On the Singularity Problem for the Euler Equations

Dongho Chae

Abstract In this expository article we discuss the finite time singularity problem for the three dimensional incompressible Euler equations. The local in time well-posedness for the 3D Euler equations for initial data in the Sobolev space $H^k(\mathbb{R}^3)$, $k > 5/2$ is well-known. The question of the spontaneous apparition of singularity(blow-up), however, is a wide-open problem in the mathematical fluid mechanics. Here we overview some of the previous results on the problem, and present their recent updates. More specifically, after a brief review of Kato's classical local well-posedness result, we present the celebrated Beale, Kato and Majda's blow-up criterion, and its recent developments. After that, we review the results related to the Type I blow-up. Finally, we present recent studies on the singularity problem for the 2D Boussinesq equations, which is regarded as a good model problem for the axisymmetric 3D Euler equations.

1 Introduction

We consider a fluid flow with mass density $\rho = \rho(x,t)$, $(x,t) \in \mathbb{R}^3 \times [0,+\infty)$, which occupies the whole domain of $\mathbb{R}^3$. The two basic functions describing the motion of the flow are the fluid velocity $u = (u_1, u_2, u_3) = u(x,t)$ and the pressure $p = p(x,t)$ The mass conservation principle applied to any fixed domain $\Omega \subset \mathbb{R}^3$ during the fluid flows is expressed by the following equation:

$$\frac{d}{dt}\int_{\Omega} \rho(x,t)dx = -\int_{\partial\Omega} \rho u \cdot \nu dS, \tag{1}$$

where ν is the outward unit normal vector on $\partial\Omega$. Indeed, the left-hand side of (1) is the mass increasing rate in time for the fluid occupying Ω, while the right-hand

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D. Chae (✉)
Department of Mathematics, Chung-Ang University, Seoul 06974, Korea
e-mail: dchae@cau.ac.kr

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side of (1) represents the total mass per unit time, escaping Ω through the boundary $\partial\Omega$, and the equality of (1) is nothing but the mass conservation for fixed domain Ω. Applying the Gauss theorem to the right-hand side of (1), we find easily

$$\int_{\Omega} \{\rho_t + \nabla \cdot (\rho u)\}\, dx = 0,$$

which holds for any domain $\Omega \subset \mathbb{R}^3$. Therefore, we have the differential form of the mass conservation law in fluid as follows:

$$\rho_t + \nabla \cdot (\rho u) = 0. \tag{2}$$

Next, we apply the momentum balance principle, which is Newton's second law of motion, to a fluid in a ball $B(x,r) = \{y \in \mathbb{R}^3 \mid |x-y| < r\}$. Given $t \geqslant 0$, let $x(t) \in$ be the position of the fluid particle. Then, the velocity of the fluid at t satisfies $\frac{dx(t)}{dt} = u(x(t),t)$, while the acceleration is given by

$$\begin{aligned}\frac{d^2x(t)}{dt^2} &= \frac{d}{dt}u(x(t),t) = \frac{\partial u}{\partial t} + \frac{dx(t)}{dt}\cdot \nabla u(x(t),t)\\ &= \frac{\partial u}{\partial t} + u\cdot\nabla u.\end{aligned}$$

Therefore, the momentum of the fluid per unit volume at (x,t) is given by

$$\rho(x,t)\frac{d^2x(t)}{dt^2} = \rho\frac{\partial u}{\partial t} + \rho u\cdot\nabla u. \tag{3}$$

The force due to the pressure on the surface $\partial B(x,r)$ is given by

$$-\int_{\partial B(x,r)} p(y,t)\nu dS, \tag{4}$$

where we consider only the force resulting from the normal directional contribution by the pressure. Actually in this consideration we use implicitly the assumption that the fluid is *ideal*. In the real physical situation we need to consider also the tangential part of the contribution of the pressure to the body force. Applying the Gauss theorem, the surface integral of (4) is transformed into

$$-\int_{B(x,r)} \nabla p(y,t)dy.$$

Hence, the force on the fluid particle at (x,t) per unit volume is given by

$$-\lim_{r\to 0}\frac{1}{|B(x,r)|}\int_{B(x,r)} \nabla p(y,t)dy = -\nabla p(x,t), \tag{5}$$

where we denote by $|A|$ the volume of $A \subset \mathbb{R}^3$. The momentum balance principle ensures the quality of (3) with (5), and we obtain

$$\rho \frac{\partial u}{\partial t} + \rho u \cdot \nabla u = -\nabla p. \tag{6}$$

The system (2) and (6) was derived first in 1755 by E. Euler in [38], and is called the Euler equations. For simplicity we further assume the homogeneity of the fluid, which means that $\rho(x,t) \equiv \text{constant} = 1$. In this case (2) reduces to the incompressibility condition $\nabla \cdot u = 0$, and the Euler equations become

$$(E) \begin{cases} u_t + u \cdot \nabla u = -\nabla p, \\ \nabla \cdot u = 0. \end{cases}$$

This is the homogeneous incompressible Euler equations for the ideal fluid. There are many nice textbooks and survey papers on the mathematical theories on the Euler equations [1, 3, 4, 26, 30–32, 46, 49]. In this article after brief studies of some of the basic properties of the equations, we review some of the classical results, and then survey recent progress on the singularity problems of the Euler equations.

Let us start by introducing the quantity $\omega = \nabla \times u$ called the vorticity, which has an important role in the incompressible fluid mechanics. Using the general vector calculus identity, $\nabla(u \cdot v) = u \cdot \nabla v + u \cdot \nabla v + u \times (\nabla \times v) + v \times (\nabla \times u)$, one can deduce

$$u \cdot \nabla u = -u \times (\nabla \times u) + \frac{1}{2}\nabla |u|^2.$$

Inserting this into the first equation of (E), we find a different form of the Euler equations

$$u_t - u \times \omega = -\nabla Q, \quad Q = \frac{1}{2}|u|^2 + p. \tag{7}$$

The quantity Q above is called the head pressure of the fluid. According to the Bernoulli theorem (see e.g. [29]) Q is constant along the stream lines. Taking curl of (7), and using the identity $\nabla \times (u \times \omega) = -u \cdot \nabla \omega - \omega \cdot \nabla u$, which holds for $\nabla \cdot u = 0$, we derive another form of the Euler equations, called the vorticity formulation.

$$\begin{cases} \omega_t + u \cdot \nabla \omega = \omega \cdot \nabla u, \\ \nabla \cdot u = 0, \ \nabla \times u = \omega. \end{cases} \tag{8}$$

The second line of (8) can be viewed as a linear elliptic system for given ω. Formally, it can be solved as follows. From $\nabla \cdot u = 0$, applying the Poincaré lemma, there exists a vector field $\psi = (\psi_1, \psi_2, \psi_3)$ such that

$$u = \nabla \times \psi, \qquad \nabla \cdot \psi = 0.$$

The second equation is imposed to remove extra degree of freedom, which is similar to the gauge fixing in physics. Hence, we obtain the Poisson equation for ψ

$$\omega = \nabla \times (\nabla \times \psi) = \nabla(\nabla \cdot \psi) - \Delta\psi = -\Delta\psi. \tag{9}$$

Assuming sufficiently fast decay of ω at spatial infinity, we can solve (9), using the Newtonian potential,

$$\psi = -\Delta^{-1}\psi = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\omega}{|x-y|}dy,$$

from which we obtain the Biot-Savart formula

$$u(x,t) = -\nabla \times \Delta^{-1}\omega = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)\times\omega(y,t)}{|x-y|^3}dy,$$

which represents the velocity in terms of the vorticity. It is also very important to see the relation between ∇u and ω. ∇u is a matrix valued singular integral operator, which can be computed as follows (see [49] for more details). For $h \in \mathbb{R}^3$ we have

$$\begin{aligned}
\nabla u\, h &= -\nabla(\nabla \times \Delta^{-1}\omega)h \\
&= -PV\int_{\mathbb{R}^3}\left\{\frac{\omega(y)\times h}{|x-y|^3} + \frac{3}{4\pi}\frac{[(x-y)\times\omega(y)]\otimes(x-y)}{|x-y|^3}h\right\}dy \\
&\quad + \frac{1}{3}\omega(x)\times h, \\
&:= PV\int_{\mathbb{R}^3} K(x-y)\omega(y)dyh + \frac{1}{3}\omega(x)\times h,
\end{aligned} \tag{10}$$

where PV means the Cauchy principal value integral defined by

$$PV\int_{\mathbb{R}^n} K(x-y)f(y)dy = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} K(x-y)f(y)dy.$$

The kernel $K(\cdot)$ in (10) is typical of the integral kernels defining singular integral operator of the Calderon-Zygmund type, which have important roles in the harmonic analysis (see e.g. [53]). We can therefore obtain the following closed form of the vorticity formulation of the Euler equations

$$\begin{cases} \omega_t + u\cdot\nabla\omega = \omega\cdot\nabla u, \\ u(x,t) = \dfrac{1}{4\pi}\displaystyle\int_{\mathbb{R}^3}\frac{(x-y)\times\omega(y,t)}{|x-y|^3}dy. \end{cases} \tag{11}$$

Now we discuss the Lagrangian formulation of the Euler equations. Given $\alpha \in \mathbb{R}^3$ and a smooth vector field $u = u(x,t)$, let $X(\alpha,t)$ be the solution of the following ordinary differential equations:

$$\begin{cases} \dfrac{\partial X(\alpha,t)}{\partial t} = u(X(\alpha,t),t), \quad t>0 \\ X(\alpha,0)=\alpha \end{cases} \tag{12}$$

The parametrized mapping $\alpha \mapsto X(\alpha,t)$ is called *the particle trajectory mapping* generated by $u = u(x,t)$. When u is the velocity field, which is a solution of (E), we say the associated $X(\alpha,t)$ the Lagrangian coordinate, and describing the dynamics of the fluid flows in terms of $X(\alpha,t)$ is called *the Lagrangian description*. Roughly speaking, it is a coordinate transform from a stationary observer to a moving observer following the flows. In terms of the Lagrangian coordinate one finds immediately that the evolution equation of (E) is written as

$$\frac{\partial^2 X(\alpha,t)}{\partial t^2} = -\nabla p(X(\alpha,t),t). \tag{13}$$

Another important equation associated with the Lagrangian coordinates is the following *Cauchy's formula*,

$$\omega(X(\alpha,t),t) = \omega_0(\alpha)\cdot\nabla X(\alpha,t), \tag{14}$$

where $\omega_0(\alpha)=\omega(\alpha,0)$ is the initial vorticity. One can regard (14) as a translation of the first equation of (11) into the Lagrangian coordinates. For the details of the proof of (14) we refer [49].

Let us consider a closed curve $\mathcal{C}_0 = \{\gamma(s)\in\mathbb{R}^3 \,:\, s\in[0,1], \gamma(0)=\gamma(1)\}$. For a solution (u,p) of (E) and the particle trajectory mapping generated by u we define

$$\mathcal{C}_t = X(\mathcal{C}_0,t) = \{X(\gamma(s),t) \,:\, s\in[0,1], \gamma(0)=\gamma(1)\}.$$

Then, from (12) and (13) we find

$$\begin{aligned}
&\frac{d}{dt}\oint_{\mathcal{C}_t} u\cdot d\ell = \frac{d}{dt}\int_0^1 \frac{\partial X(\gamma(s),t)}{\partial t}\cdot\frac{\partial X(\gamma(s),t)}{\partial s}ds \\
&= \int_0^1 \frac{\partial^2 X(\gamma(s),t),t)}{\partial t^2}\cdot\frac{\partial X(\gamma(s),t)}{\partial s}ds + \int_0^1 \frac{\partial X(\gamma(s),t),t)}{\partial t}\cdot\frac{\partial^2 X(\gamma(s),t)}{\partial t\partial s}ds \\
&= -\int_0^1 \nabla p(X(\gamma(s),t),t)\cdot\frac{\partial X(\gamma(s),t)}{\partial s}ds + \frac{1}{2}\int_0^1 \frac{\partial}{\partial s}\left|\frac{\partial X(\gamma(s),t)}{\partial t}\right|^2 ds \\
&= -\int_0^1 \frac{\partial}{\partial s}\left(p(X(\gamma(s),t),t) - \frac{1}{2}\left|\frac{\partial X(\gamma(s),t)}{\partial t}\right|^2\right) ds = 0.
\end{aligned}$$

Therefore, we obtain the following Kelvin circulation theorem:

$$\oint_{\mathcal{C}_t} u \cdot d\ell = \oint_{\mathcal{C}_0} u \cdot d\ell \quad \forall t > 0. \tag{15}$$

Let $\mathcal{C}_0$ be a vortex line of the initial vorticity $\omega_0 = \omega(\cdot, 0)$, defined by

$$\mathcal{C}_0 = \left\{ \gamma(s) \in \mathbb{R}^3 \, : \, \frac{d}{ds}\gamma(s) = \lambda(s)\omega_0(\gamma(s)), s \in [0, 1], \right\}$$

for a real valued function function $\lambda(s) > 0$ for $s \in [0, 1]$. By reparametrization we may assume without the loss of generality that $\lambda(s) \equiv 1$. Then, we first claim $\mathcal{C}_t = X(\mathcal{C}_0, t) = \{X(\gamma(s), t) \, : \, s \in [0, 1]\}$ is a vortex line at $t > 0$. Indeed, from (14) we have

$$\begin{aligned} \frac{\partial X(\gamma(s), t)}{\partial s} &= \frac{d\gamma(s)}{ds} \cdot \nabla X(\gamma(s), t) \\ &= \omega_0(\gamma(s)) \cdot \nabla X(\gamma(s), t) \\ &= \omega(X(\gamma(s), t), t), \end{aligned} \tag{16}$$

and the claim is proved. Using (11), (13) and (16), we deduce

$$\begin{aligned} &\frac{d}{dt} \int_0^1 u(X(\gamma(s), t), t) \cdot \omega(X(\gamma(s), t), t) ds \\ &= - \int_0^1 \nabla p(X(\gamma(s), t), t) \cdot \omega(X(\gamma(s), t), t) ds \\ &\quad + \int_0^1 u(X(\gamma(s), t), t) \cdot (\omega(X(\gamma(s), t), t) \cdot \nabla) \, u(X(\gamma(s), t), t) ds \\ &= - \int_0^1 \frac{\partial}{\partial s} \left(p(X(\gamma(s), t), t) - \frac{1}{2} |u(X(\gamma(s), t), t)|^2 \right) ds = 0. \end{aligned} \tag{17}$$

Therefore, we obtain the following helicity conservation along each closed vortex line

$$\int_0^1 u(X(\gamma(s), t), t) \cdot \omega(X(\gamma(s), t), t) ds = \int_0^1 u_0(\gamma(s)) \cdot \omega_0(\gamma(s)) ds \tag{18}$$

for all $t > 0$. This can be viewed as a localized version of the following helicity conservation law,

$$H = \int_{\mathbb{R}^3} u(x, t) \cdot \omega(x, t) dx = \int_{\mathbb{R}^3} u_0(x) \cdot \omega_0(x) dx \quad \forall t > 0. \tag{19}$$

The proof of (19) follows easily by taking $\frac{dH}{dt}$, and using (E), (11), and integrating by parts. The most important conservation law in the study of the Euler equation is the following energy conservation

$$E = \frac{1}{2}\int_{\mathbb{R}^3} |u(x,t)|^2 dx = \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2 dx \quad \forall t > 0. \tag{20}$$

This can be shown by multiplying (7) by u, and integrating it over $\mathbb{R}^3$, and integrating by parts. The reason why the energy is important in the mathematical fluid mechanics is that it is positive definite, while all the other conserved quantities in the 3D Euler equations have no definite signs.

