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Atsuo Kuniba

Quantum Groups in Three-Dimensional Integrability

Springer

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Preface

In integrable systems in quantum field theory in $(1 + 1)$-dimensional space time and statistical mechanical models on two-dimensional lattices, a central role is played by the Yang–Baxter equation. Its algebraic aspects are now well understood in terms of quantum groups.

An obvious challenge is to explore the higher dimensions. The first attempt in such a direction was made by A. B. Zamolodchikov in 1980 who launched the tetrahedron equation as a generalization of the Yang–Baxter equation in three dimensions. In his seminal work in 1983, Baxter referred to it as "immensely more complicated" and to Zamolodchikov's conjectural solution which he proved as "what appears to be an extraordinary feat of intuition". Over nearly forty years since the initial breakthrough, much effort has been made and many results continue to emerge despite the complexity.

This book is the first monograph devoted to the subject. It is a selective but expository introduction to a quantum group theoretical approach to the tetrahedron equations and their relatives which have been shaped during such developments. It explains the natural origin of these equations, prototypical solutions and their notable aspects. The latter half of the book also encompasses feedbacks to the two- or even lower one-dimensional systems from the viewpoint of mathematical physics.

The contents are elementary and presented in a casual style for the sake of readability. As a result, a substantial part of the text has become a collection of algebraic manipulations, which are straightforward in principle but sometimes too tedious without the help of a computer. Hopefully, such calculations are not just laborious but would be rewarding and fun for those who enjoy the programming.

The title of the book may sound somewhat odd or too strong, since admittedly it actually achieves only a glimpse into the quantum integrability in three dimensions. It is my hope, however, that it stimulates the subject, now in its adolescence, to make a transition into the next phase.

There are many people to whom I am indebted for the delightful as well as challenging opportunity to write this book. I am grateful to Masato Okado for advice on the plan of the book and collaboration on many related projects; Vladimir Mangazeev, Vincent Pasquier, Sergey Sergeev and Yasuhiko Yamada for sharing their insights and Toshiyuki Tanisaki for useful communications. Special thanks go to Rodney J. Baxter, Vladimir V. Bazhanov, Etsuro Date, Michio Jimbo, Tetsuji Miwa and the late Miki Wadati for inspiring me in the wonders of integrable systems for many years. I have had kind support on text handling from Masayuki Nakamura from Springer Japan. Last but not least, I thank my family for letting me work comfortably from home through the turbulent COVID-19 years of writing.

The first manuscript of this book, which was almost in the final form, was sent to the publisher on October 5 2021.

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Contents

Chapter 1
Introduction

Abstract This short chapter is a brief guide to the background and the topics treated in the book. We begin by recalling the key equations for integrability in two dimensions, motivate a generalization to three dimensions, digest how a class of quantum groups known as quantized coordinate rings play an important role, and mention some fruitful applications.

1.1 Quantum Integrability in Two Dimensions

In integrable systems in quantum field theories in (1 + 1)-dimensional space time [155] and in statistical mechanical models on two-dimensional lattices [10], a central role is played by the following equations[1] [30]:

$$\begin{aligned}
&\text{Yang–Baxter eq.:}\\
&\quad \mathcal{R}_{12}(x)\mathcal{R}_{13}(xy)\mathcal{R}_{23}(y) = \mathcal{R}_{23}(y)\mathcal{R}_{13}(xy)\mathcal{R}_{12}(x),\\
&\text{Reflection eq.:}\\
&\quad \mathcal{R}_{12}(x)\mathcal{K}_2(xy)\mathcal{R}_{21}(x^2y)\mathcal{K}_1(y) = \mathcal{K}_1(y)\mathcal{R}_{12}(x^2y)\mathcal{K}_2(xy)\mathcal{R}_{21}(x),\\
&G_2 \text{ reflection eq.:}\\
&\quad \mathcal{R}_{12}(x)\mathcal{X}_{132}(xy)\mathcal{R}_{23}(x^2y^3)\mathcal{X}_{213}(xy^2)\mathcal{R}_{31}(xy^3)\mathcal{X}_{321}(y)\\
&\quad = \mathcal{X}_{231}(y)\mathcal{R}_{13}(xy^3)\mathcal{X}_{123}(xy^2)\mathcal{R}_{32}(x^2y^3)\mathcal{X}_{312}(xy)\mathcal{R}_{21}(x).
\end{aligned} \tag{1.1}$$

Here $\mathcal{R}(z)$, $\mathcal{K}(z)$ and $\mathcal{X}(z)$ are matrices of amplitude for two-particle elastic scattering, one-particle boundary reflection and a three-particle special event, respectively. The indices label the particles or their world lines. The commutative variables x and y are called spectral parameters, which describe the rapidity, i.e. (exponentiated) relative angles of the world lines of the particles participating in the events. In the context of statistical mechanical models, the scattering diagrams are regarded as

[1] The G_2 reflection equation, which is less known, will be explained in some detail in Chap. 17. Its application is yet to be explored. It was written down in [85] guided by Fig. 1.2 which originates in the description in [30, p. 982].

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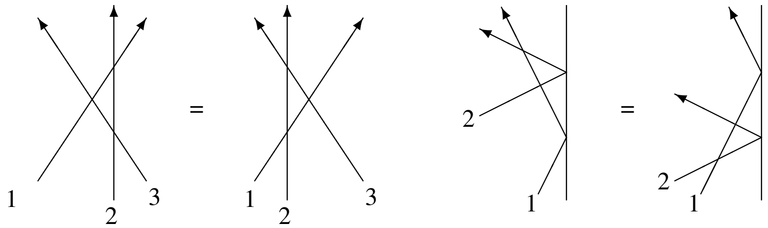

Fig. 1.1 Left: Diagram for Yang–Baxter equation. Arrows are trajectories (world lines) of particles 1, 2 and 3. $\mathcal{R}(z)$ is attached to an intersection of two arrows. Right: Diagram for reflection equation. $\mathcal{K}(z)$ is attached to a reflection by the boundary which is denoted by a vertical line.

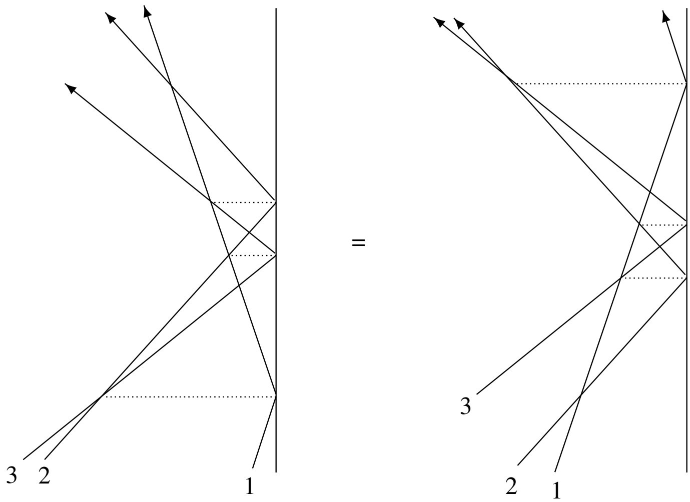

Fig. 1.2 Diagram for G_2 reflection equation. Vertical lines denote the boundary. As indicated by dotted lines, boundary reflection of a particle happens simultaneously with the collision of the other two particles, which is characteristic to the G_2 theory. Elementary geometrical consistency of such line configurations is guaranteed by the classical Desargues–Pappus theorem. See Chap. 17.

local spin configurations, and $\mathcal{R}(z)$, $\mathcal{K}(z)$ and $\mathcal{X}(z)$ are regarded as specifying their Boltzmann weights (Fig. 1.1).

When the spectral parameters tend to infinity, these equations formally reduce to the constant versions:

$$\begin{aligned}
&\text{constant Yang–Baxter eq.:}\quad L_{12}L_{13}L_{23} = L_{23}L_{13}L_{12},\\
&\text{constant reflection eq.:}\quad L_{12}G_2L_{21}G_1 = G_1L_{12}G_2L_{21},\\
&\text{constant } G_2 \text{ reflection eq.:}\ L_{12}J_{132}L_{23}J_{213}L_{31}J_{321} = J_{231}L_{13}J_{123}L_{32}J_{312}L_{21},
\end{aligned} \tag{1.2}$$

where the letters $\mathcal{R}$, $\mathcal{K}$, $\mathcal{X}$ have been replaced by L, G, J for distinction.

Each arrow carries a vector space, say V, representing a one-particle state. Thus for example L_{12} is a linear operator on $\overset{1}{V} \otimes \overset{2}{V}$, where the superscripts are just labels of the arrows in a diagram.

1.2 Quantization: Introducing the Third Dimension

Equations in the previous section are expressed in planar diagrams. Introducing further particles (arrows) would give rise to more scattering events (composition of operators), but their diagrams remain always planar. This feature is referred to as two-dimensional (2D).

Then how can we generalize things to three dimensions (3D)? A naive but natural way is to introduce an extra arrow penetrating each scattering event perpendicularly to the planar diagram and assign to it a new vector space, say $\mathcal{F}$. It implies that $L_{12} \in \mathrm{End}(\overset{1}{V} \otimes \overset{2}{V}), G_1 \in \mathrm{End}(\overset{1}{V}), J_{123} \in \mathrm{End}(\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V})$ are upgraded to $L_{12a} \in \mathrm{End}(\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{a}{\mathcal{F}}), G_{1a} \in \mathrm{End}(\overset{1}{V} \otimes \overset{a}{\mathcal{F}}), J_{123a} \in \mathrm{End}(\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V} \otimes \overset{a}{\mathcal{F}})$, where a is a label of the auxiliary space.[2] In other words, elements of L, G, J become $\mathrm{End}(\mathcal{F})$ valued or get *quantized*.

What about the corresponding generalization of the equations (1.2)? A point here is not just to demand the strict equality but to embark on the more general situation of *conjugacy equivalence*. For instance, we postulate $L_{12a}L_{13b}L_{23c}R_{abc} = R_{abc}L_{23c}L_{13b}L_{12a}$ in place of the Yang–Baxter equation by introducing an invertible operator $R = R_{abc}$ on $\overset{a}{\mathcal{F}} \otimes \overset{b}{\mathcal{F}} \otimes \overset{c}{\mathcal{F}}$. It then becomes an equality in $\mathrm{End}(\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V} \otimes \overset{a}{\mathcal{F}} \otimes \overset{b}{\mathcal{F}} \otimes \overset{c}{\mathcal{F}})$. A similar "quantization" recipe leads to

$$\begin{aligned}
&\text{quantized Yang–Baxter eq.:} \quad (L_{12}L_{13}L_{23})R = R(L_{23}L_{13}L_{12}),\\
&\text{quantized reflection eq.:} \quad (L_{12}G_2L_{21}G_1)K = K(G_1L_{12}G_2L_{21}),\\
&\text{quantized } G_2 \text{ reflection eq.:} \ (L_{12}J_{132}L_{23}J_{213}L_{31}J_{321})F = F(J_{231}L_{13}J_{123}L_{32}J_{312}L_{21}),
\end{aligned} \tag{1.3}$$

where the new objects R, K, F act on the tensor products of 3, 4, 6 auxiliary spaces whose labels have been suppressed.[3] For their full forms, see (2.15), (4.9) and (8.50).

[2] In later sections, $\mathcal{F}$ is taken slightly differently for L, G, J.

[3] The quantized Yang–Baxter equation is well known as a version of the tetrahedron equation. See Sect. 2.7 for a historical note. The quantized reflection equation and the quantized G_2 reflection equation were introduced in [85, 105].

1.3 Quantized Coordinate Ring

It has been well recognized that the group of equations (1.2) have the analogy in the Weyl group of rank-two classical simple Lie algebra $\mathfrak{g}$ [30], where the simple reflections s_1, s_2 obey the Coxeter relations $s_i^2 = 1$ and

$$\begin{aligned} A_2: \quad & s_1 s_2 s_1 = s_2 s_1 s_2, \\ B_2, C_2: \quad & s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1, \\ G_2: \quad & s_1 s_2 s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 s_2 s_1. \end{aligned} \tag{1.4}$$

One can also observe the relevance of these algebras already in (1.1), where the spectral parameters appearing on each side are in one-to-one correspondence with their positive roots. In view of such facts, if the equations (1.3) made up intuitively are to be meaningful, one should have a decent quantization of the Coxeter relations.

It turns out that such a structure is provided by the *quantized coordinate ring* $A_q(\mathfrak{g})$ for the rank-two $\mathfrak{g}$, whereby R, K, F are captured as the *intertwiner* of a certain class of representations. Let us explain the basic idea, quickly deferring the detail to the subsequent chapters.

The algebra $A_q(\mathfrak{g})$ (cf. [29, 43, 66, 76, 127, 139]) is the Hopf algebra [1] dual to the quantized universal enveloping algebra $U_q(\mathfrak{g})$ [43, 63]. One can either realize it concretely by generators and relations for some $\mathfrak{g}$ (Chaps. 3, 5 and 8), or give a universal definition independently of such presentations for any $\mathfrak{g}$ (Chap. 10). For q generic, it has the irreducible representations π_i attached to each vertex i of the Dynkin diagram of $\mathfrak{g}$. The representation space of π_i is the Fock space of the q-oscillator algebra (3.13). According to the general theory [138, 139, 146] (Theorem 3.3), one has the non-trivial equivalence of the irreducible $A_q(\mathfrak{g})$ modules:

$$\begin{aligned} A_2: \quad & \pi_1 \otimes \pi_2 \otimes \pi_1 \simeq \pi_2 \otimes \pi_1 \otimes \pi_2, \\ B_2, C_2: \quad & \pi_1 \otimes \pi_2 \otimes \pi_1 \otimes \pi_2 \simeq \pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1, \\ G_2: \quad & \pi_1 \otimes \pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1 \otimes \pi_2 \simeq \pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1 \otimes \pi_2 \otimes \pi_1. \end{aligned} \tag{1.5}$$

It turns out that they can be matched precisely with the 3D equations (1.3) by choosing L, G, J to be appropriate q-oscillator-valued scattering amplitudes (Theorems 3.21, 5.18 and 8.6).[4] The conjugation operators R, K, F in (1.3) are thereby characterized naturally as the intertwiner responsible for the equivalence (1.5). Their matrix elements are polynomials in q with integer coefficients. They are further identified with the transition coefficients of the PBW bases of the positive part of $U_q(\mathfrak{g})$ (Chap. 10).

[4] An intrinsic reason why (1.5) admits such a "physical" presentation in terms of scattering diagrams (Figs. 2.18, 4.6 and 8.1) is yet to be revealed.

1.4 Compatibility: Tetrahedron, 3D Reflection and F_4 Equations

In our argument so far, the most characteristic objects in 3D are the operators R, K, F. The quantized equations (1.3) may be regarded as the auxiliary linear problem for them. It is then natural to investigate their "associativity", which is a non-linear consistency condition among the "structure constants".[5] Such calculations in (2.22)–(2.23) and (4.19)–(4.20) and the like for the first two equations in (1.3) lead to[6]

tetrahedron eq.:

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{134}R_{123}, \tag{1.6}$$

3D reflection eqs.:

$$R_{689}K_{3579}R_{249}R_{258}K_{1478}K_{1236}R_{456} = R_{456}K_{1236}K_{1478}R_{258}R_{249}K_{3579}R_{689}, \tag{1.7}$$

$$S_{689}K_{9753}S_{249}S_{258}K_{8741}K_{6321}S_{456} = S_{456}K_{6321}K_{8741}S_{258}S_{249}K_{9753}S_{689}, \tag{1.8}$$

where $S = R|_{q\to q^2}$. The tetrahedron equation (1.6) is best known as a 3D generalization of the Yang–Baxter equation [153, 154]. The quantized Yang–Baxter equation in the foregoing argument may be regarded as a variant of it.

In the language of $A_q(\mathfrak{g})$, these equations are corollaries of the generalization of (1.5) to A_3, B_3, C_3, where one can embed the rank-two results[7] (Theorems 3.20, 5.16 and 6.7). Note that G_2 deviates from the other at this point since there is no "G_3" to play such a game.

In general the compatibility condition originating from $A_q(\mathfrak{g})$ with higher rank $\mathfrak{g}$ should be reducible to the tetrahedron and the 3D reflection equations.[8] A most curious situation of this kind is $A_q(F_4)$, where, the F_4 analogue of the tetrahedron equation takes the form

$$\begin{aligned}
&R_{14,15,16}R_{9,11,16}K_{7,8,10,16}K_{17,15,13,9}R_{4,5,16}S_{7,12,17}R_{1,2,16}S_{6,10,17}R_{9,14,18}\\
&\times K_{17,5,3,1}R_{11,15,18}K_{6,8,12,18}R_{1,4,18}R_{1,8,15}S_{7,13,19}K_{19,11,6,1}K_{19,15,12,4}S_{3,10,19}\\
&\times R_{4,8,11}K_{20,14,7,1}R_{2,5,18}S_{6,13,20}S_{3,12,20}R_{1,9,21}K_{20,15,10,2}R_{4,14,21}K_{3,8,13,21}\\
&\times R_{2,11,21}R_{2,8,14}S_{6,7,22}K_{22,4,3,2}R_{5,15,21}K_{22,14,13,11}S_{10,12,22}K_{23,9,6,2}S_{3,7,23}\\
&\times S_{19,20,22}K_{22,18,17,16}S_{10,13,23}K_{23,14,12,5}S_{3,6,24}K_{23,21,19,16}K_{24,9,7,4}S_{17,20,23}\\
&\times K_{24,11,10,5}S_{12,13,24}S_{17,19,24}K_{24,21,20,18}R_{5,8,9}S_{22,23,24}\\
&= \text{product in reverse order.}
\end{aligned} \tag{1.9}$$

[5] It is parallel with 2D, where the quantum group symmetry of the form $\mathcal{RLL} = \mathcal{LLR}$ automatically implies the Yang–Baxter equation $\mathcal{RRR} = \mathcal{RRR}$ [43, 63].

[6] See the last sections in Chaps. 2–5 for historical notes on these equations. The two versions of the 3D reflection equations correspond to types B and C. They will appear in (6.31) and (4.3).

[7] Such an approach to the tetrahedron equation was first undertaken in [77].

[8] See the argument around (3.101) and the one in Sect. 9.2.

Each side is a composition of 50 operators R, S, K which act on 24-fold tensor product of the q-oscillator Fock space. Reflecting the subalgebras $A_q(B_3)$, $A_q(C_3) \subset A_q(F_4)$, it is reduced to the composition of the two kinds of the 3D reflection equations (1.7) and (1.8) twelve times for each (Theorem 7.2).

Since the advent of quantum groups in the 1980s [43, 63, 136], algebraic studies of the Yang–Baxter equation have been done mainly along the quantized universal enveloping algebras U_q. It is also the case for the reflection equation although it requires more details on their coideal subalgebras. In contrast to this, the argument so far indicates that the dual quantum group $A_q(\mathfrak{g})$, although $\mathfrak{g}$ is hitherto limited to the classical finite types, is a clue to their 3D versions in (1.3)–(1.9). One of the main themes of this book is to highlight such utility of the quantized coordinate ring $A_q(\mathfrak{g})$ in the theory of integrable systems.

1.5 Feedback to 2D

When going from 2D to 3D, we have dropped the spectral parameters. In general it is highly non-trivial to keep them in an essential manner in 3D (cf. [11, 154]). On the other hand, one can take advantage of the 3D structure of the quantized equations (1.3) to produce rich families of solutions to the original 2D equations (1.1) including the spectral parameters.[9] In fact, all the equations (1.3) by construction admit the composition in the "third direction", i.e. auxiliary space, for arbitrary n times. Moreover, one can bring the two spectral parameters x, y back thanks to the weight conservation under the equivalence (1.5). And the last step is to evaluate R, K, F away appropriately to return to the original equations (1.1). Such a reduction is done by taking the trace or the expectation value $\langle \eta|(\cdots)|\eta'\rangle$ between the eigenvectors of R, K, F called the boundary vectors. As the result one obtains the solutions of the 2D equations (1.1) expressed by *matrix product formulas* as

$$\mathcal{R}(z) = \mathrm{Tr}(z^{\mathbf{h}} L \cdots L), \quad \mathcal{K}(z) = \mathrm{Tr}(z^{\mathbf{h}} G \cdots G), \quad \mathcal{X}(z) = \mathrm{Tr}(z^{\mathbf{h}} J \cdots J)$$

by the trace reduction, and

$$\mathcal{R}(z) = \langle \eta|z^{\mathbf{h}} L \cdots L|\eta'\rangle, \quad \mathcal{K}(z) = \langle \eta|z^{\mathbf{h}} G \cdots G|\eta'\rangle, \quad \mathcal{X}(z) = \langle \eta|z^{\mathbf{h}} J \cdots J|\eta'\rangle$$

by the boundary vector reduction. The symbol $\mathbf{h}$ denotes the q-oscillator number operator (3.14). We have n-fold matrix products of the quantized amplitudes L, G, J to evaluate the trace or $\langle \eta|(\cdots)|\eta'\rangle$ over the auxiliary q-oscillator Fock space. A similar method can be applied also to the tetrahedron equation of $RRRR = RRRR$

[9] For the Yang–Baxter equation, one may say that almost any trigonometric solution should be just the image of the universal $\mathcal{R}$ in principle (top down). True. However, to describe or construct one in a tractable manner is another problem of individual interest (bottom up). A typical recipe of the latter is the fusion construction. The 3D approach briefed in this section is another having its own intriguing scope.

type (1.6). Up to an overall scalar, these solutions are trigonometric, i.e. rational in q and z. They are characterized in a standard manner by the symmetry with respect to quantum affine algebras or their Onsager coideal subalgebras.[10]

Typically, $\mathcal{R}(z)$ are identified with the quantum R matrices of the symmetric and the anti-symmetric tensor representations of $U_p(A^{(1)}_{n-1})$ (Chaps. 11 and 13), the spin representations of $U_p(B^{(1)}_n)$, $U_p(D^{(1)}_n)$, $U_p(D^{(2)}_{n+1})$ (Chap. 12), and the q-oscillator representations of $U_p(C^{(1)}_n)$, $U_p(A^{(2)}_{2n})$, $U_p(D^{(2)}_{n+1})$ (Chap. 14) with appropriate adjustment of p and q. The matrix product formulas suit computer programming and provide us with a good practical access to those $\mathcal{R}$ and $\mathcal{K}$ matrices associated with the higher "spin" representations of the higher rank algebras.

Another beneficial insight from the matrix product structure is the interpretation of 2D systems as 3D ones. In fact, commuting row transfer matrices in the former are naturally regarded as *layer* transfer matrices in the latter, where rank n plays the role of size. An intriguing offshoot in such a direction is a matrix product formula for stationary probabilities of the 1D Markov process called the multispecies totally asymmetric simple exclusion process (TASEP) (Chap. 18). It reveals a hidden 3D structure in the 1D system, where the system size in the 3D picture is given by the number of species of the particles.

1.6 Layout of the Book

This book is intended for readers who have some familiarity or basic knowledge about quantum groups and the Yang–Baxter equation or their application to integrable systems. Rudiments of the subject can be found for example in [10, 29, 51, 65].

In Chaps. 2–8, the tetrahedron equations and their relatives are studied from the viewpoint of the quantized coordinate ring $A_q(\mathfrak{g})$ individually for $\mathfrak{g} = A, B, C, F_4, G_2$. They are based on concrete presentation by generators and relations (except for F_4). The basic flow of the argument is parallel and all the essences are contained already in the type A case (Chaps. 2 and 3). Chapter 9 is a discussion on a possible generalization to non-crystallographic Coxeter groups.

Chapter 10 is unique in that a universal definition of $A_q(\mathfrak{g})$ free from concrete presentations is given together with the basic aspects like $U_q(\mathfrak{g})$ bimodule structure and the RTT relation, etc. The main result is Theorem 10.6, which identifies the intertwiner of $A_q(\mathfrak{g})$ modules with the transition coefficients of the PBW basis of the positive part $U^+_q(\mathfrak{g})$ of $U_q(\mathfrak{g})$. It is readable without heavily consulting other parts of the book.

Chapters 11–17 describe the 3D approach to the Yang–Baxter, reflection and G_2 reflection equations. Families of solutions in matrix product forms are constructed by the trace and the boundary vector reductions. They are characterized in terms of quantum affine algebras and their representations with precise details depending on the reductions.

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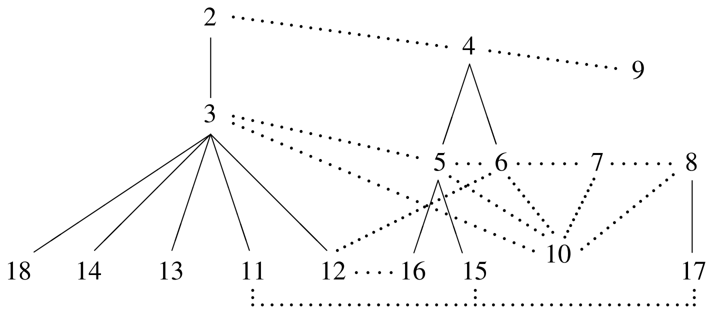
[10] To characterize $\mathcal{X}(z)$ for G_2 in such a quantum group theoretical framework is an open problem.

Finally, Chap. 18 presents a further application of the 3D approach to the multi-species TASEP, which may be viewed as a feedback to 1D. It is readable based on relevant parts in Chaps. 2 and 3 only.

These features of the chapters are roughly summarized in the following diagram:



As mentioned before, for those who wish to concentrate on the tetrahedron equation or type A case for a start, Chaps. 2 and 3 will suffice. Their applications to the Yang–Baxter equation are presented in Chaps. 11, 12, 13 and 14 with the increasing complexity in this order. Chap. 18 also provides yet another application encompassing a seemingly quite different topic.

Readers who are interested in the type BC case and the 3D reflection equations can find the basics in Chaps. 4 and 5 and slightly supplementary Chap. 6. Their applications to the 2D reflection equation are treated in Chaps. 16 and 15 which are parallel in spirit with Chaps. 11–14.

The other part consists of more or less independently readable Chaps. 8 and 17 concerning the G_2 case, Chap. 7 on F_4, Chap. 9 on non-crystallographic Coxeter groups, and Chap. 10 on the connection with the PBW basis.

Chapter 2
Tetrahedron Equation

Abstract In this chapter we introduce a few versions of tetrahedron equations with graphical representations and explain their basic features. The main players are the linear operators which we call 3D R and 3D L. They play a key role in this book.

2.1 3D *R*

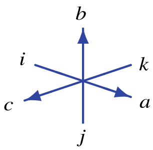
Let $\mathcal{F} = \bigoplus \mathbb{C}|m\rangle$ be a vector space with basis $\{|m\rangle\}$. It can either be finite- or infinite-dimensional, although our main example in later chapters of this book will be an infinite-dimensional Fock space of a single q-oscillator. Let R be a linear operator on the tensor cube of $\mathcal{F}$:

$$R : \mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F} \to \mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F}. \tag{2.1}$$

Its action is described as

$$R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b,c} R^{abc}_{ijk} |a\rangle \otimes |b\rangle \otimes |c\rangle \tag{2.2}$$

in terms of matrix elements R^{abc}_{ijk}. In our main example introduced in Chap. 3, $R^{abc}_{ijk} \in \mathbb{Z}[q]$ holds. Assigning a blue arrow to each $\mathcal{F}$, we depict it as

$R^{abc}_{ijk} =$ 

(2.3)

The operator R itself will be depicted by the same diagram without indices a, b, c, i, j, k. It is natural to regard R as the basic constituent of the 3D cubic lattice whose edges are assigned with the degrees of freedom of $\mathcal{F}$. In this sense R will be called 3D R[1] provided that it satisfies the tetrahedron equations explained in Sect. 2.2.

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One may describe an operator $R \in \mathrm{End}(\mathcal{F}^{\otimes 3})$ in other ways. For instance, (2.2) can also be presented as

$$R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b} |a\rangle \otimes |b\rangle \otimes R^{ab}_{ij}|k\rangle \tag{2.4}$$

in terms of an operator $R^{ab}_{ij} \in \mathrm{End}(\mathcal{F})$ such that $R^{ab}_{ij}|k\rangle = \sum_c R^{abc}_{ijk}|c\rangle$. It is equivalent to the decomposition

$$R = \sum_{a,b,i,j} E_{ai} \otimes E_{bj} \otimes R^{ab}_{ij} \tag{2.5}$$

with respect to the matrix unit $E_{ij}|k\rangle = \delta_{jk}|i\rangle$ in $\mathrm{End}(\mathcal{F})$. The formula (2.5) may be interpreted as defining a statistical model on the 2D square lattice in which the configuration a, b, i, j around a vertex is assigned with the $\mathrm{End}(\mathcal{F})$-valued Boltzmann weight R^{ab}_{ij}. Similar formulations are possible in which R^{ab}_{ij} is put in the first or the second component in (2.5).

2.2 Tetrahedron Equation of Type $RRRR = RRRR$

By tetrahedron equation of type $RRRR = RRRR$, we mean the following:

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124}. \tag{2.6}$$

See Fig. 2.1.

It is an equality in

$$\mathrm{End}(\mathcal{F}^{\otimes 6}) = \mathrm{End}(\overset{1}{\mathcal{F}} \otimes \overset{2}{\mathcal{F}} \otimes \overset{3}{\mathcal{F}} \otimes \overset{4}{\mathcal{F}} \otimes \overset{5}{\mathcal{F}} \otimes \overset{6}{\mathcal{F}}), \tag{2.7}$$

where the superscripts are just labels of the components. The indices in (2.6) specify the components on which a 3D R acts non-trivially. For instance, one has

$$\begin{aligned} &R_{124}(|i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle \otimes |m\rangle \otimes |n\rangle) \\ &= \sum_{a,b,c} R^{abc}_{ijl}|a\rangle \otimes |b\rangle \otimes |k\rangle \otimes |c\rangle \otimes |m\rangle \otimes |n\rangle. \end{aligned} \tag{2.8}$$

The indices $a, b, \ldots, k$ of R^{abc}_{ijk} referring to the bases should not be confused with those in (2.6) specifying the tensor components. In terms of matrix elements, the

[1] We prefer this nomenclature than "3D R matrix" since it is shorter and moreover it will be put in a parallelism with some other (typically, set-theoretical) version of R, which is no longer a matrix. See (3.158). A similar feature holds for 3D K in Chap. 4 and so on.

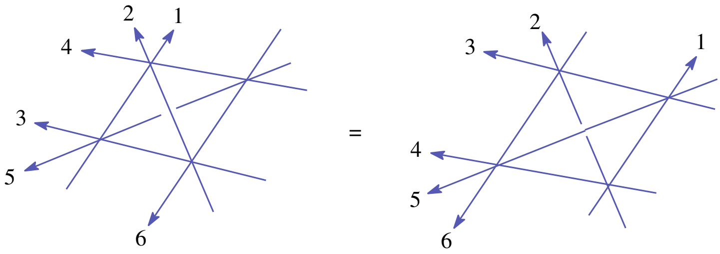


Fig. 2.1 A graphical representation of the tetrahedron equation (2.6). Each arrow is assigned with the space $\mathcal{F}$. Following the arrows leads to the designated compositions of R. The six arrows form the tetrahedron which are "reversed" in the two sides

tetrahedron equation (2.6) is expressed as

$$\sum R^{abd}_{a_1b_1d_1} R^{a_1ce}_{a_2c_1e_1} R^{b_1c_1f}_{b_2c_2f_1} R^{d_1e_1f_1}_{d_2e_2f_2} = \sum R^{def}_{d_1e_1f_1} R^{bcf_1}_{b_1c_1f_2} R^{ac_1e_1}_{a_1c_2e_2} R^{a_1b_1d_1}_{a_2b_2d_2} \tag{2.9}$$

for arbitrary a, b, c, d, e, f and $a_2, b_2, c_2, d_2, e_2, f_2$. The sums are taken over $a_1, b_1, c_1, d_1, e_1, f_1$ on both sides. So if $\mathcal{F}$ were 2-dimensional for instance, there are 2^{12} equations on 2^6 unknowns containing 2^6 summands on each side in general.

One way to remember the arrangement of many indices in (2.6) is to compare it with the Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{2.10}$$

Formally replace R_{12}, R_{13}, R_{23} here by the 3D R's R_{124}, R_{135}, R_{236} which also act on the extra *auxiliary* spaces 4, 5, 6, and relax the equality (2.10) to the similarity by the conjugation by R_{456}. The result becomes the tetrahedron equation (2.6). See also Fig. 2.4. This viewpoint of the tetrahedron equation may be regarded as a *quantization* of the Yang–Baxter equation along the direction of the auxiliary spaces. It will be a key to the applications in Chaps. 11–14. Put in the other way, the tetrahedron equation (2.6) should be reduced to the Yang–Baxter equation (2.10) as soon as the auxiliary spaces 4, 5, 6 are suppressed appropriately.

