# Two Remarks on Generalized Skew Derivations in Prime Rings

**Vincenzo De Filippis and Francesco Rania**

**Abstract** Let $R$ be a prime ring of characteristic different from 2, $Q_r$ its right Martindale quotient ring, $F$ and $G$ two non-zero generalized skew derivations of $R$, associated with the same automorphism $\alpha$ and commuting with $\alpha$. In this work we describe all possible forms of $F$ and $G$ in the following two cases: (a) there exist $a, b \in Q_r$ and a non-central Lie ideal $L$ of $R$ such that $aF(x)b = 0$, for all $x \in L$; (b) there exist $a_1, a_2, b_1, b_2 \in Q_r$ such that $a_1F(x)b_1 + a_2G(x)b_2 = 0$, for all $x \in R$.

Let $R$ be a prime ring with center $Z(R)$, $Q_r$ its *right Martindale quotient ring*, $C$ the center of $Q_r$, usually called *extended centroid* of $R$ (see [1] for more details).

An additive mapping $d\colon R \longrightarrow R$ is said to be a *derivation* of $R$ if

$$d(xy) = d(x)y + xd(y)$$

for all $x, y \in R$. An additive mapping $F\colon R \longrightarrow R$ is called a *generalized derivation* of $R$ if there exists a derivation $d$ of $R$ such that

$$F(xy) = F(x)y + xd(y)$$

for all $x, y \in R$.

Let $R$ be an associative ring and $\alpha$ be an automorphism of $R$. An additive mapping $d\colon R \longrightarrow R$ is said to be a *skew derivation* of $R$ if

$$d(xy) = d(x)y + \alpha(x)d(y)$$

V. De Filippis (✉)
De Filippis: Department of Engineering, University of Messina, Messina, Italy
e-mail: defilippis@unime.it

F. Rania
DIGES, Magna Graecia University, Catanzaro, Italy
e-mail: raniaf@unicz.it

 
M. Ashraf et al. (eds.), *Algebra and Related Topics with Applications*,
Springer Proceedings in Mathematics & Statistics 392,
https://doi.org/10.1007/978-981-19-3898-6_8

for all $x, y \in R$. The automorphisms $\alpha$ is called an *associated automorphism* of $d$. An additive mapping $F: R \longrightarrow R$ is called a *generalized skew derivation* of $R$ if there exists a skew derivation $d$ of $R$ with associated automorphism $\alpha$ such that

$$F(xy) = F(x)y + \alpha(x)d(y)$$

for all $x, y \in R$. In this case, $d$ is called an *associated skew derivation* of $F$ and $\alpha$ is called an *associated automorphism* of $F$.

In this paper we investigate some generalized differential identities involving generalized skew derivations of a prime ring of characteristic different from 2.

In [2, Theorem 2.1] Brešar describes the form of three derivations $d, g, h$ of a prime ring $R$ satisfying the condition $d(x) = ag(x) + h(x)b$, for any $x \in R$, where $a, b \in R \setminus Z(R)$. As a consequence he also studies the case when $ag(x) + h(x)b = 0$, for any $x \in R$ [2, Corollary 2.4]. More precisely, in this last case he concludes that there exists $\lambda \in C$ such that $g(x) = [\lambda b, x]$ and $h(x) = [\lambda a, x]$, for any $x \in R$. The results by Brešar extend a theorem of Herstein contained in [12].

Following this line of investigation, J.-C. Chang generalizes the previous results to the case of both skew derivation (see [3]) and generalized skew derivations (see [4]).

Here we would like to continue the study of linear differential identities having the same flavor of the above-cited ones, and involving generalized skew derivations. In this sense, the main goal of the present paper is to prove the following theorems:

**Theorem 1** *Let $R$ be a prime ring of characteristic different from 2, $F$ a non-zero generalized skew derivation of $R$, with associated automorphism $\alpha$, and $a, b$ non-zero elements of $Q_r$ such that*

$$aF(w)b = 0 \quad \forall w \in L.$$

*Then one of the following holds:*

*(a) the associated automorphism $\alpha$ is not inner and there exist $c, u \in Q_r$ be such that $F(x) = cx + \alpha(x)u$, for any $x \in R$, with $ac = ub = 0$;*
*(b) there exist $c, u, q \in Q_r$ and $\lambda \in C$ such that $F(x) = cx + \alpha(x)u$, for any $x \in R$, where $\alpha(x) = qxq^{-1}$, for any $x \in R$, with $a(c + \lambda q) = 0$ and $(\lambda + q^{-1}u)b = 0$.*