2 The Local in Time Well-Posedness

In this section we briefly review the studies on the Cauchy problem of (E). For this we recall the notions of the Sobolev spaces. Let $\Omega \subset \mathbb{R}^n$ be a measurable subset of $\mathbb{R}^n$, and f be a measurable function on Ω. We denote $|A|$ = Lebesgue measure of $A \subset \mathbb{R}^n$. Let us define the L^q-norm by

$$\|f\|_{L^q} = \begin{cases} \left(\int_\Omega |f|^q dx\right)^{\frac{1}{q}}, & \text{if} \quad 1 \leqslant q < +\infty, \\ \inf\{m \ : \ |\{x \in \mathbb{R}^n \,|\, |f(x)| > m\}| = 0\}, & \text{if} \quad q = +\infty. \end{cases}$$

Then, the Lebesgue space for $q \in [1, \infty]$ is

$$L^q(\Omega) = \{f \text{ measureable on } \Omega \ : \ \|f\|_{L^q} < +\infty\},$$

Let $\alpha = (\alpha_1, \cdots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ be a multi-index with $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then, the Sobolev space $W^{k,q}(\Omega)$ on $\Omega \subset \mathbb{R}^n$ for $k \in \mathbb{N}$, $1 \leqslant q < +\infty$ is defined as

$$W^{k,q}(\Omega) = \left\{ f \in L^q(\Omega) \ : \ \left(\sum_{|\alpha| \leqslant k} \|D^\alpha f\|^q_{L^q(\Omega)} \right)^{\frac{1}{q}} := \|f\|_{W^{k,q}(\Omega)} < +\infty \right\},$$

where the derivative D^α is in the sense of distribution. In the case $q = 2$ we also denote $W^{k,2}(\Omega) = H^k(\Omega)$. In the case $\Omega = \mathbb{R}^n$ we have an equivalent Sobolev norm in $H^k(\mathbb{R}^n)$ defined by the Fourier transform. Let $\hat{f}$ be the Fourier transform of f defined by

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi} dx, \tag{21}$$

where $i = \sqrt{-1}$. The function f is recovered from $\hat{f}$ by the inverse Fourier transform defined by

$$f(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{ix\cdot\xi} dx. \tag{22}$$

Then, $f \in H^k(\mathbb{R}^n)$ if and only if

$$\left(\int_{\mathbb{R}^n} (1+|\xi|^2)^k |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} := \|f\|_{H^k} < +\infty.$$

Using this equivalent norm, one can prove the following Sobolev inequality

$$\|f\|_{L^\infty} \leqslant C_k \|f\|_{H^k} \quad \forall k > \frac{n}{2}, \tag{23}$$

by a simple argument as follows.

Proof of (23): From (22) we find

$$\begin{aligned} |f(x)| &\leqslant \int_{\mathbb{R}^n} |\hat{f}(\xi)| d\xi = \int_{\mathbb{R}^n} (1+|\xi|^2)^{\frac{k}{2}} |\hat{f}(\xi)| (1+|\xi|^2)^{-\frac{k}{2}} |d\xi \\ &\leqslant \left(\int_{\mathbb{R}^n} (1+|\xi|^2)^k |\hat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^k} d\xi\right)^{\frac{1}{2}} \\ &\leqslant C_k \|f\|_{H^k}, \end{aligned} \tag{24}$$

where we use the fact that for $k > \frac{n}{2}$ the following holds

$$\int_{\mathbb{R}^n} \frac{1}{(1+|\xi|^2)^k} d\xi \leqslant C \int_0^\infty \frac{r^{n-1}}{(1+r^2)^k} dr < +\infty.$$

Taking the supremum of (24) over $x \in \mathbb{R}^n$, we obtain (23). ∎

A fundamental local in time well-posedness result for the Cauchy problem of (E) is the following theorem due to Kato [40].

Theorem 2.1 *Let $u_0 \in H^k(\mathbb{R}^3)$, $k > \frac{5}{2}$. Then, there exists $T = T(\|u_0\|_{H^k})$ such that a unique solution $u \in C([0,T); H^k(\mathbb{R}^3)) \cap AC(0,T; H^{k-1}(\mathbb{R}^3))$ exists with $u(\cdot, 0+) = u_0$, where $AC(a,b;X)$ denotes the class of X-valued functions u such that $t \mapsto u(t)$ is absolutely continuous.*

(Sketch of the proof) Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ a multi-index. We operate D^α on (E), and take $L^2(\mathbb{R}^3)$ inner product it with $D^\alpha u$. Then, summing over $|\alpha| \leqslant k$ we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|u\|_{H^k}^2 &= -\sum_{|\alpha|\leqslant k} \int_{\mathbb{R}^3} \{D^\alpha(u\cdot\nabla u) - (u\cdot\nabla)D^\alpha u\}\cdot D^\alpha u dx \\ &\quad - \sum_{|\alpha|\leqslant k} \int_{\mathbb{R}^3} (u\cdot\nabla)D^\alpha u \cdot D^\alpha u dx - \sum_{|\alpha|\leqslant k} \int_{\mathbb{R}^3} D^\alpha u \cdot \nabla D^\alpha p dx \\ &:= I_1 + I_2 + I_3. \end{aligned} \tag{25}$$

For I_3 we apply the integration by parts to have

$$I_3 = -\sum_{|\alpha|\leqslant k} \int_{\mathbb{R}^3} D^\alpha(\nabla \cdot u) D^\alpha p dx = 0. \tag{26}$$

Similarly, we also have for I_2,

$$I_2 = -\frac{1}{2} \sum_{|\alpha|\leqslant k} \int_{\mathbb{R}^3} (u \cdot \nabla)|D^\alpha u|^2 dx = \frac{1}{2} \sum_{|\alpha|\leqslant k} \int_{\mathbb{R}^3} (\nabla \cdot u)|D^\alpha u|^2 dx = 0. \tag{27}$$

In order to estimate I_1 we recall the following commutator estimate due to Klainerman and Majda [43]

$$\sum_{|\alpha|\leqslant k} \|D^\alpha(fg) - f D^\alpha g\|_{L^2} \\ \leqslant C_k \left\{ \|\nabla f\|_{L^\infty} \|D^{k-1} g\|_{L^2} + \|D^k f\|_{L^2} \|g\|_{L^\infty} \right\}. \tag{28}$$

Applying (28) to I_1 with $f = D^\alpha u$, $g = \nabla u$, we obtain, using the Cauchy-Schwartz inequality

$$I_1 \leqslant \sum_{|\alpha|\leqslant k} \|D^\alpha(u \cdot \nabla u) - (u \cdot \nabla) D^\alpha u\|_{L^2} \|D^\alpha u\|_{L^2} \\ \leqslant C \|\nabla u\|_{L^\infty} \|u\|^2_{H^k}. \tag{29}$$

Combining (26), (27) and (29), using (23) for $k > \frac{5}{2}$, we find

$$\frac{d}{dt} \|u\|_{H^k} \leqslant C \|\nabla u\|_{L^\infty} \|u\|_{H^k} \leqslant C \|u\|^2_{H^k}. \tag{30}$$

From this differential inequality we find that

$$\|u(t)\|_{H^k} \leqslant \frac{\|u_0\|_{H^k}}{1 - C_0 \|u_0\|_{H^k} t}, \tag{31}$$

and hence

$$\sup_{0<t<T} \|u(t)\|_{H^k} \leqslant 2\|u_0\|_{H^k} \quad \text{where} \quad T = \frac{1}{2C_0 \|u_0\|_{H^k}}. \tag{32}$$

From (E) we have

$$\left\| \frac{\partial u}{\partial t} \right\|_{H^{k-1}} \leqslant \|u \cdot \nabla u\|_{H^{k-1}} + \|\nabla p\|_{H^{k-1}} := J_1 + J_2. \tag{33}$$

In order to estimate J_1 we recall the following product estimate of the Sobolev spaces $H^m(\mathbb{R}^n)$ (see e.g. [49]).

$$\|fg\|_{H^m} \leqslant C_m(\|f\|_{L^\infty}\|g\|_{H^m} + \|g\|_{L^\infty}\|f\|_{H^m}) \quad \forall m > \frac{n}{2}. \tag{34}$$

Applying this to J_1, we find

$$J_1 \leqslant \|u\|_{L^\infty}\|u\|_{H^k} + \|\nabla u\|_{L^\infty}\|u\|_{H^{k-1}} \leqslant C\|u\|_{H^k}^2 \leqslant C\|u_0\|_{H^k}^2, \tag{35}$$

where we use the Sobolev inequality (23) for $k > \frac{5}{2}$ and (32). In order to estimate J_2 we recall the method of estimating the pressure. Taking divergence of the first equation of (E), and using the second equation of the divergence free condition, we find

$$\Delta p = -\sum_{i,j=1}^{3} \partial_i\partial_j(u_i u_j),$$

and

$$p = -\sum_{i,j=1}^{3} \Delta^{-1}\partial_i\partial_j(u_i u_j) = \sum_{i,j=1}^{3} R_i R_j(u_i u_j), \tag{36}$$

where R_j, $j = 1, 2, 3$, is the Riesz transform on $\mathbb{R}^3$. The definition of R_j is easily understood via its Fourier transform,

$$\widehat{R_j(f)}(\xi) = i\frac{\xi_j}{|\xi|}\hat{f}(\xi),$$

where $i = \sqrt{-1}$. The following Calderon-Zygmund type estimate [53] holds for the Riesz transform

$$\|R_j f\|_{H^m} \leqslant C\|f\|_{H^m} \quad \forall m \geqslant 0. \tag{37}$$

Applying (37) to J_2, we estimate

$$\begin{aligned} J_2 \leqslant \|p\|_{H^k} &\leqslant \sum_{i,j=1}^{3} \|R_i R_j(u_i u_j)\|_{H^k} \leqslant C\|u\otimes u\|_{H^k} \\ &\leqslant C\|u\|_{L^\infty}\|u\|_{H^k} \leqslant C\|u\|_{H^k}^2 \leqslant C\|u_0\|_{H^k}^2, \end{aligned} \tag{38}$$

where we use (34) and (32). Combining (35) and (38) with (33), we obtain

$$\left\|\frac{\partial u}{\partial t}\right\|_{H^{k-1}} \leqslant C\|u_0\|_{H^k}^2 \quad \forall t \in [0, T],$$

from which we have

$$\|u(t_2) - u(t_1)\|_{H^{k-1}} \leqslant \int_{t_1}^{t_2} \left\| \frac{\partial u(s)}{\partial s} \right\|_{H^{k-1}} ds$$
$$\leqslant C\|u_0\|_{H^k}^2 (t_2 - t_1) \quad \forall 0 < t_1 < t_2 < T.$$

Namely,

$$\|u\|_{Lip(0,T;H^{k-1})} \leqslant C\|u_0\|_{H^k}^2. \tag{39}$$

Once the a priori estimates (32) and (39) are obtained, the existence proof of a local in time solution is rather straightforward. We construct a sequence of the approximate solutions $\{u_m\}_{m\in\mathbb{N}}$ by mollification of (E) or by the Galerkin approximation. The sequence will be shown to satisfy the uniform estimate

$$\sup_{m\in\mathbb{N}} \|u_m\|_{L^\infty(0,T;H^k(\mathbb{R}^3))\cap Lip(0,T;H^{k-1}(\mathbb{R}^3))} \leqslant C\|u_0\|_{H^k}^2.$$

Applying the Lions-Aubin type compactness lemma (see e.g. [49]), there exist a subsequence u_{m_j} and the limit $u \in L^\infty(0,T;H^k(\mathbb{R}^3)) \cap Lip(0,T;H^{k-1}(\mathbb{R}^3))$ such that $u_{m_j} \to u$ in $L^\infty(0,T;H^{k-\varepsilon}_{\text{loc}}(\mathbb{R}^3))$ for all $\varepsilon > 0$. Using this strong convergence, we find that the limit $u \in L^\infty(0,T;H^k(\mathbb{R}^3)) \cap Lip(0,T;H^{k-1}(\mathbb{R}^3))$ satisfies (E). We now show the uniqueness. Let $u_1, u_2 \in L^\infty(0,T;H^k(\mathbb{R}^3))$ satisfy (E) with the pressure p_1 and p_2 and the initial data $u_{1,0}, u_{2,0} \in H^k(\mathbb{R}^3)$, respectively. Then, setting $u = u_1 - u_2$, $p = p_1 - p_2$, and subtracting the equation for u_2 from the one for u_1, we find that u satisfies

$$u_t + u_1 \cdot \nabla u + u \cdot \nabla u_2 = -\nabla p. \tag{40}$$

Taking L^2 inner product of (40) by u, and integrating by parts we obtain,

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 = -\int_{\mathbb{R}^3} (u\cdot\nabla)u_2 \cdot u dx$$
$$\leqslant \|\nabla u_2\|_{L^\infty}\|u\|_{L^2}^2,$$

from which we deduce

$$\|u(t)\|_{L^2} \leqslant \|u_0\|_{L^2} \exp\left(\frac{1}{2}\int_0^t \|\nabla u_2(s)\|_{L^\infty} ds\right)$$
$$\leqslant \|u_0\|_{L^2} \exp\left(C\int_0^t \|u_2(s)\|_{H^k} ds\right)$$
$$\leqslant \|u_0\|_{L^2} \exp\left(CT\|u_{2,0}\|_{H^k} ds\right), \tag{41}$$

which shows that $u_1 \equiv u_2$ on $\mathbb{R}^3 \times [0,T]$ if $u_{1,0} = u_{2,0}$. ■

After the above results Kato and Ponce proved the local well-posedness in more general Sobolev spaces $W^{s,p}(\mathbb{R}^n)$, $s > \frac{n}{p} + 1$ [41]. This local well-posedness can be extended to exotic spaces such as the Besov space [12], and Triebel-Lizorkin spaces [13, 28]. Recently, the spatial decay conditions of such function class have been relaxed to allow linear growth of the velocities [20].

3 The BKM Type Blow-Up Criterion

Let $u \in C([0,T); H^k(\mathbb{R}^3))$, $k > \frac{5}{2}$, be a smooth solution to (E). We say that solution blows up at $t = T$ if

$$\limsup_{t \to T} \|u(t)\|_{H^k} = +\infty. \tag{42}$$

The question of finite time blow-up for (E) for a smooth initial data $u_0 \in H^k(\mathbb{R}^3)$ with $k > \frac{5}{2}$ is an outstanding open problem in the mathematical fluid mechanics. There are many survey papers [1, 4, 32], and numerical results [42, 47, 48] devoted to this problem. We also mention that in the case where the domain of the fluid has a singular boundary finite time blow-up is shown in [36]. Also in [27] authors proved apparition of singularity of (E) on the boundary of a cylinder. Our main concern here is the possibility of *interior singularity* in the whole domain for smooth initial data belonging to the above Sobolev space. In this direction one of the most celebrated results is the following theorem by Beale, Kato and Majda [2].

