Chapter 10
Connection to PBW Bases of Nilpotent Subalgebra of U_q

Abstract For a finite-dimensional simple Lie algebra $\mathfrak{g}$, let $U_q^+(\mathfrak{g})$ be the positive part of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ with respect to the triangular decomposition. It has the Poincaré–Birkhoff–Witt (PBW) base labeled with the longest element of the Weyl group W of $\mathfrak{g}$. Let $A_q(\mathfrak{g})$ be the quantized coordinate ring of $\mathfrak{g}$. In this chapter, the intertwiner of the irreducible $A_q(\mathfrak{g})$ modules labeled with two different reduced expressions of W is identified with the transition matrix of the corresponding PBW bases of $U_q^+(\mathfrak{g})$. It leads to an alternative proof of the tetrahedron and 3D reflection equations within $U_q^+(\mathfrak{g})$. The boundary vectors in Sects. 3.6.1, 5.8.1 and 8.6.1 give rise to invariants of an anti-algebra involution in $U_q^+(\mathfrak{g})$ in an infinite product form.

10.1 Quantized Universal Enveloping Algebra $U_q(\mathfrak{g})$

10.1.1 Definition

In this chapter $\mathfrak{g}$ stands for a finite-dimensional simple Lie algebra. Its simple roots, simple coroots, fundamental weights are denoted by $\{\alpha_i\}_{i\in I}$,$\{h_i\}_{i\in I}$, $\{\varpi_i\}_{i\in I}$, where I is the index set of the Dynkin diagram of $\mathfrak{g}$. The weight lattice is $P = \oplus_{i\in I}\mathbb{Z}\varpi_i$ and the Cartan matrix $(a_{ij})_{i,j\in I}$ is given by $a_{ij} = \langle h_i, \alpha_j\rangle = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$.

The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is an associative algebra over $\mathbb{Q}(q)$ generated by $\{e_i, f_i, k_i^{\pm 1} \mid i \in I\}$ satisfying the relations:

$$k_i k_j = k_j k_i, \quad k_i k_i^{-1} = k_i^{-1} k_i = 1,$$

$$k_i e_j k_i^{-1} = q_i^{\langle h_i,\alpha_j\rangle} e_j, \quad k_i f_j k_i^{-1} = q_i^{-\langle h_i,\alpha_j\rangle} f_j, \quad [e_i, f_j] = \delta_{ij}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r e_i^{(r)} e_j e_i^{(1-a_{ij}-r)} = \sum_{r=0}^{1-a_{ij}} (-1)^r f_i^{(r)} f_j f_i^{(1-a_{ij}-r)} = 0 \quad (i \neq j). \qquad (10.1)$$

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Here we use the following notations: $q_i = q^{(\alpha_i,\alpha_i)/2}$, $[m]_i = (q_i^m - q_i^{-m})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{m=1}^n [m]_i$, $e_i^{(n)} = e_i^n/[n]_i!$, $f_i^{(n)} = f_i^n/[n]_i!$. We normalize the simple roots so that $q_i = q$ when α_i is a short root. The relation (10.1) is called q-Serre relation. The algebra $U_q(\mathfrak{g})$ is a Hopf algebra. For the comultiplication (or coproduct), we adopt the following[1]:

$$\Delta(k_i) = k_i \otimes k_i, \quad \Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i. \tag{10.2}$$

10.1.2 PBW Basis

Let W be the Weyl group of $\mathfrak{g}$. It is generated by simple reflections $\{s_i \mid i \in I\}$ obeying the relations: $s_i^2 = 1$, $(s_i s_j)^{m_{ij}} = 1$ $(i \neq j)$, where $m_{ij} = 2, 3, 4, 6$ for $\langle h_i, \alpha_j\rangle\langle h_j, \alpha_i\rangle = 0, 1, 2, 3$, respectively. Let w_0 be the longest element of W and fix a reduced expression $w_0 = s_{i_1} s_{i_2} \cdots s_{i_l}$. Then every positive root occurs exactly once in

$$\beta_1 = \alpha_{i_1},\ \beta_2 = s_{i_1}(\alpha_{i_2}), \ldots,\ \beta_l = s_{i_1} s_{i_2} \cdots s_{i_{l-1}}(\alpha_{i_l}). \tag{10.3}$$

Correspondingly, define elements $e_{\beta_r} \in U_q(\mathfrak{g})$ $(r = 1, \ldots, l)$ by

$$e_{\beta_r} = T_{i_1} T_{i_2} \cdots T_{i_{r-1}}(e_{i_r}). \tag{10.4}$$

Here T_i is the action of the braid group on $U_q(\mathfrak{g})$. It is an algebra automorphism and is given on the generators $\{e_j\}$ by

$$T_i(e_i) = -k_i f_i, \quad T_i(e_j) = \sum_{r=0}^{-a_{ij}} (-1)^r q_i^r e_i^{(r)} e_j e_i^{(-a_{ij}-r)} \quad (i \neq j). \tag{10.5}$$

Let $U_q^+(\mathfrak{g})$ be a subalgebra of $U_q(\mathfrak{g})$ generated by $\{e_i \mid i \in I\}$. The only relation among them is the q-Serre relation (10.1) for e_i's. It is known that $e_{\beta_r} \in U_q^+(\mathfrak{g})$ holds for any r. $U_q^+(\mathfrak{g})$ has the PBW basis. It depends on the reduced expression $s_{i_1} s_{i_2} \cdots s_{i_l}$ of w_0. Set $\mathbf{i} = (i_1, i_2, \ldots, i_l)$ and define for $A = (a_1, a_2, \ldots, a_l) \in (\mathbb{Z}_{\geq 0})^l$

$$E_{\mathbf{i}}^A = e_{\beta_1}^{(a_1)} e_{\beta_2}^{(a_2)} \cdots e_{\beta_l}^{(a_l)}. \tag{10.6}$$

Then $\{E_{\mathbf{i}}^A \mid A \in (\mathbb{Z}_{\geq 0})^l\}$ forms a basis of $U_q^+(\mathfrak{g})$. We warn that the notations e_{i_r} with $i_r \in I$ and e_{β_r} with a positive root β_r should be distinguished properly from the context. In particular $e_{\beta_r}^{(a_r)} = (e_{\beta_r})^{a_r} / \prod_{m=1}^{a_r} \frac{p_r^m - p_r^{-m}}{p_r - p_r^{-1}}$ with $p_r = q^{(\beta_r,\beta_r)/2}$.

[1] This convention will be kept throughout the book.

10.2 Quantized Coordinate Ring $A_q(\mathfrak{g})$

10.2.1 Definition

Let us give the definition of the quantized coordinate ring $A_q(\mathfrak{g})$.[2] The relation to the concrete realization by generators and relations in earlier chapters will be explained later.

Let $O_{\text{int}}(\mathfrak{g})$ be the category of integrable left $U_q(\mathfrak{g})$ modules M such that, for any $v \in M$, there exists $l \geq 0$ satisfying $e_{i_1} \cdots e_{i_l} v = 0$ for any $i_1, \ldots, i_l \in I$. Then $O_{\text{int}}(\mathfrak{g})$ is semisimple and any simple object is isomorphic to the irreducible module $V(\lambda)$ with dominant integral highest weight λ. Similarly, we can consider the category $O_{\text{int}}(\mathfrak{g}^{\text{op}})$ of integrable right $U_q(\mathfrak{g})$ modules M^r such that, for any $u \in M^r$, there exists $l \geq 0$ satisfying $u f_{i_1} \cdots f_{i_l} = 0$ for any $i_1, \ldots, i_l \in I$. The superscript op signifies "opposite". $O_{\text{int}}(\mathfrak{g}^{\text{op}})$ is also semisimple and any simple object is isomorphic to the irreducible module $V^r(\lambda)$ with dominant integral highest weight λ. Let v_λ (resp. u_λ) be a highest weight vector of $V(\lambda)$ (resp. $V^r(\lambda)$). Then there exists a unique bilinear form $(\ ,\ )$

$$V^r(\lambda) \otimes V(\lambda) \to \mathbb{Q}(q)$$

satisfying

$$(u_\lambda, v_\lambda) = 1 \quad \text{and}$$
$$(ug, v) = (u, gv) \quad \text{for } u \in V^r(\lambda),\ v \in V(\lambda),\ g \in U_q(\mathfrak{g}).$$

Let $U_q(\mathfrak{g})^*$ be $\mathrm{Hom}_{\mathbb{Q}(q)}(U_q(\mathfrak{g}), \mathbb{Q}(q))$ and $\langle\ ,\ \rangle$ be the canonical pairing between $U_q(\mathfrak{g})^*$ and $U_q(\mathfrak{g})$. The comultiplication Δ of $U_q(\mathfrak{g})$ induces a multiplication of $U_q(\mathfrak{g})^*$ by

$$\langle \varphi\varphi', g\rangle = \langle \varphi \otimes \varphi', \Delta(g)\rangle \qquad \text{for } g \in U_q(\mathfrak{g}), \tag{10.7}$$

thereby giving $U_q(\mathfrak{g})^*$ the structure of $\mathbb{Q}(q)$-algebra. It also has a $U_q(\mathfrak{g})$ bimodule structure by

$$\langle x\varphi y, g\rangle = \langle \varphi, ygx\rangle \qquad \text{for } x, y, g \in U_q(\mathfrak{g}). \tag{10.8}$$

We define the subalgebra $A_q(\mathfrak{g})$ of $U_q(\mathfrak{g})^*$ by

$$A_q(\mathfrak{g}) = \{\varphi \in U_q(\mathfrak{g})^*;\ U_q(\mathfrak{g})\varphi \text{ belongs to } O_{\text{int}}(\mathfrak{g}) \text{ and } \varphi U_q(\mathfrak{g}) \text{ belongs to } O_{\text{int}}(\mathfrak{g}^{\text{op}})\},$$

and call it the quantized coordinate ring.

The following theorem is the q-analogue of the Peter–Weyl theorem.

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[2] The definition and Theorem 10.1 are valid for any symmetrizable Kac–Moody algebra.

Theorem 10.1 *As a $U_q(\mathfrak{g})$ bimodule, $A_q(\mathfrak{g})$ is isomorphic to $\bigoplus_\lambda V^r(\lambda) \otimes V(\lambda)$, where λ runs over all dominant integral weights, by the homomorphisms*

$$\Psi_\lambda : V^r(\lambda) \otimes V(\lambda) \to A_q(\mathfrak{g})$$

given by

$$\langle \Psi_\lambda(u \otimes v), g\rangle = (u, gv)$$

for $u \in V^r(\lambda)$, $v \in V(\lambda)$, and $g \in U_q(\mathfrak{g})$.[3]

In our case of a finite-dimensional simple Lie algebra $\mathfrak{g}$, $A_q(\mathfrak{g})$ turns out to be a Hopf algebra. See for example [66, Chap. 9]. Its comultiplication is also denoted by Δ.

Let $\mathcal{R}$ be the universal R matrix for $U_q(\mathfrak{g})$. For its explicit formula see [29, p. 273] for example. For our purpose it is enough to know that

$$\mathcal{R} \in q^{(\mathrm{wt}\,\cdot,\mathrm{wt}\,\cdot)} \bigoplus_{\beta \in Q^+} (U_q^+)_\beta \otimes (U_q^-)_{-\beta}, \tag{10.9}$$

where $q^{(\mathrm{wt}\,\cdot,\mathrm{wt}\,\cdot)}$ is an operator acting on the tenor product $v_\lambda \otimes v_\mu$ of weight vectors v_λ, v_μ of weight λ, μ by $q^{(\mathrm{wt}\,\cdot,\mathrm{wt}\,\cdot)}(v_\lambda \otimes v_\mu) = q^{(\lambda,\mu)} v_\lambda \otimes v_\mu$, $Q_+ = \bigoplus_i \mathbb{Z}_{\geq 0}\alpha_i$, and $(U_q^\pm)_{\pm\beta}$ is the subspace of $U_q^\pm(\mathfrak{g})$ spanned by root vectors corresponding to $\pm\beta$.

Fix λ, let $\{u_j^\lambda\}$ and $\{v_i^\lambda\}$ be bases of $V^r(\lambda)$ and $V(\lambda)$ such that $(u_i^\lambda, v_j^\lambda) = \delta_{ij}$. Set

$$\varphi_{ij}^\lambda = \Psi_\lambda(u_i^\lambda \otimes v_j^\lambda) \in A_q(\mathfrak{g}). \tag{10.10}$$

Let R be the so-called constant R matrix for $V(\lambda) \otimes V(\mu)$. Denoting the homomorphism $U_q(\mathfrak{g}) \to \mathrm{End}(V(\lambda))$ by ρ_λ, it is given as

$$R \propto (\rho_\lambda \otimes \rho_\mu)(P\mathcal{R}), \tag{10.11}$$

where P stands for the exchange of the first and second components. The scalar multiple is determined appropriately depending on $\mathfrak{g}$. The reason we apply P is to fit the so-called RTT relation in (10.15). The dependence of R on λ and μ has been suppressed in the notation. R satisfies

$$R\Delta(g) = \Delta^{\mathrm{op}}(g)R \qquad \text{for any } g \in U_q(\mathfrak{g}), \tag{10.12}$$

where $\Delta^{\mathrm{op}} = P \circ \Delta \circ P$. Define matrix elements R_{kl}^{ij} by

$$R(v_k^\lambda \otimes v_l^\mu) = \sum_{i,j} R_{kl}^{ij} v_i^\lambda \otimes v_j^\mu. \tag{10.13}$$

[3] Of course this Ψ_λ has nothing to do with the intertwiners in (5.33), (6.22) and (7.5).

Define the right action of R on $V^r(\lambda) \otimes V^r(\mu)$ in such a way that $((u_i^\lambda \otimes u_j^\mu)R, v_k^\lambda \otimes v_l^\mu) = (u_i^\lambda \otimes u_j^\mu, R(v_k^\lambda \otimes v_l^\mu))$ holds. Then we have

$$(u_i^\lambda \otimes u_j^\mu)R = \sum_{k,l} R_{kl}^{ij} u_k^\lambda \otimes u_l^\mu. \tag{10.14}$$

Now for any $x \in U_q(\mathfrak{g})$, we have

$$\begin{aligned}
&\sum_{m,p} R_{mp}^{ij} \langle \varphi_{mk}^\lambda \varphi_{pl}^\mu, x\rangle = \sum_{m,p} R_{mp}^{ij} \langle \varphi_{mk}^\lambda \otimes \varphi_{pl}^\mu, \Delta(x)\rangle \\
&= \sum_{m,p} R_{mp}^{ij} \langle \Psi_\lambda(u_m^\lambda \otimes v_k^\lambda) \otimes \Psi_\mu(u_p^\mu \otimes v_l^\mu), \Delta(x)\rangle \\
&= \sum_{m,p} R_{mp}^{ij} (u_m^\lambda \otimes u_p^\mu, \Delta(x)(v_k^\lambda \otimes v_l^\mu)) = ((u_i^\lambda \otimes u_j^\mu)R, \Delta(x)(v_k^\lambda \otimes v_l^\mu)) \\
&= (u_i^\lambda \otimes u_j^\mu, R\Delta(x)(v_k^\lambda \otimes v_l^\mu)) = (u_i^\lambda \otimes u_j^\mu, \Delta^{\mathrm{op}}(x)R(v_k^\lambda \otimes v_l^\mu)) \\
&= \sum_{m,p} (u_i^\lambda \otimes u_j^\mu, \Delta^{\mathrm{op}}(x)(v_m^\lambda \otimes v_p^\mu))R_{kl}^{mp} = \sum_{m,p} (u_j^\mu \otimes u_i^\lambda, \Delta(x)(v_p^\mu \otimes v_m^\lambda))R_{kl}^{mp} \\
&= \sum_{m,p} \langle \varphi_{jp}^\mu \otimes \varphi_{im}^\lambda, \Delta(x)\rangle R_{kl}^{mp} = \sum_{m,p} \langle \varphi_{jp}^\mu \varphi_{im}^\lambda, x\rangle R_{kl}^{mp}.
\end{aligned}$$

Thus we get

$$\sum_{m,p} R_{mp}^{ij} \varphi_{mk}^\lambda \varphi_{pl}^\mu = \sum_{m,p} \varphi_{jp}^\mu \varphi_{im}^\lambda R_{kl}^{mp} \in A_q(\mathfrak{g}). \tag{10.15}$$

We call such a relation an RTT relation. It forms a large family containing conventional ones as the special case where $\lambda = \mu = \varpi_r$ for some specific fundamental weight ϖ_r.