**Theorem 2** *Let $R$ be a prime ring of characteristic different from 2, $F, G$ two non-zero generalized skew derivations of $R$, associated with the same automorphism $\alpha$ and commuting with $\alpha$. Let $a_1, a_2, b_1, b_2$ be non-zero elements of $Q_r$ such that*

$$a_1F(x)b_1 + a_2G(x)b_2 = 0 \quad \forall x \in R.$$

*Then one of the following cases must occur*

*(a) There exist $p, u, v, w, q \in Q_r$, where $q$ is an invertible element, such that $F(x) = px + qxq^{-1}u$, $G(x) = vx + qxq^{-1}w$, for any $x \in R$, and one of the following holds:*

1. *there exist* $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in C$ *such that* $b_1 = \alpha_1 b_2 + \alpha_2 q^{-1} w b_2$, $q^{-1} u b_1 = \alpha_3 b_2 + \alpha_4 q^{-1} w b_2$ *and* $\alpha_1 a_1 p + \alpha_3 a_1 q + a_2 v = \alpha_2 a_1 p + \alpha_4 a_1 q + a_2 q = 0$*;*
2. *there exist* $\lambda, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in C$ *such that* $q^{-1} w b_2 = \lambda b_2$, $b_1 = (\alpha_1 + \lambda\alpha_2) b_2$, $q^{-1} u b_1 = (\alpha_3 + \lambda\alpha_4) b_2$ *and* $(\alpha_1 + \lambda\alpha_2) a_1 p + (\alpha_3 + \lambda\alpha_4) a_1 q + a_2 (v + \lambda q) = 0$*;*
3. *there exist* $0 \neq \lambda \in C$ *and* $\beta_1, \beta_2 \in C$ *such that* $a_1 p = \lambda a_1 q$, $a_2 v = \beta_1 a_1 q$, $a_2 q = \beta_2 a_1 q$ *and* $\lambda b_1 + q^{-1} u b_1 + \beta_1 b_2 + \beta_2 q^{-1} w b_2 = 0$*;*
4. *there exist* $0 \neq \lambda \in C$ *and* $\mu, \eta \in C$ *such that* $a_1 p = \lambda a_1 q$, $a_2(v + \mu q) = \eta a_1 q$, $(\lambda + q^{-1} u) b_1 = -\eta b_2$ *and* $q^{-1} w b_2 = \mu b_2$.

(b) *There exist* $p, u, v, w \in Q_r$ *such that* $F(x) = px + \alpha(x) u$, $G(x) = vx + \alpha(x) w$, *for any* $x \in R$, *and one of the following holds:*

5. $a_1 p = a_2 v = u b_1 = w b_2 = 0$*;*
6. $a_1 p = a_2 v = 0$ *and there exists* $\mu \in C$ *such that* $u b_1 = \mu w b_2$ *and* $a_2 = -\mu a_1$*;*
7. $u b_1 = w b_2 = 0$ *and there exists* $\lambda \in C$ *such that* $a_1 p = \lambda a_2 v$ *and* $b_2 = -\lambda b_1$*;*
8. *there exist* $\lambda, \mu \in C$ *such that* $a_1 p = \lambda a_2 v$, $b_2 = -\lambda b_1$, $u b_1 = \mu w b_2$ *and* $a_2 = -\mu a_1$.

(c) *There exist* $p, v \in Q_r$ *and* $d, \delta$ *skew derivations of* $R$ *such that* $F(x) = px + d(x)$*,* $G(x) = vx + \delta(x)$*, for all* $x \in R$*, and one of the following holds:*

9. *there exist* $\vartheta \in C$ *and* $0 \neq \eta \in C$ *such that* $\delta(x) = \eta d(x)$*, for any* $x \in R$*,* $a_1 p = \vartheta a_2 v$, $b_2 = -\vartheta b_1$, *and* $a_1 = \vartheta\eta a_2$*;*
10. *there exist* $0 \neq \vartheta \in C$*,* $0 \neq \eta \in C$ *and* $p_0 \in Q_r$ *such that* $\delta(x) = p_0 x - \alpha(x) p_0 + \eta d(x)$*, for any* $x \in R$*,* $a_1 = \vartheta\eta a_2$*,* $b_2 = -\vartheta b_1$*,* $p_0 b_1 = 0$ *and* $\eta a_2 p - a_2(v + p_0) = 0$*;*
11. *there exist* $\vartheta \in C$*,* $0 \neq \eta \in C$ *and* $p_0, q \in Q_r$*, where* $q$ *is an invertible element, such that* $\delta(x) = p_0 x - q x q^{-1} p_0 + \eta d(x)$*, for any* $x \in R$*,* $a_1 = \vartheta\eta a_2$*,* $b_2 = -\vartheta b_1$*,* $q^{-1} p_0 b_1 = \vartheta b_1$ *and* $\eta a_2 p - a_2(v + p_0) + \vartheta a_2 q = 0$.

Let us recall some basic facts which will be useful in the sequel.