Theorem 3.1 (BKM criterion) *Let $u \in C([0,T); H^k(\mathbb{R}^3))$, $k > \frac{5}{2}$, be a local in time smooth solution to (E). Then, the solution blows at $t = T$ if and only if $\int_0^T \|\omega(t)\|_{L^\infty} dt = +\infty$.*

This theorem was later refined by Kozono and Taniuchi [45], replacing the L^∞ norm of ω by the BMO norm. See also a geometric type blow-up criterion [33, 35], controlling the blow-up in terms of the direction field $\xi = \omega/|\omega|$ of the vorticity. We recall the notion of BMO, the class of functions with bounded mean oscillations, which is first introduced by John and Nirenberg [39]. For $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ let us set

$$f_{x,r} = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy.$$

Then, BMO is defined by

$$BMO = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) \, : \, \sup_{x \in \mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y) - f_{x,r}| dy =: \|f\|_{BMO} < +\infty \right\}.$$

We observe the obvious inequality, immediately from the definition

$$\|f\|_{BMO} \leqslant 2\|f\|_{L^\infty}. \tag{43}$$

It is well-known that BMO is bounded by the mapping of the singular integral operator $\mathcal{P}$ of the Calderon-Zygmund type

$$\|\mathcal{P}(f)\|_{BMO} \leqslant C\|f\|_{BMO}. \tag{44}$$

In particular we have

$$\|\nabla u\|_{BMO} \leqslant C\|\omega\|_{BMO} \tag{45}$$

(see (10)). A refined version of Theorem 3.1 due to Kozono and Taniuchi [45] is the following.

Theorem 3.2 *Let $u \in C([0,T); H^k(\mathbb{R}^3))$, $k > \frac{5}{2}$, be a local in time smooth solution to (E). Then, the solution blows at $t = T$ if and only if $\int_0^T \|\omega(t)\|_{BMO} dt = +\infty$.*

(Sketch of the proof) We recall the following version of the logarithmic Sobolev inequality in $\mathbb{R}^n$, which is the key inequality of the proof.

$$\|f\|_{L^\infty} \leqslant C(1 + \|f\|_{BMO}) \log(e + \|f\|_{H^m}) \quad \forall m > \frac{n}{2}. \tag{46}$$

Applying (45) and (46) to the first part of the estimate (30), we find

$$\begin{aligned}\frac{d}{dt}\|u\|_{H^k} &\leqslant C\|\nabla u\|_{L^\infty}\|u\|_{H^k}\\ &\leqslant C(1 + \|\nabla u\|_{BMO}) \log(e + \|u\|_{H^k})\|u\|_{H^k}\\ &\leqslant C(1 + \|\omega\|_{BMO}) \log(e + \|u\|_{H^k})\|u\|_{H^k}.\end{aligned}$$

Therefore, setting $a(t) = 1 + \|\omega(t)\|_{BMO}$, $y(t) = e + \|u(t)\|_{H^k}$, we obtain the differential inequality

$$\frac{dy}{dt} \leqslant Ca(t)y \log y.$$

This can be solved to lead us to

$$y(t) \leqslant y_0^{\exp\left(C\int_0^t a(s)ds\right)}.$$

Hence,

$$e + \|u(t)\|_{H^k} \leqslant (e + \|u_0\|_{H^k})^{\exp\left(C\int_0^t (1+\|\omega(s)\|_{BMO})ds\right)}.$$

This shows that

$$\limsup_{t\to T} \|u(t)\|_{H^k} = +\infty \Rightarrow \int_0^T \|\omega(t)\|_{BMO} dt = +\infty.$$

On the other hand, the following inequalities

$$\int_0^T \|\omega(t)\|_{BMO}dt \leqslant 2\int_0^T \|\omega(t)\|_{L^\infty}dt \leqslant 2T \sup_{0<t<T} \|\nabla u(t)\|_{L^\infty} \quad \leqslant CT \sup_{0<t<T} \|u(t)\|_{H^k},$$

where we use the Sobolev inequality (23) in the last step, show that

$$\int_0^T \|\omega(t)\|_{BMO}dt = +\infty \Rightarrow \limsup_{t\to T} \|u(t)\|_{H^k} = +\infty.$$

■

Theorem 3.1 has been localized recently in [19]. To state the result we recall the notion of the local BMO space. For $r > 0$ and $x \in \mathbb{R}^n$ we denote $B(x, r) = \{y \in \mathbb{R}^n \mid |x - y| < r\}$, and $B(r) = B(0, r)$ below. By $BMO(B(r))$ we denote the space of all $u \in L^1(B(r))$ such that

$$|u|_{BMO(B(r))} = \sup_{\substack{z\in B(r)\\ 0<\rho\leqslant 2r}} \frac{1}{|B(z,\rho)\cap B(r)|}\int_{B(z,\rho)\cap B(r)} |u - u_{B(z,\rho)\cap B(r)}|dy < +\infty,$$

where we use the following notation for the average of u over $\Omega \subset \mathbb{R}^n$.

$$u_\Omega = \frac{1}{|\Omega|}\int_\Omega u dx.$$

The space $BMO(B(r))$ will be equipped with the norm

$$\|u\|_{BMO(B(r))} = |u|_{BMO(B(r))} + r^{-n}\|u\|_{L^1(B(r))}.$$

Note that $BMO(B(r))$ is continuously embedded into $L^q(B(r))$ for all $1 \leqslant q < +\infty$, and it holds

$$\|u\|_{L^q(B(r))} \leqslant c r^{\frac{n}{q}} \|u\|_{BMO(B(r))}.$$

The following is the localized version of Theorem 3.1.

Theorem 3.3 *Let $u \in C^1(B(\rho) \times (T-\rho, T))$ be a solution to (E) such that $u \in C([T-\rho, T); W^{2,q}(B(\rho))) \cap L^\infty(T-\rho, T; L^2(B(\rho)))$ for some $q \in (3, +\infty)$. If u satisfies*

$$\int_{T-\rho}^{T} |\omega(s)|_{BMO(B(\rho))}ds < +\infty, \tag{47}$$

then there exists no blow-up in $B(\rho) \times \{t = T\}$, namely

$$\limsup_{t\to T} \|u(t)\|_{W^{2,q}(B(r))} < +\infty \qquad \forall r \in (0, \rho).$$

(Idea of the Proof) There are three key ingredients of the proof of Theorem 3.3. The first one is the following local version of the logarithmic Sobolev inequality.

Lemma 3.4 *Let $B(r)$ be a ball in $\mathbb{R}^n$ with the radius $r > 0$. For every $u \in W^{1,q}(B(r))$, $n < q < +\infty$, the following inequality holds true*

$$\|u\|_{L^\infty(B(r))} \leqslant C(1 + \|u\|_{BMO(B(r))}) \log \left(e + c\|\nabla u\|_{L^q(B(r))} + Cr^{-1+\frac{n}{q}-\frac{n}{2}} \|u\|_{L^2(B(r))}\right) \tag{48}$$

with a constant $C > 0$ depending only on n and q.

The second key ingredient in the proof of Theorem 3.3 is the following localized version of the Kozono-Taniuchi inequality (see [44] for the global version).

Lemma 3.5 *Let $f, g \in BMO(B(r)) \cap L^q(B(r))$, $1 < q < +\infty$. Then $f \cdot g \in L^q(B(r))$ and it holds*

$$\begin{aligned} \|f \cdot g\|_{L^q(B(r))} \leqslant C\Big(&|f|_{BMO(B(r))}\|g\|_{L^q(B(r))} + |g|_{BMO(B(r))}\|f\|_{L^q(B(r))}\Big) \\ &+ Cr^{-\frac{3}{q}} \|f\|_{L^q(B(r))}\|g\|_{L^q(B(r))}, \end{aligned} \tag{49}$$

where the constant $C > 0$ depends on q only.

Using suitable sequence of cut-off functions, and using the above two lemmas one can have an iterative sequence of infinite inequalities for derivatives of the vorticity. In order to close this sequence of inequalities we establish the following Gronwall type iteration lemma.

Lemma 3.6 (Iteration lemma) *Let $a(t) \geqslant 0$ and $\beta_m : [t_0, t_1] \to \mathbb{R}$ be a sequences of bounded functions. Suppose there exists $K(t)$ such that*

$$|\beta_m(t)| < K(t)^m \qquad \forall t \in [t_0, t_1), \forall m \in \mathbb{N}.$$

Suppose

$$\beta_m(t) \leqslant Cm + \int_{t_0}^{t} a(s)\beta_{m+1}(s)ds, \quad m \in \mathbb{N} \cup \{0\}.$$

Then the following inequality holds true for all $t \in [t_0, t_1]$

$$\beta_0(t) \leqslant C \int_{t_0}^{t} a(s)ds \, e^{\int_{t_0}^{t} a(s)ds}.$$

■

We can also establish the similar continuation criterion for solutions belonging to the Hölder spaces [21]. For the precise statement of this result let us define the space $C^\alpha(\overline{\Omega})$, $0 < \alpha \leqslant 1$, containing all Hölder continuous and bounded functions $f : \overline{\Omega} \to \mathbb{R}$, $n \in \mathbb{N}$, such that

$$[f]_{\alpha,\Omega} := \sup_{\substack{x,y\in\overline{\Omega}\\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < +\infty.$$

The space $C^\alpha(\overline{\Omega})$ equipped by the norm $\|f\|_{C^\alpha(\bar{\Omega})} = \|f\|_{L^\infty(\Omega)} + [f]_{\alpha,\Omega}$ becomes a Banach space. Furthermore, by $C^{1,\alpha}(\overline{\Omega})$ we denote the space of all $f \in C^1(\overline{\Omega})$ with $\nabla f \in C^\alpha(\overline{\Omega})$.

Theorem 3.7 *Let $\Omega \subset \mathbb{R}^3$ be an open set. Let $u \in L^\infty_{loc}([0,T); C^{1,\alpha}(\Omega) \cap L^\infty(0,T; L^2(\Omega))$ be a local solution to the Euler equations. We assume that for every ball $B \subset \Omega$*

$$\int_0^T \|\omega(s)\|_{BMO(B)} ds < +\infty. \tag{50}$$

Then, $u \in L^\infty([0,T]; C^{1,\alpha}(K))$ for every compact $K \subset \Omega$.

The proof is more technical than that of Theorem 3.3.

In all of the above theorems on the blow-up criterion basically the vorticity controls the finite time blow-up for the smooth solutions. In the followings we introduce a different type of criterion, which controls the blow-up of solutions in terms of the Hessian of the pressure. These are recent results by Chae and Constantin [14, 15].

Theorem 3.8 *Let $(u,p) \in C^1(\mathbb{R}^3 \times (0,T))$ be a solution of the Euler equation (E) with $u \in C([0,T); W^{2,q}(\mathbb{R}^3))$, for some $q > 3$. If*

$$\int_0^T \exp\left(\int_0^t \int_0^s \|D^2 p(\tau)\|_{L^\infty} d\tau ds\right) dt < +\infty, \tag{51}$$

then $\limsup_{t\to T} \|u(t)\|_{W^{2,q}} < +\infty$.

(*Sketch of the proof*) By direct computation we derive the following equation from the vorticity formulation of the Euler equations.

$$\frac{D^2\omega}{Dt^2} = -(\omega\cdot\nabla)\nabla p, \quad \text{where} \quad \frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\cdot\nabla f. \tag{52}$$

Integrating twice the above along the particle trajectory, we have

$$\begin{aligned} |\omega(X(\alpha,t),t)| &\leqslant |\omega_0(\alpha)| + |\omega_0(\alpha)\cdot\nabla u_0(\alpha)|t \\ &+ \int_0^t \int_0^s |\omega(X(a,\tau),\tau)||D^2 p(X(\alpha,\tau),\tau)| d\tau ds, \end{aligned} \tag{53}$$

where we use the fact

$$\frac{\partial}{\partial t}\omega(X(\alpha,t),t)\Big|_{t=0} = \omega(X(\alpha,t),t)\cdot\nabla u(X(\alpha,t),t)\Big|_{t=0}$$
$$= \omega_0(\alpha)\cdot\nabla u_0(\alpha).$$

Then, we establish the following Gronwall type lemma for the double integral inequality [15].

Lemma 3.9 *Let $\alpha=\alpha(t)$ be a non-decreasing function, and $\beta=\beta(t)\geqslant 0$ on $[a,b]$. Suppose $y(t)\geqslant 0$ on $[a,b]$,and satisfies*

$$y(t)\leqslant \alpha(t)+\int_a^t\int_a^s \beta(\tau)y(\tau)d\tau ds \quad \forall t\in[a,b].$$

Then, for all $t\in(a,b]$ we have

$$y(t)\leqslant \alpha(t)\exp\left(\int_a^t\int_a^s \beta(\tau)d\tau ds\right).$$

Applying this lemma, we obtain

$$|\omega(X(\alpha,t),t)|\leqslant (|\omega_0(\alpha)|+|\omega_0(\alpha)\cdot\nabla u_0(\alpha)|t)\exp\left(\int_0^t\int_0^s |D^2p(X(\alpha,\tau),\tau)|d\tau ds\right),$$

and taking the supremum over $\alpha\in\mathbb{R}^3$, and integrating it over $[0,T]$, we find

$$\int_0^T \|\omega(t)\|_{L^\infty}dt\leqslant (\|\omega_0\|_{L^\infty}+T\|\omega_0(\alpha)\cdot\nabla u_0(\alpha)\|_{L^\infty})\int_0^T \exp\left(\int_0^t\int_0^s \|D^2p(\tau)\|_{L^\infty}d\tau ds\right)dt.$$

Applying the BKM criterion, we complete the proof. ■

The following is a localized version of the above theorem.

Theorem 3.10 *Let $(u,p)\in C^1(B(x_0,\rho)\times(T-\rho,T))$ be a solution to (E) with $u\in C([T-\rho,T);W^{2,q}(B(x_0,\rho)))\cap L^\infty(T-\rho,T;L^2(B(x_0,\rho)))$ for some $q\in(3,\infty)$. If*

$$\int_{T-\rho}^T \|u(t)\|_{L^\infty(B(x_0,\rho))}dt<+\infty \tag{54}$$

and

$$\int_{T-\rho}^T \exp\left(\int_{T-\rho}^t\int_{T-\rho}^s \|D^2p(\tau)\|_{L^\infty(B(x_0,\rho))}d\tau ds\right)dt<+\infty, \tag{55}$$

then for all $r\in(0,\rho)$

$$\limsup_{t\to T} \|u(t)\|_{W^{2,q}(B(x_0,r))} < +\infty. \tag{56}$$

The above two theorems are refined, using new kinematic relations between various quantities in the fluid mechanics. We associate to a solution (u, p) of the Euler system (E) the $\mathbb{R}^{3\times 3}$-valued functions $S = (S_{ij})$ and $P = (P_{ij})$, where

$$S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i), \quad P_{ij} = \partial_i \partial_j p.$$

For the vorticity $\omega = \nabla \times u$ we define the direction vectors

$$\xi = \omega/|\omega|, \qquad \zeta = S\xi/|S\xi|,$$

In the case $\omega(x,t) = 0$ we set $\alpha(x,t) = \rho(x,t) = 0$. Note that ξ is the *vorticity direction* vector, while ζ is the *vorticity stretching direction* vector. Then, we can show that the following kinematic relations hold.

