-z}$ are bounded in $z>0$, we obtain $h(x,y)\leq c_2|y|^{1/2}$ for some constant $c_2>0$. It holds $P_{(b,a,y)}^A$-a.s. that, by (ii) of Proposition 2.1,

$$\varphi^A(X_t) = h(-X_t^A, -X_t^B) \to \infty, \tag{8.11}$$

which yields $[|X_t^B| \to \infty]$. But $[P^A_{(b,a,y)}(X_t^B \to \infty) = 0]$, since $[X_t^B \to \infty]$ implies $[X_t^A = a + \int_0^t X_s^B \mathrm{d}s \to \infty]$, which contradicts the fact that $X_0^A = a < 0$ and $\tau^A = \infty$ by (i) of Proposition 2.1. Hence we obtain (8.10).

(2) Supremum penalisation. Let $f : \mathbb{R} \to [0, \infty)$ be a continuous function such that for some $-\infty < y_0 \le 0$, we have $f(y) > 0$ for $y \le y_0$ and $f(y) = 0$ for $y > y_0$. Set

$$\Gamma_t^{\sup, f} = \frac{f(X_t^A)}{f(X_0^A)} 1_{\{X_t^A \le y_0\}}, \qquad \begin{aligned} S^{\sup, f} &= \{(b, a, y) \in S : y \le y_0\} \\ &= \{(b, a, y) \in \mathbb{R}^3 : a \le y < y_0\}. \end{aligned} \tag{8.12}$$

By Profeta [10, Proposition 18 and Theorem 19], we see that all the assumptions of Proposition 5.1 are satisfied with $\rho(t) = c_1 t^{1/4}$ and

$$\varphi^{\sup, f}(b, a, y) = h(y - a, -b) + \frac{1}{f(y)} \int_y^{y_0} f(w) \frac{\partial}{\partial w} h(w - a, -b) \mathrm{d}w,$$
$$(b, a, y) \in S^{\sup, f}, \tag{8.13}$$

so that (A1) and (A2′) are satisfied. By a similar argument to that deducing (6.5), we see that $[X_t^{\sup} = X_\infty^{\sup}$ for large $t]$ $P^{\sup, f}_{(b,a,y)}$-a.s., and that $[\Gamma_t^{\sup, f} \to \Gamma_\infty^{\sup, f} > 0]$ $P^{\sup, f}_{(b,a,y)}$-a.s., which shows (A3). By the fact that $\frac{\partial h}{\partial w} \ge 0$, we have

$$\varphi^{\sup, f}(b, a, y) \le \left(\sup_{y \le w \le y_0} f(w) \right) h(y_0 - a, -b). \tag{8.14}$$

By a similar argument after (8.11), and by (ii) of Proposition 2.1, we can deduce

$$P^{\sup, f}_{(b,a,y)}\big(X_t^B \to -\infty \text{ and } X_t^A \to -\infty\big) = 1. \tag{8.15}$$

(∗) The universality class of Langevin penalisation. We would like to compare the three unweighted measures $\mathscr{P}^A_{(b,a,y)}$, $\mathscr{P}^{\sup, f}_{(b,a,y)}$ and $\mathscr{P}^B_{(b,a,y)}$. Here we write $\tau^B = \inf\{t > 0 : X_t^B \ge 0\}$ for the exit time from $(-\infty, 0)$ for the Brownian motion X^B and set

$$\Gamma_t^B = 1_{\{\tau^B > t\}}, \quad S^B = \{(b, a, y) \in S : b < 0\}. \tag{8.16}$$

The penalisation for the weight Γ^B is nothing else but the conditioning to stay negative for the Brownian motion, so that we obtain $\varphi^B(b, a, y) = -b$. The penalized probability $P^B_{(b,a,y)}$ is the minus times 3-dimensional Bessel process and the corresponding unweighted measure is given as $\mathscr{P}^B_{(b,a,y)} = (-b) P^B_{(b,a,y)}$. Since

$X_t^A = a + \int_0^t X_u^B \mathrm{d}u$, we obtain

$$P^B_{(b,a,y)}\big(X_t^B \to -\infty \text{ and } X_t^A \to -\infty\big) = 1. \tag{8.17}$$

We prove the following proposition with conjectured assumptions.

Proposition 8.1 *Set* $Z_t = \dfrac{(-X_t^B)^3}{(-X_t^A)}$. *Then the following assertions hold:*

(i) *Suppose the following conjecture is true:*

$$Z_t \xrightarrow[t\to\infty]{P^A_{(b,a,y)}} \infty \quad \text{and} \quad Z_t \xrightarrow[t\to\infty]{P^{\sup,f}_{(b,a,y)}} \infty \quad \text{for } (b,a,y) \in S^{\sup,f}. \tag{8.18}$$

Then $\mathscr{P}^{\sup,f}_{(b,a,y)}$ *and* $\mathscr{P}^A_{(b,a,y)}$ *coincide for* $(b,a,y) \in S^{\sup,f} (\subset S^A)$.

(ii) *Suppose the following conjecture is true:*

$$Z_t \xrightarrow[t\to\infty]{P^A_{(b,a,y)}} \infty \quad \text{and} \quad Z_t \xrightarrow[t\to\infty]{P^B_{(b,a,y)}} \infty \quad \text{for } (b,a,y) \in S^A \cap S^B. \tag{8.19}$$

Then $\mathscr{P}^A_{(b,a,y)}$ *and* $\mathscr{P}^B_{(b,a,y)}$ *are singular to each other for* $(b,a,y) \in S^A \cap S^B$.

Proof (i) Set $Z_t^{\sup} = \dfrac{(-X_t^B)^3}{(X_t^{\sup} - X_t^A)}$. Then $Z_t \xrightarrow[t\to\infty]{P} \infty$ both for $P = P^A_{(b,a,y)}$ and for $P = P^{\sup,f}_{(b,a,y)}$. Since $X_t^B < 0$ for large t, we have

$$\frac{h(X_t^{\sup} - X_t^A, -X_t^B)}{h(-X_t^A, -X_t^B)} = \frac{(Z_t^{\sup})^{1/6} U(\frac{1}{6}, \frac{4}{3}, Z_t^{\sup})}{(Z_t)^{1/6} U(\frac{1}{6}, \frac{4}{3}, Z_t)} \xrightarrow[t\to\infty]{P} 1 \tag{8.20}$$

by the assumption. Noting that (8.5) implies

$$\frac{\partial}{\partial x} h(x,y) = c_3 x^{-5/6} \cdot z^{7/6} U(\tfrac{7}{6}, \tfrac{4}{3}, z) \le c_4 x^{-5/6}, \quad x, y > 0, \; z = \frac{2}{9}\frac{|y|^3}{x} \tag{8.21}$$

for some constants $c_3, c_4 > 0$, we obtain

$$\frac{\varphi^{\sup,f}(X_t)}{\varphi^A(X_t)} \xrightarrow[t\to\infty]{P} 1 \tag{8.22}$$

both for $P = P^A_{(b,a,y)}$ and for $P = P^{\sup,f}_{(b,a,y)}$. We may now apply Theorem 4.1 for $\mathcal{E} = \Gamma^A$ and $\Gamma = \Gamma^{\sup,f}$, and thus we obtain the desired result.