**Fact 1** Let $R$ be a prime ring, then the following statements hold:

(a) Every generalized derivation of $R$ can be uniquely extended to $Q_r$ [14, Theorem 3].
(b) Any automorphism of $R$ can be uniquely extended to $Q_r$ [7, Fact 2].
(c) Every generalized skew derivation of $R$ can be uniquely extended to $Q_r$ [4, Lemma 2].

**Fact 2** A generalized skew derivation having associated automorphism $\alpha$ and skew derivation $d$ assumes the following form:

$$F(x) = ax + d(x) \tag{1}$$

for all $x \in R$ (see [4, Lemma 2], [5, Theorem 3.1 and Corollary 3.2]).

We also need to recall some well-known results on generalized polynomial identities for prime rings involving skew derivations and automorphisms.

**Fact 3** ([9]) If $\Phi(x_i, D(x_i))$ is a generalized polynomial identity for $R$, where $R$ is a prime ring and $D$ is an outer skew derivation of $R$, then $R$ also satisfies the generalized polynomial identity $\Phi(x_i, y_i)$, where $x_i$ and $y_i$ are distinct indeterminates.

If $\Phi(x_i, D(x_i), \alpha(x_i))$ is a generalized polynomial identity for a prime ring $R$, $D$ is an outer skew derivation of $R$ and $\alpha$ is an outer automorphism of $R$, then $R$ also satisfies the generalized polynomial identity $\Phi(x_i, y_i, z_i)$, where $x_i$, $y_i$, and $z_i$ are distinct indeterminates.

**Fact 4** ([13, Theorem 6.5.9, page 365]) Let a prime ring $R$ obey a polynomial identity of the type $f(x_j^{\alpha_i \Delta_k}) = 0$, where $f(z_j^{i,k})$ is a generalized polynomial with the coefficients from $Q_r$, $\Delta_1, \ldots, \Delta_n$ are mutually different correct words from a reduced set of skew derivations commuting with all the corresponding automorphisms, and $\alpha_1, \ldots, \alpha_m$ are mutually outer automorphisms. In this case the identity $f(z_j^{i,k}) = 0$ is valid on $Q_r$.

**Fact 5** ([8, Theorem 1]) Let $R$ be a prime ring and $I$ be a two-sided ideal of $R$. Then $I$, $R$, and $Q_r$ satisfy the same generalized polynomial identities with coefficients in $Q_r$ (see [6]). Furthermore, $I$, $R$, and $Q_r$ satisfy the same generalized polynomial identities with automorphisms.

**Fact 6** ([9, Theorem 2]) Let $R$ be a prime ring and $I$ be a two-sided ideal of $R$. Then $I$, $R$, and $Q_r$ satisfy the same generalized polynomial identities with a single skew derivation.

In the sequel, $R$ will be a non-commutative ring of characteristic different from 2, $F$ and $G$ two non-zero generalized skew derivations of $R$, associated with the same automorphism $\alpha$ and commuting with $\alpha$.

## 1 Annihilating Condition for a Single Generalized Skew Derivation

In this second section our aim will be to prove Theorem 1. More precisely, let $F$ be a generalized skew derivation of $R$ and $a, b$ are non-zero elements of $R$ such that

$$aF(w)b = 0 \quad \forall w \in L \quad \text{a non-central Lie ideal of} \quad R. \tag{2}$$

The study of this result will be useful for the proof of our main Theorem (i.e., Theorem 2).

We permit the following:

**Lemma 1** *Let $R$ be a prime and $a_i, b_i \in U$, for $1 \le i \le n$. If $\sum_{i=1}^{n} a_i[x, y]b_i = 0$, for all $x, y \in R$. If $a_i \neq 0$ for some $i$, then $b_1, \ldots, b_n$ are $C$-dependent. Similarly, if $b_i \neq 0$ for some $i$, then $a_1, \ldots, a_n$ are $C$-dependent.*

***Proof*** The result follows easily from [15, Lemma 2.2] and [16, Lemma 1].

**Lemma 2** *Let $c, u \in Q_r$ be such that $F(x) = cx + \alpha(x)u$, for any $x \in R$. If*

$$aF([r_1, r_2])b = 0 \quad \forall r_1, r_2 \in R. \tag{3}$$

*then one of the following holds:*

*(a)* $ac = ub = 0$*;*
*(b)* *there exist $q \in Q_r$ and $\lambda \in C$ such that $\alpha(x) = qxq^{-1}$, for any $x \in R$, with $a(c + \lambda q) = 0$ and $(\lambda + q^{-1}u)b = 0$.*

***Proof*** By our assumption $R$ satisfies

$$a\Big(c[x_1, x_2] + \alpha([x_1, x_2])u\Big)b. \tag{4}$$

We consider firstly the case $\alpha(x) = qxq^{-1}$, for any $x \in R$, where $q \in Q_r$ is an invertible element. In this case, by (4), $R$ satisfies

$$a\Big(c[x_1, x_2] + q[x_1, x_2]q^{-1}u\Big)b. \tag{5}$$

A direct application of Lemma 1 leads to conclusion (b).