Proposition 3.11 *Let (u, p) be a solution of (E), which belongs to $C^1(\mathbb{R}^3 \times (0,T))$. Then, the followings hold true on $\mathbb{R}^3 \times (0,T)$.*

$$D_t|S\omega| = -\zeta \cdot P\omega.$$

Using the above proposition, we can improve Theorem 3.7 as follows. Below we also use the notations $[f]_+ = \max\{f, 0\}$ and $[f]_- = \max\{-f, 0\}$.

Theorem 3.12 *Let $(u, p) \in C^1(\mathbb{R}^3 \times (0,T))$ be a solution of the Euler equation (E) with $u \in C([0,T); W^{2,q}(\mathbb{R}^3))$, for some $q > 3$. If*

$$\int_0^T \exp\left(\int_0^t \int_0^s \|[\zeta \cdot P\xi]_-(\tau)\|_{L^\infty} d\tau ds\right) dt < +\infty, \tag{57}$$

then $\limsup_{t\to T} \|u(t)\|_{W^{2,q}} < +\infty$.

Comparing the above theorem with Theorem 3.8, observing the pointwise inequality$|[\zeta \cdot P\xi]_-| \leqslant |P|$ the above theorem (and its localized version below) improve the result of Theorem 3.8. Furthermore, the above theorem implies that the dynamical changes of the signs of the scalar quantities $\zeta \cdot P\xi$ and $|S\xi|^2 - 2\alpha^2 - \rho$ are important in the phenomena of blow-up/regularity of the solutions to (E).

The following is a localized version of the above theorem.

Theorem 3.13 *Let $(u, p) \in C^1(B(x_0, r) \times (T-r, T))$ be a solution to (E) with $u \in C([T-r, T); W^{2,q}(B(x_0, r))) \cap L^\infty(T-r, T; L^2(B(x_0, r)))$ for some $q \in (3, \infty)$. We suppose*

$$\int_{T-r}^T \|u(t)\|_{L^\infty(B(x_0,r))} dt < +\infty,$$

and the following holds. Suppose

$$\int_{T-r}^{T} \exp\left(\int_0^t \int_0^s \|[\zeta \cdot P\xi]_-(\tau)\|_{L^\infty(B(x_0,r))} d\tau ds\right) dt < +\infty,$$

Then for all $\varepsilon \in (0, r)$ $\limsup_{t\to T} \|u(t)\|_{W^{2,q}(B(x_0,\varepsilon))} < +\infty$.

In the case of the Euler equations having axial symmetry there still exists the possibility of finite time blow-up. The finite time blow-up/global regularity in this case is also a wide-open question, and there are many interesting numerical results (see [47], and the references therein). Therefore, establishing a sharp blow-up criterion for this special case is also important.

Let u be an axisymmetric solution of the Euler equations if u solves (E), and can be written as

$$u = u^r(r, x_3, t)e_r + u^\theta(r, x_3, t)e_\theta + u^3(r, x_3, t)e_3,$$

where

$$e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0), \quad e_\theta = (\frac{x_2}{r}, \frac{-x_1}{r}, 0), \quad e_3 = (0, 0, 1), \quad r = \sqrt{x_1^2 + x_2^2}$$

are the canonical basis of the cylindrical coordinate system. The Euler equations for an axisymmetric solution turn into the following equations

$$\partial_t u^r + u^r \partial_r u^r + u^3 \partial_3 u^r = -\partial_r p + \frac{(u^\theta)^2}{r}, \tag{58}$$

$$\partial_t u^\theta + u^r \partial_r u^\theta + u^3 \partial_3 u^\theta = -\frac{u^r u^\theta}{r}, \tag{59}$$

$$\partial_t u^3 + u^r \partial_r u^3 + u^3 \partial_3 u^3 = -\partial_3 p, \tag{60}$$

$$\partial_r(ru^r) + \partial_3(ru^3) = 0. \tag{61}$$

Multiplying (59) by r, we see that ru^θ satisfies the transport equation

$$\partial_t(ru^\theta) + u^r \partial_r(ru^\theta) + u^3 \partial_3(ru^\theta) = 0. \tag{62}$$

For the vorticity ω we get

$$\omega = \omega^r e_r + \omega^\theta e_\theta + \omega^3 e_3,$$

where

$$\omega^r = -\partial_3 u^\theta, \quad \omega^\theta = \partial_3 u^r - \partial_r u^3, \quad \omega^3 = \frac{u^\theta}{r} + \partial_r u^\theta.$$

Applying ∂_3 to (58) and applying ∂_r to (60), and taking the difference of the two equations, we obtain the following equation for ω^θ

$$\partial_t \omega^\theta + u^r \partial_r \omega^\theta + u^3 \partial_3 \omega^\theta = \frac{u^r \omega^\theta}{r} + \partial_3 \frac{(u^\theta)^2}{r}. \tag{63}$$

This leads to the equation

$$\partial_t \left(\frac{\omega^\theta}{r}\right) + u^r \partial_r \left(\frac{\omega^\theta}{r}\right) + u^3 \partial_3 \left(\frac{\omega^\theta}{r}\right) = \frac{\partial_3 (u^\theta)^2}{r^2}. \tag{64}$$

In the region off the axis we can have substantial improvement of the BKM criterion as follows [23].

Theorem 3.14 *Let $u \in C([0,T); W^{2,q}(\mathbb{R}^3)) \cap L^\infty(0,T; L^2(\mathbb{R}^3))$, $3 < q < +\infty$, be an axisymmetric solution to (E) in $\mathbb{R}^3 \times (0,T)$. If the following condition is fulfilled*

$$\int_0^T (T-t)\|\omega(t)\|_{L^\infty(B(x_*,R_0))} dt < +\infty, \tag{65}$$

for some ball $B(x_*, R_0) \subset \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 > 0\}$, where $\omega = \nabla \times v$, then for all $0 < R < R_0$ it holds $u \in C([0,T], W^{2,q}(B(x_*, R)))$. In particular, this implies $u \in C([0,T], W^{2,q}(\mathbb{T}(x_*, R)))$. Here, $\mathbb{T}(x_*, R)$ stands for the torus generated by rotation of $B(x_*, R_0)$ around the axis, i.e.

$$\mathbb{T}(x_*, R) = \left\{ x \in \mathbb{R}^3 \, : \, \left(\sqrt{x_1^2 + x_2^2} - \rho_*\right)^2 + (x_3 - x_{3,*})^2 < R^2 \right\},$$

where $\rho_* = \sqrt{x_{1,*}^2 + x_{2,*}^2}$.

The main idea in the proof of this theorem is that the Eqs. (64) and (62) have a similar structure to the 2D Boussinesq equations (see Sect. 6 below for more concrete correspondence relations), which has a different scaling properties than the 3D Euler equations.

As an immediate consequence of the above theorem we have substantial improvement for the condition of the blow-up rate of the vorticity near the possible blow-up time as follows [23].

Theorem 3.15 *Let $u \in C([0,T); W^{2,q}(\mathbb{R}^3)) \cap L^\infty(0,T; L^2(\mathbb{R}^3))$, $3 < q < +\infty$, be an axisymmetric solution to (E) in $\mathbb{R}^3 \times (0,T)$. Suppose the following vorticity blow-up rate condition holds*

$$\sup_{t \in (0,T)} (T-t)^2 \left| \log\left(\frac{1}{T-t}\right) \right|^\alpha \|\omega(t)\|_{L^\infty(B(x_*,R_0))} < +\infty \tag{66}$$

for some $\alpha > 1$ and some ball $B(x_*, R_0) \subset \{x \in \mathbb{R}^3 \, : \, x_1^2 + x_2^2 > 0\}$. Then $u \in C([0,T]; W^{2,q}(\mathbb{T}(x_*, R))$ for all $0 < R < R_0$.

In particular, Theorem 3.14 says that there exists no singularity at $t = T$ in the off the axis region if the vorticity blow-up rate satisfies

$$\|\omega(t)\|_{L^\infty(\mathbb{R}^3)} = O\left(\frac{1}{(T-t)^\gamma}\right), \tag{67}$$

as $t \to T$ if $1 \leqslant \gamma < 2$. Due to the global BKM criterion, however, the singularity in this case should happen only on the axis. It would be interesting to compare this result with Tao's construction of a singular solution (see [54, Fig. 3, p.18]) for a *modified Euler system*, where $\gamma = 1$ and the set of singularity is a circle around the axis.

4 On the Type I Blow-Up

We observe that Euler system (E) has scaling property that if (u, p) is a solution, then for any $\lambda > 0$ and $\alpha \in \mathbb{R}$ the functions

$$u^{\lambda,\alpha}(x,t) = \lambda^\alpha u(\lambda x, \lambda^{\alpha+1}t), \quad p^{\lambda,\alpha}(x,t) = \lambda^{2\alpha} p(\lambda x, \lambda^{\alpha+1}t) \tag{68}$$

are also solutions with the initial data $u_0^{\lambda,\alpha}(x) = \lambda^\alpha u_0(\lambda x)$.

The case $\alpha = \frac{3}{2}$ is important for our analysis, since in this case the energy is scaling invariant. Indeed, by the energy conservation we have for $u^\lambda = u^{\lambda,\frac{3}{2}}$,

$$\|u^\lambda(t)\|_{L^2} = \|u(\lambda^{\frac{5}{2}}t)\|_{L^2} = \|u(t)\|_{L^2}.$$

Hereafter, we consider (E) in $\mathbb{R}^3 \times (-1, 0)$ and $t = 0$ is the possible first blow-up time.

Definition 4.1 One says that a solution u of (E) is *self-similar (SS)* with respect to $(0, 0)$ if there exists $\alpha > -1$ such that $u(x,t) = \lambda^\alpha u(\lambda x, \lambda^{\alpha+1}t)$ for all $\lambda > 1$.

Definition 4.2 A solution u is *discretely self-similar (DSS)* with respect to $(0, 0)$ if there exists $\alpha > -1$ such that $u(x,t) = \lambda^\alpha u(\lambda x, \lambda^{\alpha+1}t)$ for some $\lambda > 1$. For more specification we also say u is (λ, α)-DSS.

Definition 4.3 We say u blows up at $t = 0$ with *Type I* if

$$\limsup_{t\to 0}(-t)\|\nabla u(t)\|_{L^\infty} < +\infty. \tag{69}$$

If $\limsup_{t\to 0}(-t)\|\nabla u(t)\|_{L^\infty} = +\infty$, then we say it is of Type II.

In order to study self-similar solutions it is convenient to make self-similar transform from u on $\mathbb{R}^3 \times (-1, 0)$ to U on $\mathbb{R}^3 \times (0, +\infty)$ defined by

$$u(x,t) = \frac{1}{(-t)^{\frac{\alpha}{\alpha+1}}} U(y,s) \tag{70}$$

where

$$y = \frac{x}{(-t)^{\frac{1}{\alpha+1}}}, \quad s = -\log(-t).$$

U is called the profile. Then, (E) is transformed into equations for the profile

$$(SSE) \begin{cases} U_s + \frac{\alpha}{\alpha+1} U + \frac{1}{\alpha+1} (y \cdot \nabla) U + (U \cdot \nabla) U = -\nabla P, \\ \nabla \cdot U = 0. \end{cases}$$

Note that a SS solution of (E) is a stationary solution of (SSE), while a DSS solution of (E) is a time-periodic solution of (SSE),

$$U(y,s) = U(y, s + S_0), \qquad S_0 = (\alpha+1)\log\lambda.$$

A Type I solution of (E) is a global solution U of (SSE) with

$$\limsup_{t\to 0} (-t)\|\nabla u(t)\|_{L^\infty} = \limsup_{s\to+\infty} \|\nabla U(s)\|_{L^\infty} < +\infty.$$

SS and DSS obviously satisfy the above condition. Therefore, Type I blow-up scenario is a natural generalization of SS or DSS blow-up. There are many previous studies excluding SS or DSS blow-up (e.g. [5, 6, 9, 18]). Also, for the periodic solution of (SSE) one can show unique continuation type result [7].

In the case of DSS function having one point singularity one can have strong restriction to the spatial decay of the profile function, independent of the equations. We present it here.

Proposition 4.4 *Let be a $(\lambda,\alpha)-$DSS function with $\lambda > 1$, $\alpha \in \mathbb{R}\setminus\{-1\}$ having one point singularity. Then*

(i)
$$|f(x,t)| \leqslant \frac{C}{(|x| + |t|^{\frac{1}{\alpha+1}})^\alpha} \quad \forall (x,t) \in \mathbb{R}^n \times (-\infty, 0] \setminus \{(0,0)\},$$

where $C = C(\lambda, \alpha)$.

(ii) *Moreover, if*
$$|f(x,t)|(|x| + |t|^{\frac{1}{\alpha+1}})^\alpha = o(1),$$

as either $|x| + |t|^{\frac{1}{\alpha+1}} \to +\infty$ or $|x| + |t|^{\frac{1}{\alpha+1}} \to 0$, which means

$$\lim_{r\to\pm\infty} \sup_{e^r < |x| + |t|^{\frac{1}{\alpha+1}} < \lambda^2 e^r} |f(x,t)|(|x| + |t|^{\frac{1}{\alpha+1}})^\alpha = 0,$$

then

$$f = 0 \quad on \quad \mathbb{R}^n \times (-\infty, 0].$$

Therefore, if $f \not\equiv 0$ *non trivial DSS function, then there exist* $\{(x_k, t_k)\}$, $\{(\bar{x}_k, \bar{t}_k)\} \in \mathbb{R}^n \times (-\infty, 0] \setminus \{(0,0)\}$ *with* $(x_k, t_k) \to +\infty$ *and* $(\bar{x}_k, \bar{t}_k) \to (0,0)$ *as* $k \to +\infty$ *such that*

$$\limsup_{k\to\infty} (|x_k| + |t_k|^{\frac{1}{\alpha+1}})^\alpha |f(x_k, t_k)| > 0,$$

and

$$\limsup_{k\to 0} (|\bar{x}_k| + |\bar{t}_k|^{\frac{1}{\alpha+1}})^\alpha |f(\bar{x}_k, \bar{t}_k)| > 0.$$

(Proof) Let us define $Q_1 = B(0, \lambda) \times (-\lambda^{\alpha+1}, 0)$ and $Q_0 = B(0,1) \times (-1, 0)$, and set $A_1 = Q_1 - Q_0$. For each $(x,t) \in \mathbb{R}^n \times (-\infty, 0] \setminus \{(0,0)\}$ there exist an integer $k \in \mathbb{Z}$ and $(z, \tau) \in A_1$ such that $x = \lambda^k z$, $t = \lambda^{(\alpha+1)k}\tau$. Then, by the DSS property of f we have

$$\begin{aligned}(|x| + |t|^{\frac{1}{\alpha+1}})^\alpha f(x,t) &= (|z| + |\tau|^{\frac{1}{\alpha+1}})^\alpha \lambda^{\alpha k} f(\lambda^k z, \lambda^{(\alpha+1)k}\tau) \\ &= (|z| + |\tau|^{\frac{1}{\alpha+1}})^\alpha f(z, \tau). \qquad (71)\end{aligned}$$

For (i) we observe

$$|f(x,t)| = \left(\frac{|z| + |\tau|^{\frac{1}{\alpha+1}}}{|x| + |t|^{\frac{1}{\alpha+1}}} \right)^\alpha |f(z,\tau)| \leqslant \frac{C_1}{(|x| + |t|^{\frac{1}{\alpha+1}})^\alpha}$$

for all $(x,t) \in \mathbb{R}^n \times (-\infty, 0] \setminus \{(0,0)\}$, where we set

$$C_1 = ess \sup_{(z,\tau)\in A_1} \left\{ (|z| + |\tau|^{\frac{1}{\alpha+1}})^\alpha |f(z,\tau)| \right\}.$$

In order to show (ii) we see that (71) implies also

$$\sup_{\lambda e^r < |x| + |t|^{\frac{1}{\alpha+1}} < \lambda^2 e^r} (|x| + |t|^{\frac{1}{\alpha+1}})^\alpha |f(x,t)| = \sup_{(z,\tau)\in A_1} \left\{ (|z| + |\tau|^{\frac{1}{\alpha+1}})^\alpha |f(z,\tau)| \right\},$$

from which, passing $r \to \pm\infty$, we obtain $f = 0$. ■

Let us consider the profile $F = F(y, s)$ of $f(x,t)$ defined by

$$f(x,t) = \frac{1}{(-t)^{\frac{\alpha}{\alpha+1}}} F(y,s) \quad \text{with} \quad y = \frac{x}{(-t)^{\frac{1}{\alpha+1}}}, \quad s = -\log(-t). \qquad (72)$$

Then, by the similar argument to the above proof one can show the following:

$$\sup_{t\in(-\infty,0)} |D^m f(x,t)|(|x|+|t|^{\frac{1}{\alpha+1}})^{m+\alpha} = \sup_{s\in\mathbb{R}} |D^m F(y,s)|(|y|+1)^{m+\alpha} \tag{73}$$

for all $m \in \mathbb{N} \cup \{0\}$. Following the same argument as the above proposition we have the following.

