Chapter 1
Introduction

Abstract This short chapter is a brief guide to the background and the topics treated in the book. We begin by recalling the key equations for integrability in two dimensions, motivate a generalization to three dimensions, digest how a class of quantum groups known as quantized coordinate rings play an important role, and mention some fruitful applications.

1.1 Quantum Integrability in Two Dimensions

In integrable systems in quantum field theories in $(1+1)$-dimensional space time [155] and in statistical mechanical models on two-dimensional lattices [10], a central role is played by the following equations[1] [30]:

$$
\begin{aligned}
&\text{Yang–Baxter eq.:}\\
&\quad \mathcal{R}_{12}(x)\mathcal{R}_{13}(xy)\mathcal{R}_{23}(y) = \mathcal{R}_{23}(y)\mathcal{R}_{13}(xy)\mathcal{R}_{12}(x),\\
&\text{Reflection eq.:}\\
&\quad \mathcal{R}_{12}(x)\mathcal{K}_2(xy)\mathcal{R}_{21}(x^2y)\mathcal{K}_1(y) = \mathcal{K}_1(y)\mathcal{R}_{12}(x^2y)\mathcal{K}_2(xy)\mathcal{R}_{21}(x),\\
&G_2 \text{ reflection eq.:}\\
&\quad \mathcal{R}_{12}(x)\mathcal{X}_{132}(xy)\mathcal{R}_{23}(x^2y^3)\mathcal{X}_{213}(xy^2)\mathcal{R}_{31}(xy^3)\mathcal{X}_{321}(y)\\
&\quad = \mathcal{X}_{231}(y)\mathcal{R}_{13}(xy^3)\mathcal{X}_{123}(xy^2)\mathcal{R}_{32}(x^2y^3)\mathcal{X}_{312}(xy)\mathcal{R}_{21}(x).
\end{aligned}
\tag{1.1}
$$

Here $\mathcal{R}(z)$, $\mathcal{K}(z)$ and $\mathcal{X}(z)$ are matrices of amplitude for two-particle elastic scattering, one-particle boundary reflection and a three-particle special event, respectively. The indices label the particles or their world lines. The commutative variables x and y are called spectral parameters, which describe the rapidity, i.e. (exponentiated) relative angles of the world lines of the particles participating in the events. In the context of statistical mechanical models, the scattering diagrams are regarded as

[1] The G_2 reflection equation, which is less known, will be explained in some detail in Chap. 17. Its application is yet to be explored. It was written down in [85] guided by Fig. 1.2 which originates in the description in [30, p. 982].

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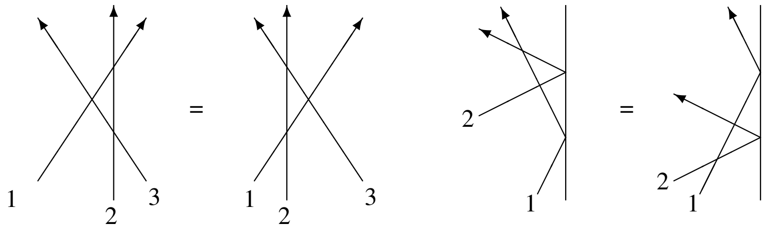

Fig. 1.1 Left: Diagram for Yang–Baxter equation. Arrows are trajectories (world lines) of particles 1, 2 and 3. $\mathcal{R}(z)$ is attached to an intersection of two arrows. Right: Diagram for reflection equation. $\mathcal{K}(z)$ is attached to a reflection by the boundary which is denoted by a vertical line.

