Chapter 17
Reductions of Quantized G_2 Reflection Equation

Abstract Spectral parameters in the Yang–Baxter and the reflection equations correspond to the positive roots of A_2 and B_2/C_2, respectively. They appear as angles, or relative rapidity, of the world lines of particles that undergo factorized scattering in integrable $(1+1)$D quantum field theories in the bulk and at the boundary. There is an analogous equation associated with G_2, which we call the G_2 *reflection equation* in this book. It describes the three-body scattering related to the geometry of the Desargues–Pappus theorem. In addition to the usual two-body collision in the bulk, it involves the special three-particle event in which a two-body collision takes place at exactly the same instant as the boundary reflection of the third particle. In this chapter we construct infinite families of trigonometric solutions to the G_2 reflection equation by the 3D approach parallel with Chaps. 11–16. We start from the quantized G_2 reflection equation and its solution in Theorem 8.6, and perform the trace and the boundary vector reductions. The resulting solutions to the G_2 reflection equation involve quantum R matrices of $A^{(1)}_{n-1}$ and $D^{(2)}_{n+1}$, and they are coupled with the scattering amplitude of the special three-particle event expressed by a matrix product formula.

17.1 Introduction

Thus far we have presented a 3D approach to the Yang–Baxter and the reflection equations, which are presented in terms of additive spectral parameters as

$$R_{12}(\alpha_1)R_{13}(\alpha_1+\alpha_2)R_{23}(\alpha_2) = R_{23}(\alpha_2)R_{13}(\alpha_1+\alpha_2)R_{12}(\alpha_1), \tag{17.1}$$

$$\begin{aligned} &R_{12}(\alpha_1)K_2(\alpha_1+\alpha_2)R_{21}(\alpha_1+2\alpha_2)K_1(\alpha_2) \\ &= K_1(\alpha_2)R_{12}(\alpha_1+2\alpha_2)K_2(\alpha_1+\alpha_2)R_{21}(\alpha_1). \end{aligned} \tag{17.2}$$

They are spectral parameter dependent versions (sometimes referred to as Yang–Baxterizations) of the cubic and the quartic Coxeter relations for the simple reflections s_1, s_2 of the root systems of A_2 and B_2/C_2:

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$$s_1s_2s_1 = s_2s_1s_2, \qquad \Delta_+ = \{\alpha_1, \alpha_1+\alpha_2, \alpha_2\},$$
$$s_1s_2s_1s_2 = s_2s_1s_2s_1, \qquad \Delta_+ = \{\alpha_1, \alpha_1+\alpha_2, \alpha_1+2\alpha_2, \alpha_2\}.$$

Here α_1, α_2 are the simple roots and Δ_+ denotes the set of positive roots which formally correspond to the spectral parameters. They are so ordered that the kth one from the left is $s_{i_1}\cdots s_{i_{k-1}}(\alpha_{i_k})$ with $i_k = 1$ (k: odd) and $i_k = 2$ (k: even). See (10.3).

In this chapter we consider a natural G_2 analogue of them as

$$\begin{aligned} &R_{12}(\alpha_1)X_{132}(\alpha_1+\alpha_2)R_{23}(2\alpha_1+3\alpha_2)X_{213}(\alpha_1+2\alpha_2)R_{31}(\alpha_1+3\alpha_2)X_{321}(\alpha_2) \\ &= X_{231}(\alpha_2)R_{13}(\alpha_1+3\alpha_2)X_{123}(\alpha_1+2\alpha_2)R_{32}(2\alpha_1+3\alpha_2)X_{312}(\alpha_1+\alpha_2)R_{21}(\alpha_1), \end{aligned} \tag{17.3}$$

which we call the G_2 *reflection equation*. Based on the results on $A_q(G_2)$ in Chap. 8, we construct infinite families of solutions by extending the 3D approach further. The basic ingredient is the quantized G_2 reflection equation (8.2):

$$(L_{12}J_{132}L_{23}J_{213}L_{31}J_{321})F = F(J_{231}L_{13}J_{123}L_{32}J_{312}L_{21}). \tag{17.4}$$

It is a generalization of the constant G_2 reflection equation $R_{12}X_{132}R_{23}X_{213}R_{31}X_{321} = X_{231}R_{13}X_{123}R_{32}X_{312}R_{21}$ to a conjugacy equivalence by the intertwiner F. The contents are parallel with those for the Yang–Baxter and the reflection equations in Chaps. 11–16.

17.2 The G_2 Reflection Equation

Let $\mathbf{V}$ be a vector space and consider the operators

$$R(z) \in \mathrm{End}(\mathbf{V}\otimes\mathbf{V}), \qquad X(z) \in \mathrm{End}(\mathbf{V}\otimes\mathbf{V}\otimes\mathbf{V}) \tag{17.5}$$

depending on the spectral parameter z. We assume that $R(z)$ satisfies the Yang–Baxter equation by itself:

$$R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x) \in \mathrm{End}(\mathbf{V}\otimes\mathbf{V}\otimes\mathbf{V}). \tag{17.6}$$

We consider the G_2 reflection equation in $\mathrm{End}(\mathbf{V}\otimes\mathbf{V}\otimes\mathbf{V})$ with multiplicative spectral parameters:[1]

$$\begin{aligned} &R_{12}(x)X_{132}(xy)R_{23}(x^2y^3)X_{213}(xy^2)R_{31}(xy^3)X_{321}(y) \\ &= X_{231}(y)R_{13}(xy^3)X_{123}(xy^2)R_{32}(x^2y^3)X_{312}(xy)R_{21}(x). \end{aligned} \tag{17.7}$$

[1] In the solutions that we will obtain later, $\mathbf{V}$ has the structure $\mathbf{V} = V^{\otimes n}$, hence bold font will be used there for the indices.