Example 10.2 Consider the simplest case $\mathfrak{g} = A_1$ with $\lambda = \mu = \varpi_1$. We write $u_i^{\varpi_1}, v_i^{\varpi_1}$ simply as u_i, v_i $(i = 1, 2)$. The $U_q(sl_2)$ module structure is

$$f_1 v_1 = v_2,\ f_1 v_2 = 0,\ e_1 v_1 = 0,\ e_1 v_2 = v_1,\ k_1 v_1 = q v_1,\ k_1 v_2 = q^{-1} v_2, \tag{10.16}$$

$$u_1 f_1 = 0,\ u_2 f_1 = u_1,\ u_1 e_1 = u_2,\ u_2 e_1 = 0,\ u_1 k_1 = q u_1,\ u_2 k_1 = q^{-1} u_2. \tag{10.17}$$

The R matrix (3.3) acts as

$$R(v_1 \otimes v_1) = qv_1 \otimes v_1, \quad R(v_1 \otimes v_2) = v_1 \otimes v_2 + (q - q^{-1})v_2 \otimes v_1, \tag{10.18}$$
$$R(v_2 \otimes v_1) = v_2 \otimes v_1, \quad R(v_2 \otimes v_2) = qv_2 \otimes v_2, \tag{10.19}$$
$$(u_1 \otimes u_1)R = qu_1 \otimes u_1, \quad (u_2 \otimes u_1)R = u_2 \otimes u_1 + (q - q^{-1})u_1 \otimes u_2, \tag{10.20}$$
$$(u_1 \otimes u_2)R = u_1 \otimes u_2, \quad (u_2 \otimes u_2)R = qu_2 \otimes u_2. \tag{10.21}$$

Set $t_{ij} = \Psi_{\omega_1}(u_i \otimes v_j) \in A_q(A_1)$. Then we have

$$\begin{aligned}
\langle t_{11}t_{22}, x\rangle &= \langle \Psi_{\omega_1}(u_1 \otimes v_1) \otimes \Psi_{\omega_1}(u_2 \otimes v_2), \Delta(x)\rangle = (u_1 \otimes u_2, \Delta(x)(v_1 \otimes v_2)) \\
&= ((u_1 \otimes u_2)R, \Delta(x)(v_1 \otimes v_2)) = (u_1 \otimes u_2, \Delta^{\mathrm{op}}(x)R(v_1 \otimes v_2)) \\
&= (u_1 \otimes u_2, \Delta^{\mathrm{op}}(x)(v_1 \otimes v_2 + (q - q^{-1})v_2 \otimes v_1)) \\
&= (u_2 \otimes u_1, \Delta(x)(v_2 \otimes v_1 + (q - q^{-1})v_1 \otimes v_2)) \\
&= \langle \Psi_{\omega_1}(u_2 \otimes v_2) \otimes \Psi_{\omega_1}(u_1 \otimes v_1) \\
&\qquad + (q - q^{-1})\Psi_{\omega_1}(u_2 \otimes v_1) \otimes \Psi_{\omega_1}(u_1 \otimes v_2), \Delta(x)\rangle \\
&= \langle t_{22} \otimes t_{11} + (q - q^{-1})t_{21} \otimes t_{12}, \Delta(x)\rangle \\
&= \langle t_{22}t_{11} + (q - q^{-1})t_{21}t_{12}, x\rangle,
\end{aligned}$$

which reproduces the relation $[t_{11}, t_{22}] = (q - q^{-1})t_{21}t_{12}$ in (3.9). Similarly, we have

$$\begin{aligned}
\langle t_{11}t_{22} - qt_{12}t_{21}, x\rangle &= \langle t_{11} \otimes t_{22} - qt_{12} \otimes t_{21}, \Delta(x)\rangle \\
&= (u_1 \otimes u_2, \Delta(x)(v_1 \otimes v_2)) - q(u_1 \otimes u_2, \Delta(x)(v_2 \otimes v_1)) \\
&= (u_1 \otimes u_2, \Delta(x)(v_1 \otimes v_2 - qv_2 \otimes v_1)).
\end{aligned}$$

Suppose $x = e_1^l k_1^m f_1^n \in U_q(sl_2)$ $(l, m, n \in \mathbb{Z}_{\geq 0})$ without loss of generality. Since $v_1^0 := v_1 \otimes v_2 - qv_2 \otimes v_1$ is a $U_q(sl_2)$-singlet annihilated either by $\Delta(e_1)$ and $\Delta(f_1)$, one has $\Delta(x)v_1^0 = \delta_{l0}\delta_{n0}v_1^0$. Thus the RHS of the above calculation is equal to $\delta_{l0}\delta_{n0}(u_1 \otimes u_2, v_1^0) = \delta_{l0}\delta_{n0} = \langle 1, x\rangle$. This yields $t_{11}t_{22} - qt_{12}t_{21} = 1$ in (3.9).

Let us mention the relation to the formulation of $A_q(\mathfrak{g})$ in earlier chapters using specific generators and relations. Suppose ϖ_l is a fundamental weight such that any $V(\lambda)$ is included in the tensor power $V(\varpi_l)^{\otimes m}$ for some m.[4] Denoting the base of $V^r(\varpi_l)$ and $V(\varpi_l)$ by u_i and v_i, set

$$t_{ij} = \Psi_{\varpi_l}(u_i \otimes v_j) \in A_q(\mathfrak{g}). \tag{10.22}$$

[4] For example, in type B, it is the *spin* representation that qualifies this postulate rather than the vector representation. For type D, the argument in the text needs a slight modification since the two kinds of spin representations $V(\varpi_{n-1})$ and $V(\varpi_n)$ are necessary, but it does not influence the results in the chapter.

We know that t_{ij} satisfies the RTT relation (10.15) whose structure constant is the constant R matrix for $\lambda = \mu = \varpi_l$. Any vectors $u \in V^r(\lambda)$ and $v \in V(\lambda)$ are expressed as linear combinations $u = \sum C_{i_1,\dots,i_m} u_{i_1} \otimes \cdots \otimes u_{im}$ and $v = \sum D_{j_1,\dots,j_m} v_{j_1} \otimes \cdots \otimes v_{j_m}$. Theorem 10.1 shows that an arbitrary element of $A_q(\mathfrak{g})$ is constructed as $\Psi_\lambda(u \otimes v)$. A calculation similar to Example 10.2 leads to $\Psi_\lambda(u \otimes v) = \sum C_{i_1,\dots,i_m} D_{j_1,\dots,j_m} t_{i_1 j_1} \cdots t_{i_m j_m}$, which says that t_{ij}'s are certainly generators. They satisfy RTT and additional relations reflecting a fine structure of the Grothendieck ring of $\mathfrak{g}$ like $V(\varpi_l)^{\otimes m} \supset V(0)$ and $V(\varpi_l)^{\otimes m} \supset V(\varpi_l)$, etc. Our individual treatment in the earlier chapters corresponds to the choice $l = 1$ for A_{n-1}, C_n, G_2 and $l = n$ for B_n.[5]

10.2.2 Right Quotient Ring $A_q(\mathfrak{g})_S$

Here we prepare the necessary ingredients for the proof of Theorem 10.6. The point is to assure the well definedness of the division in (10.39).

Recall that $w_0 \in W$ is the longest element of the Weyl group. For any $l \in I$, let $v_{w_0\varpi_l} \in V(\varpi_l)$ be a lowest weight vector. Similarly, let $u_{\varpi_l} \in V^r(\varpi_l)$ be a highest weight vector. The following element will play a key role:

$$\sigma_l = \Psi_{\varpi_l}(u_{\varpi_l} \otimes v_{w_0\varpi_l}) \in A_q(\mathfrak{g}). \tag{10.23}$$

Example 10.3 For $\mathfrak{g} = A_1$ treated in Example 10.2, one has $\sigma_1 = \Psi_{\omega_1}(u_1 \otimes v_2) = t_{12}$.

Proposition 10.4 *The commutativity $\sigma_r\sigma_s = \sigma_s\sigma_r$ holds for any $r, s \in I$.*

Proof From (10.9) and (10.11) we have

$$(u_{\varpi_r} \otimes u_{\varpi_s})R = q^{(\varpi_r,\varpi_s)} u_{\varpi_r} \otimes u_{\varpi_s}, \tag{10.24}$$

$$R(v_{w_0\varpi_r} \otimes v_{w_0\varpi_s}) = q^{(\varpi_r,\varpi_s)} v_{w_0\varpi_r} \otimes v_{w_0\varpi_s}, \tag{10.25}$$

where $(w_0\varpi_r, w_0\varpi_s) = (\varpi_r, \varpi_s)$ has been used. Consider the RTT relation (10.15) with $\lambda = \varpi_r$, $\mu = \varpi_s$, and take the indices i, j, k, l so as to specify the following bases:

$$u_i^\lambda = u_{\varpi_r}, \quad u_j^\mu = u_{\varpi_s}, \quad v_k^\lambda = v_{w_0\varpi_r}, \quad v_l^\mu = v_{w_0\varpi_s}. \tag{10.26}$$

Then (10.24) and (10.25) indicate $R^{ij}_{mp} = q^{(\varpi_r,\varpi_s)}\delta^i_m\delta^i_p$ and $R^{mp}_{kl} = q^{(\varpi_r,\varpi_s)}\delta^m_k\delta^p_l$. Thus the RTT relation (10.15) reduces to

$$\varphi^{\varpi_r}_{ik}\varphi^{\varpi_s}_{jl} = \varphi^{\varpi_s}_{jl}\varphi^{\varpi_r}_{ik}. \tag{10.27}$$

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[5] As for F_4 we did not present specific generators and relations.

The proof is finished by noting $\varphi_{ik}^{\varpi_r} = \sigma_r$ and $\varphi_{jl}^{\varpi_s} = \sigma_s$ by comparing (10.10) and (10.23). □

Since $A_q(\mathfrak{g})$ is a right $U_q(\mathfrak{g})$ module, we have an element $\sigma_i e_i \in A_q(\mathfrak{g})$. Later in Sect. 10.3.2, we will need the division $(\sigma_i e_i)/\sigma_i$ for $i \in I$. The following localization is known to be possible making sense of it.

Theorem 10.5 *Let n be the rank of $\mathfrak{g}$. For the multiplicatively closed subset $\mathcal{S} = \{\sigma_1^{m_1} \cdots \sigma_n^{m_n} \mid m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}\} \subset A_q(\mathfrak{g})$, the right quotient ring $A_q(\mathfrak{g})_{\mathcal{S}}$ exists.*

Elements of $A_q(\mathfrak{g})_{\mathcal{S}}$ are expressed in the form r/s with $r \in A_q(\mathfrak{g})$ and $s \in \mathcal{S}$. Theorem 10.5 guarantees the well-defined ring structure, namely, the addition and the multiplication of r_1/s_1 and r_2/s_2 in $A_q(\mathfrak{g})_{\mathcal{S}}$ as

$$r_1/s_1 + r_2/s_2 = (r_1 u + r_2 u')/(s_1 u), \qquad (r_1/s_1)(r_2/s_2) = (r_1 v')/(s_2 v), \quad (10.28)$$

where u, u', v, v' are so chosen that $s_1 u = s_2 u'$ $(u \in \mathcal{S}, u' \in A_q(\mathfrak{g}))$, $r_2 v = s_1 v'$ $(v \in \mathcal{S}, v' \in A_q(\mathfrak{g}))$.

10.3 Main Theorem

In this section we fix two reduced words $\mathbf{i} = (i_1, \ldots, i_l)$, $\mathbf{j} = (j_1, \ldots, j_l)$ of the longest element $w_0 = s_{i_1} \cdots s_{i_l} = s_{j_1} \cdots s_{j_l} \in W$.

10.3.1 Definitions of γ_B^A and Φ_B^A

In the $U_q(\mathfrak{g})$ side, we defined the PBW bases $E_{\mathbf{i}}^A$, $E_{\mathbf{j}}^B$ of $U_q^+(\mathfrak{g})$ in Sect. 10.1.2. We define their transition coefficient γ_B^A by

$$E_{\mathbf{i}}^A = \sum_B \gamma_B^A E_{\mathbf{j}}^B.$$

In the $A_q(\mathfrak{g})$ side, we have the intertwiner $\Phi : \mathcal{F}_{q_{i_1}} \otimes \cdots \otimes \mathcal{F}_{q_{i_l}} \to \mathcal{F}_{q_{j_1}} \otimes \cdots \otimes \mathcal{F}_{q_{j_l}}$ satisfying

$$\pi_{\mathbf{j}}(g) \circ \Phi = \Phi \circ \pi_{\mathbf{i}}(g) \qquad (\forall g \in A_q(\mathfrak{g})). \quad (10.29)$$

We take the parameters μ_i as in (3.21) and (5.19) to be 1. The intertwiner Φ is normalized by $\Phi(|0\rangle \otimes \cdots \otimes |0\rangle) = |0\rangle \otimes \cdots \otimes |0\rangle$. Under these conditions a matrix element Φ_B^A of Φ is uniquely specified by

$$\Phi|B\rangle = \sum_A \Phi_B^A |A\rangle,$$

where $A = (a_1, \ldots, a_l) \in (\mathbb{Z}_{\geq 0})^l$ and $|A\rangle = |a_1\rangle \otimes \cdots \otimes |a_l\rangle \in \mathcal{F}_{q_{j_1}} \otimes \cdots \otimes \mathcal{F}_{q_{j_l}}$ and similarly for $|B\rangle \in \mathcal{F}_{q_{i_1}} \otimes \cdots \otimes \mathcal{F}_{q_{i_l}}$. The main result of this chapter is

Theorem 10.6

$$\gamma^A_B = \Phi^A_B.$$

For any pair $(\mathbf{i}, \mathbf{j})$, from $\mathbf{i}$ one can reach $\mathbf{j}$ by applying Coxeter relations (for indices of the simple reflections). In view of the uniqueness of γ and Φ and the fact that the braid group action T_i is an algebra homomorphism, the proof of this theorem reduces to establishing the same equality for the rank 2 case $\mathfrak{g} = A_2, C_2$ and G_2.[6] This will be done in the sequel.

10.3.2 Proof of Theorem 10.6 for Rank 2 Cases

In the rank 2 cases, there are two reduced expressions $s_{i_1} \cdots s_{i_l}$ for the longest element of the Weyl group. Denote the associated sequences $\mathbf{i} = (i_1, \ldots, i_l)$ by $\mathbf{1}, \mathbf{2}$ and set $\mathbf{1}' = \mathbf{2}, \mathbf{2}' = \mathbf{1}$. Concretely, we take them as

$$A_2 : \mathbf{1} = (1, 2, 1), \qquad \mathbf{2} = (2, 1, 2), \qquad (q_1, q_2) = (q, q), \tag{10.30}$$

$$C_2 : \mathbf{1} = (1, 2, 1, 2), \qquad \mathbf{2} = (2, 1, 2, 1), \qquad (q_1, q_2) = (q, q^2), \tag{10.31}$$

$$G_2 : \mathbf{1} = (1, 2, 1, 2, 1, 2), \qquad \mathbf{2} = (2, 1, 2, 1, 2, 1), \qquad (q_1, q_2) = (q, q^3), \tag{10.32}$$

where q_i defined after (10.1) is also recalled. In order to simplify the formulas in Sect. 10.4, we use the PBW bases and the Fock states in yet another normalization as follows:

$$\tilde{E}^A_{\mathbf{i}} := ([a_1]_{i_1}! \cdots [a_l]_{i_l}!) E^A_{\mathbf{i}} = e^{a_1}_{\beta_1} \cdots e^{a_l}_{\beta_l}, \tag{10.33}$$

$$|A\rangle\!\rangle := d_{i_1,a_1} \cdots d_{i_l,a_l} |A\rangle, \quad d_{i,a} = q_i^{-a(a-1)/2} \lambda_i^a, \quad \lambda_i = (1 - q_i^2)^{-1}, \tag{10.34}$$

where $A = (a_1, \ldots, a_l)$. See after (10.1) for the symbol $[a]_i!$. The root vector e_{β_r} is defined in (10.4). Accordingly, we introduce the matrix elements $\tilde{\gamma}^A_B$ and $\tilde{\Phi}^A_B$ by

$$\tilde{E}^A_{\mathbf{i}} = \sum_B \tilde{\gamma}^A_B \tilde{E}^B_{\mathbf{i}'}, \quad \Phi|B\rangle\!\rangle = \sum_A \tilde{\Phi}^A_B |A\rangle\!\rangle, \quad (\mathbf{i} = \mathbf{1}, \mathbf{2}). \tag{10.35}$$

It follows that $\gamma^A_B = \tilde{\gamma}^A_B \prod_{k=1}^l ([b_k]_{i_k}!/[a_k]_{i_k}!)$ and $\Phi^A_B = \tilde{\Phi}^A_B \prod_{k=1}^l (d_{i_k,a_k}/d_{i_k,b_k})$ for $B = (b_1, \ldots, b_l)$. On the other hand, we know $\Phi^A_B = \Phi^B_A \prod_{k=1}^l ((q^2_{i_k})_{b_k}/(q^2_{i_k})_{a_k})$ from

[6] The B_2 case reduces to C_2 by the interchange of indices $1 \leftrightarrow 2 \in I$.