Figure 2.2 is a 5-frame cartoon producing each side of (2.6) as successive transformations induced by parallel shifts of arrows. In such a presentation, each 3D R corresponds to a reversal of a triangle.

A physical interpretation of the tetrahedron equation is provided in terms of scattering of infinitely long straight strings moving in $\mathbb{R}^3$ forming world sheets. An intersection of two and three world sheets is endowed with $\mathcal{F}$ and 3D R respectively, where the latter is regarded as the three string scattering amplitude. Then (2.6) represents that the four string scattering amplitude is independent of the order of the constituent 3D R's. It is a 3D analogue of the property known as "factorization condition" in completely integrable quantum field theory in 1+1 dimensions represented by the Yang–Baxter equation.

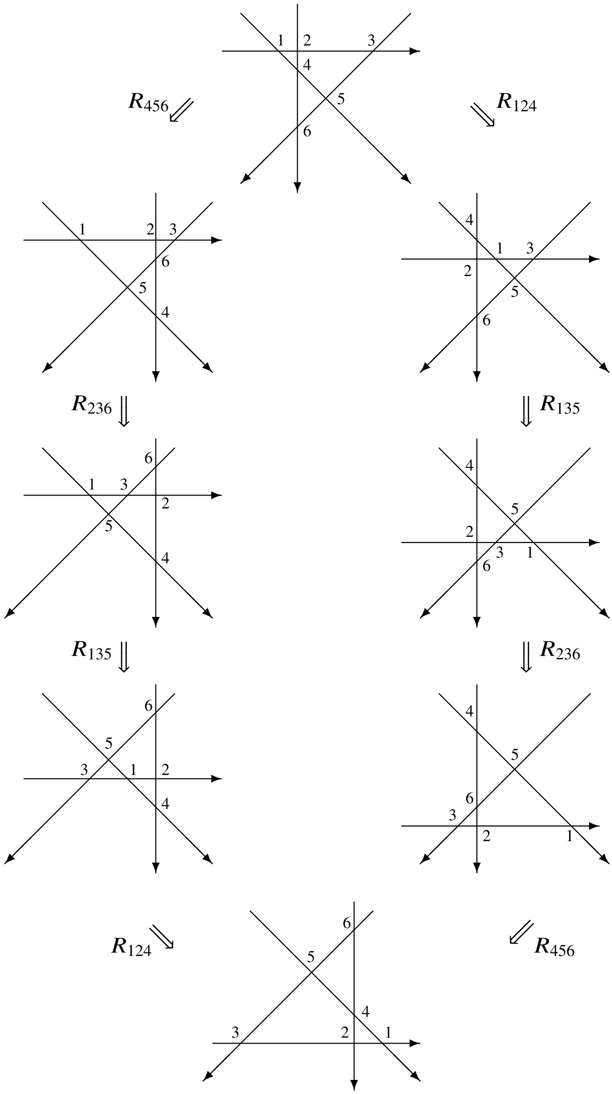


Fig. 2.2 Successive shifts of arrows leading to the composition of 3D R's in (2.6). Each intersection of arrows is assigned with a space $\mathcal{F}$. A reversal of an oriented triangle $i \to j \to k$ corresponds to R_{ijk}. The black arrows here are different in nature from the blue ones in Fig. 2.1 carrying $\mathcal{F}$. They will be assigned with another vector space V in the next section. The figure essentially depicts the manipulations in (2.22) and (2.23)

The tetrahedron equation also serves as a sufficient condition for the commutativity of layer transfer matrices in 3D lattice models. This aspect will be argued in Sects. 11.6, 12.3, 13.8 and 18.4.

2.3 3D L

Let $V = \bigoplus_i \mathbb{C}v_i$ be another vector space with basis $\{v_i\}$, which is different from $\mathcal{F}$ in Sect. 2.1 in general. Again it can either be finite or infinite dimensional although our main example in this book corresponds to the finite dimensional choice $V = \mathbb{C}^2$. We introduce a linear operator L:

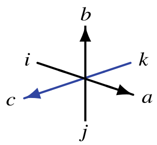
$$L : V \otimes V \otimes \mathcal{F} \to V \otimes V \otimes \mathcal{F}. \tag{2.11}$$

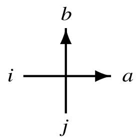
It is described either in the following two ways:

$$L(v_i \otimes v_j \otimes |k\rangle) = \sum_{a,b} v_a \otimes v_b \otimes L^{ab}_{ij}|k\rangle \tag{2.12}$$

$$= \sum_{a,b,c} L^{abc}_{ijk} v_a \otimes v_b \otimes |c\rangle \tag{2.13}$$

in terms of the operator $L^{ab}_{ij} \in \mathrm{End}(\mathcal{F})$ or the matrix element $L^{abc}_{ijk} \in \mathbb{C}$ which are connected by $L^{ab}_{ij}|k\rangle = \sum_c L^{abc}_{ijk}|c\rangle$. The sums over a, b here range over a different set from (2.2) in general although the same letters a, b have been used. We write symbolically as $L = (L^{abc}_{ijk})$ and also as $L = (L^{ab}_{ij})$ with the operator-valued elements. Assigning a blue arrow to each $\mathcal{F}$ as before and a black arrow to V, we depict the elements as

$L^{abc}_{ijk} =$ 

$L^{ab}_{ij} =$ 

(2.14)

L^{ab}_{ij} is also depicted as the left diagram of (2.26). It should be understood as an element of $\mathrm{End}(\mathcal{F})$ living at the vertex formed by the black arrows. The operator L can be considered as a unit of the 3D cubic lattice in which the two edges are assigned with V and the other one with $\mathcal{F}$. In this sense L will be called 3D L provided that it satisfies the tetrahedron equation of type $RLLL = LLLR$ explained in the next section.

2.4 Tetrahedron Equation of Type $RLLL = LLLR$

Let R be a linear operator on $\mathcal{F}^{\otimes 3}$ as in (2.1). By the tetrahedron equation of type $RLLL = LLLR$, we mean the following variant of (2.6):

$$L_{124}L_{135}L_{236}R_{456} = R_{456}L_{236}L_{135}L_{124}. \tag{2.15}$$

It is an equality in

$$\mathrm{End}(\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V} \otimes \overset{4}{\mathcal{F}} \otimes \overset{5}{\mathcal{F}} \otimes \overset{6}{\mathcal{F}}), \tag{2.16}$$

where, as in (2.7), superscripts specify the components on which the operators in (2.15) act non-trivially. For instance, one has

$$\begin{aligned} &L_{124}(v_i \otimes v_j \otimes v_k \otimes |l\rangle \otimes |m\rangle \otimes |n\rangle) \\ &= \sum_{a,b,c} L^{abc}_{ijl} v_a \otimes v_b \otimes v_k \otimes |c\rangle \otimes |m\rangle \otimes |n\rangle. \end{aligned} \tag{2.17}$$

A pictorial representation of (2.15) is obtained by replacing the blue arrows 1, 2, 3 in Fig. 2.1 with black ones as in Fig. 2.3.

In terms of the operator L^{ab}_{ij} in (2.12), the Eq. (2.15) is equivalent to

$$\sum_{\alpha,\beta,\gamma} (L^{ab}_{\alpha\beta} \otimes L^{\alpha c}_{i\gamma} \otimes L^{\beta\gamma}_{jk})R = R \sum_{\alpha,\beta,\gamma} (L^{\alpha\beta}_{ij} \otimes L^{a\gamma}_{\alpha k} \otimes L^{bc}_{\beta\gamma}) \tag{2.18}$$

for arbitrary a, b, c, i, j, k. See Fig. 2.4.

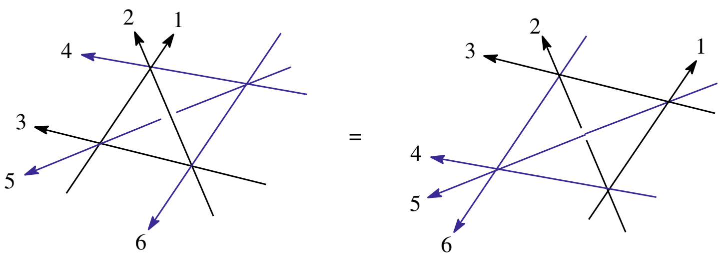


Fig. 2.3 A pictorial representation of the tetrahedron equation of type $RLLL = LLLR$ (2.15). The black arrows 1, 2, 3 and the blue ones 4,5,6 are assigned with V and $\mathcal{F}$, respectively

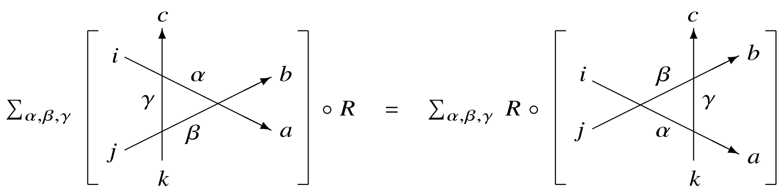


Fig. 2.4 A pictorial representation of (2.18). The operator L^{ab}_{ij} is depicted as in (2.14)

2.5 Quantized Yang–Baxter Equation

We call $RLLL = LLLR$ also the quantized Yang–Baxter equation, which implies a Yang–Baxter equation up to conjugation. As remarked after (2.10), the Eq. (2.18) possesses such a structure where the 3D R plays the role of conjugation:

$$(\overset{i}{L}_{12}\overset{j}{L}_{13}\overset{k}{L}_{23})R_{ijk} = R_{ijk}(\overset{k}{L}_{23}\overset{j}{L}_{13}\overset{i}{L}_{12}). \tag{2.19}$$

Here $\overset{i}{L}_{xy}$ denotes the matrix on $\overset{x}{V} \otimes \overset{y}{V}$ whose elements are $\mathrm{End}(\overset{i}{\mathcal{F}})$ valued.[2]

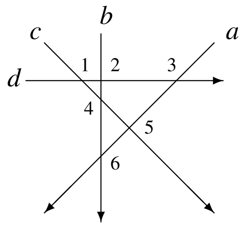
We will also frequently encounter another equivalent form in terms of $S_{ijk} = P_{ik}R^{-1}_{ijk}P_{ik}$, where P_{ik} denotes the exchange of the components $\overset{i}{\mathcal{F}}$ and $\overset{k}{\mathcal{F}}$:

$$S_{ijk}(\overset{k}{L}_{12}\overset{j}{L}_{13}\overset{i}{L}_{23}) = (\overset{i}{L}_{23}\overset{j}{L}_{13}\overset{k}{L}_{12})S_{ijk}. \tag{2.20}$$

The use of R or S is a matter of convention. In our main example in later chapters, we have $R = R^{-1} = P_{13}RP_{13}$ from Proposition 3.7, hence $S = R$.

As we shall show below, it implies the tetrahedron equation of type $RRRR = RRRR$ provided that $\mathcal{F}^{\otimes 6}$ is irreducible as a module over an algebra generated by 3D L.

Consider the composition $S_{124}S_{135}S_{236}S_{456}\overset{6}{L}_{ab}\overset{5}{L}_{ac}\overset{4}{L}_{bc}\overset{3}{L}_{ad}\overset{2}{L}_{bd}\overset{1}{L}_{cd}$, which is represented by the top diagram in Fig. 2.2, i.e.



(2.21)

where each vertex corresponds to L^{ab}_{ij} according to the right-hand diagram in (2.14). Applying (2.20) successively following the LHS of Fig. 2.2, we get

[2] $\overset{i}{L}_{xy}$ has also been denoted by L_{xyi} in (2.15). We allow the coexistence of the two notations to save space in a maneuver like (2.22)–(2.23).

$$
\begin{aligned}
&S_{124}S_{135}S_{236}S_{456}\underline{\overset{6}{L}_{ab}\overset{5}{L}_{ac}\overset{4}{L}_{bc}}\overset{3}{L}_{ad}\overset{2}{L}_{bd}\overset{1}{L}_{cd} \\
&= S_{124}S_{135}S_{236}\overset{4}{L}_{bc}\overset{5}{L}_{ac}\underline{\overset{6}{L}_{ab}\overset{3}{L}_{ad}\overset{2}{L}_{bd}}\overset{1}{L}_{cd}S_{456} \\
&= S_{124}S_{135}\overset{4}{L}_{bc}\underline{\overset{5}{L}_{ac}\overset{2}{L}_{bd}}\overset{3}{L}_{ad}\underline{\overset{6}{L}_{ab}\overset{1}{L}_{cd}}S_{236}S_{456} \\
&= S_{124}S_{135}\overset{4}{L}_{bc}\overset{2}{L}_{bd}\underline{\overset{5}{L}_{ac}\overset{3}{L}_{ad}\overset{1}{L}_{cd}}\overset{6}{L}_{ab}S_{236}S_{456} \\
&= S_{124}\underline{\overset{4}{L}_{bc}\overset{2}{L}_{bd}\overset{1}{L}_{cd}}\overset{3}{L}_{ad}\overset{5}{L}_{ac}\overset{6}{L}_{ab}S_{135}S_{236}S_{456} \\
&= \overset{1}{L}_{cd}\overset{2}{L}_{bd}\underline{\overset{4}{L}_{bc}\overset{3}{L}_{ad}}\overset{5}{L}_{ac}\overset{6}{L}_{ab}S_{124}S_{135}S_{236}S_{456}, \\
&= \overset{1}{L}_{cd}\overset{2}{L}_{bd}\overset{3}{L}_{ad}\overset{4}{L}_{bc}\overset{5}{L}_{ac}\overset{6}{L}_{ab}S_{124}S_{135}S_{236}S_{456},
\end{aligned}
\tag{2.22}
$$

where the underlines indicate the components to be rewritten by (2.20) or by the obvious commutativity. Similarly, the RHS of Fig. 2.2 leads to

$$
\begin{aligned}
&S_{456}S_{236}S_{135}S_{124}\overset{6}{L}_{ab}\overset{5}{L}_{ac}\underline{\overset{4}{L}_{bc}\overset{3}{L}_{ad}}\overset{2}{L}_{bd}\overset{1}{L}_{cd} \\
&= S_{456}S_{236}S_{135}S_{124}\overset{6}{L}_{ab}\overset{5}{L}_{ac}\overset{3}{L}_{ad}\underline{\overset{4}{L}_{bc}\overset{2}{L}_{bd}\overset{1}{L}_{cd}} \\
&= S_{456}S_{236}S_{135}\overset{6}{L}_{ab}\underline{\overset{5}{L}_{ac}\overset{3}{L}_{ad}\overset{1}{L}_{cd}}\overset{2}{L}_{bd}\overset{4}{L}_{bc}S_{124} \\
&= S_{456}S_{236}\underline{\overset{6}{L}_{ab}\overset{1}{L}_{cd}}\overset{3}{L}_{ad}\underline{\overset{5}{L}_{ac}\overset{2}{L}_{bd}}\overset{4}{L}_{bc}S_{135}S_{124} \\
&= S_{456}S_{236}\overset{1}{L}_{cd}\underline{\overset{6}{L}_{ab}\overset{3}{L}_{ad}\overset{2}{L}_{bd}}\overset{5}{L}_{ac}\overset{4}{L}_{bc}S_{135}S_{124} \\
&= S_{456}\overset{1}{L}_{cd}\overset{2}{L}_{bd}\overset{3}{L}_{ad}\underline{\overset{6}{L}_{ab}\overset{5}{L}_{ac}\overset{4}{L}_{bc}}S_{236}S_{135}S_{124} \\
&= \overset{1}{L}_{cd}\overset{2}{L}_{bd}\overset{3}{L}_{ad}\overset{4}{L}_{bc}\overset{5}{L}_{ac}\overset{6}{L}_{ab}S_{456}S_{236}S_{135}S_{124}.
\end{aligned}
\tag{2.23}
$$

From (2.22) and (2.23) we see that $(S_{124}S_{135}S_{236}S_{456})^{-1}S_{456}S_{236}S_{135}S_{124}$ commutes with $\overset{6}{L}_{ab}\overset{5}{L}_{ac}\overset{4}{L}_{bc}\overset{3}{L}_{ad}\overset{2}{L}_{bd}\overset{1}{L}_{cd}$. Therefore if $\mathcal{F}^{\otimes 6}$ is irreducible under the action of $\overset{6}{L}_{ab}\overset{5}{L}_{ac}\overset{4}{L}_{bc}\overset{3}{L}_{ad}\overset{2}{L}_{bd}\overset{1}{L}_{cd}$, Schur's lemma compels $S_{124}S_{135}S_{236}S_{456} = \text{const}\, S_{456}S_{236}S_{135}S_{124}$. We will make the irreducibility argument precise in Sect. 3.5.2 along the main example of the book.

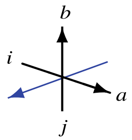
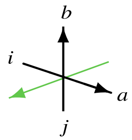
2.6 Tetrahedron Equation of Type $MMLL = LLMM$

We will also be concerned with another version of the tetrahedron equation which we call $MMLL = LLMM$ type. Consider the 3D L and its variant M both living in $\mathrm{End}(V \otimes V \otimes \mathcal{F})$ as

$$L = \sum_{a,b,i,j} E_{ai} \otimes E_{bj} \otimes L^{ab}_{ij}, \quad (2.24)$$

$$M = \sum_{a,b,i,j} E_{ai} \otimes E_{bj} \otimes M^{ab}_{ij}, \quad (2.25)$$

where $L^{ab}_{ij}, M^{ab}_{ij} \in \mathrm{End}(\mathcal{F})$, and E_{ij} is a matrix unit on V. The decomposition (2.24) is just (2.12). According to the left-hand diagram of (2.14) without the indices c, k, we depict the operators L^{ab}_{ij}, M^{ab}_{ij} as

$L^{ab}_{ij} =$  $M^{ab}_{ij} =$  (2.26)

Here, the black arrows carry V while the blue and green ones carry the space $\mathcal{F}$. As usual we will write $L_{245} = \sum 1 \otimes E_{ai} \otimes 1 \otimes E_{bj} \otimes L^{ab}_{ij} \otimes 1$, etc.

By tetrahedron equation of type $MMLL = LLMM$ we mean the following:

$$M_{126} M_{346} L_{135} L_{245} = L_{245} L_{135} M_{346} M_{126}. \quad (2.27)$$

This is an equality in $\mathrm{End}(\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V} \otimes \overset{4}{V} \otimes \overset{5}{\mathcal{F}} \otimes \overset{6}{\mathcal{F}})$ (Fig. 2.5).

Let us operate (2.27) to a vector $v_i \otimes v_j \otimes v_k \otimes v_l \otimes |X\rangle \otimes |Y\rangle$ in $V \otimes V \otimes V \otimes V \otimes \mathcal{F} \otimes \mathcal{F}$. The LHS gives

$$\begin{aligned}
& v_i \otimes v_j \otimes v_k \otimes v_l \otimes |X\rangle \otimes |Y\rangle \\
\overset{L_{245}}{\mapsto} & \sum v_i \otimes v_\beta \otimes v_k \otimes v_\delta \otimes L^{\beta\delta}_{jl} |X\rangle \otimes |Y\rangle \\
\overset{L_{135}}{\mapsto} & \sum v_\alpha \otimes v_\beta \otimes v_\gamma \otimes v_\delta \otimes L^{\alpha\gamma}_{ik} L^{\beta\delta}_{jl} |X\rangle \otimes |Y\rangle \\
\overset{M_{346}}{\mapsto} & \sum v_\alpha \otimes v_\beta \otimes v_c \otimes v_d \otimes L^{\alpha\gamma}_{ik} L^{\beta\delta}_{jl} |X\rangle \otimes M^{cd}_{\gamma\delta} |Y\rangle \\
\overset{M_{126}}{\mapsto} & \sum v_a \otimes v_b \otimes v_c \otimes v_d \otimes L^{\alpha\gamma}_{ik} L^{\beta\delta}_{jl} |X\rangle \otimes M^{ab}_{\alpha\beta} M^{cd}_{\gamma\delta} |Y\rangle.
\end{aligned} \quad (2.28)$$

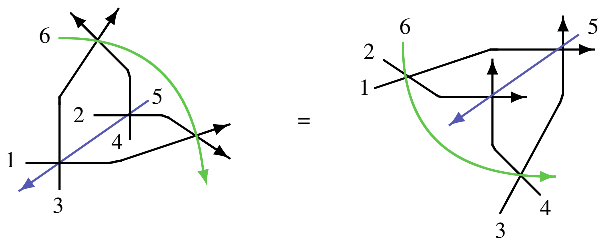


Fig. 2.5 A graphical representation of the tetrahedron equation of type $MMLL = LLMM$ (2.27)

A similar calculation for the RHS reads as

$$\begin{aligned} & v_i \otimes v_j \otimes v_k \otimes v_l \otimes |X\rangle \otimes |Y\rangle \\ \overset{M_{126}}{\mapsto} & \sum v_\alpha \otimes v_\beta \otimes v_k \otimes v_l \otimes |X\rangle \otimes M^{\alpha\beta}_{ij}|Y\rangle \\ \overset{M_{346}}{\mapsto} & \sum v_\alpha \otimes v_\beta \otimes v_\gamma \otimes v_\delta \otimes |X\rangle \otimes M^{\gamma\delta}_{kl} M^{\alpha\beta}_{ij}|Y\rangle \\ \overset{L_{135}}{\mapsto} & \sum v_a \otimes v_\beta \otimes v_c \otimes v_\delta \otimes L^{ac}_{\alpha\gamma}|X\rangle \otimes M^{\gamma\delta}_{kl} M^{\alpha\beta}_{ij}|Y\rangle \\ \overset{L_{245}}{\mapsto} & \sum v_a \otimes v_b \otimes v_c \otimes v_d \otimes L^{bd}_{\beta\delta} L^{ac}_{\alpha\gamma}|X\rangle \otimes M^{\gamma\delta}_{kl} M^{\alpha\beta}_{ij}|Y\rangle. \end{aligned} \tag{2.29}$$

The sums extend over an appropriate subset of variables $\{a, b, c, d, \alpha, \beta, \gamma, \delta\}$. Comparing the transitions $v_i \otimes v_j \otimes v_k \otimes v_l \mapsto v_a \otimes v_b \otimes v_c \otimes v_d$ in $\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V} \otimes \overset{4}{V}$ for the two sides, we see that the tetrahedron equation (2.27) is equivalent to the totality of relations

$$\sum_{\alpha,\beta,\gamma,\delta} L^{\alpha\gamma}_{ik} L^{\beta\delta}_{jl} \otimes M^{ab}_{\alpha\beta} M^{cd}_{\gamma\delta} = \sum_{\alpha,\beta,\gamma,\delta} L^{bd}_{\beta\delta} L^{ac}_{\alpha\gamma} \otimes M^{\gamma\delta}_{kl} M^{\alpha\beta}_{ij} \tag{2.30}$$

in $\mathrm{End}(\mathcal{F} \otimes \mathcal{F})$ for all the base labels a, b, c, d, i, j, k, l of V. We will see the applications of $MMLL = LLMM$ type tetrahedron equation in Chaps. 6 and 18.

2.7 Bibliographical Notes and Comments

The tetrahedron equation was first proposed in [153] as a three-dimensional analogue of the Yang–Baxter equation. It was followed by early pioneering works [11, 12, 154], where the first non-trivial solution was established and the partition function of the associated 3D lattice model was computed exactly.

There are basically three versions of tetrahedron equations depending on whether "spins" are assigned to 3D regions, 2D faces or 1D edges as in Fig. 2.1.[3] The original one [153, 154] regarded as a face version was reformulated as an IRC (Interaction Round Cube) model which corresponds to a region version [11, Eq. (2.4)]. The Eqs. (2.6) and (2.15) that will be considered in this book are the edge version, which is a natural 3D analogue of the vertex models in 2D in the sense of [10]. For the relation and transformations between various formulations, see [58] and [77, Chap. 1], where the latter reference explores a hieroglyphical description by 2-categories.

After the initial breakthrough, a new development in the 1990s was the unexpected connection [14, 72, 73, 133] of the original model [11, 153, 154] to the Chiral Potts model [5, 13] and its generalizations [15, 33]. They are associated with quantum

[3] In yet another context, there are also set-theoretical versions. Typical of them are birational (also called functional) and combinatorial (also called tropical) ones, which will be explained in Sect. 3.6.2.

groups with q at roots of unity, where the spectral parameters are effectively upgraded to the points on a higher genus algebraic curve.

The tetrahedron equation of type $RLLL = LLLR$ has appeared in numerous guises and is referred to also as a local Yang–Baxter equation, tetrahedral Zamolodchikov algebra (when one R is suppressed), quantum Korepanov equation, etc. It has been studied in various contexts in [16, 18, 42, 69–71, 74, 80, 107, 114, 115, 135]. Our naming "quantized Yang–Baxter equation" is meant to indicate the conjugacy equivalence $RRR \simeq RRR$ or $LLL \simeq LLL$. It is sometimes confused with the "quantum Yang–Baxter equation" which is often used to mean the *usual* Yang–Baxter equation $RRR = RRR$ for distinction from the *classical* Yang–Baxter equation $[r, r] + [r, r] + [r, r] = 0$ [20]. The reason we nevertheless adopt the nomenclature is that we are to encounter the *quantized reflection equation* and the *quantized G_2 reflection equation* in exactly the same vein in later chapters of the book.

The tetrahedron equation of type $MMLL = LLMM$ has appeared in [18, Eq. (34)] and [90, Theorem 3.4]. We will present its application to the proof of RTT relation of $A_q(B_n)$ in Chap. 6 and also to a multispecies totally asymmetric simple exclusion process in Chap. 18.

Generalizations of the tetrahedron equation to higher dimensions are called simplex equations. See for example [19, 28, 40, 48, 115, 118] and the references therein.

Chapter 3
3D *R* From Quantized Coordinate Ring of Type A

Abstract Let $\mathfrak{g}$ be a classical simple Lie algebra and $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}$. There is a Hopf algebra dual to $U_q(\mathfrak{g})$ which corresponds to a q deformation of the algebra of functions on the Lie group of $\mathfrak{g}$. It will be called the quantized coordinate ring and denoted by $A_q(\mathfrak{g})$ in this book. We assume that q is generic throughout. In this chapter, $A_q(\mathfrak{g})$ for $\mathfrak{g}$ of type A is treated based on a concrete realization by generators and relations, deferring a more universal formulation to Sect. 10.2. It turns out that an intertwiner of certain $A_q(\mathfrak{g})$ modules leads to a 3D R, a solution of the tetrahedron equation. It has set-theoretical and birational counterparts which satisfy the tetrahedron equation in the respective setting. The birational case admits bilinearization in terms of tau functions.

3.1 Quantized Coordinate Ring $A_q(A_{n-1})$

Let $n \geq 2$ be an integer. This chapter is devoted to the type A case $\mathfrak{g} = A_{n-1}$.[1] The quantized coordinate ring $A_q(A_{n-1})$ is a Hopf algebra [1] with n^2 generators $(t_{ij})_{1 \leq i,j \leq n}$. In terms of the n by n matrix $T = (t_{ij})$, their relations are presented in the so-called $RTT = TTR$ form and the unit quantum determinant condition:

$$\sum_{m,p} R^{ij}_{mp} t_{mk} t_{pl} = \sum_{m,p} t_{jp} t_{im} R^{mp}_{kl}, \tag{3.1}$$

$$\sum_{\sigma \in \mathfrak{S}_n} (-q)^{l(\sigma)} t_{1\sigma_1} \cdots t_{n\sigma_n} = 1. \tag{3.2}$$

The former is called the RTT relation. The symbol $\mathfrak{S}_n$ denotes the symmetric group of degree n and $l(\sigma)$ is the length of the permutation σ. The structure constant R^{ij}_{kl} is specified by

[1] Although, Theorem 3.3 is valid for general classical simple Lie algebra $\mathfrak{g}$.

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$$\sum_{i,j,k,l} R^{ij}_{kl} E_{ik} \otimes E_{jl} = q \sum_{i} E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i>j} E_{ij} \otimes E_{ji}, \tag{3.3}$$

where the indices are summed over $\{1, 2, \ldots, n\}$, and E_{ij} is a matrix unit. The matrix (3.3) is extracted as

$$\sum_{i,j,m,l} R^{ij}_{ml} E_{im} \otimes E_{jl} = q \lim_{x \to \infty} x^{-1} R(x)|_{k=q^{-1}} \tag{3.4}$$

from the quantum R matrix $R(x)$ for the vector representation of $U_q(A^{(1)}_{n-1})$ given in [64, Eq. (3.5)].[2] Explicitly, the relation (3.1) reads as

$$\begin{aligned} [t_{ik}, t_{jl}] &= \begin{cases} 0 & (i < j, k > l), \\ (q - q^{-1}) t_{jk} t_{il} & (i < j, k < l), \end{cases} \\ t_{ik} t_{jk} &= q t_{jk} t_{ik} \ (i < j), \quad t_{ki} t_{kj} = q t_{kj} t_{ki} \ (i < j). \end{aligned} \tag{3.5}$$

The coproduct or co-multiplication is given by

$$\Delta(t_{ij}) = \sum_{k} t_{ik} \otimes t_{kj}. \tag{3.6}$$

We will use the same symbol Δ flexibly to also mean the multiple coproducts like $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$, etc. The antipode S and the counit ϵ are given by

$$S(t_{ij}) = (-q)^{i-j} \sum_{\sigma \in \mathfrak{S}_{n-1}} (-q)^{l(\sigma)} t_{1,\sigma_1} \cdots t_{j-1,\sigma_{j-1}} t_{j+1,\sigma_{j+1}} \cdots t_{n,\sigma_n}, \tag{3.7}$$

$$\epsilon(t_{ij}) = \delta_{ij}. \tag{3.8}$$

The sum in (3.7) is the quantum minor which extends over permutations of $\{1, \ldots, n\} \setminus \{i\}$.

Example 3.1 The simplest case $n = 2$ is $A_q(A_1)$. It is generated by $t_{11}, t_{12}, t_{21}, t_{22}$ with the relations

$$\begin{aligned} &t_{11} t_{21} = q t_{21} t_{11}, \quad t_{12} t_{22} = q t_{22} t_{12}, \quad t_{11} t_{12} = q t_{12} t_{11}, \quad t_{21} t_{22} = q t_{22} t_{21}, \\ &[t_{12}, t_{21}] = 0, \quad [t_{11}, t_{22}] = (q - q^{-1}) t_{21} t_{12}, \quad t_{11} t_{22} - q t_{12} t_{21} = 1. \end{aligned} \tag{3.9}$$

The quantum determinant $t_{11} t_{22} - q t_{12} t_{21}$ appearing in (3.9) is central. The rule (3.6) implies that the coproduct Δ is obtained by formally replacing the product in matrix multiplication by $\otimes$ as

[2] In Chaps. 3, 5, 6 and 8, the quantum R matrices and their elements R^{ij}_{ml} appear only as the structure constant in the RTT relation. They should not be confused with those of the 3D R.

$$\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \overset{\Delta}{\mapsto} \begin{pmatrix} t_{11}\otimes t_{11} + t_{12}\otimes t_{21} & t_{11}\otimes t_{12} + t_{12}\otimes t_{22} \\ t_{21}\otimes t_{11} + t_{22}\otimes t_{21} & t_{21}\otimes t_{12} + t_{22}\otimes t_{22} \end{pmatrix}. \tag{3.10}$$

The multiple coproduct is similar. It is easy to check that Δ is an algebra homomorphism, for example, $\Delta(t_{11})\Delta(t_{21}) = q\Delta(t_{21})\Delta(t_{11})$ by using (3.9) and (3.10). A defining axiom $m \circ (1 \otimes S) \circ \Delta = \iota \circ \epsilon$ for example,[3] is checked as

$$\begin{aligned} (3.10) &\overset{1\otimes S}{\mapsto} \begin{pmatrix} t_{11}\otimes t_{22} + t_{12}\otimes(-qt_{21}) & t_{11}\otimes(-q^{-1}t_{12}) + t_{12}\otimes t_{11} \\ t_{21}\otimes t_{22} + t_{22}\otimes(-qt_{21}) & t_{21}\otimes(-q^{-1}t_{12}) + t_{22}\otimes t_{11} \end{pmatrix} \\ &\overset{m}{\mapsto} \begin{pmatrix} t_{11}t_{22} - qt_{12}t_{21} & -q^{-1}t_{11}t_{12} + t_{12}t_{11} \\ t_{21}t_{22} - qt_{22}t_{21} & -q^{-1}t_{21}t_{12} + t_{22}t_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned} \tag{3.11}$$

A sketch of "derivation" of the relations (3.9) from the dual $U_q(sl_2)$ is available in Example 10.2.

Remark 3.2 The map $t_{jk} \mapsto \xi_j^{-1}\xi_k t_{jk}$ with non-zero parameters $\xi_1, \dots, \xi_n$ is a Hopf algebra automorphism.

