Chapter 12
Boundary Vector Reductions of $RLLL = LLLR$

Abstract This chapter presents yet another reduction of an n-concatenation of the tetrahedron equation $RLLL = LLLR$ different from the previous chapter. We eliminate the 3D R not by taking the trace but by evaluation with respect to the boundary vectors using Proposition 3.28. We call it the boundary vector reduction. In contrast to the trace reduction that led to the quantum R matrices of $U_{\pm q^{-1}}(A^{(1)}_{n-1})$ (Chap. 11), it leads to the quantum R matrices for the spin representations of $U_{-q^{-1}}(B^{(1)}_n)$, $U_{-q^{-1}}(D^{(1)}_n)$ and $U_{-q^{-1}}(D^{(2)}_{n+1})$. These algebras have Dynkin diagrams with double outward arrows or double branches. It turns out that the two kinds of the boundary vectors correspond to the two choices of the end shape of the relevant Dynkin diagrams. For simplicity, we treat the reduction with respect to the q-oscillator Fock space only.

12.1 Boundary Vector Reductions

We retain the notations $\mathfrak{s}$, $\mathfrak{s}_\pm$, V, $\mathbf{V}$, $\mathbf{V}_\pm$, $v_{\mathbf{a}}$ etc. and 3D L in Sect. 11.1:

$$\begin{pmatrix} L^{00}_{00} & L^{00}_{01} & L^{00}_{10} & L^{00}_{11} \\ L^{01}_{00} & L^{01}_{01} & L^{01}_{10} & L^{01}_{11} \\ L^{10}_{00} & L^{10}_{01} & L^{10}_{10} & L^{10}_{11} \\ L^{11}_{00} & L^{11}_{01} & L^{11}_{10} & L^{11}_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -q\alpha^{-1}\mathbf{k} & \mathbf{a}^- & 0 \\ 0 & \mathbf{a}^+ & \alpha\mathbf{k} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{12.1}$$

In (11.21) we have obtained an n-concatenation of the tetrahedron equation $RLLL = LLLR$ as

$$\begin{aligned} &R_{456}\, x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5} y^{\mathbf{h}_6}(L_{1_1 2_1 4}\cdots L_{1_n 2_n 4})(L_{1_1 3_1 5}\cdots L_{1_n 3_n 5})(L_{2_1 3_1 6}\cdots L_{2_n 3_n 6})R_{456} \\ &\quad = x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5} y^{\mathbf{h}_6}(L_{2_1 3_1 6}\cdots L_{2_n 3_n 6})(L_{1_1 3_1 5}\cdots L_{1_n 3_n 5})(L_{1_1 2_1 4}\cdots L_{1_n 2_n 4}). \end{aligned} \tag{12.2}$$

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Recall the boundary vectors (3.132) and (3.133) given as[1]

$$\langle \eta_r| = \sum_{m \geq 0} \frac{\langle rm|}{(q^{r^2})_m}, \qquad |\eta_r\rangle = \sum_{m \geq 0} \frac{|rm\rangle}{(q^{r^2})_m} \qquad (r = 1, 2). \tag{12.3}$$

Sandwich (12.2) between them as

$$(\overset{4}{\langle \eta_r|} \otimes \overset{5}{\langle \eta_r|} \otimes \overset{6}{\langle \eta_r|})(\cdots)(\overset{4}{|\eta_{r'}\rangle} \otimes \overset{5}{|\eta_{r'}\rangle} \otimes \overset{6}{|\eta_{r'}\rangle}) \qquad (r, r' = 1, 2). \tag{12.4}$$

Thanks to Proposition 3.28, the two R_{456}'s disappear, leading to

$$\begin{aligned} &\langle \overset{4}{\eta_r}|x^{\mathbf{h}_4} L_{1_1 2_1 4} \cdots L_{1_n 2_n 4}|\overset{4}{\eta_{r'}}\rangle \langle \overset{5}{\eta_r}|(xy)^{\mathbf{h}_5} L_{1_1 3_1 5} \cdots L_{1_n 3_n 5}|\overset{5}{\eta_{r'}}\rangle \times \\ &\quad \times \langle \overset{6}{\eta_r}|y^{\mathbf{h}_6} L_{2_1 3_1 6} \cdots L_{2_n 3_n 6}|\overset{6}{\eta_{r'}}\rangle \\ &= \langle \overset{6}{\eta_r}|y^{\mathbf{h}_6} L_{2_1 3_1 6} \cdots L_{2_n 3_n 6}|\overset{6}{\eta_{r'}}\rangle \langle \overset{5}{\eta_r}|(xy)^{\mathbf{h}_5} L_{1_1 3_1 5} \cdots L_{1_n 3_n 5}|\overset{5}{\eta_{r'}}\rangle \times \\ &\quad \times \langle \overset{4}{\eta_r}|x^{\mathbf{h}_4} L_{1_1 2_1 4} \cdots L_{1_n 2_n 4}|\overset{4}{\eta_{r'}}\rangle. \end{aligned} \tag{12.5}$$

Let us denote the operators appearing here by

$$\begin{aligned} S^{r,r'}_{\mathbf{1},\mathbf{2}}(z) &= \varrho^{r,r'}(z) \langle \overset{4}{\eta_r}|z^{\mathbf{h}_4} L_{1_1 2_1 4} \cdots L_{1_n 2_n 4}|\overset{4}{\eta_{r'}}\rangle \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}} \otimes \overset{\mathbf{2}}{\mathbf{V}}), \\ S^{r,r'}_{\mathbf{1},\mathbf{3}}(z) &= \varrho^{r,r'}(z) \langle \overset{5}{\eta_r}|z^{\mathbf{h}_5} L_{1_1 3_1 5} \cdots L_{1_n 3_n 5}|\overset{5}{\eta_{r'}}\rangle \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}} \otimes \overset{\mathbf{3}}{\mathbf{V}}), \\ S^{r,r'}_{\mathbf{2},\mathbf{3}}(z) &= \varrho^{r,r'}(z) \langle \overset{6}{\eta_r}|z^{\mathbf{h}_6} L_{2_1 3_1 6} \cdots L_{2_n 3_n 6}|\overset{6}{\eta_{r'}}\rangle \in \mathrm{End}(\overset{\mathbf{2}}{\mathbf{V}} \otimes \overset{\mathbf{3}}{\mathbf{V}}), \end{aligned} \tag{12.6}$$

where $r, r' = 1, 2$. The normalization factor $\varrho^{r,r'}(z)$ will be specified in (12.15). They are the same operators acting on different copies of $\overset{\mathbf{1}}{\mathbf{V}}, \overset{\mathbf{2}}{\mathbf{V}}, \overset{\mathbf{3}}{\mathbf{V}}$ of $\mathbf{V}$ in (11.5) and (11.6). Now the relation (12.5) is stated as the Yang–Baxter equation:

$$S^{r,r'}_{\mathbf{1},\mathbf{2}}(x) S^{r,r'}_{\mathbf{1},\mathbf{3}}(xy) S^{r,r'}_{\mathbf{2},\mathbf{3}}(y) = S^{r,r'}_{\mathbf{2},\mathbf{3}}(y) S^{r,r'}_{\mathbf{1},\mathbf{3}}(xy) S^{r,r'}_{\mathbf{1},\mathbf{2}}(x) \quad (r, r' = 1, 2). \tag{12.7}$$

Suppressing the labels $\mathbf{1}, \mathbf{2}$ etc., we set

$$S^{r,r'}(z)(v_{\mathbf{i}} \otimes v_{\mathbf{j}}) = \sum_{\mathbf{a},\mathbf{b} \in \mathfrak{s}} S^{r,r'}(z)^{\mathbf{ab}}_{\mathbf{ij}} v_{\mathbf{a}} \otimes v_{\mathbf{b}}. \tag{12.8}$$

[1] There is no decent meaning of r^2 in this fitting formula which makes sense only for $r = 1, 2$.

