Chapter 13
Trace Reductions of $RRRR = RRRR$

Abstract Like $RLLL = LLLR$, the tetrahedron equation $RRRR = RRRR$ admits various reductions to the Yang–Baxter equation leading to several families of solutions in matrix product forms. In this chapter we focus on the trace reduction as done for $RLLL = LLLR$ in Chap. 11. We identify the solutions with quantum R matrices of $U_q(A^{(1)}_{n-1})$, present their explicit formulas, construct commuting layer transfer matrices, and demonstrate that the birational versions reproduce the distinguished example of set-theoretical solutions to the Yang–Baxter equation known as geometric R.

13.1 Preliminaries

Let $n \ge 2$ be an integer. We retain the notations for the sets $B^{(n)} = (\mathbb{Z}_{\ge 0})^n$, $B^{(n)}_k$, the vector spaces $\mathbf{W}^{(n)} = \mathcal{F}_q^{\otimes n}$ and $\mathbf{W}^{(n)}_k$ having bases $|\mathbf{a}\rangle$ labeled with n-arrays $\mathbf{a} = (a_1, \dots, a_n)$ in (11.8)–(11.13). We will also use $|\mathbf{a}| = a_1 + \cdots + a_n$, $\mathbf{a}^\vee = (a_n, \dots, a_1)$ in (11.4) and the elementary vector $\mathbf{e}_i$ in (11.1). As for the q-oscillator algebra Osc_q and the Fock space $\mathcal{F}_q$, see Sect. 3.2. Except in Sect. 13.8, n is fixed, hence the superscript "(n)" will be suppressed.

In Chap. 3, we have introduced a linear operator $R_{123} \in \mathrm{End}(\overset{1}{\mathcal{F}}_q \otimes \overset{2}{\mathcal{F}}_q \otimes \overset{3}{\mathcal{F}}_q)$ which we called a 3D R.

In Theorem 3.20 it was shown to satisfy the tetrahedron equation

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124}, \qquad (13.1)$$

which is an equality in $\mathrm{End}(\overset{1}{\mathcal{F}}_q \otimes \cdots \otimes \overset{6}{\mathcal{F}}_q)$.

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13.2 Trace Reduction Over the Third Component of *R*

The following procedure is quite parallel with that in Sect. 11.2. Consider n copies of (13.1) in which the spaces labeled with $1, 2, 3$ are replaced by $1_i, 2_i, 3_i$ with $i = 1, 2, \ldots, n$:

$$(R_{1_i 2_i 4} R_{1_i 3_i 5} R_{2_i 3_i 6})\, R_{456} = R_{456}\, (R_{2_i 3_i 6} R_{1_i 3_i 5} R_{1_i 2_i 4}).$$

Sending R_{456} to the left by applying this relation repeatedly, we get

$$\begin{aligned}&(R_{1_1 2_1 4} R_{1_1 3_1 5} R_{2_1 3_1 6}) \cdots (R_{1_n 2_n 4} R_{1_n 3_n 5} R_{2_n 3_n 6})\, R_{456} \\ &\quad = R_{456}\, (R_{2_1 3_1 6} R_{1_1 3_1 5} R_{1_1 2_1 4}) \cdots (R_{2_n 3_n 6} R_{1_n 3_n 5} R_{1_n 2_n 4}).\end{aligned} \tag{13.2}$$

One can rearrange this without changing the order of operators sharing common labels, hence by using the trivial commutativity, as

$$\begin{aligned}&(R_{1_1 2_1 4} \cdots R_{1_n 2_n 4})(R_{1_1 3_1 5} \cdots R_{1_n 3_n 5})(R_{2_1 3_1 6} \cdots R_{2_n 3_n 6}) R_{456} \\ &\quad = R_{456} (R_{2_1 3_1 6} \cdots R_{2_n 3_n 6})(R_{1_1 3_1 5} \cdots R_{1_n 3_n 5})(R_{1_1 2_1 4} \cdots R_{1_n 2_n 4}).\end{aligned} \tag{13.3}$$

The weight conservation (3.49) of the 3D R may be stated as

$$R_{456}\, x^{\mathbf{h}_4} (xy)^{\mathbf{h}_5} y^{\mathbf{h}_6} = x^{\mathbf{h}_4} (xy)^{\mathbf{h}_5} y^{\mathbf{h}_6} R_{456} \tag{13.4}$$

for arbitrary parameters x and y. See (3.14) for the definition of $\mathbf{h}$. Multiplying this by (13.3) from the left and applying $R^2 = 1$ from (3.60), we get

$$\begin{aligned}&R_{456}\, x^{\mathbf{h}_4} (R_{1_1 2_1 4} \cdots R_{1_n 2_n 4})(xy)^{\mathbf{h}_5} (R_{1_1 3_1 5} \cdots R_{1_n 3_n 5}) y^{\mathbf{h}_6} (R_{2_1 3_1 6} \cdots R_{2_n 3_n 6}) R_{456} \\ &\quad = y^{\mathbf{h}_6} (R_{2_1 3_1 6} \cdots R_{2_n 3_n 6})(xy)^{\mathbf{h}_5} (R_{1_1 3_1 5} \cdots R_{1_n 3_n 5}) x^{\mathbf{h}_4} (R_{1_1 2_1 4} \cdots R_{1_n 2_n 4}).\end{aligned} \tag{13.5}$$

This relation will also be utilized in the boundary vector reduction in Chap. 14 (Fig. 13.2).

Take the trace of (13.5) over $\overset{4}{\mathcal{F}}_q \otimes \overset{5}{\mathcal{F}}_q \otimes \overset{6}{\mathcal{F}}_q$ using the cyclicity of trace and $R^2 = 1$. The result reads as

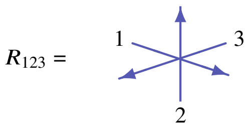

Fig. 13.1 A graphical representation of the 3D R, where 1, 2, 3 are labels of the blue arrows. Each on them carries a q-oscillator Fock space $\mathcal{F}_q$

$$\begin{aligned}&\mathrm{Tr}_4\big(x^{\mathbf{h}_4}R_{1_1 2_1 4}\cdots R_{1_n 2_n 4}\big)\mathrm{Tr}_5\big((xy)^{\mathbf{h}_5}R_{1_1 3_1 5}\cdots R_{1_n 3_n 5}\big)\mathrm{Tr}_6\big(y^{\mathbf{h}_6}R_{2_1 3_1 6}\cdots R_{2_n 3_n 6}\big)\\&=\mathrm{Tr}_6\big(y^{\mathbf{h}_6}R_{2_1 3_1 6}\cdots R_{2_n 3_n 6}\big)\mathrm{Tr}_5\big((xy)^{\mathbf{h}_5}R_{1_1 3_1 5}\cdots R_{1_n 3_n 5}\big)\mathrm{Tr}_4\big(x^{\mathbf{h}_4}R_{1_1 2_1 4}\cdots R_{1_n 2_n 4}\big).\end{aligned} \tag{13.6}$$

Let us denote the operators appearing here by

$$\begin{aligned}R^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{2}}(z)&=\mathrm{Tr}_4(z^{\mathbf{h}_4}R_{1_1 2_1 4}\cdots R_{1_n 2_n 4})\in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{W}}\otimes\overset{\mathbf{2}}{\mathbf{W}}),\\R^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{3}}(z)&=\mathrm{Tr}_5(z^{\mathbf{h}_5}R_{1_1 3_1 5}\cdots R_{1_n 3_n 5})\in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{W}}\otimes\overset{\mathbf{3}}{\mathbf{W}}),\\R^{\mathrm{tr}_3}_{\mathbf{2},\mathbf{3}}(z)&=\mathrm{Tr}_6(z^{\mathbf{h}_6}R_{2_1 3_1 6}\cdots R_{2_n 3_n 6})\in \mathrm{End}(\overset{\mathbf{2}}{\mathbf{W}}\otimes\overset{\mathbf{3}}{\mathbf{W}}).\end{aligned} \tag{13.7}$$

The superscript tr_3 indicates that the trace is taken over the *3rd* (rightmost) component of R, whereas Tr_j in RHSs signifies the *label* j of a space. A similar convention will be employed in the subsequent sections.

Those appearing in (13.7) are the same operators acting on different copies of $\mathbf{W}$ specified as $\overset{\mathbf{1}}{\mathbf{W}}=\overset{1_1}{\mathcal{F}_q}\otimes\cdots\otimes\overset{1_n}{\mathcal{F}_q}$, $\overset{\mathbf{2}}{\mathbf{W}}=\overset{2_1}{\mathcal{F}_q}\otimes\cdots\otimes\overset{2_n}{\mathcal{F}_q}$ and $\overset{\mathbf{3}}{\mathbf{W}}=\overset{3_1}{\mathcal{F}_q}\otimes\cdots\otimes\overset{3_n}{\mathcal{F}_q}$. Now the relation (13.6) is stated as the Yang–Baxter equation:

$$R^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{2}}(x)R^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{3}}(xy)R^{\mathrm{tr}_3}_{\mathbf{2},\mathbf{3}}(y)=R^{\mathrm{tr}_3}_{\mathbf{2},\mathbf{3}}(y)R^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{3}}(xy)R^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{2}}(x). \tag{13.8}$$

Suppressing the labels **1**, **2** etc., we set

$$R^{\mathrm{tr}_3}(z)(|\mathbf{i}\rangle\otimes|\mathbf{j}\rangle)=\sum_{\mathbf{a},\mathbf{b}\in B}R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}}|\mathbf{a}\rangle\otimes|\mathbf{b}\rangle. \tag{13.9}$$

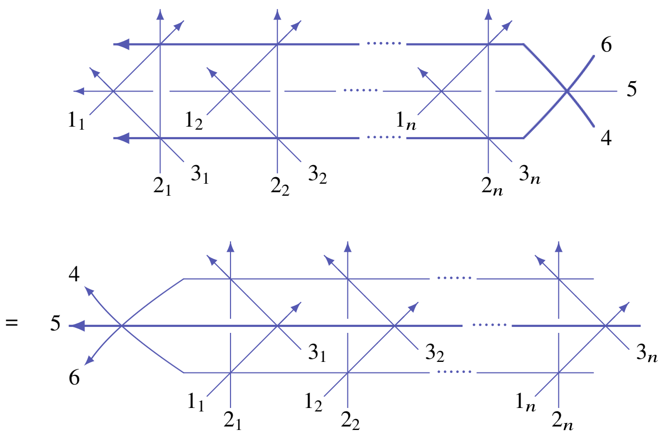

Fig. 13.2 A graphical representation of (13.2) and (13.3). It is a concatenation of Fig. 2.1 which corresponds to the basic $RRRR = RRRR$ relation. Each blue arrow carries $\mathcal{F}_q$

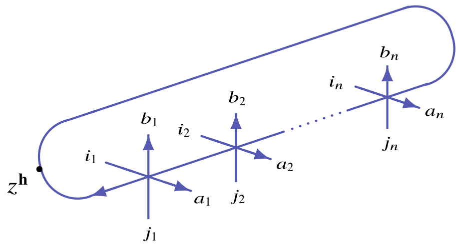

Fig. 13.3 Matrix product construction by the trace reduction (13.10) is depicted as a concatenation of Fig. 13.1 along the blue arrow corresponding to the third component of R. It is closed cyclically reflecting the trace

Then the construction (13.7) implies the matrix product formula

$$R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \mathrm{Tr}\big(z^{\mathbf{h}} R^{a_1 b_1}_{i_1 j_1} \cdots R^{a_n b_n}_{i_n j_n}\big) \tag{13.10}$$

in terms of the operator $R^{ab}_{ij} \in \mathrm{Osc}_q$ introduced in (2.4) and (2.5). In our case of the 3D R, it is explicitly given by (3.69).

By the definition, the trace is given by $\mathrm{Tr}(X) = \sum_{m\ge 0} \frac{\langle m|X|m\rangle}{\langle m|m\rangle} = \sum_{m\ge 0} \frac{\langle m|X|m\rangle}{(q^2)_m}$. See (3.12)–(3.17). Then (13.10) is evaluated by using the commutation relations of q-oscillators (3.12) and the formula (11.27). The matrix product formula (13.10) may also be presented as

$$R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \sum_{c_1,\dots,c_n\ge 0} z^{c_1} R^{a_1 b_1 c_1}_{i_1 j_1 c_2} R^{a_2 b_2 c_2}_{i_2 j_2 c_3} \cdots R^{a_n b_n c_n}_{i_n j_n c_1} \tag{13.11}$$

in terms of the elements R^{abc}_{ijk} of the 3D R in the sense of (3.47). Explicit formulas of R^{abc}_{ijk} are available in Theorems 3.11, 3.18 and (3.84) (Fig. 13.3).

From the weight conservation (3.48), c_β in (13.11) is reducible to c_1 as

$$c_\beta = c_1 + \sum_{1\le\alpha<\beta} (b_\alpha - j_\alpha), \tag{13.12}$$

therefore (13.11) is actually a *single* sum over c_1.

From (3.63), (3.48) and (3.70) it is easy to see

$$R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = 0 \text{ unless } \mathbf{a}+\mathbf{b} = \mathbf{i}+\mathbf{j} \text{ and } |\mathbf{a}| = |\mathbf{i}|,\ |\mathbf{b}| = |\mathbf{j}|, \tag{13.13}$$

$$R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = R^{\mathrm{tr}_3}(z)^{\mathbf{i}^\vee \mathbf{j}^\vee}_{\mathbf{a}^\vee \mathbf{b}^\vee} \prod_{k=1}^{n} \frac{(q^2)_{i_k}(q^2)_{j_k}}{(q^2)_{a_k}(q^2)_{b_k}}, \tag{13.14}$$

$$R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = z^{j_1-b_1} R^{\mathrm{tr}_3}(z)^{\sigma(\mathbf{a})\sigma(\mathbf{b})}_{\sigma(\mathbf{i})\sigma(\mathbf{j})}, \tag{13.15}$$

where $\sigma(\mathbf{a}) = (a_2, \dots, a_n, a_1)$ is a cyclic shift. The property (13.13) implies the decomposition

$$R^{\mathrm{tr}_3}(z) = \bigoplus_{l,m\geq 0} R^{\mathrm{tr}_3}_{l,m}(z), \qquad R^{\mathrm{tr}_3}_{l,m}(z) \in \mathrm{End}(\mathbf{W}_l \otimes \mathbf{W}_m). \tag{13.16}$$

The Yang–Baxter equation (13.8) is valid in each finite-dimensional subspace $\mathbf{W}_k \otimes \mathbf{W}_l \otimes \mathbf{W}_m$ of $\overset{1}{\mathbf{W}} \otimes \overset{2}{\mathbf{W}} \otimes \overset{3}{\mathbf{W}}$. In the current normalization we have

$$R^{\mathrm{tr}_3}_{l,m}(z)(|l\mathbf{e}_k\rangle \otimes |m\mathbf{e}_k\rangle) = \Lambda_{l,m}(z,q)\, |l\mathbf{e}_k\rangle \otimes |m\mathbf{e}_k\rangle \tag{13.17}$$

for any $1 \leq k \leq n$, where the factor $\Lambda_{l,m}(z,q)$ is given by

$$\Lambda_{l,m}(z,q) = \sum_{c\geq 0} z^c R^{lmc}_{lmc} = (-1)^m q^{m(l+1)} \frac{(q^{-l-m}z;q^2)_m}{(q^{l-m}z;q^2)_{m+1}}. \tag{13.18}$$

The second equality is shown by means of the general identity like (13.82). General elements $R^{\mathrm{tr}_3}_{l,m}(z)^{\mathbf{ab}}_{\mathbf{ij}}$ also become rational functions of q and z.

Example 13.1 Substituting the formulas in Example 3.17 into (13.10) and evaluating the trace we get

$$R^{\mathrm{tr}_3}_{m,1}(z)^{\mathbf{a}\,\mathbf{e}_b}_{\mathbf{i}\,\mathbf{e}_j} = \begin{cases} (q^{m-a_j}z - q^{a_j+1})/D & j = b, \\ z(1-q^{2a_b+2})q^{m-a_j-a_{j+1}-\cdots-a_b}/D & j < b, \\ (1-q^{2a_b+2})q^{a_{b+1}+a_{b+2}+\cdots+a_{j-1}}/D & j > b, \end{cases}$$

where $D = (1-q^{m-1}z)(1-q^{m+1}z)$, and $\mathbf{a},\mathbf{i} \in B_m$ and $\mathbf{a}+\mathbf{e}_b = \mathbf{i}+\mathbf{e}_j$ are assumed.

From the remark after (3.71), this should coincide with (11.36) divided by $\varrho^{\mathrm{tr}_3}(z)|_{\alpha=1}$ in (11.33) provided that $\mathbf{a},\mathbf{i} \in \mathfrak{s}_m$[1] and $a_j = i_j = 0$ when $j = b$. This can be checked directly.