(3.63), (5.75) and (8.30). Due to the identity $(q_i^2)_m d_{i,m} = [m]_i!$, the assertion $\gamma_B^A = \Phi_B^A$ of Theorem 10.6 is equivalent to

$$\tilde{\gamma}_B^A = \tilde{\Phi}_A^B. \tag{10.36}$$

Let $\rho_{\mathbf{i}}(x) = (\rho_{\mathbf{i}}(x)_{AB})$ be the matrix for the left multiplication of $x \in U_q^+(\mathfrak{g})$:

$$x \cdot \tilde{E}_{\mathbf{i}}^A = \sum_B \tilde{E}_{\mathbf{i}}^B \rho_{\mathbf{i}}(x)_{BA}. \tag{10.37}$$

Let further $\pi_{\mathbf{i}}(g) = (\pi_{\mathbf{i}}(g)_{AB})$ be the representation matrix of $g \in A_q(\mathfrak{g})$:

$$\pi_{\mathbf{i}}(g)|A\rangle\rangle = \sum_B |B\rangle\rangle \pi_{\mathbf{i}}(g)_{BA}. \tag{10.38}$$

The following element in the right quotient ring $A_q(\mathfrak{g})_{\mathcal{S}}$ (see Theorem 10.5) will play a key role in our proof:

$$\xi_i = \lambda_i(\sigma_i e_i)/\sigma_i \quad (i = 1, 2). \tag{10.39}$$

We recall that the general definition of σ_i is (10.23). Its concrete form in the rank 2 case will be given in Lemmas 10.10, 10.12 and 10.14. In Sect. 10.4 we will check the following statement case by case. It says that the "conjugation" of e_i by σ_i on $A_q(\mathfrak{g})$ modules $(\sigma_i e_i)/\sigma_i$ corresponds to $(1 - q_i^2)e_i$ in $U_q^+(\mathfrak{g})$.

Proposition 10.7 *For $\mathfrak{g}$ of rank 2, $\pi_{\mathbf{i}}(\sigma_i)$ is invertible and the following equality is valid:*

$$\rho_{\mathbf{i}}(e_i)_{AB} = \pi_{\mathbf{i}}(\xi_i)_{AB} \quad (i = 1, 2), \tag{10.40}$$

where the RHS means $\lambda_i \pi_{\mathbf{i}}(\sigma_i e_i)\pi_{\mathbf{i}}(\sigma_i)^{-1}$.

Proof of Theorem 10.6 for rank 2 case. We write both sides of (10.40) as M^i_{AB} and the term for $\mathbf{i}'$ instead of $\mathbf{i}$ as M'^i_{AB}. From

$$\sum_{B,C} \tilde{E}_{\mathbf{i}'}^C M'^i_{CB} \tilde{\gamma}_B^A = e_i \sum_B \tilde{E}_{\mathbf{i}'}^B \tilde{\gamma}_B^A = e_i \tilde{E}_{\mathbf{i}}^A = \sum_B \tilde{E}_{\mathbf{i}}^B M^i_{BA} = \sum_{B,C} \tilde{E}_{\mathbf{i}'}^C \tilde{\gamma}_C^B M^i_{BA}$$

we have $\sum_B M'^i_{CB} \tilde{\gamma}_B^A = \sum_B \tilde{\gamma}_C^B M^i_{BA}$. On the other hand, the actions of the two sides of (10.29) with $g = \xi_i$ and $\mathbf{j} = \mathbf{i}'$ are calculated as

$$\pi_{\mathbf{i}'}(\xi_i) \circ \Phi|A\rangle\rangle = \pi_{\mathbf{i}'}(\xi_i) \sum_B |B\rangle\rangle \tilde{\Phi}_A^B = \sum_{B,C} |C\rangle\rangle M'^i_{CB} \tilde{\Phi}_A^B$$

and

$$\Phi \circ \pi_{\mathbf{i}}(\xi_i)|A\rangle\rangle = \Phi \sum_B |B\rangle\rangle M^i_{BA} = \sum_{B,C} |C\rangle\rangle \tilde{\Phi}_B^C M^i_{BA}.$$

Hence $\sum_B M'^i_{CB}\tilde{\Phi}^B_A = \sum_B \tilde{\Phi}^C_B M^i_{BA}$. Thus $\tilde{\gamma}^A_B$ and $\tilde{\Phi}^B_A$ satisfy the same relation. Moreover, the maps $\pi_{\mathbf{i}}$ and $\rho_{\mathbf{i}}$ are both homomorphisms, i.e. $\pi_{\mathbf{i}}(gh) = \pi_{\mathbf{i}}(g)\pi_{\mathbf{i}}(h)$ and $\rho_{\mathbf{i}}(xy) = \rho_{\mathbf{i}}(x)\rho_{\mathbf{i}}(y)$. We know that Φ is the intertwiner of the irreducible $A_q(\mathfrak{g})$ modules and (10.36) obviously holds as $1 = 1$ at $A = B = (0, \dots, 0)$. Thus it is valid for arbitrary A and B. □

Remark 10.8 The equality (10.40) is valid for any $\mathfrak{g}$.

10.4 Proof of Proposition 10.7

Here we present the explicit formulas of (10.37) with $x = e_i$ and (10.38) with $g = \sigma_i, \sigma_i e_i$ that allow one to check Proposition 10.7. In each case, there are two $\mathbf{i}$-sequences, $\mathbf{1}$ and $\mathbf{2} = \mathbf{1}'$ corresponding to the two reduced words. Define

$$\chi = \text{the anti-algebra involution of } U_q^+(\mathfrak{g}) \text{ such that } \chi(e_i) = e_i. \tag{10.41}$$

Then both $E^A_{\mathbf{i}}$ in (10.6) and $\tilde{E}^A_{\mathbf{i}}$ in (10.33) satisfy

$$\chi(E^A_{\mathbf{i}}) = E^{A^\vee}_{\mathbf{i}'}, \qquad \chi(\tilde{E}^A_{\mathbf{i}}) = \tilde{E}^{A^\vee}_{\mathbf{i}'}, \tag{10.42}$$

where $A^\vee = (a_l, \dots, a_2, a_1)$ denotes the reversal of $A = (a_1, a_2, \dots, a_l)$. Applying χ to (10.37) with $x = e_i$ yields the right multiplication formula $\tilde{E}^{A^\vee}_{\mathbf{i}'} \cdot e_i = \sum_B \tilde{E}^{B^\vee}_{\mathbf{i}'} \rho_{\mathbf{i}}(e_i)_{BA}$ for the $\mathbf{i}'$-sequence. In view of this fact, we shall present the left and right multiplication formulas for $\mathbf{i} = \mathbf{2}$ only.

As for (10.38) with $g = \xi_i$ in (10.39), explicit formulas for $\sigma_i, \sigma_i e_i \in A_q(\mathfrak{g})$ and their image by both representations $\pi_{\mathbf{1}}$ and $\pi_{\mathbf{2}}$ will be given. We include an exposition on how to use these data to check (10.40) along the simplest A_2 case. The C_2 and G_2 cases are similar.

Following (10.34), we write $|m\rangle\rangle := d_{i,m}|m\rangle \in \mathcal{F}_{q_i}$ for each component. From the choice (10.30)–(10.32), the action of the q_i-oscillator on $\mathcal{F}_{q_i}$ $(i = 1, 2)$ takes the form

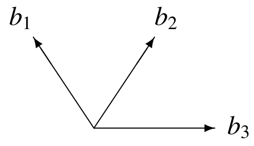
$$\begin{aligned} &\mathbf{a}^+|m\rangle\rangle = \lambda_1^{-1} q_1^m |m+1\rangle\rangle, \quad \mathbf{a}^-|m\rangle\rangle = [m]_1 |m-1\rangle\rangle, \quad \mathbf{k}|m\rangle\rangle = q_1^m |m\rangle\rangle, \\ &\mathbf{A}^+|m\rangle\rangle = \lambda_2^{-1} q_2^m |m+1\rangle\rangle, \quad \mathbf{A}^-|m\rangle\rangle = [m]_2 |m-1\rangle\rangle, \quad \mathbf{K}|m\rangle\rangle = q_2^m |m\rangle\rangle. \end{aligned} \tag{10.43}$$

See (10.34) and (3.13). We also use the shorthand

$$\langle m \rangle = q^m - q^{-m}. \tag{10.44}$$

10.4.1 Explicit Formulas for A_2

Consider $\mathfrak{g} = A_2$.



The q-Serre relations are

$$e_1^2 e_2 - [2]_1 e_1 e_2 e_1 + e_2 e_1^2 = 0, \quad e_2^2 e_1 - [2]_1 e_2 e_1 e_2 + e_1 e_2^2 = 0, \tag{10.45}$$

where $[m]_1 = \langle m\rangle/\langle 1\rangle$. For simplicity we write the positive root vectors $e_{\beta_{i_r}}$ in (10.4) with $(i_1, i_2, i_3) = \mathbf{2}$ (10.30) as

$$b_1 = e_{\beta_1} = e_2, \quad b_2 = e_{\beta_2} = e_1 e_2 - q e_2 e_1, \quad b_3 = e_{\beta_3} = e_1. \tag{10.46}$$

The corresponding positive roots are $(\beta_1, \beta_2, \beta_3) = (\alpha_2, \alpha_1 + \alpha_2, \alpha_1)$. In particular, $b_2 = T_2(e_1)$. Their commutation relations are

$$b_2 b_1 = q^{-1} b_1 b_2, \quad b_3 b_1 = b_2 + q b_1 b_3, \quad b_3 b_2 = q^{-1} b_2 b_3. \tag{10.47}$$

Lemma 10.9 *For* $\tilde{E}_{\mathbf{2}}^{a,b,c} = b_1^a b_2^b b_3^c$, *we have*

$$\begin{aligned}
\tilde{E}_{\mathbf{2}}^{a,b,c} \cdot e_1 &= \tilde{E}_{\mathbf{2}}^{a,b,c+1}, \\
\tilde{E}_{\mathbf{2}}^{a,b,c} \cdot e_2 &= q^{c-b} \tilde{E}_{\mathbf{2}}^{a+1,b,c} + [c]_1 \tilde{E}_{\mathbf{2}}^{a,b+1,c-1}, \\
e_1 \cdot \tilde{E}_{\mathbf{2}}^{a,b,c} &= q^{a-b} \tilde{E}_{\mathbf{2}}^{a,b,c+1} + [a]_1 \tilde{E}_{\mathbf{2}}^{a-1,b+1,c}, \\
e_2 \cdot \tilde{E}_{\mathbf{2}}^{a,b,c} &= \tilde{E}_{\mathbf{2}}^{a+1,b,c}.
\end{aligned}$$

Proof By induction, we have

$$\begin{aligned}
b_3 b_1^n &= q^n b_1^n b_3 + [n]_1 b_1^{n-1} b_2, \quad b_3 b_2^n = q^{-n} b_2^n b_3, \\
b_3^n b_1 &= q^n b_1 b_3^n + [n]_1 b_2 b_3^{n-1}, \quad b_2^n b_1 = q^{-n} b_1 b_2^n.
\end{aligned}$$

The lemma is a direct consequence of these formulas. □

Set $\tilde{E}_{\mathbf{1}}^{a,b,c} = \chi(\tilde{E}_{\mathbf{2}}^{c,b,a}) = \chi(b_3^a)\chi(b_2^b)\chi(b_1^c) = b_3^a b_2'^b b_1^c$, where $b_2' := \chi(b_2) = e_2 e_1 - q e_1 e_2$. By applying χ to the first two relations in Lemma 10.9, we get

$$e_1 \cdot \tilde{E}_{\mathbf{1}}^{a,b,c} = \tilde{E}_{\mathbf{1}}^{a+1,b,c}, \qquad e_2 \cdot \tilde{E}_{\mathbf{1}}^{a,b,c} = q^{a-b} E_{\mathbf{1}}^{a,b,c+1} + [a]_1 \tilde{E}_{\mathbf{1}}^{a-1,b+1,c}. \tag{10.48}$$

Thus we find $\rho_{\mathbf{i}'}(e_i) = \rho_{\mathbf{i}}(e_{3-i})$. This property is only valid for A_2 and not in C_2 and G_2.

Let u_i $(i = 1, 2, 3)$ be the bases of the right $U_q(A_2)$ module $V^r(\varpi_1)$ such that $u_j = u_1 e_1 \cdots e_{j-1} e_j$. Similarly, let v_i $(i = 1, 2, 3)$ be the bases of the left $U_q(A_2)$ module $V(\varpi_1)$ such that $v_j = f_j f_{j-1} \cdots f_1 v_1$.

k_1	k_2	$V^r(\varpi_1)$	$V(\varpi_1)$
q	1	u_1	v_1
		$\downarrow e_1$	$f_1 \downarrow$
q^{-1}	q	u_2	v_2
		$\downarrow e_2$	$f_2 \downarrow$
1	q^{-1}	u_3	v_3

The left two columns specify the weights for example as $u_2 k_1 = q^{-1} u_2, k_1 v_1 = q v_1$. For the coproduct (10.2), the bases of $V^r(\varpi_2)$ and $V(\varpi_2)$ are similarly given as

k_1	k_2	$V^r(\varpi_2)$	$V(\varpi_2)$
1	q	$u_1 \otimes u_2 - q u_2 \otimes u_1$	$v_1 \otimes v_2 - q v_2 \otimes v_1$
		$\downarrow e_2$	$f_2 \downarrow$
q	q^{-1}	$u_1 \otimes u_3 - q u_3 \otimes u_1$	$v_1 \otimes v_3 - q v_3 \otimes v_1$
		$\downarrow e_1$	$f_1 \downarrow$
q^{-1}	1	$u_2 \otimes u_3 - q u_3 \otimes u_2$	$v_2 \otimes v_3 - q v_3 \otimes v_2$

Here $g = k_i, e_i, f_i$ are to be understood as $\Delta(g)$ in (10.2).