(ii) By the assumption, we have

$$R_t := \frac{\Gamma_t^A \varphi^A(X_t)}{\varphi^B(X_t)} = \frac{\Gamma_t^A \cdot (-X_t^B)^{1/2} \cdot (Z_t)^{1/6} U(\frac{1}{6}, \frac{4}{3}, Z_t)}{(-X_t^B)} \xrightarrow[t\to\infty]{P} 0 \tag{8.23}$$

both for $P = P^A_{(b,a,y)}$ and for $P = P^B_{(b,a,y)}$. By the same argument of Theorem 4.1 with $\mathcal{E} = \Gamma^B$ and $\Gamma = \Gamma^A$, we obtain

$$P^B_{(b,a,y)}\left[F_s \cdot \frac{R_t}{1 + R_t + \Gamma_t^B}\right] = \frac{\varphi^A(b, a, y)}{\varphi^B(b, a, y)} P^A_{(b,a,y)}\left[F_s \cdot \frac{\Gamma_t^B}{1 + R_t + \Gamma_t^B}\right]. \tag{8.24}$$

Letting $t \to \infty$, we obtain $P^A_{(b,a,y)}(\Gamma^B_\infty > 0) = 0$. Since $P^B_{(b,a,y)}(\Gamma^B_\infty > 0) = 1$, we obtain the desired result. □

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Appendix: Extension of Transformed Probability Measures

We discuss in general extension of the transformed probability measures given by local absolute continuity like (2.11). Recall that $\mathbb{D}$ is the space of càdlàg paths from $[0, \infty)$ to a locally compact separable metric space S and X is the coordinate process on $\mathbb{D}$.

Theorem A.1 *Let P be a probability measure on $(\mathbb{D}, \sigma(X))$ and let $(M_t)_{t\geq 0}$ be a non-negative martingale such that $P[M_t] = 1$ for all $t \geq 0$. Then there exists a unique probability measure Q on $(\mathbb{D}, \sigma(X))$ such that*

$$Q|_{\mathcal{F}_t^X} = M_t \cdot P|_{\mathcal{F}_t^X}, \quad t \geq 0, \tag{A.1}$$

where $\mathcal{F}_t^X = \sigma(X_s : s \leq t)$ is the natural filtration of X.

Proof Since $\bigcup_{t\geq 0} \mathcal{F}_t^X$ is a π-system generating $\sigma(X)$, uniqueness of Q follows immediately from Dynkin's π-λ theorem.

Let us prove existence of Q. For $n \in \mathbb{N}$, let $\mathbb{D}_n$ denote the space of càdlàg paths from $[n-1, n)$ to S, equipped with the σ-field $\mathcal{B}_n$ generated by the coordinate process on $\mathbb{D}_n$. We thus see that $\mathbb{D}$ is the product space of $\{\mathbb{D}_n\}$:

$$\mathbb{D} = \prod_{n=1}^{\infty} \mathbb{D}_n, \quad \sigma(X) = \sigma\left(\prod_{k=1}^{n} B_k \times \prod_{k=n+1}^{\infty} \mathbb{D}_k : B_1 \in \mathcal{B}_1, \ldots, B_n \in \mathcal{B}_n;\ n \in \mathbb{N}\right). \tag{A.2}$$

Let μ_n denote the law on $\mathbb{D}_1 \times \cdots \times \mathbb{D}_n$, the space of càdlàg paths from $[0, n)$ to S, of $(X_t)_{0 \le t < n}$ under $M_n \cdot P|_{\mathcal{F}_n^X}$. We then see that $\{\mu_n\}$ is a projective sequence:

$$\mu_{n+1}(\cdot \times \mathbb{D}_{n+1}) = \mu_n, \quad n \in \mathbb{N}. \tag{A.3}$$

We may apply Daniell's extension theorem (cf. [7, Theorem 6.14]) to see that there exists a sequence of random variables $\{\xi_n\}$ defined on a probability space $(\Omega', \mathcal{F}', P')$ such that ξ_n for each n takes values in $\mathbb{D}_n$ and the joint distribution of $(\xi_1, \dots, \xi_n)$ under P' for each n coincides with μ_n.

We now define Q by the law on $\mathbb{D}$ of $(\xi_1, \xi_2, \dots)$ under P'. For any $A \in \mathcal{F}_n^X$ for each $n \in \mathbb{N}$, we can find a subset B of $\mathbb{D}_1 \times \cdots \times \mathbb{D}_n$ which belongs to $\sigma\left(\prod_{k=1}^n B_k : B_1 \in \mathcal{B}_1, \dots, B_n \in \mathcal{B}_n\right)$ such that $A = \{(X_t)_{0 \le t < n} \in B\}$, so that we obtain

$$Q(A) = P'((\xi_1, \dots, \xi_n) \in B) = \mu_n(B) = P\left[M_n; (X_t)_{0 \le t < n} \in B\right] = P[M_n; A]. \tag{A.4}$$

We thus conclude that Q is as desired. □

References

1. M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Reprint of the 1972 edn (Dover Publications Inc., New York, 1992)
2. J. Bertoin, *Lévy Processes* (Cambridge University Press, Cambridge, 1996)
3. L. Chaumont, Conditionings and path decompositions for Lévy processes. Stoch. Process. Appl. **64**(1), 39–54 (1996)
4. L. Chaumont, R.A. Doney, On Lévy processes conditioned to stay positive. Electron. J. Probab. **10**(28), 948–961 (2005)
5. R.K. Getoor, *Markov Processes: Ray Processes and Right Processes*. Lecture Notes in Mathematics, vol. 440 (Springer, Berlin, New York, 1975)
6. P. Groeneboom, G. Jongbloed, J.A. Wellner, Integrated Brownian motion, conditioned to be positive. Ann. Probab. **27**(3), 1283–1303 (1999)
7. O. Kallenberg, *Foundations of Modern Probability*. Probability and its Applications, 2nd edn. (Springer, New York, 2002)
8. J. Najnudel, B. Roynette, M. Yor, *A Global View of Brownian Penalisations*. MSJ Memoirs, vol. 19 (Mathematical Society of Japan, Tokyo, 2009)
9. H. Pantí. On Lévy processes conditioned to avoid zero. ALEA Lat. Am. J. Probab. Math. Stat. **14**(2), 657–690 (2017)
10. C. Profeta, Some limiting laws associated with the integrated Brownian motion. ESAIM Probab. Stat. **19**, 148–171 (2015)
11. C. Profeta, K. Yano, Y. Yano, Local time penalizations with various clocks for one-dimensional diffusions. J. Math. Soc. Japan **71**(1), 203–233 (2019)
12. B. Roynette, P. Vallois, M. Yor, Limiting laws associated with Brownian motion perturbed by its maximum, minimum and local time. II. Stud. Sci. Math. Hungar. **43**(3), 295–360 (2006)
13. B. Roynette, P. Vallois, M. Yor, Limiting laws associated with Brownian motion perturbed by normalized exponential weights. I. Stud. Sci. Math. Hungar. **43**(2), 171–246 (2006)