13.3 Trace Reduction Over the First Component of *R*

The following procedure is quite parallel with that in Sect. 11.3. Consider n copies of the tetrahedron equation (13.1) in which the spaces 3, 5, 6 are replaced by $3_i, 5_i, 6_i$ with $i = 1,\ldots,n$:

$$R_{45_i6_i}R_{23_i6_i}R_{13_i5_i}R_{124} = R_{124}R_{13_i5_i}R_{23_i6_i}R_{45_i6_i}.$$

Sending R_{124} to the left by applying this repeatedly, we get

[1] See (11.3) for the definition of $\mathfrak{s}_m$.

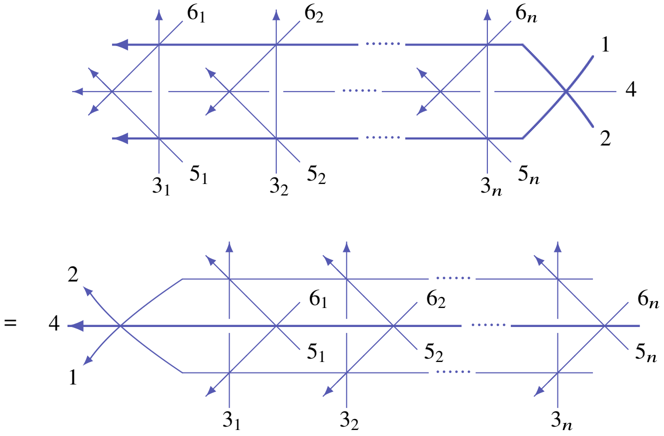

Fig. 13.4 A graphical representation of (13.19) and (13.20)

$$
\begin{aligned}
&(R_{45_16_1}R_{23_16_1}R_{13_15_1})\cdots(R_{45_n6_n}R_{23_n6_n}R_{13_n5_n})R_{124}\\
&\quad = R_{124}(R_{13_15_1}R_{23_16_1}R_{45_16_1})\cdots(R_{13_n5_n}R_{23_n6_n}R_{45_n6_n}),
\end{aligned}
\tag{13.19}
$$

which can be rearranged as (Fig. 13.4)

$$
\begin{aligned}
&(R_{45_16_1}\cdots R_{45_n6_n})(R_{23_16_1}\cdots R_{23_n6_n})(R_{13_15_1}\cdots R_{13_n5_n})R_{124}\\
&\quad = R_{124}(R_{13_15_1}\cdots R_{13_n5_n})(R_{23_16_1}\cdots R_{23_n6_n})(R_{45_16_1}\cdots R_{45_n6_n}).
\end{aligned}
\tag{13.20}
$$

Multiply $x^{\mathbf{h}_1}(xy)^{\mathbf{h}_2}y^{\mathbf{h}_4}R_{124}^{-1}$ from the left by (13.20) and take the trace over $\overset{1}{\mathcal{F}}_q \otimes \overset{2}{\mathcal{F}}_q \otimes \overset{4}{\mathcal{F}}_q$. Using the weight conservation (13.4) we get the Yang–Baxter equation.

$$
R^{\mathrm{tr}_1}_{\mathbf{5},\mathbf{6}}(y)R^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{6}}(xy)R^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{5}}(x) = R^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{5}}(x)R^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{6}}(xy)R^{\mathrm{tr}_1}_{\mathbf{5},\mathbf{6}}(y) \in \mathrm{End}(\overset{\mathbf{3}}{\mathbf{W}} \otimes \overset{\mathbf{5}}{\mathbf{W}} \otimes \overset{\mathbf{6}}{\mathbf{W}}),
\tag{13.21}
$$

where $\overset{\mathbf{3}}{\mathbf{W}} = \overset{3_1}{\mathcal{F}}_q \otimes \cdots \otimes \overset{3_n}{\mathcal{F}}_q$ $\overset{\mathbf{5}}{\mathbf{W}} = \overset{5_1}{\mathcal{F}}_q \otimes \cdots \otimes \overset{5_n}{\mathcal{F}}_q$ and $\overset{\mathbf{6}}{\mathbf{W}} = \overset{6_1}{\mathcal{F}}_q \otimes \cdots \otimes \overset{6_n}{\mathcal{F}}_q$. The superscript tr_1 signifies that the trace is taken over the 1st (leftmost) component of the 3D R as

$$
R^{\mathrm{tr}_1}_{\mathbf{5},\mathbf{6}}(z) = \mathrm{Tr}_4(z^{\mathbf{h}_4}R_{45_16_1}\cdots R_{45_n6_n}) \in \mathrm{End}(\overset{\mathbf{5}}{\mathbf{W}} \otimes \overset{\mathbf{6}}{\mathbf{W}}),
\tag{13.22}
$$

$$
R^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{5}}(z) = \mathrm{Tr}_1(z^{\mathbf{h}_1}R_{13_15_1}\cdots R_{13_n5_n}) \in \mathrm{End}(\overset{\mathbf{3}}{\mathbf{W}} \otimes \overset{\mathbf{5}}{\mathbf{W}}),
\tag{13.23}
$$

$$
R^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{6}}(z) = \mathrm{Tr}_2(z^{\mathbf{h}_2}R_{23_16_1}\cdots R_{23_n6_n}) \in \mathrm{End}(\overset{\mathbf{3}}{\mathbf{W}} \otimes \overset{\mathbf{6}}{\mathbf{W}}).
\tag{13.24}
$$

These are the same operators acting on different copies of $\mathbf{W} \otimes \mathbf{W}$. We will often suppress the labels **3**, **5** etc. The expression (13.22) has already appeared in (11.40) and it is depicted as the left diagram in Fig. 11.5. The operator $R^{\mathrm{tr}_1}(z)$ acts on the basis in (11.13) as

$$R^{\mathrm{tr}_1}(z)(|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle) = \sum_{\mathbf{a},\mathbf{b}\in B} R^{\mathrm{tr}_1}(z)^{\mathbf{ab}}_{\mathbf{ij}} |\mathbf{a}\rangle \otimes |\mathbf{b}\rangle, \tag{13.25}$$

$$R^{\mathrm{tr}_1}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \sum_{k_1,\dots,k_n\geq 0} z^{k_1} R^{k_1 a_1 b_1}_{k_2 i_1 j_1} R^{k_2 a_2 b_2}_{k_3 i_2 j_2} \cdots R^{k_n a_n b_n}_{k_1 i_n j_n}. \tag{13.26}$$

Comparing this with (13.11) and using (3.62), we find that $R^{\mathrm{tr}_1}(z)$ is simply related to $R^{\mathrm{tr}_3}(z)$ as

$$R^{\mathrm{tr}_1}(z)^{\mathbf{ab}}_{\mathbf{ij}} = R^{\mathrm{tr}_3}(z)^{\mathbf{ba}}_{\mathbf{ji}} \quad \text{i.e.} \quad R^{\mathrm{tr}_1}(z) = P R^{\mathrm{tr}_3}(z) P, \tag{13.27}$$

where $P(u \otimes v) = v \otimes u$ is the exchange of the components. Consequently, all the properties in (13.14)–(13.17) are valid beside minor changes in (13.15) and (13.17):

$$R^{\mathrm{tr}_1}(z)^{\mathbf{ab}}_{\mathbf{ij}} = 0 \text{ unless } \mathbf{a}+\mathbf{b} = \mathbf{i}+\mathbf{j} \text{ and } |\mathbf{a}| = |\mathbf{i}|, |\mathbf{b}| = |\mathbf{j}|, \tag{13.28}$$

$$R^{\mathrm{tr}_1}(z) = \bigoplus_{l,m\geq 0} R^{\mathrm{tr}_1}_{l,m}(z), \qquad R^{\mathrm{tr}_1}_{l,m}(z) \in \mathrm{End}(\mathbf{W}_l \otimes \mathbf{W}_m), \tag{13.29}$$

$$R^{\mathrm{tr}_1}_{l,m}(z)(|l\mathbf{e}_k\rangle \otimes |m\mathbf{e}_k\rangle) = \Lambda_{m,l}(z,q)\, |l\mathbf{e}_k\rangle \otimes |m\mathbf{e}_k\rangle, \tag{13.30}$$

$$R^{\mathrm{tr}_1}(z)^{\mathbf{ab}}_{\mathbf{ij}} = R^{\mathrm{tr}_1}(z)^{\mathbf{i}^\vee \mathbf{j}^\vee}_{\mathbf{a}^\vee \mathbf{b}^\vee} \prod_{k=1}^{n} \frac{(q^2)_{i_k}(q^2)_{j_k}}{(q^2)_{a_k}(q^2)_{b_k}}, \tag{13.31}$$

$$R^{\mathrm{tr}_1}(z)^{\mathbf{ab}}_{\mathbf{ij}} = z^{b_1-j_1} R^{\mathrm{tr}_1}(z)^{\sigma(\mathbf{a})\sigma(\mathbf{b})}_{\sigma(\mathbf{i})\sigma(\mathbf{j})}, \tag{13.32}$$

where $\Lambda_{m,l}(z,q)$ in (13.30) is given by $(13.18)_{l\leftrightarrow m}$. The Yang–Baxter equation (13.21) holds in each finite-dimensional subspace $\mathbf{W}_k \otimes \mathbf{W}_l \otimes \mathbf{W}_m$ of $\mathbf{W} \otimes \mathbf{W} \otimes \mathbf{W}$.

13.4 Trace Reduction Over the Second Component of R

The following procedure is quite parallel with that in Sect. 11.4. Consider n copies of the tetrahedron equation (13.1) in which the spaces 1, 4, 5 are replaced by $1_i, 4_i, 5_i$ with $i = 1, \dots, n$:

$$R_{4_i 5_i 6} R_{1_i 24_i} R_{1_i 35_i} R_{236} = R_{236} R_{1_i 35_i} R_{1_i 24_i} R_{4_i 5_i 6}.$$

Here we have relocated R by using $R = R^{-1}$ (3.60). Sending R_{236} to the left by applying this repeatedly, we get

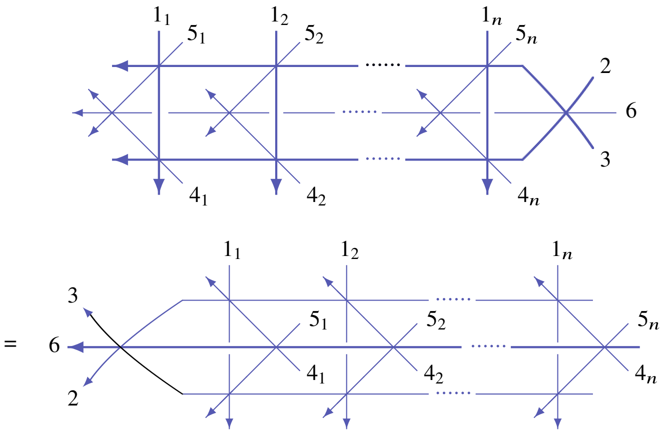

Fig. 13.5 A graphical representation of (13.33) and (13.34)

$$
\begin{aligned}
&(R_{4_15_16}R_{1_124_1}R_{1_135_1})\cdots(R_{4_n5_n6}R_{1_n24_n}R_{1_n35_n})R_{236}\\
&\quad = R_{236}(R_{1_135_1}R_{1_124_1}R_{4_15_16})\cdots(R_{1_n35_n}R_{1_n24_n}R_{4_n5_n6}),
\end{aligned}
\tag{13.33}
$$

which can be rearranged as (Fig. 13.5)

$$
\begin{aligned}
&(R_{4_15_16}\cdots R_{4_n5_n6})(R_{1_124_1}\cdots R_{1_n24_n})(R_{1_135_1}\cdots R_{1_n35_n})R_{236}\\
&\quad = R_{236}(R_{1_135_1}\cdots R_{1_n35_n})(R_{1_124_1}\cdots R_{1_n24_n})(R_{4_15_16}\cdots R_{4_n5_n6}).
\end{aligned}
\tag{13.34}
$$

Multiply $x^{\mathbf{h}_2}(xy)^{\mathbf{h}_3}y^{\mathbf{h}_6}R_{236}^{-1}$ from the left by (13.34) and take the trace over $\overset{2}{\mathcal{F}}_q \otimes \overset{3}{\mathcal{F}}_q \otimes \overset{6}{\mathcal{F}}_q$. Using the weight conservation (13.4) we get the Yang–Baxter equation.

$$
R^{\mathrm{tr}_3}_{\mathbf{4},\mathbf{5}}(y)R^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{4}}(x)R^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{5}}(xy) = R^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{5}}(xy)R^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{4}}(x)R^{\mathrm{tr}_3}_{\mathbf{4},\mathbf{5}}(y) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{W}} \otimes \overset{\mathbf{4}}{\mathbf{W}} \otimes \overset{\mathbf{5}}{\mathbf{W}}),
\tag{13.35}
$$

where $\overset{\mathbf{1}}{\mathbf{W}} = \overset{1_1}{\mathcal{F}}_q \otimes \cdots \otimes \overset{1_n}{\mathcal{F}}_q$ $\overset{\mathbf{4}}{\mathbf{W}} = \overset{4_1}{\mathcal{F}}_q \otimes \cdots \otimes \overset{4_n}{\mathcal{F}}_q$ and $\overset{\mathbf{5}}{\mathbf{W}} = \overset{5_1}{\mathcal{F}}_q \otimes \cdots \otimes \overset{5_n}{\mathcal{F}}_q$. The superscript tr_2 signifies that the trace is taken over the second (middle) component as (Fig. 13.6)

$$
R^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{4}}(z) = \mathrm{Tr}_2(z^{\mathbf{h}_2}R_{1_124_1}\cdots R_{1_n24_n}) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{W}} \otimes \overset{\mathbf{4}}{\mathbf{W}}),
\tag{13.36}
$$

$$
R^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{5}}(z) = \mathrm{Tr}_3(z^{\mathbf{h}_3}R_{1_135_1}\cdots R_{1_n35_n}) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{W}} \otimes \overset{\mathbf{5}}{\mathbf{W}}).
\tag{13.37}
$$

These are the same operators acting on different copies of $\mathbf{W} \otimes \mathbf{W}$. We will often suppress the labels like $\mathbf{1}, \mathbf{4}$. The operator $R^{\mathrm{tr}_3}(y)$ has already appeared in (11.40).

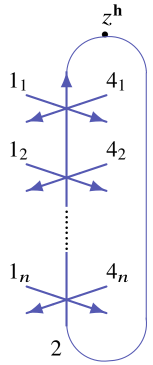

Fig. 13.6 A graphical representation of (13.36). The one for (13.37) just corresponds to a relabeling of the arrows

The operator $R^{\mathrm{tr}_2}(z)$ acts on the basis as

$$R^{\mathrm{tr}_2}(z)(|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle) = \sum_{\mathbf{a},\mathbf{b}\in B} R^{\mathrm{tr}_2}(z)^{\mathbf{ab}}_{\mathbf{ij}} |\mathbf{a}\rangle \otimes |\mathbf{b}\rangle, \tag{13.38}$$

$$R^{\mathrm{tr}_2}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \sum_{k_1,\dots,k_n \geq 0} z^{k_1} R^{a_1 k_1 b_1}_{i_1 k_2 j_1} R^{a_2 k_2 b_2}_{i_2 k_3 j_2} \cdots R^{a_n k_n b_n}_{i_n k_1 j_n}. \tag{13.39}$$

Comparing (13.39) and (13.11) using (3.86) and (3.62), we find

$$R^{\mathrm{tr}_2}(z)^{\mathbf{ab}}_{\mathbf{ij}} = (-q)^{-l+\sum_{k=1}^{n} k(j_k - b_k)} \left(\prod_{k=1}^{n} \frac{(q^2)_{j_k}}{(q^2)_{b_k}} \right) R^{\mathrm{tr}_3}((-q)^n z)^{\mathbf{ja}}_{\mathbf{bi}} \tag{13.40}$$

for $\mathbf{a}, \mathbf{i} \in B_l$ and $\mathbf{b}, \mathbf{j} \in B_m$. One can derive properties similar to $R^{\mathrm{tr}_1}(z)$ as follows:

$$R^{\mathrm{tr}_2}(z)^{\mathbf{ab}}_{\mathbf{ij}} = 0 \text{ unless } \mathbf{a} - \mathbf{b} = \mathbf{i} - \mathbf{j} \text{ and } |\mathbf{a}| = |\mathbf{i}|, |\mathbf{b}| = |\mathbf{j}|, \tag{13.41}$$

$$R^{\mathrm{tr}_2}(z) = \bigoplus_{l,m\geq 0} R^{\mathrm{tr}_2}_{l,m}(z), \qquad R^{\mathrm{tr}_2}_{l,m}(z) \in \mathrm{End}(\mathbf{W}_l \otimes \mathbf{W}_m), \tag{13.42}$$

$$R^{\mathrm{tr}_2}_{l,m}(z)(|l\mathbf{e}_1\rangle \otimes |m\mathbf{e}_2\rangle) = \frac{|l\mathbf{e}_1\rangle \otimes |m\mathbf{e}_2\rangle}{1 + (-1)^{n+1} q^{l+m+n} z}, \tag{13.43}$$

$$R^{\mathrm{tr}_2}(z)^{\mathbf{ab}}_{\mathbf{ij}} = R^{\mathrm{tr}_2}(z)^{\mathbf{ba}}_{\mathbf{ji}}, \tag{13.44}$$

$$R^{\mathrm{tr}_2}(z)^{\mathbf{ab}}_{\mathbf{ij}} = z^{j_1 - b_1} R^{\mathrm{tr}_2}(z)^{\sigma(\mathbf{a})\sigma(\mathbf{b})}_{\sigma(\mathbf{i})\sigma(\mathbf{j})}, \tag{13.45}$$

$$R^{\mathrm{tr}_2}(z)^{\mathbf{ab}}_{\mathbf{ij}} = R^{\mathrm{tr}_2}(z)^{\mathbf{i}^\vee \mathbf{j}^\vee}_{\mathbf{a}^\vee \mathbf{b}^\vee} \prod_{k=1}^{n} \frac{(q^2)_{i_k} (q^2)_{j_k}}{(q^2)_{a_k} (q^2)_{b_k}}. \tag{13.46}$$

13.5 Explicit Formulas of $R^{\mathrm{tr}_1}(z)$, $R^{\mathrm{tr}_2}(z)$, $R^{\mathrm{tr}_3}(z)$

The main result of this section is the explicit formulas in Theorem 13.3 which are derived from the matrix product construction by a direct calculation. The detail of the proof will not be used elsewhere and can be skipped. It is included in the light of the fact that the relevant quantum R matrices (Theorems 13.10, 13.11 and 13.12) are very fundamental examples associated with higher rank type A quantum groups with higher "spin" representations.

