Bipartite Consensus for Discrete-Time Signed Networks Subject to Saturation Constraints

Baicheng Chen[1(✉)], Hui Yao[1], Mingjun Du[1], and Zhiguo Yan[1,2]

[1] School of Electrical Engineering and Automation, Qilu University of Technology (Shandong Academy of Science), Jinan 250353, Shandong, People's Republic of China
19736120011@sina.com

[2] School of Control Science and Engineering, Shandong University, Jinan 250061, Shandong, People's Republic of China

Abstract. The discrete-time bipartite consensus problem under signed network with saturation constraints is investigated. With the information of neighbors, a distributed control protocol is given. By the properties of the signed digraph, a model transformation is given. With the help of model transformation, the discrete bipartite consensus issue is converted into a discrete stability issue. By the Lyapunov stability theory, the discrete stability issues of the corresponding system are demonstrated. Through numerical simulation examples, we prove the validity of our results.

Keywords: Discrete-time · Undirected graph · Bipartite consensus · Saturation constraints · Signed network · Structural balance

1 Introduction

For the past few years, the cooperation between multiple individuals has attracted more and more attention in the industry and military, such as UAV formation control [1], vehicle formation [2] and distributed sensor networks [3], which leads to the distributed control problem about multi-agents. As a basic problem in the field of multi-agents, the consensus problem has attracted the attention of many scholars [4–6]. Network system is composed of a series of agents. Besides, consensus indicates that all state of network system tends to a common value. In the traditional network, there is only cooperative interaction among agents. Because of physical limitations, the actual control system always inevitably suffers from saturation constraints. For the consensus problem of traditional networks under saturation constraints, we refer to the work of these scholars [7–10].

Signed network is a new class of network system. Different traditional networks, the signed network has the existence of antagonistic interactions between agents. The bipartite consensus indicates that all state of the signed network tends to a common modulus. For a signed network, structural balance is a crucial

Z. Ren et al. (Eds.): *Proceedings of 2021 5th Chinese Conference on Swarm Intelligence and Cooperative Control*, LNEE 934, pp. 566–573, 2023.
https://doi.org/10.1007/978-981-19-3998-3_54

property. A signed network with structural balance can reach bipartite consensus in [11]. Many scholars have acquired lots of significant results in the bipartite consensus, such as bipartite consensus of high-order signed network [12], interval bipartite consensus [13], finite-time bipartite consensus [14], discrete-time consensus [15]. It is worth studying for the discrete-time bipartite consensus under the signed network with saturation constraints based on the above analysis.

Motivated by the above discussion, our goal is to investigate the discrete-time bipartite consensus problem of the signed network under actuator saturation. Based on [5], we designed a distributed control protocol. For a signed network subject to actuator saturation, the model of the signed network is given. The bipartite consensus problem can convert into the stability problem by model transformation. We demonstrate the stability of the corresponding system to show that the bipartite consensus can be reached. Besides, two numerical examples demonstrate our results.

The rest of this article involves five sections. In Sect. 2, some knowledge for signed digraphs is introduced. In Sect. 3, we propose the problem statements about the discrete-time bipartite consensus of the signed network under saturation constraints. In Sect. 4, we prove the bipartite consensus can be reached under the signed network with saturation constraints. In Sect. 5, we give two numerical simulation examples. In Sect. 6, the conclusion is given.

Notations: We denote $\mathscr{F}_n = \{1, 2, \cdots, n\}$ and $1_n = [1, 1, \cdots, 1]^T$. Let $diag\{\Delta_1, \Delta_2, \cdots, \Delta_n\}$ represents a diagonal matrix, and its ith diagonal elements is Δ_i. For a square matrix Q, Q is positive (respectively, negative) definite matrix if $Q > 0$ (respectively, < 0). For a real number a, we denote $|a|$ as the modulus of a and $sgn(a)$ as the sign of a. The saturation function is defined as $\delta_p(n) = sgn(n)min\{|n|, p\}$. The modulus and saturation function of the vector η are expressed as follows: $|\eta| = [|\eta_1|, |\eta_2|, \cdots, |\eta_n|]^T$ and $\delta(\eta) = [\delta(\eta_1), \delta(\eta_2), \cdots, \delta(\eta_n)]^T$.