14. B. Roynette, P. Vallois, M. Yor, Some penalisations of the Wiener measure. Jpn. J. Math. **1**(1), 263–290 (2006)
15. B. Roynette, M. Yor, *Penalising Brownian Paths*. Lecture Notes in Mathematics, vol. 1969 (Springer, Berlin, 2009)
16. S. Takeda, K. Yano, Local time penalizations with various clocks for Lévy processes. Preprint, arXiv:2203.08428
17. K. Yano, Two kinds of conditionings for stable Lévy processes, in *Probabilistic Approach to Geometry*. Advanced Studies in Pure Mathematics, vol. 57 (Mathematical Society of Japan, Tokyo, 2010), pp. 493–503
18. K. Yano, On harmonic function for the killed process upon hitting zero of asymmetric Lévy processes. J. Math. Ind. **5A**, 17–24 (2013)
19. K. Yano, Y. Yano, On h-transforms of one-dimensional diffusions stopped upon hitting zero, in *In Memoriam Marc Yor—Séminaire de Probabilités XLVII*. Lecture Notes in Mathematics, vol. 2137 (Springer, Cham, 2015), pp. 127–156
20. K. Yano, Y. Yano, M. Yor, Penalising symmetric stable Lévy paths. J. Math. Soc. Japan **61**(3), 757–798 (2009)
21. K. Yano, Y. Yano, M. Yor, Penalisation of a stable Lévy process involving its one-sided supremum. Ann. Inst. Henri Poincaré Probab. Stat. **46**(4), 1042–1054 (2010)
22. Y. Yano, A remarkable σ-finite measure unifying supremum penalisations for a stable Lévy process. Ann. Inst. Henri Poincaré Probab. Stat. **49**(4), 1014–1032 (2013)

Asymptotic Behavior of Spectral Functions for Schrödinger Forms with Signed Measures

Masaki Wada

Abstract Let $\{X_t\}_{t\geq 0}$ be the rotationally invariant α-stable process and define the Schrödinger forms by two methods. In one method, the perturbation is given by $-(\mu_0 + \lambda\nu)$ $(\lambda \geq 0)$, where both μ_0 and ν are positive and μ_0 is critical. In the other method, the perturbation is given by $\lambda\mu$ $(\lambda \in \mathbb{R})$, where μ is a critical signed measure. In this paper we consider the asymptotic behavior of the spectral functions defined from these Schrödinger forms. The results are consistent with the differentiability of the spectral functions given in Nishimori (Tohoku Math J 65:467–494, 2013, [5]) or Takeda and Tsuchida (Trans Amer Math 359:4031–4054, 2007, [7]).

Keywords Schrödinger form · Critical · Spectral function · Asymptotic behavior

Mathematics Subject Classification 60J45 · 60J40 · 35J10

1 Introduction

Let $\{X_t\}_{t\geq 0}$ be the transient, rotationally invariant α-stable process on $\mathbb{R}^d$ with the generator $(-\Delta)^{\alpha/2}$ $(0 < \alpha < 2)$. Denote by $\mathcal{K}_\infty^0$ a certain class of positive measure on $\mathbb{R}^d$. Let $(\mathcal{E}, \mathcal{F})$ be the Dirichlet form corresponding to $\{X_t\}_{t\geq 0}$ and suppose μ is a signed measure in $\mathcal{K}_\infty^0 - \mathcal{K}_\infty^0$, i.e. both positive and negative parts of μ belongs to $\mathcal{K}_\infty^0$. Then we define the Schrödinger form by

$$\mathcal{E}(u,u) - \lambda \int_{\mathbb{R}^d} u^2(x)\mu(dx), \quad \lambda \in \mathbb{R}$$

and the spectral function $C(\lambda)$ is given by

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$$C(\lambda) = -\inf \left\{ \mathcal{E}(u,u) - \lambda \int_{\mathbb{R}^d} u^2(x)\mu(dx) \;\middle|\; \int_{\mathbb{R}^d} u^2(x)dx = 1 \right\}. \tag{1}$$

There are quite a few of studies about the spectral functions. In [7], it is shown that the spectral function $C(\lambda)$ is differentiable if and only if $1 < d/\alpha \leq 2$. They especially show the differentiability at $\lambda = \lambda_0 = \sup\{\lambda \mid C(\lambda) = 0\}$. The measure $\lambda_0\mu$ is said to be critical. In [9] we gave a precise asymptotic behavior of $C(\lambda)$ at $\lambda = \lambda_0$ under the condition that $\mu = V \cdot m$ for $V \in C_0^\infty(\mathbb{R}^d)$ and the Lebesgue measure m. Nishimori [5] showed the differentiability of the spectral function defined differently from (1). More precisely, he considered the Schrödinger form whose perturbation consists of two different measures, i.e.

$$\mathcal{E}(u,u) - \int_{\mathbb{R}^d} u^2(x)\mu_0(dx) - \lambda \int_{\mathbb{R}^d} u^2(x)\nu(dx), \quad \lambda \in \mathbb{R},$$

where both μ_0 and ν belong to $\mathcal{K}_\infty^0$. He showed that the spectral function defined by

$$D(\lambda) = -\inf \left\{ \mathcal{E}(u,u) - \int_{\mathbb{R}^d} u^2(x)\mu_0(dx) - \lambda \int_{\mathbb{R}^d} u^2(x)\nu(dx) \;\middle|\; \int_{\mathbb{R}^d} u^2(x)dx = 1 \right\} \tag{2}$$

is differentiable if and only if $1 < d/\alpha \leq 2$.

In this paper, we give the asymptotic behavior of the spectral functions $C(\lambda)$ (resp. $D(\lambda)$) at $\sup\{\lambda \mid C(\lambda) = 0\}$ (resp. $\sup\{\lambda \mid D(\lambda) = 0\}$). We first give the asymptotic behavior of $D(\lambda)$ and the main result is as follows:

Theorem 1 *Suppose $\mu_0, \nu \in \mathcal{K}_\infty^0$ and μ_0 satisfies criticality. Then the spectral function $D(\lambda)$ admits the following asymptotics as $\lambda \downarrow 0$:*

$$D(\lambda) \sim \left(\frac{\alpha \Gamma(\frac{d}{2}) |\sin(\frac{d}{\alpha}\pi)| \langle h_0, h_0 \rangle_\nu}{2^{1-d}\pi^{1-\frac{d}{2}} \langle \mu_0, h_0 \rangle^2} \lambda \right)^{\frac{\alpha}{d-\alpha}} \quad (1 < d/\alpha < 2),$$

$$D(\lambda) \sim \frac{\Gamma(\alpha+1) \langle h_0, h_0 \rangle_\nu}{2^{1-d}\pi^{-\frac{d}{2}} \langle \mu_0, h_0 \rangle^2} \cdot \frac{\lambda}{\log \lambda^{-1}} \quad (d/\alpha = 2),$$

$$D(\lambda) \sim \frac{\langle h_0, h_0 \rangle_\nu \langle \mu_0, h_0 \rangle^2}{\langle h_0, h_0 \rangle_m \langle h_0, h_0 \rangle_{\mu_0}} \cdot \lambda \quad (d/\alpha > 2).$$

Here $A \sim B$ stands for $B/A \to 1$.

