Chapter 16
Boundary Vector Reductions of $(LGLG)K = K(GLGL)$

Abstract This chapter is a continuation of the 3D approach to the reflection equation from the previous one. We start from the n-concatenation of the quantized reflection equation $(LGLG)K = K(GLGL)$ and perform the boundary vector reduction. The L part gives rise to the quantum R matrices for the spin representations of $\mathfrak{g}^{r,r'} = B_n^{(1)}, D_n^{(1)}, D_n^{(2)}, \tilde{B}_n^{(1)}$, which have been detailed in Chap. 12. The G part generates the companion K matrices that satisfy the reflection equation. They are expressed by a matrix product formula in terms of G and characterized as the intertwiners of various Onsager coideals of the quantum affine algebras $U_p(\mathfrak{g}^{r,r'})$. The final list of the solutions is summarized in Table 16.1.

16.1 Preliminaries

We keep the setting in Sect. 15.1 and continue to work with the solution (L, G, K) to the quantized reflection equation $L_{123}G_{24}L_{215}G_{16}K_{3456} = K_{3456}G_{16}L_{125}G_{24}L_{213}$ summarized there. Thus L and G are given by

$$\begin{pmatrix} L_{00}^{00} & L_{01}^{00} & L_{10}^{00} & L_{11}^{00} \\ L_{00}^{01} & L_{01}^{01} & L_{10}^{01} & L_{11}^{01} \\ L_{00}^{10} & L_{01}^{10} & L_{10}^{10} & L_{11}^{10} \\ L_{00}^{11} & L_{01}^{11} & L_{10}^{11} & L_{11}^{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -q^2\alpha^{-1}\mathbf{K} & \mathbf{A}^- & 0 \\ 0 & \mathbf{A}^+ & \alpha\mathbf{K} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} G_0^0 & G_1^0 \\ G_0^1 & G_1^1 \end{pmatrix} = \begin{pmatrix} \mathbf{a}^+ & -q\beta^{-1}\mathbf{k} \\ \beta\mathbf{k} & \mathbf{a}^- \end{pmatrix}. \tag{16.1}$$

The 3D K has been detailed in Chap. 5. Note that

$$(L \text{ in } (16.1)) = (L \text{ in } (11.14))|_{q\to q^2} = (L \text{ in } (12.1))|_{q\to q^2}. \tag{16.2}$$

Our starting point is the n-concatenation of the quantized reflection equation

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$$
\begin{aligned}
&(L_{1_1 2_1 3} \cdots L_{1_n 2_n 3})(G_{2_1 4} \cdots G_{2_n 4})(L_{2_1 1_1 5} \cdots L_{2_n 1_n 5})(G_{1_1 6} \cdots G_{1_n 6})K_{3456} \\
&\quad = K_{3456}(G_{1_1 6} \cdots G_{1_n 6})(L_{1_1 2_1 5} \cdots L_{1_n 2_n 5})(G_{2_1 4} \cdots G_{2_n 4})(L_{2_1 1_1 3} \cdots L_{2_n 1_n 3})
\end{aligned} \tag{16.3}
$$

and the weight conservation of the 3D K

$$
K_{3456}(xy^{-1})^{\mathbf{h}_3} x^{\mathbf{h}_4} (xy)^{\mathbf{h}_5} y^{\mathbf{h}_6} = (xy^{-1})^{\mathbf{h}_3} x^{\mathbf{h}_4} (xy)^{\mathbf{h}_5} y^{\mathbf{h}_6} K_{3456}, \tag{16.4}
$$

which are quoted from (15.10) and (15.11). The operator $\mathbf{h}_i$ is the number operator $\mathbf{h}$ (3.14) acting on the ith Fock space.

16.2 Boundary Vector Reduction

Recall the boundary vectors in (5.118) and (5.119):

$$
\langle \eta_r | = \sum_{m \geq 0} \frac{\langle rm|}{(q^{r^2})_m}, \quad |\eta_r\rangle = \sum_{m \geq 0} \frac{|rm\rangle}{(q^{r^2})_m}, \tag{16.5}
$$

$$
\langle \chi_r | = \sum_{m \geq 0} \frac{\langle rm|}{(q^{2r^2})_m}, \quad |\chi_r\rangle = \sum_{m \geq 0} \frac{|rm\rangle}{(q^{2r^2})_m}, \tag{16.6}
$$

where $r = 1, 2$. The second line is obtained by setting $q \to q^2$ in the first line. The vectors (16.5) (resp. (16.6)) are elements of a completion of $\mathcal{F}_q^*$ and $\mathcal{F}_q$ (resp. $\mathcal{F}_{q^2}^*$ and $\mathcal{F}_{q^2}$).[1] We invoke Proposition 5.21, which states that they yield particular eigenvectors of the 3D K as

$$
\begin{aligned}
&(\langle \chi_r | \otimes \langle \eta_k | \otimes \langle \chi_r | \otimes \langle \eta_k |)K = \langle \chi_r | \otimes \langle \eta_k | \otimes \langle \chi_r | \otimes \langle \eta_k |, \\
&K(|\chi_r\rangle \otimes |\eta_k\rangle \otimes |\chi_r\rangle \otimes |\eta_k\rangle) = |\chi_r\rangle \otimes |\eta_k\rangle \otimes |\chi_r\rangle \otimes |\eta_k\rangle,
\end{aligned} \tag{16.7}
$$

where $1 \leq r \leq k \leq 2$.

