Chapter 6
3D *K* From Quantized Coordinate Ring of Type B

Abstract For the quantized coordinate ring $A_q(B_n)$, fundamental representations of the generators associated with the spin representation of B_n are presented. Reflecting the equivalence of the spin representation of B_2 and the vector representation of C_2, the equivalence $A_q(B_n) \simeq A_q(C_n)$ holds for $n = 2$ but not for $n \geq 3$. In particular $A_q(B_3)$ leads to another solution to the 3D reflection equation different from Chap. 5. The RTT relation for the fundamental representations are proved by making use of the tetrahedron equation of type $MMLL = LLMM$ (Theorem 3.25) and a matrix product formula of the quantum R matrix for the spin representation (Chap. 12).

6.1 Quantized Coordinate Ring $A_q(B_n)$

Like $A_q(A_{n-1})$ and $A_q(C_n)$ treated in the preceding chapters, the quantized coordinate ring $A_q(B_n)$ ($n \geq 2$) we consider in this chapter is the $\mathfrak{g} = B_n$ case of the Hopf algebra $A_q(\mathfrak{g})$ defined in Sect. 10.2 in a universal manner. On general grounds, $A_q(B_n)$ has generators $t_{\mathbf{ab}}$ associated with the spin representation $V(\varpi_n)$ of $U_q(B_n)$.[1] Here the indices $\mathbf{a}, \mathbf{b}$ range over

$$\{0, 1\}^n = \{\mathbf{a} = (a_1, \dots, a_n) \mid a_1, \dots, a_n \in \{0, 1\}\}, \tag{6.1}$$

which is a natural labeling set of the base of $V(\varpi_n)$. A feature that distinguishes it from the A_{n-1} and C_n cases is that the complete set of defining relations among the $2^n \times 2^n$ generators $T = (t_{\mathbf{ab}})$ have not been identified explicitly in the literature. They include the RTT relation and the ρTT relation at least:

[1] See the explanations around (10.22). $V(\varpi_n)$ denotes the irreducible $U_q(B_n)$ module whose highest weight is the nth fundamental weight ϖ_n. $A_q(B_n)$ here is different from (in a sense "finer" than) $\mathrm{Fun}(\mathrm{SO}_q(2n+1))$ in [127] based on $(2n+1)^2$ generators associated with the vector representation.

A. Kuniba, *Quantum Groups in Three-Dimensional Integrability*, Theoretical and Mathematical Physics, https://doi.org/10.1007/978-981-19-3262-5_6

$$\sum_{\mathbf{l},\mathbf{m}} R^{\mathbf{ab}}_{\mathbf{lm}} t_{\mathbf{lc}} t_{\mathbf{md}} = \sum_{\mathbf{l},\mathbf{m}} t_{\mathbf{bm}} t_{\mathbf{al}} R^{\mathbf{lm}}_{\mathbf{c\,d}}, \tag{6.2}$$

$$\sum_{\mathbf{b}} \rho_{\mathbf{b}} t_{\mathbf{ab}} t_{\mathbf{l'b'}} = \sum_{\mathbf{c}} \rho_{\mathbf{c}} t_{\mathbf{ca}} t_{\mathbf{c'l'}} = \rho_{\mathbf{a}} \delta_{\mathbf{al}}, \tag{6.3}$$

where $\mathbf{a}'$ is defined by

$$\mathbf{a}' = (1 - a_1, \dots, 1 - a_n). \tag{6.4}$$

The RTT relation is known to be valid from the general argument leading to (10.15). The relation (6.3) originates in the fact that $V(\varpi_n) \otimes V(\varpi_n) \supset V(0)$, which is also the case for $A_q(C_n)$ as in (5.12). The structure constants $R^{\mathbf{ab}}_{\mathbf{ij}}$ and $\rho_{\mathbf{a}}$ are related by (5.12), and given as

$$R^{\mathbf{ab}}_{\mathbf{ij}} = \lim_{x\to\infty} x^{-2n} R(x)^{\mathbf{ab}}_{\mathbf{ij}}, \tag{6.5}$$

$$\rho_{\mathbf{a}} = q^{-\frac{n^2}{2}} \prod_{k=1}^{n} \left((-1)^k q^{2k-1}\right)^{a_{n+1-k}}. \tag{6.6}$$

In (6.5), $R(x)^{\mathbf{ab}}_{\mathbf{ij}}$ is an element of the quantum R matrix of the spin representation. See (6.61) and the explanation around it for a precise description. In (6.5), one is picking the coefficient of the highest order power of x from it as in (3.4) and (5.6). From $\rho_{\mathbf{a}}\rho_{\mathbf{a}'} = (-1)^{n(n+1)/2}$, (6.6) corresponds to $\varepsilon = (-1)^{n(n+1)/2}$ in (5.10).

Remark 6.1 Under the equivalence $U_q(C_2) \simeq U_q(B_2)$, the vector representation of the former corresponds to the spin representation of the latter. Reflecting this fact, $A_q(B_2)$ here is isomorphic to $A_q(C_2)$ in Chap. 5 via the rescaling of generators explained in Remark 5.1. Concretely, the indices 1, 2, 3, 4 for $A_q(C_2)$ correspond to (0, 0), (0, 1), (1, 0), (1, 1) for $A_q(B_2)$, and the generators are identified by (5.13) with $(g_1, g_2, g_3, g_4) = (i, 1, 1, i)$ satisfying (5.14).

6.2 Fundamental Representations

Let $\mathrm{Osc}_q = \langle \mathbf{a}^+, \mathbf{a}^-, \mathbf{k}, \mathbf{k}^{-1} \rangle$ be the q-oscillator algebra (3.12) and $\mathrm{Osc}_{q^2} = \langle \mathbf{A}^+, \mathbf{A}^-, \mathbf{K}, \mathbf{K}^{-1} \rangle$ be the q^2-oscillator algebra (5.15). As before, they are identified with elements of $\mathrm{End}(\mathcal{F}_{q_i})$. The embedding in Theorem 3.3 enables one to write down the fundamental representations

$$\pi_i : A_q(B_n) \to \mathrm{End}(\mathcal{F}_{q_i}) \qquad (1 \le i \le n), \tag{6.7}$$

$$q_1 = \cdots = q_{n-1} = q^2, \;\; q_n = q \tag{6.8}$$

containing a non-zero parameter μ_i. Note the difference of (6.8) from (5.18).