3.2 Representation Theory

Let $\mathrm{Osc}_q = \langle \mathbf{a}^+, \mathbf{a}^-, \mathbf{k}, \mathbf{k}^{-1} \rangle$ be the q-oscillator algebra, i.e. an associative algebra with the relations

$$\mathbf{k}\,\mathbf{a}^+ = q\,\mathbf{a}^+\mathbf{k}, \quad \mathbf{k}\,\mathbf{a}^- = q^{-1}\mathbf{a}^-\mathbf{k}, \quad \mathbf{a}^-\mathbf{a}^+ = 1 - q^2\mathbf{k}^2, \quad \mathbf{a}^+\mathbf{a}^- = 1 - \mathbf{k}^2 \tag{3.12}$$

and those following from the obvious ones $\mathbf{k}\,\mathbf{k}^{-1} = \mathbf{k}^{-1}\mathbf{k} = 1$. It has an irreducible representation on the Fock space $\mathcal{F}_q = \bigoplus_{m\ge 0} \mathbb{C}(q)|m\rangle$:

$$\mathbf{k}|m\rangle = q^m|m\rangle, \quad \mathbf{a}^+|m\rangle = |m+1\rangle, \quad \mathbf{a}^-|m\rangle = (1-q^{2m})|m-1\rangle. \tag{3.13}$$

In particular $\mathbf{a}^-|0\rangle = 0$. The generators $\mathbf{a}^\pm$ and $\mathbf{k}^{\pm 1}$ will be identified with the elements of $\mathrm{End}(\mathcal{F}_q)$ defined by (3.13) unless otherwise stated. We will also use the diagonal operators $\mathbf{h}$ and D_q such that

$$\mathbf{h}|m\rangle = m|m\rangle, \tag{3.14}$$

$$D_q|m\rangle = (q^2)_m|m\rangle. \tag{3.15}$$

Thus we may identify $\mathbf{k}$ as $\mathbf{k} = q^{\mathbf{h}}$. An eigenvalue of $\mathbf{h}$ will be referred to as a mode of the q-oscillator. For the notation $(q^2)_m = (q^2; q^2)_m$, see (3.65).

[3] ι and m are the unit and the multiplication of the Hopf algebra $A_q(A_1)$ under consideration.

We will also be concerned with the dual Fock space $\mathcal{F}_q^* = \bigoplus_{m\ge 0} \mathbb{C}(q)\langle m|$ whose pairing with $\mathcal{F}_q$ is specified by

$$\langle m|m'\rangle = (q^2)_m \delta_{m,m'}. \tag{3.16}$$

The q-oscillators act on $\mathcal{F}_q^*$ as

$$\langle m|\mathbf{k} = \langle m|q^m, \quad \langle m|\mathbf{a}^+ = \langle m-1|(1-q^{2m}), \quad \langle m|\mathbf{a}^- = \langle m+1| \tag{3.17}$$

and $\langle m|\mathbf{h} = \langle m|m$. In particular $\langle 0|\mathbf{a}^+ = 0$. They satisfy $(\langle m|X)|m'\rangle = \langle m|(X|m'\rangle)$ and

$$\langle m|X_1 \cdots X_j|m'\rangle = \langle m'|\overline{X_j} \cdots \overline{X_1}|m\rangle, \tag{3.18}$$

where $\overline{(\cdots)}$ is defined by $\overline{\mathbf{a}^{\pm}} = \mathbf{a}^{\mp}$, $\overline{\mathbf{k}} = \mathbf{k}$ and $\overline{\mathbf{h}} = \mathbf{h}$.

The algebra $A_q(A_1)$ in Example 3.1 has the irreducible representation π on $\mathcal{F}_q$ depending on a non-zero parameter μ as follows:

$$\pi : \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{a}^- & \mu\mathbf{k} \\ -q\mu^{-1}\mathbf{k} & \mathbf{a}^+ \end{pmatrix}. \tag{3.19}$$

For $A_q(A_{n-1})$, there are similar representations

$$\pi_i : A_q(A_{n-1}) \to \mathrm{End}(\mathcal{F}_q) \qquad (1 \le i \le n-1). \tag{3.20}$$

It contains a non-zero parameter μ_i and factors through (3.19) via the surjective map $A_q(A_{n-1}) \twoheadrightarrow A_q(sl_{2,i})$. Here, $sl_{2,i}$ denotes the $A_1 = sl_2$-subalgebra of A_{n-1} associated with i. It is given by

$$\pi_i : \begin{pmatrix} t_{11} & & & & & & t_{1n} \\ & \ddots & & & & & \\ & & t_{i-1,i-1} & & & & \\ & & & t_{i,i} & t_{i,i+1} & & \\ & & & t_{i+1,i} & t_{i+1,i+1} & & \\ & & & & & t_{i+2,i+2} & \\ & & & & & & \ddots \\ t_{n1} & & & & & & t_{nn} \end{pmatrix} \mapsto \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \mathbf{a}^- & \mu_i\mathbf{k} & & \\ & & & -q\mu_i^{-1}\mathbf{k} & \mathbf{a}^+ & & \\ & & & & & 1 & \\ & & & & & & \ddots \\ & & & & & & & 1 \end{pmatrix}, \tag{3.21}$$

where all the blanks on the RHS are to be understood as 0. It is easy to see that $\pi_1, \ldots, \pi_{n-1}$ are all inequivalent and irreducible. Starting from them, one can construct tensor product representations $\pi_{i_1} \otimes \cdots \otimes \pi_{i_l} : A_q(A_{n-1}) \to \mathrm{End}(\mathcal{F}_q^{\otimes l})$ via $f \mapsto (\pi_{i_1} \otimes \cdots \otimes \pi_{i_l})(\Delta(f))$ using the multiple coproduct Δ obtained by iterating (3.6) $l-1$ times. A natural question at this stage is, what is the totality of irreducible representations up to equivalence and how they can be realized. The answer has been known for $A_q(\mathfrak{g})$ associated with any classical simple Lie algebra $\mathfrak{g}$.

Theorem 3.3 *(i) For each vertex i of the Dynkin diagram of $\mathfrak{g}$, $A_q(\mathfrak{g})$ has an irreducible representation π_i factoring through (3.19) via $A_q(\mathfrak{g}) \twoheadrightarrow A_{q_i}(sl_{2,i})$.*

(ii) Irreducible representations of $A_q(\mathfrak{g})$ up to equivalence are in one-to-one correspondence with the elements of the Weyl group W of $\mathfrak{g}$.

(iii) Let $w = s_{i_1} \cdots s_{i_l} \in W$ be a reduced expression in terms of the simple reflections. Then the irreducible representation corresponding to w is isomorphic to $\pi_{i_1} \otimes \cdots \otimes \pi_{i_l}$.

In (i), $q_i = q^{(\alpha_i,\alpha_i)/2}$, where α_i is a simple root.[4] The assertions (ii) and (iii) actually hold up to the degrees of freedom of the parameters as μ_i in (3.19). See [138, 139, 146] for the detail. We call π_i ($i = 1, \dots, \operatorname{rank} \mathfrak{g}$) the *fundamental representations*. We will often denote $\pi_{i_1} \otimes \cdots \otimes \pi_{i_l}$ by $\pi_{i_1,\dots,i_l}$ for short.

Returning to the $\mathfrak{g} = A_{n-1}$ case, the representations $\pi_1, \dots, \pi_{n-1}$ defined in (3.21) are the fundamental representations of $A_q(A_{n-1})$ in the above sense. The Weyl group $W(A_{n-1}) = \langle s_1, \dots, s_{n-1} \rangle$ is generated by the simple reflections $s_1, \dots, s_{n-1}$ obeying the Coxeter relations

$$s_i^2 = 1, \qquad s_i s_j = s_j s_i \;\; (|i-j| \geq 2), \qquad s_i s_j s_i = s_j s_i s_j \;\; (|i-j| = 1). \tag{3.22}$$

From the second relation here and Theorem 3.3 (iii) it follows that $\pi_i \otimes \pi_j \simeq \pi_j \otimes \pi_i$ for $|i-j| \geq 2$. This isomorphism is simply provided as the transposition of components:

$$P(x \otimes y) = y \otimes x. \tag{3.23}$$

In order to show this, one should check that

$$P(\pi_i \otimes \pi_j)(\Delta(f)) = (\pi_j \otimes \pi_i)(\Delta(f))P \qquad (|i-j| \geq 2) \tag{3.24}$$

holds for any $f \in A_q(A_{n-1})$. Since Δ is an algebra homomorphism, it suffices to consider the $f = t_{km}$ case:

$$P\Big(\sum_l \pi_i(t_{kl}) \otimes \pi_j(t_{lm})\Big) = \Big(\sum_l \pi_j(t_{kl}) \otimes \pi_i(t_{lm})\Big)P \quad \text{for } |i-j| \geq 2, \tag{3.25}$$

which is equivalent to

$$\sum_l \pi_j(t_{lm}) \otimes \pi_i(t_{kl}) = \sum_l \pi_j(t_{kl}) \otimes \pi_i(t_{lm}) \quad \text{for } |i-j| \geq 2. \tag{3.26}$$

This indeed holds thanks to the simple and sparse structure of (3.21).

Remark 3.4 Not only for A_{n-1} but for general $\mathfrak{g}$, the equivalence of $\pi_i \otimes \pi_j \simeq \pi_j \otimes \pi_i$ for i, j such that $s_i s_j = s_j s_i$ is always assured by the transposition P in (3.23).

[4] We normalize the simple root so that $q_i = q$ when $\mathfrak{g}$ is simply-laced or α_i is short.

By virtue of Remark 3.2, all the parameters $\mu_1, \ldots, \mu_{n-1}$ in the fundamental representations $\pi_1, \ldots, \pi_{n-1}$ are removed by the choice $\xi_j = \prod_{k=1}^{j-1} \mu_k$. Henceforth we set $\mu_1 = \cdots = \mu_{n-1} = 1$ in the rest of the chapter without loss of generality.

3.3 Intertwiner for Cubic Coxeter Relation

The isomorphism of the two irreducible representations will be called the *intertwiner*. By Schur's lemma, it is unique up to the overall normalization. The transposition P in (3.23) is the intertwiner corresponding to the quadratic Coxeter relation.

Let us proceed to the cubic one. In view of the structure (3.21), it suffices to consider $A_q(A_2)$ and the equivalence $\pi_{121} \simeq \pi_{212}$ reflecting the Coxeter relation $s_1 s_2 s_1 = s_2 s_1 s_2$. Let

$$\Phi : \mathcal{F}_q \otimes \mathcal{F}_q \otimes \mathcal{F}_q \longrightarrow \mathcal{F}_q \otimes \mathcal{F}_q \otimes \mathcal{F}_q \tag{3.27}$$

be the associated intertwiner. It is characterized by the relations:

$$\Phi \circ \pi_{121}(\Delta(f)) = \pi_{212}(\Delta(f)) \circ \Phi \quad (\forall f \in A_q(A_2)), \tag{3.28}$$

$$\Phi(|0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle. \tag{3.29}$$

The latter just fixes the normalization. The absence of terms other than $|0\rangle \otimes |0\rangle \otimes |0\rangle$ in its RHS is assured by the weight conservation. See (3.48), (3.47) and (3.30).

It is convenient to work with R defined by

$$R = \Phi P_{13} : \ \mathcal{F}_q \otimes \mathcal{F}_q \otimes \mathcal{F}_q \longrightarrow \mathcal{F}_q \otimes \mathcal{F}_q \otimes \mathcal{F}_q. \tag{3.30}$$

Here P_{13} is the interchanger of the first and the third components defined before (2.20). We also call R the intertwiner. It will be shown to satisfy the tetrahedron equations of type $RRRR = RRRR$ in Theorem 3.20 (and also $RLLL = LLLR$ in Theorem 3.21), therefore R is a 3D R in the sense of Sect. 2.1. From (3.28) and (3.29), R is characterized by

$$R \circ \pi_{121}(\tilde{\Delta}(f)) = \pi_{212}(\Delta(f)) \circ R \quad (\forall f \in A_q(A_2)), \tag{3.31}$$

$$R(|0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle, \tag{3.32}$$

where $\tilde{\Delta}(f) = P_{13}(\Delta(f))P_{13}$. From (3.6) we have

$$\Delta(t_{ij}) = \sum_{1 \le l_1, l_2 \le 3} t_{il_1} \otimes t_{l_1 l_2} \otimes t_{l_2 j}, \quad \tilde{\Delta}(t_{ij}) = \sum_{1 \le l_1, l_2 \le 3} t_{l_2 j} \otimes t_{l_1 l_2} \otimes t_{il_1}. \tag{3.33}$$

According to (3.21), the image of the 9 generators $T = (t_{ij})_{1 \le i,j \le 3}$ by the fundamental representations reads as

$$\pi_1(T) = \begin{pmatrix} \mathbf{a}^- & \mathbf{k} & 0 \\ -q\mathbf{k} & \mathbf{a}^+ & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \pi_2(T) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{a}^- & \mathbf{k} \\ 0 & -q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}. \tag{3.34}$$

From (3.33), $\pi_{121}(\Delta(T))$ is expressed as

$$\begin{pmatrix} \mathbf{a}^- \otimes 1 \otimes \mathbf{a}^- - q\mathbf{k} \otimes \mathbf{a}^- \otimes \mathbf{k} & \mathbf{k} \otimes \mathbf{a}^- \otimes \mathbf{a}^+ + \mathbf{a}^- \otimes 1 \otimes \mathbf{k} & \mathbf{k} \otimes \mathbf{k} \otimes 1 \\ -q(\mathbf{a}^+ \otimes \mathbf{a}^- \otimes \mathbf{k} + \mathbf{k} \otimes 1 \otimes \mathbf{a}^-) & \mathbf{a}^+ \otimes \mathbf{a}^- \otimes \mathbf{a}^+ - q\mathbf{k} \otimes 1 \otimes \mathbf{k} & \mathbf{a}^+ \otimes \mathbf{k} \otimes 1 \\ q^2 1 \otimes \mathbf{k} \otimes \mathbf{k} & -q 1 \otimes \mathbf{k} \otimes \mathbf{a}^+ & 1 \otimes \mathbf{a}^+ \otimes 1 \end{pmatrix}. \tag{3.35}$$

$\pi_{121}(\tilde{\Delta}(T))$ is given by reversing the order of the tensor product as

$$\begin{pmatrix} \mathbf{a}^- \otimes 1 \otimes \mathbf{a}^- - q\mathbf{k} \otimes \mathbf{a}^- \otimes \mathbf{k} & \mathbf{a}^+ \otimes \mathbf{a}^- \otimes \mathbf{k} + \mathbf{k} \otimes 1 \otimes \mathbf{a}^- & 1 \otimes \mathbf{k} \otimes \mathbf{k} \\ -q(\mathbf{k} \otimes \mathbf{a}^- \otimes \mathbf{a}^+ + \mathbf{a}^- \otimes 1 \otimes \mathbf{k}) & \mathbf{a}^+ \otimes \mathbf{a}^- \otimes \mathbf{a}^+ - q\mathbf{k} \otimes 1 \otimes \mathbf{k} & 1 \otimes \mathbf{k} \otimes \mathbf{a}^+ \\ q^2 \mathbf{k} \otimes \mathbf{k} \otimes 1 & -q\mathbf{a}^+ \otimes \mathbf{k} \otimes 1 & 1 \otimes \mathbf{a}^+ \otimes 1 \end{pmatrix}. \tag{3.36}$$

$\pi_{212}(\Delta(T))$ takes the form

$$\begin{pmatrix} 1 \otimes \mathbf{a}^- \otimes 1 & 1 \otimes \mathbf{k} \otimes \mathbf{a}^- & 1 \otimes \mathbf{k} \otimes \mathbf{k} \\ -q\mathbf{a}^- \otimes \mathbf{k} \otimes 1 & \mathbf{a}^- \otimes \mathbf{a}^+ \otimes \mathbf{a}^- - q\mathbf{k} \otimes 1 \otimes \mathbf{k} & \mathbf{a}^- \otimes \mathbf{a}^+ \otimes \mathbf{k} + \mathbf{k} \otimes 1 \otimes \mathbf{a}^+ \\ q^2\mathbf{k} \otimes \mathbf{k} \otimes 1 & -q(\mathbf{k} \otimes \mathbf{a}^+ \otimes \mathbf{a}^- + \mathbf{a}^+ \otimes 1 \otimes \mathbf{k}) & \mathbf{a}^+ \otimes 1 \otimes \mathbf{a}^+ - q\mathbf{k} \otimes \mathbf{a}^+ \otimes \mathbf{k} \end{pmatrix}. \tag{3.37}$$

Thus the intertwining relation (3.31) reads as

$$t_{11}: \quad R(\mathbf{a}^- \otimes 1 \otimes \mathbf{a}^- - q\mathbf{k} \otimes \mathbf{a}^- \otimes \mathbf{k}) = (1 \otimes \mathbf{a}^- \otimes 1)R, \tag{3.38}$$

$$t_{12}: \quad R(\mathbf{a}^+ \otimes \mathbf{a}^- \otimes \mathbf{k} + \mathbf{k} \otimes 1 \otimes \mathbf{a}^-) = (1 \otimes \mathbf{k} \otimes \mathbf{a}^-)R, \tag{3.39}$$

$$t_{13}: \quad R(1 \otimes \mathbf{k} \otimes \mathbf{k}) = (1 \otimes \mathbf{k} \otimes \mathbf{k})R, \tag{3.40}$$

$$t_{21}: \quad R(\mathbf{k} \otimes \mathbf{a}^- \otimes \mathbf{a}^+ + \mathbf{a}^- \otimes 1 \otimes \mathbf{k}) = (\mathbf{a}^- \otimes \mathbf{k} \otimes 1)R, \tag{3.41}$$

$$t_{22}: \quad R(\mathbf{a}^+ \otimes \mathbf{a}^- \otimes \mathbf{a}^+ - q\mathbf{k} \otimes 1 \otimes \mathbf{k}) = (\mathbf{a}^- \otimes \mathbf{a}^+ \otimes \mathbf{a}^- - q\mathbf{k} \otimes 1 \otimes \mathbf{k})R, \tag{3.42}$$

$$t_{23}: \quad R(1 \otimes \mathbf{k} \otimes \mathbf{a}^+) = (\mathbf{a}^- \otimes \mathbf{a}^+ \otimes \mathbf{k} + \mathbf{k} \otimes 1 \otimes \mathbf{a}^+)R, \tag{3.43}$$

$$t_{31}: \quad R(\mathbf{k} \otimes \mathbf{k} \otimes 1) = (\mathbf{k} \otimes \mathbf{k} \otimes 1)R, \tag{3.44}$$

$$t_{32}: \quad R(\mathbf{a}^+ \otimes \mathbf{k} \otimes 1) = (\mathbf{k} \otimes \mathbf{a}^+ \otimes \mathbf{a}^- + \mathbf{a}^+ \otimes 1 \otimes \mathbf{k})R, \tag{3.45}$$

$$t_{33}: \quad R(1 \otimes \mathbf{a}^+ \otimes 1) = (\mathbf{a}^+ \otimes 1 \otimes \mathbf{a}^+ - q\mathbf{k} \otimes \mathbf{a}^+ \otimes \mathbf{k})R, \tag{3.46}$$

where the left column specifies the choice of f in (3.31).

The intertwiner R is regarded as a matrix $R = (R^{abc}_{ijk})$ acting on $\mathcal{F}_q^{\otimes 3}$ as

$$R(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b,c} R^{abc}_{ijk} |a\rangle \otimes |b\rangle \otimes |c\rangle. \tag{3.47}$$

The normalization condition (3.29) becomes $R^{abc}_{000} = \delta^a_0 \delta^b_0 \delta^c_0$. The simplest equations (3.40) and (3.44) imply

$$R^{abc}_{ijk} = 0 \ \text{ unless } \ (a+b, b+c) = (i+j, j+k). \tag{3.48}$$

This property will be referred to as the *weight conservation*. It may also be rephrased as

$$[R, z^{\mathbf{h}} \otimes z^{\mathbf{h}} \otimes 1] = [R, 1 \otimes z^{\mathbf{h}} \otimes z^{\mathbf{h}}] = 0, \tag{3.49}$$

where $\mathbf{h}$ is the number operator (3.14) and z is a non-zero parameter. The other equations lead to recursion relations of the matrix elements as follows:

$$t_{11}: q^{i+k+1}(1-q^{2j})R^{a,b,c}_{i,j-1,k} + (1-q^{2b+2})R^{a,b+1,c}_{i,j,k} = (1-q^{2i})(1-q^{2k})R^{a,b,c}_{i-1,j,k-1}, \tag{3.50}$$

$$t_{12}: q^{k}(1-q^{2j})R^{a,b,c}_{i+1,j-1,k} + q^{i}(1-q^{2k})R^{a,b,c}_{i,j,k-1} = q^{b}(1-q^{2c+2})R^{a,b,c+1}_{i,j,k}, \tag{3.51}$$

$$t_{21}: q^{i}(1-q^{2j})R^{a,b,c}_{i,j-1,k+1} + q^{k}(1-q^{2i})R^{a,b,c}_{i-1,j,k} = q^{b}(1-q^{2a+2})R^{a+1,b,c}_{i,j,k}, \tag{3.52}$$

$$t_{22}: q(q^{a+c}-q^{i+k})R^{a,b,c}_{i,j,k} + (1-q^{2j})R^{a,b,c}_{i+1,j-1,k+1} = (1-q^{2a+2})(1-q^{2c+2})R^{a+1,b-1,c+1}_{i,j,k}, \tag{3.53}$$

$$t_{23}: q^{j} R^{a,b,c}_{i,j,k+1} - q^{a} R^{a,b,c-1}_{i,j,k} - q^{c}(1-q^{2a+2})R^{a+1,b-1,c}_{i,j,k} = 0, \tag{3.54}$$

$$t_{32}: q^{c} R^{a-1,b,c}_{i,j,k} - q^{j} R^{a,b,c}_{i+1,j,k} + q^{a}(1-q^{2c+2})R^{a,b-1,c+1}_{i,j,k} = 0, \tag{3.55}$$

$$t_{33}: q^{a+c+1} R^{a,b-1,c}_{i,j,k} - R^{a-1,b,c-1}_{i,j,k} + R^{a,b,c}_{i,j+1,k} = 0. \tag{3.56}$$

The relations (3.54), (3.55) and (3.56) can be used to reduce k, i and j, respectively. Consequently, an arbitrary R^{abc}_{ijk} satisfying (3.48) is attributed to $R^{000}_{000} = 1$. Thus R is determined only by these relations. Since the intertwiner exists, compatibility of the reduction procedure and validity of the other relations is guaranteed. The resulting explicit formula will be presented in (3.67).

Lemma 3.5 *Set* $X_{ij} = (-q)^{i-j}(S(t_{4-j,4-i})|_{q\to q^{-1}})' \in A_q(A_2)$ $(1 \le i, j \le 3)$, *where* S *is the antipode (3.7) and the prime reverses the order of product of generators. Explicitly we have*

$$X_{11} = t_{22}t_{11} - q^{-1}t_{21}t_{12}, \qquad X_{12} = q^{-2}(t_{31}t_{12} - qt_{32}t_{11}),$$

$$X_{13} = q^{-3}(-t_{31}t_{22} + qt_{32}t_{21}), \qquad X_{21} = t_{21}t_{13} - qt_{23}t_{11},$$

$$X_{22} = t_{33}t_{11} - q^{-1}t_{31}t_{13}, \qquad X_{23} = q^{-2}(t_{31}t_{23} - qt_{33}t_{21}),$$

$$X_{31} = q(-t_{22}t_{13} + qt_{23}t_{12}), \qquad X_{32} = t_{32}t_{13} - qt_{33}t_{12},$$

$$X_{33} = t_{33}t_{22} - q^{-1}t_{32}t_{23}.$$

Then the following relations are valid:

$$\pi_{212}(\Delta(X_{ij})) = \pi_{121}(\tilde{\Delta}(t_{ij})), \quad \pi_{212}(\tilde{\Delta}(X_{ij})) = \pi_{121}(\Delta(t_{ij})). \tag{3.57}$$

Proof The two relations are equivalent by the conjugation by P_{13}. Let us illustrate a direct check of $\pi_{212}(\Delta(X_{23}) = \pi_{121}(\tilde{\Delta}(t_{23}))$. The LHS is $q^{-2}\pi_{212}(\Delta(t_{31}t_{23} - qt_{33}t_{21}))$. Substituting (3.36) and (3.37), we find that the relation to be shown is given by

$$\begin{aligned}&(\mathbf{k}\otimes\mathbf{k}\otimes 1)(\mathbf{a}^-\otimes\mathbf{a}^+\otimes\mathbf{k}+\mathbf{k}\otimes 1\otimes\mathbf{a}^+)\\&+(\mathbf{a}^+\otimes 1\otimes\mathbf{a}^+ - q\mathbf{k}\otimes\mathbf{a}^+\otimes\mathbf{k})(\mathbf{a}^-\otimes\mathbf{k}\otimes 1) = 1\otimes\mathbf{k}\otimes\mathbf{a}^+.\end{aligned}$$

To check this by (3.12) is straightforward. The other cases are similar. □

By definition, the transpose ${}^tY$ of an operator $Y \in \mathrm{End}(\mathcal{F}_q)$ is specified by ${}^tY|m\rangle = \sum_{m'} c^m_{m'}|m'\rangle$ for $Y|m\rangle = \sum_{m'} c^{m'}_m|m'\rangle$. Similar notations will be used also for operators on the tensor product of Fock spaces.

Set

$$\mathcal{D}_A = D_q \otimes D_q \otimes D_q, \tag{3.58}$$

where D_q is defined by (3.15).

Lemma 3.6 *The transposed representations are related to the original ones as*

$$\begin{aligned}{}^t(\pi_{212}(\Delta(t_{ij}))) &= \mathcal{D}_A\pi_{121}(\tilde{\Delta}(t_{j'i'}))\mathcal{D}_A^{-1},\\ {}^t(\pi_{121}(\tilde{\Delta}(t_{ij}))) &= \mathcal{D}_A\pi_{212}(\Delta(t_{j'i'}))\mathcal{D}_A^{-1}\end{aligned}$$

for $i, j \in \{1, 2, 3\}$*, where* $i' = 4 - i$*.*

Proof The two relations are equivalent. See (3.33). From (3.13) and (3.15), we see ${}^t(\mathbf{a}^\pm) = D_q\mathbf{a}^\mp D_q^{-1}$ and ${}^t\mathbf{k} = D_q\mathbf{k}D_q^{-1}$. They lead to

$${}^t\pi_1(t_{ij}) = D_q\pi_2(t_{j'i'})D_q^{-1}, \quad {}^t\pi_2(t_{ij}) = D_q\pi_1(t_{j'i'})D_q^{-1}$$

for the fundamental representations (3.34). The assertion is a corollary of this property. □

Proposition 3.7 *The intertwiner* R *has the following properties concerning the conjugation by* P_{13}*, the inverse* R^{-1} *and the transpose* ${}^tR$*:*

$$R = P_{13}RP_{13}, \tag{3.59}$$

$$R^{-1} = R, \tag{3.60}$$

$${}^tR = \mathcal{D}_A R\mathcal{D}_A^{-1}. \tag{3.61}$$

Proof These properties are proved by invoking the uniqueness of the intertwiner satisfying (3.31) and (3.32). To show (3.59), it suffices to recognize that the set of relations (3.38)–(3.46) are invariant under the conjugation by P_{13}.

Next we show (3.60). Comparison of the two choices $f = t_{ij}$ and $f = X_{ij}$ in (3.31) using Lemma 3.5 shows that R and R^{-1} satisfy the same set of intertwining relations. The normalization condition (3.32) is also invariant under the exchange $R \leftrightarrow R^{-1}$, hence (3.60) follows.

Finally, we show (3.61). Take the transpose of (3.31). From Lemma 3.6 we find that $\mathcal{D}_A^{-1}{}^t R \mathcal{D}_A$ again satisfies (3.31). The normalization condition (3.32) is also invariant under the exchange $R \leftrightarrow \mathcal{D}_A^{-1}{}^t R \mathcal{D}_A$, hence (3.61) follows. □

In terms of the matrix elements, the properties (3.59) and (3.61) are rephrased as

$$R^{abc}_{ijk} = R^{cba}_{kji}, \tag{3.62}$$

$$R^{abc}_{ijk} = \frac{(q^2)_i (q^2)_j (q^2)_k}{(q^2)_a (q^2)_b (q^2)_c} R^{ijk}_{abc}. \tag{3.63}$$

Remark 3.8 One may introduce another parameter ν by replacing the latter two formulas in (3.13) by $\mathbf{a}^+|m\rangle = \nu|m+1\rangle$, $\mathbf{a}^-|m\rangle = \nu^{-1}(1-q^{2m})|m-1\rangle$ keeping (3.12) invariant. It corresponds to changing the normalization of $|m\rangle$ depending on m. The resulting 3D R is $(1 \otimes \nu^{\mathbf{h}} \otimes 1)R(1 \otimes \nu^{-\mathbf{h}} \otimes 1)$.

Remark 3.9 If one switches from $\mathbf{k}$ to $\hat{\mathbf{k}} := q^{1/2}\mathbf{k}$ including the *zero point energy* of the q-oscillator (see (3.13)), all the “non-autonomous” q’s in (3.38)–(3.46) disappear. It opens an avenue toward another class of 3D R associated with the so-called *modular double* of q and $\tilde{q}$-oscillators. This topic is not covered in this book. See [97]. The same feature will be observed for the 3D K in Remark 5.5.

Remark 3.10 From (3.16), (3.47) and (3.63), the 3D R acts on the dual Fock space as

$$(\langle i| \otimes \langle j| \otimes \langle k|)R = \sum_{a,b,c} R^{abc}_{ijk} \langle a| \otimes \langle b| \otimes \langle c|. \tag{3.64}$$

3.4 Explicit Formula for 3D *R*

In this section we present explicit formulas of the matrix elements R^{abc}_{ijk} (3.47) of the intertwiner R characterized by (3.31) and (3.32).

We assume that q is generic and use the notation

$$(z;q)_m = \prod_{j=1}^{m}(1 - zq^{j-1}), \quad (q)_m = (q;q)_m,$$

$$\left\{ \begin{matrix} r_1, \dots, r_m \\ s_1, \dots, s_n \end{matrix} \right\}_q = \begin{cases} \frac{\prod_{i=1}^{m}(q)_{r_i}}{\prod_{i=1}^{n}(q)_{s_i}} & \forall r_i, s_i \in \mathbb{Z}_{\geq 0}, \\ 0 & \text{otherwise}, \end{cases} \tag{3.65}$$

$$\binom{m}{n}_q = \binom{m}{m-n}_q = \left\{ \begin{matrix} m \\ n, m-n \end{matrix} \right\}_q .$$

Unless stated otherwise, the abbreviation $(q)_m = (q;q)_m$ will be used also for $(q^k)_m$ with $k \in \mathbb{Z}$. Thus $(q^2)_m$ for instance means $(q^2;q^2)_m$. The two-storied symbol in the second line will be used *without* assuming a "well-poisedness" constraint $\sum_{i=1}^{m} r_i = \sum_{i=1}^{n} s_i$. The non-vanishing condition $\forall r_i, s_i \in \mathbb{Z}_{\geq 0}$ is quite important and will impose non-trivial constraints on the summation variables in what follows. The special case $\left\{ \begin{smallmatrix} j_1+\cdots+j_n \\ j_1,\dots,j_n \end{smallmatrix} \right\}_q$ is a q-multinomial coefficient belonging to $\mathbb{Z}_{\geq 0}[q]$. In particular the $n = 2$ case in the third line is called the q-binomial.

The Kronecker delta will be written either as δ_{ab} or δ^a_b. We will also use the notation

$$(x)_+ = \max(x, 0) = x - \min(x, 0) \qquad (x \in \mathbb{R}). \tag{3.66}$$

Theorem 3.11

$$R^{abc}_{ijk} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k} \sum_{\lambda+\mu=b} (-1)^\lambda q^{i(c-j)+(k+1)\lambda+\mu(\mu-k)} \frac{(q^2)_{c+\mu}}{(q^2)_c} \binom{i}{\mu}_{q^2} \binom{j}{\lambda}_{q^2}, \tag{3.67}$$

where the sum is over $\lambda, \mu \in \mathbb{Z}_{\geq 0}$ such that $\lambda + \mu = b$. (Thus (3.67) is actually a single sum over $(b - i)_+ \leq \lambda \leq \min(b, j)$ or $(b - j)_+ \leq \mu \leq \min(b, i)$.)