Multiply $(xy^{-1})^{\mathbf{h}_3} x^{\mathbf{h}_4} (xy)^{\mathbf{h}_5} y^{\mathbf{h}_6}$ from the left by (16.3) and sandwich the result by the boundary vectors as

$$
\langle \overset{3}{\chi}_r | \otimes \langle \overset{4}{\eta}_k | \otimes \langle \overset{5}{\chi}_r | \otimes \langle \overset{6}{\eta}_k | (\cdots) | \overset{3}{\chi}_{r'} \rangle \otimes | \overset{4}{\eta}_{k'} \rangle \otimes | \overset{5}{\chi}_{r'} \rangle \otimes | \overset{6}{\eta}_{k'} \rangle.
$$

Thanks to the commutativity (16.4) and the eigen-property (16.7), the 3D K disappears and the result becomes

[1] From (3.16), dual pairing of $\mathcal{F}_{q^2}^*$ and $\mathcal{F}_{q^2}$ should be calculated by $\langle m|m'\rangle = (q^4)_m \delta_{m,m'}$.

$$\langle \overset{3}{\chi}_r|(xy^{-1})^{\mathbf{h}_3} L_{1_1 2_1 3} \cdots L_{1_n 2_n 3}|\overset{3}{\chi}_{r'}\rangle \langle \overset{4}{\eta}_k|x^{\mathbf{h}_4} G_{2_1 4} \cdots G_{2_n 4}|\overset{4}{\eta}_{k'}\rangle \times$$
$$\times \langle \overset{5}{\chi}_r|(xy)^{\mathbf{h}_5} L_{2_1 1_1 5} \cdots L_{2_n 1_n 5}|\overset{5}{\chi}_{r'}\rangle \langle \overset{6}{\eta}_k|y^{\mathbf{h}_6} G_{1_1 6} \cdots G_{1_n 6}|\overset{6}{\eta}_{k'}\rangle$$
$$= \langle \overset{6}{\eta}_k|y^{\mathbf{h}_6} G_{1_1 6} \cdots G_{1_n 6}|\overset{6}{\eta}_{k'}\rangle \langle \overset{5}{\chi}_r|(xy)^{\mathbf{h}_5} L_{1_1 2_1 5} \cdots L_{1_n 2_n 5}|\overset{5}{\chi}_{r'}\rangle \times$$
$$\times \langle \overset{4}{\eta}_k|x^{\mathbf{h}_4} G_{2_1 4} \cdots G_{2_n 4}|\overset{4}{\eta}_{k'}\rangle \langle \overset{3}{\chi}_r|(xy^{-1})^{\mathbf{h}_3} L_{2_1 1_1 3} \cdots L_{2_n 1_n 3}|\overset{3}{\chi}_{r'}\rangle. \tag{16.8}$$

Up to scalar multiples, the factors $\langle \chi_r|(\cdots)|\chi_{r'}\rangle$ involving L yield $S^{r,r'}(z)|_{q\to q^2}$ in (12.6). In the identification one uses (16.2) and $\langle \chi_r| = \langle \eta_r||_{q\to q^2}$ and $|\chi_r\rangle = |\eta_r\rangle|_{q\to q^2}$ in (16.6). Since they appear frequently, we adopt the convention:

$$S^{r,r'}(z) \text{ in this chapter } = (S^{r,r'}(z) \text{ in (12.8)–(12.9)})_{q\to q^2}. \tag{16.9}$$

By Theorem $12.2|_{q\to q^2}$, we know that $S^{r,r'}(z)$ is the quantum R matrix for the spin representation of $U_p(\mathfrak{g}^{r,r'})$ at $p = -q^{-2}$.

Returning to (16.8), the other factors emerging from G have the form

$$K_{\mathbf{1}}^{k,k'}(z) = \kappa^{k,k'}(z)\langle \overset{6}{\eta}_k|z^{\mathbf{h}_6} G_{1_1 6} \cdots G_{1_n 6}|\overset{6}{\eta}_{k'}\rangle \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}}), \tag{16.10}$$
$$K_{\mathbf{2}}^{k,k'}(z) = \kappa^{k,k'}(z)\langle \overset{4}{\eta}_k|z^{\mathbf{h}_4} G_{2_1 4} \cdots G_{2_n 4}|\overset{4}{\eta}_{k'}\rangle \in \mathrm{End}(\overset{\mathbf{2}}{\mathbf{V}}), \tag{16.11}$$

where $k, k' = 1, 2$. The scalar $\kappa^{k,k'}(z)$ will be specified in (16.17). They are the same linear operators (16.13) acting on the different copies of $V^{\otimes n}$ given as $\overset{\mathbf{1}}{\mathbf{V}} = \overset{1_1}{V} \otimes \cdots \otimes \overset{1_n}{V}$ and $\overset{\mathbf{2}}{\mathbf{V}} = \overset{2_1}{V} \otimes \cdots \otimes \overset{2_n}{V}$.