2 Preliminaries

For a signed network, its signed digraph $\mathscr{G}$ includes the node set $\mathscr{V} = \{v_1, v_2, \cdots, v_n\}$, the edge set $\varepsilon = \{(v_i, v_j) : v_i, v_j \in \mathscr{V}\}$ and the adjacency matrix $A = [a_{ij}]_{n\times n}$. Besides, the element $a_{ij} \neq 0$ represents the existence of the directed edge (v_i, v_j). For a signed digraph $\mathscr{G}$, when its adjacency matrix satisfies $A = A^T$, then we call it signed undirected graph. The existence of a directed edge (v_i, v_j) indicates v_j can get information from v_i, where we call v_i as the neighbor of v_j. For any node v_j, the set of all its neighbors is denote by $N(j) = \{v_i : (v_i, v_j) \in \varepsilon\}$. In addition, $\Delta = diag\{\Delta_1, \Delta_2, \cdots, \Delta_n\}$ denotes in-degree matrix of $\mathscr{G}$. With the help of Δ, the Laplacian matrix L is defined by

$$L = \Delta - A = [l_{ij}]_{n\times n} \quad with \quad l_{ij} = \begin{cases} \sum_{m=1}^{n} |a_{im}|, & j = i \\ -a_{ij}, & j \neq i \end{cases}.$$

A directed path $\mathscr{P} = \{(v_{n0}, v_{n1}), (v_{n1}, v_{n2}), \cdots, (v_{nm-1}, v_{nm})\}$ from v_{n0} to v_{nm} consists of a series of edges, where $v_{n0}, v_{n1}, \cdots, v_{nm}$ are different nodes. When any node can connect to any other node via a direct path in the signed graph, we say it is strongly connected. Besides, if the signed digraph exists node is called root node v_m can connect any other node via a direct path, then the signed digraph contains a spanning tree. As an important property of signed digraph, we give the definition of structural balance.

Definition 1. *The all nodes of signed digraph $\mathscr{G}$ is separated into two sets $\mathscr{V}_q$ and $\mathscr{V}_p$, where $\mathscr{V}_q \cap \mathscr{V}_p = \emptyset$, $\mathscr{V}_q \cup \mathscr{V}_p = \mathscr{V}$. When $a_{ij} > 0$ for $v_i, v_j \in \mathscr{V}_q$ or $v_i, v_j \in \mathscr{V}_p$ and $a_{ij} < 0$ for $v_i \in \mathscr{V}_q$, $v_j \in \mathscr{V}_p$ or $v_i \in \mathscr{V}_p$, $v_j \in \mathscr{V}_q$, we call $\mathscr{G}$ is structurally balance. Otherwise, we call $\mathscr{G}$ is structurally unbalanced.*

In follow section, we introduce a leader v_{n+1} to construct a augmented signed digraph $\hat{\mathscr{G}} = (\hat{\mathscr{V}}, \hat{\varepsilon}, \hat{A})$, where $\hat{\mathscr{V}} = \mathscr{V} \cup v_{n+1}$, $\hat{\varepsilon} \subseteq \hat{\mathscr{V}} \times \hat{\mathscr{V}}$ and $\hat{A} = [a_{ij}]_{(n+1)\times(n+1)}$. From the definite of Laplacian matrix, $\hat{L}$ has the following construction

$$\hat{L} = \begin{bmatrix} L + |C| & -C1_n \\ 0 & 0 \end{bmatrix} \tag{1}$$

in which $C = diag\{c_1, c_2, \cdots, c_n\}$ whose element $c_i \neq 0$ means that exist a directed edge (v_i, v_{n+1}).

Assumption 1. *For a signed network, we assume its communication topology is undirected and its augmented signed digraph satisfies the condition of structural balance.*

Naturally, we can obtain the signed digraph $\mathscr{G}$ also is structurally balanced under the Assumption 1. From [11], we get a important properties.

Lemma 1. *[11] when a signed digraph $\mathscr{G}$ satisfies structural balance, we can get satisfies*

$$0 = \lambda_1(L) \leq \lambda_2(L) \leq \cdots \leq \lambda_{n+1}(L). \tag{2}$$

Obviously, under the Assumption 1, the augmented signed digraph $\hat{\mathscr{G}}$ contains a spinning tree and its the Laplacian matrix $\hat{L}$ satisfies the above lemma.