For the proof of this theorem, the precise asymptotic behavior of the resolvent kernel for $\{X_t\}_{t \geq 0}$ plays a crucial role. Thus the asymptotic behavior of $D(\lambda)$ contains precise multiple constants. Moreover, this result is the extension of that in [9]. Next we also consider the asymptotic behavior of $C(\lambda)$ and our main result is as follows:

Theorem 2 *Suppose μ is a critical signed measure in $\mathcal{K}_\infty^0 - \mathcal{K}_\infty^0$. Then the spectral function $C(\lambda)$ admits the following asymptotics as $\lambda \downarrow 1$:*

$$c_1(\lambda-1)^{\frac{\alpha}{d-\alpha}} \le C(\lambda) \le c_2(\lambda-1)^{\frac{\alpha}{d-\alpha}} \qquad (1 < d/\alpha < 2),$$
$$c_1\frac{\lambda-1}{\log(\lambda-1)^{-1}} \le C(\lambda) \le c_2\frac{\lambda-1}{\log(\lambda-1)^{-1}} \qquad (d/\alpha = 2),$$
$$c_1(\lambda-1) \le C(\lambda) \le c_2(\lambda-1) \qquad (d/\alpha > 2).$$

Here c_1 and c_2 are appropriate positive constants.

Unlike $D(\lambda)$, we cannot determine the precise multiple constants for $C(\lambda)$. This is because the measure μ contains both positive part μ^+ and negative part μ^-. For estimates of $C(\lambda)$, comparing the killed Dirichlet form $\mathcal{E}(u,u) + \int_{\mathbb{R}^d} u^2(x)\mu^-(dx)$ plays a crucial role. Differently from $\{X_t\}_{t\ge 0}$, we cannot describe the transition density function of the killed process by μ^- explicitly. However, this process can regard as an α-stable like process in the sense of Chen and Kumagai [2] through Doob's h-transformation. Thus, the transition density function of the killed process is comparable to that of $\{X_t\}_{t\ge 0}$ for all the time and the space, which enables us to obtain the two sided estimates in Theorem 2.

In the next section we review some basic properties such as the asymptotic behavior of the resolvent of $\{X_t\}_{t\ge 0}$, the classes of measures based on the Green kernel, and so on. In the Sect. 3, we first treat time changed processes by measures, and its Green operators based on [8]. Since these operators are compact, we can obtain the asymptotic behavior of the principal eigenvalue, and then prove Theorem 1. In the Sect. 4, we first show that killed processes by positive measures are α-stable like in the sense of [2]. Then we give the upper and lower estimates of the spectral function and prove Theorem 2. c_i's are unimportant positive constants which may vary from line to line.

2 Preliminaries

Let $\{X_t\}_{t\ge 0}$ be the rotationally invariant α-stable process on $\mathbb{R}^d$ with the generator $(-\Delta)^{\alpha/2}$ $(0 < \alpha < 2)$. The corresponding Dirichlet form is given by

$$\mathcal{E}(u,u) = \iint_{\mathbb{R}^d\times\mathbb{R}^d} (u(y)-u(x))^2 \frac{A_{d,\alpha}}{|x-y|^{d+\alpha}}dxdy, \quad A_{d,\alpha} = \frac{\alpha\cdot 2^{\alpha-2}\Gamma((d+\alpha)/2)}{\pi^{d/2}\Gamma(1-\alpha/2)}.$$

Denote by $p(t,x,y)$ the transition density function of $\{X_t\}_{t\ge 0}$. By the Fourier inverse transformation, we have an expression of $p(t,x,y)$ as follows:

$$p(t,x,y) = B_{d,\alpha}t^{-\frac{d}{\alpha}}g\left(\frac{|x-y|}{t^{1/\alpha}}\right), \qquad B_{d,\alpha} = \frac{\Gamma(d/\alpha)}{2^{d-1}\alpha\pi^{d/2}\Gamma(d/2)}, \tag{3}$$

where $g(w)$ is a function satisfying

$$g(0) = 1, \quad g(w) \asymp 1 \wedge w^{-d-\alpha}, \quad g(0) - g(w) \le c_2 w^2.$$

We assume the transience of $\{X_t\}_{t\ge 0}$, namely $\alpha < d$. Then we can consider the Green kernel $G(x, y)$ and define some classes of measures.

Definition 1 (1) A positive smooth measure μ is said to be in the Kato-class if for any $\epsilon > 0$ there exists $d_\epsilon > 0$ such that

$$\sup_{x\in\mathbb{R}^d} \int_{|x-y|\le d_\epsilon} G(x, y)\mu(dy) < \epsilon.$$

(2) A positive smooth measure μ is said to be Green-tight if for any $\epsilon > 0$ there exists $D_\epsilon > 0$ such that

$$\sup_{x\in\mathbb{R}^d} \int_{|y|\ge D_\epsilon} G(x, y)\mu(dy) < \epsilon.$$

(3) A positive smooth measure μ is said to be of finite 0-order energy if

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} G(x, y)\mu(dy)\mu(dx) < \infty.$$

In the sequel, the measure μ is said to be in the class $\mathcal{K}_\infty^0$ if μ satisfies the three conditions in Definition 1. We denote by $\mathcal{F}_e$ the extended Dirichlet space, i.e.

$$\mathcal{F}_e = \{u \mid \{u_n\}_{n=1}^\infty \subset \mathcal{F} \quad s.t. \lim_{n\to\infty} u_n = u \quad a.e.\}.$$

We also define the criticality of the measure as follows:

Definition 2 (1) A positive measure $\mu \in \mathcal{K}_\infty^0$ is said to be critical if it satisfies

$$\inf\left\{\mathcal{E}(u, u) \;\middle|\; u \in \mathcal{F}_e, \int_{\mathbb{R}^d} u^2 d\mu = 1\right\} = 1.$$

(2) A signed measure $\mu = \mu^+ - \mu^- \in \mathcal{K}_\infty^0 - \mathcal{K}_\infty^0$ is said to be critical if it satisfies

$$\inf\left\{\mathcal{E}(u, u) + \int_{\mathbb{R}^d} u^2 d\mu^- \;\middle|\; u \in \mathcal{F}_e, \int_{\mathbb{R}^d} u^2 d\mu^+ = 1\right\} = 1.$$

The following lemma is shown in [7] as the compact embedding theorem.

Lemma 1 *For $\mu \in \mathcal{K}_\infty^0$, the extended Dirichlet space $\mathcal{F}_e$ is compactly embedded into $L^2(\mu)$.*

For $\beta \ge 0$, $\mu \in \mathcal{K}_0^\infty$ and $f \in L^2(\mu)$, define

$$G_\beta(f\mu)(x) = \int_{\mathbb{R}^d} G_\beta(x, y) f(y)\mu(dy), \tag{4}$$

where $G_\beta(x, y)$ is the β-order resolvent kernel given by

$$G_\beta(x, y) = \int_0^\infty e^{-\beta t} p(t, x, y)dt.$$

If $\beta = 0$, we denote $G_0(x, y)$ by $G(x, y)$. Then $G_\beta(f\mu) \in \mathcal{F}_e$, which implies that (4) defines a compact operator on $L^2(\mu)$. In order to calculate the principal eigenvalue of the compact operators using the perturbation theory in [4], the asymptotic expansion of the resolvent kernel $G_\beta(x, y)$ as $\beta \to 0$ plays a crucial role. We close this section mentioning the asymptotic expansion of the resolvent kernel. For detail, see [10, Lemma 3.2].