For $1 \le i \le n-1$, let the image of the generators $(t_{\mathbf{ab}})_{\mathbf{a},\mathbf{b}\in\{0,1\}^n}$ by π_i be as follows:

$$\begin{pmatrix} t_{\alpha00\tilde\alpha,\alpha00\tilde\alpha} & t_{\alpha00\tilde\alpha,\alpha01\tilde\alpha} & t_{\alpha00\tilde\alpha,\alpha10\tilde\alpha} & t_{\alpha00\tilde\alpha,\alpha11\tilde\alpha} \\ t_{\alpha01\tilde\alpha,\alpha00\tilde\alpha} & t_{\alpha01\tilde\alpha,\alpha01\tilde\alpha} & t_{\alpha01\tilde\alpha,\alpha10\tilde\alpha} & t_{\alpha01\tilde\alpha,\alpha11\tilde\alpha} \\ t_{\alpha10\tilde\alpha,\alpha00\tilde\alpha} & t_{\alpha10\tilde\alpha,\alpha01\tilde\alpha} & t_{\alpha10\tilde\alpha,\alpha10\tilde\alpha} & t_{\alpha10\tilde\alpha,\alpha11\tilde\alpha} \\ t_{\alpha11\tilde\alpha,\alpha00\tilde\alpha} & t_{\alpha11\tilde\alpha,\alpha01\tilde\alpha} & t_{\alpha11\tilde\alpha,\alpha10\tilde\alpha} & t_{\alpha11\tilde\alpha,\alpha11\tilde\alpha} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathbf{A}^- & \mu_i \mathbf{K} & 0 \\ 0 & -q^2\mu_i^{-1}\mathbf{K} & \mathbf{A}^+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.9}$$

$$\text{otherwise } t_{\mathbf{a},\mathbf{b}} \mapsto 0, \tag{6.10}$$

where $\alpha \in \{0,1\}^{i-1}$ and $\tilde\alpha \in \{0,1\}^{n-i-1}$ are arbitrary in (6.9). For π_n, the image of the generators is specified as

$$\begin{pmatrix} t_{\alpha0,\alpha0} & t_{\alpha0,\alpha1} \\ t_{\alpha1,\alpha0} & t_{\alpha1,\alpha1} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{a}^- & \mu_n \mathbf{k} \\ -q\mu_n^{-1}\mathbf{k} & \mathbf{a}^+ \end{pmatrix}, \tag{6.11}$$

$$\text{otherwise } t_{\mathbf{a},\mathbf{b}} \mapsto 0, \tag{6.12}$$

where $\alpha \in \{0,1\}^{n-1}$ is arbitrary in (6.11).

Example 6.2 For $A_q(B_2)$, let $T = (t_{\mathbf{ab}})$ be the array with row $\mathbf{a}$ and column $\mathbf{b}$ ordered as $(0,0), (0,1), (1,0), (1,1)$ from the top let. Then its image reads as

$$\pi_1(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathbf{A}^- & \mu_1 \mathbf{K} & 0 \\ 0 & -q^2\mu_1^{-1}\mathbf{K} & \mathbf{A}^+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \pi_2(T) = \begin{pmatrix} \mathbf{a}^- & \mu_2\mathbf{k} & 0 & 0 \\ -q\mu_2^{-1}\mathbf{k} & \mathbf{a}^+ & 0 & 0 \\ 0 & 0 & \mathbf{a}^- & \mu_2\mathbf{k} \\ 0 & 0 & -q\mu_2^{-1}\mathbf{k} & \mathbf{a}^+ \end{pmatrix}. \tag{6.13}$$

Remark 6.3 Denote the fundamental representations of $A_q(B_2)$ in Example 6.2 by $\pi_1^{B_2}$ and $\pi_2^{B_2}$. Similarly, denote the fundamental representations of $A_q(C_2)$ in (5.24) by $\pi_1^{C_2}$ and $\pi_2^{C_2}$. Then $\pi_i^{B_2}$ coincides with $\pi_{3-i}^{C_2}$ via the adjustment explained in Remark 6.1 with a suitable redefinition of μ_i parameters.

From Remark 5.1, the parameters $\mu_1, \dots, \mu_n$ are removed by switching to the rescaled generators $\tilde t_{\mathbf{ab}}$ in (5.13) with $g_{\mathbf{a}} = \prod_{1\le k\le n} \mu_k^{a_1+\cdots+a_k-k/2}$ satisfying (5.14) with $g_{\mathbf{a}} g_{\mathbf{a}'} = 1$. Thus we set $\mu_1 = \cdots = \mu_n = 1$ in the rest of the chapter without loss of generality.

Example 6.4 Let $T = (t_{\mathbf{a},\mathbf{b}})$ be the 8-by-8 matrix of generators of $A_q(B_3)$, where the row index $\mathbf{a}$ and the column index $\mathbf{b}$ are ordered from the top left corner as $000, 001, 010, 011, 100, 101, 110, 111$. Then their image by the fundamental representations π_1, π_2, π_3 according to (6.9)–(6.12) reads as follows:

$$\pi_1(T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{A}^- & 0 & \mathbf{K} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{A}^- & 0 & \mathbf{K} & 0 & 0 \\ 0 & 0 & -q^2\mathbf{K} & 0 & \mathbf{A}^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & -q^2\mathbf{K} & 0 & \mathbf{A}^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.14}$$

$$\pi_2(T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{A}^- & \mathbf{K} & 0 & 0 & 0 & 0 & 0 \\ 0 & -q^2\mathbf{K} & \mathbf{A}^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{A}^- & \mathbf{K} & 0 \\ 0 & 0 & 0 & 0 & 0 & -q^2\mathbf{K} & \mathbf{A}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.15}$$

$$\pi_3(T) = \begin{pmatrix} \mathbf{a}^- & \mathbf{k} & 0 & 0 & 0 & 0 & 0 & 0 \\ -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{a}^- & \mathbf{k} & 0 & 0 & 0 & 0 \\ 0 & 0 & -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{a}^- & \mathbf{k} & 0 & 0 \\ 0 & 0 & 0 & 0 & -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{a}^- & \mathbf{k} \\ 0 & 0 & 0 & 0 & 0 & 0 & -q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}. \tag{6.16}$$

Proposition 6.5 *The image of the generators by the maps $\pi_1, \dots, \pi_n$ in (6.9)–(6.12) satisfies the RTT relation (6.2) and the ρTT relation (6.3).*

We will present an intriguing proof in Sect. 6.6 making use of the tetrahedron equation of type $MMLL = LLMM$ (3.122), where the MM part yields the structure constant and the LL part the generators.

Let us turn to the tensor products of the fundamental representations. We write $\pi_{i_1} \otimes \cdots \otimes \pi_{i_l}$ as $\pi_{i_1,\dots,i_l}$ for short. The Weyl group $W(B_n)$ is the same as $W(C_n)$ explained in (5.26) and (5.27). Then Theorem 3.3 asserts the same equivalence as (5.28)–(5.30):

$$\pi_{i,j} \simeq \pi_{j,i} \qquad (|i-j| \ge 2), \tag{6.17}$$

$$\pi_{i,i+1,i} \simeq \pi_{i+1,i,i+1} \qquad (1 \le i \le n-2), \tag{6.18}$$

$$\pi_{n-1,n,n-1,n} \simeq \pi_{n,n-1,n,n-1}. \tag{6.19}$$

6.3 Intertwiners

By Remark 3.4, the intertwiner responsible for the isomorphism (6.17) is just the exchange of components P defined in (3.23). See the explanation around (3.24).