3 Problem Statements

In this section, we describe the discrete-time bipartite consensus problem for the signed network under saturation constraints. Consider the signed network consists of n agents. In addition, we denote $x_i(t)$ as the state of v_i and $x_{n+1}(k)$ represents the state of leader. The dynamics of the leader are expressed as follows

$$x_{n+1}(k) = \xi$$

where ξ is a constant. For a discrete-time signed digraph with saturation constraints, the dynamics of any node are described by

$$x_i(k+1) = x_i(k) + h\delta(u_i(k)),\ \forall i \in \mathscr{I}_n \tag{3}$$

where the sampling time $h > 0$, $u_i(k)$ is a distributed control protocol for v_i. The discrete-time signed network achieves bipartite consensus if

$$\lim_{t\to\infty} (|x_i(k)| - |x_j(k)|) = 0,\ \forall i, j \in \mathscr{I}_n. \tag{4}$$

Base on [5], a control protocol is given as follows

$$\begin{aligned} u_i(k) =& \frac{1}{\Delta_i + |c_i|}\{-\gamma \left[\sum_{j=1}^{N} |a_{ij}| (x_i(k) - \mathrm{sgn}(a_{ij})x_j(k)) + |c_i| (x_i(k) - \mathrm{sgn}(c_i)x_{n+1}(k))\right] \\ &+ \sum_{j=1}^{N} a_{ij} \frac{x_j(k+1) - x_j(k)}{h}\},\ \forall i \in \mathscr{I}_n \end{aligned} \tag{5}$$

where the in-degree of v_i is Δ_i, the control parameter $\gamma > 0$. Because the presence of the leader v_{n+1}, the bipartite consensus problem becomes the following description.

$$\lim_{t\to\infty} (|x_i(k)| - |x_{n+1}(k)|) = 0,\ \forall i \in \mathscr{I}_n. \tag{6}$$

Let $x(k) = [x_1(k), x_2(k), \cdots, x_n(k)]^T$ and $u(k) = [u_1(k), u_2(k), \cdots, u_n(k)]^T$. Then, (3) and (5) can rewrite as a compact form

$$\begin{cases} x(k+1) = x(k) + h\delta(u(k)) \\ u(k) = (\Delta + |C|)^{-1}\{-\gamma[(L + |C|)x(k) - C1_n x_{n+1}(k)] + A\dfrac{x(k+1) - x(k)}{h}\} \end{cases} \tag{7}$$

4 Main Results

In this section, we aim to solve the bipartite consensus issue for the discrete system (7). From Lemma 1, we can get the eigenvalues of $\hat{L}$ satisfy (2). The eigenvector associated with zero eigenvalue is expressed as $\alpha = [\alpha_1, \alpha_2, \cdots, \alpha_{n+1}]^T$. With the help of α, we construct the following conversion

$$z_i(k) = x(k) - \frac{\alpha_i}{\alpha_{n+1}} x_{n+1}(k),\ \forall i \in \mathscr{I}_n. \tag{8}$$

Denote $z(k) = [z_1(k), z_2(k), \cdots, z_n(k)]^T$, then (7) can be shown as follows

$$z(k+1) = z(k) + h\delta((\Delta + |C|)^{-1}[-\gamma(L + |C|)z(k) + A\frac{z(k+1) - z(k)}{h}]). \tag{9}$$

Obviously, $\lim_{t\to\infty} |x_i(k)| - |x_{n+1}(k)| = 0 \Leftrightarrow \lim_{t\to\infty} z_i(k) = 0$. Therefore, our objective (6) for the discrete system (7) is equivalent to the stability for the discrete system (9). Besides, we can easily get $L + |C|$ is positive definite from the construction (1).

Theorem 1. *Consider a discrete-time signed network whose communication topology satisfies the Assumption 1 under saturation constraints. When the control parameter γ satisfies $2h^{-1} > \gamma > 0$, the discrete system (7) can achieve bipartite consensus with control protocol (5).*

Proof. From (2), we can get that there is a zero eigenvalue in $\hat{L}$. With (8), we prove the stability for the discrete system (9) to illustrate the bipartite consensus for the discrete system (7) can be reached. We denote $M(k)$

$$M(k) = (\varDelta + |C|)^{-1}[-\gamma(L + |C|)z(k) + A\frac{z(k+1) - z(k)}{h}].$$

Then, we obtain

$$\begin{cases} z(k+1) = z(k) + h\delta(M(k)) \\ z(k) = -\gamma^{-1}(L + |C|)^{-1}(\varDelta + |C|)M(k) + \gamma^{-1}(L + |C|)^{-1}A\delta(M(k)) \end{cases}$$

Notice that $\varDelta + |C|$ is a diagonal matrix whose element $\varDelta_i + |c_i| > 0$ and $L + |C|$ is a positive definite matrix. From [5], we know all eigenvalues of the matrix $(\varDelta + |C|)^{-1}(L + |C|)$ are positive, which means the matrix $(\varDelta + |C|)^{-1}(L + |C|)$ is invertible. We construct the following Lyapunov function

$$V(k) = z(k)^T(L + C)z(k).$$

Because $L + |C|$ is positive definite, then we can get $V(k) > 0$.