13.5.1 Function $A(z)^{\mathbf{ab}}_{\mathbf{ij}}$

For integer arrays $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k) \in \mathbb{Z}^k$ of *any* length k, we use the notation

$$|\boldsymbol{\alpha}| = \sum_{1 \le i \le k} \alpha_i, \quad \overline{\boldsymbol{\alpha}} = (\alpha_1, \dots, \alpha_{k-1}), \tag{13.47}$$

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = \sum_{1 \le i < j \le k} \alpha_i \beta_j, \quad (\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{1 \le i \le k} \alpha_i \beta_i, \tag{13.48}$$

where $|\boldsymbol{\alpha}|$ appeared also in (11.4) for $\boldsymbol{\alpha} \in \{0, 1\}^n$.

For parameters λ, μ and arrays $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k)$, $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k) \in \mathbb{Z}^k_{\ge 0}$ of *any* length k, define

$$\Phi_q(\boldsymbol{\gamma}|\boldsymbol{\beta}; \lambda, \mu) = q^{\langle \boldsymbol{\beta}-\boldsymbol{\gamma}, \boldsymbol{\gamma} \rangle} \left(\frac{\mu}{\lambda}\right)^{|\boldsymbol{\gamma}|} \overline{\Phi}_q(\boldsymbol{\gamma}|\boldsymbol{\beta}; \lambda, \mu), \tag{13.49}$$

$$\overline{\Phi}_q(\boldsymbol{\gamma}|\boldsymbol{\beta}; \lambda, \mu) = \frac{(\lambda; q)_{|\boldsymbol{\gamma}|} (\frac{\mu}{\lambda}; q)_{|\boldsymbol{\beta}|-|\boldsymbol{\gamma}|}}{(\mu; q)_{|\boldsymbol{\beta}|}} \prod_{i=1}^{k} \binom{\beta_i}{\gamma_i}_q. \tag{13.50}$$

From the definition of the q-binomial in (3.65), $\overline{\Phi}_q(\boldsymbol{\gamma}|\boldsymbol{\beta}; \lambda, \mu) = 0$ unless $\gamma_i \le \beta_i$ for all $1 \le i \le k$. We will write this condition as $\boldsymbol{\gamma} \le \boldsymbol{\beta}$.

Given n component arrays $\mathbf{a}, \mathbf{i} \in B_l$ and $\mathbf{b}, \mathbf{j} \in B_m$ (see (11.10) for the definition of B_k), we introduce a quadratic combination of (13.49) as

$$A(z)^{\mathbf{ab}}_{\mathbf{ij}} = q^{\langle \mathbf{i}, \mathbf{j} \rangle - \langle \mathbf{b}, \mathbf{a} \rangle} \times \sum_{\overline{\mathbf{k}}} \Phi_{q^2}(\overline{\mathbf{a}} - \overline{\mathbf{k}} | \overline{\mathbf{a}} + \overline{\mathbf{b}} - \overline{\mathbf{k}}; q^{m-l} z, q^{-l-m} z) \Phi_{q^2}(\overline{\mathbf{k}} | \overline{\mathbf{j}}; q^{-l-m} z^{-1}, q^{-2m}), \tag{13.51}$$

where the sum ranges over $\overline{\mathbf{k}} \in \mathbb{Z}_{\geq 0}^{n-1}$.[2] Due to the remark after (13.50), it is actually confined into the finite set $0 \leq \overline{\mathbf{k}} \leq \min(\overline{\mathbf{b}}, \overline{\mathbf{j}})$ meaning that $0 \leq k_r \leq \min(b_r, j_r)$ for $1 \leq r \leq n-1$. A characteristic feature of the formula (13.51) is that Φ_{q^2} depends on $\mathbf{a} = (a_1, \ldots, a_n) \in B_l$ via $\overline{\mathbf{a}} = (a_1, \ldots, a_{n-1})$ and l by which the last component is taken into account as $a_n = l - |\overline{\mathbf{a}}|$. Dependence on $\mathbf{b}$ and $\mathbf{j}$ is similar. Substituting (13.49) and (13.50) into (13.51) we get

$$A(z)_{\mathbf{i}\,\mathbf{j}}^{\mathbf{ab}} = (-1)^{b_n - j_n} q^{\varphi} \frac{(q^2)_{j_n}}{(q^2)_{b_n}} \sum_{\overline{\mathbf{k}}} q^{2\langle \overline{\mathbf{j}} - \overline{\mathbf{b}} - \overline{\mathbf{k}}, \overline{\mathbf{k}} \rangle + (l+m)|\overline{\mathbf{k}}|} \prod_{\alpha=1}^{n-1} \binom{a_\alpha + b_\alpha - k_\alpha}{b_\alpha}_{q^2} \binom{j_\alpha}{k_\alpha}_{q^2}$$

$$\times z^{|\overline{\mathbf{k}}|} \frac{(q^{m-l}z; q^2)_{|\overline{\mathbf{a}} - \overline{\mathbf{k}}|} (q^{l-m}z; q^2)_{|\overline{\mathbf{j}} - \overline{\mathbf{k}}|} (q^{-l-m}z^{-1}; q^2)_{|\overline{\mathbf{k}}|}}{(q^{-l-m}z; q^2)_{|\overline{\mathbf{a}} + \overline{\mathbf{b}} - \overline{\mathbf{k}}|}}, \tag{13.52}$$

$$\varphi = \langle \overline{\mathbf{i}}, \overline{\mathbf{j}} \rangle + \langle \overline{\mathbf{b}}, \overline{\mathbf{a}} \rangle + m a_n + l j_n + (b_n - j_n)(i_n + j_n + 1) - 2ml. \tag{13.53}$$

The factor $(q^2)_{j_n}/(q^2)_{b_n}$ here originates in $(q^{-2m})_{|\overline{\mathbf{b}}|}/(q^{-2m})_{|\overline{\mathbf{j}}|}$ contained in (13.51).

Remark 13.2 By an induction on k, it can be shown that

$$\sum_{\boldsymbol{\gamma} \in (\mathbb{Z}_{\geq 0})^k,\, \boldsymbol{\gamma} \leq \boldsymbol{\beta}} \Phi_q(\boldsymbol{\gamma} | \boldsymbol{\beta}; \lambda, \mu) = 1 \qquad (\forall \boldsymbol{\beta} \in (\mathbb{Z}_{\geq 0})^k). \tag{13.54}$$

This property has an application to stochastic models, where it plays the role of the total probability conservation. It can also be derived from Proposition 13.13 and (13.132).

13.5.2 $A(z)_{\mathbf{i}\,\mathbf{j}}^{\mathbf{ab}}$ as Elements of $R^{\mathrm{tr}_1}(z)$, $R^{\mathrm{tr}_2}(z)$ and $R^{\mathrm{tr}_3}(z)$

Theorem 13.3 *For $\mathbf{a}, \mathbf{i} \in B_l$, $\mathbf{b}, \mathbf{j} \in B_m$, the following formulas are valid:*

$$\Lambda_{l,m}(z,q)^{-1} R^{\mathrm{tr}_3}(z)_{\mathbf{i}\,\mathbf{j}}^{\mathbf{ab}} = \delta_{\mathbf{i}+\mathbf{j}}^{\mathbf{a}+\mathbf{b}} A(z)_{\mathbf{i}\,\mathbf{j}}^{\mathbf{ab}}, \tag{13.55}$$

$$\Lambda_{m,l}(z,q)^{-1} R^{\mathrm{tr}_1}(z)_{\mathbf{i}\,\mathbf{j}}^{\mathbf{ab}} = \delta_{\mathbf{i}+\mathbf{j}}^{\mathbf{a}+\mathbf{b}} A(z)_{\mathbf{j}\,\mathbf{i}}^{\mathbf{ba}}, \tag{13.56}$$

$$\Lambda_{m,l}((-q)^n z, q)^{-1} R^{\mathrm{tr}_2}(z)_{\mathbf{i}\,\mathbf{j}}^{\mathbf{ab}} = (-q)^{-l+\sum_{\alpha=1}^{n} \alpha(j_\alpha - b_\alpha)} \left(\prod_{\alpha=1}^{n} \frac{(q^2)_{j_\alpha}}{(q^2)_{b_\alpha}} \right) \delta_{\mathbf{b}+\mathbf{i}}^{\mathbf{a}+\mathbf{j}} A((-q)^n z)_{\mathbf{b}\,\mathbf{i}}^{\mathbf{j}\,\mathbf{a}}, \tag{13.57}$$

where $\Lambda_{l,m}(z,q)$ is defined by (13.18).

[2] $\overline{\mathbf{k}}$ is just an array of summation variables. We have not introduced an n component array $\mathbf{k}$ which is related to it as in (13.47).

13.5.3 Proof of Theorem 13.3

The formulas (13.56) and (13.57) follow from (13.55) by virtue of (13.27) and (13.40). Therefore we concentrate on (13.55) in the sequel. The following lemma is nothing but a quantum group symmetry (13.105) with $R^{\mathrm{tr}_3}(z)$ replaced by the matrix having the elements $A(z)^{\mathbf{ab}}_{\mathbf{ij}}$.

Lemma 13.4 *Suppose $n \geq 3$. For $1 \leq r \leq n-2$, the function $A(z)^{\mathbf{ab}}_{\mathbf{ij}}$ satisfies the relation*

$$\begin{aligned}&[b_{r+1}+1]_{q^2} A(z)^{\mathbf{a},\mathbf{b}-\hat{r}}_{\mathbf{i},\mathbf{j}} + q^{b_r-b_{r+1}}[a_{r+1}+1]_{q^2} A(z)^{\mathbf{a}-\hat{r},\mathbf{b}}_{\mathbf{i},\mathbf{j}}\\ &-[i_{r+1}]_{q^2} A(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i}+\hat{r},\mathbf{j}} - q^{i_r-i_{r+1}}[j_{r+1}]_{q^2} A(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}+\hat{r}} = 0\end{aligned} \tag{13.58}$$

for $\mathbf{a}+\mathbf{b} = \mathbf{i}+\mathbf{j}+\hat{r}$. Here $\hat{r} = \mathbf{e}_r - \mathbf{e}_{r+1}$ with $\mathbf{e}_r$ being an elementary vector in (11.1). The symbol $[m]_{q^2}$ is defined in (11.57).

Proof Let $\overline{\mathbf{k}} = (k_1, \ldots, k_{n-1})$ in (13.52). It turns out that (13.58) holds for the partial sum of (13.52) in which $k_\alpha (\alpha \neq r, r+1)$ and $|\overline{\mathbf{k}}|$ are fixed. Under this constraint $A(z)^{\mathbf{ab}}_{\mathbf{ij}}$ is proportional to

$$q^{\langle \bar{\mathbf{i}}, \bar{\mathbf{j}}\rangle - \langle \overline{\mathbf{b}}, \overline{\mathbf{a}}\rangle} \sum q^{2(j_r-b_r-k_r)k_{r+1}} \prod_{\alpha=r,r+1} \binom{a_\alpha+b_\alpha-k_\alpha}{b_\alpha}_{q^2} \binom{j_\alpha}{k_\alpha}_{q^2} \tag{13.59}$$

up to a common overall factor. The sum here is taken over $k_r, k_{r+1} \geq 0$ under the condition $k_r + k_{r+1} = k$ for any fixed k. There is no dependence on the spectral parameter z owing to the assumption $r \neq 0, n-1$. Substituting this into (13.58) and using $\langle \hat{r}, \mathbf{j}\rangle = j_{r+1}$ and $\langle \mathbf{b}, \hat{r}\rangle = -b_r$, we find that (13.58) follows from

$$\begin{aligned}&q^{-a_2-b_2-1}(1-q^{2b_2+2}) \sum q^{2(j_1-b_1-k_1+1)k_2}\\ &\quad\times \binom{a_1+b_1-k_1-1}{b_1-1}_{q^2} \binom{a_2+b_2-k_2+1}{b_2+1}_{q^2} \binom{j_1}{k_1}_{q^2} \binom{j_2}{k_2}_{q^2}\\ &+q^{2b_1-b_2-a_2-1}(1-q^{2a_2+2}) \sum q^{2(j_1-b_1-k_1)k_2}\\ &\quad\times \binom{a_1+b_1-k_1-1}{b_1}_{q^2} \binom{a_2+b_2-k_2+1}{b_2}_{q^2} \binom{j_1}{k_1}_{q^2} \binom{j_2}{k_2}_{q^2}\\ &-q^{j_2-i_2}(1-q^{2i_2}) \sum q^{2(j_1-b_1-k_1)k_2}\\ &\quad\times \binom{a_1+b_1-k_1}{b_1}_{q^2} \binom{a_2+b_2-k_2}{b_2}_{q^2} \binom{j_1}{k_1}_{q^2} \binom{j_2}{k_2}_{q^2}\\ &-q^{-i_2-j_2}(1-q^{2j_2}) \sum q^{2(j_1-b_1-k_1+1)k_2}\\ &\quad\times \binom{a_1+b_1-k_1}{b_1}_{q^2} \binom{a_2+b_2-k_2}{b_2}_{q^2} \binom{j_1+1}{k_1}_{q^2} \binom{j_2-1}{k_2}_{q^2} = 0,\end{aligned} \tag{13.60}$$

where we have denoted a_r, a_{r+1} by a_1, a_2 for simplicity and similarly for the other letters. Thus in particular, $a_1 + b_1 = i_1 + j_1 + 1$ and $a_2 + b_2 = i_2 + j_2 - 1$, reflecting the assumption $\mathbf{a} + \mathbf{b} = \mathbf{i} + \mathbf{j} + \hat{r}$.