Next we consider the intertwiner for (6.18), which corresponds to the cubic Coxeter relation. It is an element $\Phi^B \in \mathrm{End}(\mathcal{F}_{q^2}^{\otimes 3})$ characterized by

$$\Phi^B \circ \pi_{i,i+1,i}(\Delta(f)) = \pi_{i+1,i,i+1}(\Delta(f)) \circ \Phi^B \quad (1 \le i < n,\ \forall f \in A_q(B_n)), \tag{6.20}$$

$$\Phi^B(|0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle. \tag{6.21}$$

The latter just fixes the normalization. Set $R^B = \Phi^B P_{13}$ as in (3.30). Then the Eq. (6.20) is identical, as a set, with (3.38)–(3.46) for the 3D R with q replaced by q^2. Therefore, the intertwiner for (6.18) is provided by $\Phi^B = R^B P_{13}$ with $R^B = R|_{q \to q^2}$, where R in the RHS is the 3D R in Chap. 3. As before, R^B will also be called the intertwiner. We know that R^B satisfies the tetrahedron equation of type $RRRR = RRRR$ (2.6).

Finally, we consider the intertwiner for the equivalence (6.19), which corresponds to the quartic Coxeter relation. Due to the nested structure of the representations (6.9)–(6.12) with respect to rank n, the problem reduces to $\pi_{1212} \simeq \pi_{2121}$ for $A_q(B_2)$. Thus we consider the linear map

$$\Psi^B : \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \to \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \tag{6.22}$$

characterized by

$$\pi_{2121}(\Delta(f)) \circ \Psi^B = \Psi^B \circ \pi_{1212}(\Delta(f)) \quad (\forall f \in A_q(B_2)), \tag{6.23}$$

$$\Psi^B(|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle, \tag{6.24}$$

where the latter specifies the normalization. Set

$$K^B = \Psi^B P_{14} P_{23} \in \mathrm{End}(\mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2}), \tag{6.25}$$

where $P_{14}P_{23}$ reverses the order of the 4-fold tensor product. Note a slight difference from the 3D K of $A_q(C_2)$ in (5.36).

Theorem 6.6 *The intertwiner K^B is given by $K^B = P_{14}P_{23}KP_{14}P_{23}$, where K in the RHS is the 3D K for $A_q(C_2)$ in Chap. 5.*

Proof From Remark 6.3, the Eq. (6.23) is equivalent to $\pi^C_{1212}(\Delta(f)) \circ \Psi^B = \Psi^B \circ \pi^C_{2121}(\Delta(f))$. Comparing this with the type C case $\pi^C_{2121}(\Delta(f)) \circ \Psi = \Psi \circ \pi^C_{1212}(\Delta(f))$ in (5.34) and from the unique existence of Ψ^B, we have $\Psi^B = \Psi^{-1}$ taking the normalization into account. Thus we find $K^B P_{14}P_{23} = \Psi^B = \Psi^{-1} \overset{(5.36)}{=} (KP_{14}P_{23})^{-1} \overset{(5.72)}{=} P_{14}P_{23}K$. □

Let us summarize the relation of intertwiners that originate in the cubic and the quartic Coxeter relations exhibiting the types B and C as superscripts.

$$\begin{aligned} &\text{Cubic} && \text{Quartic} \\ &\Phi^B = \Phi^C|_{q\to q^2}, && \Psi^B = (\Psi^C)^{-1}, \qquad (6.26)\\ &R^B = R^C|_{q\to q^2}, && K^B = P_{14}P_{23}K^C P_{14}P_{23}. \qquad (6.27)\end{aligned}$$

The last result implies

$$K^B(|i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle) = \sum_{a,b,c,d} K^{dcba}_{lkji} |a\rangle \otimes |b\rangle \otimes |c\rangle \otimes |d\rangle \qquad (6.28)$$

in $\mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2}$ in terms of the matrix elements K^{abcd}_{ijkl} of $K = K^C$ in (5.57). We note that (5.72) also implies

$$K^B = (K^B)^{-1}. \qquad (6.29)$$

6.4 3D Reflection Equation

To be explicit, we set

$$S = R^B = R|_{q\to q^2}, \qquad K_{4321} = K^B_{1234} = P_{14}P_{23}K_{1234}P_{23}P_{14} \qquad (6.30)$$

according to (6.27).

Theorem 6.7 *The intertwiners R^B and K^B satisfy the 3D reflection equation (4.3), which is presented in terms of the above S and K as*

$$S_{689}K_{9753}S_{249}S_{258}K_{8741}K_{6321}S_{456} = S_{456}K_{6321}K_{8741}S_{258}S_{249}K_{9753}S_{689}. \qquad (6.31)$$

It is an equality of linear operators on

$$\overset{1}{\mathcal{F}}_q \otimes \overset{2}{\mathcal{F}}_{q^2} \otimes \overset{3}{\mathcal{F}}_q \otimes \overset{4}{\mathcal{F}}_{q^2} \otimes \overset{5}{\mathcal{F}}_{q^2} \otimes \overset{6}{\mathcal{F}}_{q^2} \otimes \overset{7}{\mathcal{F}}_q \otimes \overset{8}{\mathcal{F}}_{q^2} \otimes \overset{9}{\mathcal{F}}_{q^2}. \qquad (6.32)$$

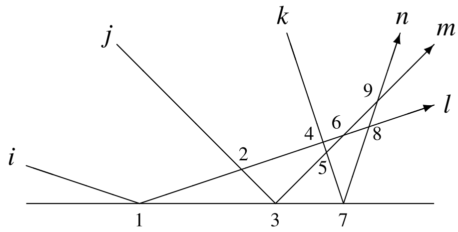
Proof Since $W(B_3) \simeq W(C_3)$, the same proof as the one for Theorem 5.16 remains valid. □

Note that the replacement $q \to q^2$ in (6.27) is done only for R. Therefore, the solution (R^B, K^B) is *not* reducible to (R, K) for type C in Chap. 5.