Following (10.22) with $l = 1$ we set

$$t_{ij} = \Psi_{\varpi_1}(u_i \otimes v_j) \tag{10.49}$$

for $1 \leq i, j \leq 3$. They satisfy the relations (3.5) and (3.2) of the earlier definition of $A_q(A_2)$. The formula (10.23) reads

$$\sigma_1 = \Psi_{\varpi_1}(u_1 \otimes v_3), \tag{10.50}$$

$$\sigma_2 = \frac{1}{1+q^2} \Psi_{\varpi_2}\big((u_1 \otimes u_2 - q u_2 \otimes u_1) \otimes (v_2 \otimes v_3 - q v_3 \otimes v_2)\big), \tag{10.51}$$

where $(1+q^2)^{-1}$ is the normalization factor.[7] Thus we see $\sigma_1 = t_{13}$. On the other hand, from

[7] The normalization of σ_i actually does not matter since only $\sigma_i e_i / \sigma_i$ will be used.

$$\begin{aligned}\langle\sigma_2, x\rangle &= \frac{(u_1\otimes u_2 - qu_2\otimes u_1,\, \Delta(x)(v_2\otimes v_3 - qv_3\otimes v_2))}{1+q^2}\\ &= \frac{\langle t_{12}\otimes t_{23} - qt_{13}\otimes t_{22} - qt_{22}\otimes t_{13} + q^2 t_{23}\otimes t_{12},\, \Delta(x)\rangle}{1+q^2}\\ &= \frac{\langle t_{12}t_{23} - qt_{13}t_{22} - qt_{22}t_{13} + q^2 t_{23}t_{12},\, x\rangle}{1+q^2}\quad (\forall x\in U_q(A_2)),\end{aligned}$$

we find $\sigma_2 = (1+q^2)^{-1}(t_{12}t_{23} - qt_{13}t_{22} - qt_{22}t_{13} + q^2 t_{23}t_{12})$.[8] Using the relations $[t_{12}, t_{23}] = (q-q^{-1})t_{22}t_{13}$ and $[t_{22}, t_{13}] = 0$ from (3.5), this is simplified into $\sigma_2 = t_{12}t_{23} - qt_{22}t_{13}$, which is the (3, 1)-quantum minor of $(t_{ij})_{1\le i,j\le 3}$.

Let us turn to $\sigma_i e_i$. First we note

$$\langle t_{ij}k_r, x\rangle = (u_i k_r, xv_j) = q^{\delta_{ir}-\delta_{i,r+1}}(u_i, xv_j) = q^{\delta_{ir}-\delta_{i,r+1}}\langle t_{ij}, x\rangle, \tag{10.52}$$

$$\langle t_{ij}e_r, x\rangle = (u_i e_r, xv_j) = \delta_{ir}(u_{i+1}, xv_j) = \delta_{ir}\langle t_{i+1,j}, x\rangle. \tag{10.53}$$

They imply

$$t_{ij}k_r = q^{\delta_{ir}-\delta_{i,r+1}}t_{ij}, \qquad t_{ij}e_r = \delta_{ir}t_{i+1,j}. \tag{10.54}$$

Using this and the coproduct Δ in (10.2), we see

$$\begin{aligned}\langle\sigma_1 e_1, x\rangle &= \langle t_{13}e_1, x\rangle = \langle t_{23}, x\rangle,\\ \langle\sigma_2 e_2, x\rangle &= \langle (t_{12}\otimes t_{23} - qt_{22}\otimes t_{13})\Delta(e_2), \Delta(x)\rangle\\ &= \langle t_{12}k_2\otimes t_{23}e_2 - qt_{22}e_2\otimes t_{13}, \Delta(x)\rangle\\ &= \langle t_{12}\otimes t_{33} - qt_{32}\otimes t_{13}, \Delta(x)\rangle = \langle t_{12}t_{33} - qt_{32}t_{13}, x\rangle.\end{aligned}$$

In these calculations, one should distinctively recognize that $t_{13}e_1$ for instance is an action of $e_1\in U_q(A_2)$ on $t_{13}\in A_q(A_2)$ viewed as an element of a right $U_q(A_2)$ module, whereas $t_{12}t_{33}$ is just a multiplication within $A_q(A_2)$. To summarize, we have shown:

Lemma 10.10 *For $A_q(A_2)$, the following relations are valid:*

$$\sigma_1 = t_{13},\quad \sigma_2 = t_{12}t_{23} - qt_{22}t_{13},\quad \sigma_1 e_1 = t_{23},\quad \sigma_2 e_2 = t_{12}t_{33} - qt_{32}t_{13}. \tag{10.55}$$

From (3.35) and Lemma 10.10, we find

$$\pi_{\mathbf{1}}(\sigma_1) = \mathbf{k}_1\mathbf{k}_2,\quad \pi_{\mathbf{1}}(\sigma_1 e_1) = \mathbf{a}_1^+\mathbf{k}_2,\quad \pi_{\mathbf{1}}(\sigma_2) = \mathbf{k}_2\mathbf{k}_3,\quad \pi_{\mathbf{1}}(\sigma_2 e_2) = \mathbf{a}_1^-\mathbf{a}_2^+\mathbf{k}_3 + \mathbf{k}_1\mathbf{a}_3^+,$$

where a notation like $\mathbf{k}_1\mathbf{a}_3^+ = \mathbf{k}\otimes 1\otimes\mathbf{a}^+$ has been used. Since $\mathbf{k}\in\mathrm{End}(\mathcal{F}_q)$ is invertible, so is $\pi_{\mathbf{i}}(\sigma_i)$ and we may write

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[8] The calculation is displayed to illustrate how this could be concluded directly from (10.51) and the definition (10.23).

$$\pi_{\mathbf{1}}(\xi_1) = \lambda_1 \mathbf{a}_1^+ \mathbf{k}_1^{-1}, \quad \pi_{\mathbf{1}}(\xi_2) = \lambda_2(\mathbf{a}_1^- \mathbf{a}_2^+ \mathbf{k}_2^{-1} + \mathbf{k}_1 \mathbf{k}_2^{-1} \mathbf{a}_3^+ \mathbf{k}_3^{-1}),$$

where $\lambda_1 = \lambda_2 = (1-q^2)^{-1}$. Thus (10.43) leads to

$$\pi_{\mathbf{1}}(\xi_1)|a,b,c\rangle\rangle = |a+1,b,c\rangle\rangle, \tag{10.56}$$

$$\pi_{\mathbf{1}}(\xi_2)|a,b,c\rangle\rangle = [a]_1|a-1,b+1,c\rangle\rangle + q^{a-b}|a,b,c+1\rangle\rangle. \tag{10.57}$$

These formulas agree with (10.48) proving (10.40) for $\mathbf{i} = \mathbf{1}$. The other case $\mathbf{i} = \mathbf{2}$ also holds due to the symmetry $\pi_{\mathbf{2}}(\xi_i) = \pi_{\mathbf{1}}(\xi_{3-i})$. Thus Proposition 10.7 is established for A_2.

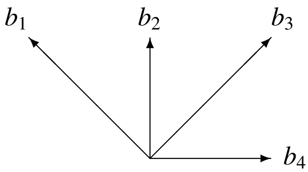
In terms of the 3DR in Chap. 3, Theorem 10.6 implies

$$E_{\mathbf{i}}^{a,b,c} = \sum_{i,j,k} R_{ijk}^{abc}\, E_{\mathbf{i}'}^{k,j,i}. \tag{10.58}$$

This is valid either for $(\mathbf{i}, \mathbf{i}') = (\mathbf{1}, \mathbf{2})$ or $(\mathbf{2}, \mathbf{1})$ thanks to (3.62). The weight conservation (3.48) assures the equality of weights of the two sides.

10.4.2 Explicit Formulas for C_2

Consider $\mathfrak{g} = C_2$.



The q-Serre relations are

$$\begin{aligned} &e_1^3 e_2 - [3]_1 e_1^2 e_2 e_1 + [3]_1 e_1 e_2 e_1^2 - e_2 e_1^3 = 0, \\ &e_2^2 e_1 - [2]_2 e_2 e_1 e_2 + e_1 e_2^2 = 0, \end{aligned} \tag{10.59}$$

where $[m]_1 = \langle m \rangle / \langle 1 \rangle$ and $[m]_2 = \langle 2m \rangle / \langle 2 \rangle$. For simplicity we write the positive root vectors $e_{\beta_{i_r}}$ in (10.4) with $(i_1, \dots, i_4) = \mathbf{2}$ (10.31) as

$$\begin{aligned} &b_1 = e_{\beta_1} = e_2, \quad b_2 = e_{\beta_2} = e_1 e_2 - q^2 e_2 e_1, \\ &b_3 = e_{\beta_3} = \frac{1}{[2]_1}(e_1 b_2 - b_2 e_1), \quad b_4 = e_{\beta_4} = e_1. \end{aligned} \tag{10.60}$$

Their commutation relations are

$$b_2b_1 = q^{-2}b_1b_2, \qquad b_3b_1 = -q^{-1}\langle 1\rangle[2]_1^{-1}b_2^2 + b_1b_3, \tag{10.61}$$
$$b_4b_1 = b_2 + q^2b_1b_4, \qquad b_3b_2 = q^{-2}b_2b_3, \tag{10.62}$$
$$b_4b_2 = [2]_1b_3 + b_2b_4, \quad b_4b_3 = q^{-2}b_3b_4. \tag{10.63}$$

Lemma 10.11 *For* $\tilde{E}_{\mathbf{2}}^{a,b,c,d} = b_1^ab_2^bb_3^cb_4^d$, *we have*

$$\begin{aligned}
\tilde{E}_{\mathbf{2}}^{a,b,c,d}\cdot e_1 &= \tilde{E}_{\mathbf{2}}^{a,b,c,d+1},\\
\tilde{E}_{\mathbf{2}}^{a,b,c,d}\cdot e_2 &= [d]_1q^{d-2c-1}\tilde{E}_{\mathbf{2}}^{a,b+1,c,d-1} + q^{2(d-b)}\tilde{E}_{\mathbf{2}}^{a+1,b,c,d}\\
&\quad - \langle 1\rangle q^{2d-2c+1}[c]_2[2]_1^{-1}\tilde{E}_{\mathbf{2}}^{a,b+2,c-1,d} + [d-1]_1[d]_1\tilde{E}_{\mathbf{2}}^{a,b,c+1,d-2},\\
e_1\cdot\tilde{E}_{\mathbf{2}}^{a,b,c,d} &= [2]_1[b]_1q^{2a-b+1}\tilde{E}_{\mathbf{2}}^{a,b-1,c+1,d} + q^{2a-2c}\tilde{E}_{\mathbf{2}}^{a,b,c,d+1} + [a]_2\tilde{E}_{\mathbf{2}}^{a-1,b+1,c,d},\\
e_2\cdot\tilde{E}_{\mathbf{2}}^{a,b,c,d} &= \tilde{E}_{\mathbf{2}}^{a+1,b,c,d}.
\end{aligned}$$

Proof By induction, we have

$$\begin{aligned}
b_4b_1^n &= b_1^nb_4q^{2n} + [n]_2b_1^{n-1}, b_2,\\
b_4b_2^n &= [2]_1[n]_1b_2^{n-1}b_3q^{-n+1} + b_2^nb_4,\\
b_4b_3^n &= q^{-2n}b_3^nb_4,\\
b_4^nb_1 &= [n]_1b_2b_4^{n-1}q^{n-1} + b_1b_4^nq^{2n} + [n-1]_1[n]_1b_3b_4^{n-2},\\
b_3^nb_1 &= -q^{1-2n}\langle 1\rangle[n]_2[2]_1^{-1}b_2^2b_3^{n-1} + b_1b_3^n,\\
b_3^nb_2 &= q^{-2n}b_2b_3^n,\\
b_2^nb_1 &= q^{-2n}b_1b_2^n.
\end{aligned}$$

The lemma is a direct consequence of these formulas. □

Set $\tilde{E}_{\mathbf{1}}^{a,b,c,d} = \chi(\tilde{E}_{\mathbf{2}}^{d,c,b,a})$. The left multiplication formula for this basis is deduced from the above lemma by applying χ.

Let u_i and v_i $(i = 1, 2, 3, 4)$ be bases of $V^r(\varpi_1)$ and $V(\varpi_1)$ such that $u_j = u_1e_1\cdots e_{j-1}e_j$ and $v_j = f_jf_{j-1}\cdots f_1v_1$, where $e_3 = e_1$, $f_3 = f_1$ just temporarily.

k_1	k_2	$V^r(\varpi_1)$	$V(\varpi_1)$
q	1	u_1	v_1
		$\downarrow e_1$	$f_1\downarrow$
q^{-1}	q	u_2	v_2
		$\downarrow e_2$	$f_2\downarrow$
q	q^{-1}	u_3	v_3
		$\downarrow e_1$	$f_1\downarrow$
q^{-1}	1	u_4	v_4

The left two columns specify the weights as in the A_2 case. For the coproduct (10.2), the bases of $V(\varpi_2)$ and $V^r(\varpi_2)$ are similarly given as

$$\begin{array}{llcc}
k_1 & k_2 & V^r(\varpi_2) & V(\varpi_2) \\
1 & q & u_1 \otimes u_2 - q u_2 \otimes u_1 & v_1 \otimes v_2 - q v_2 \otimes v_1 \\
 & & \downarrow e_2 & f_2 \downarrow \\
q^2 & q^{-1} & u_1 \otimes u_3 - q u_3 \otimes u_1 & v_1 \otimes v_3 - q v_3 \otimes v_1 \\
 & & \downarrow e_1 & f_1 \downarrow \\
1 & 1 & \begin{array}{c} u_2 \otimes u_3 + q u_1 \otimes u_4 \\ -q u_4 \otimes u_1 - q^2 u_3 \otimes u_2 \end{array} & \begin{array}{c} v_2 \otimes v_3 + q v_1 \otimes v_4 \\ -q v_4 \otimes v_1 - q^2 v_3 \otimes v_2 \end{array} \\
 & & \downarrow e_1 & f_1 \downarrow \\
q^{-2} & q & u_2 \otimes u_4 - q u_4 \otimes u_2 & v_2 \otimes v_4 - q v_4 \otimes v_2 \\
 & & \downarrow e_2 & f_2 \downarrow \\
1 & q^{-1} & u_3 \otimes u_4 - q u_4 \otimes u_3 & v_3 \otimes v_4 - q v_4 \otimes v_3
\end{array}$$

Arrows here indicate the images only up to overall normalization.