The sums in (13.60) are taken over $k_1, k_2 \geq 0$ with the constraint $k_1 + k_2 = k$ for any fixed k. Apart from this constraint, the summation variables k_1 and k_2 are coupling via the factor $q^{-2k_1k_2}$. Fortunately this can be decoupled by rewriting the q^2-binomials as

$$\begin{aligned} &\binom{a_\alpha + b_\alpha - k_\alpha}{b_\alpha}_{q^2} \binom{j_\alpha}{k_\alpha}_{q^2} \\ &= (-1)^{k_\alpha} q^{-k_\alpha^2 + (2j_\alpha - 2b_\alpha + 1)k_\alpha} \frac{(q^{2b_\alpha+2};q^2)_{a_\alpha} (q^{-2a_\alpha};q^2)_{k_\alpha} (q^{-2j_\alpha};q^2)_{a_\alpha}}{(q^2;q^2)_{a_\alpha} (q^{-2a_\alpha-2b_\alpha};q^2)_{k_\alpha} (q^2;q^2)_{k_\alpha}}. \end{aligned} \tag{13.61}$$

In fact, this converts the quadratic power of k_1 and k_2 into an overall constant $q^{-k_1^2-k_2^2-2k_1k_2} = q^{-k^2}$ which can be removed. Consequently, each sum in (13.60) is rewritten in the form $\sum_{k_1+k_2=k}(\sum_{k_1\geq 0} X_{k_1})(\sum_{k_2 \geq 0} Y_{k_2})$ for any fixed k. Thus introducing the generating series $\sum_{k\geq 0} \zeta^k(\cdots)$ decouples it into the product $(\sum_{k_1\geq 0} \zeta^{k_1} X_{k_1})(\sum_{k_2\geq 0} \zeta^{k_2} Y_{k_2})$. Each factor here becomes q^2-hypergeometric defined in (3.73). After some calculation one finds that the explicit form is given, up to an overall factor, by the LHS of (13.62) with the variables replaced as $q \to q^2$, $u_\alpha \to q^{-2a_\alpha}$, $v_\alpha \to q^{-2a_\alpha-2b_\alpha}$, $w_\alpha \to q^{-2j_\alpha}$ for $\alpha = 1, 2$. This also means $q^{-2i_1} = q^2 v_1/w_1$ and $q^{-2i_2} = q^{-2}v_2/w_2$. Therefore the proof is reduced to Lemma 13.5. □

Lemma 13.5 *The q-hypergeometric $\phi\left({a,b \atop c};\zeta\right) := {}_2\phi_1\left({a,b \atop c};q,\zeta\right)$ in (3.73) satisfies the quadratic relation involving the six parameters $u_\alpha, v_\alpha, w_\alpha (\alpha = 1, 2)$ in addition to q and ζ:*

$$\begin{aligned} &u_1(1-u_1^{-1}v_1)(q-v_2)\phi\left({u_1, w_1 \atop qv_1};q\zeta\right)\phi\left({u_2, w_2 \atop q^{-1}v_2};u_2^{-1}v_2w_2^{-1}\zeta\right) \\ &+(1-u_1)(q-v_2)\phi\left({qu_1, w_1 \atop qv_1};\zeta\right)\phi\left({q^{-1}u_2, w_2 \atop q^{-1}v_2};u_2^{-1}v_2w_2^{-1}\zeta\right) \\ &-(1-v_1)(q-v_2w_2^{-1})\phi\left({u_1, w_1 \atop v_1};\zeta\right)\phi\left({u_2, w_2 \atop v_2};u_2^{-1}v_2w_2^{-1}\zeta\right) \\ &-v_2w_2^{-1}(1-v_1)(1-w_2)\phi\left({u_1, q^{-1}w_1 \atop v_1};q\zeta\right)\phi\left({u_2, qw_2 \atop v_2};u_2^{-1}v_2w_2^{-1}\zeta\right) = 0. \end{aligned} \tag{13.62}$$

Proof First, we apply

$$\phi\left({a, b \atop c};\zeta\right) = \frac{(c-abz)}{c(1-z)}\phi\left({a, b \atop c};q\zeta\right) + \frac{z(a-c)(b-c)}{c(1-c)(1-z)}\phi\left({a, b \atop qc};q\zeta\right) \tag{13.63}$$

to the left ϕ's in the second and the third terms to change their argument from ζ to $q\zeta$ to adjust to the first and the fourth terms. The resulting sum is a linear combination of

$$X = \phi\left(\begin{matrix} u_1, w_1 \\ qv_1 \end{matrix}; q\zeta\right), \quad Y = \phi\left(\begin{matrix} qu_1, w_1 \\ qv_1 \end{matrix}; q\zeta\right), \tag{13.64}$$

$$\phi\left(\begin{matrix} qu_1, w_1 \\ q^2v_1 \end{matrix}; q\zeta\right), \quad \phi\left(\begin{matrix} u_1, w_1 \\ v_1 \end{matrix}; q\zeta\right), \quad \phi\left(\begin{matrix} u_1, q^{-1}w_1 \\ v_1 \end{matrix}; q\zeta\right). \tag{13.65}$$

Second, we express (13.65) in terms of X and Y by means of the contiguous relations:

$$\phi\left(\begin{matrix} qu_1, w_1 \\ q^2v_1 \end{matrix}; q\zeta\right) = -\frac{v_1(1-qv_1)}{u_1(qv_1-w_1)\zeta}X + \frac{(1-qv_1)(v_1-u_1w_1\zeta)}{u_1(qv_1-w_1)\zeta}Y, \tag{13.66}$$

$$\phi\left(\begin{matrix} u_1, w_1 \\ v_1 \end{matrix}; q\zeta\right) = \frac{(u_1-v_1)}{u_1(1-v_1)}X + \frac{(1-u_1)v_1}{u_1(1-v_1)}Y, \tag{13.67}$$

$$\begin{aligned}\phi\left(\begin{matrix} u_1, q^{-1}w_1 \\ v_1 \end{matrix}; q\zeta\right) &= \frac{(u_1-v_1)\big(v_1(q-w_1)-q(1-v_1)w_1\zeta\big)}{qu_1(1-v_1)(qv_1-w_1)\zeta}X \\ &+ \frac{(v_1-u_1w_1\zeta)\big((u_1-v_1)(q-w_1)-q(1-v_1)(qu_1-w_1)\zeta\big)}{qu_1(1-v_1)(qv_1-w_1)\zeta}Y.\end{aligned} \tag{13.68}$$

As the result, the LHS of (13.62) is cast into the form $AX + BY$ where A and B are *linear* combinations of the four right ϕ's all having the argument $u_2^{-1}v_2w_2^{-1}\zeta$. The coefficients of the linear combinations are Laurent polynomials of ζ. Then it is straightforward to check $A = B = 0$ by picking the coefficient of each power of ζ. □

In the remainder of this section, $(\zeta)_m$ always means $(\zeta; q^2)_m$ for any ζ.[3]

Lemma 13.6 *The formula (13.55) is valid provided that* $\mathbf{a} = (a_1, \dots, a_n)$ *has vanishing components as* $a_2 = \cdots = a_{n-1} = 0$.

Proof Throughout the proof $\mathbf{a}$ should be understood as the special one $\mathbf{a} = (a_1, 0, \dots, 0, a_n)$. We also keep assuming $\mathbf{a}, \mathbf{i} \in B_l$, $\mathbf{b}, \mathbf{j} \in B_m$ and $\mathbf{a} + \mathbf{b} = \mathbf{i} + \mathbf{j}$ following Theorem 13.3. Then we have the relations like

$$l = a_1 + a_n = i_n + |\bar{\mathbf{i}}|, \quad m = b_n + |\bar{\mathbf{b}}| = j_n + |\bar{\mathbf{j}}|, \tag{13.69}$$

$$a_\alpha + b_\alpha = i_\alpha + j_\alpha \ (\alpha = 1, n), \quad b_\alpha = i_\alpha + j_\alpha \ (\alpha \neq 1, n). \tag{13.70}$$

[3] This is cautioned since the convention (3.65) may wrongly indicate $(q^{-2k})_{k_1} = (q^{-2k}; q^{-2k})_{k_1}$ for example.

Substitute (3.87) into the sum (13.11) for $R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}}$ with $a_2 = \cdots = a_{n-1} = 0$. The result reads as

$$R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = (-1)^m q^{m-(\mathbf{a},\mathbf{j})} \sum_{c_1,k_1,k_n} (-1)^{k_1+k_n} z^{c_1} q^{\varphi_1} \prod_{\alpha=1,n} \binom{a_\alpha + b_\alpha - k_\alpha}{b_\alpha}_{q^2} \binom{j_\alpha}{k_\alpha}_{q^2}, \tag{13.71}$$

$$\varphi_1 = (\mathbf{a}+\mathbf{j}, \mathbf{c}) + \sum_{\alpha=1,n} k_\alpha(k_\alpha - 2c_\alpha - 1), \tag{13.72}$$

$$c_\beta = c_1 + \sum_{1\le\alpha<\beta} (b_\alpha - j_\alpha), \tag{13.73}$$

where the sum (13.71) extends over $c_1 \in \mathbb{Z}_{\ge 0}$ and $k_1, k_n \in \mathbb{Z}_{\ge 0}$. See (13.48) for the definition of $(\mathbf{a},\mathbf{j})$ and $(\mathbf{a}+\mathbf{j},\mathbf{c})$. The relation (13.73) is quoted from (13.12). It leads to $(\mathbf{a}+\mathbf{j},\mathbf{c}) = \langle \mathbf{b}-\mathbf{j}, \mathbf{a}+\mathbf{j}\rangle + (l+m)c_1$ and $c_n = c_1 + |\bar{\mathbf{b}}| - |\bar{\mathbf{j}}| = c_1 + j_n - b_n$ due to (13.69). Thus the sum over c_1 yields

$$R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = (-1)^m q^{\varphi_3} \sum_{k\ge 0} \frac{(-1)^k}{1 - zq^{l+m-2k}} \sum_{k_1 \ge 0} q^{\varphi_2} \prod_{\alpha=1,n} \binom{a_\alpha + b_\alpha - k_\alpha}{b_\alpha}_{q^2} \binom{j_\alpha}{k_\alpha}_{q^2}, \tag{13.74}$$

$$\varphi_2 = k_1^2 + (k-k_1)^2 - k + 2(b_n - j_n)(k - k_1), \tag{13.75}$$

$$\varphi_3 = m - (\mathbf{a},\mathbf{j}) + \langle \mathbf{b}-\mathbf{j}, \mathbf{a}+\mathbf{j}\rangle. \tag{13.76}$$

Here and in what follows, k_n is to be understood as $k_n = k - k_1$. Both sums are actually finite due to the non-vanishing condition of the q^2-binomials.[4] For example, from $k_\alpha \le \min(a_\alpha, j_\alpha)$, k is bounded as $k = k_1 + k_n \le \min(l,m) \le m$ at most.

Rewrite the q^2-binomial factor with $\alpha = n$ as

$$\binom{a_n + b_n - k_n}{b_n}_{q^2} \binom{j_n}{k_n}_{q^2} = \frac{(q^2)_{j_n} (q^{2a_n - 2k_n + 2})_{b_n}}{(q^2)_{b_n} (q^2)_{k_n} (q^2)_{j_n - k_n}}, \tag{13.77}$$

$$\frac{1}{(q^2)_{k_n}} = (-1)^{k_1} q^{k_1(2k-k_1+1)} \frac{(q^{-2k})_{k_1}}{(q^2)_k}, \tag{13.78}$$

$$\frac{1}{(q^2)_{j_n - k_n}} = (-1)^k q^{k(2m-k+1)} \frac{(q^{-2m})_k (q^{2j_n - 2k_n + 2})_{m - j_n - k_1}}{(q^2)_m}. \tag{13.79}$$

Then (13.74) is expressed as

$$R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \frac{(-1)^m q^{\varphi_3} (q^2)_{j_n}}{(q^2)_{b_n} (q^2)_m} \sum_{k=0}^{m} \frac{1}{1 - zq^{l+m-2k}} \frac{(q^{-2m})_k}{(q^2)_k} \mathcal{P}(q^{2k}), \tag{13.80}$$

[4] Conditions like $k \ge k_1$ can formally be dispensed with since the negative k_n kills $\binom{j_n}{k_n}_{q^2}$.

$$\mathcal{P}(w) = w^{m+b_n-j_n} \sum_{k_1=0}^{\min(b_1,j_1)} (-1)^{k_1} q^{k_1^2+(2j_n-2b_n+1)k_1} (w^{-1})_{k_1} \times (w^{-1}q^{2a_n+2k_1+2})_{b_n} (w^{-1}q^{2j_n+2k_1+2})_{m-j_n-k_1} \binom{a_1+b_1-k_1}{b_1}_{q^2} \binom{j_1}{k_1}_{q^2}. \tag{13.81}$$

The upper bound $k_1 \le \min(b_1, j_1)$ in (13.81) is necessary and sufficient for the q^2-binomials and $(w^{-1}q^{2j_n+2k_1+2})_{m-j_n-k_1}$ to survive individually since $m - j_n \ge j_1$ because of $\mathbf{j} \in B_m$. Obviously, $\mathcal{P}(w)$ is a polynomial of w with $\deg\mathcal{P}(w) \le m + b_n - j_n$. In Lemma 13.7 we will show $\deg\mathcal{P}(w) \le m$ even if $b_n > j_n$ due to a non-trivial cancellation. Thanks to this fact, the sum in (13.80) is taken either for $b_n \le j_n$ or $b_n > j_n$ as

$$\sum_{k=0}^{m} \frac{1}{1-zq^{l+m-2k}} \frac{(q^{-2m})_k}{(q^2)_k} \mathcal{P}(q^{2k}) = \frac{(-1)^m q^{-m(m+1)}(q^2)_m}{(zq^{l-m})_{m+1}} \mathcal{P}(zq^{l+m}), \tag{13.82}$$

which is just a partial fraction expansion. Consequently (13.80) gives

$$\Lambda_{l,m}(z,q)^{-1} R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \frac{(-1)^m q^{\varphi_3 - m(l+m+2)}(q^2)_{j_n}}{(q^2)_{b_n}(zq^{-l-m})_m} \mathcal{P}(zq^{l+m}), \tag{13.83}$$

where we have used $\Lambda_{l,m}(z,q)$ in (13.18). On the other hand, the formula (13.53) of $A^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}}$ for the special case $a_2 = \cdots = a_{n-1} = 0$ is simplified considerably. In fact the multidimensional sum over $\overline{\mathbf{k}} = (k_1, \dots, k_{n-1})$ is reduced to the single sum over k_1 entering $\overline{\mathbf{k}} = (k_1, 0, \dots, 0)$. The result reads as

$$A(z)^{\mathbf{ab}}_{\mathbf{ij}} = (-1)^{b_n-j_n} \frac{q^{\varphi}(q^2)_{j_n}}{(q^2)_{b_n}} \sum_{k_1 \ge 0} (zq^{l+m})^{k_1} \binom{a_1+b_1-k_1}{b_1}_{q^2} \binom{j_1}{k_1}_{q^2} \times \frac{(q^{m-l}z)_{l-a_n-k_1}(q^{l-m}z)_{m-j_n-k_1}(q^{-l-m}z^{-1})_{k_1}}{(q^{-l-m}z)_{l+m-a_n-b_n-k_1}}, \tag{13.84}$$

where φ is defined in (13.53). By using (13.81) and relations like

$$(\mathbf{a}, \mathbf{j}) = lm - (l-a_n)j_n - (m-j_n)a_n - \langle \overline{\mathbf{a}}, \overline{\mathbf{j}} \rangle, \quad \langle \overline{\mathbf{i}}, \overline{\mathbf{j}} \rangle = \langle \overline{\mathbf{a}} + \overline{\mathbf{b}} - \overline{\mathbf{j}}, \overline{\mathbf{j}} \rangle, \tag{13.85}$$

$$\langle \mathbf{b} - \mathbf{j}, \mathbf{a} + \mathbf{j} \rangle = (j_n - b_n)(a_n + j_n) + \langle \overline{\mathbf{b}} - \overline{\mathbf{j}}, \overline{\mathbf{j}} \rangle, \tag{13.86}$$

the two expressions (13.83) and (13.84) can be identified directly. □

Apart from q, the polynomial $\mathcal{P}(w)$ (13.81) depends on m and $a_\alpha, b_\alpha, j_\alpha$ with $\alpha = 1, n$. From (13.69) and (13.70), we have $a_1 + a_n = l \ge i_1 + i_n = a_1 + a_n + b_1 + b_n - j_1 - j_n$ and $m \ge j_1 + j_n$.

Lemma 13.7 *The polynomial* $\mathcal{P}(w)$ *(13.81) satisfies* $\deg\mathcal{P}(w) \le m$.

Proof From the preceding remark we assume

$$b_1 + b_n \le j_1 + j_n \le m, \quad b_n > j_n, \tag{13.87}$$

where the last condition selects the non-trivial case of the claim. Up to an overall factor independent of w, $\mathcal{P}(w)$ is equal to

$$\sum_{k_1 \ge 0} (-1)^{k_1} q^{k_1(k_1-1)} (wq^{-2k_1+2})_{k_1} (xwq^{-2k_1})_{b_n} (wq^{-2m})_{m-j_n-k_1} (yq^{-2k_1})_{b_1} \binom{j_1}{k_1}_{q^2} \tag{13.88}$$

at $x = q^{-2a_n-2b_n}$ and $y = q^{2a_1+2}$. This is further expanded into the powers of x and y as

$$\sum_{r=0}^{b_n} \sum_{s=0}^{b_1} (-1)^{r+s} x^r y^s q^{r(r-1)+s(s-1)} \binom{b_n}{r}_{q^2} \binom{b_1}{s}_{q^2} w^r \mathcal{F}_{r+s}(w), \tag{13.89}$$

$$\mathcal{F}_d(w) = \sum_{k_1=0}^{j_1} (-1)^{k_1} q^{k_1(k_1-1-2d)} (wq^{-2k_1+2})_{k_1} (wq^{-2m})_{m-j_n-k_1} \binom{j_1}{k_1}_{q^2}. \tag{13.90}$$

The variable d has the range $0 \le d = r + s \le b_1 + b_n \le j_1 + j_n$ due to (13.87). Thus it suffices to show $\deg\mathcal{F}_d(w) \le m - d$. The reason we consider this slightly stronger inequality rather than $\deg\mathcal{F}_d(w) \le m - r$ is of course that $\mathcal{F}_d(w)$ depends on d instead of r. It is a non-trivial claim when $j_n < d (\le j_1 + j_n)$.