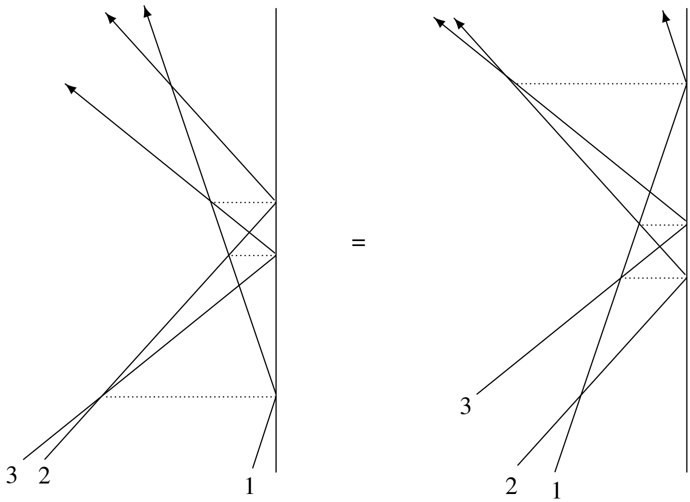

Fig. 1.2 Diagram for G_2 reflection equation. Vertical lines denote the boundary. As indicated by dotted lines, boundary reflection of a particle happens simultaneously with the collision of the other two particles, which is characteristic to the G_2 theory. Elementary geometrical consistency of such line configurations is guaranteed by the classical Desargues–Pappus theorem. See Chap. 17.

local spin configurations, and $\mathcal{R}(z)$, $\mathcal{K}(z)$ and $\mathcal{X}(z)$ are regarded as specifying their Boltzmann weights (Fig. 1.1).

When the spectral parameters tend to infinity, these equations formally reduce to the constant versions:

$$\begin{aligned}
&\text{constant Yang–Baxter eq.:} \quad L_{12}L_{13}L_{23} = L_{23}L_{13}L_{12},\\
&\text{constant reflection eq.:} \quad L_{12}G_2L_{21}G_1 = G_1L_{12}G_2L_{21},\\
&\text{constant } G_2 \text{ reflection eq.:} \; L_{12}J_{132}L_{23}J_{213}L_{31}J_{321} = J_{231}L_{13}J_{123}L_{32}J_{312}L_{21},
\end{aligned} \tag{1.2}$$

where the letters $\mathcal{R}$, $\mathcal{K}$, $\mathcal{X}$ have been replaced by L, G, J for distinction.

Each arrow carries a vector space, say V, representing a one-particle state. Thus for example L_{12} is a linear operator on $\overset{1}{V} \otimes \overset{2}{V}$, where the superscripts are just labels of the arrows in a diagram.

1.2 Quantization: Introducing the Third Dimension

Equations in the previous section are expressed in planar diagrams. Introducing further particles (arrows) would give rise to more scattering events (composition of operators), but their diagrams remain always planar. This feature is referred to as two-dimensional (2D).

Then how can we generalize things to three dimensions (3D)? A naive but natural way is to introduce an extra arrow penetrating each scattering event perpendicularly to the planar diagram and assign to it a new vector space, say $\mathcal{F}$. It implies that $L_{12} \in \mathrm{End}(\overset{1}{V} \otimes \overset{2}{V}), G_1 \in \mathrm{End}(\overset{1}{V}), J_{123} \in \mathrm{End}(\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V})$ are upgraded to $L_{12a} \in \mathrm{End}(\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{a}{\mathcal{F}}), G_{1a} \in \mathrm{End}(\overset{1}{V} \otimes \overset{a}{\mathcal{F}}), J_{123a} \in \mathrm{End}(\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V} \otimes \overset{a}{\mathcal{F}})$, where a is a label of the auxiliary space.[2] In other words, elements of L, G, J become $\mathrm{End}(\mathcal{F})$ valued or get *quantized*.

What about the corresponding generalization of the equations (1.2)? A point here is not just to demand the strict equality but to embark on the more general situation of *conjugacy equivalence*. For instance, we postulate $L_{12a}L_{13b}L_{23c}R_{abc} = R_{abc}L_{23c}L_{13b}L_{12a}$ in place of the Yang–Baxter equation by introducing an invertible operator $R = R_{abc}$ on $\overset{a}{\mathcal{F}} \otimes \overset{b}{\mathcal{F}} \otimes \overset{c}{\mathcal{F}}$. It then becomes an equality in $\mathrm{End}(\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V} \otimes \overset{a}{\mathcal{F}} \otimes \overset{b}{\mathcal{F}} \otimes \overset{c}{\mathcal{F}})$. A similar "quantization" recipe leads to