In terms of (16.10)–(16.11) and $(12.6)|_{q\to q^2}$, the relation (16.8) is stated as the reflection equation

$$S_{\mathbf{1},\mathbf{2}}^{r,r'}(xy^{-1}) K_{\mathbf{2}}^{k,k'}(x) S_{\mathbf{2},\mathbf{1}}^{r,r'}(xy) K_{\mathbf{1}}^{k,k'}(y) = K_{\mathbf{1}}^{k,k'}(y) S_{\mathbf{1},\mathbf{2}}^{r,r'}(xy) K_{\mathbf{2}}^{k,k'}(x) S_{\mathbf{2},\mathbf{1}}^{r,r'}(xy^{-1}) \tag{16.12}$$

for $1 \le r \le k \le 2$ and $1 \le r' \le k' \le 2$.

The construction (16.10)–(16.11) yields the matrix product formula for each element as

$$K^{k,k'}(z) v_{\mathbf{a}} = \sum_{\mathbf{b}\in\mathfrak{s}} K^{k,k'}(z)_{\mathbf{a}}^{\mathbf{b}} v_{\mathbf{b}},$$
$$K^{k,k'}(z)_{\mathbf{a}}^{\mathbf{b}} = \kappa^{k,k'}(z)\langle \eta_k|z^{\mathbf{h}} G_{a_1}^{b_1} \cdots G_{a_n}^{b_n}|\eta_{k'}\rangle. \tag{16.13}$$

See (11.1)–(11.7) for the notations $\mathfrak{s}$, $v_{\mathbf{a}}$, $\mathbf{V}$ etc. From (16.1), we see that it depends on the parameter β in (16.1) as the conjugation:

$$K^{k,k'}(z) = \beta^{\mathbf{h}_1+\cdots+\mathbf{h}_n}\big(K^{k,k'}(z)|_{\beta=1}\big)\beta^{-\mathbf{h}_1-\cdots-\mathbf{h}_n}. \tag{16.14}$$

From (3.18) and the fact that $\kappa^{k,k'}(z) = \kappa^{k',k}(z)$ in (16.17), it can be shown that

$$K^{k,k'}(z)^{\mathbf{b}}_{\mathbf{a}} = z^{n-|\mathbf{a}|-|\mathbf{b}|} K^{k',k}(z)^{\mathbf{e}_1+\cdots+\mathbf{e}_n-\mathbf{a}^\vee}_{\mathbf{e}_1+\cdots+\mathbf{e}_n-\mathbf{b}^\vee}, \tag{16.15}$$

where $(a_1, \ldots, a_n)^\vee = (a_n, \ldots, a_1)$ is the reverse ordered array as in (11.4). Noting the factor $\theta(j \in 2\mathbb{Z})$ in the last formula in (12.10), one can show

$$K^{2,2}(z)^{\mathbf{b}}_{\mathbf{a}} = 0 \quad \text{unless} \quad |\mathbf{a}| + |\mathbf{b}| \equiv n \mod 2 \tag{16.16}$$

by an argument similar to the one for deriving (15.20). Consequently, the direct sum decomposition

$$K^{2,2}(z) = K^{2,2}_{+}(z) \oplus K^{2,2}_{-}(z), \qquad K^{2,2}_{\sigma}(z) : \mathbf{V}_\sigma \to \mathbf{V}_{\sigma(-1)^n}$$

holds, where $\mathbf{V}_\pm$ was defined in (11.6). As for $K^{k,k'}(z)$ with $(k, k') \neq (2, 2)$, there is no selection rule like (15.20) or (16.16). We choose the scalar $\kappa^{k,k'}(z)$ as

$$\kappa^{k,k'}(z) = q^{-\frac{n}{2}} \frac{((zq^n)^t; q^{kk'})_\infty}{((-q)^s (zq^n)^t; q^{kk'})_\infty}, \qquad s = \min(k, k'), \quad t = \max(k, k'), \tag{16.17}$$

which is the inverse of $q^{\frac{n}{2}} \langle \eta_k | z^{\mathbf{h}} \mathbf{k}^n | \eta_{k'} \rangle$ calculated from (12.10). In this normalization,

$$K^{k,k'}(z) v_{\mathbf{e}_1+\cdots+\mathbf{e}_l} = (-1)^l (q^{-\frac{1}{2}}\beta)^{n-2l} v_{\mathbf{e}_{l+1}+\cdots+\mathbf{e}_n} + \cdots \tag{16.18}$$

for $0 \le l \le n,\ 1 \le k, k' \le 2$ holds, and general elements are rational functions of $\beta, q^{\frac{1}{2}}$ and z.

As seen from (16.13) and also in Example 16.1 below, the K matrix $K^{k,k'}(z)$ is *dense* in the sense that all the elements are non-zero (for $K^{2,2}$ non-zero within each sector implied by (16.16)).

Example 16.1 We present $K^{k,k'}(z)$ with $\beta = q^{\frac{1}{2}}$ for $n = 1, 2$ and $(k, k') = (1, 1)$, $(1, 2)$, $(2, 2)$. The general β case and $(k, k') = (2, 1)$ can be deduced from them by (16.14) and (16.15). We write $v_0 \otimes v_1$ as $|01\rangle$ etc.