The w-dependent factors in (13.90) are expanded as

$$\begin{aligned}
&(wq^{-2k_1+2})_{k_1} (wq^{-2m})_{m-j_n-k_1} \\
&= \sum_{t=0}^{m-j_n} w^{m-j_n-t} \sum_{\alpha+\beta=t} C_{\alpha,\beta} q^{2(j_n+\beta+1)k_1} \binom{k_1}{\alpha}_{q^2} \binom{m-j_n-k_1}{\beta}_{q^2},
\end{aligned} \tag{13.91}$$

$$\binom{k_1}{\alpha}_{q^2} = \sum_{u=0}^{\alpha} f_u q^{2uk_1}, \quad \binom{m-j_n-k_1}{\beta}_{q^2} = \sum_{v=0}^{\beta} g_v q^{-2vk_1}, \tag{13.92}$$

where $\sum_{\alpha+\beta=t}$ denotes the finite sum over $(\alpha, \beta) \in \{0, 1, \dots, t\}^2$ under the condition $\alpha + \beta = t$. In the following argument, precise forms of the coefficients $C_{\alpha,\beta}$, f_u, g_v do not matter and only the fact that they are independent of k_1 is used. Substituting (13.91) and (13.92) into (13.90) we get

$$\mathcal{F}_d(w) = \sum_{t=0}^{m-j_n} w^{m-j_n-t} \sum_{\alpha+\beta=t} \sum_{u=0}^{\alpha} \sum_{v=0}^{\beta} D_{u,v}^{\alpha,\beta} (q^{2(j_n-d+1+\beta+u-v)}; q^2)_{j_1} \tag{13.93}$$

for some coefficient $D^{\alpha,\beta}_{u,v}$. Thus it is sufficient to show that all the q^2-factorials appearing here are zero for $t = 0, 1, \ldots, d - j_n - 1$. It amounts to checking

$$\text{(i) } j_n - d + 1 + \beta + u - v \leq 0, \qquad \text{(ii) } j_1 + j_n - d + \beta + u - v \geq 0 \quad (13.94)$$

for all the terms for $t = 0, 1, \ldots, d - j_n - 1$. For (i), the most critical case is $v = 0$ and $\beta + u = t = d - j_n - 1$ for which the LHS is exactly 0. Therefore it is satisfied. For (ii), the most critical case is $\beta - v = 0$ and $u = 0$ for which the LHS is $j_1 + j_n - d$. This is indeed non-negative according to the remark after (13.90). □

Proof of Theorem 13.3. Consider the relation (13.58) with $\mathbf{a}$ replaced by $\mathbf{a} + \hat{r}$. The result is a recursion formula which reduces $\mathbf{a} = (a_1, \ldots, a_r, a_{r+1}, \ldots, a_n)$ in $A(z)^{\mathbf{a}\bullet}_{\bullet\bullet}$ to $\mathbf{a} + \hat{r} = (a_1, \ldots, a_r + 1, a_{r+1} - 1, \ldots, a_n)$ for $r = n - 2, \ldots, 2, 1$. Thus $\mathbf{a}$ can ultimately be reduced to the form $(a_1, 0, \ldots, 0, a_n)$. As remarked before Lemma 13.4, the quantum group symmetry (13.105) in Theorem 13.10 shows that $R^{\mathrm{tr}_3}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}}$ also satisfies the same relation as (13.58). Therefore Lemma 13.4 reduces the proof of Theorem 13.3 to the situation $\mathbf{a} = (a_1, 0, \ldots, 0, a_n)$. Since this has been established in Lemma 13.6, the proof is completed. □

13.6 Identification with Quantum *R* Matrices of $A^{(1)}_{n-1}$

Let $U_p(A^{(1)}_{n-1})$ be the quantum affine algebra. We keep the convention specified in the beginning of Sect. 11.5. We take $p = q$ throughout this section, hence the relevant algebra is always $U_q(A^{(1)}_{n-1})$.

Consider the n-fold tensor product $\mathrm{Osc}_q^{\otimes n}$ of q-oscillators and let $\mathbf{a}^+_i, \mathbf{a}^-_i, \mathbf{k}_i, \mathbf{k}^{-1}_i$ be the copy of the generators $\mathbf{a}^+, \mathbf{a}^-, \mathbf{k}, \mathbf{k}^{-1}$ (3.12) corresponding to its ith component. By the definition, generators with different indices are trivially commutative.

Proposition 13.8 *The following maps for $i \in \mathbb{Z}_n$ define algebra homomorphisms $U_q(A^{(1)}_{n-1}) \to \mathrm{Osc}_q^{\otimes n}$ depending on a spectral parameter x:*

$$\rho^{(3)}_x : e_i \mapsto \frac{x^{\delta_{i0}} q \mathbf{a}^+_i \mathbf{a}^-_{i+1} \mathbf{k}^{-1}_{i+1}}{1 - q^2}, \quad f_i \mapsto \frac{x^{-\delta_{i0}} q \mathbf{a}^-_i \mathbf{a}^+_{i+1} \mathbf{k}^{-1}_i}{1 - q^2}, \quad k_i \mapsto \mathbf{k}_i \mathbf{k}^{-1}_{i+1}, \quad (13.95)$$

$$\rho^{(1)}_x : e_i \mapsto \frac{x^{\delta_{i0}} q \mathbf{a}^-_i \mathbf{a}^+_{i+1} \mathbf{k}^{-1}_i}{1 - q^2} \quad f_i \mapsto \frac{x^{-\delta_{i0}} q \mathbf{a}^+_i \mathbf{a}^-_{i+1} \mathbf{k}^{-1}_{i+1}}{1 - q^2}, \quad k_i \mapsto \mathbf{k}^{-1}_i \mathbf{k}_{i+1}. \quad (13.96)$$

Proof The relations (11.56) with $p = q$ are directly checked by using (3.12). □

The maps $\rho^{(1)}_x$ and $\rho^{(3)}_x$ are interchanged via the algebra automorphism $e_i \leftrightarrow f_i, k_i \leftrightarrow k^{-1}_i$ up to the spectral parameter.

By (3.13) one can further let $\mathrm{Osc}_q^{\otimes n}$ act on $\mathbf{W} = \mathcal{F}_q^{\otimes n} = \bigoplus_{\mathbf{a} \in B} \mathbb{C}|\mathbf{a}\rangle$ in (11.11). Since (13.95) and (13.96) preserve $|\mathbf{a}|$ in (11.4), the representation space can be restricted to $\mathbf{W}_k$ (11.12) for any $k \in \mathbb{Z}_{\geq 0}$. Let us denote the resulting representations by

$$\tilde{\pi}_{k\varpi_1,x}: \; U_q(A^{(1)}_{n-1}) \xrightarrow{\rho^{(3)}_x} \mathrm{Osc}_q^{\otimes n}[x, x^{-1}] \to \mathrm{End}(\mathbf{W}_k), \tag{13.97}$$

$$\tilde{\pi}_{k\varpi_{n-1},x}: \; U_q(A^{(1)}_{n-1}) \xrightarrow{\rho^{(1)}_x} \mathrm{Osc}_q^{\otimes n}[x, x^{-1}] \to \mathrm{End}(\mathbf{W}_k), \tag{13.98}$$

where the second arrow is given by (3.13) for each component. Explicitly they are given by

$$\tilde{\pi}_{k\varpi_1,x}: \quad \begin{aligned} e_i|\mathbf{m}\rangle &= x^{\delta_{i0}}[m_{i+1}]_q|\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle, \\ f_i|\mathbf{m}\rangle &= x^{-\delta_{i0}}[m_i]_q|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle, \\ k_i|\mathbf{m}\rangle &= q^{m_i - m_{i+1}}|\mathbf{m}\rangle, \end{aligned} \tag{13.99}$$

$$\tilde{\pi}_{k\varpi_{n-1},x}: \quad \begin{aligned} e_i|\mathbf{m}\rangle &= x^{\delta_{i0}}[m_i]_q|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle, \\ f_i|\mathbf{m}\rangle &= x^{-\delta_{i0}}[m_{i+1}]_q|\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle, \\ k_i|\mathbf{m}\rangle &= q^{m_{i+1} - m_i}|\mathbf{m}\rangle \end{aligned} \tag{13.100}$$

for $\mathbf{m} \in B_k$ and $i \in \mathbb{Z}_n$.[5] As a representation of the classical part $U_q(A_{n-1})$ without $e_0, f_0, k_0^{\pm 1}$, $\tilde{\pi}_{k\varpi_1,x}$ (resp. $\tilde{\pi}_{k\varpi_{n-1},x}$) is the irreducible highest weight representation with the highest weight vector $|k\mathbf{e}_1\rangle$ (resp. $|k\mathbf{e}_n\rangle$) with highest weight $k\varpi_1$ (resp. $k\varpi_{n-1}$). They are q-analogues of the k-fold symmetric tensor of the vector and the anti-vector representations.

Remark 13.9 The representations $\tilde{\pi}_{k\varpi_1,x}$ in (13.99), (13.95) and the earlier one $\pi_{k\varpi_1,x}$ in (11.67) with $p = q$ are equivalent. In fact, by an automorphism

$$\mathbf{a}^+_j \mapsto \mathbf{a}^+_j\mathbf{k}_j, \quad \mathbf{a}^-_j \mapsto \mathbf{k}^{-1}_j\mathbf{a}^-_j, \quad \mathbf{k}_j \mapsto \mathbf{k}_j \tag{13.101}$$

of Osc_q induced by the conjugation $\mathbf{a}^\pm_j \mapsto q^{\mathbf{h}_j(\mathbf{h}_j-1)/2}\mathbf{a}^\pm_j q^{-\mathbf{h}_j(\mathbf{h}_j-1)/2}$, we get another algebra homomorphism $U_q(A^{(1)}_{n-1}) \to \mathrm{Osc}_q^{\otimes n}$ as

$$\rho^{(3)'}_x : \; e_i \mapsto \frac{x^{\delta_{i0}}q^2\mathbf{a}^+_i\mathbf{a}^-_{i+1}\mathbf{k}_i\mathbf{k}^{-2}_{i+1}}{1-q^2}, \quad f_i \mapsto \frac{x^{-\delta_{i0}}q^2\mathbf{a}^+_{i+1}\mathbf{a}^-_i\mathbf{k}^{-2}_i\mathbf{k}_{i+1}}{1-q^2}, \quad k_i \mapsto \mathbf{k}_i\mathbf{k}^{-1}_{i+1}. \tag{13.102}$$

Employing this $\rho^{(3)'}_x$ in (13.97) instead of $\rho^{(3)}_x$ yields $(11.67)|_{p=q}$.

[5] The definition of $[m]_q$ is in (11.57).

13.6.1 $R^{\mathrm{tr}_3}(z)$

Let $\tilde{\pi}_{k\varpi_1,x} : U_q(A^{(1)}_{n-1}) \to \mathrm{End}(\mathbf{W}_k)$ be the representation (13.99). Let $\Delta_{x,y} = (\tilde{\pi}_{l\varpi_1,x} \otimes \tilde{\pi}_{m\varpi_1,y}) \circ \Delta$ and $\Delta^{\mathrm{op}}_{x,y} = (\tilde{\pi}_{l\varpi_1,x} \otimes \tilde{\pi}_{m\varpi_1,y}) \circ \Delta^{\mathrm{op}}$ be the tensor product representations, where the coproducts Δ and Δ^{op} are specified in (11.58) and (11.59).

Let $\mathcal{R}_{l\varpi_1,m\varpi_1}(z) \in \mathrm{End}(\mathbf{W}_l \otimes \mathbf{W}_m)$ be the quantum R matrix of $U_q(A^{(1)}_{n-1})$ which is characterized, up to normalization, by the commutativity

$$\mathcal{R}_{l\varpi_1,m\varpi_1}(\tfrac{x}{y})\Delta_{x,y}(g) = \Delta^{\mathrm{op}}_{x,y}(g)\mathcal{R}_{l\varpi_1,m\varpi_1}(\tfrac{x}{y}) \quad (\forall g \in U_q(A^{(1)}_{n-1})), \tag{13.103}$$

where we have taken into account the obvious fact that $\mathcal{R}_{l\varpi_1,m\varpi_1}$ depends only on the ratio x/y. The relation (13.103) is a generalization of $(10.12)|_{q\to p}$ including the latter as the classical part $g \in U_q(A_{n-1})$.

Theorem 13.10 *Up to normalization, $R^{\mathrm{tr}_3}_{l,m}(z)$ by the matrix product construction (13.9)–(13.11) based on the 3D R coincides with the quantum R matrix of $U_q(A^{(1)}_{n-1})$ as*

$$R^{\mathrm{tr}_3}_{l,m}(z) = \mathcal{R}_{l\varpi_1,m\varpi_1}(z^{-1}). \tag{13.104}$$

Proof It suffices to check

$$R^{\mathrm{tr}_3}(\tfrac{y}{x})(e_r \otimes 1 + k_r \otimes e_r) = (1 \otimes e_r + e_r \otimes k_r)R^{\mathrm{tr}_3}(\tfrac{y}{x}), \tag{13.105}$$

$$R^{\mathrm{tr}_3}(\tfrac{y}{x})(1 \otimes f_r + f_r \otimes k_r^{-1}) = (f_r \otimes 1 + k_r^{-1} \otimes f_r)R^{\mathrm{tr}_3}(\tfrac{y}{x}), \tag{13.106}$$

$$R^{\mathrm{tr}_3}(\tfrac{y}{x})(k_r \otimes k_r) = (k_r \otimes k_r)R^{\mathrm{tr}_3}(\tfrac{y}{x}) \tag{13.107}$$

under the image by $\tilde{\pi}_{l\varpi_1,x} \otimes \tilde{\pi}_{m\varpi_1,y}$. Actually, they can be shown by using (13.95) instead of (13.99), which means that the commutativity holds already in $\mathrm{Osc}_q^{\otimes n} \otimes \mathrm{Osc}_q^{\otimes n}$ without taking the image in $\mathrm{End}(\mathbf{W}_l \otimes \mathbf{W}_m)$. Due to the $\mathbb{Z}_n$ symmetry of (13.95) and (13.7) up to the spectral parameter, it suffices to check this for $r = 0$.[6] The relevant part of (13.11) is $R^{a_n b_n c_n}_{i_n j_n c_1} z^{c_1} R^{a_1 b_1 c_1}_{i_1 j_1 c_2}$, which we regard as an element of the product $R_{123} z^{\mathbf{h}_3} R_{1'2'3}$ of 3D R. The indices here are labels of the corresponding spaces as in Fig. 13.7.

In terms of the labels, the image by (13.95) reads as

$$\begin{aligned}
&e_0 \otimes 1 = xd\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_{1'}^{-1}, && 1 \otimes e_0 = yd\mathbf{a}_2^+\mathbf{a}_{2'}^-\mathbf{k}_{2'}^{-1},\\
&f_0 \otimes 1 = x^{-1}d\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_1^{-1}, && 1 \otimes f_0 = y^{-1}d\mathbf{a}_{2'}^+\mathbf{a}_2^-\mathbf{k}_2^{-1},\\
&k_0 \otimes 1 = \mathbf{k}_1\mathbf{k}_{1'}^{-1}, && 1 \otimes k_0 = \mathbf{k}_2\mathbf{k}_{2'}^{-1},
\end{aligned} \tag{13.108}$$

[6] The case $r \neq 0$ corresponds to the special case $x = y = 1$.

