Prime Rings with Generalized Derivations and Power Values on Lie Ideals

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Abstract Suppose $\mathscr{R}$ is a prime ring that is non-commutative in structure and characteristic of $\mathscr{R}$ is a positive integer apart from 2 and $M = (-2)^{k-1} - 1$, where k is any odd positive integers greater than one. Let the Utumi ring of quotients be denoted by $\mathscr{Q}$, the extended centroid of $\mathscr{R}$ by $\mathscr{C}$. Consider $\mathscr{L}$ to be Lie ideal of $\mathscr{R}$ non-central in nature and $\mathscr{T}$ be a non-zero generalized derivation of $\mathscr{R}$. If $[\mathscr{T}(u^s), u^t]^m = [\mathscr{T}(u), u]$, for every $u \in \mathscr{L}$, where $m,\ s$ and t be the fixed positive integers such that $m > 1,\quad s \geq 1$ and $t \geq 1$, then one of the following situations prevails:

(i) The standard identity $s_4(x_1, \ldots, x_4)$ is satisfied by $\mathscr{R}$ and there exists $a \in \mathscr{Q}$ and $\beta \in \mathscr{C}$ such that $\mathscr{T}(x) = \beta x + ax + xa$, for every $x \in \mathscr{R}$.

(ii) there exists certain $\theta \in \mathscr{C}$ such that $\mathscr{T}(x) = \theta x$, for every $x \in \mathscr{R}$.

Keywords Prime rings · Generalized derivation · Maximal right ring of quotients · Generalized polynomial identity (GPI) · Polynomial identity (PI)

1 Introduction

In the entire article put forth, $\mathscr{R}$ always depicts prime ring that is non-commutative and associative in nature and its center is given by $\mathscr{Z}(\mathscr{R})$. Further, Q is the Martindale ring of quotients and $\mathscr{Q}$ denotes the Utumi ring of quotients with $\mathscr{C} = \mathscr{Z}(\mathscr{Q})$ as the center of $\mathscr{Q}$ called as the extended centroid of $\mathscr{R}$ and ρ the dense ideal of $\mathscr{R}$. Also, the well-known theory from [1] establishes that Q and $\mathscr{Q}$ share the same center.

A right ideal ρ is a right dense ideal if whenever $x_1, x_2 \in \rho$ with $x_1 \neq 0$, there exists $r \in \mathscr{R}$ with the condition that $x_1 r \neq 0$ and $x_2 r \in \rho$. A left dense ideal is defined likewise. An ideal ρ is called a dense ideal if it is both a left as well as a right dense ideal. Throughout the paper, by a left faithful ring we mean a ring whose

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left annihilator is zero. Similarly, right faithful ring is also defined and thus, a ring is faithful if it is both left and a right faithful. We observe that since $\mathscr{R}$ is prime, the Utumi ring of quotients $\mathscr{Q}$ is also a prime ring. With respect to a dense right ideal, the right Utumi ring of quotients of $\mathscr{R}$ can be characterized as the ring $\mathscr{Q}(\mathscr{R})$. We state some properties of $\mathscr{Q}(\mathscr{R})$ as follows:

- (i) $\mathscr{R} \subseteq \mathscr{Q}(\mathscr{R})$;
- (ii) For every $q \in \mathscr{Q}(\mathscr{R})$, there exists a right dense ideal $\mathscr{H}$ of $\mathscr{R}$ such that $q\mathscr{H} \subseteq \mathscr{R}$;
- (iii) If $q \in \mathscr{Q}(\mathscr{R})$ and for certain non-zero right dense ideal $\mathscr{H}$ of $\mathscr{R}$ with $q\mathscr{H} = 0$, then $q = 0$;
- (iv) If $\mathscr{H}$ is right dense ideal of $\mathscr{R}$ and $\mathscr{F} : \mathscr{H} \longrightarrow \mathscr{R}$ is a right $\mathscr{R}$-module map, then there exists certain $q \in \mathscr{Q}(\mathscr{R})$ such that $\mathscr{F}(x) = qx$, for every $x \in \mathscr{H}$.

These are the characterizing properties of $\mathscr{Q}(\mathscr{R})$. See [2] and [1] for the salient features of special rings like $\mathscr{Q}$, Q and $\mathscr{C}$.

For a prime ring $\mathscr{R}$, the extended centroid $\mathscr{C}$ of $\mathscr{R}$ is notably a field also called as the field of quotients of $\mathscr{Z}(\mathscr{R})$. Let $Y = \{y_1, y_2, \ldots\}$, be the set consisting of the non-commuting indeterminates say $y_1, y_2, \ldots$ which are countable. Let $\mathscr{C}\{Y\}$ be the free $\mathscr{C}$ algebra of the set Y. Consider $\mathscr{Q}\{Y\} = \mathscr{Q} *_{\mathscr{C}} \mathscr{C}\{Y\}$, the free $\mathscr{C}$-product of $\mathscr{Q}$ and $\mathscr{C}\{Y\}$. The elements of $\mathscr{Q}\{Y\}$ are called the GP (the generalized polynomials). By a non-trivial GP, we mean a non-zero element of $\mathscr{Q}\{Y\}$. Every element $w \in \mathscr{Q}\{Y\}$ is of the peculiar form $w = j_o t_1 j_1 t_2 j_2 \ldots t_n j_n$, where $\{j_o, \ldots, j_n\} \subseteq \mathscr{Q}$ and $\{t_1, \ldots, t_n\} \subseteq Y$, is called a monomial where $j_o, \ldots, j_n$ are called the coefficients of w. Each $g \in \mathscr{Q}\{Y\}$ constitutes of such monomials as a finite sum. Such representation is easily seen to be not unique. For a detailed study see [4].