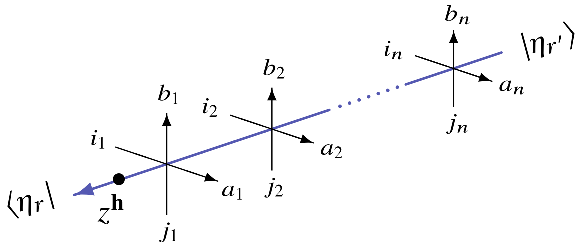

Fig. 12.1 The boundary vector reduction. The matrix product formula (12.9) is depicted as a concatenation of Fig. 11.1 along the blue arrow carrying $\mathcal{F}_q$ sandwiched by the boundary vectors $\langle\eta_r|$ and $|\eta_{r'}\rangle$ in (12.3). It is a BBQ stick with X-shaped sausages and extra caps at the two ends. The dual pairing is defined by (3.16)

Then the construction (12.6) implies the matrix product formula

$$S^{r,r'}(z)_{\mathbf{ij}}^{\mathbf{ab}} = \varrho^{r,r'}(z)\langle\eta_r|z^{\mathbf{h}} L_{i_1 j_1}^{a_1 b_1}\cdots L_{i_n j_n}^{a_n b_n}|\eta_{r'}\rangle \quad (r, r' = 1, 2) \tag{12.9}$$

in terms of the components of the 3D L in (12.1) (Fig. 12.1).

From the q-oscillator relations (3.12) and the dual pairing rule (3.16), calculation of the quantities $\langle\eta_r|(\cdots)|\eta_{r'}\rangle$ is reduced to the following:

$$\begin{aligned}
\langle\eta_r|z^{\mathbf{h}}(\mathbf{a}^{\pm})^j\mathbf{k}^m w^{\mathbf{h}}|\eta_{r'}\rangle &= \langle\eta_{r'}|w^{\mathbf{h}}\mathbf{k}^m(\mathbf{a}^{\mp})^j z^{\mathbf{h}}|\eta_r\rangle \quad (r, r' = 1, 2),\\
\langle\eta_1|z^{\mathbf{h}}(\mathbf{a}^{+})^j\mathbf{k}^m w^{\mathbf{h}}|\eta_1\rangle &= z^j(-q;q)_j\frac{(-q^{j+m+1}zw;q)_\infty}{(q^m zw;q)_\infty},\\
\langle\eta_1|z^{\mathbf{h}}(\mathbf{a}^{-})^j\mathbf{k}^m w^{\mathbf{h}}|\eta_2\rangle &= z^{-j}\sum_{i=0}^{j}(-1)^i q^{\frac{1}{2}i(i+1-2j)}\frac{(q)_j(-q^{2i+2m+1}z^2w^2;q^2)_\infty}{(q)_i(q)_{j-i}(q^{2i+2m}z^2w^2;q^2)_\infty},\\
\langle\eta_1|z^{\mathbf{h}}(\mathbf{a}^{+})^j\mathbf{k}^m w^{\mathbf{h}}|\eta_2\rangle &= z^j\sum_{i=0}^{j} q^{\frac{1}{2}i(i+1)}\frac{(q)_j(-q^{2i+2m+1}z^2w^2;q^2)_\infty}{(q)_i(q)_{j-i}(q^{2i+2m}z^2w^2;q^2)_\infty},\\
\langle\eta_2|z^{\mathbf{h}}(\mathbf{a}^{+})^j\mathbf{k}^m w^{\mathbf{h}}|\eta_2\rangle &= \theta(j\in 2\mathbb{Z})\,z^j(q^2;q^4)_{j/2}\frac{(q^{2j+2m+2}z^2w^2;q^4)_\infty}{(q^{2m}z^2w^2;q^4)_\infty}.
\end{aligned} \tag{12.10}$$

See (3.65) for the notation. The symbol θ is defined after (6.66). These formulas are easily derived from the elementary identity (3.82). From (12.9) we see

$$S^{r,r'}(z)_{\mathbf{ij}}^{\mathbf{ab}} = \alpha^{|\mathbf{a}|-|\mathbf{j}|}\left(S^{r,r'}(z)_{\mathbf{ij}}^{\mathbf{ab}}|_{\alpha=1}\right), \tag{12.11}$$

$$S^{r,r'}(z)_{\mathbf{ij}}^{\mathbf{ab}} = 0 \ \text{ unless } \ \mathbf{a}+\mathbf{b} = \mathbf{i}+\mathbf{j}, \tag{12.12}$$

$$S^{2,2}(z)_{\mathbf{ij}}^{\mathbf{ab}} = 0 \ \text{ unless } \ |\mathbf{a}| \equiv |\mathbf{i}| \text{ and } |\mathbf{b}| \equiv |\mathbf{j}| \mod 2. \tag{12.13}$$

The α-dependence (12.11) is a direct consequence of (12.1), the weight conservation (12.12) follows from (11.15) and the parity constraint (12.13) is due to the fact that the boundary vectors $\langle\eta_2|, |\eta_2\rangle$ in (12.3) contain "even modes" only. It leads to the decomposition

$$S^{2,2}(z) = \bigoplus_{\sigma,\sigma'=+,-} S^{\sigma,\sigma'}(z), \qquad S^{\sigma,\sigma'}(z) \in \mathrm{End}(\mathbf{V}_\sigma \otimes \mathbf{V}_{\sigma'}). \tag{12.14}$$

When $(r, r') = (2, 2)$, the Yang–Baxter equation (12.7) is valid in each subspace $\mathbf{V}_\sigma \otimes \mathbf{V}_{\sigma'} \otimes \mathbf{V}_{\sigma''}$ of $\mathbf{V} \otimes \mathbf{V} \otimes \mathbf{V}$. The scalar $\varrho^{2,2}(z)$ in (12.9) may be chosen as $\varrho^{\sigma,\sigma'}(z)$ depending on the summands in (12.14). We take them as

$$\begin{aligned} \varrho^{r,r'}(z) &= \frac{(z^{\max(r,r')}; q^{rr'})_\infty}{(-z^{\max(r,r')}q; q^{rr'})_\infty} \quad ((r, r') = (1, 1), (1, 2), (2, 1)), \\ \varrho^{\pm,\pm}(z) &= \frac{(z^2; q^4)_\infty}{(z^2q^2; q^4)_\infty}, \qquad \varrho^{\pm,\mp}(z) = \frac{(z^2q^2; q^4)_\infty}{(z^2q^4; q^4)_\infty}. \end{aligned} \tag{12.15}$$

Then $S^{r,r'}(z)^{\mathbf{ab}}_{\mathbf{ij}}$ becomes a rational function of q and z^r normalized as

$$S(z)(v_{\mathbf{a}} \otimes v_{\mathbf{a}}) = v_{\mathbf{a}} \otimes v_{\mathbf{a}} \quad (\mathbf{a} \in \mathfrak{s},\ S = S^{1,1}, S^{1,2}, S^{2,1}, S^{+,+}), \tag{12.16}$$

$$S^{-,-}(z)(v_{\mathbf{e}_1} \otimes v_{\mathbf{e}_1}) = v_{\mathbf{e}_1} \otimes v_{\mathbf{e}_1}, \tag{12.17}$$

$$S^{+,-}(z)(v_{\mathbf{0}} \otimes v_{\mathbf{e}_1}) = -q\alpha^{-1} v_{\mathbf{0}} \otimes v_{\mathbf{e}_1}, \quad S^{-,+}(z)(v_{\mathbf{e}_1} \otimes v_{\mathbf{0}}) = \alpha v_{\mathbf{e}_1} \otimes v_{\mathbf{0}}. \tag{12.18}$$