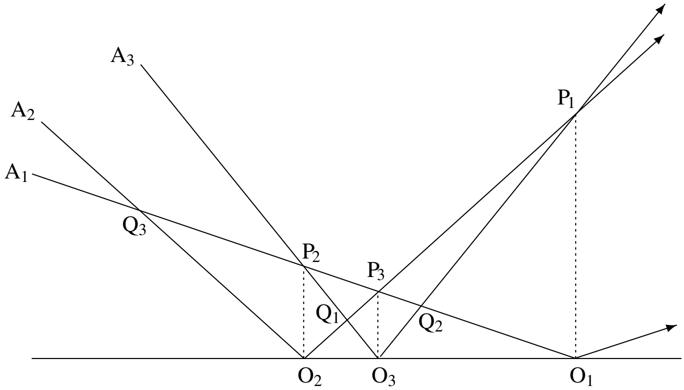


Fig. 17.1 Scattering diagram for the RHS of (17.7)

To clarify the notation, write temporarily as $R(z) = \sum r_l^{(1)} \otimes r_l^{(2)}$ and $X(z) = \sum x_l^{(1)} \otimes x_l^{(2)} \otimes x_l^{(3)}$ in terms of sums over l.[2] Then

$$\begin{aligned} R_{12}(z) &= \sum r_l^{(1)} \otimes r_l^{(2)} \otimes 1, \quad R_{21}(z) = \sum r_l^{(2)} \otimes r_l^{(1)} \otimes 1, \\ R_{13}(z) &= \sum r_l^{(1)} \otimes 1 \otimes r_l^{(2)}, \quad R_{31}(z) = \sum r_l^{(2)} \otimes 1 \otimes r_l^{(1)}, \\ R_{23}(z) &= \sum 1 \otimes r_l^{(1)} \otimes r_l^{(2)}, \quad R_{32}(z) = \sum 1 \otimes r_l^{(2)} \otimes r_l^{(1)}, \\ X_{ijk}(z) &= \sum x_l^{(i)} \otimes x_l^{(j)} \otimes x_l^{(k)}. \end{aligned} \tag{17.8}$$

Let us illustrate the special three-particle scattering diagram corresponding to the G_2 reflection equation. Consider the three particles 1,2,3 coming from A_1,A_2,A_3 and being reflected by the boundary at O_1, O_2, O_3, respectively. See Fig. 17.1. The bottom horizontal line is the boundary which may also be viewed as the time axis. The vertical direction corresponds to the 1D space. Each arrow carries $\mathbf{V}$ which specifies internal degrees of the freedom of a particle. So a three-particle state at a time is described by an element in $\mathbf{V} \otimes \mathbf{V} \otimes \mathbf{V}$.

One can arrange the three particle world lines so that the two-particle scattering P_1, P_2, P_3 happens exactly at the same instant as the boundary reflection O_1, O_2, O_3 of the other particle, respectively. This is non-trivial. For instance, suppose there were only particles 2 and 3. They already determine the reflecting points O_2, O_3 and the intersection P_1 (and Q_1) and its projection O_1 onto the boundary. Let P_2, P_3 be the points on the world lines of particles 3 and 2 whose projection are O_2 and O_3, respectively. In order to be able to draw the world line for the last particle 1, the three points P_2, P_3 and O_1 must be collinear. This is guaranteed by a special case of the Pappus theorem from the fourth century.

[2] Although these expansions do not specify $r_l^{(i)}, x_l^{(i)}$ uniquely, it suffices to make (17.8) unambiguous.

One can state it more symmetrically just by starting from $\mathrm{P}_1, \mathrm{P}_2$ and their projection $\mathrm{O}_1, \mathrm{O}_2$ onto the boundary. Let $\mathrm{P}'_1, \mathrm{P}'_2$ be the mirror image of $\mathrm{P}_1, \mathrm{P}_2$ with respect to the boundary. Then the three intersections $\overline{\mathrm{P}_1\mathrm{O}_2} \cap \overline{\mathrm{O}_1\mathrm{P}_2}$, $\overline{\mathrm{P}_1\mathrm{P}'_2} \cap \overline{\mathrm{P}'_1\mathrm{P}_2}$ and $\overline{\mathrm{O}_1\mathrm{P}'_2} \cap \overline{\mathrm{P}'_1\mathrm{O}_2}$ are collinear; in fact they are P_3, O_3 and the mirror image of P_3.

Let us call the so arranged scattering diagram a *Pappus configuration*. The reflection at O_i with the simultaneous two-particle scattering at P_i will be referred to as a *special three-particle event* ($i = 1, 2, 3$). Up to translation in the horizontal direction and the overall scale a Pappus configuration is parameterized by two real numbers, for instance, by the reflection angles $\angle \mathrm{P}_3\mathrm{O}_2\mathrm{O}_3$ and $\angle \mathrm{P}_3\mathrm{O}_1\mathrm{O}_3$. Set

$$
\begin{aligned}
u &= \angle \mathrm{P}_3\mathrm{O}_2\mathrm{O}_3, \quad w = \angle \mathrm{P}_2\mathrm{O}_3\mathrm{O}_2, \quad v = \angle \mathrm{P}_3\mathrm{O}_1\mathrm{O}_3,\\
\theta_1 &= \angle \mathrm{A}_2\mathrm{Q}_3\mathrm{A}_1, \quad \theta_2 = \angle \mathrm{A}_3\mathrm{P}_2\mathrm{A}_1, \quad \theta_3 = \angle \mathrm{A}_3\mathrm{Q}_1\mathrm{O}_2,\\
\theta_4 &= \angle \mathrm{A}_1\mathrm{P}_3\mathrm{O}_2, \quad \theta_5 = \angle \mathrm{A}_1\mathrm{Q}_2\mathrm{O}_3, \quad \theta_6 = \angle \mathrm{O}_2\mathrm{P}_1\mathrm{O}_3.
\end{aligned} \tag{17.9}
$$

Then it is elementary to see

$$\tan w = \tan u + \tan v, \tag{17.10}$$

$$\theta_1 = u - v, \quad \theta_2 = w - v, \quad \theta_3 = u + w, \quad \theta_4 = u + v, \quad \theta_5 = v + w, \quad \theta_6 = w - u. \tag{17.11}$$

We formally consider the infinitesimal angles, hence replace (17.10) by $w = u + v$. By a further substitution $u = \alpha_1 + \alpha_2$ and $v = \alpha_2$, (17.11) becomes

$$\theta_1 = \alpha_1, \ \ \theta_2 = \alpha_1 + \alpha_2, \ \ \theta_3 = 2\alpha_1 + 3\alpha_2, \ \ \theta_4 = \alpha_1 + 2\alpha_2, \ \ \theta_5 = \alpha_1 + 3\alpha_2, \ \ \theta_6 = \alpha_2. \tag{17.12}$$

Regard the symbols α_1, α_2 formally as the simple roots of G_2. They are transformed by the simple reflections s_1, s_2 of the Weyl group $W(G_2)$ as

$$s_1(\alpha_1) = -\alpha_1, \quad s_1(\alpha_2) = \alpha_1 + \alpha_2, \quad s_2(\alpha_1) = \alpha_1 + 3\alpha_2, \quad s_2(\alpha_2) = -\alpha_2. \tag{17.13}$$

Thus we find

$$\theta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), \qquad (i_1, i_2, i_3, i_4, i_5, i_6) = (1, 2, 1, 2, 1, 2), \tag{17.14}$$

and $\{\theta_1, \ldots, \theta_6\}$ yields the set of the positive roots of G_2.