In the reminder of this section we present another proof of Theorem 6.7 based on the quantized reflection equation introduced in Sect. 4.4. Define L by (5.110), G by (5.111) and $K = K^C$ to be the 3D K for type C in Chap. 5. Then Theorem 5.18 shows that L, G and K satisfy the quantized reflection equation (4.12) with $\mathcal{F}' = \mathcal{F}_{q^2}$ and $\mathcal{F} = \mathcal{F}_q$. We know $K = K^{-1}$ by (5.72). Thus J in (4.14) and (4.15) coincides with K^B in (6.27). From Theorem 3.21, (3.59) and (3.60), we see that R^B in (6.27) and the above L satisfy the quantized Yang–Baxter equations $(2.19)|_{R\to R^B}$ and $(2.20)|_{S\to R^B}$. In this way we have a concrete realization of all the operators appearing in (4.19) and (4.20) in which $R = S = R^B$ and $J = K^B$. Thus the argument leading to (4.23) proves Theorem 6.7 provided that the operators (4.18) act irreducibly on the space (4.22), which is (6.32) in the present setting.

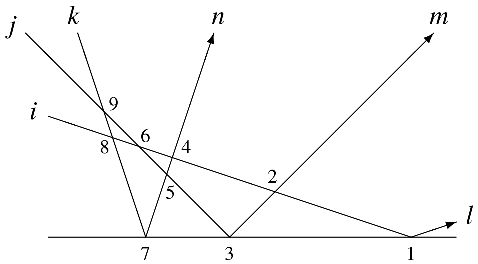
The last point of the irreducibility is established by identifying the quantized three-body reflection amplitude with the representation of $A_q(B_3)$ corresponding to the longest element of $W(B_3)$. To state it precisely, we set

$$\mathcal{M}^{lmn}_{ijk} =$$



(6.33)

$$\tilde{\mathcal{M}}^{lmn}_{ijk} =$$



(6.34)

They stand for the quantized three-body reflection amplitudes

$$\mathcal{M}^{lmn}_{ijk}, \tilde{\mathcal{M}}^{lmn}_{ijk} \in \mathrm{End}(\overset{1}{\mathcal{F}}_q \otimes \overset{2}{\mathcal{F}}_{q^2} \otimes \overset{3}{\mathcal{F}}_q \otimes \overset{4}{\mathcal{F}}_{q^2} \otimes \overset{5}{\mathcal{F}}_{q^2} \otimes \overset{6}{\mathcal{F}}_{q^2} \otimes \overset{7}{\mathcal{F}}_{q^2} \otimes \overset{8}{\mathcal{F}}_q \otimes \overset{9}{\mathcal{F}}_{q^2} \otimes \mathcal{F}_{q^2}). \tag{6.35}$$

They allow us to express the first and the last operators in (4.19) and (4.20) as

$$\overset{9}{L}_{bc}\overset{8}{L}_{ac}\overset{7}{G}_c\overset{6}{L}_{ab}\overset{5}{L}_{cb}\overset{4}{L}_{ca}\overset{3}{G}_b\overset{2}{L}_{ba}\overset{1}{G}_a = \sum \overset{a}{E}_{li} \otimes \overset{b}{E}_{mj} \otimes \overset{c}{E}_{nk} \otimes \mathcal{M}^{lmn}_{ijk}, \tag{6.36}$$

$$\overset{1}{G}_a\overset{2}{L}_{ab}\overset{3}{G}_b\overset{4}{L}_{ac}\overset{5}{L}_{bc}\overset{6}{L}_{ba}\overset{7}{G}_c\overset{8}{L}_{ca}\overset{9}{L}_{cb} = \sum \overset{a}{E}_{li} \otimes \overset{b}{E}_{mj} \otimes \overset{c}{E}_{nk} \otimes \tilde{\mathcal{M}}^{lmn}_{ijk}, \tag{6.37}$$

where the sums extend over $i, j, k, l, m, n \in \{0, 1\}$ and E_{ij} is the matrix unit on V.

Proposition 6.8 *The quantized three-body reflection amplitudes are identified with the representation of $A_q(B_3)$ corresponding to the longest element of $W(B_3)$ as follows:*

$$\mathcal{M}^{lmn}_{ijk} = (-q)^{i+j+k-l-m-n}\pi_{323121321}(\Delta(t_{\mathbf{a},\mathbf{b}})), \tag{6.38}$$

$$\tilde{\mathcal{M}}^{lmn}_{ijk} = (-q)^{i+j+k-l-m-n}\pi_{323121321}(\tilde{\Delta}(t_{\mathbf{a},\mathbf{b}})), \tag{6.39}$$

$$\mathbf{a} = (1-k, 1-j, 1-i), \quad \mathbf{b} = (1-n, 1-m, 1-l). \tag{6.40}$$

This can be verified directly. From this proposition and Theorem 3.3, it follows that $\mathcal{M}^{lmn}_{ijk}$ and $\tilde{\mathcal{M}}^{lmn}_{ijk}$ act irreducibly on (6.32). Therefore the argument in (4.21)–(4.23) proves that (R^B, K^B) satisfies the 3D reflection equation (4.3). We note that the reduced word 323121321 has been encoded in (6.33) as the sequence of "heights" of the points $1, 2, \dots, 9$, where the bottom level is set to be 3.

Example 6.9

$$\mathcal{M}^{000}_{011} = q^2\mathbf{a}^+ \otimes \mathbf{K} \otimes \mathbf{k} \otimes \mathbf{K} \otimes \mathbf{K} \otimes 1 \otimes \mathbf{k} \otimes 1 \otimes 1, \tag{6.41}$$

$$\begin{aligned}\mathcal{M}^{111}_{110} &= q^4\mathbf{a}^- \otimes 1 \otimes \mathbf{a}^- \otimes \mathbf{K} \otimes \mathbf{K} \otimes 1 \otimes \mathbf{k} \otimes 1 \otimes 1\\ &\quad - q^5\mathbf{k} \otimes \mathbf{A}^- \otimes \mathbf{k} \otimes \mathbf{K} \otimes \mathbf{K} \otimes 1 \otimes \mathbf{k} \otimes 1 \otimes 1,\end{aligned} \tag{6.42}$$

$$\begin{aligned}\tilde{\mathcal{M}}^{010}_{000} &= -q^2\mathbf{a}^+ \otimes \mathbf{A}^- \otimes \mathbf{a}^+ \otimes \mathbf{A}^+ \otimes \mathbf{K} \otimes 1 \otimes \mathbf{k} \otimes 1 \otimes 1\\ &\quad - q^2\mathbf{a}^+ \otimes \mathbf{K} \otimes \mathbf{a}^- \otimes 1 \otimes \mathbf{A}^+ \otimes 1 \otimes \mathbf{k} \otimes 1 \otimes 1\\ &\quad - q^2\mathbf{a}^+ \otimes \mathbf{K} \otimes \mathbf{k} \otimes 1 \otimes 1 \otimes 1 \otimes \mathbf{a}^+ \otimes 1 \otimes 1\\ &\quad + q^3\mathbf{k} \otimes 1 \otimes \mathbf{k} \otimes \mathbf{A}^+ \otimes \mathbf{K} \otimes 1 \otimes \mathbf{k} \otimes 1 \otimes 1.\end{aligned} \tag{6.43}$$