Proof The prefactor $\delta^{a+b}_{i+j}\delta^{b+c}_{j+k}$ represents the weight conservation (3.48). The recursion relations (3.55) and (3.56) can be iterated m times to reduce i and j indices as

$$R^{abc}_{ijk} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k} \sum_{r=0}^{m} q^{(m-r)(c-j)+r(a-j-m+r)} \frac{(q^2)_{c+r}}{(q^2)_c} \binom{m}{r}_{q^2} R^{a-m+r, b-r, c+r}_{i-m, j, k},$$

$$R^{abc}_{ijk} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k} \sum_{r=0}^{m} (-1)^r q^{r(a+c-2m+2r+1)} \binom{m}{r}_{q^2} R^{a-m+r, b-r, c-m+r}_{i, j-m, k}.$$

By combining them, general elements are reduced to R^{00k}_{00k}. The relation (3.54) shows that $R^{00k}_{00k} = R^{000}_{000} = 1$. The result of these reductions is given by (3.67). □

Example 3.12 The following is the list of all the non-zero R^{abc}_{314}.

$$\begin{aligned}
R^{041}_{314} &= -q^2(1-q^4)(1-q^6)(1-q^8),\\
R^{132}_{314} &= (1-q^6)(1-q^8)(1-q^4-q^6-q^8-q^{10}),\\
R^{223}_{314} &= q^2(1+q^2)(1+q^4)(1-q^6)(1-q^6-q^{10}),\\
R^{314}_{314} &= q^6(1+q^2+q^4-q^8-q^{10}-q^{12}-q^{14}),\\
R^{405}_{314} &= q^{12}.
\end{aligned}$$

Remark 3.13 From (3.67) we have

$$\begin{aligned}
&(-1)^b R^{abc}_{ijk}|_{q\to q^{-1}}\\
&= \delta^{a+b}_{i+j}\delta^{b+c}_{j+k}\sum_{\lambda+\mu=b} q^{i(j-c)-(k+1)b+2\lambda(\lambda-j)-2\mu a}\frac{(q^2)_{c+\mu}}{(q^2)_c}\binom{i}{\mu}_{q^2}\binom{j}{\lambda}_{q^2}.
\end{aligned} \tag{3.68}$$

From $(q^2)_{c+\mu}/(q^2)_c = (q^{2c+2};q^2)_\mu$, it follows that $(-1)^b R^{abc}_{ijk} \geq 0$ in the regime $q > 1$.

Remark 3.14 Set $R(x,y) = (1\otimes x^{\mathbf{h}}\otimes 1)R(1\otimes y^{-\mathbf{h}}\otimes 1)$, where x, y are non-zero parameters and $\mathbf{h}$ is defined by (3.14). Thanks to the weight conservation (3.49), $R(x,y)$ also satisfies the tetrahedron equation $R_{124}(x,y)R_{135}(x,y)R_{236}(x,y)R_{456}(x,y) = R_{456}(x,y)R_{236}(x,y)R_{135}(x,y)R_{124}(x,y)$. In particular, $R(-1,1)$ has the elements $(-1)^b R^{abc}_{ijk}$. Thus Remark 3.13 shows that $R(-1,1)$ is a 3D R whose elements are all non-negative for $q \geq 1$.

Example 3.15

$$R^{a0c}_{ijk} = q^{ik}\delta^a_{i+j}\delta^c_{j+k}, \qquad R^{abc}_{i0k} = q^{ac}\frac{(q^2)_i(q^2)_k}{(q^2)_a(q^2)_b(q^2)_c}\delta^{a+b}_i\delta^{b+c}_k,$$

$$R^{abc}_{0jk} = (-1)^b q^{b(k+1)}\binom{j}{b}_{q^2}\delta^{a+b}_j\delta^{b+c}_{j+k}, \qquad R^{0bc}_{ijk} = (-1)^j q^{j(c+1)}\frac{(q^2)_k}{(q^2)_c}\delta^b_{i+j}\delta^{b+c}_{j+k},$$

$$R^{11k}_{11k} = 1-(1+q^2)q^{2k}.$$

It is an easy exercise to deduce a formula for the operator $R^{ab}_{ij} \in \mathrm{End}(\mathcal{F}_q)$ in the general scheme (2.4) by comparing it with (2.2) and using Theorem 3.11. The result reads as[5]

$$R^{ab}_{ij} = \delta^{a+b}_{i+j}\sum_{\lambda+\mu=b}(-1)^\lambda q^{\lambda+\mu^2-ib}\binom{i}{\mu}_{q^2}\binom{j}{\lambda}_{q^2}(\mathbf{a}^-)^\mu(\mathbf{a}^+)^{j-\lambda}\mathbf{k}^{i+\lambda-\mu}, \tag{3.69}$$

[5] This R^{ab}_{ij} is not the structure constants in (3.1)–(3.4).

where the sum extends over $\lambda, \mu \in \mathbb{Z}_{\geq 0}$ such that $\lambda + \mu = b$. As a consequence of the weight conservation (3.49), R^{ab}_{ij} is homogeneous in the sense that

$$z^{\mathbf{h}} R^{ab}_{ij} = R^{ab}_{ij} z^{\mathbf{h}+j-b}. \tag{3.70}$$

Example 3.16

$$\begin{pmatrix} R^{00}_{00} & R^{00}_{01} & R^{00}_{10} & R^{00}_{11} \\ R^{01}_{00} & R^{01}_{01} & R^{01}_{10} & R^{01}_{11} \\ R^{10}_{00} & R^{10}_{01} & R^{10}_{10} & R^{10}_{11} \\ R^{11}_{00} & R^{11}_{01} & R^{11}_{10} & R^{11}_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -q\mathbf{k} & \mathbf{a}^- & 0 \\ 0 & \mathbf{a}^+ & \mathbf{k} & 0 \\ 0 & 0 & 0 & \mathbf{a}^-\mathbf{a}^+ - \mathbf{k}^2 \end{pmatrix}. \tag{3.71}$$

Except for the bottom right element, this coincides with the corresponding matrix from of the 3D L in $(11.14)|_{\alpha=1}$. Its consequence will be mentioned in Example 13.1.

$$\begin{pmatrix} R^{02}_{02} & R^{02}_{11} & R^{02}_{20} \\ R^{11}_{02} & R^{11}_{11} & R^{11}_{20} \\ R^{20}_{02} & R^{20}_{11} & R^{20}_{20} \end{pmatrix} = \begin{pmatrix} q^2\mathbf{k}^2 & -\mathbf{a}^-\mathbf{k} & (\mathbf{a}^-)^2 \\ -q(1+q^2)\mathbf{a}^+\mathbf{k} & \mathbf{a}^-\mathbf{a}^+ - \mathbf{k}^2 & q^{-1}(1+q^2)\mathbf{a}^-\mathbf{k} \\ (\mathbf{a}^+)^2 & \mathbf{a}^+\mathbf{k} & \mathbf{k}^2 \end{pmatrix}. \tag{3.72}$$

Reversing the order of the columns of this matrix coincides with the central three-by-three block in (8.8) up to coefficients.

Example 3.17 The following formulas will be used in Example 13.1:

$$R^{m,0}_{m,0} = \mathbf{k}^m, \qquad R^{m,1}_{m+1,0} = q^{-m}\binom{m+1}{1}_{q^2} \mathbf{a}^-\mathbf{k}^m,$$
$$R^{m+1,0}_{m,1} = \mathbf{a}^+\mathbf{k}^m, \quad R^{m,1}_{m,1} = q^{1-m}\left(\binom{m}{1}_{q^2} \mathbf{a}^-\mathbf{a}^+ - \mathbf{k}^2\right)\mathbf{k}^{m-1}.$$

Let us present another formula in terms of the q-hypergeometric function [50]:

$${}_2\phi_1\left(\begin{matrix}\alpha, \beta \\ \gamma\end{matrix}; q, w\right) = \sum_{n\geq 0} \frac{(\alpha; q)_n (\beta; q)_n}{(\gamma; q)_n (q; q)_n} w^n. \tag{3.73}$$

Theorem 3.18

$$R^{abc}_{ijk} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k} \frac{q^{(a-j)(c-j)}}{(q^2)_b} P_b(q^{2i}, q^{2j}, q^{2k}), \tag{3.74}$$

$$P_b(x, y, z) = (q^{2-2b}z; q^2)_b \, {}_2\phi_1\left(\begin{matrix}q^{-2b}, q^{2-2b}yz \\ q^{2-2b}z\end{matrix}; q^2, q^2x\right), \tag{3.75}$$

$$P_b(x,y,z) = q^{-b(b-1)}(q^2)_b \oint \frac{du}{2\pi \mathrm{i} u^{b+1}} \frac{(-q^{2-2b}xyzu;q^2)_\infty(-u;q^2)_\infty}{(-xu;q^2)_\infty(-zu;q^2)_\infty}, \quad (3.76)$$

where the integral encircles $u = 0$ anti-clockwise picking up the residue.

Proof From (3.39) and $R = R^{-1}$ we have $R(1 \otimes \mathbf{k} \otimes \mathbf{a}^-) = (\mathbf{k} \otimes 1 \otimes \mathbf{a}^- + \mathbf{a}^+ \otimes \mathbf{a}^- \otimes \mathbf{k})R$. In terms of matrix elements it reads as

$$q^j(1-q^{2k})R^{a,b,c}_{i,j,k-1} = q^a(1-q^{2c+2})R^{a,b,c+1}_{i,j,k} + q^c(1-q^{2b+2})R^{a-1,b+1,c}_{i,j,k}. \quad (3.77)$$

Substituting (3.74) into (3.77) and (3.50), we get the recursion relations

$$(1-z)P_b(x,y,q^{-2}z) = q^{-2b}x(1-q^{-2b}yz)P_b(x,y,z) + P_{b+1}(x,y,z), \quad (3.78)$$

$$q^{-2b}xz(1-y)P_b(x,q^{-2}y,z) + P_{b+1}(x,y,z) = (1-x)(1-z)P_b(q^{-2}x,y,q^{-2}z). \quad (3.79)$$

The initial condition should be set as $P_0(x,y,z) = 1$ since $R^{a0c}_{ijk} = \delta^a_{i+j}\delta^c_{j+k}q^{ik}$ from (3.67). Obviously, both formulas (3.75) and (3.76) satisfy the initial condition. The remaining task is to show that they satisfy either one of the above recursion relations. It is straightforward to check that (3.75) satisfies (3.78) by comparing coefficients of the powers of x. To show (3.76), substitute it into (3.79) and replace u by q^2u in the RHS. Then the relation to be shown becomes

$$\oint \frac{du(-q^{-2b}xyzu;q^2)_\infty(-q^2u;q^2)_\infty}{u^{b+2}(-xu;q^2)_\infty(-zu;q^2)_\infty} X = 0,$$
$$X = xz(1-y)u(1+u) - (1-x)(1-z)u + (1-q^{2b+2})(1+u).$$

By setting $f(u) = (-q^{-2-2b}xyzu;q^2)_\infty(-u;q^2)_\infty/((-xu;q^2)_\infty(-zu;q^2)_\infty)$, this is identified with the identity $\oint \frac{du}{u^{b+2}}(f(q^2u) - q^{2b+2}f(u)) = 0$. □

Note that (3.75) is a terminating series due to the entry q^{-2b}. In fact, $P_b(x,y,z)$ is a polynomial belonging to $q^{-2b(b-1)}\mathbb{Z}[q^2,x,y,z]$ with the symmetry $P_b(x,y,z) = P_b(z,y,x)$ reflecting (3.59).

Example 3.19

$$P_0(x,y,z) = 1, \quad P_1(x,y,z) = 1 - x - z + xyz,$$
$$q^4P_2(x,y,z) = x^2y^2z^2 - (1+q^2)xyz(-1+x+z)$$
$$+ q^2(q^2 - x - q^2x + x^2 - z - q^2z + xz + q^2xz + z^2),$$
$$R^{405}_{314} = q^{12}P_0(q^6,q^2,q^8), \quad R^{314}_{314} = \frac{q^6P_1(q^6,q^2,q^8)}{1-q^2}, \quad R^{223}_{314} = \frac{q^2P_2(q^6,q^2,q^8)}{(1-q^2)(1-q^4)}.$$

This agrees with Example 3.12.

The formula (3.76) is also presented in terms of the generating series:

$$\sum_{b\geq 0}\frac{q^{b(b-1)}u^b}{(q^2)_b}P_b(x,q^{2b-2}y,z)=\frac{(-xyzu;q^2)_\infty(-u;q^2)_\infty}{(-xu;q^2)_\infty(-zu;q^2)_\infty}. \tag{3.80}$$

Due to (3.76), matrix elements of the 3D R are expressed as

$$R^{abc}_{ijk}=\delta^{a+b}_{i+j}\delta^{b+c}_{j+k}\,q^{ik+b}\oint\frac{du}{2\pi \mathrm{i}u^{b+1}}\frac{(-q^{2+a+c}u;q^2)_\infty(-q^{-i-k}u;q^2)_\infty}{(-q^{a-c}u;q^2)_\infty(-q^{c-a}u;q^2)_\infty}. \tag{3.81}$$

Note that the ratio of the four infinite products equals $(-q^{-i-k}u;q^2)_i/(-q^{c-a}u;q^2)_{a+1}$ because of $a-c=i-k$. By means of the identity

$$\frac{(zx;p)_\infty}{(z;p)_\infty}=\sum_{k\geq 0}\frac{(x;p)_k}{(p;p)_k}z^k, \tag{3.82}$$

it is expanded as

$$\left(\sum_{\lambda\geq 0}\binom{\lambda+a}{\lambda}_{q^2}(-u)^\lambda q^{\lambda(c-a)}\right)\left(\sum_{0\leq\mu\leq i}\binom{i}{\mu}_{q^2}q^{\mu(\mu-i-k-1)}u^\mu\right). \tag{3.83}$$

Collecting the coefficients of u^b, one gets

$$R^{abc}_{ijk}=\delta^{a+b}_{i+j}\delta^{b+c}_{j+k}\sum_{\lambda+\mu=b}(-1)^\lambda q^{ik+b+\lambda(c-a)+\mu(\mu-i-k-1)}\binom{\lambda+a}{a}_{q^2}\binom{i}{\mu}_{q^2} \tag{3.84}$$

summed over $\lambda,\mu\in\mathbb{Z}_{\geq 0}$ under the constraint $\lambda+\mu=b$. Thus it is actually the single sum over $(b-i)_+\leq\lambda\leq b$ or $0\leq\mu\leq\min(b,i)$.

Both formulas (3.67) and (3.84) show that R^{abc}_{ijk} is a Laurent polynomial of q with integer coefficients. On the other hand, Example 3.12 suggests that it is actually a *polynomial* in q. In Lemma 3.29, a stronger claim identifying the constant term of the polynomial will be presented which will lead to further aspects.

One can express (3.84) in terms of the terminating q-hypergeometric as

$$R^{abc}_{ijk}=\delta^{a+b}_{i+j}\delta^{b+c}_{j+k}\,(-1)^b q^{ik+b(k-i+1)}\binom{a+b}{a}_{q^2}{}_2\phi_1\left(\begin{matrix}q^{-2b},q^{-2i}\\ q^{-2a-2b}\end{matrix};q^2,q^{-2c}\right), \tag{3.85}$$

which is a different formula from (3.74)–(3.75). It manifests the symmetry

$$R^{abc}_{ijk}=(-q)^{b-i}\frac{(q^2)_i(q^2)_j}{(q^2)_a(q^2)_b}R^{jic}_{bak}=(-q)^{b-i}\frac{(q^2)_k}{(q^2)_c}R^{bak}_{jic}, \tag{3.86}$$

where the second equality is due to (3.63). In Chap. 13 we will use

$$
\begin{aligned}
R^{abc}_{ijk} &= \delta^{a+b}_{i+j}\delta^{b+c}_{j+k}(-1)^j q^{(1-a)j+(a+j)c}\frac{(q^2)_k}{(q^2)_c} \\
&\times \sum_{0\le\mu\le\min(a,j)} (-1)^\mu q^{\mu(\mu-2c-1)}\binom{a+b-\mu}{b}_{q^2}\binom{j}{\mu}_{q^2},
\end{aligned}
\tag{3.87}
$$

which is derived from (3.84) by applying the latter transformation in (3.86).

3.5 Solution to the Tetrahedron Equations

Recall that we have characterized R as the intertwiner of $A_q(A_2)$ modules in (3.31) and (3.32). Various explicit formulas for it are presented in the previous section. Now we proceed to the proof of the tetrahedron equations.

3.5.1 $RRRR = RRRR$ Type

Theorem 3.20 *The intertwiner R satisfies the tetrahedron equation of $RRRR = RRRR$ type in (2.6).*

Proof Consider $A_q(A_3)$ and let π_1, π_2, π_3 be the fundamental representations given in (3.21). The Weyl group $W(A_3)$ is generated by simple reflections s_1, s_2, s_3 with the relations

$$
s_i^2 = 1, \quad s_1s_3 = s_3s_1, \quad s_1s_2s_1 = s_2s_1s_2, \quad s_2s_3s_2 = s_3s_2s_3. \tag{3.88}
$$

According to Theorem 3.3, the equivalence of the tensor product representations $\pi_{13} \simeq \pi_{31}$, $\pi_{121} \simeq \pi_{212}$ and $\pi_{232} \simeq \pi_{323}$ are valid. ($\pi_{i_1,\ldots,i_k}$ is a shorthand for $\pi_{i_1} \otimes \cdots \otimes \pi_{i_k}$ as mentioned after Theorem 3.3.) By Remark 3.4, the intertwiner for $\pi_{13} \simeq \pi_{31}$ is just the transposition of the components. Let $\Phi^{(1)}$ and $\Phi^{(2)}$ be the intertwiners for the latter two, i.e.

$$
\begin{aligned}
\Phi^{(1)} \circ \pi_{121}(\Delta(f)) &= \pi_{212}(\Delta(f)) \circ \Phi^{(1)}, \\
\Phi^{(2)} \circ \pi_{232}(\Delta(f)) &= \pi_{323}(\Delta(f)) \circ \Phi^{(2)}
\end{aligned}
\tag{3.89}
$$

for any $f \in A_q(A_3)$. By inspection of (3.21), they are both given by the same Φ as the $A_q(A_2)$ case characterized in (3.27)–(3.29). Therefore from (3.30) we get

$$
\Phi^{(1)} = RP_{13}, \quad \Phi^{(2)} = RP_{13}, \tag{3.90}
$$

which means that they are the copies of the same operator acting on the respective spaces.

Let $w_0 \in W(A_3)$ be the longest element. We pick two reduced expressions, say,

$$w_0 = s_1 s_2 s_1 s_3 s_2 s_1 = s_3 s_2 s_1 s_3 s_2 s_3, \tag{3.91}$$

where the two sides are interchanged by replacing s_i by s_{4-i} and reversing the order. According to Theorem 3.3, we have the equivalence of the two irreducible representations of $A_q(A_3)$:

$$\pi_{121321} \simeq \pi_{321323}. \tag{3.92}$$

Let P_{ij} and $\Phi^{(1)}_{ijk}$, $\Phi^{(2)}_{ijk}$ be the transposition P (3.23) and the intertwiners $\Phi^{(1)}$, $\Phi^{(2)}$ that act on the tensor components specified by the indices. These components must be adjacent (i.e. $j - i = k - j = 1$) to make the relations (3.25) and (3.89) work. With this guideline, one can construct the intertwiners for (3.92) by following the transformation of the reduced expressions by the Coxeter relations (3.88). There are two ways to achieve this. In terms of the indices, they look as follows:

$$\begin{array}{llll}
\underline{121}321 & \Phi^{(1)}_{123} & 12\underline{13}21 & P_{34} \\
21\underline{232}1 & \Phi^{(2)}_{345} & 123\underline{121} & \Phi^{(1)}_{456} \\
2\underline{13}2\underline{31} & P_{23}P_{56} & 1\underline{232}12 & \Phi^{(2)}_{234} \\
23\underline{121}3 & \Phi^{(1)}_{345} & \underline{13}2\underline{31}2 & P_{12}P_{45} \\
\underline{232}123 & \Phi^{(2)}_{123} & 3\underline{121}32 & \Phi^{(1)}_{234} \\
32\underline{31}23 & P_{34} & 321\underline{232} & \Phi^{(2)}_{456} \\
321323 & & 321323 &
\end{array} \tag{3.93}$$

The underlines indicate the components to which the intertwiners given on the right are to be applied. (Note that they are completely parallel with those in (2.22)–(2.23).) Thus the following intertwining relations are valid for any $f \in A_q(A_3)$:

$$\begin{aligned}
&P_{34}\Phi^{(2)}_{123}\Phi^{(1)}_{345}P_{23}P_{56}\Phi^{(2)}_{345}\Phi^{(1)}_{123}\pi_{121321}(\Delta(f)) \\
&\quad = \pi_{321323}(\Delta(f))P_{34}\Phi^{(2)}_{123}\Phi^{(1)}_{345}P_{23}P_{56}\Phi^{(2)}_{345}\Phi^{(1)}_{123},
\end{aligned} \tag{3.94}$$

$$\begin{aligned}
&\Phi^{(2)}_{456}\Phi^{(1)}_{234}P_{12}P_{45}\Phi^{(2)}_{234}\Phi^{(1)}_{456}P_{34}\pi_{121321}(\Delta(f)) \\
&\quad = \pi_{321323}(\Delta(f))\Phi^{(2)}_{456}\Phi^{(1)}_{234}P_{12}P_{45}\Phi^{(2)}_{234}\Phi^{(1)}_{456}P_{34}.
\end{aligned} \tag{3.95}$$

Since the representation (3.92) is irreducible, the intertwiner is unique up to an overall constant factor. The factor is one because both constructions send $|0\rangle^{\otimes 6}$ to itself by the normalization (3.29). Therefore we have

$$P_{34}\Phi^{(2)}_{123}\Phi^{(1)}_{345}P_{23}P_{56}\Phi^{(2)}_{345}\Phi^{(1)}_{123} = \Phi^{(2)}_{456}\Phi^{(1)}_{234}P_{12}P_{45}\Phi^{(2)}_{234}\Phi^{(1)}_{456}P_{34}. \tag{3.96}$$

In the current setting, (3.90) implies that both $\Phi^{(1)}_{ijk}$ and $\Phi^{(2)}_{ijk}$ are equal to $R_{ijk}P_{ik}$, leading to

$$\begin{aligned}&P_{34}R_{123}P_{13}R_{345}P_{35}P_{23}P_{56}R_{345}P_{35}R_{123}P_{13}\\&=R_{456}P_{46}R_{234}P_{24}P_{12}P_{45}R_{234}P_{24}R_{456}P_{46}P_{34}.\end{aligned}$$

Sending all the P_{ij}'s to the right by using $P_{34}R_{123}=R_{124}P_{34}$, etc., we find

$$R_{124}R_{135}R_{236}R_{456}\sigma=R_{456}R_{236}R_{135}R_{124}\sigma',$$

where $\sigma=P_{34}P_{13}P_{35}P_{23}P_{56}P_{35}P_{13}$ and $\sigma'=P_{46}P_{24}P_{12}P_{45}P_{24}P_{46}P_{34}$. One can check that $\sigma=\sigma'$, which gives the reverse ordering of the components $|m_1\rangle\otimes\cdots\otimes|m_6\rangle\mapsto|m_6\rangle\otimes\cdots\otimes|m_1\rangle$. Thus they can be canceled, completing the proof of Theorem 3.20. □

In terms of the 3D R, the intertwining relations (3.94) and (3.95) take the form:

$$R_{124}R_{135}R_{236}R_{456}\pi_{121321}(\tilde{\Delta}(f))=\pi_{321323}(\Delta(f))R_{124}R_{135}R_{236}R_{456},\tag{3.97}$$

$$R_{456}R_{236}R_{135}R_{124}\pi_{121321}(\tilde{\Delta}(f))=\pi_{321323}(\Delta(f))R_{456}R_{236}R_{135}R_{124},\tag{3.98}$$

where $\tilde{\Delta}(f)=\sigma\circ\Delta(f)\circ\sigma$. For a generator $f=t_{ij}$ it reads as

$$\tilde{\Delta}(t_{ij})=\sum_{1\le k_1,\ldots,k_5\le 4}t_{k_5j}\otimes t_{k_4k_5}\otimes t_{k_3k_4}\otimes t_{k_2k_3}\otimes t_{k_1k_2}\otimes t_{ik_1}.\tag{3.99}$$

We have started from the two particular reduced expressions of the longest element in (3.91). One can play the same game for any pair of the "most distant" reduced expressions which are related by $s_i\to s_{4-i}$ and the reverse ordering. The result can always be brought to the form (2.6) by using (3.59) and (3.60).

In general for $A_q(A_{n-1})$ with $n\ge 5$, similar compatibility conditions on the intertwiners can be derived from reduced expressions of the longest element of $W(A_{n-1})$ along the transformation $s_{i_1}\cdots s_{i_l}\to s_{n-i_l}\cdots s_{n-i_1}$ by the Coxeter relations (3.22), where $l=n(n-1)/2$. Since any reduced expression is transformed to any of the others by the Coxeter relations [119], the compatibility conditions for any $s_{j_1}\cdots s_{j_l}\to s_{n-j_l}\cdots s_{n-j_1}$ are equivalent to each other by a conjugation.

As an illustration, consider the $n=5$ case. The longest element of $W(A_4)$ has length 10 and the compatibility for $\pi_{1234123121}\simeq\pi_{4342341234}$ leads to

$$R_{123}R_{145}R_{246}R_{356}R_{178}R_{279}R_{389}R_{470}R_{580}R_{690}=\text{product in reverse order}.\tag{3.100}$$

This can be derived by using the original tetrahedron equation (2.6) five times in addition to the trivial commutativity as

$$
\begin{aligned}
&\underline{R_{123}R_{145}R_{246}R_{356}}R_{178}R_{279}R_{389}R_{470}R_{580}R_{690}\\
&= R_{356}R_{246}R_{145}\underline{R_{123}R_{178}R_{279}R_{389}}R_{470}R_{580}R_{690}\\
&= R_{356}R_{246}R_{145}R_{389}R_{279}R_{178}R_{123}R_{470}R_{580}R_{690}\\
&= R_{356}R_{246}R_{389}R_{279}\underline{R_{145}R_{178}R_{470}R_{580}}R_{690}R_{123}\\
&= R_{356}R_{246}R_{389}R_{279}R_{580}R_{470}R_{178}R_{145}R_{690}R_{123}\\
&= R_{356}R_{389}R_{580}\underline{R_{246}R_{279}R_{470}R_{690}}R_{178}R_{145}R_{123}\\
&= \underline{R_{356}R_{389}R_{580}R_{690}}R_{470}R_{279}R_{246}R_{178}R_{145}R_{123}\\
&= R_{690}R_{580}R_{389}R_{356}R_{470}R_{279}R_{246}R_{178}R_{145}R_{123}\\
&= R_{690}R_{580}R_{470}R_{389}R_{279}R_{178}R_{356}R_{246}R_{145}R_{123},
\end{aligned}
\tag{3.101}
$$

where the underlines indicate the places to which the tetrahedron equation is applied. The first and the last expressions in (3.101) fit the geometric interpretation as the transformations between the 5-line diagrams in Fig. 3.1 in the same manner as in Fig. 2.2.

For general n, the compatibility condition arising from $\pi_{i_1,\dots,i_l} \simeq \pi_{n-i_l,\dots,n-i_1}$ allows a similar geometric interpretation in terms of generic positioned n-line diagrams with $n(n-1)/2$ vertices. They are all reducible to the the tetrahedron equation. This last claim follows from [126, Theorem 2.17], which states that any non-trivial loop in a reduced expression (rex) graph (see Sect. 9.2) is generated from the loops in the one for the longest element in the parabolic subgroups of rank 3, hence A_3 in the present case.

3.5.2 *RLLL* = *LLLR* Type

Let us introduce the operator L along the scheme (2.12). In (2.11), we choose $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$ and $\mathscr{F} = \mathscr{F}_q = \bigoplus_{m\geq 0} \mathbb{C}(q)|m\rangle$ which is the Fock space introduced in (3.13) as an irreducible module over the q-oscillator algebra (3.12). Then $L = (L^{ab}_{ij})$ is specified for $a, b, i, j = 0, 1$ as

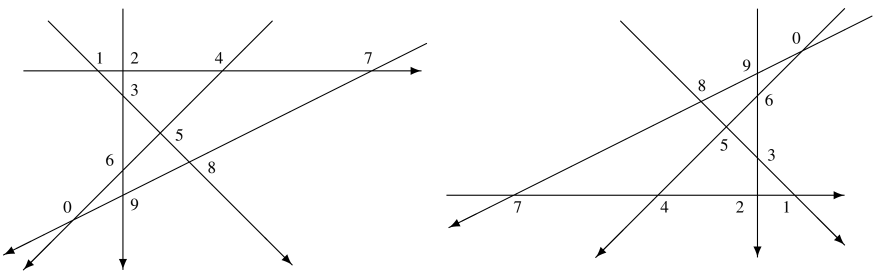


Fig. 3.1 The 5-line diagrams connected by (3.101) in the same manner as Fig. 2.2

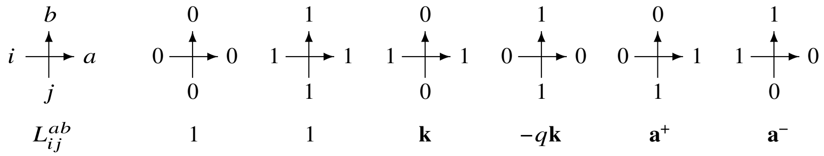


Fig. 3.2 3D L as an Osc_q-valued six-vertex model. The last two relations in (3.12) corresponds to a quantization of the so-called free Fermion condition [10, Fig. 10.1, Eq. (10.16.4)$|_{\omega_7=\omega_8=0}$]

$$L^{ab}_{ij} = 0 \quad \text{if} \;\; a+b \neq i+j, \tag{3.102}$$

$$L^{00}_{00} = L^{11}_{11} = 1, \quad L^{10}_{10} = \mathbf{k}, \quad L^{01}_{01} = -q\mathbf{k}, \quad L^{10}_{01} = \mathbf{a}^+, \quad L^{01}_{10} = \mathbf{a}^-. \tag{3.103}$$

The property

$$\mathbf{h}L^{ab}_{ij} = L^{ab}_{ij}(\mathbf{h}+a-i) \tag{3.104}$$

is valid, where $\mathbf{h}$ is the number operator (3.14). From (3.13) and (2.13), non-trivial matrix elements L^{abc}_{ijk} read as

$$\begin{aligned} &L^{0,0,c}_{0,0,k} = L^{1,1,c}_{1,1,k} = \delta^c_k, \quad L^{1,0,c}_{1,0,k} = \delta^c_k q^k, \quad L^{0,1,c}_{0,1,k} = -\delta^c_k q^{k+1}, \\ &L^{1,0,c}_{0,1,k} = \delta^c_{k+1}, \quad L^{0,1,c}_{1,0,k} = \delta^c_{k-1}(1-q^{2k}). \end{aligned} \tag{3.105}$$

The operator L may be regarded as an Osc_q-valued six-vertex model [10, Sect. 8] as in Fig. 3.2.