Corollary 4.5 *Let f be a (λ, α)-DSS function, having one point singularity, and let F be its profile defined by (72). Then, there exists a constant $C > 0$ such that*

$$\sup_{s\in\mathbb{R}} |D^m F(y,s)| \leqslant \frac{C}{(|y|+1)^{m+\alpha}} \quad \forall y \in \mathbb{R}^n. \tag{74}$$

At this moment we could not exclude general Type I blow-up scenario for the solution of (E). As we shall observe below, however, under some smallness condition, we can remove the Type I blow-up. In this direction the following result is first derived in [8].

Theorem 4.6 *Let $u \in C([-1,0); H^m(\mathbb{R}^3))$, $m > 5/2$, be a solution to the Euler equations. Suppose u satisfies the following "small Type I condition"*

$$\limsup_{t\to 0} (-t)\|\nabla u(t)\|_{L^\infty} < 1. \tag{75}$$

Then,

$$\limsup_{t\to 0} \|u(t)\|_{H^m} < +\infty. \tag{76}$$

In other words, there exist no small Type I blow-up.

(Proof) The condition (75) implies that there exists $t_0 \in (-1,0)$ and $0 < C_0 < 1$ such that

$$\sup_{t_0<s<0} (-s)\|\nabla u(s)\|_{L^\infty} \leqslant C_0.$$

We consider the particle trajectory $X(a,t)$ generated by $u = u(x,t)$, i.e.

$$\partial_t X(a,t) = u(X(a,t),t), \qquad X(a,0) = a.$$

The vorticity form of the Euler equations

$$\partial_t \omega + u\cdot\nabla\omega = \omega\cdot\nabla u$$

can be written as an equation along the particle trajectory

$$\partial_t\{\omega(X(a,t),t)\} = (\omega\cdot\nabla u)(X(a,t),t).$$

Integrating $|\omega(X(a,t),t)|$ over $[t_0,t]$ along the particle trajectory, we obtain

$$|\omega(X(a,t),t)| \leqslant |\omega(X(a,t_0),t_0)| \exp\left(\int_{t_0}^t |\nabla u(X(a,s),s)|ds\right).$$

From this we estimate

$$\begin{aligned}\|\omega(t)\|_{L^\infty} &\leqslant \|\omega(t_0)\|_{L^\infty} \exp\left(\int_{t_0}^t \|\nabla v(s)\|_{L^\infty} ds\right)\\ &\leqslant \|\omega(t_0)\|_{L^\infty} \exp\left(C_0 \int_{t_0}^t (-s)^{-1} ds\right)\\ &= \|\omega(t_0)\|_{L^\infty} \left(\frac{t_0}{t}\right)^{C_0} \quad \forall t \in (t_0, 0).\end{aligned}$$

Since $0 < C_0 < 1$, we have $\int_{t_0}^T \|\omega(t)\|_{L^\infty} dt < +\infty$, and by the BKM criterion above there exists no blow-up at T. ■

The above theorem has been localized in [24] as follows.

Theorem 4.7 *Let* $u \in L^\infty(-1,0; L^2(B(r))) \cap C([-1,0); W^{2,q}(B(r)))$ *be a solution to (E) for some* $3 < q < +\infty$. *Suppose there exists* $r_0 \in (0, r)$ *such that*

$$\limsup_{t\to 0}(-t)\|\nabla u(t)\|_{L^\infty(B(r_0))} < 1.$$

Then, $\limsup_{t\to 0} \|u(t)\|_{W^{2,q}(B(\rho))} < +\infty$ *for all* $\rho \in (0, r_0)$.

In a recent paper [14] the Type I condition of the above theorems is replaced by the condition involving the Hessian of the pressure as follows.

Theorem 4.8 *Let* $(u, p) \in C^1(\mathbb{R}^3 \times (-1, 0))$ *be a solution of the Euler equation (E) with* $u \in C([-1, 0); W^{2,q}(\mathbb{R}^3))$*, for some* $q > 3$. *If*

$$\limsup_{t\to 0} (-t)^2 \|D^2 p(t)\|_{L^\infty} < 1,$$

then $\limsup_{t\to 0} \|u(t)\|_{W^{2,q}} < +\infty$.

This is also localized in the same paper [14].

Theorem 4.9 *Let* $(u, p) \in C^1(B(x_0, \rho) \times (-\rho, 0))$ *be a solution to (E) with* $u \in C([-\rho, 0); W^{2,q}(B(x_0, \rho))) \cap L^\infty(0 - \rho, 0; L^2(B(x_0, \rho)))$ *for some* $q \in (3, \infty)$. *If*

$$\int_{-\rho}^0 \|u(t)\|_{L^\infty(B(x_0,\rho))} dt < +\infty$$

and

$$\limsup_{t\to 0}(-t)^2 \|D^2 p(t)\|_{L^\infty(B(x_0,\rho))} < 1,$$

then for all $r \in (0, \rho)$ *we have*

$$\limsup_{t\to 0} \|u(t)\|_{W^{2,q}(B(x_0,r))} < +\infty.$$

The following is a refined version of the above theorems [10, 15], considering also the sign condition for the Hessian of the pressure. We use the same notations as Proposition 3.11.

Theorem 4.10 *Let* $(u, p) \in C^1(\mathbb{R}^3 \times (-1, 0))$ *be a solution of the Euler equation (E) with* $u \in C([-1, 0); W^{2,q}(\mathbb{R}^3))$*, for some* $q > 3$*. Suppose the following holds. If either*

$$\limsup_{t\to 0} (-t)^2 \|[\zeta \cdot P\xi]_-(t)\|_{L^\infty} < 1,$$

or

$$\limsup_{t\to 0} (-t)^2 \|[|S\xi|^2 - 2\alpha^2 - \rho]_+(t)\|_{L^\infty} < 1,$$

then $\limsup_{t\to 0} \|u(t)\|_{W^{2,q}} < +\infty$.

The following is a localized version of the above theorem.

Theorem 4.11 *Let* $(u, p) \in C^1(B(x_0, r) \times (-r, 0))$ *be a solution to (E) with* $u \in C([-r, 0); W^{2,q}(B(x_0, r))) \cap L^\infty(-r, 0; L^2(B(x_0, r)))$ *for some* $q \in (3, \infty)$*. We suppose*

$$\int_{-r}^{0} \|u(t)\|_{L^\infty(B(x_0,r))} dt < +\infty.$$

If either

$$\limsup_{t\to 0} (-t)^2 \|[\zeta \cdot P\xi]_-(t)\|_{L^\infty(B(x_0,r))} < 1,$$

or

$$\limsup_{t\to 0} (-t)^2 \|[|S\xi|^2 - 2\alpha^2 - \rho]_+(t)\|_{L^\infty(B(x_0,r))} < 1,$$

then for all $\varepsilon \in (0, r)$ $\limsup_{t\to 0} \|u(t)\|_{W^{2,q}(B(x_0,\varepsilon))} < +\infty$.

5 Type I Blow-Up and the Energy Concentrations

Although we cannot exclude the possibility of Type I blow-up, we shall show in this section that under Type I condition the energy concentration in the form of atomic measure cannot happen at the blow-up time. Energy concentration in atomic form means that there exists an atomic measure μ (i.e. $\mu(\{x\}) > 0$ for some $x \in \mathbb{R}^3$) such that

$$|u(\cdot, t)|^2 dx \rightharpoonup \mu \quad \text{as} \quad t \to 0$$

in the sense of measure. A typical example is

$$|u(\cdot,t)|^2dx \rightharpoonup \sum_{k=1}^{\infty} C_k\delta_{x_k}.$$

DSS singularity in the energy conserving scale is an example of Type I blow-up with one point energy conservation. Removing this scenario has been open. Concentration phenomena in the other equations are well studied. For example for the nonlinear Schödinger equations blow-up happens with L^2 norm concentration, while in the chemotaxis equations the blow-up occurs with L^1 norm concentration.

We first remove one point energy concentration under Type I. In the case $u \in L^\infty(-1,0;L^2(\mathbb{R}^3))$, we can show that there exists a unique measure $\sigma \in \mathcal{M}(\mathbb{R}^3)$ such that

$$|u(t)|^2dx \to \sigma \quad \textit{weakly-}* \quad \text{in} \quad \mathcal{M}(\mathbb{R}^3) \quad \textit{as} \quad t \to 0. \tag{77}$$

Here, we first consider the case σ is equal to the Dirac measure $E_0\delta_0$ for some constant $0 \leqslant E_0 < +\infty$. Under the Type I condition we can exclude such one-point concentration of the energy, namely we have the following [22].

Theorem 5.1 *Let $u \in L^\infty(-1,0;L^2(\mathbb{R}^3))$ be a solution to the Euler equations. In addition, we assume that u satisfies the Type I blow-up condition* (69) *and* (82) *with $\sigma = E_0\delta_0$ for some $0 \leqslant E_0 < +\infty$. Then $u \equiv 0$.*

In the proof of the above theorem we use several decay properties of the solution to the Euler equations with respect to the space and time variables as we approach the blow-up time. The decay estimate is actually obtained under following more general condition than (69)

$$\exists\, \mu \in \left[\frac{3}{5},1\right) : \quad \sup_{t\in(-1,0)} (-t)^{\frac{5}{3}\mu}\|\nabla u(t)\|_{L^\infty} < +\infty. \tag{78}$$

The following lemma is one of the two key decay estimates used to prove Theorem 5.1.

Lemma 5.2 *Let $u \in L^2(-\infty,0;L^2(\mathbb{R}^3)) \cap L^\infty_{loc}([-1,0),W^{1,\infty}(\mathbb{R}^3))$ be a solution to the Euler equations satisfying* (69) *and* (82) *with $\sigma = E\delta_0$. Then for every $0 < \beta < 5$ there exists a constant C such that for every $t \in [-1,0)$ it holds*

$$\int_{\mathbb{R}^3} |v(t)|^2|x|^\beta dx \leqslant C(-t)^{\beta(1-\alpha)}. \tag{79}$$

(Sketch of the Proof) We first claim the estimate

$$\|u(t)\|_{L^\infty} \leqslant C(-t)^{-\frac{3}{5}}. \tag{80}$$

Indeed, by the Gagliardo-Nirenberg interpolation we obtain

$$
\begin{aligned}
(-t)^{\frac{3}{5}} \|u(t)\|_{L^\infty} &\leqslant C(-t)^{\frac{3}{5}} \|u(t)\|_{L^2}^{\frac{2}{5}} \|\nabla u(t)\|_{L^\infty}^{\frac{3}{5}} \\
&\leqslant CE(-1)^{\frac{2}{5}} \{(-t)\|\nabla u(t)\|_{L^\infty}\}^{\frac{3}{5}} < +\infty.
\end{aligned}
$$

We first prove the decay estimate for $\beta = 1$. We multiply (E) by $u|x|u\eta_R$ for a smooth cut-off η_R supported on the ball B_R, and integrate both sides over $\mathbb{R}^3 \times (t, 0)$. Integrating by parts, using the assumption of L^2-energy concentration at $x = 0$ as $t \to 0$, we have

$$
\begin{aligned}
&\frac{1}{2} \int_{\mathbb{R}^3} |u(t)|^2 |x| \eta_R dx \\
&= \int_t^0 \int_{\mathbb{R}^3} \eta_R |u|^2 u \cdot \nabla |x| dx ds + \int_t^0 \int_{\mathbb{R}^3} \eta_R p u \cdot \nabla |x| dx ds \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^3} |u(0^+)|^2 |x| \eta_R dx + \{\text{terms vanishing as } R \to +\infty\} \\
&\leqslant \int_t^0 \int_{\mathbb{R}^3} |u(s)|^2 dx \, \|v(s)\|_{L^\infty} ds + \int_t^0 \int_{\mathbb{R}^3} |p(s)||v(s)| dx ds + o(1) \\
&\leqslant CE(-1)^2 \int_t^0 (-s)^{-\frac{3}{5}} ds + \int_t^0 \|p(s)\|_{L^2} \|v(s)\|_{L^2} ds + o(1).
\end{aligned}
$$

For the pressure term estimate we use the Calderon-Zygmund inequality $\|p\|_{L^q} \leqslant \|u\|_{L^q}^2$, which follows from the well-known velocity-pressure relation $p = R_j R_k (u_j u_k)$, and estimate

$$
\begin{aligned}
\int_t^0 \|p(s)\|_{L^2} \|u(s)\|_{L^2} ds &\leqslant C \int_t^0 \||u(s)|^2\|_{L^2} \|u(s)\|_{L^2} ds \\
&\leqslant C \int_t^0 \|u(s)\|_{L^\infty} \|u(s)\|_{L^2}^2 ds \leqslant C\{E(-1)\}^2 \int_t^0 (-s)^{-\frac{3}{5}} ds.
\end{aligned}
$$

Passing $R \to \infty$, the lemma is proved for $\beta = 1$:

$$
\int_{\mathbb{R}^3} |u(t)|^2 |x| dx \leqslant C(-t)^{\frac{2}{5}}.
$$

For $\beta > 1$ we multiply (E) by $u|x|^\beta \eta_R$, and integrate by parts as the above, and use the induction argument. For the pressure estimate we use the following weighted Calderon-Zygmund inequality (A_p weight) [53]:

$$
\int_{\mathbb{R}^3} |p(s)|^2 |x|^\gamma dx \leqslant C \int_{\mathbb{R}^3} |u(s)|^4 |x|^\gamma dx \leqslant C(-s)^{-\frac{6}{5}} \int_{\mathbb{R}^3} |u(s)|^2 |x|^\gamma dx,
$$

which holds true for all $0 \leqslant \gamma < 3$. ■

The following is the second decay estimate for the Proof of Theorem 5.1.