We adopt the definition of t_{ij} in (10.22) with $l = 1$ for $1 \le i, j \le 4$. Then t_{ij}'s satisfy the relations (5.1), (5.2) of the earlier definition of $A_q(C_2)$. The formula (10.23) reads as

$$\sigma_1 = \Psi_{\varpi_1}(u_1 \otimes v_4), \tag{10.64}$$

$$\sigma_2 = \frac{1}{1+q^2}\Psi_{\varpi_2}\big((u_1 \otimes u_2 - q u_2 \otimes u_1) \otimes (v_3 \otimes v_4 - q v_4 \otimes v_3)\big). \tag{10.65}$$

By a calculation similar to $A_q(A_2)$ using the commutation relations

$$[t_{24}, t_{13}] = (q - q^{-1}) t_{23} t_{14}, \quad [t_{14}, t_{23}] = 0, \tag{10.66}$$

we get:

Lemma 10.12 *For $A_q(C_2)$, the following relations are valid:*

$$\sigma_1 = t_{14}, \quad \sigma_2 = t_{13}t_{24} - q t_{23} t_{14}, \quad \sigma_1 e_1 = t_{24}, \quad \sigma_2 e_2 = t_{13} t_{34} - q t_{33} t_{14}. \tag{10.67}$$

Images of the generators t_{ij} by the representations $\pi_{\mathbf{1}}$ and $\pi_{\mathbf{2}}$ in (10.31) are available in Sect. 5.4 as $\pi_{\mathbf{1}}(t_{ij}) = P_{14}P_{23}\pi_{2121}(\tilde{\Delta}(t_{ij}))P_{14}P_{23}$ and $\pi_{\mathbf{2}}(t_{ij}) = \pi_{2121}(\Delta(t_{ij}))$, where the conjugation by $P_{14}P_{23}$ reverses the order of the four-fold tensor product. See (5.39) and (5.40). From (5.37), the relations (5.41)–(5.56) are displaying the concrete form of $\pi_{\mathbf{2}}(t_{ij})K = K(P_{14}P_{23}\pi_{\mathbf{1}}(t_{ij})P_{14}P_{23})$. For convenience, we pick those generators appearing in Lemma 10.12:

$$\pi_{\mathbf{1}}(t_{13}) = \mathbf{a}_1^-\mathbf{k}_3\mathbf{K}_4 + \mathbf{k}_1\mathbf{A}_2^-\mathbf{a}_3^+\mathbf{K}_4 + \mathbf{k}_1\mathbf{K}_2\mathbf{a}_3^-\mathbf{A}_4^+, \tag{10.68}$$
$$\pi_{\mathbf{1}}(t_{14}) = -\mathbf{k}_1\mathbf{K}_2\mathbf{k}_3, \tag{10.69}$$
$$\pi_{\mathbf{1}}(t_{23}) = \mathbf{a}_1^+\mathbf{A}_2^-\mathbf{a}_3^+\mathbf{K}_4 + \mathbf{a}_1^+\mathbf{K}_2\mathbf{a}_3^-\mathbf{A}_4^+ - q\mathbf{k}_1\mathbf{k}_3\mathbf{K}_4, \tag{10.70}$$
$$\pi_{\mathbf{1}}(t_{24}) = -\mathbf{a}_1^+\mathbf{K}_2\mathbf{k}_3, \tag{10.71}$$
$$\pi_{\mathbf{1}}(t_{33}) = \mathbf{a}_1^-\mathbf{A}_2^+\mathbf{a}_3^-\mathbf{A}_4^+ - q^2\mathbf{a}_1^-\mathbf{K}_2\mathbf{a}_3^+\mathbf{K}_4 - q\mathbf{k}_1\mathbf{k}_3\mathbf{A}_4^+, \tag{10.72}$$
$$\pi_{\mathbf{1}}(t_{34}) = -\mathbf{a}_1^-\mathbf{A}_2^+\mathbf{k}_3 - \mathbf{k}_1\mathbf{a}_3^+, \tag{10.73}$$
$$\pi_{\mathbf{2}}(t_{13}) = \mathbf{k}_2\mathbf{K}_3\mathbf{a}_4^-, \tag{10.74}$$
$$\pi_{\mathbf{2}}(t_{14}) = -\mathbf{k}_2\mathbf{K}_3\mathbf{k}_4, \tag{10.75}$$
$$\pi_{\mathbf{2}}(t_{23}) = \mathbf{A}_1^-\mathbf{a}_2^+\mathbf{K}_3\mathbf{a}_4^- + \mathbf{K}_1\mathbf{a}_2^-\mathbf{A}_3^+\mathbf{a}_4^- - q\mathbf{K}_1\mathbf{k}_2\mathbf{k}_4, \tag{10.76}$$
$$\pi_{\mathbf{2}}(t_{24}) = -\mathbf{A}_1^-\mathbf{a}_2^+\mathbf{K}_3\mathbf{k}_4 - \mathbf{K}_1\mathbf{a}_2^-\mathbf{A}_3^+\mathbf{k}_4 - \mathbf{K}_1\mathbf{k}_2\mathbf{a}_4^+, \tag{10.77}$$
$$\pi_{\mathbf{2}}(t_{33}) = \mathbf{A}_1^+\mathbf{a}_2^-\mathbf{A}_3^+\mathbf{a}_4^- - q\mathbf{A}_1^+\mathbf{k}_2\mathbf{k}_4 - q^2\mathbf{K}_1\mathbf{a}_2^+\mathbf{K}_3\mathbf{a}_4^-, \tag{10.78}$$
$$\pi_{\mathbf{2}}(t_{34}) = -\mathbf{A}_1^+\mathbf{a}_2^-\mathbf{A}_3^+\mathbf{k}_4 - \mathbf{A}_1^+\mathbf{k}_2\mathbf{a}_4^+ + q^2\mathbf{K}_1\mathbf{a}_2^+\mathbf{K}_3\mathbf{k}_4. \tag{10.79}$$

From this and Lemma 10.12 we get

$$\begin{aligned}
\pi_{\mathbf{1}}(\sigma_1) &= -\mathbf{k}_1\mathbf{K}_2\mathbf{k}_3,\\
\pi_{\mathbf{1}}(\sigma_1 e_1) &= -\mathbf{a}_1^+\mathbf{K}_2\mathbf{k}_3,\\
\pi_{\mathbf{1}}(\sigma_2) &= -\mathbf{K}_2{\mathbf{k}_3}^2\mathbf{K}_4,\\
\pi_{\mathbf{1}}(\sigma_2 e_2) &= -\mathbf{a}_1^{-2}\mathbf{A}_2^+{\mathbf{k}_3}^2\mathbf{K}_4 - [2]_1\mathbf{a}_1^-\mathbf{k}_1\mathbf{a}_3^+\mathbf{k}_3\mathbf{K}_4 - {\mathbf{k}_1}^2\mathbf{A}_2^-\mathbf{a}_3^{+2}\mathbf{K}_4 - \mathbf{A}_4^+{\mathbf{k}_1}^2\mathbf{K}_2,\\
\lambda_1^{-1}\pi_{\mathbf{1}}(\xi_1) &= \mathbf{a}_1^+{\mathbf{k}_1}^{-1},\\
\lambda_2^{-1}\pi_{\mathbf{1}}(\xi_2) &= \mathbf{a}_1^{-2}\mathbf{A}_2^+{\mathbf{K}_2}^{-1} + {\mathbf{k}_1}^2\mathbf{A}_2^-{\mathbf{K}_2}^{-1}\mathbf{a}_3^{+2}{\mathbf{k}_3}^{-2} + [2]_1\mathbf{a}_1^-\mathbf{k}_1{\mathbf{K}_2}^{-1}\mathbf{a}_3^+{\mathbf{k}_3}^{-1}\\
&\quad + {\mathbf{k}_1}^2{\mathbf{k}_3}^{-2}\mathbf{A}_4^+{\mathbf{K}_4}^{-1},\\
\pi_{\mathbf{2}}(\sigma_1) &= -\mathbf{k}_2\mathbf{K}_3\mathbf{k}_4,\\
\pi_{\mathbf{2}}(\sigma_1 e_1) &= -\mathbf{K}_1\mathbf{k}_2\mathbf{a}_4^+ - \mathbf{K}_1\mathbf{a}_2^-\mathbf{A}_3^+\mathbf{k}_4 - \mathbf{A}_1^-\mathbf{a}_2^+\mathbf{K}_3\mathbf{k}_4,\\
\pi_{\mathbf{2}}(\sigma_2) &= -\mathbf{K}_1\mathbf{k}_2^2\mathbf{K}_3,\\
\pi_{\mathbf{2}}(\sigma_2 e_2) &= -\mathbf{A}_1^+\mathbf{k}_2^2\mathbf{K}_3,\\
\lambda_1^{-1}\pi_{\mathbf{2}}(\xi_1) &= \mathbf{A}_1^-\mathbf{a}_2^+\mathbf{k}_2^{-1} + \mathbf{K}_1\mathbf{a}_2^-\mathbf{k}_2^{-1}\mathbf{A}_3^+\mathbf{K}_3^{-1} + \mathbf{K}_1\mathbf{K}_3^{-1}\mathbf{a}_4^+\mathbf{k}_4^{-1},\\
\lambda_2^{-1}\pi_{\mathbf{2}}(\xi_2) &= \mathbf{A}_1^+{\mathbf{K}_1}^{-1}.
\end{aligned}$$

Note that $\pi_{\mathbf{i}}(\sigma_i)$ is invertible. Comparing these formulas with Lemma 10.11 by using (10.43), the equality (10.40) is directly checked. Thus Proposition 10.7 is established for C_2.

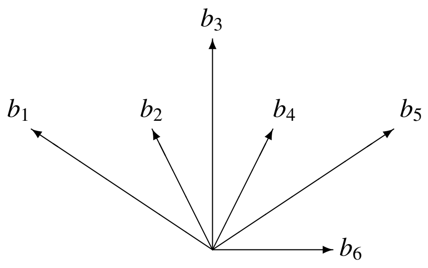
In terms of the 3D K in Chap. 5, Theorem 10.6 implies

$$E_{\mathbf{2}}^{a,b,c,d} = \sum_{i,j,k,l} K_{ijkl}^{abcd}\, E_{\mathbf{1}}^{l,k,j,i}. \tag{10.80}$$

The weight conservation (5.65) assures the equality of weights of the two sides.

10.4.3 Explicit Formulas for G_2

Consider $\mathfrak{g} = G_2$.



The q-Serre relations are

$$\begin{aligned} &e_1^4e_2 - [4]_1e_1^3e_2e_1 + [4]_1[3]_1/[2]_1^{-1}e_1^2e_2e_1^2 - [4]_1e_1e_2e_1^3 + e_2e_1^4 = 0, \\ &e_2^2e_1 - [2]_2e_2e_1e_2 + e_1e_2^2 = 0, \end{aligned} \tag{10.81}$$

where $[m]_1 = \langle m\rangle/\langle 1\rangle$ and $[m]_2 = \langle 3m\rangle/\langle 3\rangle$. For simplicity we write the positive root vectors $e_{\beta_{i_r}}$ in (10.4) with $(i_1, \ldots, i_6) = \mathbf{2}$ (10.32) as

$$\begin{aligned} &b_1 = e_{\beta_1} = e_2, \quad b_2 = e_{\beta_2} = e_1e_2 - q^3e_2e_1, \\ &b_4 = e_{\beta_3} = \frac{1}{[2]_1}(e_1b_2 - qb_2e_1), \quad b_5 = e_{\beta_4} = \frac{1}{[3]_1}(e_1b_4 - q^{-1}b_4e_1), \\ &b_3 = e_{\beta_5} = \frac{1}{[3]_1}(b_4b_2 - q^{-1}b_2b_4), \quad b_6 = e_{\beta_6} = e_1. \end{aligned} \tag{10.82}$$

Their commutation relations are

$$b_2b_1 = b_1b_2q^{-3}, \qquad b_3b_1 = \langle 1\rangle^2b_2^3q^{-3}[3]_1^{-1} + b_1b_3q^{-3}, \tag{10.83}$$

$$b_4b_1 = b_1b_4 - b_2^2\langle 1\rangle q^{-1}, \tag{10.84}$$

$$b_5b_1 = b_1b_5q^3 - b_2b_4\langle 1\rangle q^{-1} - (q^4 + q^2 - 1)b_3q^{-3}, \tag{10.85}$$

$$b_6b_1 = b_1b_6q^3 + b_2, \qquad b_3b_2 = b_2b_3q^{-3}, \tag{10.86}$$

$$b_4b_2 = b_2b_4q^{-1} + b_3[3]_1, \qquad b_5b_2 = b_2b_5 - b_4^2\langle 1\rangle q^{-1}, \tag{10.87}$$

$$b_6b_2 = qb_2b_6 + b_4[2]_1, \qquad b_4b_3 = b_3b_4q^{-3}, \tag{10.88}$$

$$b_5b_3 = \langle 1\rangle^2b_4^3q^{-3}[3]_1^{-1} + b_3b_5q^{-3}, \tag{10.89}$$

$$b_6b_3 = b_3b_6 - b_4^2\langle 1\rangle q^{-1}, \qquad b_5b_4 = b_4b_5q^{-3}, \tag{10.90}$$

$$b_6b_4 = [3]_1b_5 + b_4b_6q^{-1}, \qquad b_6b_5 = b_5b_6q^{-3}. \tag{10.91}$$

Lemma 10.13 *For* $\tilde{E}_{\mathbf{2}}^{a,b,c,d,e,f} = b_1^a b_2^b b_3^c b_4^d b_5^e b_6^f$, *we have*

$$\begin{aligned}
\tilde{E}_{\mathbf{2}}^{a,b,c,d,e,f} \cdot e_1 &= \tilde{E}_{\mathbf{2}}^{a,b,c,d,e,f+1}. \\
\tilde{E}_{\mathbf{2}}^{a,b,c,d,e,f} \cdot e_2 &= -\langle 1\rangle [e]_2 q^{-3c-d+3f-1} \tilde{E}_{\mathbf{2}}^{a,b+1,c,d+1,e-1,f} \\
&+ \langle 1\rangle^2 [e-1]_2 [e]_2 [3]_1^{-1} q^{-3e+3f+3} \tilde{E}_{\mathbf{2}}^{a,b,c,d+3,e-2,f} \\
&- \langle 3\rangle [d-1]_1 [d]_1 q^{-3c-2d+3e+3f+1} \tilde{E}_{\mathbf{2}}^{a,b+1,c+1,d-2,e,f} \\
&- \langle 1\rangle [d]_1 q^{-6c-d+3(e+f)} \tilde{E}_{\mathbf{2}}^{a,b+2,c,d-1,e,f} \\
&+ [f-1]_1 [f]_1 q^{-3e+f-2} \tilde{E}_{\mathbf{2}}^{a,b,c,d+1,e,f-2} \\
&+ [3]_1 [d]_1 [f]_1 q^{2f-2d} \tilde{E}_{\mathbf{2}}^{a,b,c+1,d-1,e,f-1} \\
&+ [f]_1 q^{-3c-d+2f-2} \tilde{E}_{\mathbf{2}}^{a,b+1,c,d,e,f-1} \\
&+ q^{-3(b+c-e-f)} \tilde{E}_{\mathbf{2}}^{a+1,b,c,d,e,f} \\
&+ \langle 1\rangle^2 [c]_2 [3]_1^{-1} q^{3(-2c+e+f+1)} \tilde{E}_{\mathbf{2}}^{a,b+3,c-1,d,e,f} \\
&- \langle 3\rangle [d-2]_1 [d-1]_1 [d]_1 q^{3(-d+e+f+2)} \tilde{E}_{\mathbf{2}}^{a,b,c+2,d-3,e,f} \\
&- \langle 1\rangle [e]_2 [f]_1 q^{-3e+2f} \tilde{E}_{\mathbf{2}}^{a,b,c,d+2,e-1,f-1} \\
&- [e]_2 q^{-3d+3f} (q^{2d+1}[3]_1 - [2]_2) \tilde{E}_{\mathbf{2}}^{a,b,c+1,d,e-1,f} \\
&+ [f-2]_1 [f-1]_1 [f]_1 \tilde{E}_{\mathbf{2}}^{a,b,c,d,e+1,f-3}. \\
e_1 \cdot \tilde{E}_{\mathbf{2}}^{a,b,c,d,e,f} &= -\langle 1\rangle [c]_2 q^{3a+b-3c+2} \tilde{E}_{\mathbf{2}}^{a,b,c-1,d+2,e,f} \\
&+ [3]_1 [b-1]_1 [b]_1 q^{3a-b+2} \tilde{E}_{\mathbf{2}}^{a,b-2,c+1,d,e,f} \\
&+ [3]_1 [d]_1 q^{3a+b-2d+2} \tilde{E}_{\mathbf{2}}^{a,b,c,d-1,e+1,f} \\
&+ q^{3a+b-d-3e} \tilde{E}_{\mathbf{2}}^{a,b,c,d,e,f+1} \\
&+ [2]_1 [b]_1 q^{3(a-c)} \tilde{E}_{\mathbf{2}}^{a,b-1,c,d+1,e,f} \\
&+ [a]_2 \tilde{E}_{\mathbf{2}}^{a-1,b+1,c,d,e,f}. \\
e_2 \cdot \tilde{E}_{\mathbf{2}}^{a,b,c,d,e,f} &= \tilde{E}_{\mathbf{2}}^{a+1,b,c,d,e,f}.
\end{aligned}$$

Proof By induction, we have

$$\begin{aligned}
b_6 b_1^n &= q^{3n} b_1^n b_6 + [n]_2 b_1^{n-1} b_2, \\
b_6 b_2^n &= [3]_1 q^{2-n} [n-1]_1 [n]_1 b_2^{n-2} b_3 + q^n b_2^n b_6 + [2]_1 [n]_1 b_2^{n-1} b_4, \\
b_4 b_3^n &= q^{-3n} b_3^n b_4, \\
b_6 b_3^n &= b_3^n b_6 - \langle 1\rangle q^{2-3n} [n]_2 b_3^{n-1} b_4 b_4, \\
b_6 b_4^n &= [3]_1 q^{2-2n} [n]_1 b_4^{n-1} b_5 + q^{-n} b_4^n b_6, \\
b_6 b_5^n &= q^{-3n} b_5^n b_6,
\end{aligned}$$

and

$$\begin{aligned}
b_6^n b_1 &= q^{n-2}[n-1]_1[n]_1 b_4 b_6^{n-2} + q^{3n} b_1 b_6^n \\
&\quad + q^{2(n-1)}[n]_1 b_2 b_6^{n-1} + [n-2]_1[n-1]_1[n]_1 b_5 b_6^{n-3}, \\
b_5^n b_1 &= \langle 1\rangle^2 q^{-3(n-1)}[n-1]_2[n]_2[3]_1^{-1} b_4^3 b_5^{n-2} + q^{3n} b_1 b_5^n \\
&\quad - q^{-3}(q^4+q^2-1)[n]_2 b_3 b_5^{n-1} - q^{-1}\langle 1\rangle [n]_2 b_2 b_4 b_5^{n-1}, \\
b_5^n b_2 &= b_2 b_5^n - \langle 1\rangle q^{2-3n}[n]_2 b_4 b_4 b_5^{n-1}, \\
b_5^n b_4 &= q^{-3n} b_4 b_5^n, \\
b_4^n b_1 &= -\langle 3\rangle q^{6-3n}[n-2]_1[n-1]_1[n]_1 b_3^2 b_4^{n-3} - \langle 1\rangle q^{-n}[n]_1 b_2^2 b_4^{n-1} \\
&\quad - \langle 3\rangle q^{1-2n}[n-1]_1[n]_1 b_2 b_3 b_4^{n-2} + b_1 b_4^n, \\
b_4^n b_2 &= [3]_1 q^{2-2n}[n]_1 b_3 b_4^{n-1} + q^{-n} b_2 b_4^n, \\
b_4^n b_3 &= q^{-3n} b_3 b_4^n, \\
b_3^n b_1 &= q^{-3n} b_1 b_3^n + \langle 1\rangle^2 q^{3-6n}[n]_2[3]_1^{-1} b_2^3 b_3^{n-1}, \\
b_3^n b_2 &= q^{-3n} b_2 b_3^n, \\
b_2^n b_1 &= q^{-3n} b_1 b_2^n.
\end{aligned}$$

The lemma is a direct consequence of these formulas. □

Let v_i ($i = 1, \ldots, 7$) be the basis of $V(\varpi_1)$ for which the representation matrix is given by (8.79)–(8.81). Its highest and lowest weight vectors are v_1 and v_7, respectively. Let $u_i \in V^r(\varpi_1)$ be the dual base of v_i.