On the other hand, for instance, we have

$$\begin{aligned}&\pi_{323121321}(\Delta(t_{001,111}))\\ &= \pi_{323121321}(t_{001,001} \otimes t_{001,010} \otimes t_{010,011} \otimes t_{011,101} \otimes t_{101,110}\\ &\qquad \otimes t_{110,110} \otimes t_{110,111} \otimes t_{111,111} \otimes t_{111,111} + \cdots)\\ &= \mathbf{a}^+ \otimes \mathbf{K} \otimes \mathbf{k} \otimes \mathbf{K} \otimes \mathbf{K} \otimes 1 \otimes \mathbf{k} \otimes 1 \otimes 1,\end{aligned} \tag{6.44}$$

where all the other $+\cdots$ terms are vanishing upon evaluation by $\pi_{323121321}$. Thus we see $\mathcal{M}^{000}_{011} = q^2\pi_{323121321}(\Delta(t_{001,111}))$.

One can similarly define the n-body quantized reflection amplitudes generalizing (6.33) and (6.34) by arranging the n reflecting arrows. They are linear operators on $\mathcal{F}_q^{\otimes n} \otimes \mathcal{F}_{q^2}^{\otimes n(n-1)}$. Thus the total number of the components is n^2, which is equal to the length of the longest element of $W(B_n)$. The formulas (6.38)–(6.40) suggest a natural extension to general n. It is an interesting problem to establish a result like Proposition 6.8 for general n hopefully more intrinsically.

6.5 Combinatorial and Birational Counterparts

In view of (6.27) it is natural to set

$$K^B_{\mathrm{combinatorial}} = P_{14}P_{23}K_{\mathrm{combinatorial}}P_{14}P_{23}, \tag{6.45}$$
$$K^B_{\mathrm{birational}} = P_{14}P_{23}K_{\mathrm{birational}}P_{14}P_{23} \tag{6.46}$$

in terms of (5.153) and (5.160) for type C. On the other hand, we simply set $R^B_{\mathrm{combinatorial}} = R_{\mathrm{combinatorial}}$ in terms of (3.150) and $R^B_{\mathrm{birational}} = R_{\mathrm{birational}}$ in terms of (3.151). These definitions lead to another triad of the 3D R analogous to (5.162):

$$K^B_{\mathrm{quantum}} \overset{q\to 0}{\longrightarrow} K^B_{\mathrm{combinatorial}} \overset{\mathrm{UD}}{\longleftarrow} K^B_{\mathrm{birational}}. \tag{6.47}$$

The 3D reflection equation also remains valid both at combinatorial and birational level. However, it should be stressed that the equation defines *different* transformations for type B and C. See Example 5.24 for comparison in the combinatorial case.

In the rest of the section, we mention a slight variant of the upper triangular matrices relevant to $K^B_{\mathrm{birational}}$ adapted to the natural (rather than spin) representation of B_n. Define the $2n+1$ by $2n+1$ upper triangular matrices

$$Y_i(x) = 1 + xE_{i,i+1} - xE_{2n+1-i,2n-i+2} \quad (1 \le i < n), \tag{6.48}$$
$$Y_n(x) = 1 + xE_{n,n+1} - xE_{n+1,n+2} - \frac{x^2}{2}E_{n,n+2}, \tag{6.49}$$

where x is a parameter and $E_{i,j}$ is a matrix unit. The matrix $Y_i(x)$ is a generator of the unipotent subgroup of $\mathrm{SO}(2n+1)$. It satisfies $Y_i(x)^{-1} = Y_i(-x)$ and $Y_i(a)Y_j(b) = Y_j(b)Y_i(a)$ for $|i-j| > 1$. Given parameters a, b, c, d, each of the matrix equations

$$Y_i(a)Y_j(b)Y_i(c) = Y_j(\tilde{c})Y_i(\tilde{b})Y_j(\tilde{a}) \quad (|i-j|=1, i, j < n), \tag{6.50}$$
$$Y_{n-1}(a)Y_n(b)Y_{n-1}(c)Y_n(d) = Y_n(d'')Y_{n-1}(c'')Y_n(b'')Y_{n-1}(a'') \tag{6.51}$$

has the unique solution. For (6.50) it is given by (3.151). The second equation (6.51) determines $K^B_{\mathrm{birational}} : (a,b,c,d) \mapsto (a'',b'',c'',d'')$. It essentially reduces to the $n=2$ case:

$$Y_1(z) = \begin{pmatrix} 1 & z & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & -z \\ & & & & 1 \end{pmatrix}, \qquad Y_2(z) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ & 1 & z & -z^2/2 & 0 \\ & & 1 & -z & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}.$$

In accordance with (6.46), the solution is given in terms of a', b', c', d' in (5.159) as

$$\begin{aligned}(a'', b'', c'', d'') &= (d', c', b', a')|_{(a,b,c,d)\to(d,c,b,a)}\\ &= \frac{ab^2c}{\mathcal{B}}, \frac{\mathcal{B}}{\mathcal{A}}, \frac{\mathcal{A}^2}{\mathcal{B}}, \frac{bcd}{\mathcal{A}},\\ \mathcal{A} = ab + ad + cd, &\quad \mathcal{B} = ab^2 + 2abd + ad^2 + cd^2.\end{aligned} \tag{6.52}$$

6.6 Proof of Proposition 6.5

6.6.1 Matrix Product Formula of the Structure Function

Let us collect the necessary definitions and facts for the proof of Proposition 6.5. Following (2.24) and (2.25) we set

$$L = \sum_{a,b,i,j} E_{ai} \otimes E_{bj} \otimes L^{ab}_{ij}, \qquad M = \sum_{a,b,i,j} E_{ai} \otimes E_{bj} \otimes M^{ab}_{ij}, \tag{6.53}$$

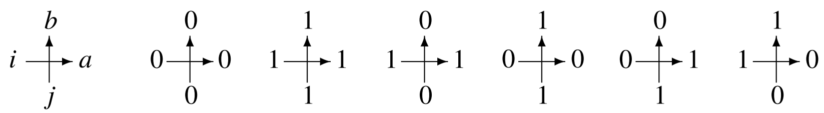
where the sums are taken over $a, b, i, j \in \{0, 1\}$ and E_{ij} is a matrix unit on $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$. The operators L^{ab}_{ij} and M^{ab}_{ij} are defined by

$$L^{ab}_{ij} = M^{ab}_{ij} = 0 \quad \text{if } a + b \neq i + j, \tag{6.54}$$

$$L^{00}_{00} = L^{11}_{11} = 1, \quad L^{10}_{10} = -q^2\mathbf{K}, \quad L^{01}_{01} = \mathbf{K}, \quad L^{10}_{01} = \mathbf{A}^+, \quad L^{01}_{10} = \mathbf{A}^-, \tag{6.55}$$

$$M^{00}_{00} = M^{11}_{11} = 1, \; M^{10}_{10} = q\tilde{\mathbf{K}}, \quad M^{01}_{01} = q\tilde{\mathbf{K}}, \; M^{10}_{01} = \mathbf{A}^+, \quad M^{01}_{10} = \mathbf{A}^-. \tag{6.56}$$