Lemma 2 *The resolvent kernel $G_\beta(x, y)$ admits the asymptotic expansion as follows:*

$$G_\beta(x, y) = G(x, y) - \frac{2^{1-d}\pi^{1-d/2}}{\alpha\Gamma(d/2)|\sin(d\pi/\alpha)|}\beta^{\frac{d}{\alpha}-1} + E_\beta(x, y), \qquad (1 < d/\alpha < 2),$$

$$G_\beta(x, y) = G(x, y) - \frac{2^{1-d}\pi^{-d/2}}{\Gamma(\alpha+1)}\beta \log\beta^{-1} + E_\beta(x, y), \qquad (d/\alpha = 2),$$

$$G_\beta(x, y) = G(x, y) - \beta\int_0^\infty tp(t, x, y)dt + E_\beta(x, y), \qquad (d/\alpha > 2),$$

where $E_\beta(x, y)$ has the smaller order of β than each second term for fixed x and y. In particular, $G_\beta(x, y)$ admits the lower estimate as follows:

$$G_\beta(x, y) \geq G(x, y) - \frac{2^{1-d}\pi^{1-d/2}}{\alpha\Gamma(d/2)|\sin(d\pi/\alpha)|}\beta^{\frac{d}{\alpha}-1}, \qquad (1 < d/\alpha < 2),$$

$$G_\beta(x, y) \geq (1 - c_1\beta)G(x, y) - \frac{2^{1-d}\pi^{-d/2}}{\Gamma(\alpha+1)}\beta\log\beta^{-1} - c_2\beta, \qquad (d/\alpha = 2),$$

$$G_\beta(x, y) \geq G(x, y) - \beta\int_0^\infty tp(t, x, y)dt, \qquad (d/\alpha > 2).$$

3 The Spectral Function of a Schrödinger Form with Positive Measures

In this section, we treat two positive measures $\mu_0, \nu \in \mathcal{K}_\infty^0$ and assume the criticality of μ_0 based on the former of Definition 2. For $\lambda \geq 0$, we define the Schrödinger form

$$\mathcal{E}(u, u) - \int_{\mathbb{R}^d} u^2 d\mu_0 - \lambda\int_{\mathbb{R}^d} u^2 d\nu$$

and consider the spectral function

$$D(\lambda) = -\inf\left\{\mathcal{E}(u,u) - \int_{\mathbb{R}^d} u^2 d\mu_0 - \lambda \int_{\mathbb{R}^d} u^2 d\nu \;\middle|\; u \in \mathcal{F}, \int_{\mathbb{R}^d} u^2(x)dx = 1\right\}. \quad (5)$$

For $\lambda > 0$, it follows that $D(\lambda) > 0$ and there exists $h_\lambda \in \mathcal{F}$ attaining the infimum of (5) by [6, Theorem 2.8]. Moreover, there exists $h_0 \in \mathcal{F}_e$ such that

$$\mathcal{E}(h_0, h_0) - \int_{\mathbb{R}^d} h_0^2 d\mu_0 = 0.$$

We can characterize the function h_λ by the following lemma:

Lemma 3 *For $\lambda \geq 0$, h_λ is the principal eigenfunction of the compact operator on $L^2(\mu_0 + \lambda\nu)$*

$$K_{D(\lambda)} f(x) = \int_{\mathbb{R}^d} G_{D(\lambda)}(x,y) f(y)(\mu_0 + \lambda\nu)(dy), \quad f \in L^2(\mu_0 + \lambda\nu).$$

Moreover, h_λ admits the principal eigenvalue 1 for all $\lambda \geq 0$.

Proof First, we have the equality

$$\mathcal{E}_{D(\lambda)}(h_\lambda, h_\lambda) = \int_{\mathbb{R}^d} h_\lambda^2 d\mu_0 + \lambda \int_{\mathbb{R}^d} h_\lambda^2 d\nu = \mathcal{E}_{D(\lambda)}(G_{D(\lambda)} h_\lambda(\mu_0 + \lambda\nu), h_\lambda).$$

Then there exists a function $k_\lambda \in L^2(\mu_0 + \lambda\nu)$ such that $G_{D(\lambda)} h_\lambda(\mu_0 + \lambda\nu) = h_\lambda + k_\lambda$ m-a.e. and $\mathcal{E}_{D(\lambda)}(h_\lambda, k_\lambda) = 0$. Moreover, we see that $k_\lambda \equiv 0$, which implies the desired result. Indeed, the property of h_λ and the spectral function $D(\lambda)$ imply

$$\mathcal{E}_{D(\lambda)}(h_\lambda + k_\lambda, h_\lambda + k_\lambda) \geq \int_{\mathbb{R}^d} (h_\lambda + k_\lambda)^2 d(\mu_0 + \lambda\nu).$$

Since $\mathcal{E}_{D(\lambda)}(h_\lambda, k_\lambda) = 0$ and $\mathcal{E}_{D(\lambda)}(h_\lambda, h_\lambda) = \int_{\mathbb{R}^d} h_\lambda^2 d(\mu_0 + \lambda\nu)$, we have

$$\begin{aligned}
\mathcal{E}_{D(\lambda)}(k_\lambda, k_\lambda) &\geq 2\int_{\mathbb{R}^d} h_\lambda k_\lambda d(\mu_0 + \lambda\nu) + \int_{\mathbb{R}^d} k_\lambda^2 d(\mu_0 + \lambda\nu) \\
&= 2\mathcal{E}_{D(\lambda)}(G_{D(\lambda)} h_\lambda(\mu_0 + \lambda\nu), k_\lambda) + \int_{\mathbb{R}^d} k_\lambda^2 d(\mu_0 + \lambda\nu) \\
&= 2\mathcal{E}_{D(\lambda)}(h_\lambda + k_\lambda, k_\lambda) + \int_{\mathbb{R}^d} k_\lambda^2 d(\mu_0 + \lambda\nu) \\
&= 2\mathcal{E}_{D(\lambda)}(k_\lambda, k_\lambda) + \int_{\mathbb{R}^d} k_\lambda^2 d(\mu_0 + \lambda\nu).
\end{aligned}$$

Hence we have $\mathcal{E}_{D(\lambda)}(k_\lambda, k_\lambda) = 0$ and consequently $k_\lambda \equiv 0$. □

In the sequel we normalize h_λ on $L^2(\mu_0 + \lambda\nu)$. $K_{D(\lambda)}$ is the Green operator for time change process of $D(\lambda)$-killed process by $\mu_0 + \lambda\nu$. Moreover we can characterize the corresponding Dirichlet form $(\check{\mathcal{E}}_\lambda, \check{\mathcal{F}}_\lambda)$ by

$$\check{\mathcal{E}}_\lambda(u,u) = \mathcal{E}_{D(\lambda)}(P_\lambda f, P_\lambda f), \tag{6}$$

$$\check{\mathcal{F}}_\lambda = \{u \in L^2(\mu_0+\lambda\nu) \mid u = f \ \mu_0+\lambda\nu\text{-a.e. on } Y_\lambda \text{ for some } f \in \mathcal{F}_e\}, \tag{7}$$

$$P_\lambda f(x) = \mathbb{E}_x^{D(\lambda)}[f(X_{\sigma_{Y_\lambda}})], \tag{8}$$

where Y_λ stands for the fine support of the measure $\mu_0+\lambda\nu$ and σ_{Y_λ} is the first hitting time of Y_λ (cf. [3, Section 6.2]). The relation between $\mathcal{F}_e$ and $\check{\mathcal{F}}_\lambda$ is as follows: For $f \in \mathcal{F}_e$, we define the restriction map by $r(f) = f\,|_{Y_\lambda} \in \check{\mathcal{F}}_\lambda$. For $u \in \check{\mathcal{F}}_\lambda$ we define the extension map by $e(u) = P_\lambda f$. Clearly we have

$$\check{\mathcal{E}}(r(f), r(f)) \le \mathcal{E}(f,f), \qquad \check{\mathcal{E}}_\lambda(u,u) = \mathcal{E}_{D(\lambda)}(e(u), e(u)).$$

In particular, the principal eigenfunction h_λ can be extended the whole space of $\mathbb{R}^d$ by harmonic method and we also denote this function by h_λ.