Therefore we may assume that $\alpha$ is not an inner automorphism of $Q_r$. Thus, by (4) and Fact 3, $R$ satisfies the generalized polynomial identity

$$a\Big(c[x_1, x_2] + [y_1, y_2]u\Big)b. \tag{6}$$

In particular $R$ satisfies both the blended components $ac[x_1, x_2]b$ and $a[y_1, y_2]ub$. Since $a \neq 0$ and $b \neq 0$ and by the primeness of $R$, we get the required conclusion $ac = ub = 0$.

***Proof*** (Proof of Theorem 1) By Fact 2, $F(x) = cx + d(x)$ for all $x \in R$, where $c \in Q_r$ and $d$ is the skew derivation associated with $F$.

Since $L$ is not central and $char(R) \neq 2$, it is well known that there exists a nonzero ideal $I$ of $R$ such that $0 \neq [I, R] \subseteq L$ (see [11, pages 4–5]). Therefore, by (2), the ideal $I$ satisfies $aF([x_1, x_2])b$. Since $R$ and $I$ satisfy the same generalized identities with automorphisms and skew derivations, we may assume that $R$ also satisfies $aF([x_1, x_2])b$, that is

$$a\Big(c[x_1, x_2] + d([x_1, x_2])\Big)b. \tag{7}$$

In case $d$ is an inner skew derivation of $R$, the conclusion follows from Lemma 2. Then we may assume that $d$ is not inner and prove that a contradiction follows. Expansion of (7) says that $R$ satisfies

$$a\Big(c[x_1, x_2] + d(x_1)x_2 + \alpha(x_1)d(x_2) - d(x_2)x_1 - \alpha(x_2)d(x_1)\Big)b. \tag{8}$$

Since $d$ is not inner and by Fact 3, (8) implies that $R$ satisfies

$$a\Big(c[x_1, x_2] + y_1x_2 + \alpha(x_1)y_2 - y_2x_1 - \alpha(x_2)y_1\Big)b \tag{9}$$

and in particular $R$ satisfies

$$a\Big(y_1x_2 - \alpha(x_2)y_1\Big)b. \tag{10}$$

If $\alpha$ is outer, relation (10) implies that $R$ satisfies

$$a\Big(y_1x_2 - z_2y_1\Big)b$$

and, in particular, $a[r_1, r_2]b = 0$, for any $r_1, r_2 \in R$. It follows that either $a = 0$ or $b = 0$, which contradicts the assumption $a, b \neq 0$.

On the other hand, if $\alpha(x) = qxq^{-1}$, where $q$ is an invertible element of $Q_r$, one may replace in (main-8) $y_1$ with $qx_1$. Hence $R$ satisfies $aq[x_1, x_2]b$. Since $q$ is invertible, once again the contradiction that either $a = 0$ or $b = 0$ follows.

## 2 Annihilating Conditions for Two Generalized Skew Derivations

We conclude our paper giving the description of two generalized skew derivations $F$ and $G$ of a prime ring $R$ satisfying the condition

$$a_1F(x)b_1 + a_2G(x)b_2 = 0 \quad \forall x \in R \tag{11}$$

where $a_1, a_2, b_1, b_2 \in Q_r$.

In light of Theorem 1, we may assume that $a_1, a_2, b_1, b_2$ are all non-zero elements of $Q_r$ and also that both $F \neq 0$ and $G \neq 0$.

We start with two useful results, that we quote as follows, by applying [6, Theorem 2]:

**Lemma 3** *Let $R$ be a prime and $a_i, b_i \in Q_r$, for $1 \le i \le n$. If $\sum_{i=1}^{n} a_i x b_i = 0$, for all $x \in R$, and $b_i \neq 0$ for some $i$, then $a_1, \dots, a_n$ are $C$-dependent (see [15, Lemma 2.2]).*

**Lemma 4** *Let $R$ be a prime and $a_i, b_i, c_i, d_i \in Q_r$ such that $\sum_{i=1}^{m} a_i x b_i + \sum_{j=1}^{n} c_j x d_j = 0$, for all $x \in R$. If $a_1, \dots, a_m$ are linearly $C$-independent then each $b_i$ is a linear combination of $d_1, \dots, d_n$ over $C$. Analogously, if $b_1, \dots, b_m$ are linearly $C$-independent then each $a_i$ is a linear combination of $c_1, \dots, c_n$ over $C$. (see [17, Lemma 1.2]).*