From (3.18), (11.15) and (11.16), we also have

$$S^{r,r'}(z)^{\mathbf{ab}}_{\mathbf{ij}} = z^{|\mathbf{j}|-|\mathbf{b}|} S^{r',r}(z)^{\mathbf{i}^\vee \mathbf{j}^\vee}_{\mathbf{a}^\vee \mathbf{b}^\vee} = S^{r,r'}(z)^{\mathbf{j}\,\mathbf{i}}_{\mathbf{b}\,\mathbf{a}}|_{\alpha \to -q\alpha^{-1}}. \tag{12.19}$$

Example 12.1 We consider the simplest case $n = 1$. $S^{r,r'}(z)$ with $(r, r') = (1, 1)$, $(1, 2)$, $(2, 1)$ are given as follows:

$$\begin{aligned} &v_i \otimes v_i \mapsto v_i \otimes v_i \quad (i = 0, 1), \\ &v_0 \otimes v_1 \mapsto -\frac{q(1 - z^s)v_0 \otimes v_1}{\alpha(1 + qz^s)} + \frac{(1 + q)z^r v_1 \otimes v_0}{1 + qz^s}, \\ &v_1 \otimes v_0 \mapsto \frac{(1 + q)z^{r'-1} v_0 \otimes v_1}{1 + qz^s} + \frac{\alpha(1 - z^s)v_1 \otimes v_0}{1 + qz^s}, \end{aligned}$$

where $s = \max(r, r')$. $S^{2,2}(z)$ with $n = 1$ reads as

$$v_i \otimes v_i \mapsto v_i \otimes v_i \ \ (i = 0, 1), \quad v_0 \otimes v_1 \mapsto -q\alpha^{-1} v_0 \otimes v_1, \quad v_1 \otimes v_0 \mapsto \alpha v_1 \otimes v_0.$$

Examples of the case $n = 2$ are available in Sect. 12.4.

12.2 Identification with Quantum R Matrices of $B_n^{(1)}$, $D_n^{(1)}$, $D_{n+1}^{(2)}$

12.2.1 Quantum Affine Algebra $U_p(\mathfrak{g}^{r,r'})$

We will be concerned with the affine Kac–Moody algebras[2]

$$\mathfrak{g}^{1,1} = D_{n+1}^{(2)}, \quad \mathfrak{g}^{2,1} = B_n^{(1)}, \quad \mathfrak{g}^{1,2} = \tilde{B}_n^{(1)}, \quad \mathfrak{g}^{2,2} = D_n^{(1)}, \tag{12.20}$$

where the notation $\mathfrak{g}^{r,r'}$ will turn out to fit to $S^{r,r'}(z)$ in the previous section.

Let $U_p = U_p(D_{n+1}^{(2)})$ $(n \geq 2), U_p(B_n^{(1)})$ $(n \geq 3), U_p(\tilde{B}_n^{(1)})$ $(n \geq 3), U_p(D_n^{(1)})$ $(n \geq 3)$ be the quantum affine algebras. They are Hopf algebras generated by $\{e_i, f_i, k_i^{\pm 1} \mid i \in \{0, 1, \ldots, n\}\}$ satisfying the relations (10.1) with q replaced by p (and the index set I there understood as $\{0, 1, \ldots, n\}$). Beside the commutativity of $k_i^{\pm 1}$ and the p-Serre relations, they include

$$k_i e_j k_i^{-1} = p_i^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = p_i^{-a_{ij}} f_j, \quad [e_i, f_j] = \delta_{i,j} \frac{k_i - k_i^{-1}}{p_i - p_i^{-1}}, \tag{12.21}$$

where the constants p_i $(0 \leq i \leq n)$ are taken as[3]

$$p_i = p \quad \text{except for} \quad p_0 = p^{r/2}, \ p_n = p^{r'/2}. \tag{12.22}$$

Thus the actual exceptions are $p_0 = p_n = p^{1/2}$ for $D_{n+1}^{(2)}$, $p_n = p^{1/2}$ for $B_n^{(1)}$ and $p_0 = p^{1/2}$ for $\tilde{B}_n^{(1)}$.

The affine Lie algebra $\tilde{B}_n^{(1)}$ is just $B_n^{(1)}$ with different enumeration of the vertices as shown in Fig. 12.2. We keep it for uniformity of the description. The Cartan matrix $(a_{ij})_{0 \leq i,j \leq n}$ is determined from the Dynkin diagrams of the relevant affine Lie algebras according to the convention of [67]. Thus for instance in $U_p(D_{n+1}^{(2)})$, one has $a_{01} = -2, a_{10} = -1$ and $k_0 e_0 = p e_0 k_0$, $k_0 e_1 = p^{-1} e_1 k_0$, $k_1 e_0 = p^{-1} e_0 k_1$ and $k_1 e_1 = p^2 e_1 k_1$. Forgetting the 0th node in the Dynkin diagrams yields the classical subalgebras $U_p(B_n) \subset U_p(D_{n+1}^{(2)})$, $U_p(B_n) \subset U_p(B_n^{(1)})$, $U_p(D_n) \subset U_p(\tilde{B}_n^{(1)})$ and $U_p(D_n) \subset U_p(D_n^{(1)})$.

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[2] Some symbols including $\mathfrak{g}^{r,r'}$ here and Sect. 14.2.1 are apparently the same, but they should be understood as redefined in each place.

[3] This normalization agrees with (14.19). The normalization mentioned after (10.1) for $U_q(\mathfrak{g})$ with non-affine $\mathfrak{g}$ has not been adopted here.

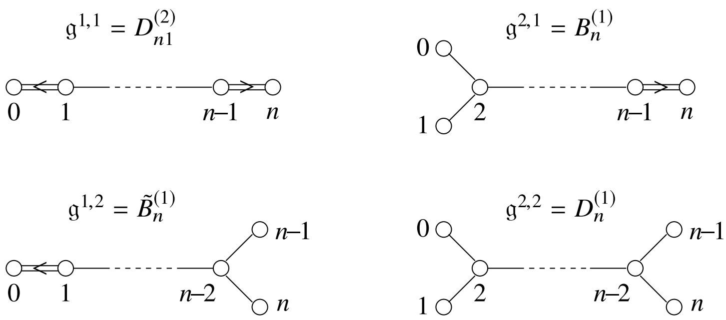

Fig. 12.2 Dynkin diagrams of (12.20) with enumeration of vertices

12.2.2 Spin Representations of $U_p(\mathfrak{g}^{r,r'})$

Let $\pi_{\varpi_n,x} : U_p(\mathfrak{g}^{r,r'}) \to \mathrm{End}(\mathbf{V})$ be the representations

$$e_0 v_{\mathbf{m}} = x v_{\mathbf{m}-\mathbf{e}_1}, \quad f_0 v_{\mathbf{m}} = x^{-1} v_{\mathbf{m}+\mathbf{e}_1}, \quad k_0 v_{\mathbf{m}} = p^{\frac{1}{2}-m_1} v_{\mathbf{m}} \quad (r = 1), \tag{12.23}$$