Theorem 3.21 *The intertwiner R and the above L satisfy the tetrahedron equation of $RLLL = LLLR$ type in (2.15).*

Proof The equations (2.18) coincide with the intertwining relations (3.38)–(3.46) for R and $R^{-1} = R$. (See (3.60).) This is shown more concretely in Lemma 3.22 below. □

Let us write the quantized Yang–Baxter equation (2.18) as

$$R\mathcal{L}^{abc}_{ijk} = \tilde{\mathcal{L}}^{abc}_{ijk}R, \tag{3.106}$$

$$\mathcal{L}^{abc}_{ijk} = \sum_{\alpha,\beta,\gamma}(L^{\alpha\beta}_{ij} \otimes L^{a\gamma}_{\alpha k} \otimes L^{bc}_{\beta\gamma}), \tag{3.107}$$

$$\tilde{\mathcal{L}}^{abc}_{ijk} = \sum_{\alpha,\beta,\gamma}(L^{ab}_{\alpha\beta} \otimes L^{\alpha c}_{i\gamma} \otimes L^{\beta\gamma}_{jk}). \tag{3.108}$$

The objects $\mathcal{L}^{abc}_{ijk}$ and $\tilde{\mathcal{L}}^{abc}_{ijk}$ are $\mathrm{End}(\mathcal{F}_q^{\otimes 3})$-valued quantized three-body scattering amplitudes. They are non-vanishing only when $a+b+c = i+j+k$ due to (3.102) and non-trivial only when $a+b+c = i+j+k = 1, 2$ due to (3.103). For example,

$$\mathscr{L}^{001}_{100} = L^{01}_{10} \otimes L^{00}_{00} \otimes L^{01}_{10} + L^{10}_{10} \otimes L^{01}_{10} \otimes L^{01}_{01} = \mathbf{a}^- \otimes 1 \otimes \mathbf{a}^- - q\mathbf{k} \otimes \mathbf{a}^- \otimes \mathbf{k},$$
$$\tilde{\mathscr{L}}^{001}_{100} = L^{00}_{00} \otimes L^{01}_{10} \otimes L^{00}_{00} = 1 \otimes \mathbf{a}^- \otimes 1.$$

Observe that these operators are exactly those appearing in the intertwining relation (3.38). This happens generally. One can directly check:

Lemma 3.22 *The quantized three-body scattering amplitudes* $\mathscr{L}^{abc}_{ijk}$ *and* $\tilde{\mathscr{L}}^{abc}_{ijk}$ *with* $a+b+c = i+j+k = 1, 2$ *coincide with the representations (3.36)–(3.37) of* $A_q(A_2)$ *as follows:*

$$\pi_{121}(\tilde{\Delta}(t_{ij})) = \tilde{\mathscr{L}}^{\bar{\mathbf{e}}_j}_{\bar{\mathbf{e}}_{4-i}} = (-q)^{i-j} \mathscr{L}^{\mathbf{e}_{4-i}}_{\mathbf{e}_j}, \tag{3.109}$$
$$\pi_{212}(\Delta(t_{ij})) = \mathscr{L}^{\bar{\mathbf{e}}_j}_{\bar{\mathbf{e}}_{4-i}} = (-q)^{i-j} \tilde{\mathscr{L}}^{\mathbf{e}_{4-i}}_{\mathbf{e}_j}. \tag{3.110}$$

Here $\mathbf{e}_i, \bar{\mathbf{e}}_i$ *are arrays of* 0, 1 *with length three specified by*

$$\mathbf{e}_i = \overbrace{0, \ldots, 0}^{i-1}, 1, 0, \ldots, 0, \quad \bar{\mathbf{e}}_i = \overbrace{1, \ldots, 1}^{i-1}, 0, 1, \ldots, 1. \tag{3.111}$$

From (3.109) and (3.110), the intertwining relation (3.31) and the tetrahedron equation (2.15) are identified.

Remark 3.23 As an equation for R, the tetrahedron equation $RLLL = LLLR$ (3.106) is invariant under the change $L^{ab}_{ij} \to \alpha^{a-j} L^{ab}_{ij}$ by a parameter α by virtue of (3.102).

Remark 3.24 Let $L_\alpha = (\alpha^{a-j} L^{ab}_{ij})$ be the 3D L in Remark 3.23 including a parameter α. It is invertible with the inverse

$$(L_\alpha)^{-1} = L_{\alpha^{-1}}, \tag{3.112}$$

This is easily verified by means of (3.12).

As an application of Theorem 3.21, let us present another proof of Theorem 3.20, i.e. $RRRR = RRRR$. We invoke the argument in Sect. 2.5 which establishes $RRRR = RRRR$ by using $RLLL = LLLR$ up to the irreducibility. For the 3D L under consideration, we can make the irreducibility argument precise. Recall the initial and final elements $\overset{6}{L}_{ab}\overset{5}{L}_{ac}\overset{4}{L}_{bc}\overset{3}{L}_{ad}\overset{2}{L}_{bd}\overset{1}{L}_{cd}$ and $\overset{1}{L}_{cd}\overset{2}{L}_{bd}\overset{4}{L}_{bc}\overset{3}{L}_{ad}\overset{5}{L}_{ac}\overset{6}{L}_{ab}$ in (2.22) and (2.23), which are linear operators on

$$\overset{a}{V} \otimes \overset{b}{V} \otimes \overset{c}{V} \otimes \overset{d}{V} \otimes \overset{1}{\mathscr{F}}_q \otimes \overset{2}{\mathscr{F}}_q \otimes \overset{3}{\mathscr{F}}_q \otimes \overset{4}{\mathscr{F}}_q \otimes \overset{5}{\mathscr{F}}_q \otimes \overset{6}{\mathscr{F}}_q.$$

Let us call their matrix elements for the transition $v_{i_1} \otimes v_{j_1} \otimes v_{k_1} \otimes v_{l_1} \mapsto v_{i_4} \otimes v_{j_4} \otimes v_{k_4} \otimes v_{l_4}$ as $\mathscr{L}^{i_4 j_4 k_4 l_4}_{i_1 j_1 k_1 l_1}$ and $\tilde{\mathscr{L}}^{i_4 j_4 k_4 l_4}_{i_1 j_1 k_1 l_1}$, respectively. Then (2.22) and (2.23) are the totality of the relations

$$R_{124} R_{135} R_{236} R_{456} \mathscr{L}^{i_4 j_4 k_4 l_4}_{i_1 j_1 k_1 l_1} = \tilde{\mathscr{L}}^{i_4 j_4 k_4 l_4}_{i_1 j_1 k_1 l_1} R_{124} R_{135} R_{236} R_{456}, \tag{3.113}$$

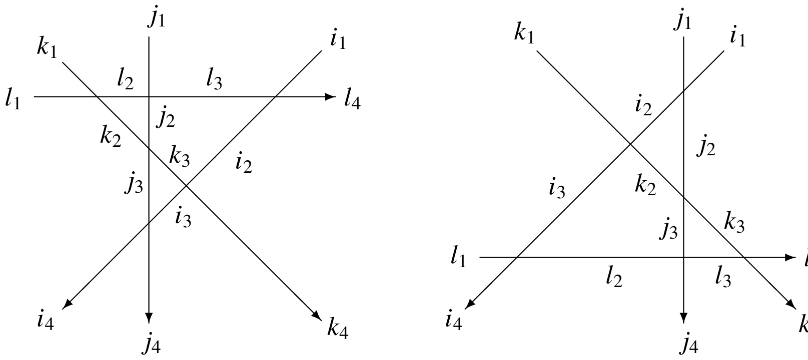
$$R_{456} R_{236} R_{135} R_{124} \mathscr{L}^{i_4 j_4 k_4 l_4}_{i_1 j_1 k_1 l_1} = \tilde{\mathscr{L}}^{i_4 j_4 k_4 l_4}_{i_1 j_1 k_1 l_1} R_{456} R_{236} R_{135} R_{124} \tag{3.114}$$

for $i_1, \dots, l_4 = 0, 1$. Here we have substituted $S = R$ for our 3D R according to the comment after (2.20). The matrix elements $\mathscr{L}^{i_4 j_4 k_4 l_4}_{i_1 j_1 k_1 l_1}$ and $\tilde{\mathscr{L}}^{i_4 j_4 k_4 l_4}_{i_1 j_1 k_1 l_1}$ are $\mathrm{End}(\mathscr{F}_q^{\otimes 6})$ valued and, from the diagrams (2.21) and (2.14), they are given by

$$\mathscr{L}^{i_4 j_4 k_4 l_4}_{i_1 j_1 k_1 l_1} = \sum L^{k_2 l_2}_{k_1 l_1} \otimes L^{j_2 l_3}_{j_1 l_2} \otimes L^{i_2 l_4}_{i_1 l_3} \otimes L^{j_3 k_3}_{j_2 k_2} \otimes L^{i_3 k_4}_{i_2 k_3} \otimes L^{i_4 j_4}_{i_3 j_3}, \tag{3.115}$$

$$\tilde{\mathscr{L}}^{i_4 j_4 k_4 l_4}_{i_1 j_1 k_1 l_1} = \sum L^{k_4 l_4}_{k_3 l_3} \otimes L^{j_4 l_3}_{j_3 l_2} \otimes L^{i_4 l_2}_{i_3 l_1} \otimes L^{j_3 k_3}_{j_2 k_2} \otimes L^{i_3 k_2}_{i_2 k_1} \otimes L^{i_2 j_2}_{i_1 j_1}, \tag{3.116}$$

where the sums are taken over $i_r, j_r, k_r, l_r = 0, 1$ for $r = 1, 2$. These are depicted as follows:



By substituting (3.102), (3.103) and using (3.99), (3.21), one can directly check

$$\pi_{121321}(\tilde{\Delta}(t_{ij})) = (-q)^{i-j} \mathscr{L}^{\bar{\mathbf{e}}_i}_{\bar{\mathbf{e}}_{5-j}}, \quad \pi_{321323}(\Delta(t_{ij})) = (-q)^{i-j} \tilde{\mathscr{L}}^{\bar{\mathbf{e}}_i}_{\bar{\mathbf{e}}_{5-j}} \quad (1 \le i, j \le 4), \tag{3.117}$$

where $\bar{\mathbf{e}}_i$ is length four array given by (3.111). Since the representations π_{121321} and π_{321323} are irreducible by Theorem 3.3, and the relations (3.97)–(3.98) with generators $f = t_{ij}$ are reproduced, the equality $R_{124} R_{135} R_{236} R_{456} = R_{456} R_{236} R_{135} R_{124}$ follows.

3.5.3 $MMLL = LLMM$ Type

Let us present a solution to the tetrahedron equation of type $MMLL = LLMM$ in Sect. 2.6. We take $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1, \mathcal{F} = \mathcal{F}_q$ in the setting therein and consider a slight generalization of (2.24)–(2.25) including a spectral parameter:

$$L(z) = \sum_{a,b,i,j} E_{ai} \otimes E_{bj} \otimes L(z)^{ab}_{ij}, \tag{3.118}$$

$$M(z) = \sum_{a,b,i,j} E_{ai} \otimes E_{bj} \otimes M(z)^{ab}_{ij}, \tag{3.119}$$

where the sums extend over $\{0, 1\}^4$ and both belong to $\mathrm{End}(V \otimes V \otimes \mathcal{F}_q)$. The operators $L(z)^{ab}_{ij}, M(z)^{ab}_{ij} \in \mathrm{End}(\mathcal{F}_q)$, which are nonzero only when $a + b = i + j$, are specified by

b	0	1	0	1	0	1
$i \to a$	$0 \to 0$	$1 \to 1$	$1 \to 1$	$0 \to 0$	$0 \to 1$	$1 \to 0$
j	0	1	0	1	1	0
$L(z)^{ab}_{ij}$	1	1	$\mu\mathbf{k}$	$-q\mu^{-1}\mathbf{k}$	$z\,\mathbf{a}^+$	$z^{-1}\mathbf{a}^-$
$M(z)^{ab}_{ij}$	1	1	$\nu\tilde{\mathbf{k}}$	$q\nu^{-1}\tilde{\mathbf{k}}$	$z\,\mathbf{a}^+$	$z^{-1}\mathbf{a}^-$

(3.120)

Here $\mathbf{a}^{\pm}$, $\mathbf{k}$ are q-oscillators in (3.13), and $\tilde{\mathbf{k}}$ is $\mathbf{k}$ with q replaced by $-q$, i.e.

$$\tilde{\mathbf{k}}|m\rangle = (-q)^m|m\rangle. \tag{3.121}$$

See (3.13). In (3.120), μ, ν are fixed parameters and suppressed in the notation. On the other hand, z will play a similar role to the spectral parameter below. We note a simple relation $M(z) = L(z)|_{q \to -q, \mu \to \nu}$.

Theorem 3.25 *For any μ, ν, the operators $L(z)$ and $M(z)$ defined in (3.118)–(3.121) satisfy the tetrahedron equation of type $MMLL = LLMM$ in $\mathrm{End}(V^{\otimes 4} \otimes \mathcal{F}_q^{\otimes 2})$ as*

$$\begin{aligned} &M_{126}(z_{12})M_{346}(z_{34})L_{135}(z_{13})L_{245}(z_{24}) \\ &= L_{245}(z_{24})L_{135}(z_{13})M_{346}(z_{34})M_{126}(z_{12}), \end{aligned} \tag{3.122}$$

where $z_{ij} = z_i/z_j$.

See Fig. 2.5 for a graphical representation.

Proof A direct calculation. As an illustration, let us compare $X \in \mathrm{End}(\overset{5}{\mathcal{F}_q} \otimes \overset{6}{\mathcal{F}_q})$ occurring in (LHS or RHS)$(v_0 \otimes v_0 \otimes v_1 \otimes v_1 \otimes 1 \otimes 1) = v_1 \otimes v_0 \otimes v_0 \otimes v_1 \otimes X + \cdots$. The X is given by

$$L(z_{13})^{01}_{01}L(z_{24})^{10}_{01} \otimes M(z_{12})^{10}_{01}M(z_{34})^{01}_{10} + L(z_{13})^{10}_{01}L(z_{24})^{01}_{01} \otimes M(z_{12})^{10}_{10}M(z_{34})^{01}_{01}$$
$$= -q\mu^{-1}z_{13}(\mathbf{ka}^+ \otimes \mathbf{a}^+\mathbf{a}^- + q\mathbf{a}^+\mathbf{k} \otimes \tilde{\mathbf{k}}^2)$$

for the LHS and

$$L(z_{24})^{01}_{01}L(z_{13})^{10}_{01} \otimes M(z_{34})^{11}_{11}M(z_{12})^{00}_{00} = -q\mu^{-1}z_{13}\mathbf{ka}^+ \otimes 1$$

for the RHS. Their difference is proportional to $\mathbf{ka}^+ \otimes \mathbf{a}^+\mathbf{a}^- + q\mathbf{a}^+\mathbf{k} \otimes \tilde{\mathbf{k}}^2 - \mathbf{ka}^+ \otimes 1$, which is zero due to (3.12), (3.13) and (3.121). □

Theorem 3.25 will be utilized for $A_q(B_n)$ in Chap. 6 and for multispecies TASEP in Chap. 18.

The solution in Theorem 3.25 consists of the 3D L and its slight variant M. There is a parallel solution consisting of the 3D R and its variant, which we write as S below.[6] Set

$$R(z)_{123} = z^{-\mathbf{h}_2}R_{123}z^{\mathbf{h}_2} = z^{\mathbf{h}_1}R_{123}z^{-\mathbf{h}_1}, \quad S(z)_{123} = z^{-\mathbf{h}_2}R_{213}z^{\mathbf{h}_2} = z^{\mathbf{h}_1}R_{213}z^{-\mathbf{h}_1}, \tag{3.123}$$

where $\mathbf{h}$ is defined in (3.14), and the second equalities are due to the weight conservation (3.49). The indices 1, 2, 3 specify the components in $\overset{1}{\mathcal{F}_q} \otimes \overset{2}{\mathcal{F}_q} \otimes \overset{3}{\mathcal{F}_q}$. In the notation (3.47), they are described as

$$R(z)(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b,c} z^{j-b}R^{abc}_{ijk}|a\rangle \otimes |b\rangle \otimes |c\rangle, \tag{3.124}$$

$$S(z)(|i\rangle \otimes |j\rangle \otimes |k\rangle) = \sum_{a,b,c} z^{j-b}R^{bac}_{jik}|a\rangle \otimes |b\rangle \otimes |c\rangle. \tag{3.125}$$

Theorem 3.26 *$R(z)$ and $S(z)$ satisfy the tetrahedron equation of type $MMLL = LLMM$ in $\mathrm{End}(\mathcal{F}_q^{\otimes 6})$ as*

$$\begin{aligned} &S(z_{12})_{126}S(z_{34})_{346}R(z_{13})_{135}R(z_{24})_{245} \\ &= R(z_{24})_{245}R(z_{13})_{135}S(z_{34})_{346}S(z_{12})_{126}, \end{aligned} \tag{3.126}$$

where $z_{ij} = z_i/z_j$.

Proof By substituting (3.123) into (3.126) and applying (3.49), one finds that the similarity transformation $z_{13}^{-\mathbf{h}_1}z_{23}^{-\mathbf{h}_2}z_{34}^{\mathbf{h}_4}(3.126)z_{13}^{\mathbf{h}_1}z_{23}^{\mathbf{h}_2}z_{34}^{-\mathbf{h}_4}$ removes the z-dependence, completely reducing it to $R_{216}R_{436}R_{135}R_{245} = R_{245}R_{135}R_{436}R_{216}$. Exchanging the

[6] This S will not be used elsewhere. It is different from the one in (2.20).

indices as $1 \leftrightarrow 5, 2 \leftrightarrow 4$ gives $R_{456}R_{236}R_{531}R_{421} = R_{421}R_{531}R_{236}R_{456}$. From (3.59) this is equivalent to $R_{456}R_{236}R_{135}R_{124} = R_{124}R_{135}R_{236}R_{456}$, which is indeed valid due to Theorem 3.20. □

3.6 Further Aspects of 3D *R*

Let us quote (3.38)–(3.46) in the form of the adjoint action of the 3D R:

$$R^{-1}\mathbf{k}_2\mathbf{a}_1^{\pm}R = \mathbf{k}_3\mathbf{a}_1^{\pm} + \mathbf{k}_1\mathbf{a}_2^{\pm}\mathbf{a}_3^{\mp}, \tag{3.127}$$

$$R^{-1}\mathbf{a}_2^{\pm}R = \mathbf{a}_1^{\pm}\mathbf{a}_3^{\pm} - q\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^{\pm}, \tag{3.128}$$

$$R^{-1}\mathbf{k}_2\mathbf{a}_3^{\pm}R = \mathbf{k}_1\mathbf{a}_3^{\pm} + \mathbf{k}_3\mathbf{a}_1^{\mp}\mathbf{a}_2^{\pm}, \tag{3.129}$$

$$R^{-1}(\mathbf{a}_1^{\pm}\mathbf{a}_2^{\mp}\mathbf{a}_3^{\pm} - q\mathbf{k}_1\mathbf{k}_3)R = \mathbf{a}_1^{\mp}\mathbf{a}_2^{\pm}\mathbf{a}_3^{\mp} - q\mathbf{k}_1\mathbf{k}_3, \tag{3.130}$$

$$R^{-1}\mathbf{k}_1\mathbf{k}_2R = \mathbf{k}_1\mathbf{k}_2, \quad R^{-1}\mathbf{k}_2\mathbf{k}_3R = \mathbf{k}_2\mathbf{k}_3. \tag{3.131}$$

The fact that $R = R^{-1}$ (3.60) has been taken into account. We have written $\mathbf{a}^+ \otimes \mathbf{k} \otimes 1$ as $\mathbf{k}_2\mathbf{a}_1^+$ for example. Thus the q-oscillator operators with different indices are commutative.

3.6.1 Boundary Vector

We define

$$|\eta_1\rangle = \sum_{m\ge 0}\frac{|m\rangle}{(q)_m}, \qquad |\eta_2\rangle = \sum_{m\ge 0}\frac{|2m\rangle}{(q^4)_m}, \tag{3.132}$$

$$\langle\eta_1| = \sum_{m\ge 0}\frac{\langle m|}{(q)_m}, \qquad \langle\eta_2| = \sum_{m\ge 0}\frac{\langle 2m|}{(q^4)_m}, \tag{3.133}$$

and call them *boundary vectors*. They will play an important role in the reduction procedure in Chaps. 12–17. They actually belong to a completion of $\mathcal{F}_q$ and $\mathcal{F}_q^*$ since infinite sums are involved. Nonetheless, we will refer to them as $|\eta_s\rangle \in \mathcal{F}_q$ and $\langle\eta_s| \in \mathcal{F}_q^*$ for simplicity.

Lemma 3.27 *Up to normalization, the boundary vector $|\eta_1\rangle$ is characterized by any one of the following three equivalent conditions:*

$$(\mathbf{a}^+ - 1 + \mathbf{k})|\eta_1\rangle = 0, \tag{3.134}$$

$$(\mathbf{a}^- - 1 - q\mathbf{k})|\eta_1\rangle = 0, \tag{3.135}$$

$$(\mathbf{a}^- + q\mathbf{a}^+ - 1 - q)|\eta_1\rangle = 0. \tag{3.136}$$

Similarly, the boundary vector $|\eta_2\rangle$ is characterized, up to normalization, by

$$(\mathbf{a}^+ - \mathbf{a}^-)|\eta_2\rangle = 0. \tag{3.137}$$

Proof Substituting $|\eta_s\rangle = \sum_m c_m|m\rangle$ into these conditions and using (3.13), one can check that c_m/c_0 is determined uniquely as in (3.132). □

A linear combination of (3.134) and (3.135) leads to (3.136). However, the lemma includes a less trivial reverse that (3.136) implies the preceding two.

From (3.17) the dual boundary vectors (3.133) have similar characterizations:

$$\langle\eta_1|(\mathbf{a}^- - 1 + \mathbf{k}) = 0, \tag{3.138}$$
$$\langle\eta_1|(\mathbf{a}^+ - 1 - q\mathbf{k}) = 0, \tag{3.139}$$
$$\langle\eta_1|(\mathbf{a}^+ + q\mathbf{a}^- - 1 - q) = 0, \tag{3.140}$$
$$\langle\eta_2|(\mathbf{a}^- - \mathbf{a}^+) = 0. \tag{3.141}$$

Proposition 3.28 *The 3D R and the boundary vectors satisfy the following relations:*

$$(\langle\eta_s| \otimes \langle\eta_s| \otimes \langle\eta_s|)R = \langle\eta_s| \otimes \langle\eta_s| \otimes \langle\eta_s| \quad (s = 1, 2), \tag{3.142}$$
$$R(|\eta_s\rangle \otimes |\eta_s\rangle \otimes |\eta_s\rangle) = |\eta_s\rangle \otimes |\eta_s\rangle \otimes |\eta_s\rangle \quad (s = 1, 2). \tag{3.143}$$

Proof From Remark 3.10, it suffices to prove (3.143). First we consider the case $s = 1$. By Lemma 3.27, it suffices to check

$$(\mathbf{a}_2^- + q\mathbf{a}_2^+ - 1 - q)R|\eta_1\rangle^{\otimes 3} = 0, \tag{3.144}$$
$$(\mathbf{a}_1^+ - 1 + \mathbf{k}_1)R|\eta_1\rangle^{\otimes 3} = (\mathbf{a}_3^+ - 1 + \mathbf{k}_3)R|\eta_1\rangle^{\otimes 3} = 0. \tag{3.145}$$

To show (3.144), we multiply R^{-1} from the left and apply (3.128) to convert the LHS into

$$\big(\mathbf{a}_1^-\mathbf{a}_3^- - q\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^- + q(\mathbf{a}_1^+\mathbf{a}_3^+ - q\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^+) - 1 - q\big)|\eta_1\rangle^{\otimes 3}. \tag{3.146}$$

From (3.134) and (3.135), one may set $\mathbf{a}_i^+ = 1 - \mathbf{k}_i$ and $\mathbf{a}_i^- = 1 + q\mathbf{k}_i$ here. The resulting polynomial in $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ vanishes identically, proving (3.144). By Lemma 3.27, it follows that $(\mathbf{a}_2^+ - 1 + \mathbf{k}_2)R|\eta_1\rangle^{\otimes 3} = 0$ has also been proved. Multiplying R^{-1} again by it and applying (3.128), (3.134), (3.135), we get

$$\big(-\mathbf{k}_1 - \mathbf{k}_3 + (1-q)\mathbf{k}_1\mathbf{k}_3 + q\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3 + \mathbf{k}_2'\big)|\eta_1\rangle^{\otimes 3} = 0, \tag{3.147}$$

where $\mathbf{k}_2' = R^{-1}\mathbf{k}_2 R$. This enables us to show (3.145). In fact, by multiplying $R^{-1}\mathbf{k}_2$ by the first relation, its LHS becomes $(\mathbf{k}_3\mathbf{a}_1^+ + \mathbf{k}_1\mathbf{a}_2^+\mathbf{a}_3^- - \mathbf{k}_2' + \mathbf{k}_1\mathbf{k}_2)|\eta_1\rangle^{\otimes 3}$ owing to (3.127). Substitution of $\mathbf{a}_i^+ = 1 - \mathbf{k}_i$ and $\mathbf{a}_i^- = 1 + q\mathbf{k}_i$ leads to the same expression as (3.147), hence zero. The second relation in (3.145) can be verified in the same manner.

Next we consider the case $s = 2$. From Lemma 3.27, it suffices to check $\mathbf{k}_2(\mathbf{a}_i^+ - \mathbf{a}_i^-)R|\eta_2\rangle^{\otimes 3} = 0$ $(i = 1, 3)$ and $(\mathbf{a}_2^+ - \mathbf{a}_2^-)R|\eta_2\rangle^{\otimes 3} = 0$. The proof is similar to the $s = 1$ case and actually simpler in that an intermediate identity like (3.147) need not be prepared. So we demonstrate the last identity only. By multiplying R^{-1} and using (3.128), its LHS becomes

$$\big((\mathbf{a}_1^+\mathbf{a}_3^+ - q\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^+) - (\mathbf{a}_1^-\mathbf{a}_3^- - q\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^-)\big)|\eta_2\rangle^{\otimes 3}.$$

From (3.137), we may set $\mathbf{a}_i^+ = \mathbf{a}_i^-$ here. □

3.6.2 Combinatorial and Birational Counterparts

As remarked after (3.84), we know $R^{abc}_{ijk} \in \mathbb{Z}[q, q^{-1}]$. Actually a stronger property holds.

Lemma 3.29 *R^{abc}_{ijk} is a polynomial in q with the constant term given by*

$$R^{abc}_{ijk}\big|_{q=0} = R^{ijk}_{abc}\big|_{q=0} = \delta^a_{j+(i-k)_+}\delta^b_{\min(i,k)}\delta^c_{j+(k-i)_+}. \tag{3.148}$$

See (3.66) for the definition of the symbol $(x)_+$.

Proof First we show $R^{abc}_{ijk} \in \mathbb{Z}[q]$. Let A be a ring of rational functions of q regular at $q = 0$. In view of $\mathbb{Z}[q, q^{-1}] \cap A = \mathbb{Z}[q]$, it suffices to show $R^{abc}_{ijk} \in A$. From (3.50) we have $R^{abc}_{ijk} \in AR^{a,b-1,c}_{i,j-1,k} + AR^{a,b-1,c}_{i-1,j,k-1}$. By induction on b, this attributes the claim to $R^{a,0,c}_{ijk} \in A$ for arbitrary a, c, i, j, k. But this is obviously true since $R^{a,0,c}_{ijk} = \delta^a_{i+j}\delta^c_{j+k}q^{ik}$ either from (3.67) or (3.74).

Next we show (3.148). The first equality is due to (3.63). Setting $q = 0$ in (3.50) and (3.56), we get

$$R^{a,b,c}_{i-1,j,k-1}\big|_{q=0} = R^{a,b+1,c}_{i,j,k}\big|_{q=0}, \quad R^{a-1,b,c-1}_{i,j,k}\big|_{q=0} = R^{a,b,c}_{i,j+1,k}\big|_{q=0}. \tag{3.149}$$

From the symmetry (3.62), it suffices to verify the $i \le k$ case. Then the first relation shows that $R^{abc}_{ijk}\big|_{q=0} = 0$ if $b > i$. For $b \le i$, we have $R^{abc}_{ijk}\big|_{q=0} = R^{a,0,c}_{i-b,j,k-b}\big|_{q=0} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k}q^{(i-b)(k-b)}\big|_{q=0}$. This is non-vanishing only if $b = i$ because otherwise $b < i \le k$. Thus we conclude $R^{abc}_{ijk}\big|_{q=0} = \delta^{a+b}_{i+j}\delta^{b+c}_{j+k}\delta^b_i = \delta^a_j\delta^b_i\delta^c_{j+k-i}$. □

Lemma 3.29 shows that 3D R at $q = 0$ maps a monomial to another monomial as $R\big|_{q=0}(|i\rangle \otimes |j\rangle \otimes |k\rangle) = |j + (i-k)_+\rangle \otimes |\min(i,k)\rangle \otimes |j + (k-i)_+\rangle$. Motivated by this fact, we define the *combinatorial* 3D R to be a map on $(\mathbb{Z}_{\ge 0})^3$ given by

$$R_{\text{combinatorial}} : (a, b, c) \mapsto (b + (a-c)_+, \min(a, c), b + (c-a)_+). \tag{3.150}$$

Corollary 3.30 *The combinatorial 3D R (3.150) is an involution on* $(\mathbb{Z}_{\geq 0})^3$. *It satisfies the tetrahedron equation of type* $RRRR = RRRR$ *on* $(\mathbb{Z}_{\geq 0})^6$.

Proof The assertions follow from (3.60) and Theorem 3.20 by setting $q = 0$ and using Lemma 3.29. □

Example 3.31 An example of the tetrahedron equation (2.6) for the combinatorial 3D R. The map R here denotes $R_{\text{combinatorial}}$ in (3.150). The first SW arrow R_{124} is due to $R_{\text{combinatorial}} : (3, 1, 4) \mapsto (1, 3, 2)$, which can be seen in Example 3.12.

$$
\begin{array}{ccc}
 & (315416) & \\
R_{124} \swarrow & & \searrow R_{456} \\
(135216) & & (315143) \\
R_{135} \downarrow & & \downarrow R_{236} \\
(531256) & & (351147) \\
R_{236} \downarrow & & \downarrow R_{135} \\
(513254) & & (153127) \\
R_{456} \searrow & & \swarrow R_{124} \\
 & (513527) &
\end{array}
$$

Let us proceed to the third 3D R. Regarding a, b, c as indeterminates, we introduce the map

$$R_{\text{birational}} : (a, b, c) \mapsto (\tilde{a}, \tilde{b}, \tilde{c}) = \left(\frac{ab}{a+c}, \, a + c, \, \frac{bc}{a+c}\right). \tag{3.151}$$

We called it the *birational* 3D R in the current context. The combinatorial 3D R (3.150) is reproduced from it by the *tropical variable change*

$$ab \to a + b, \quad \frac{a}{b} \to a - b, \quad a + b \to \min(a, b), \tag{3.152}$$

which keeps the distributive law since $a(b + c) = ab + ac$ is replaced by $a + \min(b, c) = \min(a + b, a + c)$. One way to materialize (3.152) is a transformation to logarithmic variables via

$$
\begin{aligned}
-\lim_{\varepsilon \to +0} \varepsilon \log(e^{-\frac{a}{\varepsilon}} e^{\mp\frac{b}{\varepsilon}}) &= a \pm b, \\
-\lim_{\varepsilon \to +0} \varepsilon \log(e^{-\frac{a}{\varepsilon}} + e^{-\frac{b}{\varepsilon}}) &= \min(a, b),
\end{aligned}
\tag{3.153}
$$

supposing $a, b \in \mathbb{R}$. In this context, (3.152) is also called the *ultradiscretization* (UD).

Set

$$Z_i(x) = 1 + xE_{i,i+1}, \tag{3.154}$$

where x is a parameter and $E_{i,j}$ is the n-by-n matrix unit whose only non-zero element is 1 at the ith row and the jth column. $Z_i(x)$ is a generator of the unipotent subgroup of SL(n). The birational 3D R (3.151) is characterized as the unique solution to the matrix equation

$$Z_i(a)Z_j(b)Z_i(c) = Z_j(\tilde{c})Z_i(\tilde{b})Z_j(\tilde{a}) \qquad (|i-j|=1). \tag{3.155}$$

It essentially reduces to the $n = 3$, $(i, j) = (1, 2)$ case:

$$\begin{pmatrix}1 & a & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & c & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & \tilde{c}\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & \tilde{b} & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & \tilde{a}\\ 0 & 0 & 1\end{pmatrix}. \tag{3.156}$$

The $R_{\text{birational}}$ is birational due to $R^{-1}_{\text{birational}} = R_{\text{birational}}$. It preserves ab and bc. The intertwining relation (3.28) is a quantization of (3.155) (with $(i, j) = (1, 2)$). Note that $Z_i(a)Z_j(b) = Z_j(b)Z_i(a)$ for $|i - j| > 1$ also holds analogously to the Coxeter relations.

Given a Weyl group element $w \in W(A_{n-1})$ (not necessarily longest), assign a matrix $M = Z_{i_1}(x_1)\cdots Z_{i_r}(x_r)$ to a reduced expression $w = s_{i_1}\cdots s_{i_r}$. Then to any reduced expression $w = s_{j_1}\cdots s_{j_r}$ one can assign the expression $M = Z_{j_1}(\tilde{x}_1)\cdots Z_{j_r}(\tilde{x}_r)$, where $\tilde{x}_k$ is determined independently of the intermediate steps applying (3.155). This property is the source of the tetrahedron equation for $R_{\text{birational}}$ and forms a birational counterpart of the previous calculation (3.93). In fact, the uniqueness of the map $(a, b, c, d, e, f) \mapsto (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{e}, \tilde{f})$ defined by

$$Z_1(a)Z_2(b)Z_1(c)Z_3(d)Z_2(e)Z_1(f) = Z_3(\tilde{f})Z_2(\tilde{e})Z_1(\tilde{d})Z_3(\tilde{c})Z_2(\tilde{b})Z_3(\tilde{a}) \tag{3.157}$$

implies the tetrahedron equation of type $RRRR = RRRR$ for $R_{\text{birational}}$. To summarize, we have:

Proposition 3.32 *The birational 3D R (3.151) is an involutive map on the ring of rational functions of three variables. It satisfies the tetrahedron equation of type $RRRR = RRRR$.*

Let us denote the 3D R detailed in Sects. 3.3 and 3.4 by R_{quantum}. Then we have the triad of the 3D R's whose relation is summarized as

$$R_{\text{quantum}} \overset{q\to 0}{\longrightarrow} R_{\text{combinatorial}} \overset{\text{UD}}{\longleftarrow} R_{\text{birational}}. \tag{3.158}$$

$R_{\text{combinatorial}}$ and $R_{\text{birational}}$ (and R^{λ} below) are typical set-theoretical solutions to the tetrahedron equation.