For $n = 1$, $K^{k,k'}(z)|_{\beta=q^{\frac{1}{2}}}$ acts on the basis as

$$K^{1,1}(z) : |0\rangle \mapsto -\frac{q^{-\frac{1}{2}}(1+q)z|0\rangle}{-1+z} + |1\rangle, \qquad |1\rangle \mapsto -|0\rangle - \frac{q^{-\frac{1}{2}}(1+q)|1\rangle}{-1+z},$$

$$K^{1,2}(z) : |0\rangle \mapsto -\frac{q^{-\frac{1}{2}}(1+q)z|0\rangle}{-1+z^2} + |1\rangle, \qquad |1\rangle \mapsto -|0\rangle - \frac{q^{-\frac{1}{2}}(1+q)z|1\rangle}{-1+z^2},$$

$$K^{2,2}(z) : |0\rangle \mapsto |1\rangle, \qquad |1\rangle \mapsto -|0\rangle.$$

For $n = 2$, $K^{1,1}(z)|_{\beta=q^{\frac{1}{2}}}$ acts on the basis as

$$|00\rangle \mapsto \frac{q^{-1}(1+q)(1+q^2)z^2|00\rangle}{(-1+z)(-1+qz)} - \frac{q^{-\frac{1}{2}}(1+q)z|01\rangle}{(-1+qz)} - \frac{q^{\frac{1}{2}}(1+q)z|10\rangle}{-1+qz} + |11\rangle,$$

$$|01\rangle \mapsto \frac{q^{-\frac{1}{2}}(1+q)z|00\rangle}{-1+qz} + \frac{q^{-1}(1+q)z(1+q-qz+q^2z)|01\rangle}{(-1+z)(-1+qz)}$$
$$- |10\rangle - \frac{q^{-\frac{1}{2}}(1+q)|11\rangle}{-1+qz},$$

$$|10\rangle \mapsto \frac{q^{\frac{1}{2}}(1+q)z|00\rangle}{-1+qz} - |01\rangle + \frac{q^{-1}(1+q)(1-q+qz+q^2z)|10\rangle}{(-1+z)(-1+qz)}$$
$$- \frac{q^{\frac{1}{2}}(1+q)|11\rangle}{-1+qz},$$

$$|11\rangle \mapsto |00\rangle + \frac{q^{-\frac{1}{2}}(1+q)|01\rangle}{-1+qz} + \frac{q^{\frac{1}{2}}(1+q)|10\rangle}{-1+qz} + \frac{q^{-1}(1+q)(1+q^2)|11\rangle}{(-1+z)(-1+qz)}.$$

$K^{1,2}(z)|_{\beta=q^{\frac{1}{2}}}$ acts on the basis as

$$|00\rangle \mapsto \frac{q^{-1}(1+q)z^2(1+q^2-q^2z^2+q^3z^2)|00\rangle}{(-1+z^2)(-1+q^2z^2)} - \frac{q^{-\frac{1}{2}}(1+q)z|01\rangle}{-1+q^2z^2}$$
$$- \frac{q^{\frac{1}{2}}(1+q)z|10\rangle}{-1+q^2z^2} + |11\rangle,$$

$$|01\rangle \mapsto \frac{q^{-\frac{1}{2}}(1+q)z|00\rangle}{-1+q^2z^2} + \frac{q^{-1}(1+q)z^2(1+q^2-q^2z^2+q^3z^2)|01\rangle}{(-1+z^2)(-1+q^2z^2)}$$
$$- |10\rangle - \frac{q^{\frac{1}{2}}(1+q)z|11\rangle}{-1+q^2z^2},$$

$$|10\rangle \mapsto \frac{q^{\frac{1}{2}}(1+q)z|00\rangle}{-1+q^2z^2} - |01\rangle + \frac{q^{-1}(1+q)(1-q+qz^2+q^3z^2)|10\rangle}{(-1+z^2)(-1+q^2z^2)}$$
$$- \frac{q^{\frac{3}{2}}(1+q)z|11\rangle}{-1+q^2z^2},$$

$$|11\rangle \mapsto |00\rangle + \frac{q^{\frac{1}{2}}(1+q)z|01\rangle}{-1+q^2z^2} + \frac{q^{\frac{3}{2}}(1+q)z|10\rangle}{-1+q^2z^2}$$
$$+ \frac{q^{-1}(1+q)(1-q+qz^2+q^3z^2)|11\rangle}{(-1+z^2)(-1+q^2z^2)}.$$

$K^{2,2}(z)|_{\beta=q^{\frac{1}{2}}}$ acts on the basis as

$$|00\rangle \mapsto \frac{q^{-1}(-1+q^2)z^2|00\rangle}{-1+z^2} + |11\rangle, \qquad |01\rangle \mapsto \frac{q^{-1}(-1+q^2)z^2|01\rangle}{-1+z^2} - |10\rangle,$$
$$|10\rangle \mapsto -|01\rangle + \frac{q^{-1}(-1+q^2)|10\rangle}{-1+z^2}, \qquad |11\rangle \mapsto |00\rangle + \frac{q^{-1}(-1+q^2)|11\rangle}{-1+z^2}.$$