We can also define another compact operator on $L^2(\mu_0)$ by

$$\tilde{K}_{D(\lambda)} f(x) = \int_{\mathbb{R}^d} G_{D(\lambda)}(x,y) f(y)\mu_0(dy), \quad f \in L^2(\mu_0)$$

Denote by $e_{D(\lambda)}$ and $\tilde{h}_\lambda$ the principal eigenvalue and eigenfunction for $\tilde{K}_{D(\lambda)}$. We also assume that the function $\tilde{h}_\lambda$ is normalized on $L^2(\mu_0)$. Since $\tilde{K}_{D(\lambda)}$ is the Green operator for time change process of $D(\lambda)$-killed process by μ_0, the same argument in $K_{D(\lambda)}$ enables us to obtain the corresponding Dirichlet form, harmonic extension of $\tilde{h}_\lambda$, and so on.

Now we compare two equations as follows:

$$G_{D(\lambda)}(h_\lambda\mu_0) + \lambda G_{D(\lambda)}(h_\lambda\nu) = h_\lambda, \tag{9}$$

$$G_{D(\lambda)}(\tilde{h}_\lambda\mu_0) = e_{D(\lambda)}\tilde{h}_\lambda. \tag{10}$$

Note that $h_\lambda, \tilde{h}_\lambda \in \mathcal{F}_e \subset L^2(\mu_0) \cap L^2(\nu)$ and consider the $L^2(\mu_0)$-inner product of them. Then we have

$$\begin{aligned}\langle h_\lambda, \tilde{h}_\lambda\rangle_{\mu_0} &= \langle G_{D(\lambda)} h_\lambda(\mu_0+\lambda\nu), \tilde{h}_\lambda\rangle_{\mu_0} = \langle G_{D(\lambda)}\tilde{h}_\lambda(\mu_0+\lambda\nu), h_\lambda\rangle_{\mu_0}\\ &= e_{D(\lambda)}\langle \tilde{h}_\lambda, h_\lambda\rangle_{\mu_0} + \lambda\langle G_{D(\lambda)}(\tilde{h}_\lambda\nu), h_\lambda\rangle_{\mu_0}\\ &= e_{D(\lambda)}\langle h_\lambda, \tilde{h}_\lambda\rangle_{\mu_0} + \lambda e_{D(\lambda)}\langle h_\lambda, \tilde{h}_\lambda\rangle_\nu.\end{aligned}$$

Hence we obtain

$$\frac{1-e_{D(\lambda)}}{\lambda e_{D(\lambda)}} = \frac{\langle h_\lambda, \tilde{h}_\lambda\rangle_\nu}{\langle h_\lambda, \tilde{h}_\lambda\rangle_{\mu_0}} \tag{11}$$

and the following lemma:

Lemma 4 *As* $\lambda \to 0$ *in (11), it follows that*

$$\lim_{\lambda\to 0}\frac{1-e_{D(\lambda)}}{\lambda e_{D(\lambda)}}=\frac{\langle h_0,h_0\rangle_\nu}{\langle h_0,h_0\rangle_{\mu_0}}. \tag{12}$$

Proof Since the compact operators $\tilde{K}_{D(0)}$ and $K_{D(0)}$ are the same, it follows that $\tilde{h}_0 = h_0$. By [8, Lemma 3.5], the eigenfunction $\tilde{h}_\lambda$ converges to h_0 $\mathcal{E}$-weakly as $\lambda \to 0$. Since $\mathcal{F}_e$ is compactly embedded into both $L^2(\mu_0)$ and $L^2(\nu)$, this convergence is strongly in $L^2(\mu_0)$ and $L^2(\nu)$.

In the same way of [8, Lemma 3.5], The eigenfunction h_λ converges to h_0 $\mathcal{E}$-weakly, so does $L^2(\mu_0)$-strongly and $L^2(\nu)$-strongly as $\lambda \to 0$. Thus we have (12).

Next we recall that the asymptotic behavior of principal eigenvalues e_β is given via that of the resolvent kernel, i.e. Lemma 2.

Lemma 5 *The principal eigenvalue e_β admits asymptotics as follows:*

$$\lim_{\beta\to 0}\frac{1-e_\beta}{\beta^{d/\alpha-1}}=\frac{2^{1-d}\pi^{1-\frac{d}{2}}\langle\mu_0,h_0\rangle^2}{\alpha\Gamma(\frac{d}{2})|\sin(\frac{d}{\alpha}\pi)|\langle h_0,h_0\rangle_{\mu_0}},\qquad (1<d/\alpha<2),$$

$$\lim_{\beta\to 0}\frac{1-e_\beta}{\beta\log\beta^{-1}}=\frac{2^{1-d}\pi^{-\frac{d}{2}}\langle\mu_0,h_0\rangle^2}{\Gamma(\alpha+1)\langle h_0,h_0\rangle_{\mu_0}},\qquad (d/\alpha=2),$$

$$\lim_{\beta\to 0}\frac{1-e_\beta}{\beta}=\frac{\langle h_0,h_0\rangle_m}{\langle\mu_0,h_0\rangle^2},\qquad (d/\alpha>2).$$

For details, see [10, Theorem 3.6].

(Proof of Theorem 1)

Combining the formula (12) and Lemma 5, we obtain the desired result. □

In particular, if $\nu = \mu_0 = V\cdot m$ for $V \in C_0(\mathbb{R}^d)$, we obtain [9, Theorem 1.1].

4 The Spectral Function of a Schrödinger Form with a Signed Measure

In this section we consider the Schrödinger form with a signed measure $\mu = \mu^+ - \mu^-$. For $\lambda \in \mathbb{R}$ define the Schrödinger form and its spectral function as follows;

$$C(\lambda)=-\inf\left\{\mathcal{E}(u,u)-\lambda\int_{\mathbb{R}^d}u^2(x)\mu(dx)\ \middle|\ \int_{\mathbb{R}^d}u^2(x)dx=1\right\}. \tag{13}$$

By [7], there exist a negative number λ_- and a positive number λ_+ such that $C(\lambda)=0$ is equivalent to $\lambda_- \le \lambda \le \lambda_+$. Without loss of the generality, we can assume $\lambda_+ = 1$, that is, the criticality of μ mentioned in the latter of Definition 2. We here consider the behavior of the spectral function $C(\lambda)$ at the neighborhood of $\lambda = 1$ by estimating the upper and lower bounds.