The representation $V(\varpi_2)$ is the adjoint representation with dimension 14. Its lowest weight vector is $v_{14}^{(14)}$ in (8.84), which is $v_6 \otimes v_7 - q v_7 \otimes v_6$ in the notation here. The highest weight vector of $V^r(\varpi_2)$ is $u_1 \otimes u_2 - q u_2 \otimes u_1$. From these facts we have

$$\sigma_1 = \Psi_{\varpi_1}(u_1 \otimes v_7), \tag{10.92}$$

$$\sigma_2 = \frac{1}{1+q^2}\Psi_{\varpi_2}\big((u_1 \otimes u_2 - q u_2 \otimes u_1) \otimes (v_6 \otimes v_7 - q v_7 \otimes v_6)\big). \tag{10.93}$$

We define t_{ij} by the formula (10.22) with $l = 1$ for $1 \le i, j \le 7$. They satisfy the relations (8.3) and (8.4) of the earlier definition of $A_q(G_2)$. By a calculation similar to $A_q(A_2)$ using the commutation relations

$$[t_{16}, t_{27}] = (q - q^{-1}) t_{26} t_{17}, \quad [t_{17}, t_{26}] = 0, \tag{10.94}$$

we get[9]

[9] σ_2 and $\sigma_2 e_2$ in [102, Eq. (42)] are $(-q)$ times those in Lemma 10.14.

Lemma 10.14 *For $A_q(G_2)$, the following relations are valid:*

$$\sigma_1 = t_{17}, \quad \sigma_2 = t_{16}t_{27} - qt_{27}t_{16}, \quad \sigma_1 e_1 = t_{27}, \quad \sigma_2 e_2 = t_{16}t_{37} - qt_{36}t_{17}. \quad (10.95)$$

Images of the generators t_{ij} by the representations $\pi_{\mathbf{1}}$ and $\pi_{\mathbf{2}}$ in (10.31) are available from (8.11) and (8.12). For convenience, we present explicit formulas for those appearing in Lemma 10.14:

$$\begin{aligned}
\pi_{\mathbf{1}}(t_{16}) &= \mathbf{a}_1^-\mathbf{k}_3\mathbf{K}_4\mathbf{k}_5^2\mathbf{K}_6 + \mathbf{k}_1\mathbf{A}_2^-\mathbf{a}_3^+\mathbf{K}_4\mathbf{k}_5^2\mathbf{K}_6 + \mathbf{k}_1\mathbf{K}_2\mathbf{a}_3^{-2}\mathbf{A}_4^+\mathbf{k}_5^2\mathbf{K}_6 \\
&\quad + [2]_1\mathbf{k}_1\mathbf{K}_2\mathbf{a}_3^-\mathbf{k}_3\mathbf{a}_5^+\mathbf{k}_5\mathbf{K}_6 + \mathbf{k}_1\mathbf{K}_2\mathbf{k}_3^2\mathbf{A}_4^-\mathbf{a}_5^{+2}\mathbf{K}_6 + \mathbf{k}_1\mathbf{K}_2\mathbf{k}_3^2\mathbf{K}_4\mathbf{a}_5^-\mathbf{A}_6^+, \\
\pi_{\mathbf{1}}(t_{17}) &= \mathbf{k}_1\mathbf{K}_2\mathbf{k}_3^2\mathbf{K}_4\mathbf{k}_5, \\
\pi_{\mathbf{1}}(t_{27}) &= \mathbf{a}_1^+\mathbf{K}_2\mathbf{k}_3^2\mathbf{K}_4\mathbf{k}_5, \\
\pi_{\mathbf{1}}(t_{36}) &= \mathbf{a}_1^{-2}\mathbf{A}_2^+\mathbf{a}_2^{-2}\mathbf{A}_4^+\mathbf{k}_5^2\mathbf{K}_6 + [2]_1\mathbf{a}_1^{-2}\mathbf{A}_2^+\mathbf{a}_3^-\mathbf{k}_3\mathbf{a}_5^+\mathbf{k}_5\mathbf{K}_6 + \mathbf{a}_1^{-2}\mathbf{A}_2^+\mathbf{k}_3^2\mathbf{A}_4^-\mathbf{a}_5^{+2}\mathbf{K}_6 \\
&\quad + \mathbf{a}_1^{-2}\mathbf{A}_2^+\mathbf{k}_3^2\mathbf{K}_4\mathbf{a}_5^-\mathbf{A}_6^+ - q^3\mathbf{a}_1^{-2}\mathbf{K}_2\mathbf{a}_3^+\mathbf{K}_4\mathbf{k}_5^2\mathbf{K}_6 + [2]_1\mathbf{a}_1^-\mathbf{k}_1\mathbf{a}_5^+\mathbf{k}_5\mathbf{K}_6 \\
&\quad - [2]_1\mathbf{a}_1^-\mathbf{k}_1\mathbf{a}_3^-\mathbf{k}_3\mathbf{A}_4^+\mathbf{k}_5^2\mathbf{K}_6 + [2]_1\mathbf{a}_1^-\mathbf{k}_1\mathbf{a}_3^+\mathbf{k}_3\mathbf{A}_4^-\mathbf{a}_5^{+2}\mathbf{K}_6 + [2]_1\mathbf{a}_1^-\mathbf{k}_1\mathbf{a}_3^+\mathbf{k}_3\mathbf{K}_4\mathbf{a}_5^-\mathbf{A}_6^+ \\
&\quad - q[2]_1^2\mathbf{a}_1^-\mathbf{k}_1\mathbf{k}_3^2\mathbf{a}_5^+\mathbf{k}_5\mathbf{K}_6 + \mathbf{k}_1^2\mathbf{A}_2^-\mathbf{a}_3^{+2}\mathbf{A}_4^-\mathbf{a}_5^{+2}\mathbf{K}_6 + \mathbf{k}_1^2\mathbf{A}_2^-\mathbf{a}_3^{+2}\mathbf{K}_4\mathbf{a}_5^-\mathbf{A}_6^+ \\
&\quad - q^2[2]_1\mathbf{k}_1^2\mathbf{A}_2^-\mathbf{a}_3^+\mathbf{k}_3\mathbf{a}_5^+\mathbf{k}_5\mathbf{K}_6 + q^2\mathbf{k}_1^2\mathbf{A}_2^-\mathbf{k}_3^2\mathbf{A}_4^+\mathbf{k}_5^2\mathbf{K}_6 + \mathbf{k}_1^2\mathbf{K}_2\mathbf{a}_3^-\mathbf{A}_4^+\mathbf{a}_5^-\mathbf{A}_6^+ \\
&\quad - q^3\mathbf{k}_1^2\mathbf{K}_2\mathbf{a}_3^-\mathbf{K}_4\mathbf{a}_5^{+2}\mathbf{K}_6 - q\mathbf{k}_1^2\mathbf{K}_2\mathbf{k}_3\mathbf{k}_5\mathbf{A}_6^+, \\
\pi_{\mathbf{1}}(t_{37}) &= \mathbf{a}_1^{-2}\mathbf{A}_2^+\mathbf{k}_3^2\mathbf{K}_4\mathbf{k}_5 + [2]_1\mathbf{a}_1^-\mathbf{k}_1\mathbf{a}_3^+\mathbf{k}_3\mathbf{K}_4\mathbf{k}_5 + \mathbf{k}_1^2\mathbf{A}_2^-\mathbf{a}_3^{+2}\mathbf{K}_4\mathbf{k}_5 \\
&\quad + \mathbf{k}_1^2\mathbf{K}_2\mathbf{a}_3^-\mathbf{A}_4^+\mathbf{k}_5 + \mathbf{k}_1^2\mathbf{K}_2\mathbf{k}_3\mathbf{a}_5^+, \\
\pi_{\mathbf{2}}(t_{16}) &= \mathbf{k}_2\mathbf{K}_3\mathbf{k}_4^2\mathbf{K}_5\mathbf{a}_6^-, \\
\pi_{\mathbf{2}}(t_{17}) &= \mathbf{k}_2\mathbf{K}_3\mathbf{k}_4^2\mathbf{K}_5\mathbf{k}_6, \\
\pi_{\mathbf{2}}(t_{27}) &= \mathbf{A}_1^-\mathbf{a}_2^+\mathbf{K}_3\mathbf{k}_4^2\mathbf{K}_5\mathbf{k}_6 + \mathbf{K}_1\mathbf{a}_2^{-2}\mathbf{A}_3^+\mathbf{k}_4^2\mathbf{K}_5\mathbf{k}_6 + [2]_1\mathbf{K}_1\mathbf{a}_2^-\mathbf{k}_2\mathbf{a}_4^+\mathbf{k}_4\mathbf{K}_5\mathbf{k}_6 \\
&\quad + \mathbf{K}_1\mathbf{k}_2^2\mathbf{A}_3^-\mathbf{a}_4^{+2}\mathbf{K}_5\mathbf{k}_6 + \mathbf{K}_1\mathbf{k}_2^2\mathbf{K}_3\mathbf{a}_4^-\mathbf{A}_5^+\mathbf{k}_6 + \mathbf{K}_1\mathbf{k}_2^2\mathbf{K}_3\mathbf{k}_4\mathbf{a}_6^+, \\
\pi_{\mathbf{2}}(t_{36}) &= \mathbf{A}_1^+\mathbf{a}_2^{-2}\mathbf{A}_3^+\mathbf{k}_4^2\mathbf{K}_5\mathbf{a}_6^- + [2]_1\mathbf{A}_1^+\mathbf{a}_2^-\mathbf{k}_2\mathbf{a}_4^+\mathbf{k}_4\mathbf{K}_5\mathbf{a}_6^- + \mathbf{A}_1^+\mathbf{k}_2^2\mathbf{A}_3^-\mathbf{a}_4^{+2}\mathbf{K}_5\mathbf{a}_6^- \\
&\quad + \mathbf{A}_1^+\mathbf{k}_2^2\mathbf{K}_3\mathbf{a}_4^-\mathbf{A}_5^+\mathbf{a}_6^- - q\mathbf{A}_1^+\mathbf{k}_2^2\mathbf{K}_3\mathbf{k}_4\mathbf{k}_6 - q^3\mathbf{K}_1\mathbf{a}_2^+\mathbf{K}_3\mathbf{k}_4^2\mathbf{K}_5\mathbf{a}_6^-, \\
\pi_{\mathbf{2}}(t_{37}) &= \mathbf{A}_1^+\mathbf{a}_2^{-2}\mathbf{A}_3^+\mathbf{k}_4^2\mathbf{K}_5\mathbf{k}_6 + [2]_1\mathbf{A}_1^+\mathbf{a}_2^-\mathbf{k}_2\mathbf{a}_4^+\mathbf{k}_4\mathbf{K}_5\mathbf{k}_6 + \mathbf{A}_1^+\mathbf{k}_2^2\mathbf{A}_3^-\mathbf{a}_4^{+2}\mathbf{K}_5\mathbf{k}_6 \\
&\quad + \mathbf{A}_1^+\mathbf{k}_2^2\mathbf{K}_3\mathbf{a}_4^-\mathbf{A}_5^+\mathbf{k}_6 + \mathbf{A}_1^+\mathbf{k}_2^2\mathbf{K}_3\mathbf{k}_4\mathbf{a}_6^+ - q^3\mathbf{K}_1\mathbf{a}_2^+\mathbf{K}_3\mathbf{k}_4^2\mathbf{K}_5\mathbf{k}_6.
\end{aligned}$$