16.3 Characterization as the Intertwiner of the Onsager Coideal

We keep the definitions of the quantum affine algebras $U_p(\mathfrak{g}^{r,r'})$ $(r, r' = 1, 2)$ in Sect. 12.2, where

$$\mathfrak{g}^{1,1} = D^{(2)}_{n+1}, \quad \mathfrak{g}^{2,1} = B^{(1)}_n, \quad \mathfrak{g}^{1,2} = \tilde{B}^{(1)}_n, \quad \mathfrak{g}^{2,2} = D^{(1)}_n \tag{16.19}$$

as in (12.20). We use the spin representation $\pi_{\varpi_n,x} : U_p(\mathfrak{g}^{r,r'}) \to \mathrm{End}(\mathbf{V})$ in (12.23)–(12.27), which we quote here for convenience:

$$e_0 v_{\mathbf{m}} = x v_{\mathbf{m}-\mathbf{e}_1}, \qquad f_0 v_{\mathbf{m}} = x^{-1} v_{\mathbf{m}+\mathbf{e}_1}, \quad k_0 v_{\mathbf{m}} = p^{\frac{1}{2}-m_1} v_{\mathbf{m}} \qquad (r = 1), \tag{16.20}$$

$$e_0 v_{\mathbf{m}} = x^2 v_{\mathbf{m}-\mathbf{e}_1-\mathbf{e}_2}, \quad f_0 v_{\mathbf{m}} = x^{-2} v_{\mathbf{m}+\mathbf{e}_1+\mathbf{e}_2}, \quad k_0 v_{\mathbf{m}} = p^{1-m_1-m_2} v_{\mathbf{m}} \quad (r = 2), \tag{16.21}$$

$$e_i v_{\mathbf{m}} = v_{\mathbf{m}+\mathbf{e}_i-\mathbf{e}_{i+1}}, \quad f_i v_{\mathbf{m}} = v_{\mathbf{m}-\mathbf{e}_i+\mathbf{e}_{i+1}}, \quad k_i v_{\mathbf{m}} = p^{m_i-m_{i+1}} v_{\mathbf{m}} \quad (0 < i < n), \tag{16.22}$$

$$e_n v_{\mathbf{m}} = v_{\mathbf{m}+\mathbf{e}_n}, \qquad f_n v_{\mathbf{m}} = v_{\mathbf{m}-\mathbf{e}_n}, \qquad k_n v_{\mathbf{m}} = p^{m_n-\frac{1}{2}} v_{\mathbf{m}} \qquad (r' = 1), \tag{16.23}$$

$$e_n v_{\mathbf{m}} = v_{\mathbf{m}+\mathbf{e}_{n-1}+\mathbf{e}_n}, \quad f_n v_{\mathbf{m}} = v_{\mathbf{m}-\mathbf{e}_{n-1}-\mathbf{e}_n}, \quad k_n v_{\mathbf{m}} = p^{m_n+m_{n-1}-1} v_{\mathbf{m}} \quad (r' = 2), \tag{16.24}$$

where $\mathbf{m} \in \mathfrak{s}$. As mentioned before, it is irreducible except for $\mathfrak{g}^{2,2} = D^{(1)}_n$, where $\mathbf{V} = \mathbf{V}_+ \oplus \mathbf{V}_-$ as defined in (11.6) corresponding to the two kinds of spin representations.

According to the remark after (16.9), we will be concerned with $U_p(\mathfrak{g}^{r,r'})$ with $p = -q^{-2}$. In the rest of the chapter we set

$$p^{\frac{1}{2}} = -\mathrm{i}\epsilon q^{-1}, \qquad \epsilon = \pm 1 \tag{16.25}$$

and allow the coexistence of the letters p, q and ϵ.

16.3.1 Generalized p-Onsager Algebra $O_p(\mathfrak{g}^{r,r'})$

For each $\mathfrak{g}^{r,r'}$ in (16.19) we consider the quantum affine algebra $U_p(\mathfrak{g}^{r,r'})$ (12.20) and the Onsager algebra $O_p(\mathfrak{g}^{r,r'})$.

For comparison we write down the p-Serre relations in $U_p(\mathfrak{g}^{r,r'})$ which were not displayed together with (12.21):

$$e_i e_j - e_j e_i = 0 \quad (a_{ij} = 0), \tag{16.26}$$

$$e_i^2 e_j - (p + p^{-1}) e_i e_j e_i + e_j e_i^2 = 0 \quad (a_{ij} = -1), \tag{16.27}$$

$$e_i^3 e_j - (p + 1 + p^{-1}) e_i^2 e_j e_i + (p + 1 + p^{-1}) e_i e_j e_i^2 - e_j e_i^3 = 0 \quad (a_{ij} = -2). \tag{16.28}$$

The same relations are imposed also for f_j's. The data $(a_{ij})_{0 \le i,j \le n}$ is the Cartan matrix of the affine Lie algebra $\mathfrak{g}^{r,r'}$. The Onsager algebra $O_p(\mathfrak{g}^{r,r'})$ is generated by $\mathfrak{b}_0, \ldots, \mathfrak{b}_n$ obeying the modified p-Serre relations:

$$\mathfrak{b}_i \mathfrak{b}_j - \mathfrak{b}_j \mathfrak{b}_i = 0 \quad (a_{ij} = 0), \tag{16.29}$$

$$\mathfrak{b}_i^2 \mathfrak{b}_j - (p + p^{-1}) \mathfrak{b}_i \mathfrak{b}_j \mathfrak{b}_i + \mathfrak{b}_j \mathfrak{b}_i^2 = \mathfrak{b}_j \quad (a_{ij} = -1), \tag{16.30}$$

$$\begin{aligned} &\mathfrak{b}_i^3 \mathfrak{b}_j - (p + 1 + p^{-1}) \mathfrak{b}_i^2 \mathfrak{b}_j \mathfrak{b}_i + (p + 1 + p^{-1}) \mathfrak{b}_i \mathfrak{b}_j \mathfrak{b}_i^2 - \mathfrak{b}_j \mathfrak{b}_i^3 \\ &= (p^{\frac{1}{2}} + p^{-\frac{1}{2}})^2 (\mathfrak{b}_i \mathfrak{b}_j - \mathfrak{b}_j \mathfrak{b}_i) \quad (a_{ij} = -2). \end{aligned} \tag{16.31}$$