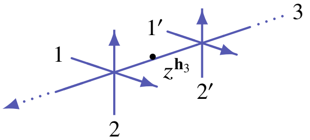

Fig. 13.7 The part of the matrix product construction (13.11) relevant to the commutation relations with e_0, f_0, k_0

where $d = q(1-q^2)^{-1}$. Then (13.105)–(13.107) are attributed to

$$Rz^{\mathbf{h}_3}R'(x\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_{1'}^{-1} + y\mathbf{k}_1\mathbf{k}_{1'}^{-1}\mathbf{a}_2^+\mathbf{a}_{2'}^-\mathbf{k}_{2'}^{-1})$$
$$= (y\mathbf{a}_2^+\mathbf{a}_{2'}^-\mathbf{k}_{2'}^{-1} + x\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_{1'}^{-1}\mathbf{k}_2\mathbf{k}_{2'}^{-1})Rz^{\mathbf{h}_3}R', \tag{13.109}$$
$$Rz^{\mathbf{h}_3}R'(y^{-1}\mathbf{a}_{2'}^+\mathbf{a}_2^-\mathbf{k}_2^{-1} + x^{-1}\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_1^{-1}\mathbf{k}_2^{-1}\mathbf{k}_{2'})$$
$$= (x^{-1}\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_1^{-1} + y^{-1}\mathbf{k}_1^{-1}\mathbf{k}_{1'}\mathbf{a}_{2'}^+\mathbf{a}_2^-\mathbf{k}_2^{-1})Rz^{\mathbf{h}_3}R', \tag{13.110}$$
$$Rz^{\mathbf{h}_3}R'\mathbf{k}_1\mathbf{k}_{1'}^{-1}\mathbf{k}_2\mathbf{k}_{2'}^{-1} = \mathbf{k}_1\mathbf{k}_{1'}^{-1}\mathbf{k}_2\mathbf{k}_{2'}^{-1}Rz^{\mathbf{h}_3}R', \tag{13.111}$$

where $z = yx^{-1}$ and we have set $R = R_{123}$ and $R' = R_{1'2'3}$ for short. To show these relations we invoke the intertwining relations (3.127)–(3.131),[7] i.e.

$$R\,\mathbf{k}_2\mathbf{a}_1^+ = (\mathbf{k}_3\mathbf{a}_1^+ + \mathbf{k}_1\mathbf{a}_2^+\mathbf{a}_3^-)R, \qquad R\,\mathbf{k}_2\mathbf{a}_1^- = (\mathbf{k}_3\mathbf{a}_1^- + \mathbf{k}_1\mathbf{a}_2^-\mathbf{a}_3^+)R, \tag{13.112}$$
$$R\,\mathbf{a}_2^+ = (\mathbf{a}_1^+\mathbf{a}_3^+ - q\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^+)R, \qquad R\,\mathbf{a}_2^- = (\mathbf{a}_1^-\mathbf{a}_3^- - q\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^-)R, \tag{13.113}$$
$$R\,\mathbf{k}_2\mathbf{a}_3^+ = (\mathbf{k}_1\mathbf{a}_3^+ + \mathbf{k}_3\mathbf{a}_1^-\mathbf{a}_2^+)R, \qquad R\,\mathbf{k}_2\mathbf{a}_3^- = (\mathbf{k}_1\mathbf{a}_3^- + \mathbf{k}_3\mathbf{a}_1^+\mathbf{a}_2^-)R, \tag{13.114}$$
$$R\,\mathbf{k}_1\mathbf{k}_2 = \mathbf{k}_1\mathbf{k}_2R, \qquad R\,\mathbf{k}_2\mathbf{k}_3 = \mathbf{k}_2\mathbf{k}_3R \tag{13.115}$$

and their copy where R and the indices $1, 2$ are replaced with R' and $1', 2'$. The relation (13.111) follows from (13.115) immediately. By multiplying $\mathbf{k}_{1'}\mathbf{k}_{2'}$ from the right by (13.109) and $\mathbf{k}_1\mathbf{k}_2$ from the left to (13.110) and using the commutativity with R and R' by (13.115), they are slightly simplified into

$$Rz^{\mathbf{h}_3}R'(x\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_{2'} + y\mathbf{k}_1\mathbf{a}_2^+\mathbf{a}_{2'}^-) = (y\mathbf{a}_2^+\mathbf{a}_{2'}^-\mathbf{k}_{1'} + x\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_2)Rz^{\mathbf{h}_3}R', \tag{13.116}$$
$$Rz^{\mathbf{h}_3}R'(y^{-1}\mathbf{a}_{2'}^+\mathbf{a}_2^-\mathbf{k}_1 + x^{-1}\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_{2'}) = (x^{-1}\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_2 + y^{-1}\mathbf{k}_{1'}\mathbf{a}_{2'}^+\mathbf{a}_2^-)Rz^{\mathbf{h}_3}R'. \tag{13.117}$$

To get (13.117) we have used $\mathbf{k}_j\mathbf{a}_j^\pm = q^{\pm 1}\mathbf{a}_j^\pm\mathbf{k}_j$. All the terms appearing here can be brought to the form $Rz^{\mathbf{h}_3}(\cdots)R'$ by means of $z^{\mathbf{h}_3}\mathbf{a}^\pm = \mathbf{a}^\pm z^{\mathbf{h}_3\pm 1}$, $R = R^{-1}$, (13.112)–(13.115) and the corresponding relations for R'. Explicitly, we have the following for (13.116):

[7] The relation (3.130) can be dispensed with.

$$
\begin{aligned}
Rz^{\mathbf{h}_3}R'x\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_{2'} &= xRz^{\mathbf{h}_3}\mathbf{a}_1^+(\underline{\mathbf{k}_3\mathbf{a}_{1'}^-} + \underline{\mathbf{k}_{1'}\mathbf{a}_{2'}^-\mathbf{a}_3^+})R',\\
Rz^{\mathbf{h}_3}R'y\mathbf{k}_1\mathbf{a}_2^+\mathbf{a}_{2'}^- &= yRz^{\mathbf{h}_3}\mathbf{k}_1\mathbf{a}_2^+(\underline{\mathbf{a}_{1'}^-\mathbf{a}_3^-} - q\underline{\mathbf{k}_{1'}\mathbf{k}_3\mathbf{a}_{2'}^-})R',\\
y\mathbf{a}_2^+\mathbf{a}_{2'}^-\mathbf{k}_{1'}Rz^{\mathbf{h}_3}R' &= yR(\mathbf{a}_1^+\mathbf{a}_3^+ - q\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^+)z^{\mathbf{h}_3}\mathbf{a}_{2'}^-\mathbf{k}_{1'}R'\\
&= yRz^{\mathbf{h}_3}(z^{-1}\underline{\mathbf{a}_1^+\mathbf{a}_3^+} - q\underline{\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^+})\mathbf{a}_{2'}^-\mathbf{k}_{1'}R',\\
x\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_2Rz^{\mathbf{h}_3}R' &= xR(\mathbf{k}_3\mathbf{a}_1^+ + \mathbf{k}_1\mathbf{a}_2^+\mathbf{a}_3^-)z^{\mathbf{h}_3}\mathbf{a}_{1'}^-R'\\
&= xRz^{\mathbf{h}_3}(\underline{\mathbf{k}_3\mathbf{a}_1^+} + z\underline{\mathbf{k}_1\mathbf{a}_2^+\mathbf{a}_3^-})\mathbf{a}_{1'}^-R'.
\end{aligned}
$$

As shown by the underlines, (13.116) is indeed valid at $z = yx^{-1}$. A similar calculation casts the four terms in (13.117) into

$$
\begin{aligned}
Rz^{\mathbf{h}_3}R'y^{-1}\mathbf{a}_{2'}^+\mathbf{a}_2^-\mathbf{k}_1 &= y^{-1}Rz^{\mathbf{h}_3}\mathbf{a}_2^-\mathbf{k}_1(\underline{\mathbf{a}_{1'}^+\mathbf{a}_3^+} - q\underline{\mathbf{k}_{1'}\mathbf{k}_3\mathbf{a}_{2'}^+})R',\\
Rz^{\mathbf{h}_3}R'x^{-1}\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_{2'} &= x^{-1}Rz^{\mathbf{h}_3}\mathbf{a}_1^-(\underline{\mathbf{k}_3\mathbf{a}_{1'}^+} + \underline{\mathbf{k}_{1'}\mathbf{a}_{2'}^+\mathbf{a}_3^-})R',\\
x^{-1}\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_2Rz^{\mathbf{h}_3}R' &= x^{-1}Rz^{\mathbf{h}_3}(\underline{\mathbf{k}_3\mathbf{a}_1^-} + z^{-1}\underline{\mathbf{k}_1\mathbf{a}_2^-\mathbf{a}_3^+})\mathbf{a}_{1'}^+R',\\
y^{-1}\mathbf{k}_{1'}\mathbf{a}_{2'}^+\mathbf{a}_2^-Rz^{\mathbf{h}_3}R' &= y^{-1}Rz^{\mathbf{h}_3}(z\underline{\mathbf{a}_1^-\mathbf{a}_3^-} - q\underline{\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^-})\mathbf{k}_{1'}\mathbf{a}_{2'}^+R',
\end{aligned}
$$

which are again valid at $z = yx^{-1}$. □

13.6.2 $R^{\mathrm{tr}_1}(z)$

Let $\tilde{\pi}_{k\varpi_{n-1},x} : U_q(A_{n-1}^{(1)}) \to \mathrm{End}(\mathbf{W}_k)$ be the representation (13.100). Let $\Delta_{x,y} = (\tilde{\pi}_{l\varpi_{n-1},x} \otimes \tilde{\pi}_{m\varpi_{n-1},y}) \circ \Delta$ and $\Delta_{x,y}^{\mathrm{op}} = (\tilde{\pi}_{l\varpi_{n-1},x} \otimes \tilde{\pi}_{m\varpi_{n-1},y}) \circ \Delta^{\mathrm{op}}$ be the tensor product representations, where the coproducts Δ and Δ^{op} are specified in (11.58) and (11.59).

Let $\mathcal{R}_{l\varpi_{n-1},m\varpi_{n-1}}(z) \in \mathrm{End}(\mathbf{W}_l \otimes \mathbf{W}_m)$ be the quantum R matrix of $U_q(A_{n-1}^{(1)})$ which is characterized, up to normalization, by the commutativity

$$
\mathcal{R}_{l\varpi_{n-1},m\varpi_{n-1}}(\tfrac{x}{y})\Delta_{x,y}(g) = \Delta_{x,y}^{\mathrm{op}}(g)\mathcal{R}_{l\varpi_{n-1},m\varpi_{n-1}}(\tfrac{x}{y}) \quad (\forall g \in U_q(A_{n-1}^{(1)})), \tag{13.118}
$$

where we have taken into account the fact that $\mathcal{R}_{l\varpi_{n-1},m\varpi_{n-1}}$ depends only on the ratio x/y.

Theorem 13.11 *Up to normalization, $R_{l,m}^{\mathrm{tr}_1}(z)$ by the matrix product construction (13.25)–(13.26) and (13.29) based on the 3D R coincides with the quantum R matrix of $U_q(A_{n-1}^{(1)})$ as*

$$
R_{l,m}^{\mathrm{tr}_1}(z) = \mathcal{R}_{l\varpi_{n-1},m\varpi_{n-1}}(z^{-1}). \tag{13.119}
$$

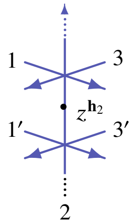

Fig. 13.8 The part of the matrix product construction (13.39) relevant to the commutation relations with e_0, f_0, k_0

Proof This follows from the relation (13.27), Theorem 13.10, the commutativity (13.105)–(13.107) and the fact that $\tilde{\pi}_{k\varpi_1,x}$ (13.99) and $\tilde{\pi}_{k\varpi_{n-1},x^{-1}}$ (13.100) are interchanged via the algebra automorphism $e_i \leftrightarrow f_i, k_i \leftrightarrow k_i^{-1}$. □

13.6.3 $R^{\mathrm{tr}_2}(z)$

Let $\tilde{\pi}_{k\varpi_1,x}$ and $\tilde{\pi}_{k\varpi_{n-1},x}$ be the representations $U_q(A^{(1)}_{n-1}) \to \mathrm{End}(\mathbf{W}_k)$ in (13.99) and (13.100). Let $\Delta_{x,y} = (\tilde{\pi}_{l\varpi_1,x} \otimes \tilde{\pi}_{m\varpi_{n-1},y}) \circ \Delta$ and $\Delta^{\mathrm{op}}_{x,y} = (\tilde{\pi}_{l\varpi_1,x} \otimes \tilde{\pi}_{m\varpi_{n-1},y}) \circ \Delta^{\mathrm{op}}$ be the tensor product representations, where the coproducts Δ and Δ^{op} are specified in (11.58) and (11.59).

Let $\mathcal{R}_{l\varpi_1,m\varpi_{n-1}}(z) \in \mathrm{End}(\mathbf{W}_l \otimes \mathbf{W}_m)$ be the quantum R matrix of $U_q(A^{(1)}_{n-1})$ which is characterized, up to normalization, by the commutativity

$$\mathcal{R}_{l\varpi_1,m\varpi_{n-1}}(\tfrac{x}{y})\Delta_{x,y}(g) = \Delta^{\mathrm{op}}_{x,y}(g)\mathcal{R}_{l\varpi_1,m\varpi_{n-1}}(\tfrac{x}{y}) \quad (\forall g \in U_q(A^{(1)}_{n-1})), \tag{13.120}$$

where we have taken into account the fact that $\mathcal{R}_{l\varpi_1,m\varpi_{n-1}}$ depends only on the ratio x/y.

Theorem 13.12 *Up to normalization, $R^{\mathrm{tr}_2}_{l,m}(z)$ by the matrix product construction (13.38)–(13.39) and (13.42) based on the 3D R coincides with the quantum R matrix of $U_q(A^{(1)}_{n-1})$ as*

$$R^{\mathrm{tr}_2}_{l,m}(z) = \mathcal{R}_{l\varpi_1,m\varpi_{n-1}}(z). \tag{13.121}$$

Proof The proof is similar to the one for Theorem 13.10. So we shall list the corresponding formulas along the labeling in Fig. 13.8 without a detailed explanation.

We are to investigate the commutation relation of $Rz^{\mathbf{h}_2}R' = R_{123}z^{\mathbf{h}_2}R_{1'23'}$ and

$$\begin{aligned} e_0 \otimes 1 &= xd\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_{1'}^{-1}, & 1 \otimes e_0 &= yd\mathbf{a}_{3'}^+\mathbf{a}_3^-\mathbf{k}_3^{-1}, \\ f_0 \otimes 1 &= x^{-1}d\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_1^{-1}, & 1 \otimes f_0 &= y^{-1}d\mathbf{a}_3^+\mathbf{a}_{3'}^-\mathbf{k}_{3'}^{-1}, \\ k_0 \otimes 1 &= \mathbf{k}_1\mathbf{k}_{1'}^{-1}, & 1 \otimes k_0 &= \mathbf{k}_3^{-1}\mathbf{k}_{3'}, \end{aligned} \tag{13.122}$$

where $d = q(1-q^2)^{-1}$. The relation (13.118) with $g = e_0$ becomes, after multiplying $\mathbf{k}_{1'}\mathbf{k}_2\mathbf{k}_3$ from the right,

$$Rz^{\mathbf{h}_2}R'(x\mathbf{k}_2\mathbf{k}_3\mathbf{a}_1^+\mathbf{a}_{1'}^- + y\mathbf{k}_1\mathbf{k}_2\mathbf{a}_{3'}^+\mathbf{a}_3^-) = (y\mathbf{a}_{3'}^+\mathbf{a}_3^-\mathbf{k}_2\mathbf{k}_{1'} + x\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_2\mathbf{k}_{3'})Rz^{\mathbf{h}_2}R'. \tag{13.123}$$

The four terms here are rewritten by means of (13.112)–(13.115) as

$$\begin{aligned}
Rz^{\mathbf{h}_2}R'x\mathbf{k}_2\mathbf{k}_3\mathbf{a}_1^+\mathbf{a}_{1'}^- &= xRz^{\mathbf{h}_2}\mathbf{k}_3\mathbf{a}_1^+(\underline{\mathbf{k}_{3'}\mathbf{a}_{1'}^-} + \underline{\mathbf{k}_{1'}\mathbf{a}_2^-\mathbf{a}_{3'}^+})R',\\
Rz^{\mathbf{h}_2}R'y\mathbf{k}_1\mathbf{k}_2\mathbf{a}_{3'}^+\mathbf{a}_3^- &= yRz^{\mathbf{h}_2}\mathbf{k}_1\mathbf{a}_3^-(\underline{\mathbf{k}_{1'}\mathbf{a}_{3'}^+} + \underline{\mathbf{k}_{3'}\mathbf{a}_{1'}^-\mathbf{a}_2^+})R',\\
y\mathbf{a}_{3'}^+\mathbf{a}_3^-\mathbf{k}_2\mathbf{k}_{1'}Rz^{\mathbf{h}_2}R' &= yRz^{\mathbf{h}_2}(\underline{\mathbf{k}_1\mathbf{a}_3^-} + z\underline{\mathbf{k}_3\mathbf{a}_1^+\mathbf{a}_2^-})\mathbf{a}_{3'}^+\mathbf{k}_{1'}R',\\
x\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_2\mathbf{k}_{3'}Rz^{\mathbf{h}_2}R' &= xRz^{\mathbf{h}_2}(\underline{\mathbf{k}_3\mathbf{a}_1^+} + z^{-1}\underline{\mathbf{k}_1\mathbf{a}_2^+\mathbf{a}_3^-})\mathbf{a}_{1'}^-\mathbf{k}_{3'}R'.
\end{aligned}$$