For a lucid explanation of the notion of a non-triviality of a GPI, let us look at the following simple example.

Exam: Let W be the ring of real quaternions and $\mathscr{Z}(W) = \mathbb{R}$ be its center, that is the ring of real numbers. Then, for every $w \in W$, where $w = w_o + w_1 i + w_2 j + w_3 k$ and $w_i \in \mathbb{R}$, for every $i \in \{0, 1, 2, 3\}$, the following relation holds $w^2 = 2w_o w - w\overline{w}$. Follow-up of the above relation, results in the identity below,

$$w^2 iwi - wiw^2 i + iw^2 iw - iwiw^2 = 0, \text{ for every } w \in W,$$

called the non-trivial GPI satisfied by W where we can see $i \neq 0$.

The following definitions concerning commutators and prime rings shall be utilized in the present paper without emphasizing specifically each time. The commutator for every $x, y \in \mathscr{R}$ is given as $[x, y] =: xy - yx$ and anticommutators is given by $xoy =: xy + yx$ and the definition of a prime ring $\mathscr{R}$ viz. if $a\mathscr{R}b = (0)$, where $a, b \in \mathscr{R}$ then $a = 0$ or $b = 0$. Similarly, a ring $\mathscr{R}$ in which $a\mathscr{R}a = (0)$, then it implies that a equates to zero, is termed as semiprime ring.

An additive map $\mathscr{F} : \mathscr{R} \to \mathscr{R}$ is called as derivation if $\mathscr{F}(xy) = x\mathscr{F}(y) + \mathscr{F}(x)y$ stands true for every $x, y \in \mathscr{R}$. By saying derivation is inner derivation induced by an element $q \in \mathscr{R}$, we mean an additive map $\mathscr{G} : \mathscr{R} \to \mathscr{R}$ such that $\mathscr{G}(x) = [q, x]$ for every $x \in \mathscr{R}$. Furthermore, many researchers authored papers in the scenario of generalized derivations satisfying special identities in the presence of prime rings. An additive map $\mathscr{S}$ on $\mathscr{R}$ defined by $\mathscr{S}(w) = aw + wb$, for every $w \in \mathscr{R}$ and for some fixed $a, b \in \mathscr{R}$ is called the generalized inner derivation on the ring $\mathscr{R}$. Such maps prompt the definition of generalized derivation say $\mathscr{T} : \mathscr{R} \to \mathscr{R}$ which is expressed as the following,

$$\mathscr{T}(xy) = x[y, b] + \mathscr{T}(x)y = xI_b(y) + \mathscr{T}(x)y, \text{ for every } x,\ y \in \mathscr{R}$$

where I_b is an inner derivation induced by b. Further, we pen down the definition of a generalized derivation say $\mathscr{T}$ related to a derivation μ of $\mathscr{R}$ as following $\mathscr{T}(xy) = x\mu(y) + \mathscr{T}(x)y$, for every $x, y \in \mathscr{R}$. It is obvious that every inner generalized derivation is a generalized derivation and if $\mu = \mathscr{T}$ in the above definition of a generalized derivation then, $\mathscr{T}$ is an ordinary derivation.

We remark that Lee in [15] discussed the extension of a generalized derivation on any right dense ideal ρ to $\mathscr{Q}$, the Utumi ring of quotients. That is, if we have $\mathscr{T} : \rho \to \mathscr{Q}$, then due to Lee $\mathscr{T}$ is uniquely extended as $\mathscr{T} : \mathscr{Q} \to \mathscr{Q}$ and has the form $\mathscr{T}(x) = \mu(x) + ax$, for every $x \in \mathscr{Q}$, where $a \in \mathscr{Q}$. This definition of a generalized derivation shall be used in the entire paper without any special reference.

Throughout the paper, we will study the situation when the generalized derivations are acted upon the Lie ideals. By a Lie ideal $\mathscr{L}$, we mean an additive group $\mathscr{L}$ where the commutator $[\mathscr{L}, \mathscr{R}]$ is contained in $\mathscr{L}$. Obviously $[\mathscr{L}, \mathscr{L}]$ is a Lie ideal. We will consider only a non-central Lie ideal $\mathscr{L}$. A Lie ideal is called non-central if the commutator $[\mathscr{L}, \mathscr{L}]$ is not zero.

The prolific work in this direction clearly dictates that the global structure of the ring $\mathscr{R}$ is intimately related to the action of additive maps defined on the ring $\mathscr{R}$. For instance, derivation equipped with some properties intrigued many authors to investigate the structure of ring-like commutativity and even characterization of such additive maps. Some of the important works in this direction include [6] and [7].