$$\begin{aligned} \triangle V(k+1) &= V(k+1) - V(k) \\ &= z(k+1)^T(L + |C|)z(k+1) - z(k)^T(L + |C|)z(k) \\ &= (z(k) + h\delta(M(k)))^T(L + |C|)(z(k) + h\delta(M(k)) - z(k)^T(L + |C|)z(k) \\ &= \; - h(2\gamma^{-1}M(k)^T(\varDelta + |C|)\delta(M(k)) + (2\gamma^{-1} - h)\delta(M(k))^T L\delta(M(k)) \\ &\quad - 2\gamma^{-1}\delta(M(k))^T\varDelta\delta(M(k)) - h\delta(M(k))^T|C|\delta(M(k))) \end{aligned}$$

Based on above analysis, if the control parameter γ satisfies $2h^{-1} > \gamma > 0$, then we can get

$$\begin{cases} 2\gamma^{-1}[M(k)^T(\varDelta + |C|)\delta(M(k)) - \delta(M(k))^T\varDelta\delta(M(k))] - h\delta(M(k))^T|C|\delta(M(k))) > 0 \\ (2\gamma^{-1} - h)\delta(M(k))^T L\delta(M(k)) \geq 0 \end{cases}$$

Apparently, the sampling time $h > 0$. It means

$$\triangle V(k) < 0.$$

The above analysis indicates the discrete system (9) can achieve stability. Meanwhile, it illustrates the bipartite consensus for the discrete system (7) can be achieved.

5 Simulations

Example 1. For the discrete system (7) under the signed digraph $\mathscr{G}$, its communication topology of $\mathscr{G}$ and its augmented signed digraph $\hat{\mathscr{G}}$ are represented by Fig. 1 and Fig. 2 respectively. We provide the initial state of all nodes by

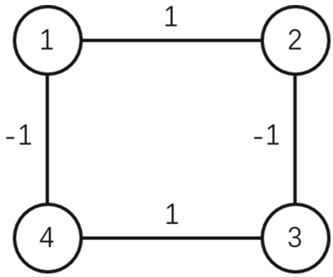

Fig. 1. The communication topology of $\mathscr{G}$.

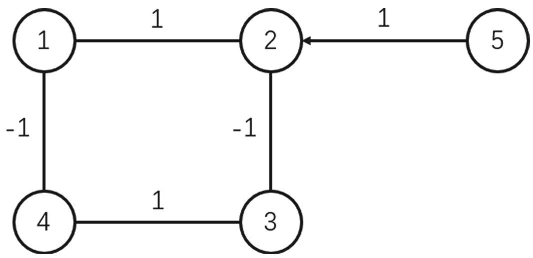

Fig. 2. The communication topology of $\hat{\mathscr{G}}$.

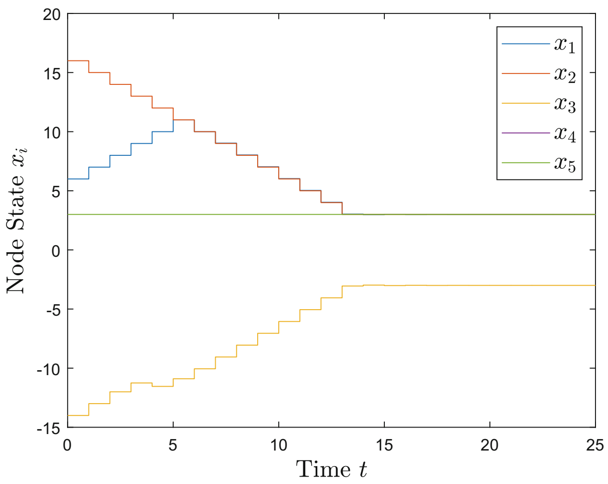

Fig. 3. Example 1 of actuator saturation: $h = 1$ and $\gamma = 1.5$.

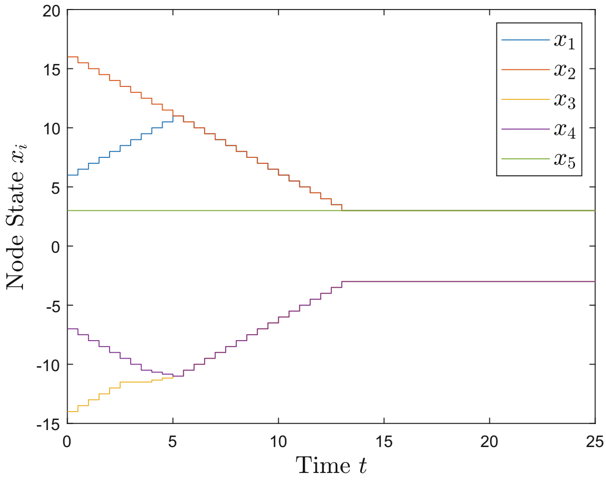

Fig. 4. Example 2 of actuator saturation: $h = 0.5$ and $\gamma = 3.5$.