Remark 3.33 Define a map R^λ involving a parameter λ by

$$R^\lambda : (a,b,c) \mapsto \left(\frac{ab}{a+c+\lambda abc},\ a+c+\lambda abc,\ \frac{bc}{a+c+\lambda abc}\right) \qquad (3.159)$$

The birational 3D R (3.151) corresponds to $\lambda = 0$ or equivalently infinitesimal a, b, c. Then the inversion relation $R^\lambda = (R^\lambda)^{-1}$ and the tetrahedron equation

$$R^\lambda_{124} R^\lambda_{135} R^\lambda_{236} R^\lambda_{456} = R^\lambda_{456} R^\lambda_{236} R^\lambda_{135} R^\lambda_{124} \qquad (3.160)$$

are valid.

3.6.3 Bilinearization and Geometric Interpretation

The map (3.159) is bilinearized in the following sense. Parameterize a, b, c in terms of "tau functions" as

$$a = \frac{\tau\tau_{12}}{\tau_1\tau_2}, \quad b = \frac{\tau_2\tau_{123}}{\tau_{12}\tau_{23}}, \quad c = \frac{\tau\tau_{23}}{\tau_2\tau_3}, \qquad (3.161)$$

where indices signify the shifts of independent variables of the tau functions in the respective directions., say, $\tau = \tau(\mathbf{x})$, $\tau_{12} = \tau(\mathbf{x} + \mathbf{e}_1 + \mathbf{e}_2)$ etc. Suppose the tau function satisfies the bilinear equation

$$\tau_1\tau_{23} - \tau_2\tau_{13} + \tau_3\tau_{12} + \lambda\tau\tau_{123} = 0. \qquad (3.162)$$

Then the image $(a', b', c') = R^\lambda\big((a,b,c)\big)$ in the RHS of (3.159) is expressed in the same format as (3.161) as follows:

$$a' = \frac{\tau_3\tau_{123}}{\tau_{13}\tau_{23}}, \quad b' = \frac{\tau\tau_{13}}{\tau_1\tau_3}, \quad c' = \frac{\tau_1\tau_{123}}{\tau_{12}\tau_{13}}. \qquad (3.163)$$

The change $(a,b,c) \mapsto (a',b',c')$ corresponds to the shift $(+3, -2, +1)$ of the argument of the tau functions. It is interpreted as a transformation of the three back faces of a cube to the front ones as in Fig. 3.3.

The tetrahedron equation (3.160) is bilinearized by using tau functions living on a four-dimensional cube. We prepare τ_I with I running over the power set of $\{1, 2, 3, 4\}$. They are supposed to obey

$$\tau_i\tau_{jk} - \tau_j\tau_{ik} + \tau_k\tau_{ij} + \lambda\tau\tau_{ijk} = 0, \qquad (3.164)$$

$$\tau_{il}\tau_{jkl} - \tau_{jl}\tau_{ikl} + \tau_{kl}\tau_{ijl} + \lambda\tau_l\tau_{ijkl} = 0, \qquad (3.165)$$

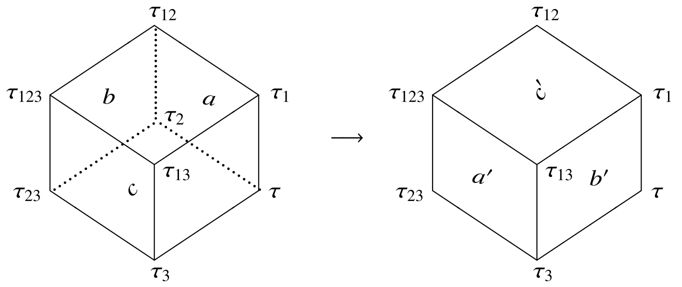


Fig. 3.3 Birational 3D R corresponds to a transformation generating a cube

where $\{i, j, k, l\} = \{1, 2, 3, 4\}$.[7] The latter is a *translation* of the former in the l direction.

Now the LHS of the tetrahedron equation (3.160) is described as the successive transformations

$$\begin{aligned}
&\left(\frac{\tau\tau_{12}}{\tau_1\tau_2}, \frac{\tau_2\tau_{123}}{\tau_{12}\tau_{23}}, \frac{\tau_{23}\tau_{1234}}{\tau_{123}\tau_{234}}, \frac{\tau\tau_{23}}{\tau_2\tau_3}, \frac{\tau_3\tau_{234}}{\tau_{23}\tau_{34}}, \frac{\tau\tau_{34}}{\tau_3\tau_4}\right)\\
\xmapsto{R^{\lambda}_{456}} &\left(\frac{\tau\tau_{12}}{\tau_1\tau_2}, \frac{\tau_2\tau_{123}}{\tau_{12}\tau_{23}}, \frac{\tau_{23}\tau_{1234}}{\tau_{123}\tau_{234}}, \frac{\tau_4\tau_{234}}{\tau_{24}\tau_{34}}, \frac{\tau\tau_{24}}{\tau_2\tau_4}, \frac{\tau_2\tau_{234}}{\tau_{23}\tau_{24}}\right)\\
\xmapsto{R^{\lambda}_{236}} &\left(\frac{\tau\tau_{12}}{\tau_1\tau_2}, \frac{\tau_{24}\tau_{1234}}{\tau_{124}\tau_{234}}, \frac{\tau_2\tau_{124}}{\tau_{12}\tau_{24}}, \frac{\tau_4\tau_{234}}{\tau_{24}\tau_{34}}, \frac{\tau\tau_{24}}{\tau_2\tau_4}, \frac{\tau_{12}\tau_{1234}}{\tau_{123}\tau_{124}}\right)\\
\xmapsto{R^{\lambda}_{135}} &\left(\frac{\tau_4\tau_{124}}{\tau_{14}\tau_{24}}, \frac{\tau_{24}\tau_{1234}}{\tau_{124}\tau_{234}}, \frac{\tau\tau_{14}}{\tau_1\tau_4}, \frac{\tau_4\tau_{234}}{\tau_{24}\tau_{34}}, \frac{\tau_1\tau_{124}}{\tau_{12}\tau_{14}}, \frac{\tau_{12}\tau_{1234}}{\tau_{123}\tau_{124}}\right)\\
\xmapsto{R^{\lambda}_{124}} &\left(\frac{\tau_{34}\tau_{1234}}{\tau_{134}\tau_{234}}, \frac{\tau_4\tau_{134}}{\tau_{14}\tau_{34}}, \frac{\tau\tau_{14}}{\tau_1\tau_4}, \frac{\tau_{14}\tau_{1234}}{\tau_{124}\tau_{134}}, \frac{\tau_1\tau_{124}}{\tau_{12}\tau_{14}}, \frac{\tau_{12}\tau_{1234}}{\tau_{123}\tau_{124}}\right).
\end{aligned} \tag{3.166}$$

Similarly, the RHS of (3.160) is realized as

$$\begin{aligned}
&\left(\frac{\tau\tau_{12}}{\tau_1\tau_2}, \frac{\tau_2\tau_{123}}{\tau_{12}\tau_{23}}, \frac{\tau_{23}\tau_{1234}}{\tau_{123}\tau_{234}}, \frac{\tau\tau_{23}}{\tau_2\tau_3}, \frac{\tau_3\tau_{234}}{\tau_{23}\tau_{34}}, \frac{\tau\tau_{34}}{\tau_3\tau_4}\right)\\
\xmapsto{R^{\lambda}_{124}} &\left(\frac{\tau_3\tau_{123}}{\tau_{13}\tau_{23}}, \frac{\tau\tau_{13}}{\tau_1\tau_3}, \frac{\tau_{23}\tau_{1234}}{\tau_{123}\tau_{234}}, \frac{\tau_1\tau_{123}}{\tau_{12}\tau_{13}}, \frac{\tau_3\tau_{234}}{\tau_{23}\tau_{34}}, \frac{\tau\tau_{34}}{\tau_3\tau_4}\right)\\
\xmapsto{R^{\lambda}_{135}} &\left(\frac{\tau_{34}\tau_{1234}}{\tau_{134}\tau_{234}}, \frac{\tau\tau_{13}}{\tau_1\tau_3}, \frac{\tau_3\tau_{134}}{\tau_{13}\tau_{34}}, \frac{\tau_1\tau_{123}}{\tau_{12}\tau_{13}}, \frac{\tau_{13}\tau_{1234}}{\tau_{123}\tau_{134}}, \frac{\tau\tau_{34}}{\tau_3\tau_4}\right)\\
\xmapsto{R^{\lambda}_{236}} &\left(\frac{\tau_{34}\tau_{1234}}{\tau_{134}\tau_{234}}, \frac{\tau_4\tau_{134}}{\tau_{14}\tau_{34}}, \frac{\tau\tau_{14}}{\tau_1\tau_4}, \frac{\tau_1\tau_{123}}{\tau_{12}\tau_{13}}, \frac{\tau_{13}\tau_{1234}}{\tau_{123}\tau_{134}}, \frac{\tau_1\tau_{134}}{\tau_{13}\tau_{14}}\right)\\
\xmapsto{R^{\lambda}_{456}} &\left(\frac{\tau_{34}\tau_{1234}}{\tau_{134}\tau_{234}}, \frac{\tau_4\tau_{134}}{\tau_{14}\tau_{34}}, \frac{\tau\tau_{14}}{\tau_1\tau_4}, \frac{\tau_{14}\tau_{1234}}{\tau_{124}\tau_{134}}, \frac{\tau_1\tau_{124}}{\tau_{12}\tau_{14}}, \frac{\tau_{12}\tau_{1234}}{\tau_{123}\tau_{124}}\right).
\end{aligned} \tag{3.167}$$

[7] τ_I is supposed to be independent of the ordering of the indices in I.

The initial and the final six components correspond to the faces 12, 13, 14, 23, 24, 34 of the 4D cube up to translation. Their tau functions are simply related by the interchange $\tau_I \leftrightarrow \tau_{\{1,2,3,4\}\setminus I}$. It means that the two sides of the tetrahedron equation represent transformations of the "back" six faces of a 4D cube to the "front" ones as compositions of elementary transformations associated with the 3D cube in Fig. 3.3. This 4D picture is rather transparent. On the other hand, one can also describe it in 3D space as a dissection of a rhombic dodecahedron into four quadrilatelal hexahedra. After all, the 3D R in this chapter provides a quantization of the transformation of the geometric data associated with such objects.

3.7 Bibliographical Notes and Comments

The RTT realization of the quantized coordinate rings has been presented in many publications. See for example [43, 127] and [29, Chap. 7]. The fundamental Theorem 3.3 on the representations of $A_q(\mathfrak{g})$ was obtained in [138, 139, 146]. Its application to the tetrahedron equation was found in [77]. In fact, Sect. 3.3, Theorems 3.11 and 3.5 form an exposition of it along [93, Sect. 2]. In particular, the formula (3.67) is a correction of that for S^{abc}_{ijk} on [77, p. 194] which contained an unfortunate misprint. The solution of the tetrahedron equation of type $RRRR = RRRR$ was derived later also from a quantum geometry consideration [16, 18]. It was shown to coincide with the 3D R in [77] (with the correction of the misprint) at [93, Eq. (2.29)]. The operator version R^{ab}_{ij} (3.69) of the 3D R was introduced in [84, Eq. (8)]. A similar operator with respect to the second component of the 3D R is given in [86, Eqs. (2.68) and (2.70)]. The integral formula (3.76) and Theorem 3.21 are due to [18, 132], respectively. The solution to the tetrahedron equation of type $MMLL = LLMM$ (Theorem 3.25) is due to [90, Theorem 3.4] and [18, Eq. (34)] with some conventional adjustment. Theorem 3.26 is taken from [92, Theorem 3.1]. They have applications to the multispecies totally asymmetric simple exclusion process (Chap. 18) and multispecies totally asymmetric zero rage process. More comments on them are available in Sect. 18.6. Proposition 3.28 for the boundary vector was obtained in [107, Proposition 4.1].

As for the birational and combinatorial 3D R, there are many relevant publications. The map (3.151) is a member of a wider list in [70, 71, 130]. It has also appeared in [112, Proposition 2.5] and [21, Theorem 3.1] for example. It is characterized as the transition map of parameterizations of the totally positive part of the special linear group $\mathrm{SL}(3)$. Such transition maps have been described explicitly for any semisimple Lie groups, and they all admit the combinatorial counterparts via the tropical variable change [22, 113]. The deformation (3.159) involving a cubic term (see [69]) has been linked to "electrical" Lie groups [110]. Sect. 3.6.3 is an exposition of the classical geometry aspects with an additional perspective concerning tau functions. For related topics, see [16, 24, 69, 78] and the references therein.

Chapter 4
3D Reflection Equation and Quantized Reflection Equation

Abstract This chapter is a brief introduction to the 3D reflection equation and the quantized reflection equation. They are both fundamental and will work coherently in later chapters of the book. In addition to the 3D R satisfying the tetrahedron equation, a central role is played by a linear operator which we call 3D K.

4.1 Introduction

In the preceding chapters, we have been concerned with the tetrahedron equations of type $RRRR = RRRR$ and $RLLL = LLLR$. Roughly speaking, the latter, which we also called the quantized Yang–Baxter equation, serves as the auxiliary linear problem characterizing the 3D R whereby the former is captured as its compatibility condition. In a sense, $RLLL = LLLR$ provided a linearization scheme of $RRRR = RRRR$. Both equations happened to possess the same quartic form.

In the presence of a boundary, things are parallel and yet apparently different. We will need to cope with

$$\text{3D reflection equation: } RKRRKKR = RKKRRKR,$$
$$\text{quantized reflection equation: } (LGLG)K = K(GLGL)$$

as natural analogues of $RRRR = RRRR$ and $RLLL = LLLR$, respectively. Here K is the most characteristic operator which we call 3D K. Although the two equations look quite different, the latter (together with $RLLL = LLLR$) serves as an auxiliary linear problem of the 3D K and the former emerges from its compatibility as in the previous story. Through several chapters of the book, we will demonstrate how the two equations fit the quantized coordinate ring of type BC, and how they lead to a 3D approach to the usual reflection equation in 2D.

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4.2 3D K

Let $\mathcal{F}' = \bigoplus \mathbb{C}|m\rangle'$ be a vector space with basis $\{|m\rangle'\}$.[1] In our main example in this book, $\mathcal{F}' \simeq \mathcal{F}$ as vector spaces but they will be equipped with different module structures with respect to some quantum algebra. Let K be a linear operator

$$K : \mathcal{F}' \otimes \mathcal{F} \otimes \mathcal{F}' \otimes \mathcal{F} \to \mathcal{F}' \otimes \mathcal{F} \otimes \mathcal{F}' \otimes \mathcal{F} \tag{4.1}$$

which is described as

$$K(|i\rangle' \otimes |j\rangle \otimes |k\rangle' \otimes |l\rangle) = \sum_{a,b,c,d} K^{abcd}_{ijkl} |a\rangle' \otimes |b\rangle \otimes |c\rangle' \otimes |d\rangle \tag{4.2}$$

in terms of matrix elements $K^{abcd}_{ijkl} \in \mathbb{C}$. In order to explain their graphical representation, we label the spaces appearing in (4.1) as $\overset{1}{\mathcal{F}'} \otimes \overset{2}{\mathcal{F}} \otimes \overset{3}{\mathcal{F}'} \otimes \overset{4}{\mathcal{F}}$.

Consider a book 1 having only the front page 1^+ and the back page 1^-. Similarly, let 3 be another book with the front and the back pages 3^+ and 3^-. Imagine that they are put on a desk as in Fig. 4.1. These pages are world sheets of the strings 1 and 3 that are being reflected on the desk surface. They are actually infinitely large hence intersecting although Fig. 4.1 displays finite portions to avoid complexity. The books should be *upright* since the incident and the reflecting angles are the same, but otherwise angles are arbitrary, including the one between the two broken arrows in Fig. 4.1.

We attach the spaces $\overset{1}{\mathcal{F}'}$ and $\overset{3}{\mathcal{F}'}$ to the spines of the books 1 and 3 as depicted by broken arrows. The operator $K \in \mathrm{End}(\overset{1}{\mathcal{F}'} \otimes \overset{2}{\mathcal{F}} \otimes \overset{3}{\mathcal{F}'} \otimes \overset{4}{\mathcal{F}})$ lives at P which is the intersection of them. Beside the two spines, there are four half lines 1^+3^+, 1^+3^-, 1^-3^-, 1^-3^+ emanating from P, where $1^\epsilon 3^{\epsilon'}$ denotes the intersections of the book

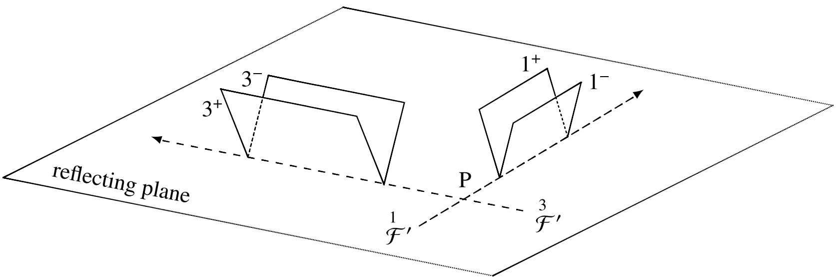


Fig. 4.1 World sheets of the two strings 1 and 3 being reflected by a plane. They actually consist of four half infinite planes 1^+, 1^-, 3^+ and 3^-. The operator K lives at P

[1] In later chapters, the actual coefficient field will be taken as $\mathbb{C}(q^{\frac{1}{2}})$, etc.

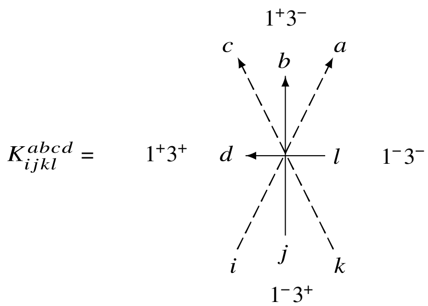


Fig. 4.2 Graphical representation of the matrix element K^{abcd}_{ijkl} in (4.2) projected onto the reflecting plane. The central point is P. The two broken arrows which correspond to $\overset{1}{\mathcal{F}}{}'$ and $\overset{3}{\mathcal{F}}{}'$ can intersect with any angle. On the other hand, the two solid arrows (projection of intersection of the book pages) always intersect with the right angle by construction and correspond to $\overset{2}{\mathcal{F}}$ and $\overset{4}{\mathcal{F}}$. They bisect the angles so that $\angle c\mathrm{P}d = \angle i\mathrm{P}d = \angle a\mathrm{P}l = \angle k\mathrm{P}l$ and $\angle c\mathrm{P}b = \angle a\mathrm{P}b = \angle i\mathrm{P}j = \angle k\mathrm{P}j$. The intersections $1^{\epsilon}3^{\epsilon'}$ of the book pages are also indicated

pages 1^{ϵ} and $3^{\epsilon'}$. We assign $\overset{2}{\mathcal{F}}$ to the concatenation of the arrows approaching P along 1^-3^+ and that departing from P along 1^+3^-. Similarly, the concatenation of the arrows approaching P along 1^-3^- and that departing from P along 1^+3^+ is assigned to $\overset{4}{\mathcal{F}}$. Such concatenations are natural since the two half lines $1^{\epsilon}3^{\epsilon'}$ and $1^{-\epsilon}3^{-\epsilon'}$ match up to a single *straight* line when projected onto the desk surface. See Fig. 4.2.

We call the operator (4.1) 3D K when it satisfies the 3D reflection equation explained in the next section.

4.3 3D Reflection Equation

Suppose that R is a 3D R, i.e. it satisfies the tetrahedron equation of type $RRRR = RRRR$ in (2.6). By 3-dimensional (3D) reflection equation, we mean the following:

$$R_{689}K_{3579}R_{249}R_{258}K_{1478}K_{1236}R_{456} = R_{456}K_{1236}K_{1478}R_{258}R_{249}K_{3579}R_{689}. \quad (4.3)$$

It is an equality in

$$\mathrm{End}(\overset{1}{\mathcal{F}}{}' \otimes \overset{2}{\mathcal{F}} \otimes \overset{3}{\mathcal{F}}{}' \otimes \overset{4}{\mathcal{F}} \otimes \overset{5}{\mathcal{F}} \otimes \overset{6}{\mathcal{F}} \otimes \overset{7}{\mathcal{F}}{}' \otimes \overset{8}{\mathcal{F}} \otimes \overset{9}{\mathcal{F}}), \quad (4.4)$$

where, as in (2.7) and (2.16), superscripts specify the components on which the operators in (4.3) act non-trivially. One sees that the ordering of 3D R's and 3D K's in the two sides of (4.3) are reversed. A solution of the 3D reflection equation means a pair of 3D R and 3D K. In terms of matrix elements, the 3D reflection equation (4.3) is expressed as

$$\begin{aligned}&\sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} R^{bdi_2}_{b_1d_1i_3} R^{b_1e_1h_1}_{b_2e_2h_2} K^{ad_1g_1h_2}_{a_1d_2g_2h_3} K^{a_1b_2c_1f_1}_{a_2b_3c_2f_2} R^{d_2e_2f_2}_{d_3e_3f_3}\\ &= \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} K^{a_1d_1gh}_{a_2d_2g_1h_1} R^{b_1e_1h_1}_{b_2e_2h_2} R^{b_2d_2i}_{b_3d_3i_1} K^{c_1e_2g_1i_1}_{c_2e_3g_2i_2} R^{f_2h_2i_2}_{f_3h_3i_3}\end{aligned} \tag{4.5}$$

for any $a, b, c, d, e, f, g, h, i$ and $a_2, b_3, c_2, d_3, e_3, f_3, g_2, h_3, i_3$. The sums are taken over 15 indices $a_1, b_1, b_2, c_1, d_1, d_2, e_1, e_2, f_1, f_2, g_1, h_1, h_2, i_1, i_2$ on both sides. So if all the spaces were 2-dimensional for instance, there are 2^{18} equations on 2^8 unknowns containing 2^{15} terms on each side in general even if R^{abc}_{ijk}'s are known.

Despite the horrible looking forms (4.3) and (4.5), the 3D reflection equation admits an elegant geometric interpretation using *three* intersecting books. To draw them artistically, however, is beyond the skill of the author. So let us present their image projected onto the desk, i.e. the reflecting plane in Fig. 4.3.

Consider the three books whose spines are indicated by the dotted arrows 1, 3 and 7 in the LHS of Fig. 4.3. To their intersections P, Q and R, we attach a diagram according to Fig. 4.2 (without indices like $a, b, \ldots$) . It amounts to introducing the solid arrows which are labeled as 2, 4, 5, 6, 8 and 9. By the construction, the arrows 2, 5, 8 are perpendicular to the arrows 6, 9, 4, and they bisect the angles ∠QRP, ∠PQR, ∠RPQ, respectively. Let the intersections of the arrows 4, 6 and 9 be A, B and C. The essential fact which validates Fig. 4.3 is the elementary geometry theorem:

$$\text{orthocenter of } \triangle\text{ABC} = \text{inner center of } \triangle\text{PQR}. \tag{4.6}$$

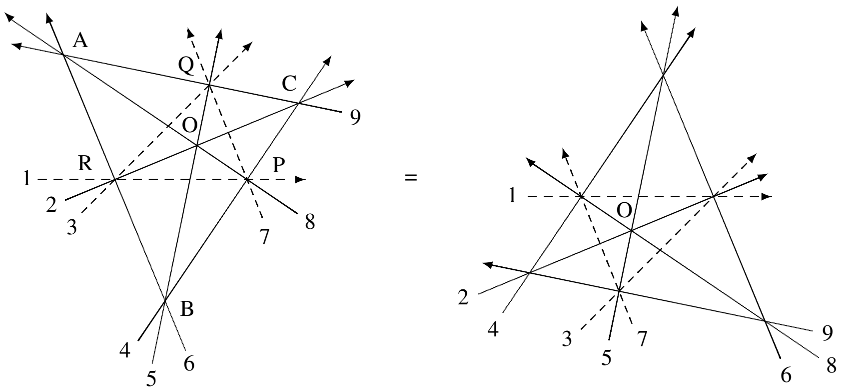


Fig. 4.3 Pictorial representation of the 3D reflection equation (4.3) projected onto the reflecting plane. In the LHS, the vertices and the corresponding operators are A: K_{689}, B: R_{456}, C: R_{249}, O: R_{258}, P: K_{1478}, Q: K_{3579}, R: K_{1236}

It has been denoted by O. Now one can readily reconstruct the LHS of (4.3). Proceed faithfully along the arrows as B → R → P → O → C → Q → A, and form the composition of the corresponding 3D R's and 3D K's. The RHS is similar. Note that the triangle PQR formed by the book spines gets reversed in the two sides like the Yang–Baxter equation. In this sense one can simply say that the 3D reflection equation is a decorated Yang–Baxter equation associated with the Yang–Baxter move of the spines of the three upright books on a desk.

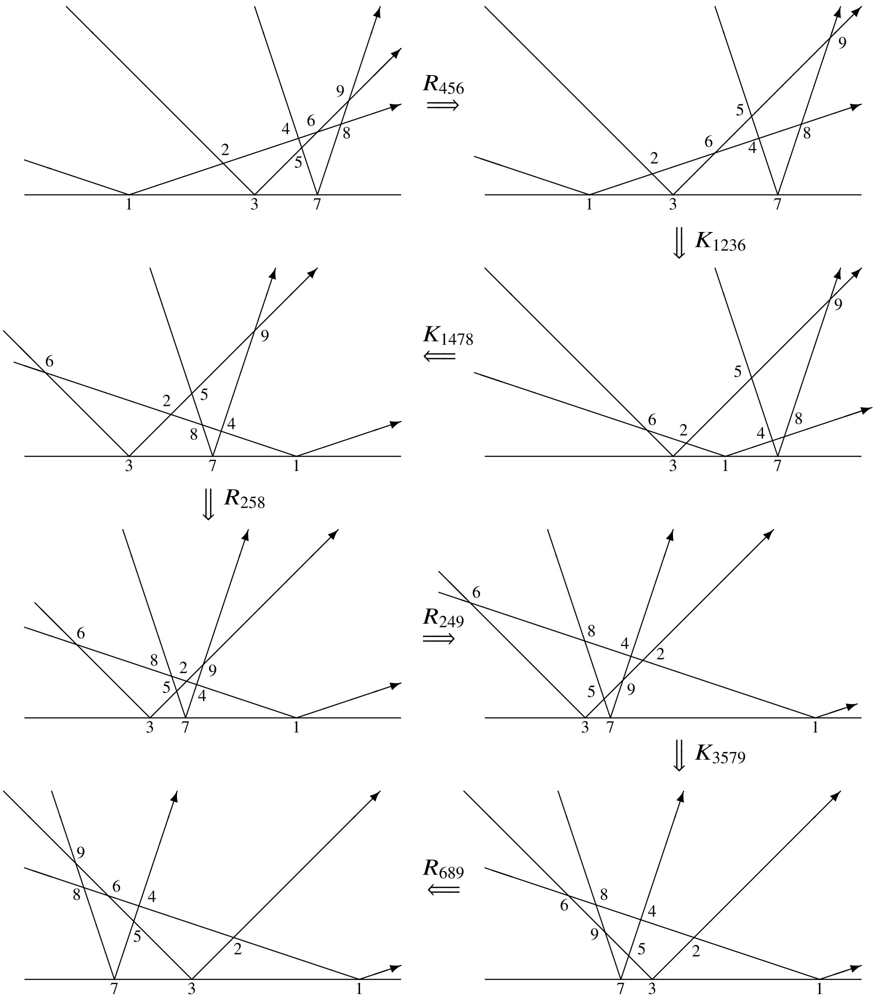


Fig. 4.4 The process that generates LHS of (4.3) is depicted by using slices of the three intersecting books on a desk in successive instances. The bottom horizontal line is the edge view of the desk surface. A triangle $i \to j \to k$ and a zigzag $i \to j \to k \to l$ give rise to R_{ijk} and K_{ijkl} respectively when they get reversed. The vertices 123456789 aligned in this order horizontally in the top left diagram have been reversed in the bottom left diagram.

One can also draw a 8-frame cartoon for each side of the 3D reflection equation (4.3) in a manner analogous to Fig. 2.2 for the tetrahedron equation. For the LHS it is given by Fig. 4.4.

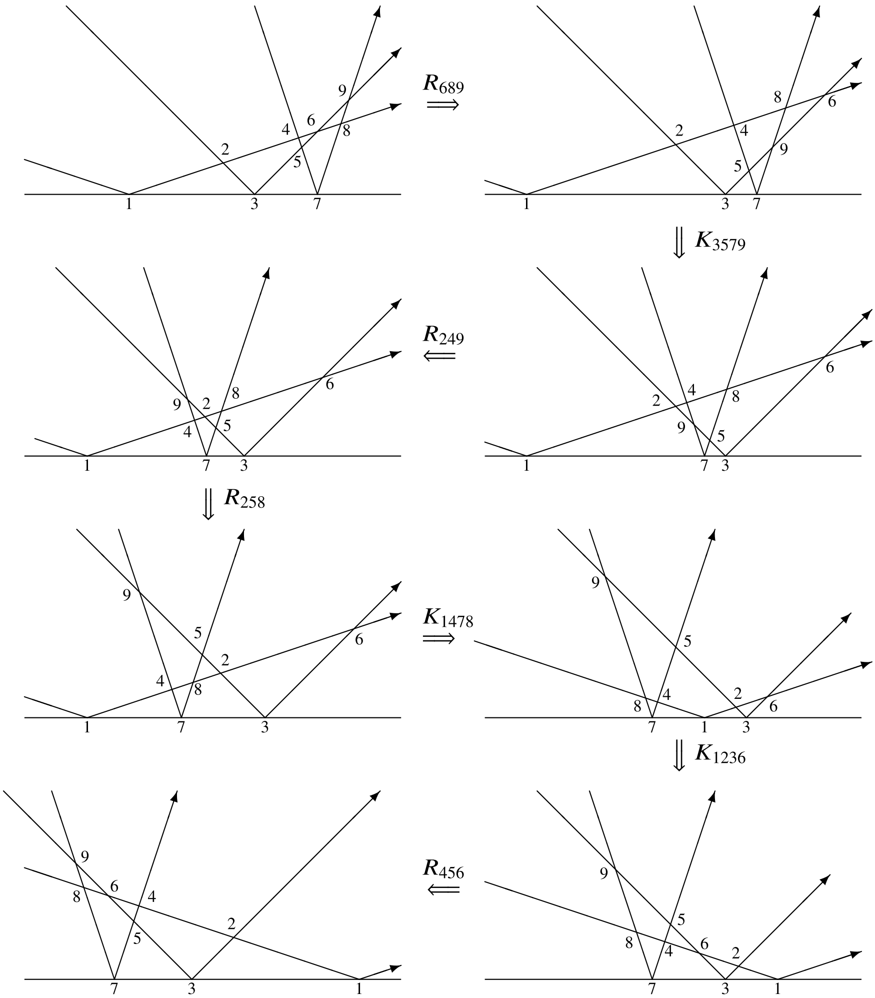


Fig. 4.5 The process that generates RHS of (4.3) similar to Fig. 4.4

4.4 Quantized Reflection Equation

Let $L \in \mathrm{End}(V \otimes V \otimes \mathcal{F}')$ be a 3D L, where we have replaced $\mathcal{F}$ by $\mathcal{F}'$ in (2.11) for reasons of convention.[2] This in particular implies that the L obeys the quantized Yang–Baxter equation (2.19) and (2.20) with $R, S \in \mathrm{End}(\mathcal{F}' \otimes \mathcal{F}' \otimes \mathcal{F}')$.