In [13], the work of Lanski gives that, if for a derivation d of $\mathscr{R}$ such that $d(x)$ is n-nilpotent for some n a positive integer that is $d(x)^n = 0$ for every x from $\mathscr{L}$ a non-central Lie ideal, then d is vanishing on $\mathscr{R}$. An analogue fair result was developed by Lee in [15] for the more complex case of generalized derivations. More precisely, he contributed that if G is a generalized derivation on the prime ring $\mathscr{R}$ and $\mathscr{L}$ a Lie ideal which is non-central. If $G(x)^n = 0$ for every $x \in \mathscr{L}$, for n a certain positive integer, then $\mathscr{R}$ is bound to be commutative.

De Filippis and Carini in [3] studied a non-zero derivation d of $\mathscr{R}$ such that n-power of commutator $[d(x), x]$ is vanishing viz., $[d(x), x]^n = 0$, for every x in a Lie ideal $\mathscr{L}$ of $\mathscr{R}$ which is non-central, for certain positive integer n. They made the conclusion that if $\text{char}(R) \neq 2$, then $\mathscr{R}$ is forced to be commutative.

De Filippis later on in [5] improved this result by taking a generalized derivation G in the attempt to give a more general result instead of d. He concluded that either $\mathscr{R}$ satisfies the standard identity in four non-commuting variables s_4, and there exist

$a \in \mathscr{Q}$, $\alpha \in \mathscr{C}$ such that $G(x) = ax + xa + \alpha x$, for every $x \in \mathscr{R}$ or for particular $\gamma \in \mathscr{C}$, $G(x) = \gamma x$, for every x from $\mathscr{R}$.

Recently, Scudo and Ansari [18] took the task of investigating the set $A = \{[G(u), u] : u \in L\}$. They proved that if $A \neq \{0\}$, then it is void of any non-trivial idempotent element of $\mathscr{R}$. In this investigation, they focused the study of a generalized derivation on prime rings and provided the following result:

Theorem 1 ([18, Theorem]) *Let $\mathscr{R}$ be a prime ring that is non-commutative and char($\mathscr{R}$) $\neq 2$. Suppose associated with $\mathscr{R}$, the Utumi ring of quotients and the extended centroid of $\mathscr{R}$ is denoted by $\mathscr{Q}$ and $\mathscr{C}$, respectively. Then for $\mathscr{L}$ the non-central Lie ideal of $\mathscr{R}$ and $\mathscr{T}$ the non-zero generalized derivation of $\mathscr{R}$,*

if $[\mathscr{T}(u), u]^m = [\mathscr{T}(u), u]$, for every $u \in \mathscr{L}$, where m is a fixed positive integer such that $m > 1$, then one of the following conditions stands true:

- (i) *The standard identity $s_4(x_1, \ldots, x_4)$ is satisfied by $\mathscr{R}$ and there exists certain $a \in \mathscr{Q}$ and $\beta \in \mathscr{C}$ such that $\mathscr{T}(x) = \beta x + ax + xa$, for every $x \in \mathscr{R}$.*
- (ii) *There exists $\theta \in \mathscr{C}$ such that $\mathscr{T}(x) = \theta x$, for every $x \in \mathscr{R}$.*

In view of the results above, it is reasonable to raise the following question.

Question: What can we say about the ring $\mathscr{R}$ admitting the generalized derivation $\mathscr{T}$ of $\mathscr{R}$ and $[\mathscr{T}(u^s), u^t]^m = [\mathscr{T}(u), u]$, for every $u \in \mathscr{L}$, where m, s and t are the fixed positive integers such that $m > 1$, $s \geq 1$ and $t \geq 1$, where $\mathscr{L}$ is a Lie ideal which is not central?

Here, in non-commutative prime ring, characteristic of $\mathscr{R}$ different from two, if we choose $s = t = 1$, then we have the same as case of [18, Theorem]. Our goal is to answer the above question in two cases.

Firstly, the case of inner generalized derivation, which we deal in Sect. 3 and then the study of general case, that is, the case of any generalized derivation, which we will discuss in the last section of this paper (Sect. 4).

2 Preliminary Results

Recall that Der($\mathscr{Q}$) is the set of all derivations on Utumi quotient ring $\mathscr{Q}$. The derivation word is an additive map δ given by $\delta = d^1 d^2 \ldots d^m$, with each $d^i \in$ Der($\mathscr{Q}$). Every differential polynomial is a GP taking coefficients out of $\mathscr{Q}$, of peculiar type $\psi(\delta^j x_i)$ utilizing non-commuting indeterminates x_i on which the δ^j words acts as the unary operator. The polynomial $\psi(\delta^j x_i)$ is termed as a DI (differential identity) over a subset T of $\mathscr{Q}$ if whenever a value is assigned from T to x_i, $\psi(\delta^j x_i)$ vanishes to zero. We refer the reader to [1, Chaps. 6–7] for the in depth and presentable description of GPI-Theory involving derivations. The $\mathscr{C}$-subspace of Der($\mathscr{Q}$) that is D_{int} comprises of all inner derivations on $\mathscr{Q}$. For d a nonzero derivation d on $\mathscr{R}$. By [12, Theorem 2], we pen down the non-trivial Kharchenkos effect. See also [14, Theorem 1].