$$e_0 v_{\mathbf{m}} = x^2 v_{\mathbf{m}-\mathbf{e}_1-\mathbf{e}_2}, \quad f_0 v_{\mathbf{m}} = x^{-2} v_{\mathbf{m}+\mathbf{e}_1+\mathbf{e}_2}, \quad k_0 v_{\mathbf{m}} = p^{1-m_1-m_2} v_{\mathbf{m}} \quad (r = 2), \tag{12.24}$$

$$e_i v_{\mathbf{m}} = v_{\mathbf{m}+\mathbf{e}_i-\mathbf{e}_{i+1}}, \quad f_i v_{\mathbf{m}} = v_{\mathbf{m}-\mathbf{e}_i+\mathbf{e}_{i+1}}, \quad k_i v_{\mathbf{m}} = p^{m_i-m_{i+1}} v_{\mathbf{m}} \quad (0 < i < n), \tag{12.25}$$

$$e_n v_{\mathbf{m}} = v_{\mathbf{m}+\mathbf{e}_n}, \quad f_n v_{\mathbf{m}} = v_{\mathbf{m}-\mathbf{e}_n}, \quad k_n v_{\mathbf{m}} = p^{m_n-\frac{1}{2}} v_{\mathbf{m}} \quad (r' = 1), \tag{12.26}$$

$$e_n v_{\mathbf{m}} = v_{\mathbf{m}+\mathbf{e}_{n-1}+\mathbf{e}_n}, \quad f_n v_{\mathbf{m}} = v_{\mathbf{m}-\mathbf{e}_{n-1}-\mathbf{e}_n}, \quad k_n v_{\mathbf{m}} = p^{m_n+m_{n-1}-1} v_{\mathbf{m}} \quad (r' = 2), \tag{12.27}$$

where $\mathbf{m} \in \mathfrak{s}$. See Sect. 11.1 for the definitions of $\mathbf{V}$, $v_{\mathbf{m}}$, $\mathfrak{s}$ and $\mathbf{e}_i$. In the LHSs, e_i for example actually means $\pi_{\varpi_n,x}(e_i)$. In the RHSs, the vector $v_{\mathbf{m}}$ should be understood as 0 if $\mathbf{m} \notin \mathfrak{s}$. The choice of $x^{\pm r}$ rather than $x^{\pm 1}$ in (12.23) and (12.24) is the option leading to a uniform description of the results in Theorem 12.2.

The algebras $U_p(\mathfrak{g}^{1,1})$ and $U_p(\mathfrak{g}^{2,1})$ have a common classical subalgebra $U_p(B_n)$ without $e_0, f_0, k_0^{\pm}$. As a $U_p(B_n)$ module, $\mathbf{V}$ is already irreducible and is isomorphic to the highest weight module $V(\varpi_n)$ in the notation of Sect. 10.1.1 with highest weight vector $v_{\mathbf{e}_1+\cdots+\mathbf{e}_n}$. It is called the spin representation.[4]

The algebras $U_p(\mathfrak{g}^{1,2})$ and $U_p(\mathfrak{g}^{2,2})$ have a common classical subalgebra $U_p(D_n)$ without $e_0, f_0, k_0^{\pm}$. As a $U_p(D_n)$ module, $\mathbf{V}$ decomposes into two irreducible

[4] As a $U_p(\mathfrak{g}^{1,1})$ or $U_p(\mathfrak{g}^{2,1})$ module, it is a Kirillov–Reshetikhin module $W^{(n)}_1$ up to specification of the spectral parameter.

components $\mathbf{V}_+$ and $\mathbf{V}_-$ in (11.6). The space $\mathbf{V}_{(-1)^n}$ (resp. $\mathbf{V}_{(-1)^{n-1}}$) is isomorphic to the highest weight module $V(\varpi_n)$ (resp. $V(\varpi_{n-1})$) in the notation of Sect. 10.1.1 whose highest weight vector is $v_{\mathbf{e}_1+\cdots+\mathbf{e}_n}$ (resp. $v_{\mathbf{e}_1+\cdots+\mathbf{e}_{n-1}}$). Both $V(\varpi_n)$ and $V(\varpi_{n-1})$ are called spin representations. As a $U_p(\mathfrak{g}^{1,2})$ module, $\mathbf{V}$ is irreducible. As a $U_p(\mathfrak{g}^{2,2})$ module, each $\mathbf{V}_\pm$ remains individually irreducible since the parity of $|\mathbf{m}| = m_1 + \cdots + m_n$ in $v_\mathbf{m}$ is preserved.[5] We will simply refer to $\pi_{\varpi_n,x}$ as the spin representation of $U_p(\mathfrak{g}^{r,r'})$.

12.2.3 $S^{r,r'}(z)$ as Quantum R Matrices for Spin Representations

Let $\Delta_{x,y} = (\pi_{\varpi_n,x} \otimes \pi_{\varpi_n,y}) \circ \Delta$ and $\Delta^{\mathrm{op}}_{x,y} = (\pi_{\varpi_n,x} \otimes \pi_{\varpi_n,y}) \circ \Delta^{\mathrm{op}}$ be the tensor product of the spin representations, where Δ and Δ^{op} are the coproduct (11.58) and its opposite (11.59). For $(r, r') \neq (2, 2)$, let $\mathcal{R}^{r,r'}(z) \in \mathrm{End}(\mathbf{V} \otimes \mathbf{V})$ be a quantum R matrix of $U_p(\mathfrak{g}^{r,r'})$ which is characterized, up to normalization, by the commutativity

$$\mathcal{R}^{r,r'}(\tfrac{x}{y})\Delta_{x,y}(g) = \Delta^{\mathrm{op}}_{x,y}(g)\mathcal{R}^{r,r'}(\tfrac{x}{y}) \quad (\forall g \in U_p(\mathfrak{g}^{r,r'})). \tag{12.28}$$

For $(r, r') = (2, 2)$, we set

$$\mathcal{R}^{2,2}(z) = \mathcal{R}^{+,+}(z) \oplus \mathcal{R}^{+,-}(z) \oplus \mathcal{R}^{-,+}(z) \oplus \mathcal{R}^{-,-}(z) \in \mathrm{End}(\mathbf{V} \otimes \mathbf{V}), \tag{12.29}$$

where $\mathcal{R}^{\varepsilon,\varepsilon'}(z) \in \mathrm{End}(\mathbf{V}_\varepsilon \otimes \mathbf{V}_{\varepsilon'})$ $(\varepsilon, \varepsilon' = \pm)$ is a quantum R matrix of $U_p(\mathfrak{g}^{2,2})$ which is characterized, up to normalization, by the commutativity

$$\mathcal{R}^{\varepsilon,\varepsilon'}(\tfrac{x}{y})\Delta_{x,y}(g) = \Delta^{\mathrm{op}}_{x,y}(g)\mathcal{R}^{\varepsilon,\varepsilon'}(\tfrac{x}{y}) \quad (\forall g \in U_p(\mathfrak{g}^{2,2})). \tag{12.30}$$

We have taken the obvious fact that the R matrices depend only on the ratio x/y into account. The relations (12.28) and (12.30) are generalizations of (10.12) including the latter as the classical part. The main result in this chapter is the following.