Lemma 5.3 *Let $u \in L^2(-1, 0; L^2_\sigma(\mathbb{R}^3)) \cap L^\infty_{loc}([-1, 0), W^{1,\infty}(\mathbb{R}^3))$ be a solution to the Euler equations satisfying (69) for some $\mu \in [\frac{3}{5}, 1)$ and (82) with $\sigma_0 = E\delta_0$. Then for all $k \in \mathbb{N} \cup \{0\}$ and $0 < r < +\infty$ it holds*

$$\|\mathbb{P}_r(v(t))\|^2_{L^2(B(r)^c)} \leqslant C_0^k 4^{k^2} (-t)^{(1-\mu)k} r^{-k} \quad \forall t \in (-1, 0),$$

where $\mathbb{P}_r$ is the Helmholtz projection operator on $B(r)$.

(Sketch of the proof of Theorem 5.1) We choose θ small enough: $0 < \theta < \frac{1}{5}$. For a solution u to the Euler equations we transform: $u \mapsto w$,

$$w(x, t) = u((-t)^\theta x, t).$$

Then, w solves the transformed Euler system,

$$w_t + \theta(-t)^{-1} x \cdot \nabla w + (-t)^{-\theta}(w \cdot \nabla)w = -\nabla\pi,$$
$$\nabla \cdot w = 0.$$

Using the two decay lemmas above, one can show that there exists $t_0 \in (-1, 0)$ such that

$$\nabla \times w(t) = 0 \quad \text{on} \quad B(1)^c \quad \forall t_0 < t < 0.$$

Transforming back to the original vorticity, $\omega(t) = \nabla \times u(t)$, we find

$$\operatorname{supp} \omega(t) \subset B((-t)^\theta) \quad \forall t_0 < t < 0.$$

Since the measure of $\operatorname{supp} \omega(t)$ is preserved due to the Cauchy formula,

$$\omega(X(a, t), t) = \nabla_a X(a, t)\omega_0(a),$$

we have

$$\text{meas}\{\operatorname{supp} \omega(t_0)\} = \text{meas}\{\operatorname{supp} \omega(t)\} \leqslant C(-t)^{3\theta} \to 0$$

as $t \to 0$. Whence, $\omega(t_0) \equiv 0$, and $u(t_0)$ is harmonic. Since $u(t_0) \in L^2(\mathbb{R}^3)$, we conclude that $u(t_0) \equiv 0$, and hence $u \equiv 0$, which is a contradiction. Therefore, one point energy concentration under the Type I condition is impossible. ■

As an immediate corollary of the above theorem we establish the following.

Corollary 5.4 *Let $u \in L^\infty(-1, 0; L^2(\mathbb{R}^3)) \cap L^\infty_{loc}([-1, 0), W^{1,\infty}(\mathbb{R}^3))$ be a DSS solution to the Euler equation, i.e. there exists $\lambda > 1$ such that*

$$u(x,t)=\lambda^{\frac{3}{2}}u(\lambda x,\lambda^{\frac{5}{2}}t)\quad\forall\,(x,t)\in\mathbb{R}^3\times(-1,0).$$

Then $u\equiv 0$.

For the proof we refer to [22].

Next, we shall use the blow-up argument to remove more general form of atomic concentration under local Type I condition. More specifically, we have the following.

Theorem 5.5 *Let* $u\in L^\infty(-1,0;L^2(\mathbb{R}^3))\cap L^\infty_{loc}([-1,0);W^{1,\infty}(\mathbb{R}^3))$ *be a solution of the Euler equations satisfying the Type I condition,*

$$\sup_{t\in(-1,0)}(-t)\|\nabla u(t)\|_{L^\infty}<+\infty.$$

Suppose there exists $\sigma_0\in\mathcal{M}(\mathbb{R}^3)$ *such that*

$$|u(t)|^2dx\to\sigma_0\quad as\quad t\to 0^-.$$

Then, σ_0 *is non-atomic.*

We first recall the notion of suitable weak solution (u,p) of (E), a weak solution satisfying the local energy inequality:

$$\int_{\mathbb{R}^3}|u(t)|^2\phi dx\leqslant\int_{\mathbb{R}^3}|u(s)|^2\phi dx+\int_s^t\int_{\mathbb{R}^3}(|v|^2+2p)u\cdot\nabla\phi dxd\tau$$

for all $\phi\in C_c^\infty(\mathbb{R}^3)$ and for a.e. $-1\leqslant t<s<0$. Below we denote the 'parabolic cylinder' consistent with the energy conserving scale by $Q(R):=B(R)\times(-R^{5/2},0)$. Then we establish the following criterion of energy non-concentration in terms of a Morrey norm.

Theorem 5.6 *We set the cylinder* $Q(R)=B(R)\times(-R^{\frac{5}{2}},0)$. *Let* $u\in L^\infty(-R^{\frac{5}{2}},0;L^2(B(R)))\cap L^3(Q(R))$ *be a local suitable weak solution to (E) such that the local energy inequality is satisfied. Furthermore, we assume that*

$$\sup_{0<r\leqslant R}r^{-1}\|u\|^3_{L^3(Q(r))}<+\infty,\quad\liminf_{r\to 0^+}r^{-1}\|u\|^3_{L^3(Q(r))}=0.\tag{81}$$

Then, there is no energy concentration at the point $x=0$ *as* $t\to 0$.

Remark In [52] Shvydkoy showed that if $u\in L^q(-1,0;L^\infty(\Omega))\cap L^\infty(-1,0;L^2(\Omega))$, $q=\frac{5}{3}$, is a suitable weak solution, then there is no atomic concentration in Ω. This actually follows from the above theorem immediately. Let $Q(r)\subset\Omega\times(-1,0)$. Then

$$
\begin{aligned}
r^{-1}\|u\|^3_{L^3(Q(r))} &= r^{-1}\int_{-r^{\frac{5}{2}}}^{0}\int_{B(r)}|u|^3 dxdt \\
&\leqslant \|u\|^2_{L^\infty(-1,0;L^2(\Omega))} r^{-1}\int_{-r^{\frac{5}{2}}}^{0}\|u\|_{L^\infty(B(r))}dt \\
&\leqslant \|u\|^2_{L^\infty(-1,0;L^2(\Omega))}\left(\int_{-r^{\frac{5}{2}}}^{0}\|u\|^{\frac{5}{3}}_{L^\infty(B(r))}dt\right)^{\frac{3}{5}} \to 0
\end{aligned}
$$

as $r \to 0$.

(Sketch of the proof of Theorem 5.6) We shall use the blow-up argument for the proof of the theorem. Let us first note the following interpolation inequality,

$$
r^{-1}\|u\|^3_{L^3(Q(r))} \leqslant CK_0 r^{-\frac{5}{2}}\|u\|^2_{L^2(Q(r))} + CK_0^{\frac{1}{2}}K_1^{\frac{3}{2}}\left(r^{-\frac{5}{2}}\|u\|^2_{L^2(Q(r))}\right)^{\frac{1}{2}}, \tag{82}
$$

where we set

$$
K_0 := \|u(t)\|_{L^\infty(-R^{5/2},0);L^2(B(R))}, \quad K_1 := \sup_{t\in(-R^{\frac{5}{2}},0)}(-t)\|\nabla u(t)\|_{L^\infty(B(R))},
$$

which are bounded constants by the hypothesis. Suppose there exists an atomic concentration. Then Theorem 5.6, combined with the above interpolation inequality (82) implies that there exists $\varepsilon > 0$ and a sequence $r_k \to 0$ such that

$$
r_k^{-\frac{5}{2}}\|u\|^2_{L^2(Q(r_k))} \geqslant \varepsilon \quad \forall k \in \mathbb{N}.
$$

We define a (blow-up) sequence

$$
u_k(x,t) = r_k^{\frac{3}{2}} u(r_k x, r_k^{\frac{5}{2}} t), \qquad k \in \mathbb{N}.
$$

Using Type I condition and the energy conservation, we can deduce the following uniform bound for $\{u_k\}$,

$$
\|u_k\|_{L^\infty(-1,0;L^2(\mathbb{R}^3))} + \|u_k\|_{L^3([-1,0);\dot{W}^{\theta,3}(\mathbb{R}^3))} \leqslant C
$$

for all $0 < \theta < \frac{1}{3}$. Here, we use the following norm for the fractional derivatives (Sobolev-Slobodeckij semi-norm) in $\mathbb{R}^3$,

$$
|f|_{\dot{W}^{\theta,p}} := \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|f(x)-f(y)|^p}{|x-y|^{\theta p+3}}dxdy\right)^{\frac{1}{p}}.
$$

Taking the limit for a sub-sequence (by compactness lemma), one can construct a non-trivial suitable weak solution to (E),

$$u^* \in L^\infty(-1,0; L^2_\sigma(\mathbb{R}^3)) \cap L^3([-1,0); \dot{W}^{\theta,3}(\mathbb{R}^3))$$

satisfying the following 'weaker-norm version' of local Type I condition

$$\sup_{r\in(0,R)} \frac{1}{r^{1-3\theta}} \int_{-r^{\frac{5}{2}}}^{0} |u^*(t)|^3_{\dot{W}^{\theta,3}(B(r))} dt < +\infty.$$

Indeed, we have the following another interpolation inequality:

$$\sup_{r\in(0,R)} \frac{1}{r^{1-3\theta}} \int_{-r^{\frac{5}{2}}}^{0} |u(t)|^3_{\dot{W}^{\theta,3}(B(r))} dt \leqslant C \sup_{r\in(0,R)} r^{-1}\|u\|^3_{L^3(Q(r))}$$
$$+ C \sup_{-R^{\frac{5}{2}}<t<0} (-t)^3 \|\nabla u(t)\|^3_{L^\infty(B(R))} < +\infty$$

by (82) and the Type I condition respectively. Moreover, for such limiting solution u^* one can choose a sequence of time $\{t_k\} \subset [-1,0)$ and a positive constant $c_0 > 0$ such that

$$|u^*(x,t_k)|^2 dx \rightharpoonup C_0\delta_0 \quad \text{as} \quad k \to +\infty$$

in the sense of measure, namely one point concentration in $\mathbb{R}^3$ for blow-up limiting solution. Our previous exclusion theorem for one point energy concentration in $\mathbb{R}^3$ with Type I blow-up condition implies $C_0 = 0$, namely no atomic concentration. ■

6 The Boussinesq Equations

We consider the Boussinesq equations in the space time cylinder $\mathbb{R}^2 \times (0,\infty)$

$$(B)\begin{cases} \partial_t u + (u\cdot\nabla)u = e_2\theta - \nabla p, \\ \partial_t\theta + (u\cdot\nabla)\theta = 0, \\ \nabla\cdot u = 0, \quad u(x,0) = u_0(x),\ \theta(x,0) = \theta_0(x) \end{cases}$$

where $u = (u_1(x,t), u_2(x,t))$, $(x,t) \in \mathbb{R}^2 \times (0,+\infty)$ is the fluid velocity, while $\theta = \theta(x,t)$ represents the temperature of the fluid, and $e_2 = (0,1)$. The system (B) is a fundamental system of equations describing the motion of atmosphere (see e.g. [49, 50]). Besides its importance in application to the atmospheric sciences another reason why the Boussinesq equation attracted many mathematicians is that the system (B) has strong similarity to the 3D axisymmetric Euler equations, thus providing a good model problem for the Euler equations. To see this resemblance between the two equations more closely we consider the following vorticity equation, obtained by operating $\nabla^\perp \cdot$ on the first equation of (B):

$$\partial_t \omega + u \cdot \nabla \omega = \partial_1 \theta, \quad \omega = \partial_1 u_2 - \partial_2 u_1. \tag{83}$$

Setting $\Theta = (ru^\theta)^2$ and $W = \frac{\omega^\theta}{r}$, the axisymmetric Euler system (64), (62) and (61) can be written as

$$\begin{cases} W_t + u^r \partial_r W + u^3 \partial_3 W = \dfrac{1}{r^4} \partial_3 \Theta, \\ \Theta_t + u^r \partial_r \Theta + u^3 \partial_3 \Theta = 0, \quad \partial_r (ru^r) + \partial_3 (ru^3) = 0. \end{cases} \tag{84}$$

Therefore, if we consider the system (84) off the axis region($r > 0$) the system (B) has the almost same structure as (84) with the correspondence

$$(x_1, x_2) \Leftrightarrow (r, x_3), \quad (u_1, u_2) \Leftrightarrow (u^r, u^\theta), \quad (\omega, \theta) \Leftrightarrow (W, \Theta).$$

Let us consider the particle trajectory $X(\alpha, t)$ generated by $u = u(x, t)$. Then, the second equation of (B) implies the conservation

$$f(\theta(X(\alpha, t), t)) = f(\theta_0(\alpha)) \quad \forall f \in C^1(\mathbb{R}).$$

The following proposition shows that a certain quantity, which corresponds to the Helicity of the 3D Euler equations, has localized conservation law.

Proposition 6.1 *Let f be a $C^1(\mathbb{R})$, and (u, θ) be a smooth solution to (B), and C_t, $t \geqslant 0$ be a level curve of $\theta(\cdot, t)$. Then,*

$$\oint_{C_t} u \cdot \nabla^\perp f(\theta) ds = \oint_{C_0} u_0 \cdot \nabla^\perp f(\theta_0) ds \quad \forall t > 0. \tag{85}$$

(Proof) From the second equation of (B) we have

$$\frac{D}{Dt} \nabla^\perp \theta = (\partial_t + u \cdot \nabla) \nabla^\perp \theta = \nabla^\perp \theta \cdot \nabla u. \tag{86}$$

Let

$$C_0 = \{\gamma(s) \, : \, \theta_0(\gamma(s)) = \lambda, s \in [0, 1], \gamma(0) = \gamma(1)\}$$

be a closed level curve for θ_0. Define

$$C_t = X(C_0, t) = \{X(\gamma(s), t) \, : \, 0 \leqslant s \leqslant 1\}.$$

Then, for any $f \in C^1(\mathbb{R})$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 u(X(\gamma(s),t),t)\cdot\nabla^\perp f(\theta(X(\gamma(s),t),t))ds\\
&\quad= f'(\lambda)\int_0^1 \frac{D}{Dt}u(X(\gamma(s),t),t)\cdot\nabla^\perp\theta(X(\gamma(s),t),t)ds\\
&\quad+ f'(\lambda)\int_0^1 u(X(\gamma(s),t),t)\cdot\frac{D}{Dt}\nabla^\perp\theta(X(\gamma(s),t),t)ds\\
&\quad= -f'(\lambda)\int_0^1 \nabla p(X(\gamma(s),t),t)\cdot\nabla^\perp\theta(X(\gamma(s),t),t)ds\\
&\quad+ f'(\lambda)\int_0^1 \theta(X(\gamma(s),t),t)e_2\cdot\nabla^\perp\theta(X(\gamma(s),t),t)ds\\
&\quad+ f'(\lambda)\int_0^1 u(X(\gamma(s),t),t)\cdot\nabla^\perp\theta(X(\gamma(s),t),t)\cdot\nabla u(X(\gamma(s),t),t)ds\\
&\quad= K_1+K_2+K_3.
\end{aligned}$$

We compute each term separately. First,

$$\begin{aligned}
K_1 &= -f'(\lambda)\int_0^1 \nabla p(X(\gamma(s),t),t)\cdot\frac{\partial}{\partial s}X(\gamma(s),t)ds\\
&= -f'(\lambda)\int_0^1 \frac{\partial}{\partial s}p(X(\gamma(s),t),t)ds = 0.
\end{aligned}$$

Second,

$$K_2 = f'(\lambda)\lambda e_2\cdot\int_0^1 \frac{\partial X(\gamma(s),t),t)}{\partial s}ds = 0.$$

Finally,

$$\begin{aligned}
K_3 &= f'(\lambda)\int_0^1 u(X(\gamma(s),t),t)\cdot\left(\frac{\partial}{\partial s}X(\gamma(s),t)\cdot\nabla\right)u(X(\gamma(s),t),t)ds\\
&= \frac{1}{2}f'(\lambda)\int_0^1 \left(\frac{\partial}{\partial s}X(\gamma(s),t)\cdot\nabla\right)|u(X(\gamma(s),t),t)|^2ds\\
&= \frac{1}{2}f'(\lambda)\int_0^1 \frac{\partial}{\partial s}|u(X(\gamma(s),t),t)|^2ds = 0.
\end{aligned}$$

Combining the calculations for K_1, K_2, K_3 above, we find that

$$\frac{d}{dt}\int_0^1 u(X(\gamma(s),t),t)\cdot\nabla^\perp f(\theta(X(\gamma(s),t),t))ds = 0.$$

This completes the proof of the proposition. ■

Regarding the Cauchy problem for the system (B) for the initial data in $H^k(\mathbb{R}^2)$, $k > 2$, the local-in-time existence of solution and the Beale-Kato-Majda type blow-up criterion are first obtained in [16].