The RHS of the G_2 reflection equation (17.7) is obtained by attaching $R(\mathrm{e}^{\theta_k})$ to the two particle scattering at Q_i and $G(\mathrm{e}^{\theta_k})$ to the special three particle event at $\mathrm{P}_i\mathrm{O}_i$ if it is the kth event starting from the left in Fig. 17.1. Setting $\mathrm{e}^{\alpha_1} = x$ and $\mathrm{e}^{\alpha_2} = y$, the assignment reads as

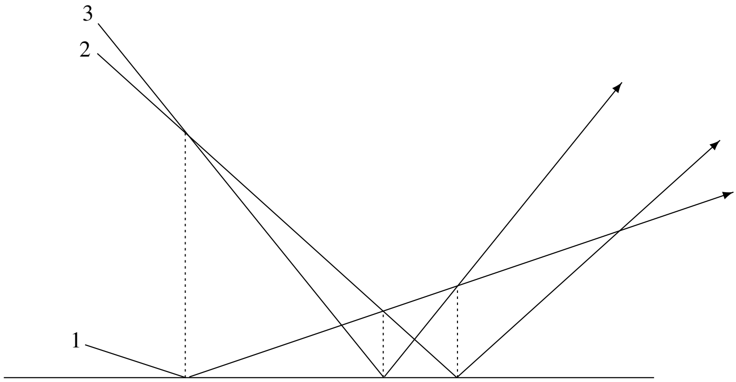


Fig. 17.2 Scattering diagram for the LHS of (17.7)

$$
\begin{aligned}
R_{21}(x) &: \text{two-particle scattering at } Q_3,\\
X_{312}(xy) &: \text{special three-particle event at } P_2O_2,\\
R_{32}(x^2y^3) &: \text{two-particle scattering at } Q_1,\\
X_{123}(xy^2) &: \text{special three-particle event at } P_3O_3,\\
R_{13}(xy^3) &: \text{two-particle scattering at } Q_2,\\
X_{231}(y) &: \text{special three-particle event at } P_1O_1.
\end{aligned}
$$

The indices for each operator correspond to the ordering of the relevant particles before the process. For instance, just before the special three-particle event at P_2O_2, the incoming particles are 3,1,2 from the top to the bottom, which is encoded in $X_{312}(xy)$. The LHS of the G_2 reflection equation (17.7) represents the Pappus configuration in which the time ordering of the processes are reversed. See Fig. 17.2.

Applications of the G_2 reflection equation to integrable systems are yet to be explored.

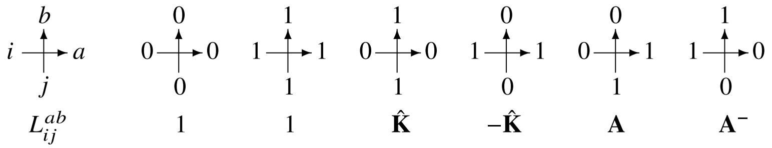
17.3 Quantized G_2 Reflection Equation

Let us recall the quantized G_2 reflection equation and its solution obtained in Sect. 8.5. The quantized G_2 reflection equation (8.50) is

$$L_{124}J_{1325}L_{236}J_{2137}L_{318}J_{3219}F_{456789} = F_{456789}J_{2319}L_{138}J_{1237}L_{326}J_{3125}L_{214}. \tag{17.15}$$

It is an equality of linear operators on $\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V} \otimes \overset{4}{\mathcal{F}}_{q^3} \otimes \overset{5}{\mathcal{F}}_q \otimes \overset{6}{\mathcal{F}}_{q^3} \otimes \overset{7}{\mathcal{F}}_q \otimes \overset{8}{\mathcal{F}}_{q^3} \otimes \overset{9}{\mathcal{F}}_q$.

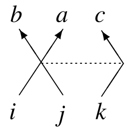
Let us recall L, J and F appearing here. First, $L \in \mathrm{End}(V \otimes V \otimes \mathcal{F}_{q^3})$ is the 3D L in (8.32)–(8.33) depicted as



(17.16)

$\mathbf{A}^{\pm}$ and $\hat{\mathbf{K}}$ are q^3-oscillators (8.7) including the zero point energy as in (8.13). This L is precisely equal to $((11.14)|_{\alpha=q^{1/2}})|_{q \to q^3}$.

Second, $J \in \mathrm{End}(V \otimes V \otimes V \otimes \mathcal{F}_q)$ is the quantized G_2 scattering operator. It is a collection of the operators $J^{abc}_{ijk} \in \mathrm{End}(\mathcal{F}_q)$ expressed by q-oscillators with zero point energy as (8.40)–(8.44). The quantized amplitude J^{abc}_{ijk} is depicted by the diagram which corresponds to the 90° rotation of the special three-particle events in Figs. 17.1 and 17.2:

$J^{abc}_{ijk} =$ 

(17.17)

Finally, $F \in \mathrm{End}(\mathcal{F}_{q^3} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^3} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^3} \otimes \mathcal{F}_q)$ is the intertwiner of the $A_q(G_2)$ modules detailed in Sect. 8.4.

17.4 Reduction of the Quantized G_2 Reflection Equation

Starting from the quantized G_2 reflection equation (17.15), one can perform two kinds of reductions to construct solutions to the G_2 reflection equation (17.7) in the matrix product form.