Theorem 12.2 *Up to normalization, $S^{r,r'}(z)$ by the matrix product construction (12.8)–(12.9) based on the 3D L (12.1) with $\alpha = p^{-1/2}$ coincides with the quantum R matrix of $U_p(\mathfrak{g}^{r,r'})$ as*

$$S^{r,r'}(z) = \mathcal{R}^{r,r'}(z^{-1}) \quad \text{at } q = -p^{-1}. \tag{12.31}$$

[5] They are Kirillov–Reshetikhin modules $W_1^{(n)}$ and $W_1^{(n-1)}$ up to specification of the spectral parameters.

Proof It suffices to check

$$S^{r,r'}(\tfrac{y}{x})(e_s \otimes 1 + k_s \otimes e_s) = (1 \otimes e_s + e_s \otimes k_s)S^{r,r'}(\tfrac{y}{x}), \tag{12.32}$$

$$S^{r,r'}(\tfrac{y}{x})(1 \otimes f_s + f_s \otimes k_s^{-1}) = (f_s \otimes 1 + k_s^{-1} \otimes f_s)S^{r,r'}(\tfrac{y}{x}), \tag{12.33}$$

$$S^{r,r'}(\tfrac{y}{x})(k_s \otimes k_s) = (k_s \otimes k_s)S^{r,r'}(\tfrac{y}{x}) \tag{12.34}$$

under the image by $\pi_{\varpi_n,x} \otimes \pi_{\varpi_n,y}$ for $0 \le s \le n$. For $0 < s < n$, the formula (12.25) coincides with (11.60) for $i \neq 0$. Therefore it is indeed valid if $q = -p^{-1}$ thanks to the proof of Theorem 11.3. Let us illustrate the proof of (12.32) for $s = n$ using the properties of the boundary vectors. The other relations can be treated similarly.

First we consider the case $r' = 1$. Then up to the normalization factor $\varrho^{r,1}(\frac{y}{x})$, the vector $S^{r,1}(\frac{y}{x})(e_n \otimes 1 + k_n \otimes e_n)(v_{\mathbf{i}} \otimes v_{\mathbf{j}})$ is calculated by using (12.26) as

$$\begin{aligned} &\langle \eta_r | \left(\tfrac{y}{x}\right)^{\mathbf{h}} L_{1_1 2_1} \cdots L_{1_n 2_n} | \eta_1 \rangle (v_{\mathbf{i}+\mathbf{e}_n} \otimes v_{\mathbf{j}} + p^{i_n - \frac{1}{2}} v_{\mathbf{i}} \otimes v_{\mathbf{j}+\mathbf{e}_n}) \\ &= \sum_{\mathbf{a},\mathbf{b} \in \mathfrak{s}} \langle \eta_r | X (L^{a_n,b_n}_{i_n+1,j_n} + p^{i_n - \frac{1}{2}} L^{a_n,b_n}_{i_n,j_n+1}) | \eta_1 \rangle v_{\mathbf{a}} \otimes v_{\mathbf{b}}, \end{aligned} \tag{12.35}$$

where $X = \left(\frac{y}{x}\right)^{\mathbf{h}} L^{a_1,b_1}_{i_1,j_1} \cdots L^{a_{n-1},b_{n-1}}_{i_{n-1},j_{n-1}}$. Similarly, $(1 \otimes e_n + e_n \otimes k_n)S^{r,1}(\frac{y}{x})(v_{\mathbf{i}} \otimes v_{\mathbf{j}})$ yields

$$\begin{aligned} &(1 \otimes e_n + e_n \otimes k_n) \sum_{\mathbf{a},\mathbf{b} \in \mathfrak{s}} \langle \eta_r | X L^{a_n,b_n}_{i_n,j_n} | \eta_1 \rangle (v_{\mathbf{a}} \otimes v_{\mathbf{b}}) \\ &= \sum_{\mathbf{a},\mathbf{b} \in \mathfrak{s}} \langle \eta_r | X L^{a_n,b_n}_{i_n,j_n} | \eta_1 \rangle (v_{\mathbf{a}} \otimes v_{\mathbf{b}+\mathbf{e}_n} + p^{b_n - \frac{1}{2}} v_{\mathbf{a}+\mathbf{e}_n} \otimes v_{\mathbf{b}}). \end{aligned} \tag{12.36}$$

From the comparison of the coefficient of $v_{\mathbf{a}} \otimes v_{\mathbf{b}}$, it suffices to show

$$(L^{a,b}_{i+1,j} + p^{i-\frac{1}{2}} L^{a,b}_{i,j+1} - L^{a.b-1}_{i,j} - p^{b-\frac{1}{2}} L^{a-1,b}_{i,j}) | \eta_1 \rangle = 0, \tag{12.37}$$

where a_n, b_n, i_n, j_n are denoted by a, b, i, j. As an example for $(a, b, i, j) = (1, 1, 0, 1)$, it reads, from (11.14), as

$$0 = (L^{11}_{11} - L^{10}_{01} - p^{\frac{1}{2}} L^{01}_{01}) | \eta_1 \rangle = (1 - \mathbf{a}^+ - p^{\frac{1}{2}}(-q\alpha^{-1}\mathbf{k})) | \eta_1 \rangle. \tag{12.38}$$

From $q = -p^{-1}$ this is indeed valid at $\alpha = p^{-1/2}$ due to the property (3.134) of the boundary vector $|\eta_1\rangle$. With the choice $(q, \alpha) = (-p^{-1}, p^{-1/2})$, all the other relations in (12.37) can be similarly checked by also using (3.135).

Next we consider the case $r' = 2$. The main difference from the $r' = 1$ case is that (12.27) concerns the "two boundary sites" $n - 1$ and n. Thus a similar argument comparing the coefficients of $v_{\ldots,a_{n-1},a_n} \otimes v_{\ldots,b_{n-1},b_n}$ leads to a quadratic relation

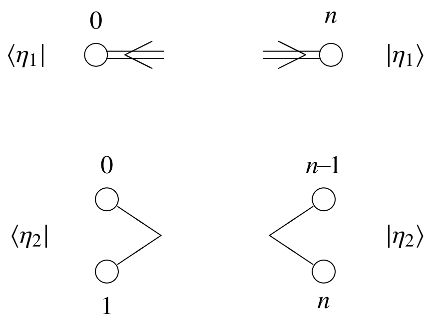

Fig. 12.3 Correspondence between the boundary vectors and the end shape of the Dynkin diagrams. These data are relevant to the LHS and the RHS of (12.31) in Theorem 12.2, respectively

$$\begin{aligned}(L^{a,b}_{i+1,j}L^{a',b'}_{i'+1,j'} + p^{i+i'-1}L^{a,b}_{i,j+1}L^{a',b'}_{i',j'+1}\\ - L^{a,b-1}_{i,j}L^{a',b'-1}_{i',j'} - p^{b+b'-1}L^{a-1,b}_{i,j}L^{a'-1,b'}_{i',j'})|\eta_2\rangle = 0.\end{aligned} \tag{12.39}$$

Consider the LHS for $(a,a',b,b',i,i',j,j') = (1,1,1,1,0,0,1,1)$ for example:

$$(L^{11}_{11}L^{11}_{11} - L^{10}_{01}L^{10}_{01} - pL^{01}_{01}L^{01}_{01})|\eta_2\rangle = (1-(\mathbf{a}^+)^2 - p(-q\alpha^{-1}\mathbf{k})^2)|\eta_2\rangle. \tag{12.40}$$

From $(q,\alpha) = (-p^{-1}, p^{-1/2})$, this is evaluated as

$$(1-(\mathbf{a}^+)^2 - \mathbf{k}^2)|\eta_2\rangle \overset{(3.137)}{=} (1-\mathbf{a}^+\mathbf{a}^- - \mathbf{k}^2)|\eta_2\rangle \overset{(3.12)}{=} 0. \tag{12.41}$$

All the other relations in (12.39) can be checked similarly by using (3.137) and (3.12). □

Remark 12.3 Theorem 12.2 suggests the following correspondence between the boundary vectors $\langle\eta_r|, |\eta_{r'}\rangle$ in (12.3) and the end shape of the Dynkin diagrams in Fig. 12.2: (Fig. 12.3)

A similar correspondence is observed in Remark 11.4 and 14.3.