Except for (16.31) which are void for the simply-laced case $\mathfrak{g}^{2,2} = D_n^{(1)}$, these relations are formally the same with (15.25) for $O_p(A_{n-1}^{(1)})$.

In terms of commutators $[X, Y] = [X, Y]_1$, $[X, Y]_r = XY - rYX$, the relations (16.29)–(16.31) are written more compactly as

$$[\mathfrak{b}_i, \mathfrak{b}_j] = 0 \quad (a_{ij} = 0), \tag{16.32}$$

$$[\mathfrak{b}_i, [\mathfrak{b}_i, \mathfrak{b}_j]_p]_{p^{-1}} = \mathfrak{b}_j \quad (a_{ij} = -1), \tag{16.33}$$

$$[\mathfrak{b}_i, [\mathfrak{b}_i, [\mathfrak{b}_i, \mathfrak{b}_j]_p]_{p^{-1}}] = (p^{\frac{1}{2}} + p^{-\frac{1}{2}})^2 [\mathfrak{b}_i, \mathfrak{b}_j] \quad (a_{ij} = -2). \tag{16.34}$$

There is an embedding $O_p(\mathfrak{g}^{r,r'}) \hookrightarrow U_p(\mathfrak{g}^{r,r'})$, depending on integer indices k, k' satisfying $1 \le r \le k \le 2$ and $1 \le r' \le k' \le 2$, given by

$$\mathfrak{b}_0 \mapsto g_0 := e_0 + p^{r/2} k_0 f_0 + d^r_k k_0, \tag{16.35}$$

$$\mathfrak{b}_i \mapsto g_i := e_i + p k_i f_i + \frac{1}{q+q^{-1}} k_i \quad (0 < i < n), \tag{16.36}$$

$$\mathfrak{b}_n \mapsto g_n := e_n + p^{r'/2} k_n f_n + d^{r'}_{k'} k_n, \tag{16.37}$$

$$d^1_1 = \epsilon \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{q+q^{-1}}, \quad d^1_2 = 0, \quad d^2_2 = \frac{1}{q+q^{-1}}, \tag{16.38}$$

where $\epsilon = \pm 1$ has been introduced in (16.25). Define

$$\mathcal{B}^{r,r'}_{k,k'} = \text{ the subalgebra of } U_p(\mathfrak{g}^{r,r'}) \text{ generated by } g_0, \dots, g_n \text{ in (16.35)–(16.37).} \tag{16.39}$$

By the remark on (15.28), it becomes a left coideal; $\Delta \mathcal{B}^{r,r'}_{k,k'} \subset U_p(\mathfrak{g}^{r,r'}) \otimes \mathcal{B}^{r,r'}_{k,k'}$. Henceforth $\mathcal{B}^{r,r'}_{k,k'}$ will be referred to as an Onsager coideal.

16.3.2 $K^{k,k'}(z)$ as the Intertwiner of Onsager Coideal

Recall that $\pi_{\varpi_n,x} : U_p(\mathfrak{g}^{r,r'}) \to \mathrm{End}(\mathbf{V})$ denotes the spin representation in (16.20)–(16.24).

Theorem 16.2 *The K matrix (16.13) with $\beta = \mathrm{i}q^{\frac{1}{2}}$ is characterized, up to normalization, as the intertwiner of the Onsager coideal $\mathcal{B}^{r,r'}_{k,k'} \subset U_p(\mathfrak{g}^{r,r'})$ at $p^{\frac{1}{2}} = -\mathrm{i}\epsilon q^{-1}$ (16.25) as*

$$K^{k,k'}(z) \pi_{\varpi_n, z^{-1}}(g) = \pi_{\varpi_n,z}(g) K^{k,k'}(z) \qquad (\forall g \in \mathcal{B}^{r,r'}_{k,k'}), \tag{16.40}$$

where $1 \le r \le k$ and $1 \le r' \le k'$.

Proof We focus on the existence referring to [104, Sect. 5.2] for the uniqueness. There are seven cases in (16.40) to verify:

(i) $g = g_i \ (0 < i < n)$,
(ii) $g = g_0,\ (r, k) = (1, 2)$,
(iii) $g = g_0,\ (r, k) = (1, 1)$,
(iv) $g = g_0,\ (r, k) = (2, 2)$,
(v) $g = g_n,\ (r', k') = (1, 2)$,
(vi) $g = g_n,\ (r', k') = (1, 1)$,
(vii) $g = g_n,\ (r', k') = (2, 2)$.

Thanks to (3.18), the cases (v), (vi) and (vii) are attributed to (ii), (iii), and (iv) at $z = 1$, respectively. Thus we consider (i)–(iv) below. The case (i) reduces to the already shown identity (15.36).