We introduce a new linear operator

$$G : V \otimes \mathcal{F} \to V \otimes \mathcal{F}. \tag{4.7}$$

The vector spaces $\mathcal{F}$ and $\mathcal{F}'$ are those appearing in $K \in \mathrm{End}(\mathcal{F}' \otimes \mathcal{F} \otimes \mathcal{F}' \otimes \mathcal{F})$ in (4.1). By quantized reflection equation we mean a reflection equation up to conjugation:

$$(L_{12}G_2L_{21}G_1)K = K(G_1L_{12}G_2L_{21}). \tag{4.8}$$

To explain the precise meaning of this abbreviated notation, let us exhibit the spaces on which the operators act by labels as $\overset{i}{L}_{xy} \in \mathrm{End}(\overset{x}{V} \otimes \overset{y}{V} \otimes \overset{i}{\mathcal{F}'})$, $\overset{i}{G}_x \in \mathrm{End}(\overset{x}{V} \otimes \overset{i}{\mathcal{F}})$ and $K_{ijkl} \in \mathrm{End}(\overset{i}{\mathcal{F}'} \otimes \overset{j}{\mathcal{F}} \otimes \overset{k}{\mathcal{F}'} \otimes \overset{l}{\mathcal{F}})$. Then (4.8) actually means

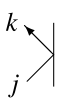
$$(\overset{i}{L}_{12}\overset{j}{G}_2\overset{k}{L}_{21}\overset{l}{G}_1)K_{ijkl} = K_{ijkl}(\overset{l}{G}_1\overset{k}{L}_{12}\overset{j}{G}_2\overset{i}{L}_{21}). \tag{4.9}$$

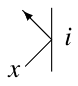
This is an equality in $\mathrm{End}(\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{i}{\mathcal{F}'} \otimes \overset{j}{\mathcal{F}} \otimes \overset{k}{\mathcal{F}'} \otimes \overset{l}{\mathcal{F}})$. The operator L_{21} is defined to be $L_{21} = P_{12}L_{12}P_{12}$, where P_{12} interchanges the first and the second tensor component from the left in $\mathrm{End}(V \otimes V \otimes \mathcal{F}')$.

One way to describe the operator G is to refer to the base of $V = \bigoplus_i \mathbb{C}v_i$ as

$$G(v_j \otimes |m\rangle) = \sum_k v_k \otimes G^k_j|m\rangle \tag{4.10}$$

for arbitrary $|m\rangle \in \mathcal{F}$. Here G^k_j is the $\mathrm{End}(\mathcal{F})$-valued matrix element of G with respect to the base $\{v_i\}$ of V. It corresponds to the decomposition $G = \sum_{a,i} E_{ai} \otimes G^a_i$ with respect to the matrix unit on V. We depict either (a) the element G^k_j or (b) the fact that $\overset{i}{G}_x \in \mathrm{End}(\overset{x}{V} \otimes \overset{i}{\mathcal{F}})$ as

(a) G^k_j = 

(b) $\overset{i}{G}_x$ = 

(4.11)

[2] It suits type B instead of type C.

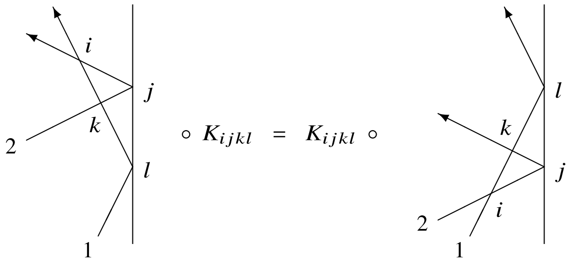


Fig. 4.6 Graphical representation of the quantized reflection equation (4.9)

They can be distinguished safely from the context. These are formally identical with the diagrams which are used conventionally for the boundary reflection amplitudes or boundary Boltzmann weights in 2D integrable systems. The vertical line signifies the boundary. The present ones, however, should be recognized as End($\overset{i}{\mathcal{F}}$)-valued which acts in the direction perpendicular to the diagrams along an (invisible) arrow going through the vertex i. In (b), the label x is attached to the arrow, whereas i is assigned to the vertex (reflection point) on the boundary rather than the boundary line.

Now with the convention (b) in (4.11), the quantized reflection equation (4.9) is depicted as follows:

The indices 1, 2 here are assigned to the arrows like x in (4.11), whereas i, j, k, l are attached to the vertices. One may regard K_{ijkl} in the left-hand (right-hand) side as a point in the back (front) of the diagram where the four arrows going toward (coming from) the vertices i, j, k, l intersect. The 3D L's have been depicted according to the right-hand diagram in (2.14).

In terms of L^{ab}_{ij} in (2.12) and G^{k}_{j} in (4.11), the component of Fig. 4.6 for the transition $v_c \otimes v_d \mapsto v_a \otimes v_b \in \overset{1}{V} \otimes \overset{2}{V}$ is expressed as

$$\sum_{i_1,i_2,j_1,j_2} (L^{ab}_{i_2 j_2} \otimes G^{j_2}_{j_1} \otimes L^{j_1 i_2}_{d\, i_1} \otimes G^{i_1}_{c}) K = \sum_{i_1,i_2,j_1,j_2} K (L^{j_1 i_1}_{d\, c} \otimes G^{j_2}_{j_1} \otimes L^{i_2 b}_{i_1 j_2} \otimes G^{a}_{i_2}). \tag{4.12}$$

See (5.108). This is an equality in End($\overset{i}{\mathcal{F}'} \otimes \overset{j}{\mathcal{F}} \otimes \overset{k}{\mathcal{F}'} \otimes \overset{l}{\mathcal{F}}$) for each choice of a, b, c, d which are labels of the base of V.

Set $G^{k}_{j}|m\rangle = \sum_{m'} G^{km'}_{jm}|m'\rangle$ in (4.10). Then by further taking the coefficients of the transition $|e_1\rangle' \otimes |f_1\rangle \otimes |g_1\rangle' \otimes |h_1\rangle \mapsto |e_3\rangle' \otimes |f_3\rangle \otimes |g_3\rangle' \otimes |h_3\rangle$ according to (4.2) and (2.13), the Eq. (4.12) is regarded as a collection of

$$\sum L^{abe_3}_{i_2 j_2 e_2} G^{j_2 f_3}_{j_1 f_2} L^{j_1 i_2 g_3}_{d\, i_1 g_2} G^{i_1 h_3}_{c\, h_2} K^{e_2 f_2 g_2 h_2}_{e_1 f_1 g_1 h_1} = \sum K^{e_3 f_3 g_3 h_3}_{e_2 f_2 g_2 h_2} L^{j_1 i_1 e_2}_{d\, c\, h_1} G^{j_2 f_2}_{j_1 g_1} L^{i_2 b\, g_2}_{i_1 j_2 f_1} G^{a\, h_2}_{i_2 e_1} \tag{4.13}$$

with respect to $a, b, c, d, e_1, f_1, g_1, h_1, e_3, f_3, g_3, h_3$, where the sums in both sides extend over $i_1, i_2, j_1, j_2, e_2, f_2, g_2, h_2$.

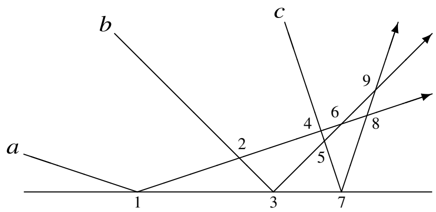


Fig. 4.7 The first diagrams in Figs. 4.4 and 4.5 with labeled arrows a, b, c

We will also work with another version of 3D K defined by

$$J_{1234} = P_{14}P_{23}K^{-1}_{1234}P_{14}P_{23} \in \mathrm{End}(\mathcal{F} \otimes \mathcal{F}' \otimes \mathcal{F} \otimes \mathcal{F}') \tag{4.14}$$
$$= K_{4321} \qquad (\text{if } K^{-1} = K). \tag{4.15}$$

Here P_{ij} is the exchanger of the component i and j, hence $P_{14}P_{23}$ reverses the order of the tensor product. Thus the second line means, when $K = K^{-1}$, that the rule corresponding to (4.2) is given by

$$J(|l\rangle \otimes |k\rangle' \otimes |j\rangle \otimes |i\rangle') = \sum_{a,b,c,d} K^{abcd}_{ijkl} |d\rangle \otimes |c\rangle' \otimes |b\rangle \otimes |a\rangle'. \tag{4.16}$$

The quantized reflection equation for J takes the form

$$J_{ijkl}(\overset{l}{L}_{12}\overset{k}{G}_2\overset{j}{L}_{21}\overset{i}{G}_1) = (\overset{i}{G}_1\overset{j}{L}_{12}\overset{k}{G}_2\overset{l}{L}_{21})J_{ijkl}. \tag{4.17}$$

Let us look at the common first diagrams in Figs. 4.4 and 4.5. We label the three arrows in it with a, b, c as follows:

Naturally, Fig. 4.7 is attached with the operator

$$\begin{aligned} &\overset{9}{L}_{bc}\overset{8}{L}_{ac}\overset{7}{G}_c\overset{6}{L}_{ab}\overset{5}{L}_{cb}\overset{4}{L}_{ca}\overset{3}{G}_b\overset{2}{L}_{ba}\overset{1}{G}_a \\ &\in \mathrm{End}(\overset{a}{V} \otimes \overset{b}{V} \otimes \overset{c}{V} \otimes \overset{1}{\mathcal{F}} \otimes \overset{2}{\mathcal{F}'} \otimes \overset{3}{\mathcal{F}} \otimes \overset{4}{\mathcal{F}'} \otimes \overset{5}{\mathcal{F}'} \otimes \overset{6}{\mathcal{F}'} \otimes \overset{7}{\mathcal{F}} \otimes \overset{8}{\mathcal{F}'} \otimes \overset{9}{\mathcal{F}'}). \end{aligned} \tag{4.18}$$

Applying the quantized Yang–Baxter equations (2.19), (2.20) and the quantized reflection equation (4.17) successively, we get

$$
\begin{aligned}
&S_{689}J_{3579}R_{249}S_{258}J_{1478}J_{1236}R_{456}\overset{9}{L_{bc}}\overset{8}{L_{ac}}\overset{7}{G_{c}}\underline{\overset{6}{L_{ab}}\overset{5}{L_{cb}}\overset{4}{L_{ca}}}\overset{3}{G_{b}}\overset{2}{L_{ba}}\overset{1}{G_{a}}\\
&=S_{689}J_{3579}R_{249}S_{258}J_{1478}J_{1236}\overset{9}{L_{bc}}\overset{8}{L_{ac}}\overset{7}{G_{c}}\overset{4}{L_{ca}}\overset{5}{L_{cb}}\underline{\overset{6}{L_{ab}}\overset{3}{G_{b}}\overset{2}{L_{ba}}\overset{1}{G_{a}}}R_{456}\\
&=S_{689}J_{3579}R_{249}S_{258}J_{1478}\overset{9}{L_{bc}}\overset{8}{L_{ac}}\overset{7}{G_{c}}\overset{4}{L_{ca}}\underline{\overset{5}{L_{cb}}\overset{1}{G_{a}}}\overset{2}{L_{ab}}\overset{3}{G_{b}}\overset{6}{L_{ba}}J_{1236}R_{456}\\
&=S_{689}J_{3579}R_{249}S_{258}J_{1478}\overset{9}{L_{bc}}\underline{\overset{8}{L_{ac}}\overset{7}{G_{c}}\overset{4}{L_{ca}}\overset{1}{G_{a}}}\overset{5}{L_{cb}}\overset{2}{L_{ab}}\overset{3}{G_{b}}\overset{6}{L_{ba}}J_{1236}R_{456}\\
&=S_{689}J_{3579}R_{249}S_{258}\overset{9}{L_{bc}}\overset{1}{G_{a}}\overset{4}{L_{ac}}\overset{7}{G_{c}}\underline{\overset{8}{L_{ca}}\overset{5}{L_{cb}}\overset{2}{L_{ab}}}\overset{3}{G_{b}}\overset{6}{L_{ba}}J_{1478}J_{1236}R_{456}\\
&=S_{689}J_{3579}R_{249}\underline{\overset{9}{L_{bc}}\overset{1}{G_{a}}}\overset{4}{L_{ac}}\underline{\overset{7}{G_{c}}\overset{2}{L_{ab}}}\overset{5}{L_{cb}}\underline{\overset{8}{L_{ca}}\overset{3}{G_{b}}}\overset{6}{L_{ba}}S_{258}J_{1478}J_{1236}R_{456}\\
&=S_{689}J_{3579}R_{249}\overset{1}{G_{a}}\underline{\overset{9}{L_{bc}}\overset{4}{L_{ac}}\overset{2}{L_{ab}}}\overset{7}{G_{c}}\overset{5}{L_{cb}}\overset{3}{G_{b}}\overset{8}{L_{ca}}\overset{6}{L_{ba}}S_{258}J_{1478}J_{1236}R_{456}\\
&=S_{689}J_{3579}\overset{1}{G_{a}}\overset{2}{L_{ab}}\overset{4}{L_{ac}}\underline{\overset{9}{L_{bc}}\overset{7}{G_{c}}\overset{5}{L_{cb}}\overset{3}{G_{b}}}\overset{8}{L_{ca}}\overset{6}{L_{ba}}R_{249}S_{258}J_{1478}J_{1236}R_{456}\\
&=S_{689}\overset{1}{G_{a}}\overset{2}{L_{ab}}\overset{4}{L_{ac}}\overset{3}{G_{b}}\overset{5}{L_{bc}}\overset{7}{G_{c}}\underline{\overset{9}{L_{cb}}\overset{8}{L_{ca}}\overset{6}{L_{ba}}}J_{3579}R_{249}S_{258}J_{1478}J_{1236}R_{456}\\
&=\overset{1}{G_{a}}\overset{2}{L_{ab}}\underline{\overset{4}{L_{ac}}\overset{3}{G_{b}}}\overset{5}{L_{bc}}\underline{\overset{7}{G_{c}}\overset{6}{L_{ba}}}\overset{8}{L_{ca}}\overset{9}{L_{cb}}S_{689}J_{3579}R_{249}S_{258}J_{1478}J_{1236}R_{456}\\
&=\overset{1}{G_{a}}\overset{2}{L_{ab}}\overset{3}{G_{b}}\overset{4}{L_{ac}}\overset{5}{L_{bc}}\overset{6}{L_{ba}}\overset{7}{G_{c}}\overset{8}{L_{ca}}\overset{9}{L_{cb}}S_{689}J_{3579}R_{249}S_{258}J_{1478}J_{1236}R_{456}.
\end{aligned}
\tag{4.19}
$$

The underlines show the changing components. These transformations follow Fig. 4.4. Similarly, we have

$$
\begin{aligned}
&S_{456}J_{1236}J_{1478}R_{258}S_{249}J_{3579}R_{689}\overset{9}{L_{bc}}\overset{8}{L_{ac}}\underline{\overset{7}{G_{c}}\overset{6}{L_{ab}}}\overset{5}{L_{cb}}\underline{\overset{4}{L_{ca}}\overset{3}{G_{b}}}\overset{2}{L_{ba}}\overset{1}{G_{a}}\\
&=S_{456}J_{1236}J_{1478}R_{258}S_{249}J_{3579}R_{689}\underline{\overset{9}{L_{bc}}\overset{8}{L_{ac}}\overset{6}{L_{ab}}}\overset{7}{G_{c}}\overset{5}{L_{cb}}\overset{3}{G_{b}}\overset{4}{L_{ca}}\overset{2}{L_{ba}}\overset{1}{G_{a}}\\
&=S_{456}J_{1236}J_{1478}R_{258}S_{249}J_{3579}\overset{6}{L_{ab}}\overset{8}{L_{ac}}\underline{\overset{9}{L_{bc}}\overset{7}{G_{c}}\overset{5}{L_{cb}}\overset{3}{G_{b}}}\overset{4}{L_{ca}}\overset{2}{L_{ba}}\overset{1}{G_{a}}R_{689}\\
&=S_{456}J_{1236}J_{1478}R_{258}S_{249}\overset{6}{L_{ab}}\overset{8}{L_{ac}}\overset{3}{G_{b}}\overset{5}{L_{bc}}\overset{7}{G_{c}}\underline{\overset{9}{L_{cb}}\overset{4}{L_{ca}}\overset{2}{L_{ba}}}\overset{1}{G_{a}}J_{3579}R_{689}\\
&=S_{456}J_{1236}J_{1478}R_{258}\overset{6}{L_{ab}}\underline{\overset{8}{L_{ac}}\overset{3}{G_{b}}}\overset{5}{L_{bc}}\underline{\overset{7}{G_{c}}\overset{2}{L_{ba}}}\overset{4}{L_{ca}}\underline{\overset{9}{L_{cb}}\overset{1}{G_{a}}}S_{249}J_{3579}R_{689}\\
&=S_{456}J_{1236}J_{1478}R_{258}\overset{6}{L_{ab}}\overset{3}{G_{b}}\underline{\overset{8}{L_{ac}}\overset{5}{L_{bc}}\overset{2}{L_{ba}}}\overset{7}{G_{c}}\overset{4}{L_{ca}}\overset{1}{G_{a}}\overset{9}{L_{cb}}S_{249}J_{3579}R_{689}\\
&=S_{456}J_{1236}J_{1478}\overset{6}{L_{ab}}\overset{3}{G_{b}}\overset{2}{L_{ba}}\overset{5}{L_{bc}}\underline{\overset{8}{L_{ac}}\overset{7}{G_{c}}\overset{4}{L_{ca}}\overset{1}{G_{a}}}\overset{9}{L_{cb}}R_{258}S_{249}J_{3579}R_{689}\\
&=S_{456}J_{1236}\overset{6}{L_{ab}}\overset{3}{G_{b}}\overset{2}{L_{ba}}\underline{\overset{5}{L_{bc}}\overset{1}{G_{a}}}\overset{4}{L_{ac}}\overset{7}{G_{c}}\overset{8}{L_{ca}}\overset{9}{L_{cb}}J_{1478}R_{258}S_{249}J_{3579}R_{689}\\
&=S_{456}J_{1236}\underline{\overset{6}{L_{ab}}\overset{3}{G_{b}}\overset{2}{L_{ba}}\overset{1}{G_{a}}}\overset{5}{L_{bc}}\overset{4}{L_{ac}}\overset{7}{G_{c}}\overset{8}{L_{ca}}\overset{9}{L_{cb}}J_{1478}R_{258}S_{249}J_{3579}R_{689}\\
&=S_{456}\overset{1}{G_{a}}\overset{2}{L_{ab}}\overset{3}{G_{b}}\underline{\overset{6}{L_{ba}}\overset{5}{L_{bc}}\overset{4}{L_{ac}}}\overset{7}{G_{c}}\overset{8}{L_{ca}}\overset{9}{L_{cb}}J_{1236}J_{1478}R_{258}S_{249}J_{3579}R_{689}\\
&=\overset{1}{G_{a}}\overset{2}{L_{ab}}\overset{3}{G_{b}}\overset{4}{L_{ac}}\overset{5}{L_{bc}}\overset{6}{L_{ba}}\overset{7}{G_{c}}\overset{8}{L_{ca}}\overset{9}{L_{cb}}S_{456}J_{1236}J_{1478}R_{258}S_{249}J_{3579}R_{689}.
\end{aligned}
\tag{4.20}
$$

These transformations follow Fig. 4.5. We note that the underlines in (4.19) and (4.20) exactly correspond to those in (5.106). From (4.19) and (4.20) we see that

$$(S_{456}J_{1236}J_{1478}R_{258}S_{249}J_{3579}R_{689})^{-1}(S_{689}J_{3579}R_{249}S_{258}J_{1478}J_{1236}R_{456}) \tag{4.21}$$

commutes with the operator (4.18). Therefore, if the collection of (4.18) with respect to the components of $\mathrm{End}(\overset{a}{V} \otimes \overset{b}{V} \otimes \overset{c}{V})$ acts irreducibly on

$$\overset{1}{\mathcal{F}} \otimes \overset{2}{\mathcal{F}'} \otimes \overset{3}{\mathcal{F}} \otimes \overset{4}{\mathcal{F}'} \otimes \overset{5}{\mathcal{F}'} \otimes \overset{6}{\mathcal{F}'} \otimes \overset{7}{\mathcal{F}} \otimes \overset{8}{\mathcal{F}'} \otimes \overset{9}{\mathcal{F}'}, \tag{4.22}$$

the operator (4.21) must be a scalar by Schur's Lemma. Thus with suitable normalization of S, R and J, we have a slight variant of the 3D reflection equation

$$S_{689}J_{3579}R_{249}S_{258}J_{1478}J_{1236}R_{456} = S_{456}J_{1236}J_{1478}R_{258}S_{249}J_{3579}R_{689}. \tag{4.23}$$

In Sect. 6.4 we will make the irreducibility argument precise for the type B case.

The quantized reflection equation is an analogue of the quantized Yang–Baxter equation in Sect. 2.5 in the presence of a boundary, where the usual reflection equation in 2D systems is relaxed to the conjugacy equivalence. In later chapters of this book, we will construct solutions of these equations and present rich applications.

4.5 Bibliographical Notes and Comments

The reflection equation in the $(1+1)$D or 2D setting takes the form $RKRK = KRKR$, where R is a solution to the Yang–Baxter equation. It has been popularized by [30, 53, 82, 137] for example, and extensive results have been obtained by now.

The 3D reflection equation (4.23) with the indices replaced as $(1, 2, 3, 4, 5, 6, 7, 8, 9) \rightarrow (\tilde{x}, 6, \tilde{y}, 3, 4, 5, \tilde{z}, 2, 1)$ essentially coincides with [62, Eq. (17)], where it was proposed as the tetrahedron reflection equation. Its illustration in terms of "books" is due to a private communication with the first author of [62]. The projected Fig. 4.3 appeared in [94]. The quantized reflection equation (4.8) was first introduced in [105, Sect. 2.2].

One of the main topics in this book is a 3D approach to the reflection equation initiated in [105]. It will be treated in Chaps. 15 and 16, where the 3D reflection equation and the quantized reflection equation will function coherently.

Chapter 5
3D K From Quantized Coordinate Ring of Type C

Abstract We introduce the quantized coordinate ring $A_q(\mathfrak{g})$ for $\mathfrak{g}$ of type C based on generators and relations. Intertwiners of the quantized coordinate ring are constructed explicitly. They lead to solutions of the 3D reflection equation and the quantized reflection equation. The 3D K admits the set-theoretical and birational counterparts. These features are parallel with Chap. 3 for type A.

5.1 Quantized Coordinate Ring $A_q(C_n)$

The quantized coordinate ring $A_q(C_n)$ $(n \geq 2)$ is a Hopf algebra generated by $(2n)^2$ generators $T = (t_{ij})_{1 \leq i,j \leq 2n}$ with the relations of the form $RTT = TTR$ and $TC\,{}^tTC^{-1} = C\,{}^tTC^{-1}T = I$. Here ${}^tM$ denotes the transpose of a matrix M. Concretely, they read as

$$\sum_{m,p} R^{ij}_{mp} t_{mk} t_{pl} = \sum_{m,p} t_{jp} t_{im} R^{mp}_{kl}, \tag{5.1}$$

$$\sum_{j,k,l} C_{jk} C_{lm} t_{ij} t_{lk} = \sum_{jkl} C_{ij} C_{kl} t_{kj} t_{lm} = -\delta_{im}. \tag{5.2}$$

The structure constants $(R^{ij}_{kl})_{1 \leq i,j,k,l \leq 2n}$ and $C = -C^{-1} = (C_{ij})_{1 \leq i,j \leq 2n}$ are specified by

$$\begin{aligned}\sum_{i,j,k,l} R^{ij}_{kl} E_{ik} \otimes E_{jl} &= q \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j,j'} E_{ii} \otimes E_{jj} + q^{-1} \sum_i E_{ii} \otimes E_{i'i'} \\ &\quad + (q - q^{-1}) \sum_{i>j} E_{ij} \otimes E_{ji} - (q - q^{-1}) \sum_{i>j} \epsilon_i \epsilon_j q^{\varrho_i - \varrho_j} E_{ij} \otimes E_{i'j'},\end{aligned} \tag{5.3}$$

$$C_{ij} = \begin{cases} \delta_{i,j'} q^{\varrho_j}, 1 \leq j \leq n, \\ -\delta_{i,j'} q^{\varrho_j}, n < j \leq n, \end{cases} \qquad i' = 2n + 1 - i, \tag{5.4}$$

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$$(\varrho_1, \dots, \varrho_{2n}) = (n, n-1, \dots, 1, -1, \dots, -n+1, -n). \tag{5.5}$$

The indices in (5.1), (5.2) and (5.3) are summed over $\{1, 2, \dots, 2n\}$. The structure constant R^{ij}_{kl} is extracted as

$$\sum_{1\le i,j,m,l\le 2n} R^{ij}_{ml} E_{im} \otimes E_{jl} = q \lim_{x\to\infty} x^{-2} R(x)|_{k=q^{-1}} \tag{5.6}$$

from the quantum R matrix $R(x)$ for the vector representation of $U_q(C_n^{(1)})$ given in [64, Eq. (3.6)]. For $n = 2$, the matrix C reads as

$$C = \begin{pmatrix} 0 & 0 & 0 & q^{-2} \\ 0 & 0 & q^{-1} & 0 \\ 0 & -q & 0 & 0 \\ -q^2 & 0 & 0 & 0 \end{pmatrix}. \tag{5.7}$$

The RTT relation (5.1) is formally the same with (3.1). The coproduct and the counit are again given by (3.6) and (3.8). The antipode takes the form

$$S(T) = C\,{}^tTC^{-1} \tag{5.8}$$

in terms of $T = (t_{ij})$. The matrix C is related to the R matrix (5.3) as[1]

$$R = (C\otimes 1)({}^{t_1}R)^{-1}(C^{-1}\otimes 1) = (1\otimes C)({}^{t_2}(R^{-1}))(1\otimes C^{-1}), \tag{5.9}$$

where t_1 and t_2 denote the transpose in the left and the right components, respectively.

Let us consider a slightly more general situation of the anti-diagonal C such that $C^2 = \varepsilon I$ with $\varepsilon = \pm 1$. Thus we set

$$C = (C_{ij}), \quad C_{ij} = \delta_{i,j'}\rho_i, \quad \rho_i\rho_{i'} = \varepsilon. \tag{5.10}$$

The current setting corresponds to $\varepsilon = -1$. The relation (5.9) is expressed as

$$(R^{-1})^{ab}_{ij} = \rho_a\rho_i^{-1} R^{i'b}_{a'j} = \rho_j\rho_b^{-1} R^{aj'}_{ib'}. \tag{5.11}$$

The relation $TC\,{}^tTC^{-1} = C\,{}^tTC^{-1}T = I$ takes the form

$$\sum_j \rho_j t_{ij} t_{m'j'} = \rho_i\delta_{im}, \quad \sum_k \rho_k t_{ki} t_{k'm'} = \rho_i\delta_{im}. \tag{5.12}$$

[1] The symbol R is of temporary use, and it should not be confused with the 3D R.

Remark 5.1 Denote by $A_q(\{t_{ij}\}; R, C)$ the Hopf algebra with generators $\{t_{ij}\}$ obeying the relations (5.1) and (5.12), where the structure constants $R = (R^{ij}_{kl})$ satisfies the Yang–Baxter equation and $C = (C_{ij})$ is of the form (5.10) and related to R by (5.11). The coproduct, counit and antipode are defined by (3.6), (3.8) and (5.8). Then $A_q(\{t_{ij}\}; R, C) \simeq A_q(\{\tilde{t}_{ij}\}; \tilde{R}, \tilde{C})$ holds under the simple rescaling of the generators and associated data as follows:

$$\tilde{t}_{ij} = g_i g_j^{-1} t_{ij}, \quad \tilde{R}^{ij}_{kl} = g_i g_j (g_k g_l)^{-1} R^{ij}_{kl}, \quad \tilde{C}_{ij} = \varepsilon_i C_{ij}. \tag{5.13}$$

Here g_k is a non-zero parameter and ε_k is a sign factor satisfying

$$\varepsilon_k = \varepsilon_{k'} = g_k g_{k'} = \pm 1. \tag{5.14}$$

In particular, one has $\tilde{C}_{ij} = \delta_{i,j'} \tilde{\rho}_i$ with $\tilde{\rho}_i = \varepsilon_i \rho_i$.

5.2 Fundamental Representations

Let $\mathrm{Osc}_q = \langle \mathbf{a}^+, \mathbf{a}^-, \mathbf{k}, \mathbf{k}^{-1} \rangle$ be the q-oscillator algebra introduced in (3.12). Here we also use $\mathrm{Osc}_{q^2} = \langle \mathbf{A}^+, \mathbf{A}^-, \mathbf{K}, \mathbf{K}^{-1} \rangle$, where the generators obey

$$\mathbf{K}\,\mathbf{A}^+ = q^2\,\mathbf{A}^+\mathbf{K}, \quad \mathbf{K}\,\mathbf{A}^- = q^{-2}\mathbf{A}^-\mathbf{K}, \quad \mathbf{A}^-\mathbf{A}^+ = \mathbf{1} - q^4\mathbf{K}^2, \quad \mathbf{A}^+\mathbf{A}^- = \mathbf{1} - \mathbf{K}^2 \tag{5.15}$$

and those following from the obvious ones $\mathbf{K}\,\mathbf{K}^{-1} = \mathbf{K}^{-1}\mathbf{K} = 1$. It has an irreducible representation on the Fock space $\mathcal{F}_{q^2} = \bigoplus_{m \ge 0} \mathbb{C}(q)|m\rangle$:

$$\mathbf{K}|m\rangle = q^{2m}|m\rangle, \;\; \mathbf{A}^+|m\rangle = |m+1\rangle, \;\; \mathbf{A}^-|m\rangle = (1-q^{4m})|m-1\rangle. \tag{5.16}$$

We will use the same symbol $|m\rangle$ to denote a base either for $\mathcal{F}_q$ or $\mathcal{F}_{q^2}$ as they can be distinguished from the context.[2]

The general result on the representations stated in Theorem 3.3 applies to $A_q(C_n)$. We have the fundamental representations

$$\pi_i : A_q(C_n) \to \mathrm{End}(\mathcal{F}_{q_i}) \qquad (1 \le i \le n), \tag{5.17}$$

$$q_1 = \cdots = q_{n-1} = q, \;\; q_n = q^2 \tag{5.18}$$

containing a non-zero parameter μ_i. For π_i with $1 \le i \le n-1$, the image of the generators $(t_{jk})_{1 \le j,k \le 2n}$ is specified as follows:

[2] In particular from (3.16), $\langle m|m'\rangle$ is $(q^2)_m \delta_{m,m'}$ or $(q^4)_m \delta_{m,m'}$ depending on $\mathcal{F}_q$ or $\mathcal{F}_{q^2}$.

$$\begin{pmatrix} t_{i,i} & t_{i,i+1} \\ t_{i+1,i} & t_{i+1,i+1} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{a}^- & \mu_i\,\mathbf{k} \\ -q\mu_i^{-1}\,\mathbf{k} & \mathbf{a}^+ \end{pmatrix}, \quad \begin{pmatrix} t_{(i+1)',(i+1)'} & t_{(i+1)',i'} \\ t_{i',(i+1)'} & t_{i',i'} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{a}^- & -\mu_i\,\mathbf{k} \\ q\mu_i^{-1}\,\mathbf{k} & \mathbf{a}^+ \end{pmatrix}, \tag{5.19}$$

$$t_{jj} \mapsto 1 \ (j \neq i, i+1, (i+1)', i'), \quad \text{otherwise } t_{jk} \mapsto 0. \tag{5.20}$$

See (5.4) for the definition of i'. For π_n, the image of the generators is specified as follows:

$$\begin{pmatrix} t_{n,n} & t_{n,n+1} \\ t_{n+1,n} & t_{n+1,n+1} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{A}^- & \mu_n\,\mathbf{K} \\ -q^2\mu_n^{-1}\,\mathbf{K} & \mathbf{A}^+ \end{pmatrix}, \tag{5.21}$$

$$t_{jj} \mapsto 1 \ (j \neq n, n+1), \quad \text{otherwise } t_{jk} \mapsto 0. \tag{5.22}$$

Due to the presence of creation and annihilation operators, there is no non-trivial invariant subspace within the Fock spaces, hence $\pi_1, \ldots, \pi_n$ are indeed irreducible. The claim (i) in Theorem 3.3 is reflected in the fact that the 2-by-2 blocks of q-oscillator operators in (5.19) and (5.21) are identical with (3.19) up to μ_i parameters and the replacement $q \to q^2$ for π_n.

Example 5.2 The image of the 16 generators $T = (t_{ij})_{1\le i,j\le 4}$ of $A_q(C_2)$ by the fundamental representations reads as

$$\pi_1(T) = \begin{pmatrix} \mathbf{a}^- & \mu_1\,\mathbf{k} & 0 & 0 \\ -q\mu_1^{-1}\,\mathbf{k} & \mathbf{a}^+ & 0 & 0 \\ 0 & 0 & \mathbf{a}^- & -\mu_1\,\mathbf{k} \\ 0 & 0 & q\mu_1^{-1}\,\mathbf{k} & \mathbf{a}^+ \end{pmatrix}, \quad \pi_2(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathbf{A}^- & \mu_2\mathbf{K} & 0 \\ 0 & -q^2\mu_2^{-1}\mathbf{K} & \mathbf{A}^+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.23}$$

Remark 5.3 All the parameters $\mu_1, \ldots, \mu_n$ in the fundamental representations can be absorbed into the rescaling of the generators by taking $g_k^{-1} = g_{k'} = \mu_n^{1/2} \prod_{k\le j<n} \mu_j$ for $1 \le k \le n$ in Remark 5.1. Therefore, one may set $\mu_1 = \cdots = \mu_n = 1$ in the reminder of this chapter except for their use in the proof of Lemma 5.7 and Proposition 5.8.