If $\psi(x_1, \dots x_n, d(x_1), \dots, d(x_n))$ is a DI on $\mathscr{R}$, then one of the result listed below stands true:

1. The derivation d is inner i.e. $d \in D_{int}$;
2. The following GPI is satisfied by $\mathscr{R}$, $\psi(x_1, \dots, x_n, y_1, \dots, y_n)$.

Before we commence to establish our results, we pen down some well-known facts. Precisely we shall use the following facts incessantly wherever applicable.

Fact 2 *Every generalized derivation of $\mathscr{R}$ can be uniquely extended to a generalized derivation of $\mathscr{Q}$ and assumes the form that $\mathscr{T}(x) = \mu(x) + ax$, for some $a \in \mathscr{Q}$ and a derivation μ of $\mathscr{Q}$. That is, every generalized derivation of $\mathscr{R}$ can be defined on $\mathscr{Q}$ explicitly. See [15].*

Fact 3 *Let A be an ideal of $\mathscr{R}$. Then A, $\mathscr{R}$ and $\mathscr{Q}$ satisfy the same GPI with coefficients in $\mathscr{Q}$. See [4].*

Fact 4 *Let A be an ideal of $\mathscr{R}$. Then A, $\mathscr{R}$ and $\mathscr{Q}$ satisfy the same DI with coefficients in $\mathscr{Q}$. See [14].*

Fact 5 *Let $\mathscr{L}$ be a non-central Lie ideal of $\mathscr{R}$. If char$(\mathscr{R}) \neq 2$ or $\mathscr{R}$ does not satisfy s_4, then there exists a non-zero ideal $\mathscr{H}$ of $\mathscr{R}$ such that $0 \neq [\mathscr{H}, \mathscr{R}] \subseteq \mathscr{L}$. For a simple ring $\mathscr{R}$, $[\mathscr{R}, \mathscr{R}] \subseteq \mathscr{L}$ drawing arguments from [10, pp. 4–5] and also see [8, Lemma 2, Proposition 1].*

Fact 6 *For a prime ring $\mathscr{R}$ having $\mathscr{C}$ as the extended centroid, the following equivalent conditions shall hold:*

(a) *The linear space $\mathscr{R}\mathscr{C}$ over $\mathscr{C}$ has atmost four dimension;*
(b) *The standard identity $s_4(x_1, \dots, x_4)$ is satisfied by $\mathscr{R}$;*
(c) *For a certain field $\mathscr{F}$, $\mathscr{R}$ embeds in $M_2(\mathscr{F})$ or $\mathscr{R}$ is commutative;*
(d) *Degree of algebraicness of $\mathscr{R}$ over $\mathscr{C}$ is two;*
(e) *$\mathscr{R}$ satisfies the PI $[[x^2, y], [x, y]] = 0$.*

3 When $\mathscr{T}$ Is Generalized Inner Derivations

We consider in this segment of the proof that, every generalized inner derivation induced by the elements $a, b \in \mathscr{Q}$ takes the form as $\mathscr{T}(x) = ax + xb$, for every $x \in \mathscr{R}$. Henceforth, it is supposed that the following GPI is satisfied by $\mathscr{R}$

$$\Pi(w_1, w_2) = \{[a[w_1, w_2]^s + [w_1, w_2]^s b, [w_1, w_2]^t]\}^m - [a[w_1, w_2] + [w_1, w_2]b, [w_1, w_2]].$$

In an attempt to establish the main result of the article, we shall need the support of the following crucial fact.

Fact 7 *By the pivotal assumption of the article, the following relation*

$$\{[a[w_1, w_2]^s + [w_1, w_2]^s b, [w_1, w_2]^t]\}^m - [a[w_1, w_2] + [w_1, w_2]b, [w_1, w_2]] = 0,$$

holds for every $w_1, w_2 \in \mathscr{R}$. Further, for every automorphism ψ of $\mathscr{R}$ which is inner, we have

$$[\psi(a)[w_1, w_2]^s + [w_1, w_2]^s \psi(b), [w_1, w_2]^t]^m - [\psi(a)[w_1, w_2] + [w_1, w_2]\psi(b), [w_1, w_2]] = 0,$$

holds for every $w_1, w_2 \in \mathscr{R}$. Clearly, $a + b, a - b, a, b$ are the central elements of the ring $\mathscr{R}$ if and only if $\psi(a + b)$, $\psi(a - b)$, $\psi(a)$, $\psi(b)$ are the central elements of the ring $\mathscr{R}$. Thus, whenever it is demanded, we can use $\psi(a)$ and $\psi(b)$ instead of a and b, respectively.