$$\begin{aligned}
&\text{quantized Yang–Baxter eq.:} \quad (L_{12}L_{13}L_{23})R = R(L_{23}L_{13}L_{12}),\\
&\text{quantized reflection eq.:} \quad (L_{12}G_2L_{21}G_1)K = K(G_1L_{12}G_2L_{21}),\\
&\text{quantized } G_2 \text{ reflection eq.:} \ (L_{12}J_{132}L_{23}J_{213}L_{31}J_{321})F = F(J_{231}L_{13}J_{123}L_{32}J_{312}L_{21}),
\end{aligned} \tag{1.3}$$

where the new objects R, K, F act on the tensor products of 3, 4, 6 auxiliary spaces whose labels have been suppressed.[3] For their full forms, see (2.15), (4.9) and (8.50).

[2] In later sections, $\mathcal{F}$ is taken slightly differently for L, G, J.

[3] The quantized Yang–Baxter equation is well known as a version of the tetrahedron equation. See Sect. 2.7 for a historical note. The quantized reflection equation and the quantized G_2 reflection equation were introduced in [85, 105].

1.3 Quantized Coordinate Ring

It has been well recognized that the group of equations (1.2) have the analogy in the Weyl group of rank-two classical simple Lie algebra $\mathfrak{g}$ [30], where the simple reflections s_1, s_2 obey the Coxeter relations $s_i^2 = 1$ and

$$\begin{aligned} A_2: &\quad s_1 s_2 s_1 = s_2 s_1 s_2, \\ B_2, C_2: &\quad s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1, \\ G_2: &\quad s_1 s_2 s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 s_2 s_1. \end{aligned} \tag{1.4}$$

One can also observe the relevance of these algebras already in (1.1), where the spectral parameters appearing on each side are in one-to-one correspondence with their positive roots. In view of such facts, if the equations (1.3) made up intuitively are to be meaningful, one should have a decent quantization of the Coxeter relations.

It turns out that such a structure is provided by the *quantized coordinate ring* $A_q(\mathfrak{g})$ for the rank-two $\mathfrak{g}$, whereby R, K, F are captured as the *intertwiner* of a certain class of representations. Let us explain the basic idea, quickly deferring the detail to the subsequent chapters.

The algebra $A_q(\mathfrak{g})$ (cf. [29, 43, 66, 76, 127, 139]) is the Hopf algebra [1] dual to the quantized universal enveloping algebra $U_q(\mathfrak{g})$ [43, 63]. One can either realize it concretely by generators and relations for some $\mathfrak{g}$ (Chaps. 3, 5 and 8), or give a universal definition independently of such presentations for any $\mathfrak{g}$ (Chap. 10). For q generic, it has the irreducible representations π_i attached to each vertex i of the Dynkin diagram of $\mathfrak{g}$. The representation space of π_i is the Fock space of the q-oscillator algebra (3.13). According to the general theory [138, 139, 146] (Theorem 3.3), one has the non-trivial equivalence of the irreducible $A_q(\mathfrak{g})$ modules:

$$\begin{aligned} A_2: &\quad \pi_1 \otimes \pi_2 \otimes \pi_1 \simeq \pi_2 \otimes \pi_1 \otimes \pi_2, \\ B_2, C_2: &\quad \pi_1 \otimes \pi_2 \otimes \pi_1 \otimes \pi_2 \simeq \pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1, \\ G_2: &\quad \pi_1 \otimes \pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1 \otimes \pi_2 \simeq \pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1. \end{aligned} \tag{1.5}$$

It turns out that they can be matched precisely with the 3D equations (1.3) by choosing L, G, J to be appropriate q-oscillator-valued scattering amplitudes (Theorems 3.21, 5.18 and 8.6).[4] The conjugation operators R, K, F in (1.3) are thereby characterized naturally as the intertwiner responsible for the equivalence (1.5). Their matrix elements are polynomials in q with integer coefficients. They are further identified with the transition coefficients of the PBW bases of the positive part of $U_q(\mathfrak{g})$ (Chap. 10).