(ii) From (16.35) and (16.38), the Eq. (16.40) reads as

$$K^{2,k'}(z) \pi_{\varpi_n,z^{-1}}(e_0 + p^{\frac{1}{2}} k_0 f_0) = \pi_{\varpi_n,z}(e_0 + p^{\frac{1}{2}} k_0 f_0) K^{2,k'}(z). \tag{16.41}$$

From (16.20), this is translated to the relation of the coefficients for the transition $v_{\mathbf{a}} \to v_{\mathbf{b}}$ ($\mathbf{a}, \mathbf{b} \in \mathfrak{s}$ in (11.1)) as

$$z^{-1}K^{2,k'}(z)^{\mathbf{b}}_{\mathbf{a}-\mathbf{e}_1} + zp^{-a_1}K^{2,k'}(z)^{\mathbf{b}}_{\mathbf{a}+\mathbf{e}_1} = zK^{2,k'}(z)^{\mathbf{b}+\mathbf{e}_1}_{\mathbf{a}} + z^{-1}p^{1-b_1}K^{2,k'}(z)^{\mathbf{b}-\mathbf{e}_1}_{\mathbf{a}}. \tag{16.42}$$

One can drop the factors p^{-a_1} and p^{1-b_1} since the attached terms are non-vanishing only for $\mathbf{a}+\mathbf{e}_1, \mathbf{b}-\mathbf{e}_1 \in \mathfrak{s}$ compelling $a_1 = 0$ and $b_1 = 1$. Then, in view of the matrix product formula (16.13), the relation in question follows from

$$\langle \eta_2 | z^{\mathbf{h}}(z^{-1}G^b_{a-1} + zG^b_{a+1}) = \langle \eta_2 | z^{\mathbf{h}}(zG^{b+1}_a + z^{-1}G^{b-1}_a) \tag{16.43}$$

for $a, b = 0, 1$. From (15.6) this is further reduced to the z-independent relation

$$\langle \eta_2 | (G^b_{a-1} + G^b_{a+1}) = \langle \eta_2 | (G^{b+1}_a + G^{b-1}_a). \tag{16.44}$$

It contains two non-trivial cases

$$0 = \langle \eta_2 | (G^0_0 - G^1_1) = \langle \eta_2 | (\mathbf{a}^+ - \mathbf{a}^-), \tag{16.45}$$

$$0 = \langle \eta_2 | (G^1_0 - G^0_1) = \langle \eta_2 | (\beta + q\beta^{-1})\mathbf{k}, \tag{16.46}$$

where (16.1) is substituted. The first equality holds due to (3.141) and the second does from the assumption $\beta = \mathrm{i}q^{\frac{1}{2}}$ of the theorem.

(iii) By an argument parallel with (ii), the proof reduces to showing

$$\begin{aligned} &z^{-1}K^{1,k'}(z)^{\mathbf{b}}_{\mathbf{a}-\mathbf{e}_1} + zp^{-a_1}K^{1,k'}(z)^{\mathbf{b}}_{\mathbf{a}+\mathbf{e}_1} + d^1_1 p^{\frac{1}{2}-a_1}K^{1,k'}(z)^{\mathbf{b}}_{\mathbf{a}} \\ &= zK^{1,k'}(z)^{\mathbf{b}+\mathbf{e}_1}_{\mathbf{a}} + z^{-1}p^{1-b_1}K^{1,k'}(z)^{\mathbf{b}-\mathbf{e}_1}_{\mathbf{a}} + d^1_1 p^{\frac{1}{2}-b_1}K^{1,k'}(z)^{\mathbf{b}}_{\mathbf{a}}. \end{aligned} \tag{16.47}$$

The matrix product formula (16.13) and (15.6) attribute it to

$$\langle \eta_1 | (G^b_{a-1} + G^b_{a+1} + d^1_1 p^{\frac{1}{2}-a}G^b_a) = \langle \eta_1 | (G^{b+1}_a + G^{b-1}_a + d^1_1 p^{\frac{1}{2}-b}G^b_a) \tag{16.48}$$

for $a, b = 0, 1$, where d^1_1 is specified in (16.38). This can be checked case by case by using $\beta = \mathrm{i}q^{\frac{1}{2}}$, (16.25) and the property of $\langle \eta_1 |$ given in (3.138) and (3.139).

(iv) By a parallel argument with respect to the representation (16.21), the proof reduces to showing

$$\begin{aligned} &z^{-2}K^{2,k'}(z)^{\mathbf{b}}_{\mathbf{a}-\mathbf{e}_1-\mathbf{e}_2} + z^2p^{-a_1-a_2}K^{2,k'}(z)^{\mathbf{b}}_{\mathbf{a}+\mathbf{e}_1+\mathbf{e}_2} + \frac{p^{1-a_1-a_2}}{q+q^{-1}}K^{2,k'}(z)^{\mathbf{b}}_{\mathbf{a}} \\ &= z^2K^{2,k'}(z)^{\mathbf{b}+\mathbf{e}_1+\mathbf{e}_2}_{\mathbf{a}} + z^{-2}p^{2-b_1-b_2}K^{2,k'}(z)^{\mathbf{b}-\mathbf{e}_1-\mathbf{e}_2}_{\mathbf{a}} + \frac{p^{1-b_1-b_2}}{q+q^{-1}}K^{2,k'}(z)^{\mathbf{b}}_{\mathbf{a}}. \end{aligned} \tag{16.49}$$