Proposition 1 *Suppose $\mathscr{R}$ is a prime ring that is non-commutative in structure. If char($\mathscr{R}$) is a positive integer apart from 2 and $M = (-2)^{k-1} - 1$ where $k(> 1)$ is any odd positive integers. Let the Utumi ring of quotients be denoted by $\mathscr{Q}$, the extended centroid of $\mathscr{R}$ by $\mathscr{C}$. Consider $\mathscr{L}$ to be Lie ideal of $\mathscr{R}$ non-central in nature and $\mathscr{T}$ be a non-zero inner generalized derivation of $\mathscr{R}$ induced by elements $a, b \in \mathscr{Q}$. If $[\mathscr{T}(u^s), u^t]^m = [\mathscr{T}(u), u]$, for every $u \in \mathscr{L}$, where m, s and t be the fixed positive integers such that $m > 1$, $s \geq 1$ and $t \geq 1$, then one of the following situations prevails:*

- (i) *The standard identity $s_4(x_1, \ldots, x_4)$ is satisfied by $\mathscr{R}$ and there exists $a \in \mathscr{Q}$ and $\beta \in \mathscr{C}$ such that $\mathscr{T}(x) = \beta x + ax + xa$, for every $x \in \mathscr{R}$.*
- (ii) *there exists certain $\theta \in \mathscr{C}$ such that $\mathscr{T}(x) = \theta x$, for every $x \in \mathscr{R}$.*

Proof The following GPI is satisfied by $\mathscr{R}$

$$\Pi(w_1, w_2) = \{[a[w_1, w_2]^s + [w_1, w_2]^s b, [w_1, w_2]^t]\}^m - [a[w_1, w_2] + [w_1, w_2]b, [w_1, w_2]],$$

for every $w_1, w_2 \in \mathscr{R}$. Owing to Beidar [2, Theorem 2] and also by Fact 3, this GPI is satisfied by $\mathscr{Q}$ also. When $\mathscr{C}$ is infinite, then $\Pi(r_1, r_2) = 0$, for every $r_1, r_2 \in \mathscr{Q} \otimes \overline{\mathscr{C}}$, where $\overline{\mathscr{C}}$ is the algebraic closure of $\mathscr{C}$. We note that since both $\mathscr{Q} \otimes \overline{\mathscr{C}}$ and $\mathscr{Q}$ are centrally closed (see [9, Theorems 2.5 and 3.5]), we may replace $\mathscr{R}$ by $\mathscr{Q} \otimes \overline{\mathscr{C}}$ or $\mathscr{Q}$, in accordance with the situation whether $\mathscr{C}$ is infinite or finite. Thus we may assume that $\mathscr{R}$ is centrally closed over $\mathscr{C}$ which is either algebraically closed or finite.
Case I: If $a, b \in \mathscr{C}$, then for certain $\theta \in \mathscr{C}$, $\mathscr{T}(x) = \theta x$ for every $x \in [\mathscr{R}, \mathscr{R}]$.
Case II: If either $a \notin \mathscr{C}$ or $b \notin \mathscr{C}$, in this situation by [4], $\Pi(w_1, w_2)$ is a non-trivial GPI satisfied by $\mathscr{R}$. Hence, by Martindale's strong result known as Martindale's Theorem [16], $\mathscr{R}$ is bound to be primitive ring with non-zero socle $\mathscr{S}$ with $\mathscr{C}$ as the division ring. Under the awe of Jacobson's Theorem [11, p. 75], $\mathscr{R}$ is isomorphic to a dense ring of linear transformation on certain linear space $\mathscr{V}$ over $\mathscr{C}$.

If $dim_{\mathscr{C}}(\mathscr{V}) = 2$, then $\mathscr{R} \cong M_2(C)$, the ring of all 2×2 matrices over $\mathscr{C}$. Thus, we may observe from the Fact 6 that, $s_4(x_1, \ldots, x_4)$ is satisfied by $\mathscr{R}$. Now take $b - a = \sum_{ij} \gamma_{ij} e_{ij}$, where $\gamma_{ij} \in \mathscr{C}$ and e_{ij} are the unit standard matrices.

Let $[w_1, w_2] = [e_{ii}, e_{ij}] = e_{ij}$, for any i different from j, from our assumption and suppose χ denote the following

$$\chi = [a(e_{ij})^s + (e_{ij})^s b, (e_{ij})^t]^m - [ae_{ij} + e_{ij}b, e_{ij}].$$

When $s = t = 1$. Then $\chi = \{e_{ij}(b - a)e_{ij}\}^m - \{e_{ij}(b - a)e_{ij}\} = 0$. Consequently, we have that

$$e_{ij}(b - a)e_{ij} = 0, \text{ since } m > 1. \tag{1}$$

This gives that $\gamma_{ji} = 0$, for every i different from j. That is, off-diagonal entries of $b - a$ vanishes to zero. Hence, $b - a$ is diagonal matrix. Also, when $s > 1,\ t = 1$ we get the same conclusion that $b - a$ is a diagonal matrix. Indeed, we can easily see that χ gives the following relation

$$[ae_{ij} + e_{ij}b, e_{ij}] = 0.$$

This implies that

$$e_{ij}(b - a)e_{ij} = 0, \quad \text{which is Eq. (1) thus it gives the same conclusion.}$$

Lastly, for every s and $t > 1$, there still holds the same conclusion by similar tactic as above.