Pictorially, the non-zero cases look like



L^{ab}_{ij}	1	1	$-q^2\mathbf{K}$	$\mathbf{K}$	$\mathbf{A}^+$	$\mathbf{A}^-$
M^{ab}_{ij}	1	1	$q\tilde{\mathbf{K}}$	$q\tilde{\mathbf{K}}$	$\mathbf{A}^+$	$\mathbf{A}^-$

(6.57)

The operators $\mathbf{A}^{\pm}$, $\mathbf{K}$ are q^2-oscillators in (5.15) and (5.16), and $\tilde{\mathbf{K}}$ is given by

$$\tilde{\mathbf{K}}|m\rangle = (-q^2)^m|m\rangle. \tag{6.58}$$

Thus we have $L, M \in \mathrm{End}(V \otimes V \otimes \mathcal{F}_{q^2})$. These definitions are related to the earlier ones as

$$L^{ab}_{ij} = L(1)^{ab}_{ij}|_{q\to q^2,\mu\to -q^2}, \qquad M^{ab}_{ij} = M(1)^{ab}_{ij}|_{q\to q^2,\nu\to q}, \tag{6.59}$$

where the RHSs are those in (3.120) and (3.121).

We will further need the boundary vectors

$$\langle\tilde{\eta}_2| = \sum_{m\geq 0} \frac{\langle 2m|}{(q^8;q^8)_m}, \qquad |\tilde{\eta}_1\rangle = \sum_{m\geq 0} \frac{|m\rangle}{(-q^2;-q^2)_m}, \tag{6.60}$$

whose pairing is specified by $\langle m|m'\rangle = (q^4;q^4)_m \delta_{m,m'}$. These are obtained from the earlier ones in (3.132)–(3.133) by replacing q with $-q^2$.

Now the function $R^{\mathbf{ab}}_{\mathbf{ij}}(x)$ appearing in (6.5) is given by the matrix product formula

$$R(x)^{\mathbf{ab}}_{\mathbf{ij}} = (\text{scalar}) \times \langle\tilde{\eta}_2|x^{\mathbf{h}} M^{a_1b_1}_{i_1j_1}\cdots M^{a_nb_n}_{i_nj_n}|\tilde{\eta}_1\rangle, \tag{6.61}$$

where $\mathbf{h}$ is the number operator (3.14) and $\mathbf{a} = (a_1,\dots,a_n)$, etc. according to (6.1). The scalar can be chosen so that $R(x)^{\mathbf{ab}}_{\mathbf{ij}}$ becomes a polynomial in x^2 with maximal degree exactly n in the following sense[2]:

$$\max\left\{\deg_{x^2}\left(R(x)^{\mathbf{ab}}_{\mathbf{ij}}\right) \mid \mathbf{a},\mathbf{b},\mathbf{i},\mathbf{j}\in\{0,1\}^n\right\} = n. \tag{6.62}$$

Otherwise the normalization is not essential, being required only to validate (5.11) which does not influence (6.2) and (6.3).

Up to normalization and gauge, $R(x)^{\mathbf{ab}}_{\mathbf{ij}}$ is an element of the quantum R matrix of the spin representation of $B^{(1)}_n$ with spectral parameter x.[3] This fact and the related results will be explained in detail in Chapter 12.

What we need here is the tetrahedron equation of type $MMLL = LLMM$:

$$M_{126}M_{346}L_{135}L_{245} = L_{245}L_{135}M_{346}M_{126}. \tag{6.63}$$

This is a corollary of Theorem 3.25 and (6.59). It is an identity in $\mathrm{End}(\overset{1}{V}\otimes\overset{2}{V}\otimes\overset{3}{V}\otimes\overset{4}{V}\otimes\overset{5}{\mathcal{F}}_{q^2}\otimes\overset{6}{\mathcal{F}}_{q^2})$.

Another necessary fact is a property of the boundary vector

$$(\mathbf{A}^+ - 1 + \tilde{\mathbf{K}})|\tilde{\eta}_1\rangle = 0, \qquad (\mathbf{A}^- - 1 + q^2\tilde{\mathbf{K}})|\tilde{\eta}_1\rangle = 0. \tag{6.64}$$

This is a corollary of (3.134) and (3.135) and the origin of $|\tilde{\eta}_1\rangle$ mentioned after (6.60).

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[2] It is guaranteed from the fact that the spectral decomposition of the R matrix consists of $(n+1)$ eigenvalues. One can also check the claim in the example of $S^{2,1}(z)$ for $n=2$ in Sect. 12.4.

[3] Equation (6.61) is related to (12.9) as $R(x)^{\mathbf{ab}}_{\mathbf{ij}} = (\text{scalar})\, S^{2,1}(x)^{\mathbf{ab}}_{\mathbf{ij}}|_{q\to -q^2,\alpha\to q}$, where α originates in (12.1). Therefore from Theorem 12.2, it is an element of the quantum R matrix of the spin representation of $U_{q^{-2}}(B^{(1)}_n)$.

6.6.2 *RTT* Relation

This subsection is devoted to the proof of the RTT relation (6.2). According the remark after Remark 6.3, we take the parameter μ_i in π_i to be 1.

Lemma 6.10 *The image of the generator $t_{\mathbf{ab}} \in A_q(B_n)$ by π_i in (6.9)–(6.12) with $\mu_i = 1$ (also denoted by $t_{\mathbf{ab}}$ for simplicity) satisfies the RTT relation including the spectral parameter:*

$$\sum_{\mathbf{l},\mathbf{m}} R(x)^{\mathbf{ab}}_{\mathbf{lm}} t_{\mathbf{lc}} t_{\mathbf{md}} = \sum_{\mathbf{l},\mathbf{m}} t_{\mathbf{bm}} t_{\mathbf{al}} R(x)^{\mathbf{lm}}_{\mathbf{cd}} \tag{6.65}$$

In view of (6.5), the defining RTT relation of $A_q(B_n)$ in (6.2) follows from this lemma by picking the highest order terms in x since π_i's are independent of it.