From this and Lemma 10.14 we get

$$
\begin{aligned}
\pi_{\mathbf{1}}(\sigma_1) &= \mathbf{k}_1\mathbf{K}_2\mathbf{k}_3^2\mathbf{K}_4\mathbf{k}_5,\\
\pi_{\mathbf{1}}(\sigma_2) &= \mathbf{K}_2\mathbf{k}_3^3\mathbf{K}_4^2\mathbf{k}_5^3\mathbf{K}_6,\\
\pi_{\mathbf{1}}(\sigma_1 e_1) &= \mathbf{a}_1^+\mathbf{K}_2\mathbf{k}_3^2\mathbf{K}_4\mathbf{k}_5,\\
\pi_{\mathbf{1}}(\sigma_2 e_2) &= \mathbf{k}_1^3\mathbf{K}_2^2\mathbf{k}_3^3\mathbf{K}_4\mathbf{A}_6^+ + [2]_2\mathbf{k}_1^3\mathbf{A}_2^-\mathbf{K}_2\mathbf{A}_4^+\mathbf{K}_4\mathbf{k}_5^3\mathbf{K}_6 + \mathbf{a}_1^{-3}\mathbf{A}_2^+\mathbf{k}_3^3\mathbf{K}_4^2\mathbf{k}_5^3\mathbf{K}_6\\
&\quad + [3]_1\mathbf{a}_1^{-2}\mathbf{k}_1\mathbf{a}_3^+\mathbf{k}_3^2\mathbf{K}_4^2\mathbf{k}_5^3\mathbf{K}_6 + [3]_1\mathbf{a}_1^-\mathbf{k}_1^2\mathbf{K}_2\mathbf{k}_3^2\mathbf{K}_4\mathbf{a}_5^+\mathbf{k}_5^2\mathbf{K}_6\\
&\quad - q[3]_1\mathbf{k}_1^3\mathbf{A}_2^-\mathbf{K}_2\mathbf{k}_3^2\mathbf{A}_4^+\mathbf{K}_4\mathbf{k}_5^3\mathbf{K}_6 + [3]_1\mathbf{k}_1^3\mathbf{K}_2^2\mathbf{a}_3^-\mathbf{k}_3^2\mathbf{a}_5^{+2}\mathbf{k}_5\mathbf{K}_6\\
&\quad + \mathbf{k}_1^3\mathbf{K}_2^2\mathbf{k}_3^3\mathbf{A}_4^-\mathbf{a}_5^{+3}\mathbf{K}_6 + [3]_1\mathbf{a}_1^-\mathbf{k}_1^2\mathbf{A}_2^-\mathbf{a}_3^{+2}\mathbf{k}_3\mathbf{K}_4^2\mathbf{k}_5^3\mathbf{K}_6\\
&\quad + [3]_1\mathbf{a}_1^-\mathbf{k}_1^2\mathbf{K}_2\mathbf{a}_3^-\mathbf{k}_3\mathbf{A}_4^+\mathbf{K}_4\mathbf{k}_5^3\mathbf{K}_6 + \mathbf{k}_1^3\mathbf{A}_2^{-2}\mathbf{a}_3^{+3}\mathbf{K}_4^2\mathbf{k}_5^3\mathbf{K}_6\\
&\quad + [3]_1\mathbf{k}_1^3\mathbf{A}_2^-\mathbf{K}_2\mathbf{a}_3^+\mathbf{k}_3\mathbf{K}_4\mathbf{a}_5^+\mathbf{k}_5^2\mathbf{K}_6 + \mathbf{k}_1^3\mathbf{K}_2^2\mathbf{a}_3^{-3}\mathbf{A}_4^{+2}\mathbf{k}_5^3\mathbf{K}_6\\
&\quad + [3]_1\mathbf{k}_1^3\mathbf{K}_2^2\mathbf{a}_3^{-2}\mathbf{k}_3\mathbf{A}_4^+\mathbf{a}_5^+\mathbf{k}_5^2\mathbf{K}_6,\\
\lambda_1^{-1}\pi_{\mathbf{1}}(\xi_1) &= \mathbf{a}_1^+\mathbf{k}_1^{-1},\\
\lambda_2^{-1}\pi_{\mathbf{1}}(\xi_2) &= \mathbf{a}_1^{-3}\mathbf{A}_2^+\mathbf{K}_2^{-1} + [2]_2\mathbf{k}_1^3\mathbf{A}_2^-\mathbf{k}_3^{-3}\mathbf{A}_4^+\mathbf{K}_4^{-1} - q[3]_1\mathbf{k}_1^3\mathbf{A}_2^-\mathbf{k}_3^{-1}\mathbf{A}_4^+\mathbf{K}_4^{-1}\\
&\quad + [3]_1\mathbf{a}_1^{-2}\mathbf{k}_1\mathbf{K}_2^{-1}\mathbf{a}_3^+\mathbf{k}_3^{-1} + [3]_1\mathbf{a}_1^-\mathbf{k}_1^2\mathbf{a}_3^-\mathbf{k}_3^{-2}\mathbf{A}_4^+\mathbf{K}_4^{-1}\\
&\quad + [3]_1\mathbf{a}_1^-\mathbf{k}_1^2\mathbf{k}_3^{-1}\mathbf{K}_4^{-1}\mathbf{a}_5^+\mathbf{k}_5^{-1} + \mathbf{k}_1^3\mathbf{K}_2\mathbf{K}_4^{-1}\mathbf{k}_5^{-3}\mathbf{A}_6^+\mathbf{K}_6^{-1}\\
&\quad + [3]_1\mathbf{a}_1^-\mathbf{k}_1^2\mathbf{A}_2^-\mathbf{K}_2^{-1}\mathbf{a}_3^{+2}\mathbf{k}_3^{-2} + [3]_1\mathbf{k}_1^3\mathbf{A}_2^-\mathbf{a}_3^+\mathbf{k}_3^{-2}\mathbf{K}_4^{-1}\mathbf{a}_5^+\mathbf{k}_5^{-1}\\
&\quad + \mathbf{k}_1^3\mathbf{A}_2^{-2}\mathbf{K}_2^{-1}\mathbf{a}_3^{+3}\mathbf{k}_3^{-3} + [3]_1\mathbf{k}_1^3\mathbf{K}_2\mathbf{a}_3^-\mathbf{k}_3^{-1}\mathbf{K}_4^{-2}\mathbf{a}_5^{+2}\mathbf{k}_5^{-2}\\
&\quad + \mathbf{k}_1^3\mathbf{K}_2\mathbf{A}_4^-\mathbf{K}_4^{-2}\mathbf{a}_5^{+3}\mathbf{k}_5^{-3} + \mathbf{k}_1^3\mathbf{K}_2\mathbf{a}_3^{-3}\mathbf{k}_3^{-3}\mathbf{A}_4^{+2}\mathbf{K}_4^{-2}\\
&\quad + [3]_1\mathbf{k}_1^3\mathbf{K}_2\mathbf{a}_3^{-2}\mathbf{k}_3^{-2}\mathbf{A}_4^+\mathbf{K}_4^{-2}\mathbf{a}_5^+\mathbf{k}_5^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\pi_{\mathbf{2}}(\sigma_1) &= \mathbf{k}_2\mathbf{K}_3\mathbf{k}_4^2\mathbf{K}_5\mathbf{k}_6,\\
\pi_{\mathbf{2}}(\sigma_2) &= \mathbf{K}_1\mathbf{k}_2^3\mathbf{K}_3^2\mathbf{k}_4^3\mathbf{K}_5,\\
\pi_{\mathbf{2}}(\sigma_1 e_1) &= \mathbf{K}_1\mathbf{k}_2^2\mathbf{K}_3\mathbf{k}_4\mathbf{a}_6^+ + \mathbf{A}_1^-\mathbf{a}_2^+\mathbf{K}_3\mathbf{k}_4^2\mathbf{K}_5\mathbf{k}_6 + \mathbf{K}_1\mathbf{k}_2^2\mathbf{K}_3\mathbf{a}_4^-\mathbf{A}_5^+\mathbf{k}_6\\
&\quad + \mathbf{K}_1\mathbf{a}_2^{-2}\mathbf{A}_3^+\mathbf{k}_4^2\mathbf{K}_5\mathbf{k}_6 + [2]_1\mathbf{K}_1\mathbf{a}_2^-\mathbf{k}_2\mathbf{a}_4^+\mathbf{k}_4\mathbf{K}_5\mathbf{k}_6 + \mathbf{K}_1\mathbf{k}_2^2\mathbf{A}_3^-\mathbf{a}_4^{+2}\mathbf{K}_5\mathbf{k}_6,\\
\pi_{\mathbf{2}}(\sigma_2 e_2) &= \mathbf{A}_1^+\mathbf{k}_2^3\mathbf{K}_3^2\mathbf{k}_4^3\mathbf{K}_5,\\
\lambda_1^{-1}\pi_{\mathbf{2}}(\xi_1) &= \mathbf{A}_1^-\mathbf{a}_2^+\mathbf{k}_2^{-1} + [2]_1\mathbf{K}_1\mathbf{a}_2^-\mathbf{K}_3^{-1}\mathbf{a}_4^+\mathbf{k}_4^{-1} + \mathbf{K}_1\mathbf{a}_2^{-2}\mathbf{k}_2^{-1}\mathbf{A}_3^+\mathbf{K}_3^{-1}\\
&\quad + \mathbf{K}_1\mathbf{k}_2\mathbf{a}_4^-\mathbf{k}_4^{-2}\mathbf{A}_5^+\mathbf{K}_5^{-1} + \mathbf{K}_1\mathbf{k}_2\mathbf{k}_4^{-1}\mathbf{K}_5^{-1}\mathbf{a}_6^+\mathbf{k}_6^{-1} + \mathbf{K}_1\mathbf{k}_2\mathbf{A}_3^-\mathbf{K}_3^{-1}\mathbf{a}_4^{+2}\mathbf{k}_4^{-2},\\
\lambda_2^{-1}\pi_{\mathbf{2}}(\xi_2) &= \mathbf{A}_1^+\mathbf{K}_1^{-1}.
\end{aligned}
$$

Note that $\pi_{\mathbf{i}}(\sigma_i)$ is invertible. Comparing these formulas with Lemma 10.13 by using (10.43), the equality (10.40) is directly checked. Thus Proposition 10.7 is established for G_2.

In terms of the intertwiner F in Chap. 8, Theorem 10.6 implies

$$
E_{\mathbf{2}}^{a,b,c,d,e,f} = \sum_{i,j,k,l,m,n} F_{ijklmn}^{abcdef}\, E_{\mathbf{1}}^{n,m,l,k,j,i}. \tag{10.96}
$$

The weight conservation (8.29) assures the equality of weights of the two sides.

10.5 Tetrahedron and 3D Reflection Equations from PBW Bases

The relation (10.58) serves as an auxiliary linear system by which the tetrahedron equation (2.6) is established as the non-linear consistency condition. To see this, consider a PBW basis (10.6) of $U_q^+(A_3)$ having the form $E^{a,b,c,d,e}_{1,2,3,1,2,1}$. In addition to $E^{\dots ab\dots}_{\dots 13\dots} = E^{\dots ba\dots}_{\dots 31\dots}$, we may apply (10.58) as

$$E^{\dots abc\dots}_{\dots 121\dots} = \sum R^{abc}_{ijk} E^{\dots kji\dots}_{\dots 212\dots}, \quad E^{\dots abc\dots}_{\dots 212\dots} = \sum R^{abc}_{ijk} E^{\dots kji\dots}_{\dots 121\dots} \tag{10.97}$$

reflecting the $U_q^+(A_2)$ subalgebra structure. Then we have

$$\begin{aligned}
&E^{a,b,c,d,e,f}_{1,2,3,1,2,1} = E^{a,b,d,c,e,f}_{1,2,1,3,2,1} = \sum R^{abd}_{a_1b_1d_1} E^{d_1,b_1,a_1,c,e,f}_{2,1,2,3,2,1}\\
&= \sum R^{abd}_{a_1b_1d_1} R^{a_1ce}_{a_2c_1e_1} E^{d_1,b_1,e_1,c_1,a_2,f}_{2,1,3,2,3,1}\\
&= \sum R^{abd}_{a_1b_1d_1} R^{a_1ce}_{a_2c_1e_1} E^{d_1,e_1,b_1,c_1,f,a_2}_{2,3,1,2,1,3}\\
&= \sum R^{abd}_{a_1b_1d_1} R^{a_1ce}_{a_2c_1e_1} R^{b_1c_1f}_{b_2c_2f_1} E^{d_1,e_1,f_1,c_2,b_2,a_2}_{2,3,2,1,2,3}\\
&= \sum R^{abd}_{a_1b_1d_1} R^{a_1ce}_{a_2c_1e_1} R^{b_1c_1f}_{b_2c_2f_1} R^{d_1e_1f_1}_{d_2e_2f_2} E^{f_2,e_2,d_2,c_2,b_2,a_2}_{3,2,3,1,2,3}.
\end{aligned}$$

There is another route going from $E^{a,b,c,d,e,f}_{1,2,3,1,2,1}$ to $E^{f_2,e_2,d_2,c_2,b_2,a_2}_{3,2,3,1,2,3}$ as

$$\begin{aligned}
&E^{a,b,c,d,e,f}_{1,2,3,1,2,1} = \sum R^{def}_{d_1e_1f_1} E^{a,b,c,f_1,e_1,d_1}_{1,2,3,2,1,2}\\
&= \sum R^{def}_{d_1e_1f_1} R^{bcf_1}_{b_1c_1f_2} E^{a,f_2,c_1,b_1,e_1,d_1}_{1,3,2,3,1,2}\\
&= \sum R^{def}_{d_1e_1f_1} R^{bcf_1}_{b_1c_1f_2} E^{f_2,a,c_1,e_1,b_1,d_1}_{3,1,2,1,3,2}\\
&= \sum R^{def}_{d_1e_1f_1} R^{bcf_1}_{b_1c_1f_2} R^{ac_1e_1}_{a_1c_2e_2} E^{f_2,e_2,c_2,a_1,b_1,d_1}_{3,2,1,2,3,2}\\
&= \sum R^{def}_{d_1e_1f_1} R^{bcf_1}_{b_1c_1f_2} R^{ac_1e_1}_{a_1c_2e_2} R^{a_1b_1d_1}_{a_2b_2d_2} E^{f_2,e_2,c_2,d_2,b_2,a_2}_{3,2,1,3,2,3}\\
&= \sum R^{def}_{d_1e_1f_1} R^{bcf_1}_{b_1c_1f_2} R^{ac_1e_1}_{a_1c_2e_2} R^{a_1b_1d_1}_{a_2b_2d_2} E^{f_2,e_2,d_2,c_2,b_2,a_2}_{3,2,3,1,2,3}.
\end{aligned}$$

Comparison of them leads to

$$\sum R^{abd}_{a_1b_1d_1} R^{a_1ce}_{a_2c_1e_1} R^{b_1c_1f}_{b_2c_2f_1} R^{d_1e_1f_1}_{d_2e_2f_2} = \sum R^{def}_{d_1e_1f_1} R^{bcf_1}_{b_1c_1f_2} R^{ac_1e_1}_{a_1c_2e_2} R^{a_1b_1d_1}_{a_2b_2d_2} \tag{10.98}$$

for arbitrary a, b, c, d, e, f and $a_2, b_2, c_2, d_2, e_2, f_2$. The sums are over $a_1, b_1, c_1, d_1, e_1, f_1 \in \mathbb{Z}_{\geq 0}$ on both sides. They are finite sums due to the weight conservation (3.48). The identity (10.98) reproduces the tetrahedron equation (2.9).

A similar proof of the 3D reflection equation (4.3) is possible based on (10.80). We now start from a PBW basis (10.6) of $U_q^+(C_3)$ having the form $E^{a,b,c,d,e,f,g,h,i}_{3,2,3,1,2,1,3,2,1}$ and apply (10.97) and $E^{\dots abcd\dots}_{\dots 3232\dots} = \sum K^{abcd}_{ijkl} E^{\dots lkji\dots}_{\dots 2323\dots}$. The two routes are as follows:

$$
\begin{aligned}
& E^{a,b,c,d,e,f,g,h,i}_{3,2,3,1,2,1,3,2,1} = E^{a,b,d,c,e,g,f,h,i}_{3,2,1,3,2,3,1,2,1} = \sum R^{fhi}_{f_1h_1i_1} E^{a,b,d,c,e,g,i_1,h_1,f_1}_{3,2,1,3,2,3,2,1,2} \\
& = \sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} E^{a,b,d,i_2,g_1,e_1,c_1,h_1,f_1}_{3,2,1,2,3,2,3,1,2} \\
& = \sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} R^{bdi_2}_{b_1d_1i_3} E^{a,i_3,d_1,b_1,g_1,e_1,c_1,h_1,f_1}_{3,1,2,1,3,2,3,1,2} \\
& = \sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} R^{bdi_2}_{b_1d_1i_3} E^{i_3,a,d_1,g_1,b_1,e_1,h_1,c_1,f_1}_{1,3,2,3,1,2,1,3,2} \\
& = \sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} R^{bdi_2}_{b_1d_1i_3} R^{b_1e_1h_1}_{b_2e_2h_2} E^{i_3,a,d_1,g_1,h_2,e_2,b_2,c_1,f_1}_{1,3,2,3,2,1,2,3,2} \\
& = \sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} R^{bdi_2}_{b_1d_1i_3} R^{b_1e_1h_1}_{b_2e_2h_2} K^{ad_1g_1h_2}_{a_1d_2g_2h_3} E^{i_3,h_3,g_2,d_2,a_1,e_2,b_2,c_1,f_1}_{1,2,3,2,3,1,2,3,2} \\
& = \sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} R^{bdi_2}_{b_1d_1i_3} R^{b_1e_1h_1}_{b_2e_2h_2} K^{ad_1g_1h_2}_{a_1d_2g_2h_3} E^{i_3,h_3,g_2,d_2,e_2,a_1,b_2,c_1,f_1}_{1,2,3,2,1,3,2,3,2} \\
& = \sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} R^{bdi_2}_{b_1d_1i_3} R^{b_1e_1h_1}_{b_2e_2h_2} K^{ad_1g_1h_2}_{a_1d_2g_2h_3} K^{a_1b_2c_1f_1}_{a_2b_3c_2f_2} E^{i_3,h_3,g_2,d_2,e_2,f_2,c_2,b_3,a_2}_{1,2,3,2,1,2,3,2,3} \\
& = \sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} R^{bdi_2}_{b_1d_1i_3} R^{b_1e_1h_1}_{b_2e_2h_2} K^{ad_1g_1h_2}_{a_1d_2g_2h_3} K^{a_1b_2c_1f_1}_{a_2b_3c_2f_2} R^{d_2e_2f_2}_{d_3e_3f_3} E^{i_3,h_3,g_2,f_3,e_3,d_3,c_2,b_3,a_2}_{1,2,3,1,2,1,3,2,3}
\end{aligned}
$$