Thus (13.123) is valid at $z = xy^{-1}$. The relation (13.118) with $g = f_0$ becomes, after multiplying $\mathbf{k}_1\mathbf{k}_2\mathbf{k}_{3'}$ from the left,

$$Rz^{\mathbf{h}_2}R'(y^{-1}\mathbf{k}_1\mathbf{k}_2\mathbf{a}_3^+\mathbf{a}_{3'}^- + x^{-1}\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_2\mathbf{k}_3) = (x^{-1}\mathbf{k}_2\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_{3'} + y^{-1}\mathbf{k}_2\mathbf{k}_{1'}\mathbf{a}_3^+\mathbf{a}_{3'}^-)Rz^{\mathbf{h}_2}R'. \tag{13.124}$$

The four terms here are rewritten by means of (13.112)–(13.115) as

$$\begin{aligned}
Rz^{\mathbf{h}_2}R'y^{-1}\mathbf{k}_1\mathbf{k}_2\mathbf{a}_3^+\mathbf{a}_{3'}^- &= y^{-1}Rz^{\mathbf{h}_2}\mathbf{k}_1\mathbf{a}_3^+(\underline{\mathbf{k}_{1'}\mathbf{a}_{3'}^-} + \underline{\mathbf{k}_{3'}\mathbf{a}_{1'}^+\mathbf{a}_2^-})R',\\
Rz^{\mathbf{h}_2}R'x^{-1}\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_2\mathbf{k}_3 &= x^{-1}Rz^{\mathbf{h}_2}\mathbf{a}_1^-\mathbf{k}_3(\underline{\mathbf{k}_{3'}\mathbf{a}_{1'}^+} + \underline{\mathbf{k}_{1'}\mathbf{a}_2^+\mathbf{a}_{3'}^-})R',\\
x^{-1}\mathbf{k}_2\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_{3'}Rz^{\mathbf{h}_2}R' &= x^{-1}Rz^{\mathbf{h}_2}(\underline{\mathbf{k}_3\mathbf{a}_1^-} + z\underline{\mathbf{k}_1\mathbf{a}_2^-\mathbf{a}_3^+})\mathbf{a}_{1'}^+\mathbf{k}_{3'}R,\\
y^{-1}\mathbf{k}_2\mathbf{k}_{1'}\mathbf{a}_3^+\mathbf{a}_{3'}^-Rz^{\mathbf{h}_2}R' &= y^{-1}Rz^{\mathbf{h}_2}(\underline{\mathbf{k}_1\mathbf{a}_3^+} + z^{-1}\underline{\mathbf{k}_3\mathbf{a}_1^-\mathbf{a}_2^+})\mathbf{k}_{1'}\mathbf{a}_{3'}^-R'.
\end{aligned}$$

Thus (13.124) is valid at $z = xy^{-1}$. □

We note that (13.113) has not been used in the above proof.

13.7 Stochastic R Matrix

This section is a small digression on a special gauge of the R matrix. For $l, m \in \mathbb{Z}_{\geq 1}$, we introduce $\mathcal{S}(z) \in \mathrm{End}(\mathbf{W}_l \otimes \mathbf{W}_m)$ by

$$\mathcal{S}(z)(|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle) = \sum_{\mathbf{a}\in B_l, \mathbf{b}\in B_m} \mathcal{S}(z)_{\mathbf{ij}}^{\mathbf{ab}}|\mathbf{a}\rangle \otimes |\mathbf{b}\rangle, \tag{13.125}$$

$$\mathcal{S}(z)_{\mathbf{ij}}^{\mathbf{ab}} = \delta_{\mathbf{i}+\mathbf{j}}^{\mathbf{a}+\mathbf{b}}\mathcal{A}(z)_{\mathbf{ij}}^{\mathbf{ab}}, \tag{13.126}$$

where $\mathcal{A}(z)_{\mathbf{ij}}^{\mathbf{ab}}$ is a slight modification of $A(z)_{\mathbf{ij}}^{\mathbf{ab}}$ (13.51):

$$
\begin{aligned}
\mathscr{A}(z)^{\mathbf{ab}}_{\mathbf{ij}} &= q^{\langle \mathbf{b},\mathbf{a}\rangle - \langle \mathbf{i},\mathbf{j}\rangle} A(z)^{\mathbf{ab}}_{\mathbf{ij}} \\
&= \sum_{\overline{\mathbf{k}}} \Phi_{q^2}(\overline{\mathbf{a}} - \overline{\mathbf{k}} | \overline{\mathbf{a}} + \overline{\mathbf{b}} - \overline{\mathbf{k}}; q^{m-l}z, q^{-l-m}z) \Phi_{q^2}(\overline{\mathbf{k}} | \overline{\mathbf{j}}; q^{-l-m}z^{-1}, q^{-2m}).
\end{aligned}
\tag{13.127}
$$

From (13.17), (13.55) and Theorem 13.10, $\mathcal{S}(z)$ satisfies

$$\text{Yang–Baxter relation: } \mathcal{S}_{12}(x)\mathcal{S}_{13}(xy)\mathcal{S}_{23}(y) = \mathcal{S}_{23}(y)\mathcal{S}_{13}(xy)\mathcal{S}_{12}(x), \tag{13.128}$$

$$\text{Inversion relation: } \mathcal{S}(z)P\mathcal{S}(z^{-1})P = \mathrm{id}, \tag{13.129}$$

$$\text{Normalization: } \mathcal{S}(z)(|l\mathbf{e}_k\rangle \otimes |m\mathbf{e}_k\rangle) = |l\mathbf{e}_k\rangle \otimes |m\mathbf{e}_k\rangle, \tag{13.130}$$

where $P(u \otimes v) = v \otimes u$ and $k \in \mathbb{Z}_n$ is arbitrary. In fact, it is easy to check that the extra factor $q^{\langle \mathbf{b},\mathbf{a}\rangle - \langle \mathbf{i},\mathbf{j}\rangle}$ in (13.127) does not spoil these properties.[8]

A notable feature of this gauge is the *sum to unity* property:

Proposition 13.13

$$\sum_{\mathbf{a}\in B_l, \mathbf{b}\in B_m} \mathcal{S}(z)^{\mathbf{ab}}_{\mathbf{ij}} = 1 \qquad (\forall (\mathbf{i},\mathbf{j}) \in B_l \times B_m). \tag{13.131}$$

$\mathcal{S}(z)$ has an application to stochastic models where Proposition 13.13 plays the role of the total probability conservation. In such a context, it is called a *stochastic* R matrix.[9]

From (13.49) and (13.50), one sees $\Phi_{q^2}(\boldsymbol{\gamma}|\boldsymbol{\beta}, \lambda = 1, \mu) = \delta_{\boldsymbol{\gamma},0}$. Therefore $\mathcal{S}(z)$ has a factorized special value:

$$\mathcal{S}(z = q^{l-m})^{\mathbf{ab}}_{\mathbf{ij}} = \delta^{\mathbf{a}+\mathbf{b}}_{\mathbf{i}+\mathbf{j}} \, \Phi_{q^2}(\overline{\mathbf{a}} | \overline{\mathbf{j}}; q^{-2l}, q^{-2m}). \tag{13.132}$$

The specialization of (13.131) to (13.132) agrees with (13.54).

13.8 Commuting Layer Transfer Matrices and Duality

This section is parallel with Sect. 11.6. Let $m, n \geq 2$ and consider the composition of $m \times n$ 3D R's as follows:

At the intersection of 1_i and 2_j, we have the 3D R $L_{1_i,2_j,3_{ij}}$ as in Fig. 13.1, where the arrow 3_{ij} corresponds to the vertical arrows carrying $\mathcal{F}_q$. We take the parameters μ_i and ν_j as

$$\mu_i = xu_i \ (i = 1, \dots, m), \qquad \nu_j = yw_j \ (j = 1, \dots, n). \tag{13.133}$$

[8] See [87, Proposition 4].

[9] For reasons of convention, the R matrix $R^{\mathrm{tr}3}_{l,m}(z) = \mathcal{R}_{l\varpi_1, m\varpi_1}(z^{-1})$ in (13.104) of this book is proportional to $R(z)$ in [87, Eq. (6)].

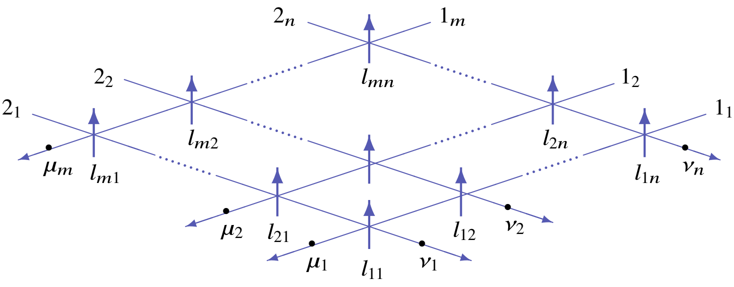

Fig. 13.9 Graphical representation of the layer transfer matrix $T(x, y)$. There are $m + n$ horizontal arrows $1_1, \dots, 1_m$ and $2_1, \dots, 2_n$ carrying $\mathcal{F}_q$ and being traced out, which corresponds to the periodic boundary condition. The mark • with μ_i and ν_j signifies an operator $\mu_i^{\mathbf{h}}$ and $\nu_j^{\mathbf{h}}$ attached to 1_i and 2_j, respectively. At the intersection of 1_i and 2_j, there is a q-oscillator Fock space $\mathcal{F}_q$ depicted with a vertical arrow

Tracing out the horizontal degrees of freedom leaves us with a linear operator acting along vertical arrows. We write the resulting *layer* transfer matrix in the third direction as[10]

$$T(x, y) = T(x, y|\mathbf{u}, \mathbf{w}) \in \mathrm{End}(\mathcal{F}_q^{\otimes mn}), \tag{13.134}$$

$$\mathbf{u} = (u_1, \dots, u_m), \quad \mathbf{w} = (w_1, \dots, w_n). \tag{13.135}$$

Figure 13.9 shows its action on the basis $\bigotimes_{1 \le i \le m, 1 \le j \le n} |l_{ij}\rangle \in \mathcal{F}_q^{\otimes mn}$.

We exhibit the n-dependence in the notations in Sect. 11.1 as $B^{(n)}, \mathbf{W}^{(n)}, \mathbf{W}_k^{(n)}$, etc. In what follows, $\mathbf{u}^H$ for $\mathbf{u} \in \mathbb{C}^m$ should be understood as the linear diagonal operator $u_1^{\mathbf{h}_1} \cdots u_m^{\mathbf{h}_m}$, i.e.[11]

$$\mathbf{u}^H : |\mathbf{a}\rangle \mapsto u_1^{a_1} \cdots u_m^{a_m} |\mathbf{a}\rangle \quad \text{for } \mathbf{a} = (a_1, \dots, a_m) \in B^{(m)}. \tag{13.136}$$

Viewing Fig. 13.9 from the SW, or taking the traces over $1_1, \dots, 1_m$ first, we find that it represents the trace of the product of $(y\mathbf{w})^H$ and $R^{\mathrm{tr}_1}(\mu_1), \dots, R^{\mathrm{tr}_1}(\mu_m)$:

$$\begin{aligned} T(x, y) &= \mathrm{Tr}_{\overset{2}{\mathbf{W}}^{(n)}} \left((y\mathbf{w})^H R^{\mathrm{tr}_1}(xu_1) \cdots R^{\mathrm{tr}_1}(xu_m) \right) \\ &= \sum_{k \ge 0} y^k \mathrm{Tr}_{\overset{2}{\mathbf{W}}_k^{(n)}} \left(\mathbf{w}^H R^{\mathrm{tr}_1}(xu_1) \cdots R^{\mathrm{tr}_1}(xu_m) \right) \in \mathrm{End}\big((\mathbf{W}^{(n)})^{\otimes m}\big), \end{aligned} \tag{13.137}$$

where the matrix product constructed $R^{\mathrm{tr}_1}(xu_i) \in \mathrm{End}(\overset{2}{\mathbf{W}}^{(n)} \otimes \mathbf{W}^{(n)})$ is a quantum R matrix of $U_q(A_{n-1}^{(1)})$ due to Theorem 13.11 and (13.29). The product is taken with respect to $\overset{2}{\mathbf{W}}^{(n)} = \overset{2_1}{\mathcal{F}}_q \otimes \cdots \otimes \overset{2_n}{\mathcal{F}}_q$, which corresponds to the first (left) component of R^{tr_1}'s.

[10] $T(x, y)$ here is different from the one in (11.85).

[11] For H we do not exhibit the number of components m, n as $H^{(m)}$ or $H^{(n)}$.

Alternatively, Fig. 13.9 viewed from the SE or first taking the traces over $2_1, \dots, 2_n$ gives rise to the trace of the product of $(x\mathbf{u})^H$ and $R^{\mathrm{tr}_2}(\nu_1), \dots, R^{\mathrm{tr}_2}(\nu_n)$:

$$\begin{aligned} T(x,y) &= \mathrm{Tr}_{\mathbf{W}^{(m)}}\big((x\mathbf{u})^H R^{\mathrm{tr}_2}(yw_1)\cdots R^{\mathrm{tr}_2}(yw_n)\big) \\ &= \sum_{k\geq 0} x^k \mathrm{Tr}_{\mathbf{W}^{(m)}_k}\big(\mathbf{u}^H R^{\mathrm{tr}_2}(yw_1)\cdots R^{\mathrm{tr}_2}(yw_n)\big) \in \mathrm{End}\big((\mathbf{W}^{(m)})^{\otimes n}\big), \end{aligned} \tag{13.138}$$

where the matrix product constructed $R^{\mathrm{tr}_2}(yw_j) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{W}}{}^{(m)} \otimes \mathbf{W}^{(m)})$ is a quantum R matrix of $U_q(A^{(1)}_{m-1})$ due to Theorem 13.12 and (13.42). The product is taken with respect to $\overset{\mathbf{1}}{\mathbf{W}}{}^{(m)} = \overset{1_1}{\mathscr{F}}_q \otimes \cdots \otimes \overset{1_m}{\mathscr{F}}_q$ in Fig. 13.9, which corresponds to the first (left) component of R^{tr_2}'s.

The identifications (13.137) and (13.138) correspond to the two complementary pictures $\mathscr{F}_q^{\otimes mn} = (\mathbf{W}^{(n)})^{\otimes m} = (\mathbf{W}^{(m)})^{\otimes n}$. In either case, $R^{\mathrm{tr}_1}(z)$ and $R^{\mathrm{tr}_2}(z)$ satisfy the Yang–Baxter equations, which implies the two-parameter commutativity

$$[T(x,y), T(x',y')] = 0 \tag{13.139}$$

for fixed $\mathbf{u}$ and $\mathbf{w}$.

Due to the weight conservations (13.28) and (13.41), the layer transfer matrix $T(x,y)$ has many invariant subspaces. The resulting decomposition is again described as (11.91)–(11.95) for another layer transfer matrix $T(x,y)$ considered in Sect. 11.6.

Consequently, each summand in (13.137) and (13.138) is further decomposed as

$$\begin{aligned} &\mathrm{Tr}_{\mathbf{W}^{(n)}_k}\big(\mathbf{w}^H R^{\mathrm{tr}_1}(xu_1)\cdots R^{\mathrm{tr}_1}(xu_m)\big) \\ &= \bigoplus_{I_1,\dots,I_m\geq 0} \mathrm{Tr}_{\mathbf{W}^{(n)}_k}\big(\mathbf{w}^H R^{\mathrm{tr}_1}_{k,I_1}(xu_1)\cdots R^{\mathrm{tr}_1}_{k,I_m}(xu_m)\big), \end{aligned} \tag{13.140}$$

$$\begin{aligned} &\mathrm{Tr}_{\mathbf{W}^{(m)}_k}\big(\mathbf{u}^H R^{\mathrm{tr}_2}(yw_1)\cdots R^{\mathrm{tr}_2}(yw_n)\big) \\ &= \bigoplus_{J_1,\dots,J_n\geq 0} \mathrm{Tr}_{\mathbf{W}^{(m)}_k}\big(\mathbf{u}^H R^{\mathrm{tr}_2}_{k,J_1}(yw_1)\cdots R^{\mathrm{tr}_2}_{k,J_n}(yw_n)\big). \end{aligned} \tag{13.141}$$

In the terminology of the quantum inverse scattering method, each summand in the RHS of (13.140) is a row transfer matrix of the $U_q(A^{(1)}_{n-1})$ vertex model of size m whose auxiliary space is $\mathbf{W}^{(n)}_k$ and the quantum space is $\mathbf{W}^{(n)}_{I_1} \otimes \cdots \otimes \mathbf{W}^{(n)}_{I_m}$ having the spectral parameter x with inhomogeneity $u_1, \dots, u_m$ and the "horizontal" boundary electric/magnetic field $\mathbf{w}$. It forms a commuting family with respect to x provided that the other parameters are fixed. In the dual picture (13.141), the role of these data is interchanged as $m \leftrightarrow n, x \leftrightarrow y, \mathbf{u} \leftrightarrow \mathbf{w}$. This is another example of duality between rank and size, spectral inhomogeneity and field in addition to the one demonstrated in Sect. 11.6.