[4] An intrinsic reason why (1.5) admits such a "physical" presentation in terms of scattering diagrams (Figs. 2.18, 4.6 and 8.1) is yet to be revealed.

1.4 Compatibility: Tetrahedron, 3D Reflection and F_4 Equations

In our argument so far, the most characteristic objects in 3D are the operators R, K, F. The quantized equations (1.3) may be regarded as the auxiliary linear problem for them. It is then natural to investigate their "associativity", which is a non-linear consistency condition among the "structure constants".[5] Such calculations in (2.22)–(2.23) and (4.19)–(4.20) and the like for the first two equations in (1.3) lead to[6]

tetrahedron eq.:

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{134}R_{123}, \tag{1.6}$$

3D reflection eqs.:

$$R_{689}K_{3579}R_{249}R_{258}K_{1478}K_{1236}R_{456} = R_{456}K_{1236}K_{1478}R_{258}R_{249}K_{3579}R_{689}, \tag{1.7}$$

$$S_{689}K_{9753}S_{249}S_{258}K_{8741}K_{6321}S_{456} = S_{456}K_{6321}K_{8741}S_{258}S_{249}K_{9753}S_{689}, \tag{1.8}$$

where $S = R|_{q\to q^2}$. The tetrahedron equation (1.6) is best known as a 3D generalization of the Yang–Baxter equation [153, 154]. The quantized Yang–Baxter equation in the foregoing argument may be regarded as a variant of it.

In the language of $A_q(\mathfrak{g})$, these equations are corollaries of the generalization of (1.5) to A_3, B_3, C_3, where one can embed the rank-two results[7] (Theorems 3.20, 5.16 and 6.7). Note that G_2 deviates from the other at this point since there is no "G_3" to play such a game.

In general the compatibility condition originating from $A_q(\mathfrak{g})$ with higher rank $\mathfrak{g}$ should be reducible to the tetrahedron and the 3D reflection equations.[8] A most curious situation of this kind is $A_q(F_4)$, where, the F_4 analogue of the tetrahedron equation takes the form

$$\begin{aligned}
&R_{14,15,16}R_{9,11,16}K_{7,8,10,16}K_{17,15,13,9}R_{4,5,16}S_{7,12,17}R_{1,2,16}S_{6,10,17}R_{9,14,18}\\
&\times K_{17,5,3,1}R_{11,15,18}K_{6,8,12,18}R_{1,4,18}R_{1,8,15}S_{7,13,19}K_{19,11,6,1}K_{19,15,12,4}S_{3,10,19}\\
&\times R_{4,8,11}K_{20,14,7,1}R_{2,5,18}S_{6,13,20}S_{3,12,20}R_{1,9,21}K_{20,15,10,2}R_{4,14,21}K_{3,8,13,21}\\
&\times R_{2,11,21}R_{2,8,14}S_{6,7,22}K_{22,4,3,2}R_{5,15,21}K_{22,14,13,11}S_{10,12,22}K_{23,9,6,2}S_{3,7,23}\\
&\times S_{19,20,22}K_{22,18,17,16}S_{10,13,23}K_{23,14,12,5}S_{3,6,24}K_{23,21,19,16}K_{24,9,7,4}S_{17,20,23}\\
&\times K_{24,11,10,5}S_{12,13,24}S_{17,19,24}K_{24,21,20,18}R_{5,8,9}S_{22,23,24}\\
&= \text{product in reverse order.}
\end{aligned} \tag{1.9}$$

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[5] It is parallel with 2D, where the quantum group symmetry of the form $\mathcal{RLL} = \mathcal{LLR}$ automatically implies the Yang–Baxter equation $\mathcal{RRR} = \mathcal{RRR}$ [43, 63].

[6] See the last sections in Chaps. 2–5 for historical notes on these equations. The two versions of the 3D reflection equations correspond to types B and C. They will appear in (6.31) and (4.3).

[7] Such an approach to the tetrahedron equation was first undertaken in [77].

[8] See the argument around (3.101) and the one in Sect. 9.2.