Thus in all, let $\psi(x) = (1 + e_{ij})x(1 - e_{ij})$, for every $x \in \mathscr{R}$, be an inner automorphism induced by matrix $(1 + e_{ij})$ where i different from j. By Fact 7, $\psi(b - a)$ is also a diagonal matrix. Therefore, the (i, j) entry of $\psi(b - a)$ is indeed zero.

$$0 = [\psi(b - a)]_{ij} = \gamma_{jj} - \gamma_{ii}, \text{ that is } \gamma_{jj} = \gamma_{ii}.$$

The above calculation establishes that $b - a \in \mathscr{C}$. Thus, we take opportunity to write the generalized derivation $\mathscr{T}(x) = ax + xb$ as, $\mathscr{T}(x) = ax + \beta x + xa$, where $\beta = b - a \in \mathscr{C}$.

Now we assume that $\dim_{\mathscr{C}}(\mathscr{V}) \geq 3$. The following relation holds for every $u \in [\mathscr{R}, \mathscr{R}]$

$$\{au^{s+t} + u^s bu^t - u^t au^s - u^{s+t}b\}^m - \{au^2 + ubu - uau - u^2 b\} = 0. \tag{2}$$

Further, we claim that for every $v \in \mathscr{V}$, the set $\{v, bv\}$ is linearly $\mathscr{C}$-dependent. For that we suppose on contrary that there exits non-zero $v_o \in \mathscr{V}$ such that $\{v_o, bv_o\}$ are linearly $\mathscr{C}$-independent. Since $\dim_{\mathscr{C}}(\mathscr{V}) \geq 3$, there exists non-zero $w_o \in \mathscr{V}$ such that $\{v_0, bv_o, w_o\}$ are linearly $\mathscr{C}$-independent. With the gratitude towards Jacobson's Theorem, we see that there exists $u_1, u_2 \in \mathscr{R}$ so that the following relations hold

$u_1 v_o = 0,\ \ u_2 v_o = v_o,\ \ u_1 bv_o = w_o;$

$u_2 bv_o = bv_o,\ \ u_1 w_o = -2v_o$ and $u_2 w_o = 0.$

Hence, for some $u \in [\mathscr{R}, \mathscr{R}]$ say $u = [u_1, u_2]$, we have that

$uv_o = 0, \ ubv_o = w_o, \quad uw_o = 2v_o.$
Now right multiplying by v_o in relation (2), we obtain that

$$\{au^{s+t} + u^s bu^t - u^t au^s - u^{s+t}b\}^m v_o - \{au^2 + ubu - uau - u^2 b\}v_o = 0.$$

If either $s > 1$ and $t \geq 1$ or $s = 1$ and $t > 1$, we face the same contradiction that $2v_o = 0$. Now, for $s = t = 1$, we have

$$\{au^2 + ubu - uau - u^2 b\}^m v_o - \{au^2 + ubu - uau - u^2 b\}v_o = 0.$$

Making proper use of Density theorem, one can see there exists certain u_1 and $u_2 \in \mathscr{R}$ due to which
$u_1 v_o = 0, \ u_2 v_o = v_o, \ u_1 bv_o = w_o;$
$u_2 bv_o = bv_o, \ u_1 w_o = -\alpha v_o,$ where $0 \neq \alpha \in \mathscr{C}$ and $u_2 w_o = 0.$
Hence, for some $u \in [\mathscr{R}, \mathscr{R}]$ say $u = [u_1, u_2]$, we have that
$uv_o = 0, \ ubv_o = w_o, \quad uw_o = \alpha v_o.$
Now right multiplying by v_o in above relation, we obtain

$$\{au^2 + ubu - uau - u^2 b\}^{m-1}\{au^2 + ubu - uau - u^2 b\}v_o - \{au^2 + ubu - uau - u^2 b\}v_o = 0.$$

This implies that

$$\{au^2 + ubu - uau - u^2 b\}^{m-1}(-\alpha v_o) + \alpha v_o = 0.$$

Applying this linear transformation $m - 1$ times on v_o, we have the following simple consequence

$$((-\alpha)^m + \alpha)v_o = 0.$$

Since $\alpha \in \mathscr{C}$, for $\alpha = 1$ we get $((-1)^m + 1)v_o = 0$, which gives contradiction for every even positive integers as char$(\mathscr{R}) \neq 2$. For $\alpha = 1 + 1$ (say 2), $((-2)^{m-1} - 1)v_o = 0$, wherein a contradiction is prompted as char$(\mathscr{R}) \neq M$. Therefore, for every $v \in \mathscr{V}$, the set of vectors $\{bv, v\}$ is linearly $\mathscr{C}$-dependent and for every $v \in \mathscr{V}, \ bv = \beta_v v$, for certain $\beta_v \in \mathscr{C}$. It is easy consequence that β_v does not depends on the vector $v \in \mathscr{V}$ and thus we consider $bv = \beta v$, for every $v \in \mathscr{V}$ and for some fixed $\beta \in \mathscr{C}$. Further, assume that for every $u \in \mathscr{R}$ and for any $v \in \mathscr{V}$, we have

$$[u, b]v = u(bv) - b(uv) = u(\beta v) - \beta uv = 0.$$

Hence, we have $[u, b]\mathscr{V} = 0$, as $[u, b]$ is a linear transformation that acts faithfully on the linear space $\mathscr{V}$. Therefore, $[u, b] = 0$, for every $u \in \mathscr{R}$. Thus, $b \in \mathscr{Z}(\mathscr{R}) \subseteq \mathscr{C}$. Therefore, relation (2) reduces to the following relation

$$\{au^{s+t} - u^t au^s\}^m - \{au^2 - uau\} = 0, \ \text{for every } u \in [\mathscr{R}, \mathscr{R}]. \tag{3}$$