Consider the cube of size $l \times m \times n$ formed by concatenating Fig. 13.9 vertically for l times. As in Remark 11.8, one can formulate further two versions of the duality on the layer transfer matrices in the first and the second directions, which correspond to the interchanges $l \leftrightarrow m$ and $l \leftrightarrow n$.

13.9 Geometric R From Trace Reductions of Birational 3D R

We have constructed solutions to the Yang–Baxter equation by the trace reduction of the compositions of the 3D R. They were identified with the quantum R matrices for specific representations of $U_q(A^{(1)}_{n-1})$. Here we present a parallel story for the birational 3D R in Sect. 3.6.2 without going into the detailed proof.

Let us write the birational 3D R $R_{\text{birational}}$ in (3.151) simply as

$$R : (a, b, c) \mapsto \left(\frac{ab}{a+c},\ a+c,\ \frac{bc}{a+c}\right). \tag{13.142}$$

Given arrays of n variables $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ and an extra single variable z_{n+1}, we construct $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$, $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_n)$ and $z_1, \dots, z_n$ by postulating the following relations successively in the order $i = n, n-1, \dots, 1$:

$$R : (x_i, y_i, z_{i+1}) \mapsto (\tilde{x}_i, \tilde{y}_i, z_i). \tag{13.143}$$

See Fig. 13.10.

By the construction, z_1 is expressed as

$$z_1 = \frac{z_{n+1}\prod_{j=1}^n y_j}{\prod_{j=1}^n x_j + z_{n+1}Q_0(x, y)} \tag{13.144}$$

in terms of $Q_0(x, y)$ which will be given in (13.146). Reflecting the “trace”, we impose the periodic boundary condition $z_1 = z_{n+1}$. This determines z_{n+1} hence every

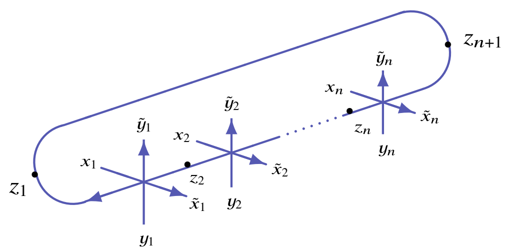

Fig. 13.10 Trace reduction of the birational 3D R along the third component. Each vertex is defined by (13.143) and (13.142). The periodic boundary condition $z_1 = z_{n+1}$ is imposed

z_i in terms of x and y. Explicitly, we get $z_i = (\prod_{k=1}^{n} y_k - \prod_{k=1}^{n} x_k)/Q_{i-1}(x, y)$. Substituting it back to $\tilde{x}$ and $\tilde{y}$, we obtain a map of $2n$ variables

$$\mathcal{R}^{(3)} : (x, y) \mapsto (\tilde{y}, \tilde{x}), \quad \tilde{x}_i = x_i \frac{Q_i(x, y)}{Q_{i-1}(x, y)}, \quad \tilde{y}_i = y_i \frac{Q_{i-1}(x, y)}{Q_i(x, y)}, \tag{13.145}$$

where the superscript (3) signifies that the third component is used for the trace reduction. The function $Q_i(x, y)$ is defined by

$$Q_i(x, y) = \sum_{k=1}^{n} \Big(\prod_{j=1}^{k-1} x_{i+j}\Big)\Big(\prod_{j=k+1}^{n} y_{i+j}\Big). \tag{13.146}$$

The indices of $Q_i, x_i, y_i, \tilde{x}_i, \tilde{y}_i$ are to be understood as belonging to $\mathbb{Z}_n$.

Example 13.14 For $n = 2, 3$, we have

$$n = 2: \quad Q_0(x, y) = x_2 + y_1, \quad Q_1(x, y) = x_1 + y_2, \tag{13.147}$$
$$n = 3: \quad Q_0(x, y) = x_1x_2 + x_1y_3 + y_2y_3, \tag{13.148}$$
$$Q_1(x, y) = x_2x_3 + x_2y_1 + y_1y_3, \tag{13.149}$$
$$Q_2(x, y) = x_1x_3 + x_3y_2 + y_1y_2. \tag{13.150}$$

One can construct similar maps $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ by replacing the elementary step (13.143) by

$$R : (z_{i+1}, x_i, y_i) \mapsto (z_i, \tilde{x}_i, \tilde{y}_i), \tag{13.151}$$
$$R : (x_i, z_{i+1}, y_i) \mapsto (\tilde{x}_i, z_i, \tilde{y}_i), \tag{13.152}$$

respectively, and applying them still in the order $i = n, n-1, \ldots, 1$. For (13.151), z_1 is given by (13.144) with the interchange $x \leftrightarrow y$ reflecting the symmetry (3.59) of the birational 3D R (13.142). Thus we have

$$\mathcal{R}^{(1)} : (x, y) \mapsto (\tilde{y}, \tilde{x}); \quad \tilde{x}_i = x_i \frac{Q_{i-1}(y, x)}{Q_i(y, x)}, \quad \tilde{y}_i = y_i \frac{Q_i(y, x)}{Q_{i-1}(y, x)}. \tag{13.153}$$

For (13.152), z_i becomes much simpler as $z_i = x_i + y_i$, leading to

$$\mathcal{R}^{(2)} : (x, y) \mapsto (\tilde{y}, \tilde{x}); \quad \tilde{x}_i = x_i \frac{x_{i+1} + y_{i+1}}{x_i + y_i}, \quad \tilde{y}_i = y_i \frac{x_{i+1} + y_{i+1}}{x_i + y_i}. \tag{13.154}$$

We also introduce

$$\mathcal{R}^{\vee(2)} : (x, y) \mapsto (\tilde{y}, \tilde{x}); \quad \tilde{x}_i = x_i \frac{x_{i-1} + y_{i-1}}{x_i + y_i}, \quad \tilde{y}_i = y_i \frac{x_{i-1} + y_{i-1}}{x_i + y_i}. \tag{13.155}$$

It is obtained by the reverse procedure for $\mathcal{R}^{(2)}$ where $R : (x_i, z_i, y_i) \mapsto (\tilde{x}_i, z_{i+1}, \tilde{y}_i)$ is applied in the order $i = 1, 2, \dots, n$ followed by $z_{n+1} = z_1$. It is related to $\mathcal{R}^{(2)}$ as

$$\mathcal{R}^{(2)} : (x^\vee, y^\vee) \mapsto (u, v) \quad \Leftrightarrow \quad \mathcal{R}^{\vee(2)} : (x, y) \to (u^\vee, v^\vee), \tag{13.156}$$

where $\vee$ denotes the reverse ordering of the n component arrays as in (11.4).

The maps $\mathcal{R}^{(1)}, \mathcal{R}^{(2)}, \mathcal{R}^{\vee(2)}$ and $\mathcal{R}^{(3)}$ are examples of *geometric R* of type A.[12] They satisfy the inversion relations and the Yang–Baxter equations. To describe them uniformly, we introduce a temporary notation

$$\mathcal{R}^{3,3} = \mathcal{R}^{(3)}, \quad \mathcal{R}^{1,3} = \mathcal{R}^{(2)}, \quad \mathcal{R}^{3,1} = \mathcal{R}^{\vee(2)}, \quad \mathcal{R}^{1,1} = \mathcal{R}^{(1)}. \tag{13.157}$$

Then the inversion relations read as

$$\mathcal{R}^{\alpha,\beta}\mathcal{R}^{\beta,\alpha} = \mathrm{id} \tag{13.158}$$

for $\alpha, \beta \in \{1, 3\}$. Thus these geometric R's are birational maps. They form set-theoretical solutions to the eight types of the Yang–Baxter equations

$$\big(1 \otimes \mathcal{R}^{\alpha,\beta}\big)\big(\mathcal{R}^{\alpha,\gamma} \otimes 1\big)\big(1 \otimes \mathcal{R}^{\beta,\gamma}\big) = \big(\mathcal{R}^{\beta,\gamma} \otimes 1\big)\big(1 \otimes \mathcal{R}^{\alpha,\gamma}\big)\big(\mathcal{R}^{\alpha,\beta} \otimes 1\big) \tag{13.159}$$

labeled with $\alpha, \beta, \gamma \in \{1, 3\}$. Here for instance $\big(1 \otimes \mathcal{R}^{\alpha,\beta}\big)(u, x, y) = (u, \tilde{y}, \tilde{x})$ and $\big(\mathcal{R}^{\alpha,\beta} \otimes 1\big)(x, y, u) = (\tilde{y}, \tilde{x}, u)$ in terms of the $\tilde{x}$ and $\tilde{y}$ corresponding to $\mathcal{R}^{\alpha,\beta}$ given by (13.145), (13.153), (13.154) or (13.155). One can bilinearize $\mathcal{R}^{\alpha,\beta}$ in terms of tau functions by incorporating the result in Sect. 3.6.3 into the trace reduction here.

Remark 13.15 The trace reduction considered here admits a two-parameter deformation leading to $\mathcal{R}^{\alpha,\beta}(\lambda, \omega)$. The parameter λ is introduced by replacing the birational 3D R (13.142) with the λ-deformed one in (3.159). The parameter ω is introduced by replacing the periodicity $z_1 = z_{n+1}$ of the auxiliary variable by the *quasi*-periodicity condition $z_1 = \omega z_{n+1}$. Then the inversion relation $\mathcal{R}^{\alpha,\beta}(\lambda, \omega)\mathcal{R}^{\beta,\alpha}(\lambda, \omega) = \mathrm{id}$ persists for any λ and ω. The Yang–Baxter equations remain valid for $\mathcal{R}^{\alpha,\beta}(\lambda, 1)$ and $\mathcal{R}^{\alpha,\beta}(0, \omega)$.

13.10 Bibliographical Notes and Comments

The trace reduction of the 3D R with respect to the first component was considered in [18, Eq. (36)], and the identification with the type A quantum R matrices for symmetric tensor representations was announced in [18, Eq. (54)]. See also [75]. A proof of a similar identification concerning the third component was given in

[12] Some early publications refer to them as "tropical R".

[96, Proposition 17]. This chapter provides a unified treatment of the trace reductions along the three possible directions. They are symbolically expressed, for $n = 3$, as

$$\mathrm{Tr}_{\bullet}\Bigl(z^{\mathbf{h}_{\bullet}} R_{\bullet\circ\circ} R_{\bullet\circ\circ} R_{\bullet\circ\circ}\Bigr), \quad \mathrm{Tr}_{\bullet}\Bigl(z^{\mathbf{h}_{\bullet}} R_{\circ\bullet\circ} R_{\circ\bullet\circ} R_{\circ\bullet\circ}\Bigr). \quad \mathrm{Tr}_{\bullet}\Bigl(z^{\mathbf{h}_{\bullet}} R_{\circ\circ\bullet} R_{\circ\circ\bullet} R_{\circ\circ\bullet}\Bigr).$$

Other variations mixing the components like $\mathrm{Tr}_{\bullet}\Bigl(z^{\mathbf{h}_{\bullet}} R_{\bullet\circ\circ} R_{\circ\circ\bullet} R_{\bullet\circ\circ}\Bigr)$ also yield solutions to the Yang–Baxter equation. Their quantum group symmetry has been described in [86] using the appropriate automorphisms of q-oscillator algebra interchanging the creation and the annihilation operators.

Even if the auxiliary Fock space $\bullet$ to take the trace is limited to the third component, there are more significant generalizations mixing the 3D R and 3D L as

$$\mathrm{Tr}\bigl(z^{\mathbf{h}}\mathcal{R}^{(\epsilon_1)}\cdots\mathcal{R}^{(\epsilon_n)}\bigr), \qquad (\mathcal{R}^{(0)} = R,\ \ \mathcal{R}^{(1)} = L) \tag{13.160}$$

for $\epsilon_1, \ldots, \epsilon_n = 0, 1$. These 2^n objects are easily seen to satisfy the Yang–Baxter equation by a mixed usage of the tetrahedron equations of type $RRRR = RRRR$ and $RLLL = LLLR$ [95, Theorem 12]. Chapter 11 and the present one correspond to the two special cases without the coexistence of the 3D L and 3D R. In order to characterize them as the intertwiner, one is naturally led to an algebra $\mathcal{U}_A(\epsilon_1, \ldots, \epsilon_n)$ interpolating $\mathcal{U}_A(0, \ldots, 0) = U_{-q^{-1}}(A^{(1)}_{n-1})$ in Theorem 11.3 and $\mathcal{U}_A(1, \ldots, 1) = U_q(A^{(1)}_{n-1})$ in Theorem 13.10 via some quantum superalgebras in between [98]. The algebra $\mathcal{U}_A(\epsilon_1, \ldots, \epsilon_n)$ has been identified as an example of *generalized quantum groups*. This notion emerged in [56] through the classification of pointed Hopf algebras [2, 55] and it has been studied further in [3, 6, 9, 57]. For recent developments related to the content of this book, see [108, 109].

The algebra homomorphism from U_q to q-oscillators as in Proposition 13.8 goes back to [54] for example. The proof of Theorem 13.10 utilizing such a homomorphism is simpler and is due to [97].

The explicit formula $A(z)^{\mathbf{ab}}_{\mathbf{ij}}$ in Theorem 13.3 was presented in [26]. Unfortunately the derivation therein has a gap when $|\mathbf{i}| > |\mathbf{i}'|$ in [26, Eq. (3.15)]. Section 13.5.3 provides the first complete proof of (13.55). It fills the gap effectively by Lemma 13.7, and provides a new insight that the quantum group symmetry is translated into a bilinear identity of q-hypergeometric as in Lemma 13.5.

Section 13.7 is based on [87], where the building block Φ_q (13.49) of the R matrices was extracted which plays the role of local hopping rate of an integrable Markov process of multispecies particles subject to a particular zero-range-type interaction. The case $n = 2$ of Φ_q first appeared in [123]. See also [25, 81, 100] for the subsequent developments.

The 3D lattice model in Sect. 13.8 has been considered in [17]. The layer transfer matrix corresponds to a quantization of the earlier work [68], where the 3D R is replaced by the birational 3D R and the description in terms of geometric R was adopted in accordance with Sect. 13.9. In such a setting, the duality shows up as the $W(A^{(1)}_{m-1} \times A^{(1)}_{n-1})$ symmetry.

One of the earliest appearances of the birational map $\mathcal{R}^{(1)}$ is [150]. The maps $\mathcal{R}^{(3)}$, $\mathcal{R}^{(2)}$, $\mathcal{R}^{\vee(2)}$ and $\mathcal{R}^{(1)}$ in (13.145)–(13.155) are the geometric lifts of R, $R^{\vee}$, ${}^{\vee}R$ and $R^{\vee\vee}$ in [101, Eqs. (2.1)–(2.4)], respectively. $\mathcal{R}^{(3)}$, $\mathcal{R}^{(2)}$ and $\mathcal{R}^{(1)}$ are also contained in the first example of set-theoretical solutions to the reflection equation [101, Appendix A]. Associated with the type A Kirillov–Reshetikhin (KR) module $W_s^{(r)}$ with $1 \leq r \leq n-1$, $s \geq 1$, one has the geometric crystal $\mathcal{B}^{(r)}$. The most general geometric R $R^{r,r'} : \mathcal{B}^{(r)} \times \mathcal{B}^{(r')} \to \mathcal{B}^{(r')} \times \mathcal{B}^{(r)}$ has been constructed in [49]. See also [99]. The four examples in Sect. 13.9 are the special cases of it as $\mathcal{R}^{3,3} = R^{1,1}, \mathcal{R}^{3,1} = R^{1,n-1}, \mathcal{R}^{1,3} = R^{n-1,1}, \mathcal{R}^{1,1} = R^{n-1,n-1}$. Set-theoretical solutions to the Yang–Baxter equation are also called Yang–Baxter maps [145]. Geometric R's form an important class in it having the quantum and combinatorial counterparts which are connected to the KR modules and integrable soliton cellular automata known as (generalized) box–ball systems [60].