12.3 Commuting Layer Transfer Matrix

This section is a continuation from Sect. 11.6 from which we will borrow some terminology. Given parameters $\mathbf{u} = (u_1,\dots,u_m)$ and $\mathbf{w} = (w_1,\dots,w_n)$, consider the row transfer matrix of the vertex model associated with the spin representation of $U_p(\mathfrak{g}^{r,r'})$:

$$T(x|\mathbf{u},\mathbf{w}) = \mathrm{Tr}_{\mathbf{1}}\big(\mathbf{w}^H S^{r,r'}_{\mathbf{1},\mathbf{2}_1}(xu_1)\cdots S^{r,r'}_{\mathbf{1},\mathbf{2}_m}(xu_m)\big) \in \mathrm{End}(\overset{\mathbf{2}_1}{\mathbf{V}}\otimes\cdots\otimes\overset{\mathbf{2}_m}{\mathbf{V}}). \tag{12.42}$$

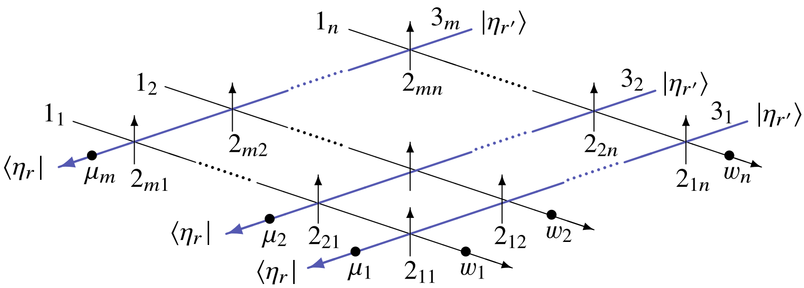

Fig. 12.4 A layer transfer matrix interpretation of the row transfer matrix of the $U_p(\mathfrak{g}^{r,r'})$ vertex model associated with the spin representation. There are n black horizontal arrows $1_1, \ldots, 1_n$ carrying $V \simeq \mathbb{C}^2$ which are being traced out corresponding to the periodic boundary condition. There are also m blue horizontal arrows $3_1, \ldots, 3_m$ carrying $\mathcal{F}_q$ which are to be evaluated between the boundary vectors. The mark • with z signifies an operator $z^{\mathbf{h}}$. At the intersection of 1_i and 3_j, there is a vertical black arrow 2_{ij} carrying V, which corresponds to a 3D L $L_{1_i,2_{ij},3_j}$. The parameter μ_i is taken as $\mu_i = xu_i$ as in (11.84)

To each $S^{r,r'}(xu_i)$, labels have been attached indicating the spaces it acts. The label $\mathbf{1} = (1_1, \ldots, 1_n)$ is the one for the auxiliary space $\overset{\mathbf{1}}{\mathbf{V}} = \overset{1_1}{V} \otimes \cdots \otimes \overset{1_n}{V}$ and $\mathbf{2}_j$ is the one for the jth component $\overset{\mathbf{2}_j}{\mathbf{V}} = \overset{2_{j1}}{V} \otimes \cdots \otimes \overset{2_{jn}}{V}$ in the quantum space $\overset{\mathbf{2}_1}{\mathbf{V}} \otimes \cdots \otimes \overset{\mathbf{2}_m}{\mathbf{V}}$. For the symbol $\mathbf{w}^H$, see (11.87). The parameters x, $\mathbf{u}$ and $\mathbf{w}$ are spectral parameters, their inhomogeneity and the boundary field. From the Yang–Baxter equation (12.7) and the weight conservation (12.12), it forms a commuting family:

$$[T(x|\mathbf{u}, \mathbf{w}), T(x'|\mathbf{u}, \mathbf{w})] = 0. \tag{12.43}$$

Theorem 12.2 endows $T(x|\mathbf{u}, \mathbf{w})$ with an interpretation as a layer transfer matrix of a 3D lattice model with a special boundary condition explained below.

The formula (12.42) corresponds to looking at Fig. 12.4 from the SW, or evaluating $\langle\eta_r|(\cdots)|\eta_{r'}\rangle$ first. On the other hand, one can look at it from the SE or first take the trace over $1_1, \ldots, 1_n$. From (11.41), it leads to an alternative interpretation:

$$T(x|\mathbf{u}, \mathbf{w}) = \langle\eta_r|^{\otimes m}(x\mathbf{u})^H S^{\mathrm{tr}_1}_{\mathbf{2}_1,\mathbf{3}}(w_1) \cdots S^{\mathrm{tr}_1}_{\mathbf{2}_n,\mathbf{3}}(w_n)|\eta_{r'}\rangle^{\otimes m} \in \mathrm{End}(V^{\otimes mn}). \tag{12.44}$$

Here $\mathbf{3} = (3_1, \ldots, 3_m)$ is the label of the auxiliary space $\overset{\mathbf{3}}{\mathbf{W}} = \overset{3_1}{\mathcal{F}}_q \otimes \cdots \otimes \overset{3_m}{\mathcal{F}}_q$ along which the product of S^{tr_1} is taken and $\langle\eta_r|^{\otimes m}(\cdots)|\eta_{r'}\rangle^{\otimes m}$ is evaluated. The label $\mathbf{2}_j = (2_{1j}, \ldots, 2_{mj})$ signifies the jth component $\overset{\mathbf{2}_j}{\mathbf{V}} = \overset{2_{1j}}{V} \otimes \cdots \overset{2_{mj}}{V}$ of the quantum space $\overset{\mathbf{2}_1}{\mathbf{V}} \otimes \cdots \otimes \overset{\mathbf{2}_n}{\mathbf{V}} \simeq V^{\otimes mn}$.

The operator (12.44) arises from the dual pairing between $\mathbf{W}^{(m)} = \mathcal{F}_q^{\otimes m} = \bigoplus_{k \geq 0} \mathbf{W}_k^{(m)}$ in (11.11) and its dual. From the weight conservation (11.45) and the decomposition (11.46), it is expanded as

$$T(x|\mathbf{u}, \mathbf{w}) = \bigoplus_{k\geq 0,\, \mathbf{l}=(l_1,\dots,l_n)\in\{0,1,\dots,m\}^n} x^k T_{k,\mathbf{l}}(\mathbf{u}, \mathbf{w}), \tag{12.45}$$

$$T_{k,\mathbf{l}}(\mathbf{u}, \mathbf{w}) = \langle \eta^m_{r,k}| \mathbf{u}^H S^{\mathrm{tr}_1}_{l_1,k}(w_1)\cdots S^{\mathrm{tr}_1}_{l_n,k}(w_n)|\eta^m_{r',k}\rangle \in \mathrm{End}(\mathbf{V}^{(m)}_{l_1}\otimes\cdots\otimes\mathbf{V}^{(m)}_{l_n}), \tag{12.46}$$

where the vector $|\eta^m_{r',k}\rangle$ is the projection of $|\eta_{r'}\rangle^{\otimes m}$ onto $\mathbf{W}^{(m)}_k$ in (11.11). The vector $\langle\eta^m_{r,k}|$ is the dual of $|\eta^m_{r,k}\rangle$. From (12.3) they are explicitly given as *finite* sums:

$$|\eta^m_{r',k}\rangle = \sum_{(d_1,\dots,d_m)\in B^{(m)}_k} \frac{|r'd_1\rangle\otimes\cdots\otimes|r'd_m\rangle}{(q^{r'^2})_{d_1}\cdots(q^{r'^2})_{d_m}}, \tag{12.47}$$

$$\langle\eta^m_{r,k}| = \sum_{(d_1,\dots,d_m)\in B^{(m)}_k} \frac{\langle rd_1|\otimes\cdots\otimes\langle rd_m|}{(q^{r^2})_{d_1}\cdots(q^{r^2})_{d_m}}. \tag{12.48}$$