and

$$
\begin{aligned}
& E^{a,b,c,d,e,f,g,h,i}_{3,2,3,1,2,1,3,2,1} = \sum R^{def}_{d_1e_1f_1} E^{a,b,c,f_1,e_1,d_1,g,h,i}_{3,2,3,2,1,2,3,2,1} \\
& = \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} E^{f_2,c_1,b_1,a_1,e_1,d_1,g,h,i}_{2,3,2,3,1,2,3,2,1} \\
& = \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} E^{f_2,c_1,b_1,e_1,a_1,d_1,g,h,i}_{2,3,2,1,3,2,3,2,1} \\
& = \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} K^{a_1d_1gh}_{a_2d_2g_1h_1} E^{f_2,c_1,b_1,e_1,h_1,g_1,d_2,a_2,i}_{2,3,2,1,2,3,2,3,1} \\
& = \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} K^{a_1d_1gh}_{a_2d_2g_1h_1} R^{b_1e_1h_1}_{b_2e_2h_2} E^{f_2,c_1,h_2,e_2,b_2,g_1,d_2,a_2,i}_{2,3,1,2,1,3,2,3,1} \\
& = \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} K^{a_1d_1gh}_{a_2d_2g_1h_1} R^{b_1e_1h_1}_{b_2e_2h_2} E^{f_2,h_2,c_1,e_2,g_1,b_2,d_2,i,a_2}_{2,1,3,2,3,1,2,1,3} \\
& = \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} K^{a_1d_1gh}_{a_2d_2g_1h_1} R^{b_1e_1h_1}_{b_2e_2h_2} R^{b_2d_2i}_{b_3d_3i_1} E^{f_2,h_2,c_1,e_2,g_1,i_1,d_3,b_3,a_2}_{2,1,3,2,3,2,1,2,3} \\
& = \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} K^{a_1d_1gh}_{a_2d_2g_1h_1} R^{b_1e_1h_1}_{b_2e_2h_2} R^{b_2d_2i}_{b_3d_3i_1} K^{c_1e_2g_1i_1}_{c_2e_3g_2i_2} E^{f_2,h_2,i_2,g_2,e_3,c_2,d_3,b_3,a_2}_{2,1,2,3,2,3,1,2,3} \\
& = \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} K^{a_1d_1gh}_{a_2d_2g_1h_1} R^{b_1e_1h_1}_{b_2e_2h_2} R^{b_2d_2i}_{b_3d_3i_1} K^{c_1e_2g_1i_1}_{c_2e_3g_2i_2} R^{f_2h_2i_2}_{f_3h_3i_3} E^{i_3,h_3,f_3,g_2,e_3,c_2,d_3,b_3,a_2}_{1,2,1,3,2,3,1,2,3} \\
& = \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} K^{a_1d_1gh}_{a_2d_2g_1h_1} R^{b_1e_1h_1}_{b_2e_2h_2} R^{b_2d_2i}_{b_3d_3i_1} K^{c_1e_2g_1i_1}_{c_2e_3g_2i_2} R^{f_2h_2i_2}_{f_3h_3i_3} E^{i_3,h_3,g_2,f_3,e_3,d_3,c_2,b_3,a_2}_{1,2,3,1,2,1,3,2,3}.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
& \sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} R^{bdi_2}_{b_1d_1i_3} R^{b_1e_1h_1}_{b_2e_2h_2} K^{ad_1g_1h_2}_{a_1d_2g_2h_3} K^{a_1b_2c_1f_1}_{a_2b_3c_2f_2} R^{d_2e_2f_2}_{d_3e_3f_3} \\
& = \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} K^{a_1d_1gh}_{a_2d_2g_1h_1} R^{b_1e_1h_1}_{b_2e_2h_2} R^{b_2d_2i}_{b_3d_3i_1} K^{c_1e_2g_1i_1}_{c_2e_3g_2i_2} R^{f_2h_2i_2}_{f_3h_3i_3}
\end{aligned}
\tag{10.99}
$$

for any $a, b, c, d, e, f, g, h, i$ and $a_2, b_3, c_2, d_3, e_3, f_3, g_2, h_3, i_3$. The sums are over $a_1, b_1, b_2, c_1, d_1, d_2, e_1, e_2, f_1, f_2, g_1, h_1, h_2, i_1, i_2 \in \mathbb{Z}_{\geq 0}$ on both sides. They are finite sums due to the weight conservation (3.48) and (5.65). The identity (10.99) reproduces the 3D reflection equation (4.5). By a parallel argument for $U_q^+(B_3)$, the 3D reflection equation of type B (6.31) can also be derived.

10.6 χ-Invariants

Theorem 10.6 implies non-trivial identities in (a completion of) $U_q^+(\mathfrak{g})$. They are stated as invariance of some infinite products under the anti-involution χ introduced in (10.42). Here we illustrate the derivation along $\mathfrak{g} = A_2$ and present the results for C_2 and G_2. The point is to translate the boundary vectors in Sects. 3.6.1, 5.8.1 and 8.6.1 in terms of the PBW basis.

Let us write the boundary vectors (3.132) as

$$|\eta_s\rangle = \sum_{m\geq 0} \eta_{s,m}|m\rangle \quad (s = 1, 2). \tag{10.100}$$

By comparing the coefficient of $|a\rangle \otimes |b\rangle \otimes |c\rangle$ on the two sides of (3.143) using (3.47), we get

$$\sum_{i,j,k} \eta_{s,i}\eta_{s,j}\eta_{s,k}R^{abc}_{ijk} = \eta_{s,a}\eta_{s,b}\eta_{s,c}. \tag{10.101}$$

In view of (3.63), this is equivalent to

$$\sum_{a,b,c} \hat{\eta}_{s,a}\hat{\eta}_{s,b}\hat{\eta}_{s,c}R^{abc}_{ijk} = \hat{\eta}_{s,i}\hat{\eta}_{s,j}\hat{\eta}_{s,k}, \qquad \hat{\eta}_{s,a} = (q^2)_a\eta_{s,a}. \tag{10.102}$$

Multiply this by $E_{\mathbf{2}}^{k,j,i}$ and sum over $i, j, k \in \mathbb{Z}_{\geq 0}$. From (10.46) and (10.6), the RHS gives

$$\sum_{i,j,k} E_{\mathbf{2}}^{k,j,i}\hat{\eta}_{s,i}\hat{\eta}_{s,j}\hat{\eta}_{s,k} = \Big(\sum_k \hat{\eta}_{s,k}\frac{(b_1)^k}{[k]_1!}\Big)\Big(\sum_j \hat{\eta}_{s,j}\frac{(b_2)^j}{[j]_1!}\Big)\Big(\sum_i \hat{\eta}_{s,i}\frac{(b_3)^i}{[i]_1!}\Big). \tag{10.103}$$

As for the LHS, we have

$$\begin{aligned}\sum_{a,b,c}\Big(\sum_{i,j,k} R^{abc}_{ijk}E_{\mathbf{2}}^{k,j,i}\Big)\hat{\eta}_{s,a}\hat{\eta}_{s,b}\hat{\eta}_{s,c} &= \sum_{a,b,c} E_{\mathbf{1}}^{a,b,c}\hat{\eta}_{s,a}\hat{\eta}_{s,b}\hat{\eta}_{s,c} \\ &= \chi\Big(\sum_{a,b,c} E_{\mathbf{2}}^{c,b,a}\hat{\eta}_{s,a}\hat{\eta}_{s,b}\hat{\eta}_{s,c}\Big).\end{aligned} \tag{10.104}$$

The first equality is due to (10.58) which is the A_2 case of the main theorem of this chapter. The second equality is (10.42). The quantity within χ in (10.104) is equal to (10.103). Thus we find that (10.103) is χ-invariant. To describe the result neatly we introduce a quantum-dilogarithm-type infinite product:

$$\Theta_q(z) = \sum_m \frac{q^{m(m-1)/2}z^m}{(q)_m} = (-z;q)_\infty. \tag{10.105}$$

Then a direct calculation using (3.132) yields

$$\sum_m \hat{\eta}_{s,m} \frac{z^m}{[m]_1!} = \begin{cases} \Theta_q((1-q^2)z) & s=1, \\ \Theta_{q^4}(q(1-q^2)^2 z^2) & s=2. \end{cases} \tag{10.106}$$

Thus we get a corollary of Theorem 10.6 and Proposition 3.28.

Corollary 10.15 *Set $c_i = (1-q^2)b_i$, $c_i' = \chi(c_i) \in U_q^+(A_2)$ $(i=1,2,3)$ using b_i in (10.46) and the anti-algebra involution χ in (10.41). Then the following equalities are valid:*

$$\Theta_q(c_1)\Theta_q(c_2)\Theta_q(c_3) = \Theta_q(c_3')\Theta_q(c_2')\Theta_q(c_1'), \tag{10.107}$$

$$\Theta_{q^4}(qc_1^2)\Theta_{q^4}(qc_2^2)\Theta_{q^4}(qc_3^2) = \Theta_{q^4}(qc_3'^2)\Theta_{q^4}(qc_2'^2)\Theta_{q^4}(qc_1'^2). \tag{10.108}$$

Remark 10.16 By the rescaling $e_1 \to xe_1, e_2 \to ye_2$ with parameters x, y, the identity (10.107) is seemingly generalized to

$$\Theta_q(xc_1)\Theta_q(xyc_2)\Theta_q(yc_3) = \Theta_q(yc_3')\Theta_q(xyc_2')\Theta_q(xc_1')$$

containing x, y in the same manner as spectral parameters in the Yang–Baxter equation. The same holds for (10.108). Similar remarks apply to the C_2 and G_2 cases in the sequel where the parameters arranged along the positive roots fit the spectral parameters in the reflection and the G_2 reflection equations.

The product (10.107) is expanded as

$$\begin{aligned} &\Theta_q(c_1)\Theta_q(c_2)\Theta_q(c_3) \\ &= 1 + (1+q)(e_1+e_2) + q(1+q)(e_1^2+e_2^2) + (1+q)(e_1e_2+e_2e_1) \\ &+ (1+q)^2(e_1e_2e_1+e_2e_1e_2) + \frac{q^3(1-q^2)^2(e_1^3+e_2^3)}{(1-q)(1-q^3)} + \frac{q^6(1-q^2)^3(e_1^4+e_2^4)}{(1-q)(1-q^3)(1-q^4)} \\ &+ \frac{q^2(1-q^2)^2(e_1e_2e_1^2+e_1^2e_2e_1+e_2e_1e_2^2+e_2^2e_1e_2)}{(1-q)(1-q^3)} \\ &+ \frac{q(1-q^2)^2\big(q(e_1^2e_2^2+e_2^2e_1^2)+(1+q)^2e_1e_2^2e_1-q(1+q^2)e_2e_1^2e_2\big)}{(1-q)(1-q^4)} + \cdots, \end{aligned} \tag{10.109}$$

where the q-Serre relation (10.45) has been used to make it manifestly invariant under χ. Similarly, (10.108) is expanded as

$$\begin{aligned} &\Theta_{q^4}(qc_1^2)\Theta_{q^4}(qc_2^2)\Theta_{q^4}(qc_3^2) \\ &= 1 + \frac{q(1-q^2)^2(e_1^2+e_2^2)}{1-q^4} + \frac{q^6(1-q^2)^4(e_1^4+e_2^4)}{(1-q^4)(1-q^8)} \\ &+ \frac{q^2(1-q^2)^3(e_1^2e_2^2+e_2^2e_1^2-(1+q^2)e_2e_1^2e_2)}{(1-q^4)^2} + \cdots. \end{aligned} \tag{10.110}$$

For C_2, the relevant results are (10.80) and Proposition 5.21 concerning the boundary vectors in (5.118)–(5.120). There are three identities corresponding to the choices of (r, k) in (5.136).

Corollary 10.17 *Set* $c_i = (1-q^4)b_i$ $(i = 1, 3)$, $c_i = (1-q^2)b_i$ $(i = 2, 4)$ *and* $c'_i = \chi(c_i) \in U_q^+(C_2)$ $(i = 1, 2, 3, 4)$ *using* b_i *in (10.60) and the anti-algebra involution* χ *in (10.41). Then the following equalities are valid:*

$$\Theta_{q^2}(c_1)\Theta_q(c_2)\Theta_{q^2}(c_3)\Theta_q(c_4) = \Theta_q(c'_4)\Theta_{q^2}(c'_3)\Theta_q(c'_2)\Theta_{q^2}(c'_1), \tag{10.111}$$

$$\Theta_{q^2}(c_1)\Theta_{q^4}(qc_2^2)\Theta_{q^2}(c_3)\Theta_{q^4}(qc_4^2) = \Theta_{q^4}(qc_4'^2)\Theta_{q^2}(c'_3)\Theta_{q^4}(qc_2'^2)\Theta_{q^2}(c'_1), \tag{10.112}$$

$$\Theta_{q^8}(q^2c_1^2)\Theta_{q^4}(qc_2^2)\Theta_{q^8}(q^2c_3^2)\Theta_{q^4}(qc_4^2) = \Theta_{q^4}(qc_4'^2)\Theta_{q^8}(q^2c_3'^2)\Theta_{q^4}(qc_2'^2)\Theta_{q^8}(q^2c_1'^2). \tag{10.113}$$

For G_2, the relevant result is Conjecture 8.9 for the boundary vector (8.61) and (10.96).

Corollary 10.18 *Set* $c_i = (1-q^6)b_i$ $(i = 1, 3, 5)$, $c_i = (1-q^2)b_i$ $(i = 2, 4, 6)$ *and* $c'_i = \chi(c_i) \in U_q^+(G_2)$ $(i = 1, \dots, 6)$ *using* b_i *in (10.82) and the anti-algebra involution* χ *in (10.41). If Conjecture 8.9 holds, the following equality is valid:*

$$\begin{aligned}&\Theta_{q^3}(c_1)\Theta_q(c_2)\Theta_{q^3}(c_3)\Theta_q(c_4)\Theta_{q^3}(c_5)\Theta_q(c_6)\\ &= \Theta_q(c'_6)\Theta_{q^3}(c'_5)\Theta_q(c'_4)\Theta_{q^3}(c'_3)\Theta_q(c'_2)\Theta_{q^3}(c'_1).\end{aligned} \tag{10.114}$$

10.7 Bibliographical Notes and Comments

This chapter is an extended exposition of [102]. The braid group action (10.5) is introduced in [111]. The formulation of quantized coordinate ring in this chapter follows [76, 139]. See also [43] and [29, Chap. 7]. For quantum cluster algebra structure of quantized coordinate rings, see [52].

The Peter–Weyl-type Theorem 10.1 is taken from [76, Proposition 7.2.2]. Proposition 10.4 is a special case of [66, Corollary 9.1.4]. In [149, Theorem 7], $U_q^+(\mathfrak{g})$ has been identified with an explicit subalgebra of $A_q(\mathfrak{g})_{\mathcal{S}}$. A proof of Theorem 10.5 adapted to the present setting has been given in [102, Sect. 3.2]. The main result, Theorem 10.6, is due to [102, Theorem 5]. The case $\mathfrak{g} = A_2$ was obtained earlier in the pioneering work [131]. Remark 10.8 is due to [141], where a unified conceptual proof of Theorem 10.6 has been attained. See also [128] for yet another proof using the representation theory of q-boson algebra and the Drinfeld pairing of $U_q(\mathfrak{g})$. The multiplication rule on the PBW bases like Lemmas 10.9, 10.11 and 10.13 plays an important role also in the study of the positive principal series representations and modular double [61]. For type C_2, one can adjust the definition of $E_{\mathbf{i}}^A$ in (10.6) with that in [148] by setting $v = q^{-1}$. Some of the results like Lemma 10.13 have also been obtained in [147]. An analogue of Sect. 10.5 for quantum superalgebras has been argued in [151].