Let us now try to prove that $\{av, v\}$ are linearly $\mathscr{C}$-dependent. In that attempt, we suppose on contrary that for some non-zero $v' \in \mathscr{V}$, $\{av', v'\}$ are linearly $\mathscr{C}$-independent. Since $\dim_{\mathscr{C}}(\mathscr{V}) \geq 3$, there exists $w' \in \mathscr{V}$ such that $\{av', v', w'\}$ are linearly $\mathscr{C}$-independent. Owing to Jacobson's Theorem, there exists $u_1', u_2' \in \mathscr{R}$ so that following relations hold

$u_1'v' = v',\ \ u_2'v' = v',\ \ u_1'av' = -2v'$;
$u_2'av' = 2v',\ \ u_1'w' = -v'$ and $u_2'w' = v'$.

Hence, for some $u \in [\mathscr{R}, \mathscr{R}]$, say $u = [u_1', u_2']$, we have $uv' = [u_1', u_2']v' = 0$, $uav' = [u_1', u_2']av' = 4v'$, $uw' = [u_1', u_2']w' = 2v'$. Now right multiplying by w' in relation (3), we obtain that

$$\{au^{s+t} - u^t a u^s\}^m w' - \{au^2 - uau\}w' = 0,\ \text{for every } u \in [\mathscr{R}, \mathscr{R}]. \tag{4}$$

When $s = t = 1$. Then we obtain from the above relation that, $8v' = 0$, which is a contradiction. Now if either $s > 1,\ t \geq 1$ or $s \geq 1,\ t > 1$, we arrive at the same contradiction. Thus, for every vector $v \in \mathscr{V}$, the set of vectors $\{av, v\}$ is linearly $\mathscr{C}$-dependent and by same technique utilized to show $b \in \mathscr{C}$, we get $a \in \mathscr{C}$. Thus in all, for $\dim_{\mathscr{C}}(\mathscr{V}) \geq 3$, we get a contradiction that both a and b are in $\mathscr{C}$.

Finally, we conclude that if $\dim_{\mathscr{C}}(\mathscr{V}) = 2$, then $s_4(x_1, \ldots, x_4)$ is satisfied by $\mathscr{R}$ and $\mathscr{T}(x) = \beta x + ax + xa$, for every $x \in \mathscr{R}$, where $\beta = b - a \in \mathscr{C}$.

4 The Study of General Case

In this segment of the proof, we begin by considering that $\mathscr{T}$ is a generalized derivation. In an attempt to prove the main result, we consider for certain $a \in \mathscr{Q}$ and μ a derivation of $\mathscr{R}$, we have $\mathscr{T}(x) = \mu(x) + ax$ by using Lee [15].

Theorem 8 *Suppose $\mathscr{R}$ is a prime ring that is non-commutative in structure and characteristic of $\mathscr{R}$ is a positive integer apart from 2 and $M = (-2)^{k-1} - 1$ where k is any odd positive integers greater than one. Let the Utumi ring of quotients be denoted by $\mathscr{Q}$, the extended centroid of $\mathscr{R}$ by $\mathscr{C}$. Consider $\mathscr{L}$ to be Lie ideal of $\mathscr{R}$ non-central in nature and $\mathscr{T}$ be a non-zero generalized derivation of $\mathscr{R}$. If $[\mathscr{T}(u^s), u^t]^m = [\mathscr{T}(u), u]$, for every $u \in \mathscr{L}$, where $m,\ s$ and t be the fixed positive integers such that $m > 1,\ \ s \geq 1$ and $t \geq 1$, then one of the following situations prevails:*

(*i*) *The standard identity $s_4(x_1, \ldots, x_4)$ is satisfied by $\mathscr{R}$ and there exists $a \in \mathscr{Q}$ and $\beta \in \mathscr{C}$ such that $\mathscr{T}(x) = \beta x + ax + xa$, for every $x \in \mathscr{R}$.*
(*ii*) *there exists certain $\theta \in \mathscr{C}$ such that $\mathscr{T}(x) = \theta x$, for every $x \in \mathscr{R}$.*

Proof We find that for certain $a \in \mathscr{Q}$ and μ a derivation of $\mathscr{R}$ related to $\mathscr{T}$ a generalized derivation where $\mathscr{T}(x) = \mu(x) + ax$, for every $x \in \mathscr{R}$. Owing to the Fact 2, we may extend the definition of a generalized derivation on $\mathscr{R}$ to that on the Utumi ring of quotients $\mathscr{Q}$. Further, by the Fact 5, there exists a non-zero ideal $\mathscr{H}$ of $\mathscr{R}$

such that $0 \neq [\mathscr{H}, \mathscr{R}] \subseteq \mathscr{L}$. Also by Fact 4, $\mathscr{H}$ and $\mathscr{Q}$ satisfy the same differential identity, hence for every $w_1, w_2 \in \mathscr{Q}$,

$$[a[w_1, w_2]^s + \mu([w_1, w_2]^s), [w_1, w_2]^t]^m = [a[w_1, w_2] + \mu([w_1, w_2]), [w_1, w_2]]. \quad (5)$$

Under the effect of Kharchenko theory (See [12]), we bifurcate our situation as follows.