Theorem 6.2 *Let $(u_0, \theta_0) \in H^k(\mathbb{R}^2)$, $k > 2$. Then, there exists $T = T(\|u_0\|_{H^k}, \|\theta_0\|_{H^k})$ such that a unique solution $u \in C([0,T); H^k(\mathbb{R}^2))$ exists. Furthermore,*

$$\limsup_{t\to T}(\|u(t)\|_{H^k} + \|\theta(t)\|_{H^k}) = +\infty \quad \textit{if and only if}$$
$$\int_0^T \|\nabla\theta(t)\|_{L^\infty} dt = +\infty. \tag{87}$$

The finite time blow-up question for the Boussinesq system with a smooth initial data is also a wide-open problem. We mention that for domain with cusp singularity finite time blow-up at the boundary point is obtained recently in [37], and also in [27] the authors show singularity on the boundary point of a cylinder. Our main concern here is the possibility of interior singularity in the whole domain of $\mathbb{R}^2$. It is also worth mentioning that if we add viscosity term to either one of the velocity or the temperature equations of (B), the finite time blow-up question was posed by Moffatt in [51] as one of the millennium problems in the fluid mechanics, for which there was a partial result due to Córdoba, Fefferman and LLave in [34], removing "squirt" singularities. The problem is fully resolved in [11], which shows that there exists no finite time singularities in this partially viscous case.

A similar result to Theorem 6.2 in the setting of the Hölder space is proved in [17]. For the BKM type criterion an improvement of (87) has been obtained in [25] as follows.

Theorem 6.3 *Let $(u, \theta) \in C([0,T); W^{2,q}(\mathbb{R}^2))$, $q > 2$, be a solution of (B). If*

$$\int_0^T (T-t)\|\nabla\theta(t)\|_{L^\infty} dt < +\infty, \tag{88}$$

then there exists no blow-up at $t = T$, and thus both u and θ belong to $C([0,T]; W^{2,q}(\mathbb{R}^2))$.

(*Proof*) For convenience we shift in time so that $[0,T] \mapsto [-1,0]$.
Step (i) We first show that

$$\int_{-1}^0 \|\omega(t)\|_{L^\infty} dt + \int_{-1}^0 (-t)\|\nabla\theta(t)\|_{L^\infty} dt < +\infty \tag{89}$$

implies no blow-up at $t = 0$. Let $q > 2$. We apply the operator ∂_i to the vorticity equation, multiplying the resultant equation by $\partial_i\omega|\nabla\omega|^{q-1}$, and integrating it over

$\mathbb{R}^2$. Then, after the integration by parts and using the Hölder inequality, we are led to

$$\frac{d}{dt}\|\nabla\omega\|_{L^q} \leqslant \|\nabla u\|_{L^\infty}\|\nabla\omega\|_{L^q} + \|\nabla^2\theta\|_{L^q}$$
$$= \|\nabla u\|_{L^\infty}\|\nabla\omega\|_{L^q} + (-t)^{-1}(-t)\|\nabla^2\theta\|_{L^q}. \tag{90}$$

Next, we apply the operator $\partial_i\partial_j$ to both sides of the θ equation, multiply both sides by $\partial_i\partial_j\theta|\nabla^2\theta|^{q-2}$, and sum over $i, j = 1, 2, 3$, and the integrate it over $\mathbb{R}^2$. This, applying the integration by part and the Hölder inequality, yields the following inequality

$$\frac{d}{dt}\|\nabla^2\theta\|_{L^q} \leqslant 2\|\nabla u\|_{L^\infty}\|\nabla^2\theta\|_{L^q} + \|\nabla\theta\|_{L^\infty}\|\nabla^2 u\|_{L^q}. \tag{91}$$

Multiplying both sides of (91) by $(-t)$, we see that

$$\frac{d}{dt}(-t)\|\nabla^2\theta\|_{L^q} + \|\nabla^2\theta\|_{L^q}$$
$$\leqslant 2\|\nabla u\|_{L^\infty}(-t)\|\nabla^2\theta\|_{L^q} + (-t)\|\nabla\theta\|_{L^\infty}\|\nabla^2 u\|_{L^q}$$
$$\leqslant 2\|\nabla u\|_{L^\infty}(-t)\|\nabla^2\theta\|_{L^q} + c_{cz}(-t)\|\nabla\theta\|_{L^\infty}\|\nabla\omega\|_{L^q}. \tag{92}$$

Now define

$$\Psi(t) := \|\nabla\omega\|_{L^q} + (-t)\|\nabla^2\theta\|_{L^q}, \quad t \in (-1, 0).$$

Adding the last two inequalities (90) and (92), we are led to

$$\Psi' \leqslant \Big(2\|\nabla u(t)\|_{L^\infty} + (-t)^{-1} + c_{cz}(-t)\|\nabla\theta(t)\|_{L^\infty}\Big)\Psi. \tag{93}$$

By means of the logarithmic Sobolev embedding of the Beale-Kato-Majda type, we find

$$\|\nabla v(t)\|_{L^\infty} \leqslant C\left\{1 + \|\omega(t)\|_{L^\infty}\log(e + \|\nabla^2 u(t)\|_{L^q})\right\}$$
$$\leqslant C\left\{1 + \|\omega(t)\|_{L^\infty}\log(e + \Psi(t))\right\}. \tag{94}$$

Inserting (94) into (93), it follows

$$\Psi' \leqslant \left\{C\left[1 + (\|\omega(t)\|_{L^\infty} + (-t)\|\nabla\theta(t)\|_{L^\infty})\log(e + \Psi(t))\right] + (-t)^{-1}\right\}\Psi(t). \tag{95}$$

Setting $y(t) = \log(e + \Psi(t))$, we infer from (95) the differential inequality

$$y' \leqslant Ca(t)y + C(-t)^{-1}, \qquad a(t) = \|\omega(t)\|_{L^\infty} + (-t)\|\nabla\theta(t)\|_{L^\infty} \tag{96}$$

which can be solved as

$$
\begin{aligned}
y(t) &= \log(e + \Psi(t)) \\
&\leqslant y(t_0) e^{C \int_{t_0}^{t} a(s) ds} + C \int_{t_0}^{t} (-s)^{-1} e^{c \int_s^t a(\tau) d\tau} ds.
\end{aligned} \tag{97}
$$

We now choose t_0 so that $e^{C \int_{t_0}^{0} a(s) ds} < 2$. Then, (97) implies

$$
\log(e + \Psi(t)) \leqslant C \log(e + \Psi(t_0)) + C \log(-1/t) \qquad \forall t \in (t_0, 0), \tag{98}
$$

where $c > 2$ is another constant. From θ-equation of (B) we have immediately

$$
\frac{\partial}{\partial t} |\nabla \theta| + (u \cdot \nabla) |\nabla \theta| \leqslant |\nabla u| |\nabla \theta|. \tag{99}
$$

Let $t \in (-1, 0)$ be arbitrarily chosen but fixed. Let $x_0 \in \mathbb{R}^2$. By $X(x_0, t)$ we denote the trajectory of the particle which is located at x_0 at time $t = t_0$, defined by the following ODE

$$
\frac{dX(x_0, t)}{dt} = u(X(x_0, t), t) \quad \text{in} \quad [-1, 0), \quad X(x_0, t_0) = x_0. \tag{100}
$$

Then, (99) can be written as

$$
\frac{\partial}{\partial t} |\nabla \theta(X(x_0, t), t)| \leqslant |\nabla u(X(x_0, t), t)| |\nabla \theta(X(x_0, t), t)|, \tag{101}
$$

which can be integrated along the trajectories as

$$
|\nabla \theta(X(x_0, t), t)| \leqslant |\nabla \theta(x_0)| \exp\left(\int_{t_0}^{t} |\nabla u(X(x_0, s), s)| ds \right).
$$

Therefore, we estimate, using (94) as

$$
\begin{aligned}
\|\nabla \theta(t)\|_{L^\infty} &\leqslant \|\nabla \theta(t_0)\|_{L^\infty} \exp\left(\int_{t_0}^{t} \|\nabla u\|_{L^\infty} ds \right) \\
&\leqslant \|\nabla \theta(t_0)\|_{L^\infty} \exp\left(C \int_{t_0}^{t} \left\{ \|\omega(s)\|_{L^\infty} \left[\log(e + \Psi(t_0)) + \log(-1/s) \right] + 1 \right\} ds \right) \\
&\leqslant \|\nabla \theta(t_0)\|_{L^\infty} \times \\
&\quad \times \exp\left(C \left\{ \log(e + \Psi(t_0)) + \log(-1/t) \right\} \int_{t_0}^{t} \|\omega(s)\|_{L^\infty} ds + c(t - t_0) \right).
\end{aligned} \tag{102}
$$

Choosing $t_0 \in (-1,0)$ so that $C\int_{t_0}^0 \|\omega(s)\|_{L^\infty} ds < \frac{1}{2}$, we deduce from (102) that

$$\|\nabla\theta(t)\|_{L^\infty} \leqslant \|\nabla\theta(t_0)\|_{L^\infty}(e+\Psi(t_0))^C e^C(-t)^{-\frac{1}{2}} \quad \forall t \in (t_0, 0).$$

Therefore, $\int_{-1}^0 \|\nabla\theta\|_{L^\infty} dt < +\infty$. Applying the well-known blow-up criterion in [5], we obtain the desired result of (89).

<u>*Step (ii)*</u> Here we show the estimate:

$$\int_{-1}^{t} \|\omega(s)\|_{L^\infty} ds + \int_{-1}^{t} (-s)\|\nabla\theta(s)\|_{L^\infty} ds \leqslant \|\omega(-1)\|_{L^\infty} + 2\int_{-1}^{0} (-s)\|\nabla\theta(s)\|_{L^\infty} ds < +\infty, \tag{103}$$

thus finishing the proof, combining this with (89). We recall the vorticity equation from (B).

$$\partial_t\omega + u\cdot\nabla\omega = \partial_1\theta \quad \text{in} \quad \mathbb{R}^2\times[-1,0), \tag{104}$$

where $\omega = \partial_1 u_2 - \partial_2 u_1$. Using the particle trajectories with $X(x_0,-1) = x_0$ as the above, we have from (104)

$$\frac{d}{dt}|\omega(X(x_0,t),t)| \leqslant |\partial_1\theta(X(x_0,t),t)| \quad \text{in} \quad [-1,0), \tag{105}$$

which implies that

$$\|\omega(s)\|_{L^\infty} \leqslant \|\omega(-1)\|_{L^\infty} + \int_{-1}^{s} \|\partial_1\theta(\tau)\|_{L^\infty} d\tau. \tag{106}$$

Integrating both sides of (106) over $[-1,t)$, $t \in (-1,0)$ with respect to s, and applying integration by parts, we get

$$\begin{aligned}
\int_{-1}^{t} \|\omega(s)\|_{L^\infty} ds &\leqslant (1+t)\|\omega(-1)\|_{L^\infty} + \int_{-1}^{t}\int_{-1}^{s} \|\partial_1\theta(\tau)\|_{L^\infty} d\tau ds \\
&= (1+t)\|\omega(-1)\|_{L^\infty} + \int_{-1}^{t} \left\{\frac{d}{ds}(s)\int_{-1}^{s} \|\partial_1\theta(\tau)\|_{L^\infty} d\tau\right\} ds \\
&= (1+t)\|\omega(-1)\|_{L^\infty} + \int_{-1}^{t} (-s)\|\partial_1\theta(s)\|_{L^\infty} ds + t\int_{-1}^{t} \|\partial_1\theta(s)\|_{L^\infty} ds \\
&\leqslant \|\omega(-1)\|_{L^\infty} + \int_{-1}^{t} (-s)\|\partial_1\theta(s)\|_{L^\infty} ds.
\end{aligned}$$

■

The above theorem has been also localized in [23] as follows.

Theorem 6.4 *Let $B(r) \subset \mathbb{R}^2$ be the unit ball, $2 < q < +\infty$, and*

$$(u, \theta) \in C([0, T); W^{2,q}(B(r))) \times C([0, T); W^{2,q}(B(r)))$$

be a solution to (B). Suppose that

$$u \in L^\infty(0, T; L^2(B(r)))$$

and

$$\int_0^T (T - t)\|\nabla\theta(t)\|_{L^\infty(B(r))} dt < +\infty, \quad \int_0^T \|u(t)\|_{L^\infty(B(r))} dt < +\infty.$$

Then $u, \theta \in C([0, T], W^{2,q}(B(\rho)))$ for all $0 < \rho < r$.

The blow-up criterion in terms of the Hessian of the pressure is also recently obtained as follows. For a solution (u, p, θ) of the system (B) let us introduce the $\mathbb{R}^{2\times 2}$-valued functions $U = (\partial_i u_j)$ and $P = (\partial_i \partial_j p)$. For the vector field $\nabla^\perp\theta = (-\partial_2\theta, \partial_1\theta)$ we define the direction vectors

$$\xi = \nabla^\perp\theta/|\nabla^\perp\theta|, \qquad \zeta = U\nabla^\perp\theta/|U\nabla^\perp\theta|.$$

We note that contrary to the case of Euler equations U is not the symmetric part of the velocity gradient matrix. Then, the following blow-up criterion in terms of the Hessian of the pressure is proved in [14].

Theorem 6.5 *Let $(u, p) \in C^1(\mathbb{R}^2 \times (0, T))$ be a solution of the Boussinesq equation (B) with $u \in C([0, T); W^{2,q}(\mathbb{R}^2))$, for some $q > 2$. Suppose the following holds. Either*

$$\int_0^T (T - t) \exp\left(\int_0^t \int_0^s \|[\zeta \cdot P\xi]_-(\tau)\|_{L^\infty} d\tau ds\right) dt < +\infty,$$

or

$$\limsup_{t\to T} (T - t)^2 \|[\zeta \cdot P\xi]_-(t)\|_{L^\infty} < 2.$$

Then $\limsup_{t\to T} \|u(t)\|_{W^{2,q}} < +\infty$.