Each side is a composition of 50 operators R, S, K which act on 24-fold tensor product of the q-oscillator Fock space. Reflecting the subalgebras $A_q(B_3)$, $A_q(C_3) \subset A_q(F_4)$, it is reduced to the composition of the two kinds of the 3D reflection equations (1.7) and (1.8) twelve times for each (Theorem 7.2).

Since the advent of quantum groups in the 1980s [43, 63, 136], algebraic studies of the Yang–Baxter equation have been done mainly along the quantized universal enveloping algebras U_q. It is also the case for the reflection equation although it requires more details on their coideal subalgebras. In contrast to this, the argument so far indicates that the dual quantum group $A_q(\mathfrak{g})$, although $\mathfrak{g}$ is hitherto limited to the classical finite types, is a clue to their 3D versions in (1.3)–(1.9). One of the main themes of this book is to highlight such utility of the quantized coordinate ring $A_q(\mathfrak{g})$ in the theory of integrable systems.

1.5 Feedback to 2D

When going from 2D to 3D, we have dropped the spectral parameters. In general it is highly non-trivial to keep them in an essential manner in 3D (cf. [11, 154]). On the other hand, one can take advantage of the 3D structure of the quantized equations (1.3) to produce rich families of solutions to the original 2D equations (1.1) including the spectral parameters.[9] In fact, all the equations (1.3) by construction admit the composition in the "third direction", i.e. auxiliary space, for arbitrary n times. Moreover, one can bring the two spectral parameters x, y back thanks to the weight conservation under the equivalence (1.5). And the last step is to evaluate R, K, F away appropriately to return to the original equations (1.1). Such a reduction is done by taking the trace or the expectation value $\langle \eta | (\cdots) | \eta' \rangle$ between the eigenvectors of R, K, F called the boundary vectors. As the result one obtains the solutions of the 2D equations (1.1) expressed by *matrix product formulas* as

$$\mathcal{R}(z) = \mathrm{Tr}(z^{\mathbf{h}} L \cdots L), \quad \mathcal{K}(z) = \mathrm{Tr}(z^{\mathbf{h}} G \cdots G), \quad \mathcal{X}(z) = \mathrm{Tr}(z^{\mathbf{h}} J \cdots J)$$

by the trace reduction, and

$$\mathcal{R}(z) = \langle \eta | z^{\mathbf{h}} L \cdots L | \eta' \rangle, \quad \mathcal{K}(z) = \langle \eta | z^{\mathbf{h}} G \cdots G | \eta' \rangle, \quad \mathcal{X}(z) = \langle \eta | z^{\mathbf{h}} J \cdots J | \eta' \rangle$$

by the boundary vector reduction. The symbol $\mathbf{h}$ denotes the q-oscillator number operator (3.14). We have n-fold matrix products of the quantized amplitudes L, G, J to evaluate the trace or $\langle \eta | (\cdots) | \eta' \rangle$ over the auxiliary q-oscillator Fock space. A similar method can be applied also to the tetrahedron equation of $RRRR = RRRR$

[9] For the Yang–Baxter equation, one may say that almost any trigonometric solution should be just the image of the universal $\mathcal{R}$ in principle (top down). True. However, to describe or construct one in a tractable manner is another problem of individual interest (bottom up). A typical recipe of the latter is the fusion construction. The 3D approach briefed in this section is another having its own intriguing scope.

type (1.6). Up to an overall scalar, these solutions are trigonometric, i.e. rational in q and z. They are characterized in a standard manner by the symmetry with respect to quantum affine algebras or their Onsager coideal subalgebras.[10]

Typically, $\mathcal{R}(z)$ are identified with the quantum R matrices of the symmetric and the anti-symmetric tensor representations of $U_p(A^{(1)}_{n-1})$ (Chaps. 11 and 13), the spin representations of $U_p(B^{(1)}_n)$, $U_p(D^{(1)}_n)$, $U_p(D^{(2)}_{n+1})$ (Chap. 12), and the q-oscillator representations of $U_p(C^{(1)}_n)$, $U_p(A^{(2)}_{2n})$, $U_p(D^{(2)}_{n+1})$ (Chap. 14) with appropriate adjustment of p and q. The matrix product formulas suit computer programming and provide us with a good practical access to those $\mathcal{R}$ and $\mathcal{K}$ matrices associated with the higher "spin" representations of the higher rank algebras.