17.4.1 Concatenation of Quantized G_2 Reflection Equation

Consider n copies of (17.15) in which the spaces labeled with 1, 2, 3 are replaced by $1_i, 2_i, 3_i$ with $i = 1, 2, \ldots, n$:

$$\begin{aligned}&(L_{1_i 2_i 4} J_{1_i 3_i 2_i 5} L_{2_i 3_i 6} J_{2_i 1_i 3_i 7} L_{3_i 1_i 8} J_{3_i 2_i 1_i 9}) F_{456789}\\ &= F_{456789}(J_{2_i 3_i 1_i 9} L_{1_i 3_i 8} J_{1_i 2_i 3_i 7} L_{3_i 2_i 6} J_{3_i 1_i 2_i 5} L_{2_i 1_i 4}).\end{aligned} \tag{17.18}$$

Write this as $Z_i F_{456789} = F_{456789}\tilde{Z}_i$. Then repeated use of it leads to $Z_1 Z_2 \cdots Z_n F_{456789} = F_{456789}\tilde{Z}_1\tilde{Z}_2\cdots\tilde{Z}_n$, namely,

$$\begin{aligned}&(L_{1_1 2_1 4} J_{1_1 3_1 2_1 5} L_{2_1 3_1 6} J_{2_1 1_1 3_1 7} L_{3_1 1_1 8} J_{3_1 2_1 1_1 9}) \cdots\\ &\qquad \cdots (L_{1_n 2_n 4} J_{1_n 3_n 2_n 5} L_{2_n 3_n 6} J_{2_n 1_n 3_n 7} L_{3_n 1_n 8} J_{3_n 2_n 1_n 9}) F_{456789}\\ &= F_{456789}(J_{2_1 3_1 1_1 9} L_{1_1 3_1 8} J_{1_1 2_1 3_1 7} L_{3_1 2_1 6} J_{3_1 1_1 2_1 5} L_{2_1 1_1 4}) \cdots\\ &\qquad \cdots (J_{2_n 3_n 1_n 9} L_{1_n 3_n 8} J_{1_n 2_n 3_n 7} L_{3_n 2_n 6} J_{3_n 1_n 2_n 5} L_{2_n 1_n 4}).\end{aligned} \tag{17.19}$$

This can be rearranged without changing the order of operators sharing common labels as

$$\begin{aligned}&(L_{1_1 2_1 4}\cdots L_{1_n 2_n 4})(J_{1_1 3_1 2_1 5}\cdots J_{1_n 3_n 2_n 5})(L_{2_1 3_1 6}\cdots L_{2_n 3_n 6})\\ &\quad\times (J_{2_1 1_1 3_1 7}\cdots J_{2_n 1_n 3_n 7})(L_{3_1 1_1 8}\cdots L_{3_n 1_n 8})(J_{3_1 2_1 1_1 9}\cdots J_{3_n 2_n 1_n 9}) F_{456789}\\ &= F_{456789}(J_{2_1 3_1 1_1 9}\cdots J_{2_n 3_n 1_n 9})(L_{1_1 3_1 8}\cdots L_{1_n 3_n 8})(J_{1_1 2_1 3_1 7}\cdots J_{1_n 2_n 3_n 7})\\ &\quad\times (L_{3_1 2_1 6}\cdots L_{3_n 2_n 6})(J_{3_1 1_1 2_1 5}\cdots J_{3_n 1_n 2_n 5})(L_{2_1 1_1 4}\cdots L_{2_n 1_n 4}).\end{aligned} \tag{17.20}$$

Now we utilize the weight conservation (8.21) of F in the form

$$\begin{aligned}&F^{-1}_{456789} x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5}(x^2y^3)^{\mathbf{h}_6}(xy^2)^{\mathbf{h}_7}(xy^3)^{\mathbf{h}_8} y^{\mathbf{h}_9}\\ &= y^{\mathbf{h}_9}(xy^3)^{\mathbf{h}_8}(xy^2)^{\mathbf{h}_7}(x^2y^3)^{\mathbf{h}_6}(xy)^{\mathbf{h}_5} x^{\mathbf{h}_4} F^{-1}_{456789}.\end{aligned} \tag{17.21}$$

Multiply it by (17.20) side by side from the left. The result reads as

$$\begin{aligned}&F^{-1}_{456789}\big(x^{\mathbf{h}_4} L_{1_1 2_1 4}\cdots L_{1_n 2_n 4}\big)\big((xy)^{\mathbf{h}_5} J_{1_1 3_1 2_1 5}\cdots J_{1_n 3_n 2_n 5}\big)\\ &\quad\times\big((x^2y^3)^{\mathbf{h}_6} L_{2_1 3_1 6}\cdots L_{2_n 3_n 6}\big)\big((xy^2)^{\mathbf{h}_7} J_{2_1 1_1 3_1 7}\cdots J_{2_n 1_n 3_n 7}\big)\\ &\quad\times\big((xy^3)^{\mathbf{h}_8} L_{3_1 1_1 8}\cdots L_{3_n 1_n 8}\big)\big(y^{\mathbf{h}_9} J_{3_1 2_1 1_1 9}\cdots J_{3_n 2_n 1_n 9}\big) F_{456789}\\ &= \big(y^{\mathbf{h}_9} J_{2_1 3_1 1_1 9}\cdots J_{2_n 3_n 1_n 9}\big)\big((xy^3)^{\mathbf{h}_8} L_{1_1 3_1 8}\cdots L_{1_n 3_n 8}\big)\\ &\quad\times\big((xy^2)^{\mathbf{h}_7} J_{1_1 2_1 3_1 7}\cdots J_{1_n 2_n 3_n 7}\big)\big((x^2y^3)^{\mathbf{h}_6} L_{3_1 2_1 6}\cdots L_{3_n 2_n 6}\big)\\ &\quad\times\big((xy)^{\mathbf{h}_5} J_{3_1 1_1 2_1 5}\cdots J_{3_n 1_n 2_n 5}\big)\big(x^{\mathbf{h}_4} L_{2_1 1_1 4}\cdots L_{2_n 1_n 4}\big).\end{aligned} \tag{17.22}$$

17.4.2 Trace Reduction

Taking the trace of (17.22) over $\overset{4}{\mathcal{F}}_{q^3} \otimes \overset{5}{\mathcal{F}}_q \otimes \overset{6}{\mathcal{F}}_{q^3} \otimes \overset{7}{\mathcal{F}}_q \otimes \overset{8}{\mathcal{F}}_{q^3} \otimes \overset{9}{\mathcal{F}}_q$, we obtain