**Lemma 5** *Let $F$ and $G$ be inner generalized skew derivations of $R$ defined as*

$$F(x) = px + qxq^{-1}u, \quad G(x) = vx + qxq^{-1}w, \quad \forall x \in R$$

*where $p, u, v, w, q \in Q_r$ and $q$ is an invertible element. If $R$ satisfies (11), one of the following holds:*

*(a) there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in C$ such that $b_1 = \alpha_1 b_2 + \alpha_2 q^{-1} w b_2$, $q^{-1} u b_1 = \alpha_3 b_2 + \alpha_4 q^{-1} w b_2$ and $\alpha_1 a_1 p + \alpha_3 a_1 q + a_2 v = \alpha_2 a_1 p + \alpha_4 a_1 q + a_2 q = 0$;*
*(b) there exist $\lambda, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in C$ such that $q^{-1} w b_2 = \lambda b_2$, $b_1 = (\alpha_1 + \lambda\alpha_2) b_2$, $q^{-1} u b_1 = (\alpha_3 + \lambda\alpha_4) b_2$ and $(\alpha_1 + \lambda\alpha_2) a_1 p + (\alpha_3 + \lambda\alpha_4) a_1 q + a_2(v + \lambda q) = 0$;*
*(c) there exist $0 \neq \lambda \in C$ and $\beta_1, \beta_2 \in C$ such that $a_1 p = \lambda a_1 q$, $a_2 v = \beta_1 a_1 q$, $a_2 q = \beta_2 a_1 q$ and $\lambda b_1 + q^{-1} u b_1 + \beta_1 b_2 + \beta_2 q^{-1} w b_2 = 0$;*
*(d) there exist $0 \neq \lambda \in C$ and $\mu, \eta \in C$ such that $a_1 p = \lambda a_1 q$, $a_2(v + \mu q) = \eta a_1 q$, $(\lambda + q^{-1} u) b_1 = -\eta b_2$ and $q^{-1} w b_2 = \mu b_2$.*

***Proof*** By our main hypothesis

$$a_1 F(x) b_1 + a_2 G(x) b_2 = 0 \quad \forall x \in R.$$

Under the assumptions of the present Lemma, we have that $R$ satisfies the generalized identity

$$a_1\big(px + qxq^{-1}u\big)b_1 + a_2\big(vx + qxq^{-1}w\big)b_2 \tag{12}$$

that is

$$(a_1 p)xb_1 + (a_1 q)x(q^{-1}ub_1) + (a_2 v)xb_2 + (a_2 q)x(q^{-1}wb_2). \tag{13}$$

By Lemma 3 and since $a_1, a_2, b_1, b_2$ are all non-zero we may divide the proof in two cases.

**Case 1**. $\{a_1p, a_1q\}$ **is a linearly** $C$**-independent set**

Application of Lemma 4 implies that there exist $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in C$ such that

$$\begin{aligned} b_1 &= \alpha_1 b_2 + \alpha_2 q^{-1}wb_2 \\ q^{-1}ub_1 &= \alpha_3 b_2 + \alpha_4 q^{-1}wb_2. \end{aligned} \tag{14}$$

Thus, by (13), $R$ satisfies

$$(a_1p)x(\alpha_1 b_2 + \alpha_2 q^{-1}wb_2) + (a_1q)x(\alpha_3 b_2 + \alpha_4 q^{-1}wb_2) + (a_2v)xb_2 + (a_2q)x(q^{-1}wb_2)$$

that is

$$(\alpha_1 a_1p + \alpha_3 a_1q + a_2v)xb_2 + (\alpha_2 a_1p + \alpha_4 a_1q + a_2q)xq^{-1}wb_2. \tag{15}$$

Firstly we note that, if $\alpha_2 a_1p + \alpha_4 a_1q + a_2q = 0$ then, by the primeness of $R$ and since $b_2 \neq 0$, (15) implies $\alpha_1 a_1p + \alpha_3 a_1q + a_2v = 0$. Hence, in consideration of what is stated in relations (14), we get conclusion (a) of the present Lemma.
On the other hand, if $\alpha_2 a_1p + \alpha_4 a_1q + a_2q \neq 0$ and by Lemma 3, there is $\lambda \in C$ such that $q^{-1}wb_2 = \lambda b_2$. Thus (15) reduces to

$$(\alpha_1 a_1p + \alpha_3 a_1q + a_2v)xb_2 + \lambda(\alpha_2 a_1p + \alpha_4 a_1q + a_2q)xb_2. \tag{16}$$

Again by the primeness of $R$ and since $b_2 \neq 0$, $\alpha_1 a_1p + \alpha_3 a_1q + a_2v + \lambda(\alpha_2 a_1p + \alpha_4 a_1q + a_2q) = 0$ follows.