Another beneficial insight from the matrix product structure is the interpretation of 2D systems as 3D ones. In fact, commuting row transfer matrices in the former are naturally regarded as *layer* transfer matrices in the latter, where rank n plays the role of size. An intriguing offshoot in such a direction is a matrix product formula for stationary probabilities of the 1D Markov process called the multispecies totally asymmetric simple exclusion process (TASEP) (Chap. 18). It reveals a hidden 3D structure in the 1D system, where the system size in the 3D picture is given by the number of species of the particles.

1.6 Layout of the Book

This book is intended for readers who have some familiarity or basic knowledge about quantum groups and the Yang–Baxter equation or their application to integrable systems. Rudiments of the subject can be found for example in [10, 29, 51, 65].

In Chaps. 2–8, the tetrahedron equations and their relatives are studied from the viewpoint of the quantized coordinate ring $A_q(\mathfrak{g})$ individually for $\mathfrak{g} = A, B, C, F_4, G_2$. They are based on concrete presentation by generators and relations (except for F_4). The basic flow of the argument is parallel and all the essences are contained already in the type A case (Chaps. 2 and 3). Chapter 9 is a discussion on a possible generalization to non-crystallographic Coxeter groups.

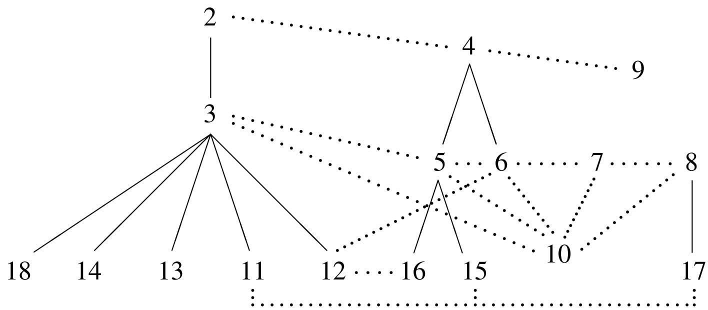
Chapter 10 is unique in that a universal definition of $A_q(\mathfrak{g})$ free from concrete presentations is given together with the basic aspects like $U_q(\mathfrak{g})$ bimodule structure and the RTT relation, etc. The main result is Theorem 10.6, which identifies the intertwiner of $A_q(\mathfrak{g})$ modules with the transition coefficients of the PBW basis of the positive part $U^+_q(\mathfrak{g})$ of $U_q(\mathfrak{g})$. It is readable without heavily consulting other parts of the book.

Chapters 11–17 describe the 3D approach to the Yang–Baxter, reflection and G_2 reflection equations. Families of solutions in matrix product forms are constructed by the trace and the boundary vector reductions. They are characterized in terms of quantum affine algebras and their representations with precise details depending on the reductions.

[10] To characterize $\mathcal{X}(z)$ for G_2 in such a quantum group theoretical framework is an open problem.

Finally, Chap. 18 presents a further application of the 3D approach to the multi-species TASEP, which may be viewed as a feedback to 1D. It is readable based on relevant parts in Chaps. 2 and 3 only.

These features of the chapters are roughly summarized in the following diagram:



As mentioned before, for those who wish to concentrate on the tetrahedron equation or type A case for a start, Chaps. 2 and 3 will suffice. Their applications to the Yang–Baxter equation are presented in Chaps. 11, 12, 13 and 14 with the increasing complexity in this order. Chap. 18 also provides yet another application encompassing a seemingly quite different topic.

Readers who are interested in the type BC case and the 3D reflection equations can find the basics in Chaps. 4 and 5 and slightly supplementary Chap. 6. Their applications to the 2D reflection equation are treated in Chaps. 16 and 15 which are parallel in spirit with Chaps. 11–14.

The other part consists of more or less independently readable Chaps. 8 and 17 concerning the G_2 case, Chap. 7 on F_4, Chap. 9 on non-crystallographic Coxeter groups, and Chap. 10 on the connection with the PBW basis.