$$\begin{aligned}
&\mathrm{Tr}_4\big(x^{\mathbf{h}_4} L_{1_1 2_1 4} \cdots L_{1_n 2_n 4}\big)\mathrm{Tr}_5\big((xy)^{\mathbf{h}_5} J_{1_1 3_1 2_1 5} \cdots J_{1_n 3_n 2_n 5}\big)\\
&\times \mathrm{Tr}_6\big((x^2y^3)^{\mathbf{h}_6} L_{2_1 3_1 6} \cdots L_{2_n 3_n 6}\big)\mathrm{Tr}_7\big((xy^2)^{\mathbf{h}_7} J_{2_1 1_1 3_1 7} \cdots J_{2_n 1_n 3_n 7}\big)\\
&\times \mathrm{Tr}_8\big((xy^3)^{\mathbf{h}_8} L_{3_1 1_1 8} \cdots L_{3_n 1_n 8}\big)\mathrm{Tr}_9\big(y^{\mathbf{h}_9} J_{3_1 2_1 1_1 9} \cdots J_{3_n 2_n 1_n 9}\big)\\
&= \mathrm{Tr}_9\big(y^{\mathbf{h}_9} J_{2_1 3_1 1_1 9} \cdots J_{2_n 3_n 1_n 9}\big)\mathrm{Tr}_8\big((xy^3)^{\mathbf{h}_8} L_{1_1 3_1 8} \cdots L_{1_n 3_n 8}\big)\\
&\times \mathrm{Tr}_7\big((xy^2)^{\mathbf{h}_7} J_{1_1 2_1 3_1 7} \cdots J_{1_n 2_n 3_n 7}\big)\mathrm{Tr}_6\big((x^2y^3)^{\mathbf{h}_6} L_{3_1 2_1 6} \cdots L_{3_n 2_n 6}\big)\\
&\times \mathrm{Tr}_5\big((xy)^{\mathbf{h}_5} J_{3_1 1_1 2_1 5} \cdots J_{3_n 1_n 2_n 5}\big)\mathrm{Tr}_4\big(x^{\mathbf{h}_4} L_{2_1 1_1 4} \cdots L_{2_n 1_n 4}\big).
\end{aligned} \tag{17.23}$$

Here $\mathrm{Tr}_4(\cdots)$, $\mathrm{Tr}_6(\cdots)$, $\mathrm{Tr}_8(\cdots)$ involving the 3D L are identified with

$$S^{\mathrm{tr}}(z) := (S^{\mathrm{tr}_3}(z) \text{ in } (11.26))|_{q\to q^3} \tag{17.24}$$

up to a scalar multiple. The replacement $q \to q^3$ takes into account the comment after (17.16). It satisfies the Yang–Baxter equation (11.24) and is identified with the quantum R matrix of $U_{-q^{-3}}(A^{(1)}_{n-1})$ for the anti-symmetric tensor representations according to (Theorem 11.3)$|_{q\to q^3}$.

The other factors emerging from J have the form

$$X^{\mathrm{tr}}_{\mathbf{123}}(z) = \mathrm{Tr}_a(z^{\mathbf{h}_a} J_{1_1 2_1 3_1 a} \cdots J_{1_n 2_n 3_n a}) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}} \otimes \overset{\mathbf{2}}{\mathbf{V}} \otimes \overset{\mathbf{3}}{\mathbf{V}}), \tag{17.25}$$

where $\overset{\mathbf{k}}{\mathbf{V}} = \overset{k_1}{V} \otimes \cdots \otimes \overset{k_n}{V} \simeq (\mathbb{C}^2)^{\otimes n}$ for $\mathbf{k} = \mathbf{1}, \mathbf{2}, \mathbf{3}$. The trace is taken over $\overset{a}{\mathcal{F}}_q$ and evaluated by means of (3.12) and (11.27). Now the relation (17.23) is rephrased as

$$\begin{aligned}
&S^{\mathrm{tr}}_{\mathbf{12}}(x)X^{\mathrm{tr}}_{\mathbf{132}}(xy)S^{\mathrm{tr}}_{\mathbf{23}}(x^2y^3)X^{\mathrm{tr}}_{\mathbf{213}}(xy^2)S^{\mathrm{tr}}_{\mathbf{31}}(xy^3)X^{\mathrm{tr}}_{\mathbf{321}}(y)\\
&= X^{\mathrm{tr}}_{\mathbf{231}}(y)S^{\mathrm{tr}}_{\mathbf{13}}(xy^3)X^{\mathrm{tr}}_{\mathbf{123}}(xy^2)S^{\mathrm{tr}}_{\mathbf{32}}(x^2y^3)X^{\mathrm{tr}}_{\mathbf{312}}(xy)S^{\mathrm{tr}}_{\mathbf{21}}(x).
\end{aligned} \tag{17.26}$$

Thus the pair $(S^{\mathrm{tr}}(z), X^{\mathrm{tr}}(z))$ yields a solution to the G_2 reflection equation (17.7) for any $n \geq 1$. Elements of $X^{\mathrm{tr}}(z)$ are rational functions of $q^{1/2}$ and z.

17.4.3 Boundary Vector Reduction

Recall the boundary vectors in (8.60) and (8.61):

$$\langle \eta_1 | = \sum_{m\geq 0} \frac{\langle m|}{(q)_m}, \qquad |\eta_1\rangle = \sum_{m\geq 0} \frac{|m\rangle}{(q)_m}, \tag{17.27}$$

$$\langle\xi| = \sum_{m\ge 0}\frac{\langle m|}{(q^3)_m},\qquad |\xi\rangle = \sum_{m\ge 0}\frac{|m\rangle}{(q^3)_m}. \tag{17.28}$$

Sandwich the relation (17.22) between $\overset{4}{\langle\xi|}\otimes\overset{5}{\langle\eta_1|}\otimes\overset{6}{\langle\xi|}\otimes\overset{7}{\langle\eta_1|}\otimes\overset{8}{\langle\xi|}\otimes\overset{9}{\langle\eta_1|}$ and $\overset{4}{|\xi\rangle}\otimes\overset{5}{|\eta_1\rangle}\otimes\overset{6}{|\xi\rangle}\otimes\overset{7}{|\eta_1\rangle}\otimes\overset{8}{|\xi\rangle}\otimes\overset{9}{|\eta_1\rangle}$. Assuming Conjecture 8.9 and using $F = F^{-1}$ (8.22), we get