See (11.10) for the notation $B^{(m)}_k$. Now the commutativity (12.43) implies

$$[T_{k,\mathbf{l}}(\mathbf{u}, \mathbf{w}), T_{k',\mathbf{l}}(\mathbf{u}, \mathbf{w})] = 0 \quad (k, k' \in \mathbb{Z}_{\geq 0}). \tag{12.49}$$

In 2D terminology, the 3D picture in Fig. 12.4 and Theorem 11.5, 12.2 show the equivalence of the spectral problem for row transfer matrices of the vertex models associated with the spin representations of $U_{-q^{-1}}(B^{(1)}_n)$, $U_{-q^{-1}}(D^{(1)}_n)$, $U_{-q^{-1}}(D^{(2)}_{n+1})$ on length m system with the periodic boundary condition and the $U_{q^{-1}}(A^{(1)}_{m-1})$ vertex model associated with the (anti-symmetric tensor rep.) $\otimes$ (symmetric tensor rep.) on a length n system with a special boundary condition.

12.4 Examples of $S^{1,1}(z)$, $S^{2,1}(z)$, $S^{2,2}(z)$ for $n = 2$

Let us present explicit formulas of $S^{r,r'}(z)$ for $n = 2$.

$S^{1,1}(z)$ is given as follows:

$$v_{ij}\otimes v_{ij} \mapsto v_{ij}\otimes v_{ij} \quad (i, j \in \{0, 1\}),$$

$$v_{00}\otimes v_{01} \mapsto \frac{q(z-1)v_{00}\otimes v_{01}}{\alpha(qz+1)} + \frac{(q+1)zv_{01}\otimes v_{00}}{qz+1},$$

$$v_{00}\otimes v_{10} \mapsto \frac{q(z-1)v_{00}\otimes v_{10}}{\alpha(qz+1)} + \frac{(q+1)zv_{10}\otimes v_{00}}{qz+1},$$

$$v_{00}\otimes v_{11} \mapsto \frac{(z-1)(qz-1)q^2 v_{00}\otimes v_{11}}{\alpha^2(qz+1)\left(zq^2+1\right)} + \frac{q^2(q+1)(z-1)zv_{01}\otimes v_{10}}{\alpha(qz+1)\left(zq^2+1\right)}$$
$$+ \frac{q(q+1)(z-1)zv_{10}\otimes v_{01}}{\alpha(qz+1)\left(zq^2+1\right)} + \frac{(q+1)\left(q^2+1\right)z^2 v_{11}\otimes v_{00}}{(qz+1)\left(zq^2+1\right)},$$

$$v_{01} \otimes v_{00} \mapsto \frac{(q+1)v_{00} \otimes v_{01}}{qz+1} - \frac{\alpha(z-1)v_{01} \otimes v_{00}}{qz+1},$$
$$v_{01} \otimes v_{10} \mapsto \frac{q(q+1)(z-1)v_{00} \otimes v_{11}}{\alpha(qz+1)\left(zq^2+1\right)} - \frac{q(z-1)(qz-1)v_{01} \otimes v_{10}}{(qz+1)\left(zq^2+1\right)}$$
$$+ \frac{(q+1)z\left(zq^2-zq+q+1\right)v_{10} \otimes v_{01}}{(qz+1)\left(zq^2+1\right)} - \frac{(q+1)\alpha(z-1)zv_{11} \otimes v_{00}}{(qz+1)\left(zq^2+1\right)},$$
$$v_{01} \otimes v_{11} \mapsto \frac{q(z-1)v_{01} \otimes v_{11}}{\alpha(qz+1)} + \frac{(q+1)zv_{11} \otimes v_{01}}{qz+1},$$
$$v_{10} \otimes v_{00} \mapsto \frac{(q+1)v_{00} \otimes v_{10}}{qz+1} - \frac{\alpha(z-1)v_{10} \otimes v_{00}}{qz+1},$$
$$v_{10} \otimes v_{01} \mapsto \frac{q^2(q+1)(z-1)v_{00} \otimes v_{11}}{\alpha(qz+1)\left(zq^2+1\right)} - \frac{q(z-1)(qz-1)v_{10} \otimes v_{01}}{(qz+1)\left(zq^2+1\right)}$$
$$- \frac{q(q+1)\alpha(z-1)zv_{11} \otimes v_{00}}{(qz+1)\left(zq^2+1\right)} + \frac{(q+1)\left(zq^2+zq-q+1\right)v_{01} \otimes v_{10}}{(qz+1)\left(zq^2+1\right)},$$
$$v_{10} \otimes v_{11} \mapsto \frac{q(z-1)v_{10} \otimes v_{11}}{\alpha(qz+1)} + \frac{(q+1)zv_{11} \otimes v_{10}}{qz+1},$$
$$v_{11} \otimes v_{00} \mapsto \frac{\alpha^2(z-1)(qz-1)v_{11} \otimes v_{00}}{(qz+1)\left(zq^2+1\right)} - \frac{q(q+1)\alpha(z-1)v_{01} \otimes v_{10}}{(qz+1)\left(zq^2+1\right)}$$
$$- \frac{(q+1)\alpha(z-1)v_{10} \otimes v_{01}}{(qz+1)\left(zq^2+1\right)} + \frac{(q+1)\left(q^2+1\right)v_{00} \otimes v_{11}}{(qz+1)\left(zq^2+1\right)},$$
$$v_{11} \otimes v_{01} \mapsto \frac{(q+1)v_{01} \otimes v_{11}}{qz+1} - \frac{\alpha(z-1)v_{11} \otimes v_{01}}{qz+1},$$
$$v_{11} \otimes v_{10} \mapsto \frac{(q+1)v_{10} \otimes v_{11}}{qz+1} - \frac{\alpha(z-1)v_{11} \otimes v_{10}}{qz+1}.$$