(1) **When μ is an inner derivation.**
Then there exists c from $\mathscr{Q}$ such that μ can be expressed as $\mu(w) = [c, w]$ for every $w \in \mathscr{R}$. Hence $\mathscr{T}(w) = (a + c)w - wc$, for every $w \in \mathscr{R}$. Therefore, from differential identity (5), we have
$[(a + c)[w_1, w_2]^s - [w_1, w_2]^s c, [w_1, w_2]^t]^m = [(a + c)[w_1, w_2] - [w_1, w_2]c, [w_1, w_2]]$. Hence, on using Proposition 1 for $(a + c)$ and c, we are done.

(2) **When μ is an outer derivation.**
We observe that

$$\mu([w_1, w_2]^s) = \sum_{i=0}^{s-1} [w_1, w_2]^i \{[\mu(w_1), w_2] + [w_1, \mu(w_2)]\}[w_1, w_2]^{s-i-1}. \quad (6)$$

Using relation (6) in differential identity (5), we have

$$[(a[w_1, w_2]^s + \sum_{i=0}^{s-1} [w_1, w_2]^i \{[\mu(w_1), w_2] + [w_1, \mu(w_2)]\}[w_1, w_2]^{s-i-1}, [w_1, w_2]^t]^m$$
$$= [a[w_1, w_2] + [\mu(w_1), w_2] + [w_1, \mu(w_2)], [w_1, w_2]].$$

The following GPI is satisfied by $\mathscr{Q}$

$$[(a[w_1, w_2]^s + \sum_{i=0}^{s-1} [w_1, w_2]^i \{[y_1, w_2] + [w_1, y_2]\}[w_1, w_2]^{s-i-1}, [w_1, w_2]^t]^m \quad (7)$$

$$= [a[w_1, w_2] + [y_1, w_2] + [w_1, y_2], [w_1, w_2]].$$

Assume $y_1 = y_2 = 0$, thus, using Proposition 1, we recollect that a is central. Under this privilege, we rewrite relation (7) as the following

$$\left[\sum_{i=0}^{s-1} [w_1, w_2]^i \{[y_1, w_2] + [w_1, y_2]\}[w_1, w_2]^{s-i-1}, [w_1, w_2]^t\right]^m \quad (8)$$

$$= [[y_1, w_2] + [w_1, y_2], [w_1, w_2]].$$

The above relation (8), is a PI for $\mathscr{R}$, then by a well-known Posner's result [17], we observe that there exists certain field $\mathscr{F}$ and an integer $l \geq 1$ such that $\mathscr{Q}$

and $M_l(\mathscr{F})$ satisfy the same PI (polynomial identities). It is evident that $l \geq 2$ as $\mathscr{R}$ is non-commutative. From now onwards, we will consider the following matrices

$$w_1 = e_{pp}, \quad w_2 = e_{qp}, \quad y_1 = 0, \quad y_2 = \gamma e_{pq}, \text{ where } p \neq q \text{ and } 0 \neq \gamma \in \mathscr{F}.$$

Therefore,

$$[w_1, w_2] = [e_{pp}, e_{qp}] = -e_{qp} \text{ and } [w_1, y_2] = [e_{pp}, \gamma e_{pq}] = \gamma e_{pq}.$$

Thus relation (8) reduces to the following

$$[\sum_{i=0}^{s-1}(-e_{qp})^i\{\gamma e_{pq}\}(-e_{qp})^{s-i-1}, (-e_{qp})^t]^m = [\gamma e_{pq}, -e_{qp}].$$

In above equation, put $s = t = 1$, we have

$$[\gamma e_{pq}, -e_{qp}]^m = [\gamma e_{pq}, -e_{qp}].$$

Now, after a simple calculation, we obtain that $\{-\gamma(e_{pp} - e_{qq})\}^m = -\gamma(e_{pp} - e_{qq})$, right multiplying above equation by e_{pp} we are in the receipt of the following relation $((-\gamma)^m + \gamma)e_{pp} = 0$. When γ is chosen as 1, we get a contradiction for every even positive integers as char$(\mathscr{R}) \neq 2$. When $\gamma = 1 + 1$ (say 2) we again face a contradiction $((-2)^m + 2)e_{pp} = 0$ as char$(\mathscr{R}) \neq M$. In the same way, put $s = 2$ and $t = 1$ in above, we get

$$[\{\gamma e_{pq}\}(-e_{qp}) + (-e_{qp})\{\gamma e_{pq}\}, (-e_{qp})]^m = [\gamma e_{pq}, -e_{qp}].$$

This implies that

$$0 = -\gamma(e_{pp} - e_{qq}).$$

That is, $e_{pp} = e_{qq}$ if and only if $p = q$, which is a contradiction to our assumption. Similarly, for $s \geq 3$ and $t = 1$ gives contradiction that can be easily verified. At last, it can be easily seen that when $t > 1$, $0 = [\gamma e_{pq}, -e_{qp}]$ which still gives a contradiction.

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