$$\begin{aligned}
&\overset{4}{\langle\xi|}x^{\mathbf{h}_4}L_{1_12_14}\cdots L_{1_n2_n4}\overset{4}{|\xi\rangle}\overset{5}{\langle\eta_1|}(xy)^{\mathbf{h}_5}J_{1_13_12_15}\cdots J_{1_n3_n2_n5}\overset{5}{|\eta_1\rangle}\\
&\quad\times\overset{6}{\langle\xi|}(x^2y^3)^{\mathbf{h}_6}L_{2_13_16}\cdots L_{2_n3_n6}\overset{6}{|\xi\rangle}\overset{7}{\langle\eta_1|}(xy^2)^{\mathbf{h}_7}J_{2_11_13_17}\cdots J_{2_n1_n3_n7}\overset{7}{|\eta_1\rangle}\\
&\quad\times\overset{8}{\langle\xi|}(xy^3)^{\mathbf{h}_8}L_{3_11_18}\cdots L_{3_n1_n8}\overset{8}{|\xi\rangle}\overset{9}{\langle\eta_1|}y^{\mathbf{h}_9}J_{3_12_11_19}\cdots J_{3_n2_n1_n9}\overset{9}{|\eta_1\rangle}\\
&=\overset{9}{\langle\eta_1|}y^{\mathbf{h}_9}J_{2_13_11_19}\cdots J_{2_n3_n1_n9}\overset{9}{|\eta_1\rangle}\overset{8}{\langle\xi|}(xy^3)^{\mathbf{h}_8}L_{1_13_18}\cdots L_{1_n3_n8}\overset{8}{|\xi\rangle}\\
&\quad\times\overset{7}{\langle\eta_1|}(xy^2)^{\mathbf{h}_7}J_{1_12_13_17}\cdots J_{1_n2_n3_n7}\overset{7}{|\eta_1\rangle}\overset{6}{\langle\xi|}(x^2y^3)^{\mathbf{h}_6}L_{3_12_16}\cdots L_{3_n2_n6}\overset{6}{|\xi\rangle}\\
&\quad\times\overset{5}{\langle\eta_1|}(xy)^{\mathbf{h}_5}J_{3_11_12_15}\cdots J_{3_n1_n2_n5}\overset{5}{|\eta_1\rangle}\overset{4}{\langle\xi|}x^{\mathbf{h}_4}L_{2_11_14}\cdots L_{2_n1_n4}\overset{4}{|\xi\rangle}.
\end{aligned}\tag{17.29}$$

The operators arising from $\langle\xi|(\cdots)|\xi\rangle$ involving L are identified, up to a scalar multiple, with

$$S^{\mathrm{bv}}(z) := (S^{1,1}(z)\ \mathrm{in}\ (12.9))|_{q\to q^3}, \tag{17.30}$$

where the superscript "bv" indicates the boundary vector reduction. The relation of the boundary vectors (17.28) = (12.3)$|_{r=1,q\to q^3}$ has also been used for the identification. The result (12.7)$|_{r=r'=1}$ shows that $S^{\mathrm{bv}}(z)$ satisfies the Yang–Baxter equation. It is identified with the quantum R matrix of $U_p(D^{(2)}_{n+1})$ for the spin representation at $p = -q^{-3}$ according to Theorem 12.2.

The other factors emerging from J have the form

$$X^{\mathrm{bv}}_{\mathbf{123}}(z) = \kappa^{\mathrm{bv}}(z)\overset{a}{\langle\eta_1|}z^{\mathbf{h}_a}J_{1_12_13_1a}\cdots J_{1_n2_n3_na}\overset{a}{|\eta_1\rangle} \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}}\otimes\overset{\mathbf{2}}{\mathbf{V}}\otimes\overset{\mathbf{3}}{\mathbf{V}}), \tag{17.31}$$

$$\kappa^{\mathrm{bv}}(z) = \frac{(z;q)_\infty}{(-qz;q)_\infty}, \tag{17.32}$$

where the normalization factor $\kappa^{\mathrm{bv}}(z)$ is introduced to make elements of $X^{\mathrm{bv}}(z)$ rational functions of $q^{1/2}$ and z. Now the relation (17.29) is rephrased as

$$\begin{aligned}
&R^{\mathrm{bv}}_{\mathbf{12}}(x)X^{\mathrm{bv}}_{\mathbf{132}}(xy)R^{\mathrm{bv}}_{\mathbf{23}}(x^2y^3)X^{\mathrm{bv}}_{\mathbf{213}}(xy^2)R^{\mathrm{bv}}_{\mathbf{31}}(xy^3)X^{\mathrm{bv}}_{\mathbf{321}}(y)\\
&= X^{\mathrm{bv}}_{\mathbf{231}}(y)R^{\mathrm{bv}}_{\mathbf{13}}(xy^3)X^{\mathrm{bv}}_{\mathbf{123}}(xy^2)R^{\mathrm{bv}}_{\mathbf{32}}(x^2y^3)X^{\mathrm{bv}}_{\mathbf{312}}(xy)R^{\mathrm{bv}}_{\mathbf{21}}(x).
\end{aligned}\tag{17.33}$$

Thus the pair $(R^{\mathrm{bv}}(z), X^{\mathrm{bv}}(z))$ provides another solution to the G_2 reflection equation (17.7) for any $n \geq 1$ provided that Conjecture 8.9 holds.

17.5 Properties of $X^{\mathrm{tr}}(z)$ and $X^{\mathrm{bv}}(z)$

We use notations like $\mathfrak{s} = \{0, 1\}^n$, $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{e}_k$, $|\mathbf{a}| = a_1 + \dots + a_n$, $v_{\mathbf{a}} \in \mathbf{V}$ and $\mathbf{V}_k \subset \mathbf{V}$ introduced in (11.1)–(11.7). The construction (17.25) and (17.31) imply the matrix product formula for the elements as

$$X(z)(v_{\mathbf{i}} \otimes v_{\mathbf{j}} \otimes v_{\mathbf{k}}) = \sum_{\mathbf{a},\mathbf{b},\mathbf{c} \in \mathfrak{s}} X(z)^{\mathbf{abc}}_{\mathbf{ijk}} v_{\mathbf{a}} \otimes v_{\mathbf{b}} \otimes v_{\mathbf{c}} \quad (X = X^{\mathrm{tr}}, X^{\mathrm{bv}}), \tag{17.34}$$

$$X^{\mathrm{tr}}(z)^{\mathbf{abc}}_{\mathbf{ijk}} = \mathrm{Tr}\big(z^{\mathbf{h}} J^{a_1,b_1,c_1}_{i_1,j_1,k_1} \cdots J^{a_n,b_n,c_n}_{i_n,j_n,k_n}\big), \tag{17.35}$$

$$X^{\mathrm{bv}}(z)^{\mathbf{abc}}_{\mathbf{ijk}} = \kappa^{\mathrm{bv}}(z)\langle \eta_1 | z^{\mathbf{h}} J^{a_1,b_1,c_1}_{i_1,j_1,k_1} \cdots J^{a_n,b_n,c_n}_{i_n,j_n,k_n} | \eta_1 \rangle \tag{17.36}$$

in terms of J^{abc}_{ijk} specified in (8.39)–(8.44). They are rational functions of z and $q^{1/2}$.