$S^{2,1}(z)$ depends on z only via z^2. It is given as follows:

$$v_{ij} \otimes v_{ij} \mapsto v_{ij} \otimes v_{ij} \quad (i, j \in \{0, 1\}),$$
$$v_{00} \otimes v_{01} \mapsto \frac{(q+1)z^2 v_{01} \otimes v_{00}}{qz^2+1} + \frac{q(z^2-1)v_{00} \otimes v_{01}}{\alpha\left(qz^2+1\right)},$$
$$v_{00} \otimes v_{10} \mapsto \frac{(q+1)z^2 v_{10} \otimes v_{00}}{qz^2+1} + \frac{q(z^2-1)v_{00} \otimes v_{10}}{\alpha\left(qz^2+1\right)},$$
$$v_{00} \otimes v_{11} \mapsto \frac{q^3(q+1)z^2(z^2-1)v_{01} \otimes v_{10}}{\alpha\left(qz^2+1\right)\left(z^2q^3+1\right)} + \frac{q^2(z^2-1)(q^2z^2-1)v_{00} \otimes v_{11}}{\alpha^2\left(qz^2+1\right)\left(z^2q^3+1\right)}$$
$$+ \frac{q^2(q+1)z^2(z^2-1)v_{10} \otimes v_{01}}{\alpha\left(qz^2+1\right)\left(z^2q^3+1\right)} + \frac{(q+1)z^2\left(z^2q^3+z^2q-q+1\right)v_{11} \otimes v_{00}}{\left(qz^2+1\right)\left(z^2q^3+1\right)},$$
$$v_{01} \otimes v_{00} \mapsto \frac{(q+1)v_{00} \otimes v_{01}}{qz^2+1} - \frac{\alpha(z^2-1)v_{01} \otimes v_{00}}{qz^2+1},$$

$$
\begin{aligned}
v_{01}\otimes v_{10} \mapsto\ & \frac{(q+1)z^2\left(z^2q^3-z^2q^2+q^2+1\right)v_{10}\otimes v_{01}}{\left(qz^2+1\right)\left(z^2q^3+1\right)} - \frac{q(q+1)\alpha z^2(z^2-1)v_{11}\otimes v_{00}}{\left(qz^2+1\right)\left(z^2q^3+1\right)}\\
&+ \frac{q(q+1)(z^2-1)v_{00}\otimes v_{11}}{\alpha\left(qz^2+1\right)\left(z^2q^3+1\right)} - \frac{q(z^2-1)(q^2z^2-1)v_{01}\otimes v_{10}}{\left(qz^2+1\right)\left(z^2q^3+1\right)},\\
v_{01}\otimes v_{11} \mapsto\ & \frac{(q+1)v_{11}\otimes v_{01}z^2}{qz^2+1} + \frac{q(z^2-1)v_{01}\otimes v_{11}}{\alpha\left(qz^2+1\right)},\\
v_{10}\otimes v_{00} \mapsto\ & \frac{(q+1)v_{00}\otimes v_{10}}{qz^2+1} - \frac{\alpha(z^2-1)v_{10}\otimes v_{00}}{qz^2+1},\\
v_{10}\otimes v_{01} \mapsto\ & \frac{q^2(q+1)(z^2-1)v_{00}\otimes v_{11}}{\alpha\left(qz^2+1\right)\left(z^2q^3+1\right)} - \frac{q^2(q+1)\alpha z^2(z^2-1)v_{11}\otimes v_{00}}{\left(qz^2+1\right)\left(z^2q^3+1\right)}\\
&- \frac{q(z^2-1)(q^2z^2-1)v_{10}\otimes v_{01}}{\left(qz^2+1\right)\left(z^2q^3+1\right)} + \frac{(q+1)\left(z^2q^3+z^2q-q+1\right)v_{01}\otimes v_{10}}{\left(qz^2+1\right)\left(z^2q^3+1\right)},\\
v_{10}\otimes v_{11} \mapsto\ & \frac{(q+1)z^2v_{11}\otimes v_{10}}{qz^2+1} + \frac{q(z^2-1)v_{10}\otimes v_{11}}{\alpha\left(qz^2+1\right)},\\
v_{11}\otimes v_{00} \mapsto\ & \frac{\alpha^2(z^2-1)(q^2z^2-1)v_{11}\otimes v_{00}}{\left(qz^2+1\right)\left(z^2q^3+1\right)} - \frac{q(q+1)\alpha(z^2-1)v_{01}\otimes v_{10}}{\left(qz^2+1\right)\left(z^2q^3+1\right)}\\
&- \frac{(q+1)\alpha(z^2-1)v_{10}\otimes v_{01}}{\left(qz^2+1\right)\left(z^2q^3+1\right)} + \frac{(q+1)\left(z^2q^3-z^2q^2+q^2+1\right)v_{00}\otimes v_{11}}{\left(qz^2+1\right)\left(z^2q^3+1\right)},\\
v_{11}\otimes v_{01} \mapsto\ & \frac{(q+1)v_{01}\otimes v_{11}}{qz^2+1} - \frac{\alpha(z^2-1)v_{11}\otimes v_{01}}{qz^2+1},\\
v_{11}\otimes v_{10} \mapsto\ & \frac{(q+1)v_{10}\otimes v_{11}}{qz^2+1} - \frac{\alpha(z^2-1)v_{11}\otimes v_{10}}{qz^2+1}.
\end{aligned}
$$

$S^{1,2}(z)$ can be obtained from the above $S^{2,1}(z)$ by applying (12.19).

$S^{2,2}(z)$ depends on z and α only via z^2 and α^2 up to overall $\alpha^{\pm 1}$. It is given as follows:

$$
\begin{aligned}
& v_{ij}\otimes v_{ij} \mapsto v_{ij}\otimes v_{ij} \quad (i, j \in \{0, 1\}),\\
& v_{00}\otimes v_{01} \mapsto -q\alpha^{-1}v_{00}\otimes v_{01}, \quad v_{00}\otimes v_{10} \mapsto -q\alpha^{-1}v_{00}\otimes v_{10}, \quad v_{01}\otimes v_{00} \mapsto \alpha v_{01}\otimes v_{00},\\
& v_{00}\otimes v_{11} \mapsto \frac{q^2(z^2-1)v_{00}\otimes v_{11}}{\alpha^2(q^2z^2-1)} + \frac{(q^2-1)z^2v_{11}\otimes v_{00}}{q^2z^2-1},\\
& v_{01}\otimes v_{10} \mapsto \frac{(q^2-1)z^2v_{10}\otimes v_{01}}{q^2z^2-1} - \frac{q(z^2-1)v_{01}\otimes v_{10}}{q^2z^2-1},\\
& v_{01}\otimes v_{11} \mapsto -q\alpha^{-1}v_{01}\otimes v_{11}, \quad v_{10}\otimes v_{00} \mapsto \alpha v_{10}\otimes v_{00},\\
& v_{10}\otimes v_{01} \mapsto \frac{(q^2-1)v_{01}\otimes v_{10}}{q^2z^2-1} - \frac{q(z^2-1)v_{10}\otimes v_{01}}{q^2z^2-1},\\
& v_{10}\otimes v_{11} \mapsto -q\alpha^{-1}v_{10}\otimes v_{11}, \quad v_{11}\otimes v_{01} \mapsto \alpha v_{11}\otimes v_{01}, \quad v_{11}\otimes v_{10} \mapsto \alpha v_{11}\otimes v_{10},\\
& v_{11}\otimes v_{00} \mapsto \frac{\alpha^2(z^2-1)v_{11}\otimes v_{00}}{q^2z^2-1} + \frac{(q^2-1)v_{00}\otimes v_{11}}{q^2z^2-1}.
\end{aligned}
$$

12.5 Bibliographical Notes and Comments

The content of this chapter, save for Sect. 12.3, is based on [107] where the reduction using the boundary vectors was introduced with a proof of Proposition 3.28. As for the relevant quantum R matrices for the spin representations, a description in terms of spectral decomposition was shown earlier for $U_p(B_n^{(1)})$ and $U_p(D_n^{(1)})$ in [121]. The eigenvalues in the spectral decomposition for the $U_p(D_{n+1}^{(2)})$ case is available in [107, Eq. (6.15)]. The matrix product form (12.9) provides a most handy and programmable formula for these R matrices via (12.31). It indicates a recursive structure of the R matrices with respect to rank n observed in earlier works including [121].