Note the relaxed smallness condition for the nonexistence of Type I blow-up compared to the case of 3D Euler equations. This is due to the extra factor, $(T - t)$ in the integral $\int_0^T (T - t)\|\nabla^\perp\theta(t)\|_{L^\infty} dt < +\infty$ in Theorem 6.3.

(*Proof of the first part of Theorem 6.5*) Let (u, p, θ) be a solution of (B), which belongs to $C^1(\mathbb{R}^2 \times (0, T))$. We first claim the following formula.

$$D_t|U\nabla^\perp\theta| = -\zeta \cdot P\nabla^\perp\theta. \tag{107}$$

Indeed, ∇ on the first equation of (B), we find

$$D_t U + U^2 = -P + \nabla(\theta e_2).$$

Taking $\nabla^\perp$ on the second equation of (B), we obtain

$$D_t \nabla^\perp \theta = U \nabla^\perp \theta.$$

Let us compute

$$\begin{aligned} D_t^2 \nabla^\perp \theta &= D_t U \nabla^\perp \theta + U D_t \nabla^\perp \theta \\ &= -U^2 \nabla^\perp \theta - P \nabla^\perp \theta + U^2 \nabla^\perp \theta + \nabla^\perp \theta \cdot \nabla(\theta e_2) \\ &= -P \nabla^\perp \theta, \end{aligned} \tag{108}$$

where we use the fact

$$\nabla^\perp \theta \cdot \nabla(\theta e_2) = 0.$$

We multiply (110) by $D_t \nabla^\perp \theta$ to have

$$\begin{aligned} |D_t \nabla^\perp \theta| D_t |D_t \nabla^\perp \theta| = \frac{1}{2} D_t \left(|D_t \nabla^\perp \theta|^2 \right) &= D_t \nabla^\perp \theta \cdot D_t^2 \nabla^\perp \theta \\ &= -U \nabla^\perp \theta \cdot P \nabla^\perp \theta. \end{aligned} \tag{109}$$

the left-hand side of which can be re written as

$$\frac{1}{2} D_t |U \nabla^\perp \theta|^2 = |U \nabla^\perp \theta| \, D_t |U \nabla^\perp \theta|.$$

Hence, dividing the both sides of (109) by $|U \nabla^\perp \theta|$, we obtain the formula (107), and the claim is proved.

Now, integrating (107) along the trajectory for $t \in [0, s]$, we obtain

$$\begin{aligned} &\frac{\partial}{\partial s} |\nabla^\perp \theta(X(\alpha, s), s)| \leqslant \left| \frac{\partial}{\partial s} \nabla^\perp \theta(X(\alpha, s), s) \right| \\ &= |(D_s \nabla^\perp \theta)(X(\alpha, s), s)| = |U \nabla^\perp \theta(X(\alpha, s), s)| \\ &\quad = |S_0(\alpha) \omega_0(\alpha)| - \int_0^s (\zeta \cdot P\xi)(X(\alpha, \tau), \tau) |\omega(X(\alpha, \tau), \tau| d\tau. \end{aligned}$$

After integrating this again with respect to s over $[0, t]$, we find

$$\begin{aligned} |\nabla^\perp \theta(X(\alpha, t), t)| &\leqslant |\nabla^\perp \theta_0(\alpha)| + |\nabla^\perp \theta_0(\alpha) \cdot \nabla u_0(\alpha)| t \\ &\quad + \int_0^t \int_0^s [\zeta \cdot P\xi]_-(X(\alpha, \tau), \tau) |\nabla^\perp \theta(X(\alpha, \tau), \tau)| d\tau ds. \end{aligned}$$

Thanks to Theorem 3.8 we find

$$|\nabla^{\perp}\theta(X(\alpha,t),t)| \leqslant (|\nabla^{\perp}\theta_0(\alpha)| + |\nabla^{\perp}\theta_0 \cdot \nabla u_0(\alpha)|t)\times \\ \times \exp\left(\int_0^t \int_0^s [\zeta \cdot P\xi]_-(X(\alpha,\tau),\tau)d\tau ds\right).$$

Taking the supremum over $a \in \mathbb{R}^2$, and integrating it with respect to t over $[0, T]$ after multiplying by $T - t$, we obtain

$$\int_0^T (T-t)\|\nabla^{\perp}\theta(t)\|_{L^\infty}dt \leqslant (\|\nabla^{\perp}\theta_0\|_{L^\infty} + \|\nabla^{\perp}\theta_0 \cdot \nabla u_0\|_{L^\infty}T)\times \\ \times \int_0^T (T-t)\exp\left(\int_0^t \int_0^s \|[\zeta \cdot P\xi(\tau)]_-\|_{L^\infty}d\tau ds\right)dt.$$

Applying the blow-up criterion of Theorem 6.3, we obtain the desired conclusion. ■

The following is a localized version of Theorem 6.5.

Theorem 6.6 *Let* $(u, p) \in C^1(B(x_0, r) \times (T-r, T))$ *be a solution to (E) with* $u \in C([T-r, T); W^{2,q}(B(x_0, r))) \cap L^\infty(T-r, T; L^2(B(x_0, r)))$ *for some* $q \in (2, \infty)$. *Let us assume*

$$\int_{T-r}^T \|u(t)\|_{L^\infty(B(x_0,r))}dt < +\infty. \tag{110}$$

If either

$$\int_{T-r}^T (T-t)\exp\left(\int_0^t \int_0^s \|[\zeta \cdot P\xi]_-(\tau)\|_{L^\infty(B(x_0,r))}d\tau ds\right)dt < +\infty,$$

or

$$\limsup_{t\to T} (T-t)^2\|[\zeta \cdot P\xi]_-(t)\|_{L^\infty(B(x_0,r))} < 2,$$

then for all $\varepsilon \in (0, r)$ $\limsup_{t\to T} \|u(t)\|_{W^{2,q}(B(x_0,\varepsilon))} < +\infty$.

For the proof we first show that the condition (110) implies that the mapping $t \mapsto X(\alpha, t)$ belongs to $C([T-r, T]; \mathbb{R}^3)$ for all $\alpha \in B(x_0, r)$. Then, the other part of the proof follows by applying the continuity argument. For more details we refer to [15].

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References

1. C. Bardos, E.S. Titi, Euler equations for an ideal incompressible fluid. Russ. Math. Surv. **62**(3), 409–451 (2007)
2. J.T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3-D Euler equations. Comm. Math. Phys. **94**, 61–66 (1984)
3. D. Chae, *Incompressible Euler Equations: Mathematical Theory, Encyclopedia of Mathematical Physics*, vol. 3 (Elsevier/Academic Press, Oxford, 2006), pp. 10–17
4. D. Chae, *Incompressible Euler Equations: The Blow-Up Problem and Related Results, Handbook of Differential Equations: Evolutionary Equations*, vol. IV (Elsevier/North-Holland, Amsterdam, 2008), pp. 1–55
5. D. Chae, Nonexistence of self-similar singularities for the 3D incompressible Euler equations. Comm. Math. Phys. **273**(1), 203–215 (2007)
6. D. Chae, Euler's equations and the maximum principle. Math. Ann. **361**, 51–66 (2015)
7. D. Chae, Unique continuation type theorem for the self-similar Euler equations. Adv. Math. **283**, 143–154 (2015)
8. D. Chae, On the generalized self-similar singularities for the Euler and the Navier-Stokes equations. J. Funct. Anal. **258**(9), 2865–2883 (2010)
9. D. Chae, Nonexistence of asymptotically self-similar singularities in the Euler and the Navier-Stokes equations. Math. Ann. **338**(2), 435–449 (2007)
10. D. Chae, On the finite-time singularities of the 3D incompressible Euler equations. Commun. Pure Appl. Math. **LX**, 0597–0617 (2007)
11. D. Chae, Global regularity for the 2D Boussinesq equations with partial viscosity terms. Adv. Math. **203**(2), 497–513 (2006)
12. D. Chae, Local existence and blow-up criterion for the Euler equations in the Besov spaces. Asymptot. Anal. **38**(3–4), 339–358 (2004)
13. D. Chae, On the well-posedness of the Euler equations in the Triebel-Lizorkin spaces. Commun. Pure Appl. Math. **55**(5), 654–678 (2002)
14. D. Chae, P. Constantin, On a Type I singularity condition in terms of the pressure for the Euler eq in $\mathbb{R}^3$, Int. Math. Res. Notices (to appear)
15. D. Chae, P. Constantin, Remarks on type I blow-up for the 3D Euler equations and the 2D Boussinesq equations, submitted
16. D. Chae, H.-S. Nam, Local existence and blow-up criterion for the Boussinesq equations. Proc. Roy. Soc. Edinburgh Sect. A **127**(5), 935–94 (1997)
17. D. Chae, S.-K. Kim, H.S. Nam, Local existence and blow-up criterion of Hölder continuous solutions of the Boussinesq equations. Nagoya Math. J. **155**, 55–80 (1999)
18. D. Chae, R. Shvydkoy, On formation of a locally self-similar collapse in the incompressible Euler equations. Arch. Ration. Mech. Anal. **209**(3), 999–1017 (2013)
19. D. Chae, J. Wolf, Localized non blow-up criterion of the Beale-Kato-Majda type for the 3D Euler equations. Math. Ann. (to appear)
20. D. Chae, J. Wolf, The Euler equations in a critical case of the generalized Campanato space. Ann. Inst. H. Poincare Anal. Non Linéaire **38**(2), 201–241 (2021)
21. D. Chae, J. Wolf, *Local non blow-up condition of $C^{1,\alpha}$ solutions of the 3D incompressible Euler equations,* (in preparation)
22. D. Chae, J. Wolf, Energy concentrations and Type I blow-up for the 3D Euler equations. Comm. Math. Phys. **376**(2), 1627–1669 (2020)
23. D. Chae, J. Wolf, Removing type II singularities off the axis for the three dimensional axisymmetric Euler equations. Arch. Ration. Mech. Anal. **23**(3), 1041–1089 (2019)
24. D. Chae, J. Wolf, On the local Type I conditions for the 3D Euler equations. Arch. Rat. Mech. Anal. **2**, 641–663 (2018)
25. D. Chae, J. Wolf, On the regularity of solutions to the 2D Boussinesq equations satisfying type I conditions. J. Nonlinear Sci. **29**(2), 643–654 (2019)
26. J.-Y. Chemin, *Perfect Incompressible Fluids*, Oxford Lecture Series in Mathematics and its Applications, vol. 14. (Oxford Univ. Press, 1998)

27. J. Chen, T.-Y. Hou, Finite time blow-up of 2D Boussinesq and 3D Euler equations with velocity and boundary. Arch. Ration. Mech. Anal. **383**(3), 1559–1667 (2021)
28. Q. Chen, C. Miao, Z. Zhang, On the well-posedness of the ideal MHD equations in the Triebel-Lizorkin spaces. Arch. Ration. Mech. Anal. **195**(2), 561–578 (2010)
29. A.J. Chorin, J.E. Marsden, *A Mathematical Introduction to Fluid Mechanics* (Springer, 1990)
30. P. Constantin, *Analysis of Hydrodynamic Models*, CBMS-NSF Regional Conf. Ser. in Applied Mathematics, vol. 90. (SIAM, 2016)
31. P. Constantin, On the Euler equations of incompressible fluids. Bull. Amer. Math. Soc. **44**(4), 603–621 (2007)
32. P. Constantin, Geometric statistics in turbulence. SIAM Rev. **36**, 73–98 (1994)
33. P. Constantin, C. Fefferman, A. Majda, Geometric constraints on potential singularity formulation in the 3-D Euler equations. Comm. P.D.E. **21**(3–4), 559–571 (1996)
34. D. Cordoba, C. Fefferman, R. De La LLave, On squirt singularities in hydrodynamics. SIAM J. Math. Anal. **36**(1), 204–213 (2004)
35. J. Deng, T.Y. Hou, X. Yu, Improved geometric conditions for non-blow-up of the 3D incompressible Euler equations. Comm. P.D.E. **31**(1–3), 293–306 (2006)
36. T. Elgindi, I-J. Jeong, Finite-time singularity formation for strong solutions to the Boussinesq system. Ann. PDE **6**(1), Paper No. 5, 50 pp. (2020)
37. T. Elgindi, I-J. Jeong, Finite-time singularity formation for strong solutions to the axi-symmetric 3D Euler equations. Ann. PDE **5**(2), Paper No. 16, 51 pp. (2019)
38. L. Euler, Principes généraux du mouvement des fluides. Mémoires de l'académie des sciences de Berlin **11**, 274–315 (1755)
39. F. John, L. Nirenberg, On functions of bounded mean oscillation. Commun. Pure Appl. Math. **343**, 415–426 (1983)
40. T. Kato, Nonstationary flows of viscous and ideal fluids in $\mathbb{R}^3$. J. Funct. Anal. **9**, 296–305 (1972)
41. T. Kato, G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations. Commun. Pure Appl. Math. **41**(7), 891–907 (1988)
42. R.M. Kerr, Evidence for a singularity of the three-dimensional, incompressible Euler equations. Phys. Fluids, A **5**(7), 1725–1746 (1993)
43. S. Klainerman, A. Majda, Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. Commun. Pure Appl. Math. **34**, 481–524 (1981)
44. H. Kozono, Y. Taniuchi, Bilinear estimates in BMO and the Navier-Stokes equations. Math. Z. **235**(1), 173–194 (2000)
45. H. Kozono, Y. Taniuchi, Limiting case of the Sobolev inequality in BMO, with applications to the Euler equations. Comm. Math. Phys. **214**, 191–200 (2000)
46. P.-L. Lions, *Mathematical Topics in Fluid Mechanics, Vol 1. Incompressible Models*, Oxford Lecture Series in Mathematics and its Applications, vol. 13. (Oxford Univ. Press, 1996)
47. G. Luo, T.-Y. Hou, Formation of finite-time singularities in the 3D axisymmetric Euler equations: a numerics guided study. SIAM Rev. **61**(4), 793–835 (2019)
48. G. Luo, T.-Y. Hou, Toward the finite-time blowup of the 3D axisymmetric Euler equations: a numerical investigation. Multiscale Model. Simul. **12**(4), 1722–1776 (2014)
49. A. Majda, A. Bertozzi, *Vorticity and Incompressible Flow* (Cambridge Univ. Press, 2002)
50. A. Majda, *Introduction to PDEs and Waves for the Atmsphere and Ocean*, Courant Lecture Note Series N. 9 (AMS, 2003)
51. H.K. Moffatt, Some remarks on topological fluid mechanics, in *An Introduction to the Geometry and Topology of Fluid Flows*. ed. by R.L. Ricca (Kluwer Academic Publishers, Dordrecht, The Netherlands, 2001), pp. 3–10
52. R. Shvydkoy, A study of energy concentration and drain in incompressible fluids. Nonlinearity **26**, 425–438 (2013)
53. E.M. Stein, *Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals* (Princeton University Press, 1993)
54. T. Tao, Finite time blowup for Lagrangian modifications of the three-dimensional Euler equation. Ann. PDE **2**(9), 1–79 (2016)