From (8.46) and (8.47), $X^{\mathrm{tr}}(z)$ has the selection rule

$$X^{\mathrm{tr}}(z)^{\mathbf{abc}}_{\mathbf{ijk}} = 0 \quad \text{unless} \quad \mathbf{a} + \mathbf{b} = \mathbf{i} + \mathbf{j} \in \mathbb{Z}^n \quad \text{and} \quad n + |\mathbf{j}| - |\mathbf{k}| = |\mathbf{b}| + |\mathbf{c}| \tag{17.37}$$

or equivalently the direct sum decomposition:

$$\begin{aligned} X^{\mathrm{tr}}(z) &= \bigoplus_{l,m,k} X^{\mathrm{tr}}(z)_{l,m,k}, \\ X^{\mathrm{tr}}(z)_{l,m,k} &: \mathbf{V}_l \otimes \mathbf{V}_m \otimes \mathbf{V}_k \to \bigoplus_{k'} \mathbf{V}_{l+k+k'-n} \otimes \mathbf{V}_{m-k-k'+n} \otimes \mathbf{V}_{k'}, \end{aligned} \tag{17.38}$$

where the sums extend over $l, m, k, k' \in [0, n]$ such that the indices $l + k + k' - n$ and $m - k - k' + n$ also belong to $[0, n]$.

Similarly, (8.46) leads to the selection rule of $X^{\mathrm{bv}}(z)$ as

$$X^{\mathrm{bv}}(z)^{\mathbf{abc}}_{\mathbf{ijk}} = 0 \quad \text{unless} \quad \mathbf{a} + \mathbf{b} = \mathbf{i} + \mathbf{j} \in \mathbb{Z}^n. \tag{17.39}$$

Example 17.1 We temporarily write $v_{\mathbf{a}}$ as $|\mathbf{a}\rangle$ to magnify the array $\mathbf{a}$. We set $\mathbf{e}_{[1,m]} = \mathbf{e}_1 + \dots + \mathbf{e}_m$. In particular, $|\mathbf{0}\rangle = |0, \dots, 0\rangle$ and $|\mathbf{1}\rangle = |\mathbf{e}_{[1,n]}\rangle = |1, \dots, 1\rangle$.

$$\begin{aligned} &X^{\mathrm{tr}}(z)(|\mathbf{e}_{[1,l]}\rangle \otimes |\mathbf{e}_{[1,m]}\rangle \otimes |\mathbf{0}\rangle) \\ &= \frac{(q^{\frac{1}{2}})^{m-l+n}}{1 - zq^{m-l+n}} |\mathbf{e}_{[1,l]}\rangle \otimes |\mathbf{e}_{[1,m]}\rangle \otimes |\mathbf{1}\rangle + \cdots \quad (l \leq m), \end{aligned} \tag{17.40}$$

$$X^{\mathrm{tr}}(z)(|\mathbf{e}_{[1,l]}\rangle \otimes |\mathbf{e}_{[1,m]}\rangle \otimes |\mathbf{1}\rangle)$$
$$= \frac{(-q^{\frac{1}{2}})^{l-m+n}}{1 - zq^{l-m+n}} |\mathbf{e}_{[1,l]}\rangle \otimes |\mathbf{e}_{[1,m]}\rangle \otimes |\mathbf{0}\rangle + \cdots \quad (l \geq m), \tag{17.41}$$
$$X^{\mathrm{bv}}(z)(|\mathbf{e}_{[1,l]}\rangle \otimes |\mathbf{e}_{[1,m]}\rangle \otimes |\mathbf{0}\rangle)$$
$$= q^{\frac{m-l+n}{2}} \frac{(z;q)_{m-l+n}}{(-qz;q)_{m-l+n}} |\mathbf{e}_{[1,l]}\rangle \otimes |\mathbf{e}_{[1,m]}\rangle \otimes |\mathbf{1}\rangle + \cdots \quad (l \leq m), \tag{17.42}$$
$$X^{\mathrm{bv}}(z)(|\mathbf{e}_{[1,l]}\rangle \otimes |\mathbf{e}_{[1,m]}\rangle \otimes |\mathbf{1}\rangle)$$
$$= (-q^{\frac{1}{2}})^{l-m+n} \frac{(z;q)_{l-m+n}}{(-qz;q)_{l-m+n}} |\mathbf{e}_{[1,l]}\rangle \otimes |\mathbf{e}_{[1,m]}\rangle \otimes |\mathbf{0}\rangle + \cdots \quad (l \geq m). \tag{17.43}$$

Example 17.2 Let us present examples of $X^{\mathrm{tr}}(z)$. We temporarily write $v_{\mathbf{a}} \otimes v_{\mathbf{b}} \otimes v_{\mathbf{c}}$ as $|\mathbf{a}, \mathbf{b}, \mathbf{c}\rangle$ for short. For $n = 1$, $X^{\mathrm{tr}}(z)$ acts on $\mathbf{V}^{\otimes 3} = V^{\otimes 3}$ as

$$|0,0,0\rangle \mapsto \frac{q^{\frac{1}{2}}|0,0,1\rangle}{1-qz}, \quad |0,0,1\rangle \mapsto -\frac{q^{\frac{1}{2}}|0,0,0\rangle}{1-qz}, \quad |0,1,0\rangle \mapsto \frac{q|0,1,1\rangle}{1-q^2z},$$
$$|0,1,1\rangle \mapsto -\frac{u_1u_3(q^2-z)|0,1,0\rangle}{\rho(1-z)(1-q^2z)} - \frac{u_3u_4(q^2-z)|1,0,1\rangle}{\rho(1-z)(1-q^2z)},$$
$$|1,0,0\rangle \mapsto -\frac{u_1u_2(q^2-z)|0,1,0\rangle}{\rho(1-z)(1-q^2z)} - \frac{u_2u_4(q^2-z)|1,0,1\rangle}{\rho(1-z)(1-q^2z)},$$
$$|1,0,1\rangle \mapsto \frac{q|1,0,0\rangle}{1-q^2z}, \quad |1,1,0\rangle \mapsto \frac{q^{\frac{1}{2}}|1,1,1\rangle}{1-qz}, \quad |1,1,1\rangle \mapsto -\frac{q^{\frac{1}{2}}|1,1,0\rangle}{1-qz},$$

where ρ defined in (8.45) and u_1, u_2, u_3, u_4 are to obey (8.10). The two kinds of the denominators $1 - qz$ and $1 - q^2z$ originate in $J^{001}_{000} = \hat{\mathbf{k}}$ and $J^{011}_{010} = \hat{\mathbf{k}}^2$.