Proof First we treat π_i with $1 \le i \le n-1$. Then it is easy to see

$$t_{\mathbf{ab}} = \theta(a_k = b_k \text{ for } k \neq i, i+1) L^{a_i a_{i+1}}_{b_{i+1} b_i}, \tag{6.66}$$

where $\theta(\text{true}) = 1$ and $\theta(\text{false}) = 0$. Compare $(6.9)|_{\mu_i=1} - (6.10)$ with (6.55). Thus $\mathbf{l}$ and $\mathbf{m}$ in the LHS (resp. RHS) of (6.65) are restricted to those $(l_k, m_k) = (c_k, d_k)$ (resp. $(l_k, m_k) = (a_k, b_k)$) for $k \neq i, i+1$.

Let us write down the $\mathrm{End}(\overset{5}{\mathcal{F}}_{q^2} \otimes \overset{6}{\mathcal{F}}_{q^2})$ component of the tetrahedron equation (6.63) corresponding to the transition

$$v_{c_{i+1}} \otimes v_{d_{i+1}} \otimes v_{c_i} \otimes v_{d_i} \to v_{a_i} \otimes v_{b_i} \otimes v_{a_{i+1}} \otimes v_{b_{i+1}}.$$

The result reads as

$$\sum M^{a_i b_i}_{l_i m_i} M^{a_{i+1} b_{i+1}}_{l_{i+1} m_{i+1}} L^{l_i l_{i+1}}_{c_{i+1} c_i} L^{m_i m_{i+1}}_{d_{i+1} d_i} = \sum L^{b_i b_{i+1}}_{m_{i+1} m_i} L^{a_i a_{i+1}}_{l_{i+1} l_i} M^{l_i m_i}_{c_i d_i} M^{l_{i+1} m_{i+1}}_{c_{i+1} d_{i+1}}, \tag{6.67}$$

where the sums are taken over $l_i, m_i, l_{i+1}, m_{i+1} \in \{0, 1\}$ on both sides. The operators $L^{\bullet\bullet}_{\bullet\bullet}$ and $M^{\bullet\bullet}_{\bullet\bullet}$ act on a different components $\overset{5}{\mathcal{F}}_{q^2}$ and $\overset{6}{\mathcal{F}}_{q^2}$, respectively. One can check that (6.67) agrees with (2.30).

Putting (6.67) in a sandwich in the space $\overset{6}{\mathcal{F}}_{q^2}$ as

$$\langle \tilde{\eta}_2 | x^{\mathbf{h}} M^{a_1 b_1}_{c_1 d_1} \cdots M^{a_{i-1} b_{i-1}}_{c_{i-1} d_{i-1}} (\cdots) M^{a_{i+2} b_{i+2}}_{c_{i+2} d_{i+2}} \cdots M^{a_n b_n}_{c_n d_n} | \tilde{\eta}_1 \rangle \tag{6.68}$$

and applying (6.66) and (6.61), we obtain (6.65).

The essence of this derivation is that the matrix product structure (6.61) makes things *local* with respect to i, and the local structure is exactly the tetrahedron equation of type $MMLL = LLMM$. Up to this point, the boundary vectors $\langle \tilde{\eta}_2 | x^{\mathbf{h}}$ and $|\tilde{\eta}_1\rangle$ have not played any role.

Next we consider π_n. From (6.11) and (6.12) we only have to concern the last factor $M^{a_n b_n}_{i_n j_n}$ in the matrix product formula (6.61) and the effect of the boundary vector $|\tilde{\eta}_1\rangle$. Moreover, the common indices α in (6.11) can be suppressed. Therefore it suffices to check

$$\sum_{l_n, m_n} \mathrm{t}_{l_n c_n} \mathrm{t}_{m_n d_n} \otimes M^{a_n b_n}_{l_n m_n} |\tilde{\eta}_1\rangle = \sum_{l_n, m_n} \mathrm{t}_{b_n m_n} \mathrm{t}_{a_n l_n} \otimes M^{l_n m_n}_{c_n d_n} |\tilde{\eta}_1\rangle \tag{6.69}$$

for the temporarily defined symbols

$$\begin{pmatrix} \mathrm{t}_{00} & \mathrm{t}_{01} \\ \mathrm{t}_{10} & \mathrm{t}_{11} \end{pmatrix} := \begin{pmatrix} t_{\alpha 0, \alpha 0} & t_{\alpha 0, \alpha 1} \\ t_{\alpha 1, \alpha 0} & t_{\alpha 1, \alpha 1} \end{pmatrix} = \begin{pmatrix} \mathbf{a}^- & \mathbf{k} \\ -q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}, \tag{6.70}$$

which is independent of $\alpha \in \{0, 1\}^{n-1}$. We have set $\mu_n = 1$ in (6.11). The relation (6.69) represents 16 identities in $\mathrm{End}(\mathcal{F}_q) \otimes \mathcal{F}_{q^2}$ corresponding to the choices $(a_n, b_n, c_n, d_n) \in \{0, 1\}^4$. It is elementary to verify them case by case. Here we shall only illustrate the two instructive examples. The other cases are similar.

Case $(a_n, b_n, c_n, d_n) = (0, 1, 0, 0)$. The difference LHS – RHS is

$$\begin{aligned} &\mathrm{t}_{00}\mathrm{t}_{10} \otimes M^{01}_{01} |\tilde{\eta}_1\rangle + \mathrm{t}_{10}\mathrm{t}_{00} \otimes M^{01}_{10} |\tilde{\eta}_1\rangle - \mathrm{t}_{10}\mathrm{t}_{00} \otimes M^{00}_{00} |\tilde{\eta}_1\rangle \\ &= \mathbf{a}^-(-q\mathbf{k}) \otimes q\tilde{\mathbf{K}} |\tilde{\eta}_1\rangle + (-q\mathbf{k})\mathbf{a}^- \otimes \mathbf{A}^- |\tilde{\eta}_1\rangle - (-q\mathbf{k}\mathbf{a}^-) \otimes |\tilde{\eta}_1\rangle \\ &= q\mathbf{k}\mathbf{a}^- \otimes (-q^2\tilde{\mathbf{K}} - \mathbf{A}^- + 1) |\tilde{\eta}_1\rangle = 0, \end{aligned}$$

where the last equality is due to (6.64).