For the proof of the upper bound, we consider another spectral function

$$C_1(\lambda) = -\inf\left\{\mathcal{E}(u,u) + \int_{\mathbb{R}^d} u^2(x)\mu^-(dx) - \lambda\int_{\mathbb{R}^d} u^2(x)\mu^+(dx) \;\Big|\; \int_{\mathbb{R}^d} u^2(x)dx = 1\right\},$$

Noting the inequality

$$\mathcal{E}(u,u) - \lambda\int_{\mathbb{R}^d} u^2(x)\mu(dx) \geq \mathcal{E}(u,u) + \int_{\mathbb{R}^d} u^2(x)\mu^-(dx) - \lambda\int_{\mathbb{R}^d} u^2(x)\mu^+(dx)$$

for $\lambda \geq 1$, we have $C(\lambda) \leq C_1(\lambda)$. In the sequel we consider the upper estimate of $C_1(\lambda)$. The spectral function $C_1(\lambda)$ is based on the Schrödinger form whose killing term is fixed by μ^- for $\lambda \geq 0$. Thus, it is easier to obtain the upper estimate than doing it for $C(\lambda)$.

Denote by A_t^μ the additive functional in the Revuz correspondence with the measure μ. The following lemma is based on the Sect. 4 of [7].

Lemma 6 *The Schrödinger form* $\mathcal{E}(u,u) + \int_{\mathbb{R}^d} u^2(x)\mu^-(dx)$ *admits the gauge function*

$$a(x) = \mathbb{E}_x[\exp(-A_\infty^{\mu^-})].$$

Moreover, Jensen's inequality implies

$$a(x) = \mathbb{E}_x[\exp(-A_\infty^{\mu^-})] \geq \exp(-\mathbb{E}_x[A_\infty^{\mu^-}]) \geq \exp(-\sup_{x\in\mathbb{R}^d}\mathbb{E}_x[A_\infty^{\mu^-}]) > 0.$$

In particular, there exists a positive constant c_0 such that

$$c_0 \leq a(x) \leq 1. \tag{14}$$

The existence of the fundamental solution for the Schrödinger form $\mathcal{E}(u,u) + \int_{\mathbb{R}^d} u^2(x)\mu^-(dx)$ follows from [1] and denote it by $p^{\mu^-}(t,x,y)$. Using Doob's h-transformation as $h(x) = a(x)$, we can define the bilinear form on $L^2(a^2\cdot m)$ as follows:

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} (u(x)-u(y))^2 \frac{A_{d,\alpha}a(x)a(y)}{|x-y|^{d+\alpha}}dxdy. \tag{15}$$

The corresponding Markov process admits the transition density function $\dfrac{p^{\mu^-}(t,x,y)}{a(x)a(y)}$ with respect to the measure $a^2\cdot m$. Moreover, noting (14), we can regard (15) as a Dirichlet form on $L^2(a^2\cdot m) = L^2(\mathbb{R}^d)$. This Dirichlet form is an α-stable like process in the sense of [2]. Thus [2, Theorem 1.1] implies the two-sided estimate

$$c_1 p(t,x,y) \leq \frac{p^{\mu^-}(t,x,y)a(y)}{a(x)} \leq c_2 p(t,x,y).$$

Using (14) again, we have the following lemma:

Lemma 7 *The fundamental solution* $p^{\mu^-}(t,x,y)$ *admits the two-sided estimates*

$$c_3 p(t,x,y) \le p^{\mu^-}(t,x,y) \le c_4 p(t,x,y).$$

for some positive constants c_3 *and* c_4.

Denote by $G^{\mu^-}(x,y)$ the corresponding Green kernel. Lemma 7 implies that $G^{\mu^-}(x,y)$ is comparable to $G(x,y)$, i.e. there exist positive constants c_3 and c_4 such that

$$c_3 G(x,y) \le G^{\mu^-}(x,y) \le c_4 G(x,y).$$

Thus, for $\beta \ge 0$, we can define the compact operator on $L^2(\mu^+)$ by

$$L_\beta f(x) = \int_{\mathbb{R}^d} G_\beta^{\mu^-}(x,y) f(y) \mu^+(dy), \tag{16}$$

where

$$G_\beta^{\mu^-}(x,y) = \int_0^\infty e^{-\beta t} p^{\mu^-}(t,x,y)dt.$$

Denote by γ_β the principal eigenvalue of the operator L_β. Note that $\gamma_0 = 1$ by the criticality of μ and we obtain the asymptotic behavior of γ_β as follows:

Lemma 8 *The principal eigenvalue* γ_β *satisfies the asymptotic behavior as follows:*

$$c_5 \le \liminf_{\beta \to 0} \frac{1-\gamma_\beta}{l(\beta)},$$

where $l(\beta)$ *is defined by*

$$l(\beta) = \begin{cases} \beta^{d/\alpha - 1} & (1 < d/\alpha < 2) \\ \beta \log \beta^{-1} & (d/\alpha = 2) \\ \beta & (d/\alpha > 2) \end{cases} .$$

Proof First, we consider the upper estimate of $G_\beta^{\mu^-}(x,y)$. We obtain

$$\begin{aligned} G_\beta^{\mu^-}(x,y) &= G_0^{\mu^-}(x,y) - \int_0^\infty (1-e^{-\beta t}) p^{\mu^-}(t,x,y)dt \\ &\le G_0^{\mu^-}(x,y) - c_3 \int_0^\infty (1-e^{-\beta t}) p(t,x,y)dt \\ &= G_0^{\mu^-}(x,y) - c_3(G(x,y) - G_\beta(x,y)). \end{aligned} \tag{17}$$

By Lemma 2 and [10, Lemma 3.3], we see that γ_β is at most $1 - c_5 l(\beta)$ and the desired result follows.

Noting that $\gamma_{C_1(\lambda)} = \frac{1}{\lambda}$, we obtain the following lemma similarly to Theorem 1.

Lemma 9 *The spectral function* $C_1(\lambda)$ *admits the upper estimate*

$$C_1(\lambda) \leq \begin{cases} c_6(\lambda - 1)^{\frac{\alpha}{d-\alpha}} & (1 < d/\alpha < 2) \\ c_6(\lambda - 1)/\log(\lambda - 1)^{-1} & (d/\alpha = 2) \\ c_6(\lambda - 1) & (d/\alpha > 2) \end{cases} .$$

Since $C(\lambda) \leq C_1(\lambda)$, we obtain the upper bound for $C(\lambda)$.