For $n = 2$, it is too lengthy to present all the data. So we give just a few examples:

$$|00,00,00\rangle \mapsto \frac{q|00,00,11\rangle}{1-q^2z}, \quad |00,00,01\rangle \mapsto \frac{(1-q^2)z|00,00,01\rangle}{(1-z)(1-q^2z)} - \frac{q|00,00,10\rangle}{1-q^2z},$$
$$|00,10,11\rangle \mapsto \frac{q^{\frac{3}{2}}u_1u_3(q-z)|00,10,00\rangle}{\rho(1-qz)(1-q^3z)} - \frac{q^{\frac{1}{2}}(1-q^2)u_3z|10,00,01\rangle}{(1-qz)(1-q^3z)}$$
$$+ \frac{q^{\frac{3}{2}}u_3u_4(q-z)|10,00,10\rangle}{\rho(1-qz)(1-q^3z)},$$
$$|10,01,01\rangle \mapsto \frac{u_1^2u_2u_3(q^4+z-2q^2z-2q^4z+q^6z+q^2z^2)|00,11,00\rangle)}{\rho^2(1-z)(1-q^2z)(1-q^4z)}$$
$$+ \frac{u_1u_2u_3u_4(q^4+z-2q^2z-2q^4z+q^6z+q^2z^2)|01,10,01\rangle}{\rho^2(1-z)(1-q^2z)(1-q^4z)}$$
$$- \frac{q(1-q^2)u_2u_3|01,10,10\rangle}{(1-q^2z)(1-q^4z)} - \frac{q(1-q^2)u_2u_3z|10,01,01\rangle}{(1-q^2z)(1-q^4z)}$$

$$+\frac{u_1u_2u_3u_4(q^4+z-2q^2z-2q^4z+q^6z+q^2z^2)|10,01,10\rangle}{\rho^2(1-z)(1-q^2z)(1-q^4z)}$$
$$+\frac{u_2u_3u_4^2(q^4+z-2q^2z-2q^4z+q^6z+q^2z^2)|11,00,11\rangle}{\rho^2(1-z)(1-q^2z)(1-q^4z)}.$$

Example 17.3 $S^{\mathrm{bv}}(z)$ with $n=1$ is available in Example 12.1 with $r=r'=1$ and the replacement $q\to q^3$. Let us present examples of $X^{\mathrm{bv}}(z)$ with $n=1$ using the same notation as Example 17.2. It acts on $\mathbf{V}^{\otimes 3}=V^{\otimes 3}$ as

$$|0,0,0\rangle \mapsto \frac{(1+q)z|0,0,0\rangle}{1+qz}+\frac{q^{\frac{1}{2}}(1-z)|0,0,1\rangle}{1+qz},$$
$$|0,0,1\rangle \mapsto -\frac{q^{\frac{1}{2}}(1-z)|0,0,0\rangle}{1+qz}+\frac{(1+q)|0,0,1\rangle}{1+qz},$$
$$|0,1,1\rangle \mapsto \frac{q^{\frac{3}{2}}(1+q)u_1(1-z)z|0,1,0\rangle}{(1+qz)(1+q^2z)}+\frac{q(1-z)(1-qz)|0,1,1\rangle}{(1+qz)(1+q^2z)}$$
$$+\frac{(1+q)(1+q^2)z^2|1,0,0\rangle}{(1+qz)(1+q^2z)}+\frac{q^{\frac{3}{2}}(1+q)u_4(1-z)z|1,0,1\rangle}{(1+qz)(1+q^2z)},$$
$$|0,1,1\rangle \mapsto \frac{u_3(-q^2+z+2qz+2q^2z+q^3z-qz^2)(u_1|0,1,0\rangle+u_4|1,0,1\rangle)}{\rho(1+qz)(1+q^2z)}$$
$$+\frac{q^{\frac{1}{2}}(1+q)u_3(1-z)(|0,1,1\rangle-z|1,0,0\rangle)}{(1+qz)(1+q^2z)},$$
$$|1,0,0\rangle \mapsto \frac{u_2(-q^2+z+2qz+2q^2z+q^3z-qz^2)(u_1|0,1,0\rangle+u_4|1,0,1\rangle)}{\rho(1+qz)(1+q^2z)}$$
$$+\frac{q^{\frac{1}{2}}(1+q)u_2(1-z)(|0,1,1\rangle-z|1,0,0\rangle)}{(1+qz)(1+q^2z)},$$
$$|1,0,1\rangle \mapsto -\frac{q^{\frac{3}{2}}(1+q)u_1(1-z)|0,1,0\rangle}{(1+qz)(1+q^2z)}+\frac{(1+q)(1+q^2)|0,1,1\rangle}{(1+qz)(1+q^2z)}$$
$$+\frac{q(1-z)(1-qz)|1,0,0\rangle}{(1+qz)(1+q^2z)}-\frac{q^{\frac{3}{2}}(1+q)u_4(1-z)|1,0,1\rangle}{(1+qz)(1+q^2z)},$$
$$|1,1,0\rangle \mapsto \frac{(1+q)z|1,1,0\rangle}{1+qz}+\frac{q^{\frac{1}{2}}(1-z)|1,1,1\rangle}{1+qz},$$
$$|1,1,1\rangle \mapsto -\frac{q^{\frac{1}{2}}(1-z)|1,1,0\rangle}{1+qz}+\frac{(1+q)|1,1,1\rangle}{1+qz}.$$

17.6 Bibliographical Notes and Comments

This chapter is based on [85]. The G_2 reflection equation (17.3) or (17.7) up to spectral parameters was suggested on [30, p. 982], where the Desargues–Pappus geometry of the G_2 scattering diagram was mentioned instead of the equation itself. The equation of the form (17.3) for generic symbols R and X without assuming a tensor product structure of their representation space (i.e. without indices) has appeared as a defining relation of the *root algebra* of type G_2 in [31, Sect. 2].

The reduction procedures in Sect. 17.4 are parallel with earlier chapters. The intertwiner F of $A_q(G_2)$ is eliminated in an early stage but it controls the matrix product construction essentially.

It is an outstanding problem whether the solution $X^{\mathrm{tr}}(z)$ and the conjectural solution $X^{\mathrm{bv}}(z)$ admit a characterization analogous to Theorems 15.3 and 16.2 by some sort of quantum group theoretical structure like coideals.