$$x(0) = [6, 16, -14, -7].$$

Besides, we select $x_5 = 3$ as the state of the leader. From the Theorem 1, the discrete system (7) can reach bipartite consensus if the control parameter γ and sampling time h satisfy $2h^{-1} > \gamma > 0$. Therefore, we firstly select $\gamma = 1.5$ and $h = 1$. The Fig. 3 displays the dynamics of the discrete system (7) with the control protocol (5).

Example 2. In this example, we select the same initial state as the example 1. Different with the example 1, we select $\gamma = 3.5$ and $h = 0.5$. Similarly, the dynamics of the discrete system (7) is shown by Fig. 4.

It is shown that the discrete system (7) can reach our objective (6) under the distributed control protocol (5) in Fig. 3 and Fig. 4.

6 Conclusions

We have investigated the discrete-time bipartite consensus for signed networks with saturation constraints. To acquire our objective, we have proposed a control protocol for each node. With the right eigenvector associated with the augmented Laplace matrix, the bipartite consensus problem has been transformed into a stability problem. The result has shown that a connected and undirected signed network whose the augment signed digraph satisfies structural balance under saturation constraints can achieve bipartite consensus. Moreover, we have illustrated the validity of our result through two numerical simulation examples.

Acknowledgments. This work was supported in part by the National Natural Science Foundation of China (No. 61877062, No. 61977043).

References

1. Wang, X., Yadav, V., Balakrishnan, S.N.: Cooperative UAV formation flying with obstacle/collision avoidance. IEEE Trans. Control Syst. Technol. **15**(4), 672–679 (2007)
2. Fax, J.A., Murra, R.M.: Information flow and cooperative control of vehicle formations. IEEE Trans. Autom. Control **49**(9), 1465–1476 (2004)
3. Su, X., Wu, L., Shi, P.: Senor networks with random link failures: distributed filtering for TCS fuzzy systems. IEEE Trans. Industr. Inf. **9**(3), 1739–1750 (2013)
4. Olfati-Saber, R., Murray, R.M.: Consensus problems in networks of agents with switching topology and time-delays. IEEE Trans. Autom. Control **49**(9), 1520–1533 (2004)
5. Liu, S., Xie, L.: Synchronization of multi-agent systems with delayed control input information from neighbors. Automatica **47**, 2152–2164 (2011)
6. Ren, W.: Multi-vehicle consensus with a time-varying reference state. Syst. Control Lett. **56**, 474–483 (2007)
7. Ren, W.: On consensus algorithms for double-integrator dynamics. IEEE Trans. Autom. Control **53**(6), 1503–1509 (2008)
8. Meng, Z., Zhao, Z., Lin, Z.: On global leader-following consensus of identical linear dynamic systems subject to actuator saturation. Automatica **62**, 132–142 (2013)
9. Su, H., Chen, M.Z.Q., Lam, J., Lin, Z.: Semi-global leader-following consensus of linear multi-agent systems with input saturation via low gain feedback. IEEE Trans. Circ. Syst. **60**(7), 1881–1889 (2013)
10. Wang, Q., Peng, C., Gao, H., Basin, M.: Global consensus of single-integrator agents subject to saturation constraints. IET Control Theory Appl. **8**(9), 765–771 (2014)
11. Altafini, C.: Consensus problems on networks with antagonistic interactions. IEEE Trans. Autom. Control **58**(4), 935–946 (2013)
12. Valcher, M.E., Misra, P.: On the consensus and bipartite consensus in high-order multi-agent dynamical systems with antagonistic interactions. Syst. Control Lett. **66**, 94–103 (2014)
13. Meng, D., Du, M., Jia, Y.: Interval bipartite consensus of networked agents associated with signed digraphs. IEEE Trans. Autom. Control **61**(12), 3755–3770 (2016)
14. Lu, J., Wang, Y., Shi, X., Cao, J.: Finite-time bipartite consensus for multiagent systems under detail-balanced antagonistic interactions. Automatica **51**(6), 3867–3875 (2021)
15. Meng, Z., Shi, G., Johansson, K.H., Cao, M., Hong, Y.: Behaviors of networks with antagonistic interactions and switching topologies. IEEE Trans. Autom. Control **73**, 110–116 (2016)