**Case 2**. $a_1p = \lambda a_1q$, $0 \neq \lambda \in C$

In this case, again by (13), $R$ satisfies

$$a_1qx(\lambda b_1 + q^{-1}ub_1) + (a_2v)xb_2 + (a_2q)x(q^{-1}wb_2). \tag{17}$$

Notice that, in case $\{b_2, q^{-1}wb_2\}$ is a linearly $C$-independent set, by (17) and Lemma 3, it follows

$$a_2v = \beta_1 a_1q, \quad a_2q = \beta_2 a_1q \quad \beta_1, \beta_2 \in C$$

and (17) reduces to

$$a_1qx(\lambda b_1 + q^{-1}ub_1 + \beta_1 b_2 + \beta_2 q^{-1}wb_2).$$

Therefore, since $a_1q \neq 0$, we get $\lambda b_1 + q^{-1}ub_1 + \beta_1 b_2 + \beta_2 q^{-1}wb_2 = 0$.

Assume finally that $\{b_2, q^{-1}wb_2\}$ is a linearly $C$-dependent set.

Without loss of generality we may write $q^{-1}wb_2 = \mu b_2$, for a suitable $\mu \in C$. Hence, by (17), $R$ satisfies

$$a_1 q x(\lambda b_1 + q^{-1} u b_1) + (a_2 v + \mu a_2 q) x b_2 \tag{18}$$

implying that there exists $\eta \in C$ such that

$$a_2 v + \mu a_2 q = \eta a_1 q$$

$$\lambda b_1 + q^{-1} u b_1 = -\eta b_2.$$

**Lemma 6** *Let $F$ and $G$ be inner generalized skew derivations of $R$ defined as*

$$F(x) = px + \alpha(x)u, \quad G(x) = vx + \alpha(x)w, \quad \forall x \in R$$

*where $p, u, v, w \in Q_r$ and $\alpha$ is an outer automorphism of $R$. If $R$ satisfies (11), one of the following holds:*

*(a)* $a_1 p = a_2 v = u b_1 = w b_2 = 0$;
*(b)* $a_1 p = a_2 v = 0$ *and there exists* $\mu \in C$ *such that* $u b_1 = \mu w b_2$ *and* $a_2 = -\mu a_1$;
*(c)* $u b_1 = w b_2 = 0$ *and there exists* $\lambda \in C$ *such that* $a_1 p = \lambda a_2 v$ *and* $b_2 = -\lambda b_1$;
*(d)* *there exist* $\lambda, \mu \in C$ *such that* $a_1 p = \lambda a_2 v$, $b_2 = -\lambda b_1$, $u b_1 = \mu w b_2$ *and* $a_2 = -\mu a_1$.

***Proof*** Here $R$ satisfies

$$a_1\big(px + \alpha(x)u\big)b_1 + a_2\big(vx + \alpha(x)w\big)b_2. \tag{19}$$

Since $\alpha$ is outer, by (19), it follows that $R$ satisfies the generalized identity

$$a_1\big(px_1 + x_2 u\big)b_1 + a_2\big(vx_1 + x_2 w\big)b_2. \tag{20}$$

In particular, both

$$a_1 p x_1 b_1 + a_2 v x_1 b_2 \tag{21}$$

and

$$a_1 x_2 u b_1 + a_2 x_2 w b_2 \tag{22}$$

are satisfied by $R$. Relation (21) implies that

- either $a_1 p = a_2 v = 0$
- or there exists $\lambda \in C$ such that $a_1 p = \lambda a_2 v$ and $b_2 = -\lambda b_1$.

Analogously, (22) implies that

- either $u b_1 = w b_2 = 0$
- or there exists $\mu \in C$ such that $u b_1 = \mu w b_2$ and $a_2 = -\mu a_1$.

Putting together all the previous informations, one of the following cases must occur:

(a) $a_1 p = a_2 v = ub_1 = wb_2 = 0$;
(b) $a_1 p = a_2 v = 0$ and there exists $\mu \in C$ such that $ub_1 = \mu wb_2$ and $a_2 = -\mu a_1$;
(c) $ub_1 = wb_2 = 0$ and there exists $\lambda \in C$ such that $a_1 p = \lambda a_2 v$ and $b_2 = -\lambda b_1$;
(d) there exist $\lambda, \mu \in C$ such that $a_1 p = \lambda a_2 v$, $b_2 = -\lambda b_1$, $ub_1 = \mu wb_2$ and $a_2 = -\mu a_1$.