One may trivialize the coefficients of the middle terms as $p^{-a_1-a_2} = p^{2-b_1-b_2} = 1$ for the non-zero contributions. The matrix product formula (16.13) and (15.6) attribute the resulting relation to

$$\begin{aligned}&\langle\eta_2|(G^b_{a-1}G^{b'}_{a'-1} + G^b_{a+1}G^{b'}_{a'+1} + \frac{p^{1-a-a'}}{q+q^{-1}}G^b_aG^{b'}_{a'})\\ &= \langle\eta_2|(G^{b+1}_aG^{b'+1}_{a'} + G^{b-1}_aG^{b'-1}_{a'} + \frac{p^{1-b-b'}}{q+q^{-1}}G^b_aG^{b'}_{a'})\end{aligned} \tag{16.50}$$

for $a, a', b, b' = 0, 1$. We have set $(a, a', b, b') = (a_1, a_2, b_1, b_2)$. This can be checked similarly by using $\beta = \mathrm{i}q^{\frac{1}{2}}$, (16.25) and the property of $\langle\eta_2|$ in (3.141). In particular it involves a maneuver like $\langle\eta_2|(\mathbf{a}^+)^2 = \langle\eta_2|\mathbf{a}^-\mathbf{a}^+ = \langle\eta_2|(1 - q^2\mathbf{k}^2)$, etc. □

One can give an alternative derivation of the reflection equation (16.12) based on the Onsager coideal $\mathcal{B}^{r,r'}_{k,k'}$ by an argument parallel with Sect. 15.4.3.

Let us summarize the solutions to the reflection equation obtained by the 3D approach in Chaps. 15 and 16. There are nine cases in (16.12), where the conditions $1 \le r \le k \le 2$ and $1 \le r' \le k' \le 2$ originate in Proposition 5.21.

Table 16.1 The quantum affine algebra $U_p(\mathfrak{g})$ with $\mathfrak{g} = A^{(1)}_{n-1}$ and $\mathfrak{g}^{r,r'}$ (16.19), the associated R matrices $S^{\mathrm{tr}}(z)$ and $S^{r,r'}(z)$, the associated K matrices $K^{\mathrm{tr}}(z)$ and $K^{k,k'}(z)$. There are a few choices of $K^{k,k'}(z)$ that can be paired with $S^{r,r'}(z)$ to jointly constitute a solution to the reflection equation depending on (r, r')

$\mathfrak{g}$	R matrix	K matrix
$A^{(1)}_{n-1}$	$S^{\mathrm{tr}}(z)$	$K^{\mathrm{tr}}(z)$
$D^{(2)}_{n+1}$	$S^{1,1}(z)$	$K^{1,1}(z)$, $K^{1,2}(z)$, $K^{2,1}(z)$, $K^{2,2}(z)$
$B^{(1)}_n$	$S^{2,1}(z)$	$K^{2,1}(z)$, $K^{2,2}(z)$
$\tilde{B}^{(1)}_n$	$S^{1,2}(z)$	$K^{1,2}(z)$, $K^{2,2}(z)$
$D^{(1)}_n$	$S^{2,2}(z)$	$K^{2,2}(z)$

16.4 Bibliographical Notes and Comments

The boundary vector reduction of the quantized reflection equation was introduced in [105], where the property (16.7) of the boundary vector remained as a conjecture. The first proof of the reflection equation (16.12) was done independently in the quantum group framework based on the Onsager coideal $\mathcal{B}^{r,r'}_{k,k'}$ and the argument like Sect. 15.4.3 [104]. Later the property (16.7) was proved in [106, Appendix B], which completed the 3D approach. Its detail has been reproduced in Proposition 5.21 of this book.

In the 3D approach to the reflection equation, either by the trace reduction (Chap. 15) or the boundary vector reduction (this chapter), the 3D K disappears at an early stage. In fact "reduction" more or less means eliminating it to return to 2D from 3D. However, the 3D K essentially controls the construction behind the scene in the sense that it guides precisely how the local operators L and G should be combined, how the spectral parameters should be arranged and what kind of boundary vectors are acceptable.

Concerning the generalized Onsager algebras, the quartic relation of the form (16.34) with $p^2 = 1$ is often referred to as the Dolan–Grady condition [41]. It is typical for the situation $a_{ij} = -2$, which was utilized to reformulate the original Onsager algebra for $A_1^{(1)}$ [122] by only a few generators. The Onsager algebra $O_p(D_n^{(1)})$ with $p = 1$ was introduced in [34]. It is an interesting open question if there is an analogue of Remark 15.2 for $\mathfrak{g} \neq A_{n-1}^{(1)}$ related to a boundary extension of the Temperley–Lieb algebra like [35].

Generalized Onsager algebras $O_p(\mathfrak{g}^{r,r'})$ have a natural classical part without the generator $\mathfrak{b}_0$. The commutativity in Theorem 16.2 interchanging z and z^{-1} implies the usual commutativity with the classical part. The corresponding spectral decomposition of $K^{r,r'}(z)$ has been described in [106, Sec.10,11].