Case $(a_n, b_n, c_n, d_n) = (0, 1, 0, 1)$. The difference LHS – RHS is

$$\begin{aligned} &\mathrm{t}_{00}\mathrm{t}_{11} \otimes M^{01}_{01} |\tilde{\eta}_1\rangle + \mathrm{t}_{10}\mathrm{t}_{01} \otimes M^{01}_{10} |\tilde{\eta}_1\rangle - \mathrm{t}_{11}\mathrm{t}_{00} \otimes M^{01}_{01} |\tilde{\eta}_1\rangle - \mathrm{t}_{10}\mathrm{t}_{01} \otimes M^{10}_{01} |\tilde{\eta}_1\rangle \\ &= (\mathbf{a}^-\mathbf{a}^+ - \mathbf{a}^+\mathbf{a}^-) \otimes q\tilde{\mathbf{K}} |\tilde{\eta}_1\rangle - q\mathbf{k}^2 \otimes (\mathbf{A}^- - \mathbf{A}^+) |\tilde{\eta}_1\rangle \\ &= (1 - q^2)\mathbf{k}^2 \otimes q\tilde{\mathbf{K}} |\tilde{\eta}_1\rangle - q\mathbf{k}^2 \otimes (1 - q^2)\tilde{\mathbf{K}} |\tilde{\eta}_1\rangle = 0, \end{aligned}$$

where the second equality is due to (3.12) and (6.64). □

6.6.3 *ρTT Relations*

The remaining task for the proof of Proposition 6.5 is to show:

Lemma 6.11 *The image of the generator* $t_{\mathbf{ab}} \in A_q(B_n)$ *by* π_i *in (6.9)–(6.12) with* $\mu_i = 1$ *satisfies (6.3).*

Proof We show

$$\sum_{\mathbf{b}} \rho_{\mathbf{b}}\, t_{\mathbf{ab}}\, t_{\mathbf{l}'\mathbf{b}'} = \rho_{\mathbf{a}}\, \delta_{\mathbf{bl}}. \tag{6.71}$$

Another relation in (6.3) can be verified similarly. See (6.4) for the notation $\mathbf{a}'$.

First we treat π_i with $1 \le i \le n-1$. From (6.9)–(6.10), the relation (6.71) is trivially valid as $0 = 0$ unless $a_k = l_k$ for $k \ne i, i+1$. So we focus on this situation which can be classified into the four cases $(a_i, a_{i+1}) \in \{0, 1\}^2$. The case $a_i = a_{i+1}$ is trivially valid as $0 = 0$ or $\rho_{\mathbf{a}}\, t_{\mathbf{aa}}\, t_{\mathbf{a}'\mathbf{a}'} = \rho_{\mathbf{a}}$. A similar fact holds also for (l_i, l_{i+1}). Thus the only non-trivial case is

$$\sum_{b_1, b_2} r_{b_1 b_2} \mathrm{t}_{a_1 a_2, b_1 b_2} \mathrm{t}_{l_1' l_2', b_1' b_2'} = r_{a_1 a_2} \delta_{a_1 l_1} \delta_{a_2 l_2} \tag{6.72}$$

for $(a_1, a_2), (l_1, l_2) \in \{(0, 1), (1, 0)\}$, where $b_1' = 1 - b_1$, etc. The sum extends over $(b_1, b_2) = (0, 1), (1, 0)$ only. Here we have written a_i, a_{i+1} as a_1, a_2, etc. for simplicity and introduced the temporary notations

$$\begin{pmatrix} \mathrm{t}_{01,01} & \mathrm{t}_{01,10} \\ \mathrm{t}_{10,01} & \mathrm{t}_{10,10} \end{pmatrix} := \begin{pmatrix} \mathbf{A}^- & \mathbf{K} \\ -q^2\mathbf{K} & \mathbf{A}^+ \end{pmatrix}, \tag{6.73}$$

$$r_{01} = 1, \quad r_{10} = -q^2. \tag{6.74}$$

The former is taken from (6.9) and the latter reflects $\rho_{\ldots 10 \ldots}/\rho_{\ldots 01 \ldots} = -q^2$ according to (6.6). Now (6.72) reads as

$$\begin{aligned} &\mathrm{t}_{01,01}\mathrm{t}_{10,10} - q^2\mathrm{t}_{01,10}\mathrm{t}_{10,01} = 1, \quad \mathrm{t}_{01,01}\mathrm{t}_{01,10} - q^2\mathrm{t}_{01,10}\mathrm{t}_{01,01} = 0, \\ &\mathrm{t}_{10,01}\mathrm{t}_{10,10} - q^2\mathrm{t}_{10,10}\mathrm{t}_{10,01} = 0, \quad \mathrm{t}_{10,01}\mathrm{t}_{01,10} - q^2\mathrm{t}_{10,10}\mathrm{t}_{01,01} = -q^2. \end{aligned} \tag{6.75}$$

These relations can be checked directly by means of (5.15). For instance, the first one reduces to $\mathbf{A}^-\mathbf{A}^+ + q^4\mathbf{K}^2 = 1$.

Next we consider π_n. From (6.11)–(6.12), the relation (6.71) is trivially valid as $0 = 0$ unless $a_j = l_j$ for $j \ne n$. So we focus on this situation which consists of the four cases $(a_n, l_n) \in \{0, 1\}^2$. In terms of the symbols in (6.70), they read as

$$\begin{aligned} &\mathrm{t}_{00}\mathrm{t}_{11} - q\mathrm{t}_{01}\mathrm{t}_{10} = 1, \quad \mathrm{t}_{00}\mathrm{t}_{01} - q\mathrm{t}_{01}\mathrm{t}_{00} = 0, \\ &\mathrm{t}_{10}\mathrm{t}_{11} - q\mathrm{t}_{11}\mathrm{t}_{10} = 0, \quad \mathrm{t}_{10}\mathrm{t}_{01} - q\mathrm{t}_{11}\mathrm{t}_{00} = -q, \end{aligned} \tag{6.76}$$

where the coefficient $-q$ reflects $\rho_{\ldots 1}/\rho_{\ldots 0} = -q$ from (6.6). Comparison of (6.73) and (6.70) shows that (6.76) is exactly reduced to (6.75) by replacing q with q^2. □

This completes a proof of Proposition 6.5. We did not use the property of the boundary vector $\langle \tilde{\eta}_2|$ directly. However, its effect is working implicitly via the fact that the normalization (6.62) is possible. We note that the representation π_i satisfies further relations obtained by taking the coefficients of the non-highest powers of x in (6.65).

6.7 Bibliographical Notes and Comments

This chapter is an extended exposition of [93, Sect. 4] and [94]. As mentioned in the beginning of Sect. 6.1, the algebra $A_q(B_n)$ in this chapter is different from $\mathrm{Fun}(\mathrm{SO}_q(2n+1))$ in [127, Definition 11] which is defined with $(2n+1)^2$ generators based on the vector representation. For $n = 2$, an explicit embedding $A_q(B_2)(\simeq A_q(C_2)) \hookrightarrow \mathrm{Fun}(\mathrm{SO}_{q^2}(5))$ is shown in [94, Theorem 2.1]. The concrete forms of the fundamental representations (6.9)–(6.12), Proposition 6.5 and its proof based on the tetrahedron equation $MMLL = LLMM$ are presented for the first time in Sect. 6.6.