Before proceeding with the proof of our main result, we need to recall the following:

**Lemma 7** *Let $R$ be a prime ring, $\alpha, \beta \in \mathrm{Aut}(Q_r)$ and $d, \delta : R \to R$ be two skew derivations, associated with the same automorphism $\alpha$. If there exist $0 \neq \eta \in C$, and $u \in Q_r$ such that*

$$\delta(x) = \Big(ux - \beta(x)u\Big) + \eta d(x), \quad \forall x \in R \tag{23}$$

*then either $\alpha = \beta$ or $\delta(x) = \eta d(x)$, for all $x \in R$.*

***Proof*** By the definition of $\delta$ we have

$$\delta(xy) = uxy - \beta(x)\beta(y)u + \eta d(x)y + \eta\alpha(x)d(y). \tag{24}$$

On the other hand, right multiplying relation (23) by $y \in R$, it follows that

$$\delta(x)y = uxy - \beta(x)uy + \eta d(x)y \quad \forall x, y \in R. \tag{25}$$

Therefore, subtracting relation (25) from (24), and using again (23), we get

$$\Big\{\alpha(x) - \beta(x)\Big\} \cdot \Big\{uy - \beta(y)u\Big\} = 0 \quad \forall x, y \in R. \tag{26}$$

Replacing $y$ by $yt$ in (26) and then using (26) we have

$$\Big\{\alpha(x) - \beta(x)\Big\} \cdot \beta(y) \cdot \Big\{\beta(t)u - ut\Big\} = 0 \quad \forall x, y, t \in R. \tag{27}$$

Then, by the primeness of $R$, above relation yields either $\alpha(x) - \beta(x) = 0$ for any $x \in R$, or $\beta(t)u - ut = 0$ for any $t \in R$. The last case and (23) imply $\delta(x) = \eta d(x)$, for all $x \in R$, as required.

**Lemma 8** ([10, Lemma 3.2]) *Let $R$ be a prime ring, $\alpha, \beta \in \mathrm{Aut}(Q_r)$ and $d : R \to R$ be a skew derivation, associated with the automorphism $\alpha$. If there exist $0 \neq \theta \in C$, $0 \neq \eta \in C$ and $u, b \in Q_r$ such that*

$$d(x) = \theta\Big(ux - \alpha(x)u\Big) + \eta\Big(bx - \beta(x)b\Big), \quad \forall x \in R$$

*then $d$ is an inner skew derivation of $R$. More precisely, either $b = 0$ or $\alpha = \beta$.*

***Proof*** (Proof of Theorem 2) For sake of clearness we recall that we may write $F(x) = px + d(x)$ and $G(x) = vx + \delta(x)$, for all $x \in R$ and suitable $p, v \in Q_r$ and $d, \delta$ skew derivations associated with the same automorphism $\alpha$. Moreover we also recall that both $d$ and $\delta$ commute with $\alpha$.

We also remind that, by our main hypothesis $R$ satisfies

$$a_1\Big(px + d(x)\Big)b_1 + a_2\Big(vx + \delta(x)\Big)b_2. \tag{28}$$

**The case $d = 0$ and $\delta \neq 0$**

We firstly study the case $F(x) = px$ and $G(x) = vx + \delta(x)$, for all $x \in R$. Since $F \neq 0$, we may assume in what follows $p \neq 0$. Moreover $\delta$ is not an inner skew derivation of $R$, otherwise the conclusion follows by Lemmas 5 and 6. In this situation, by (28) we have that $R$ satisfies

$$a_1 p x_1 b_1 + a_2\Big(vx_1 + x_2\Big)b_2.$$

In particular $a_2 y b_2 = 0$, for any $y \in R$, which is a contradiction, since both $a_2 \neq 0$ and $b_2 \neq 0$.

Analogously, we get a contradiction in the case we assume $\delta = 0$ and $d \neq 0$.

**The case $d \neq 0, \delta \neq 0$**

Here we study the case when $F(x) = px + d(x)$ and $G(x) = vx + \delta(x)$, for all $x \in R$. We start with the case $d, \delta$ are linearly $C$-independent modulo inner skew derivations. Hence, by (28),

$$a_1\Big(px_1 + x_2\Big)b_1 + a_2\Big(vx_1 + x_3\Big)b_2 \tag{29}$$

is satisfied by $R$. In particular, $a_1 x_2 b_1$ is a generalized identity for $R$, which is a contradiction, since both $a_1 \neq 0$ and $b_1 \neq 0$.

Thus we assume that $\{d, \delta\}$ are linearly $C$-dependent modulo inner skew derivations. Hence there exist $\lambda, \mu \in C$, $u \in Q_r$ and an automorphism $\beta$ of $R$ such that $\lambda d(x) + \mu\delta(x) = ux - \beta(x)u$, for any $x \in R$.

If $\lambda = 0$ and $\mu \neq 0$, we write

$$\delta(x) = \Big(p_0 x - \beta(x) p_0\Big), \quad \forall x \in R$$

where $p_0 = \mu^{-1}u$. Since the automorphism associated with a skew derivation is unique, in this case $\alpha = \beta$.