(Proof of Theorem 2)

We complete the proof of Theorem 2 by showing the lower bound of $C(\lambda)$. Denote by $g_\lambda \in \mathcal{F}$ which attains the infimum of (13). We show that g_λ corresponds to the principal eigenfunction of the compact operator on $L^2(|\mu|)$ defined by

$$S_{C(\lambda)} f(x) = \int_{\mathbb{R}^d} G_{C(\lambda)}(x, y) f(y) \mu(dy). \tag{18}$$

Indeed, g_λ satisfies the equality

$$\mathcal{E}_{C(\lambda)}(g_\lambda, g_\lambda) = \lambda \int_{\mathbb{R}^d} g_\lambda^2(x) \mu(dx) = \lambda \mathcal{E}_{C(\lambda)}(G_{C(\lambda)}(g_\lambda \mu), g_\lambda)$$

and $G_{C(\lambda)} g_\lambda \mu = \frac{1}{\lambda} g_\lambda$ by the similar argument of Lemma 3. We can also obtain the equality $\frac{\langle S_{C(\lambda)} g_\lambda, g_\lambda \rangle_{|\mu|}}{\langle g_\lambda, g_\lambda \rangle_{|\mu|}} = \frac{1}{\lambda}$ and g_λ is an eigenfunction. We show that $\frac{1}{\lambda}$ is the maximal eigenvalue of $S_{C(\lambda)}$. Let γ and $\tilde{g}_\lambda$ be the maximal eigenvalue and its eigenfunction respectively. If $\gamma > \frac{1}{\lambda}$, the definition of the spectral function $C(\lambda)$ implies that

$$\begin{aligned} \mathcal{E}_{C(\lambda)}(\tilde{g}_\lambda, \tilde{g}_\lambda) \geq \lambda \int_{\mathbb{R}^d} \tilde{g}_\lambda^2(x) \mu(dx) &= \lambda \mathcal{E}_{C(\lambda)}(G_{C(\lambda)}(\tilde{g}_\lambda \mu), \tilde{g}_\lambda) \\ &= \lambda \gamma \mathcal{E}_{C(\lambda)}(\tilde{g}_\lambda, \tilde{g}_\lambda) > \mathcal{E}_{C(\lambda)}(\tilde{g}_\lambda, \tilde{g}_\lambda), \end{aligned}$$

and this is a contradiction.

In the sequel, we consider the lower estimate for the principal eigenvalue of the compact operator

$$S_\beta f(x) = \int_{\mathbb{R}^d} G_\beta(x, y) f(y) \mu(dy)$$

for simplicity. Denoting by s_β the principal eigenvalue of S_β, we have $s_{C(\lambda)} = \dfrac{1}{\lambda}$. Then, we obtain the inequality

$$s_\beta \geq \langle S_\beta h_0, h_0 \rangle_{|\mu|} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_\beta(x, y) h_0(y) \mu(dy) h_0(x) |\mu|(dx), \tag{19}$$

where $h_0(x)$ is the ground state (normalized in $L^2(|\mu|)$) of the critical Schrödignger form $\mathcal{E}(u, u) - \int_{\mathbb{R}^d} u^2(x)\mu(dx)$. Note that $h_0(x)$ is a positive function from [7, Section 4]. The right hand side of (19) admits the lower bound according to the value of d/α. From Lemma 2, we see that for $1 < d/\alpha < 2$, the resolvent kernel $G_\beta(x, y)$ has two-sided estimate

$$G(x, y) - c_7 \beta^{\frac{d}{\alpha} - 1} \leq G_\beta(x, y) \leq G(x, y).$$

Hence we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G_\beta(x, y) h_0(y) \mu(dy) h_0(x) |\mu|(dx) \\
&\geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (G(x, y) - c_7 \beta^{\frac{d}{\alpha} - 1} 1_E(y)) h_0(y) \mu(dy) h_0(x) |\mu|(dx) \\
&= 1 - c_7 \beta^{\frac{d}{\alpha} - 1} \langle \mu^+, h_0 \rangle \langle |\mu|, h_0 \rangle = 1 - c_8 \beta^{\frac{d}{\alpha} - 1},
\end{aligned}$$

where E is the support of the measure μ^+. This inequality implies

$$1 - \frac{1}{\lambda} \leq c_8 C(\lambda)^{\frac{d}{\alpha} - 1}.$$

Combining the upper estimate of $C(\lambda)$, we obtain the asymptotic behavior

$$C(\lambda) \asymp (\lambda - 1)^{\frac{\alpha}{d - \alpha}} \quad (\lambda \to 1).$$

Similarly, for $d/\alpha = 2$, the resolvent kernel $G_\beta(x, y)$ has two-sided estimate

$$(1 - c_9 \beta) G(x, y) - c_{10} \beta \log \beta^{-1} \leq G_\beta(x, y) \leq G(x, y)$$

from Lemma 2. Hence, for $\beta < e^{-1}$, we obtain

$$s_\beta \geq 1 - c_9 \beta - c_{11} \beta \log \beta^{-1} \geq 1 - c_{12} \beta \log \beta^{-1}$$

and

$$1 - \frac{1}{\lambda} \leq c_{12} C(\lambda) \log C(\lambda)^{-1}.$$

Combining the upper estimate of $C(\lambda)$, we obtain the asymptotic behavior

$$C(\lambda) \asymp \frac{(\lambda - 1)}{\log(\lambda - 1)^{-1}} \quad (\lambda \to 1).$$

Finally, for $d/\alpha > 2$, the resolvent kernel $G_\beta(x, y)$ has two-sided estimate

$$G(x, y) - \beta \int_0^\infty tp(t, x, y)dt \leq G_\beta(x, y) \leq G(x, y)$$

from Lemma 2. Hence we obtain

$$s_\beta \geq 1 - c_{13}\beta$$

and

$$1 - \frac{1}{\lambda} \leq c_{13}C(\lambda).$$

Combining the upper estimate of $C(\lambda)$, we obtain the asymptotic behavior

$$C(\lambda) \asymp \lambda - 1 \quad (\lambda \to 1). \qquad \square$$

Now we obtained the asymptotic behavior of the spectral function as $\lambda \downarrow \lambda_+$. The similar arguments enable us to obtain the asymptotic behavior as $\lambda \uparrow \lambda_-$, which admits the same order of λ as Theorem 2. By these asymptotic behaviors, we know that the differentiability of $C(\lambda)$ is equivalent to the condition $1 < d/\alpha \leq 2$. Therefore our results are consistent with those of [7, Theorem 6.2 and Remark 6.3].

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References

1. S. Albeverio, P. Blanchard, Z.-M. Ma, Feynman-Kac semigroups in terms of signed smooth measures, In Random partial differential equations (Ober-wolfach, 1989), Birkhäuser. Inter. Ser. Num. Math. **102**, 1–31 (1991)
2. Z.-Q. Chen, T. Kumagai, Heat kernel estimates for stable-like processes on *d*-sets. Stochastic Process. Their Appl. **108**, 27–62 (2003)
3. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, 2nd ed. (Walter de Gruyter GmbH & Co. KG, Berlin, 2011)
4. T. Kato, *Perturbation Theory for Operators* (Springer, Berlin, 1980)
5. Y. Nishimori, Large deviations for symmetric stable processes with Feynman-Kac functionals and its application to pinned polymers. Tohoku Math. J. **65**, 467–494 (2013)

6. M. Takeda, Large deviations for additive functionals of symmetric stable processes. J. Theor. Prob. **21**, 336–355 (2008)
7. M. Takeda, K. Tsuchida, Differentiability of spectral functions for symmetric α-stable processes. Trans. Amer. Math. **359**, 4031–4054 (2007)
8. M. Takeda, M. Wada, Large time asymptotics of Feynman-Kac functionals for symmetric stable processes. Math. Nachr. **289**, 2069–2082 (2016)
9. M. Wada, Asymptotic expansion of resolvent kernels and behavior of spectral functions for symmetric stable processes. J. Math. Soc. Japan **69**, 673–692 (2017)
10. M. Wada, Feynman-Kac penalization problem for critical measures of symmetric α-stable processes. Elect. Com. Probab. **21**(79), 14 (2016)