If $d$ is also inner, the conclusion follows from Lemmas 5 and 6. Hence we may assume that $d$ is not inner. Thus, by (28), $R$ satisfies

$$a_1\Big(px_1 + x_2\Big)b_1 + a_2\Big(vx_1 + p_0x_1 - \beta(x_1)p_0\Big)b_2 \tag{30}$$

and in particular $a_1x_2b_1$ is an identity for $R$, which is a contradiction.

Similarly, we get a contradiction in the case $\mu = 0$ and $\lambda \neq 0$.

Hence, in the sequel we assume that both $\lambda \neq 0$ and $\mu \neq 0$. We may write

$$\delta(x) = \Big(p_0x - \beta(x)p_0\Big) + \eta d(x), \quad \forall x \in R \tag{31}$$

where $\eta = -\lambda\mu^{-1} \neq 0$ and, as above, $p_0 = \mu^{-1}u$. By Lemma 7, either $\alpha = \beta$ or $p_0 = 0$ and $\delta(x) = \eta d(x)$, for all $x \in R$.

Moreover, by Lemma 8, if $d$ is an inner skew derivation, then also $\delta$ is inner and the conclusion follows again from Lemmas 5 and 6.

Therefore, in what follows we assume that $0 \neq d$ is outer.

In the case $\delta = \eta d$, (28) reduces to

$$a_1\Big(px + d(x)\Big)b_1 + a_2\Big(vx + \eta d(x)\Big)b_2. \tag{32}$$

Thus, since $d$ is not inner, $R$ satisfies

$$a_1\Big(px_1 + x_2\Big)b_1 + a_2\Big(vx_1 + \eta x_2\Big)b_2. \tag{33}$$

In particular, both

$$a_1px_1b_1 + a_2vx_1b_2 \tag{34}$$

and

$$a_1x_2b_1 + \eta a_2x_2b_2 \tag{35}$$

are identities for $R$. Those relations imply that there exists $\vartheta \in C$ such that

$$a_1p = \vartheta a_2v \quad b_2 = -\vartheta b_1 \quad a_1 = \vartheta\eta a_2.$$

Suppose now $\alpha = \beta$. By relations (31) and (28) $R$ satisfies

$$a_1\Big(px + d(x)\Big)b_1 + a_2\Big(vx + p_0x - \alpha(x)p_0 + \eta d(x)\Big)b_2. \tag{36}$$

Since $d$ is not inner, it follows that

$$a_1\Big(px_1 + x_2\Big)b_1 + a_2\Big(vx_1 + p_0x_1 - \alpha(x_1)p_0 + \eta x_2\Big)b_2 \tag{37}$$

is a generalized identity for $R$. Hence $R$ satisfies both

$$a_1 p x_1 b_1 + a_2\Big(v x_1 + p_0 x_1 - \alpha(x_1) p_0\Big) b_2 \tag{38}$$

and

$$a_1 x_2 b_1 + \eta a_2 x_2 b_2. \tag{39}$$

By (39) and applying Lemma 3, we have that there exists $0 \neq \vartheta \in C$ such that

$$a_1 = \vartheta \eta a_2 \quad b_2 = -\vartheta b_1.$$

Substituting $a_1$ and $b_2$ in relation (38), it follows that

$$\vartheta \eta a_2 p x_1 b_1 - \vartheta a_2\Big(v x_1 + p_0 x_1 - \alpha(x_1) p_0\Big) b_1. \tag{40}$$

If $\alpha$ is not inner, by (40) we have that $R$ satisfies

$$\vartheta \eta a_2 p x_1 b_1 - \vartheta a_2\Big(v x_1 + p_0 x_1 - x_2 p_0\Big) b_1. \tag{41}$$

Thus both $a_2 x_2 p_0 b_1$ and

$$\Big(\vartheta \eta a_2 p - \vartheta a_2(v + p_0)\Big) x_1 b_1$$

are identities for $R$, implying $p_0 b_1 = 0$ and $\eta a_2 p - a_2(v + p_0) = 0$.

On the other hand, if $\alpha(x) = q x q^{-1}$, for any $x \in R$, by (40) it follows that

$$\Big(\eta a_2 p - a_2(v + p_0)\Big) x_1 b_1 + a_2 q x_1 q^{-1} p_0 b_1$$

is a generalized identity for $R$. Thus, there exists $\vartheta \in C$ such that

$$q^{-1} p_0 b_1 = \vartheta b_1 \quad \eta a_2 p - a_2(v + p_0) + \vartheta a_2 q = 0.$$

**Acknowledgements** We would like to express our gratitude to the anonymous referees for their careful reading of the manuscript and for insightful comments and suggestions, which significantly helped us improve the final presentation of this article.

**Funding** The work of the first author was partially supported by a grant of the italian "National Group for Algebraic and Geometric Structures, and their Applications" (GNSAGA-INDAM).

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