Example 5.4 The image of the 36 generators $T = (t_{ij})_{1\le i,j\le 6}$ of $A_q(C_3)$ by the fundamental representations reads as

$$\pi_1(T) = \begin{pmatrix} \mathbf{a}^- & \mathbf{k} & 0 & 0 & 0 & 0 \\ -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{a}^- & -\mathbf{k} \\ 0 & 0 & 0 & 0 & q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}, \quad \pi_2(T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{a}^- & \mathbf{k} & 0 & 0 & 0 \\ 0 & -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{a}^- & -\mathbf{k} & 0 \\ 0 & 0 & 0 & q\mathbf{k} & \mathbf{a}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.24}$$

$$\pi_3(T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{A}^- & \mathbf{K} & 0 & 0 \\ 0 & 0 & -q^2\mathbf{K} & \mathbf{A}^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{5.25}$$

where we have set $\mu_1 = \mu_2 = \mu_3 = 1$ in (5.19)–(5.22).

Let us turn to the tensor products of the fundamental representations. As in the type A case, we often write $\pi_{i_1} \otimes \cdots \otimes \pi_{i_l}$ as $\pi_{i_1,\dots,i_l}$ for short. The Weyl group $W(C_n) = \langle s_1, \dots, s_n \rangle$ is generated by the simple reflections $s_1, \dots, s_n$ obeying the Coxeter relations

$$s_i^2 = 1, \qquad s_i s_j = s_j s_i \quad (|i-j| \geq 2), \tag{5.26}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad (1 \leq i \leq n-2), \quad s_{n-1} s_n s_{n-1} s_n = s_n s_{n-1} s_n s_{n-1}. \tag{5.27}$$

According to Theorem 3.3, these relations imply that the following isomorphism is valid:

$$\pi_{i,j} \simeq \pi_{j,i} \qquad (|i-j| \geq 2), \tag{5.28}$$

$$\pi_{i,i+1,i} \simeq \pi_{i+1,i,i+1} \qquad (1 \leq i \leq n-2), \tag{5.29}$$

$$\pi_{n-1,n,n-1,n} \simeq \pi_{n,n-1,n,n-1}. \tag{5.30}$$

5.3 Interwtiners for Quadratic and Cubic Coxeter Relations

By Remark 3.4, the intertwiner responsible for the isomorphism (5.28) is just the exchange of components P defined in (3.23). See the explanation around (3.24).

Next we consider the intertwiner for (5.29). Namely, we seek $\Phi \in \mathrm{End}(\mathcal{F}_q^{\otimes 3})$ characterized by

$$\Phi \circ \pi_{i,i+1,i}(\Delta(f)) = \pi_{i+1,i,i+1}(\Delta(f)) \circ \Phi \quad (1 \leq i < n, \ \forall f \in A_q(C_n)), \tag{5.31}$$

$$\Phi(|0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle, \tag{5.32}$$

where the latter just fixes the normalization. We set $R = \Phi P_{13}$ following (3.30). Then a little manipulation using the representation (5.19) and (5.20) shows that the Eq. (5.31) is independent of μ_i parameters and is identical, as a set, with (3.38)–(3.46) for the 3D R. Thus we conclude that the intertwiner for (5.29) is provided by the 3D R in Chap. 3 by $\Phi = RP_{13}$. As mentioned before, R will also be called the intertwiner.

5.4 Intertwiner for Quartic Coxeter Relation

Let us construct the intertwiner responsible for the isomorphism (5.30). From the representations (5.19)–(5.22), it is not difficult to see that the problem is attributed to the $A_q(C_2)$ case. Thus we consider the linear map

$$\Psi : \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \longrightarrow \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \tag{5.33}$$

characterized by

$$\pi_{2121}(\Delta(f)) \circ \Psi = \Psi \circ \pi_{1212}(\Delta(f)) \quad (\forall f \in A_q(C_2)), \tag{5.34}$$

$$\Psi(|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle, \tag{5.35}$$

where the latter just specifies the normalization. The absence of terms other than $|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle$ in its RHS is assured by the weight conservation. See (5.57) and (5.65). We introduce K by

$$K = \Psi P_{14} P_{23} \in \mathrm{End}(\mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q), \tag{5.36}$$

where $P_{14}P_{23} : x \otimes y \otimes z \otimes w \mapsto w \otimes z \otimes y \otimes x$ reverses the ordering of the 4-fold tensor product. It will be shown to satisfy the 3D reflection equation in Theorem 5.16 and also in the latter half of Sect. 6.4, therefore K is a 3D K in the sense of Sect. 4.2.

The conditions (5.34) and (5.35) are translated into

$$\pi_{2121}(\Delta(f)) \circ K = K \circ \pi_{2121}(\tilde{\Delta}(f)) \quad (\forall f \in A_q(C_2)), \tag{5.37}$$

$$K(|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle, \tag{5.38}$$

where $\tilde{\Delta}(f) = P_{14}P_{23}\Delta(f)P_{23}P_{14}$. From (3.6) we have

$$\Delta(t_{ij}) = \sum_{1 \le l_1, l_2, l_3 \le 4} t_{il_1} \otimes t_{l_1 l_2} \otimes t_{l_2 l_3} \otimes t_{l_3 j}. \tag{5.39}$$

$$\tilde{\Delta}(t_{ij}) = \sum_{1 \le l_1, l_2, l_3 \le 4} t_{l_3 j} \otimes t_{l_2 l_3} \otimes t_{l_1 l_2} \otimes t_{il_1}. \tag{5.40}$$

Below, K will also be referred to as the intertwiner.

Thanks to Remark 5.3, the intertwining relation (5.37) is free from the parameters μ_1, μ_2 in (5.24). By a direct calculation using (5.39), (5.40) and (5.24), it is given explicitly as follows:

$$t_{11}: \quad [1 \otimes \mathbf{a}^- \otimes 1 \otimes \mathbf{a}^- - q 1 \otimes \mathbf{k} \otimes \mathbf{A}^- \otimes \mathbf{k},\ K] = 0, \tag{5.41}$$

$$\begin{aligned} t_{12}: \quad & (1 \otimes \mathbf{a}^- \otimes 1 \otimes \mathbf{k} + 1 \otimes \mathbf{k} \otimes \mathbf{A}^- \otimes \mathbf{a}^+)K \\ & = K(\mathbf{A}^- \otimes \mathbf{a}^+ \otimes \mathbf{A}^- \otimes \mathbf{k} + \mathbf{A}^- \otimes \mathbf{k} \otimes 1 \otimes \mathbf{a}^- - q^2 \mathbf{K} \otimes \mathbf{a}^- \otimes \mathbf{K} \otimes \mathbf{k}), \end{aligned} \tag{5.42}$$

$$t_{13}: \quad (1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^-)K = K(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^-), \tag{5.43}$$

$$t_{14}: \quad [1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{k}, K] = 0, \tag{5.44}$$

$$t_{21}: \quad (\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{A}^-\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^- - q^2\mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k})K$$
$$= K(1\otimes\mathbf{a}^-\otimes 1\otimes\mathbf{k} + 1\otimes\mathbf{k}\otimes\mathbf{A}^-\otimes\mathbf{a}^+), \tag{5.45}$$

$$t_{22}: \quad [\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{a}^+ - q\mathbf{A}^-\otimes\mathbf{k}\otimes 1\otimes\mathbf{k} - q^2\mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{a}^+,\ K] = 0, \tag{5.46}$$

$$t_{23}: \quad (\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{a}^- + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{a}^- - q\mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{k})K$$
$$= K(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{a}^+ + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{a}^+ - q\mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{k}), \tag{5.47}$$

$$t_{24}: \quad (\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^+)K = K(1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^+), \tag{5.48}$$

$$t_{31}: \quad (\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^-)K = K(1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^-), \tag{5.49}$$

$$t_{32}: \quad (\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{K}\otimes\mathbf{a}^+ + \mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{A}^-\otimes\mathbf{a}^+ - q\mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{k})K$$
$$= K(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{a}^- + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{a}^- - q\mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{k}), \tag{5.50}$$

$$t_{33}: \quad [\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{a}^- - q\mathbf{A}^+\otimes\mathbf{k}\otimes 1\otimes\mathbf{k} - q^2\mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{a}^-,\ K] = 0, \tag{5.51}$$

$$t_{34}: \quad (\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{A}^+\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^+ - q^2\mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k})K$$
$$= K(1\otimes\mathbf{a}^+\otimes 1\otimes\mathbf{k} + 1\otimes\mathbf{k}\otimes\mathbf{A}^+\otimes\mathbf{a}^-), \tag{5.52}$$

$$t_{41}: \quad [1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{k},\ K] = 0 \quad \text{(same as } t_{14}\text{)}, \tag{5.53}$$

$$t_{42}: \quad (1\otimes\mathbf{k}\otimes\mathbf{K}\otimes\mathbf{a}^+)K = K(\mathbf{A}^-\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{K}\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^+), \tag{5.54}$$

$$t_{43}: \quad (1\otimes\mathbf{a}^+\otimes 1\otimes\mathbf{k} + 1\otimes\mathbf{k}\otimes\mathbf{A}^+\otimes\mathbf{a}^-)K$$
$$= K(\mathbf{A}^+\otimes\mathbf{a}^-\otimes\mathbf{A}^+\otimes\mathbf{k} + \mathbf{A}^+\otimes\mathbf{k}\otimes 1\otimes\mathbf{a}^+ - q^2\mathbf{K}\otimes\mathbf{a}^+\otimes\mathbf{K}\otimes\mathbf{k}), \tag{5.55}$$

$$t_{44}: \quad [1\otimes\mathbf{a}^+\otimes 1\otimes\mathbf{a}^+ - q1\otimes\mathbf{k}\otimes\mathbf{A}^+\otimes\mathbf{k},\ K] = 0. \tag{5.56}$$

Here t_{ij} in the left column specifies the choice of f in (5.37).

Remark 5.5 If one switches from $\mathbf{k}$ and $\mathbf{K}$ to $\hat{\mathbf{k}} = q^{1/2}\mathbf{k}$ and $\hat{\mathbf{K}} = q\mathbf{K}$ including the *zero point energy* (see (3.13) and (5.16)), all the "non-autonomous" q's in (5.41)–(5.56) disappear, which is a parallel feature with 3D R in Remark 3.9.

The intertwiner K is regarded as a matrix $K = (K^{abcd}_{ijkl})$ acting on $\mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q$ as

$$K(|i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle) = \sum_{a,b,c,d} K^{abcd}_{ijkl} |a\rangle \otimes |b\rangle \otimes |c\rangle \otimes |d\rangle. \tag{5.57}$$

The normalization condition (5.38) becomes $K^{abcd}_{0000} = \delta^a_0\delta^b_0\delta^c_0\delta^d_0$.

Example 5.6 For later use, we write down a few examples of recursion relations among the matrix elements derived from (5.41)–(5.56):

$$
\begin{aligned}
t_{11}: &(1-q^{2b+2})(1-q^{2d+2})K^{a,b+1,c,d+1}_{i,j,k,l} - q^{1+b+d}(1-q^{4c+4})K^{a,b,c+1,d}_{i,j,k,l}\\
&= (1-q^{2j})(1-q^{2l})K^{a,b,c,d}_{i,j-1,k,l-1} - q^{1+j+l}(1-q^{4k})K^{a,b,c,d}_{i,j,k-1,l}, \qquad (5.58)\\
t_{12}: &q^d(1-q^{2b+2})K^{a,b+1,c,d}_{i,j,k,l} + q^b(1-q^{4c+4})K^{a,b,c+1,d-1}_{i,j,k,l}\\
&= q^l(1-q^{4i})(1-q^{4k})K^{a,b,c,d}_{i-1,j+1,k-1,l} + q^j(1-q^{4i})(1-q^{2l})K^{a,b,c,d}_{i-1,j,k,l-1}\\
&\quad - q^{2+2i+2k+l}(1-q^{2j})K^{a,b,c,d}_{i,j-1,k,l}, \qquad (5.59)\\
t_{13}: &q^{b+2c}(1-q^{2d+2})K^{a,b,c,d+1}_{i,j,k,l}\\
&= q^{2k+l}(1-q^{2j})K^{a,b,c,d}_{i+1,j-1,k,l} + q^{2i+l}(1-q^{4k})K^{a,b,c,d}_{i,j+1,k-1,l} + q^{2i+j}(1-q^{2l})K^{a,b,c,d}_{i,j,k,l-1}, \qquad (5.60)\\
t_{24}: &q^{2c+d}(1-q^{4a+4})K^{a+1,b-1,c,d}_{i,j,k,l} + q^{2a+d}(1-q^{2b+2})K^{a,b+1,c-1,d}_{i,j,k,l} + q^{2a+b}K^{a,b,c,d-1}_{i,j,k,l}\\
&= q^{j+2k}K^{a,b,c,d}_{i,j,k,l+1}, \qquad (5.61)\\
t_{44}: &K^{a,b-1,c,d-1}_{i,j,k,l} - q^{1+b+d}K^{a,b,c-1,d}_{i,j,k,l} = K^{a,b,c,d}_{i,j+1,k,l+1} - q^{1+j+l}K^{a,b,c,d}_{i,j,k+1,l}, . \qquad (5.62)
\end{aligned}
$$

By a direct calculation we find[3]

$$\pi_{2121}(\Delta(Y)) = \pi_{2121}(\tilde{\Delta}(Y)) = \mathbf{K} \otimes \mathbf{k}^2 \otimes \mathbf{K} \otimes 1 \qquad (5.63)$$

for $Y = \mu_1^{-2}\mu_2^{-2}(q^{-1}t_{23}t_{14} - t_{24}t_{13}) \in A_q(C_2)$. From (5.44) = (5.53) and (5.63), the intertwiner K commutes with the diagonal operators as

$$[K, \mathbf{K} \otimes \mathbf{k}^2 \otimes \mathbf{K} \otimes 1] = [K, 1 \otimes \mathbf{k} \otimes \mathbf{K} \otimes \mathbf{k}] = 0. \qquad (5.64)$$

From the definitions of $\mathbf{k}$ in (3.13) and $\mathbf{K}$ in (5.16), it implies

$$K^{abcd}_{ijkl} = 0 \ \text{ unless } \ (a+b+c, b+2c+d) = (i+j+k, j+2k+l). \qquad (5.65)$$

This property will be referred to as *weight conservation*. It may also be rephrased as

$$[K, z^{\mathbf{h}} \otimes z^{\mathbf{h}} \otimes z^{\mathbf{h}} \otimes 1] = [K, 1 \otimes z^{\mathbf{h}} \otimes z^{2\mathbf{h}} \otimes z^{\mathbf{h}}] = 0, \qquad (5.66)$$

where $\mathbf{h}$ is the number operator (3.14) and z is a non-zero parameter. The recursion relations in Example 5.6 are closed among the elements satisfying (5.65).

Let us introduce the type C analogue of (3.58) as

$$\mathcal{D}_C = D_{q^2} \otimes D_q \otimes D_{q^2} \otimes D_q, \qquad (5.67)$$

where D_q is defined by (3.15).

[3] The result for $\pi_{2121}(\Delta(Y))$ also appears in (5.123).

Lemma 5.7 *The transposed representations are related to the original ones as*

$$^t(\pi_{2121}(\Delta(t_{ij}))) = \mathcal{D}_C \pi_{2121}(\tilde{\Delta}(t_{j'i'}))\mathcal{D}_C^{-1}|_{\mu_1\to-\mu_1}, \tag{5.68}$$

$$^t(\pi_{2121}(\tilde{\Delta}(t_{ij}))) = \mathcal{D}_C \pi_{2121}(\Delta(t_{j'i'}))\mathcal{D}_C^{-1}|_{\mu_1\to-\mu_1} \tag{5.69}$$

for $i, j \in \{1, 2, 3, 4\}$*, where* $i' = 5 - i$*.*

Proof By using $^t(\mathbf{A}^{\pm}) = D_{q^2}\mathbf{A}^{\mp}D_{q^2}^{-1}$, $^t\mathbf{K} = D_{q^2}\mathbf{K}D_{q^2}^{-1}$ and similar formulas in the proof of Lemma 3.6, we get

$$^t\pi_1(t_{ij}) = D_q\pi_1(t_{j'i'})D_q^{-1}|_{\mu_1\to-\mu_1}, \tag{5.70}$$

$$^t\pi_2(t_{ij}) = D_{q^2}\pi_2(t_{j'i'})D_{q^2}^{-1} \tag{5.71}$$

for the fundamental representations (5.24). The claim can be verified by applying this to the definitions (5.39) and (5.40). □

Proposition 5.8

$$K^{-1} = K, \tag{5.72}$$

$$^tK = \mathcal{D}_C K \mathcal{D}_C^{-1}. \tag{5.73}$$

Proof These properties are proved by invoking the uniqueness of the intertwiner satisfying (5.37) and (5.38). Set $X_{ij} = \kappa_i^{-1}\kappa_j t_{ji}$, where $(\kappa_1, \kappa_2, \kappa_3, \kappa_4) = (1, -q^{-1}\mu_1^2, q^{-3}\mu_1^2\mu_2^2, -q^{-4}\mu_1^4\mu_2^2)$. Then it is easy to check

$$\pi_{2121}(\Delta(X_{ij})) = \pi_{2121}(\tilde{\Delta}(t_{ij})), \quad \pi_{2121}(\tilde{\Delta}(X_{ij})) = \pi_{2121}(\Delta(t_{ij})). \tag{5.74}$$

From this, comparison of the two choices $f = t_{ij}$ and $f = X_{ij}$ in (5.37) shows that K and K^{-1} satisfy the same set of intertwining relations. The normalization condition (5.38) is also invariant under the exchange $K \leftrightarrow K^{-1}$, hence (5.72) follows. To show (5.73), take the transpose of (5.37) and replace μ_1 by $-\mu_1$. From Lemma 5.7 and the fact that (5.37) is actually independent of μ_1, we find that $\mathcal{D}_C^{-1}{}^tK\mathcal{D}_C$ again satisfies (5.37). The normalization condition (5.38) is also invariant under the exchange $K \leftrightarrow \mathcal{D}_C^{-1}{}^tK\mathcal{D}_C$, hence (5.73) follows. □

In terms of the matrix elements, the property (5.73) is rephrased as

$$K_{ijkl}^{abcd} = \frac{(q^4)_i(q^2)_j(q^4)_k(q^2)_l}{(q^4)_a(q^2)_b(q^4)_c(q^2)_d}K_{abcd}^{ijkl}, \tag{5.75}$$

where $(q^2)_m = (q^2; q^2)_m$ and $(q^4)_m = (q^4; q^4)_m$ according to (3.65).

5.5 Explicit Formula for 3D K

In this section we present explicit formulas of the matrix elements K^{abcd}_{ijkl} (5.57) of the intertwiner K characterized by (5.37) and (5.38). We use the notation (3.65).

Theorem 5.9

$$K^{abcd}_{ijkl} = \delta^{a+b+c}_{i+j+k}\,\delta^{b+2c+d}_{j+2k+l}\,\frac{(q^4)_i}{(q^4)_a}\sum_{\alpha,\beta,\gamma\geq 0}\frac{(-1)^{\alpha+\gamma}}{(q^4)_{c-\beta}}q^{\phi_1}$$
$$\times K^{i,j+k-\alpha-\beta-\gamma,0,k+l-\alpha-\beta-\gamma}_{a,b+c-\alpha-\beta-\gamma,0,c+d-\alpha-\beta-\gamma}\left\{\begin{matrix} k,c-\beta,j+k-\alpha-\beta,k+l-\alpha-\beta \\ \alpha,\beta,\gamma,b-\alpha,d-\alpha,k-\alpha-\beta,c-\beta-\gamma\end{matrix}\right\}_{q^2}, \tag{5.76}$$

$$\phi_1 = \alpha(\alpha+2c-2\beta-1)+(2\beta-c)(b+c+d)+\gamma(\gamma-1)-k(j+k+l), \tag{5.77}$$

where the special case $K^{\bullet,\bullet,0,\bullet}_{\bullet,\bullet,0,\bullet}$ *appearing in the sum is given by*

$$K^{ab0d}_{ij0l} = \delta^{a+b}_{i+j}\,\delta^{b+d}_{j+l}\sum_{\lambda\geq 0}(-1)^{b+\lambda}\frac{(q^4)_{a+\lambda}}{(q^4)_a}q^{\phi_2}\binom{l}{\lambda}_{q^2}\binom{j}{b-\lambda}_{q^2}, \tag{5.78}$$

$$\phi_2 = (i+a+1)(b+l-2\lambda)+b-l, \tag{5.79}$$

where the sum is actually restricted to $\max(0,b-j)\leq\lambda\leq\min(l,b)$.

By the definition (3.65), all the sums appearing in the theorem and the proof below are actually *finite* ones by the non-vanishing condition that all the suffixes of K and the entries of the symbols $\left\{\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\right\}_{q^2}$ and $\left(\begin{smallmatrix}\cdots\\ \cdots\end{smallmatrix}\right)_{q^2}$ are non-negative.

Proof First we reduce K^{abcd}_{ijkl} to $k=0$ by means of (5.62):

$$K^{abcd}_{ijkl} = q^{-j-l-1}\Big(-K^{a,b-1,c,d-1}_{i,j,k-1,l}+K^{a,b,c,d}_{i,j+1,k-1,l+1}+q^{b+d+1}K^{a,b,c-1,d}_{i,j,k-1,l}\Big). \tag{5.80}$$

This can be fitted to a recursion relation of q^2-trinomial coefficients. The solution reads as

$$K^{abcd}_{ijkl} = \delta^{a+b+c}_{i+j+k}\,\delta^{b+2c+d}_{j+2k+l}\sum_{\alpha,\beta\geq 0}\left\{\begin{matrix} k \\ \alpha,\beta,k-\alpha-\beta\end{matrix}\right\}_{q^2}(-1)^{\alpha}$$
$$\times\, q^{(\alpha+\beta-k)(\alpha+\beta+k-1)+(b+d-2\alpha+1)\beta-(j+l+1)k}K^{a,b-\alpha,c-\beta,d-\alpha}_{i,j+k-\alpha-\beta,0,l+k-\alpha-\beta}, \tag{5.81}$$

where the sum is actually finite. Second we reduce c by means of $(5.58)|_{k=0}$:

$$K^{abcd}_{ij0l} = \frac{q^{-b-d-1}}{1-q^{4c}}\Big((1-q^{2b+2})(1-q^{2d+2})K^{a,b+1,c-1,d+1}_{i,j,0,l} - (1-q^{2j})(1-q^{2l})K^{a,b,c-1,d}_{i,j-1,0,l-1}\Big). \tag{5.82}$$

By considering the combination $\frac{(q^4)_c(q^2)_b(q^2)_d}{(q^2)_j(q^2)_l}K^{abcd}_{ij0l}$, this is fitted with a recursion relation of q^2-binomial coefficients. The solution reads as

$$K^{abcd}_{ij0l} = \delta^{a+b+c}_{i+j}\delta^{b+2c+d}_{j+l}\frac{q^{-(b+d+1)c}}{(q^4)_c}\begin{Bmatrix} j,l \\ b,d \end{Bmatrix}_{q^2}\sum_{\gamma\ge 0}\begin{Bmatrix} c, b+c-\gamma, d+c-\gamma \\ \gamma, c-\gamma, j-\gamma, l-\gamma \end{Bmatrix}_{q^2} \times (-1)^\gamma q^{(\gamma-c)(\gamma+c-1)}K^{a,b+c-\gamma,0,d+c-\gamma}_{i,j-\gamma,0,l-\gamma}, \tag{5.83}$$

where the sum is actually finite. Combining (5.81) and (5.83), we obtain

$$K^{abcd}_{ijkl} = \delta^{a+b+c}_{i+j+k}\delta^{b+2c+d}_{j+2k+l}\sum_{\alpha,\beta,\gamma\ge 0}\frac{(-1)^{\alpha+\gamma}}{(q^4)_{c-\beta}}q^{\phi_1}K^{a,b+c-\alpha-\beta-\gamma,0,d+c-\alpha-\beta-\gamma}_{i,j+k-\alpha-\beta-\gamma,0,l+k-\alpha-\beta-\gamma}$$
$$\times\begin{Bmatrix} k, c-\beta, j+k-\alpha-\beta, l+k-\alpha-\beta, b+c-\alpha-\beta-\gamma, d+c-\alpha-\beta-\gamma \\ \alpha,\beta,\gamma, b-\alpha, d-\alpha, k-\alpha-\beta, c-\beta-\gamma, j+k-\alpha-\beta-\gamma, l+k-\alpha-\beta-\gamma \end{Bmatrix}_{q^2}, \tag{5.84}$$

where ϕ_1 is given in (5.77).

Next we reduce d and l in K^{ab0d}_{ij0l} to 0, keeping the already attained two zeros. Such recursion relations are available from $(5.60)|_{k=c=0}$ and $(5.61)|_{k=c=0}$:

$$K^{ab0d}_{ij0l} = \frac{1}{1-q^{2d}}\Big(q^{2i+j-b}(1-q^{2l})K^{a,b,0,d-1}_{i,j,0,l-1} + q^{l-b}(1-q^{2j})K^{a,b,0,d-1}_{i+1,j-1,0,l}\Big), \tag{5.85}$$

$$K^{ab0d}_{ij0l} = q^{2a-j+b}K^{a,b,0,d-1}_{i,j,0,l-1} + q^{d-j}(1-q^{4a+4})K^{a+1,b-1,0,d}_{i,j,0,l-1}. \tag{5.86}$$

Note that either $(5.85)|_{l=0}$ or $(5.86)|_{d=0}$ leads to

$$K^{ij00}_{ij00} = (-1)^j q^{2(i+1)j}$$

with the help of the conservation law (5.65) and $K^{0000}_{0000} = 1$. It is easy to solve (5.85) and (5.86) with the above initial condition. The solution is given by (5.78). It fulfills the symmetry

$$K^{ab0d}_{ij0l} = \frac{(q^4)_i}{(q^4)_a}\begin{Bmatrix} j,l \\ b,d \end{Bmatrix}_{q^2} K^{ij0l}_{ab0d} \tag{5.87}$$

in accordance with (5.75). This is seen by replacing λ with $i-a+\lambda$ in (5.78). Finally, (5.76) is obtained by applying (5.87) in (5.84). Although this last step is optional, it cancels four entries in the symbol $\begin{Bmatrix} \cdots \\ \cdots \end{Bmatrix}_{q^2}$ in (5.84). □

Example 5.10 The following is the list of all the non-zero K^{abcd}_{2110}.

$$\begin{aligned}
K^{1300}_{2110} &= q^8(1-q^8),\\
K^{2110}_{2110} &= -q^4(1-q^8+q^{14}),\\
K^{2201}_{2110} &= -q^6(1+q^2)(1-q^2+q^4-q^6-q^{10}),\\
K^{3011}_{2110} &= 1-q^8+q^{14},\\
K^{3102}_{2110} &= -q^{10}(1-q+q^2)(1+q+q^2),\\
K^{4003}_{2110} &= q^4.
\end{aligned}$$

Remark 5.11 By counting the number of factors of the form $(1-q^n)$ mod 2 in Theorem 5.9 for $K^{abcd}_{ijkl}|_{q\to q^{-1}}$, it is easy to see $(-1)^b K^{abcd}_{ijkl} \geq 0$ for $q > 1$. This leads to a *positive* solution to the 3D reflection equation. See also Remark 5.17.

Let us proceed to another formula analogous to Theorem 3.18. Define a family of polynomials $\{Q_{b,c}(x,y,z,w) \mid b, c \in \mathbb{Z}_{\geq 0}\}$ in four variables x, y, z, w including q as a parameter by the initial condition $Q_{0,0}(x,y,z,w) = 1$ and the recursion relations decreasing b and c as

$$\begin{aligned}
Q_{b,c}(x,y,z,w) &= wy(z-1)q^{4b+8c-4}Q_{b-1,c}(x,y,q^{-4}z,w)\\
&\quad + wx(y-1)yzq^{4b+4c-4}Q_{b-1,c}(x,q^{-2}y,z,w)\\
&\quad + (w-1)(y-1)q^{6b+8c-6}Q_{b-1,c}(x,q^{-2}y,z,q^{-2}w)\\
&\quad + w(x-1)y(z-1)q^{4b+8c-4}Q_{b-1,c}(q^{-4}x,q^2y,q^{-4}z,w)\\
&\quad + (w-1)(x-1)yq^{6b+8c-6}Q_{b-1,c}(q^{-4}x,y,z,q^{-2}w), \qquad (5.88)\\
Q_{b,c}(x,y,z,w) &= -w^2y(z-1)zq^{4b+8c-8}Q_{b,c-1}(x,y,q^{-4}z,w)\\
&\quad + wx(y-1)zq^{4b+4c-6}(q^{2(b+2c)}-q^2wyz)Q_{b,c-1}(x,q^{-2}y,z,w)\\
&\quad - (w-1)w(y-1)zq^{6b+8c-8}Q_{b,c-1}(x,q^{-2}y,z,q^{-2}w)\\
&\quad + w(x-1)(z-1)q^{4b+8c-10}(q^{2(b+2c)}-q^2wyz)Q_{b,c-1}(q^{-4}x,q^2y,q^{-4}z,w)\\
&\quad + (w-1)(x-1)q^{6b+8c-10}(q^{2(b+2c)}-q^2wyz)Q_{b,c-1}(q^{-4}x,y,z,q^{-2}w). \qquad (5.89)
\end{aligned}$$

From these equations we see $Q_{b,c}(x,y,z,w) \in \mathbb{Z}[q^2,q^{-2},x,y,z,w]$.

Example 5.12 Here are a few examples of $Q_{b,c}(x,y,z,w)$'s with small b, c:

$$\begin{aligned}
Q_{1,0}(x,y,z,w) &= wxy^2z - w - xy + 1,\\
Q_{0,1}(x,y,z,w) &= q^2(wxyz - wz - x + 1) - wz\left(wxy^2z - w - xy + 1\right),\\
Q_{2,0}(x,y,z,w) &= q^6(w-1)(xy-1) + q^4\left(-w^2xy^2z + w^2 + wxy - w + xy^2 - xy\right)\\
&\quad - q^2xy^2(wxyz - wz - x + 1) + wxy^2z\left(wxy^2z - w - xy + 1\right)\\
&\quad - q^4(w - w^2 + xy - xyw - xy^2 + xy^2zw^2),
\end{aligned}$$

$$
\begin{aligned}
Q_{1,1}(x,y,z,w) &= q^{10}(w-1)(x-1) - q^{8}(w-1)wz(xy-1)\\
&+ q^{6}\left(-w^2xyz + w^2z - wx^2y^2z + 2wxyz - wz + x^2y - xy\right)\\
&+ q^{4}wz\left(w^2xy^2z - w^2 + wx^2y^3z - wxy^2z - wxy + w - x^2y^2 + xy\right)\\
&+ q^{2}wxy^2z(wxyz - wz - x + 1) - w^2xy^2z^2\left(wxy^2z - w - xy + 1\right).
\end{aligned}
$$

As these examples indicate, $Q_{b,c}(x,y,z,w)$ is actually a polynomial in q^2. This is indeed the case. See the remarks after Theorem 5.15.

Theorem 5.13 *Matrix elements of the 3D K are expressed as follows:*

$$
K^{abcd}_{ijkl} = \delta^{a+b+c}_{i+j+k}\delta^{b+2c+d}_{j+2k+l}\frac{q^{\phi_K-\varphi_{b,c}}}{(q^2)_b(q^4)_c}Q_{b,c}(q^{4i},q^{2j},q^{4k},q^{2l}), \tag{5.90}
$$

$$
\phi_K = (a-k)(d-j) + (b-l)(c-i) - 2(b-j)(c-k), \tag{5.91}
$$

$$
\varphi_{b,c} = 3b(b-1) + 2c(3c-2) + 8bc. \tag{5.92}
$$

Proof Substituting (5.90) into (5.58) and (5.59), we find that they are translated into

$$
\begin{aligned}
& yq^{-2(4b+6c+1)}Q_{b,c+1}(x,y,z,w) + q^{-2(4b+6c+1)}(wyz - q^{2b+4c+2})Q_{b+1,c}(x,y,z,w)\\
&+ (w-1)(y-1)Q_{b,c}(x,q^{-2}y,z,q^{-2}w) + wy(z-1)q^{-2b}Q_{b,c}(x,y,q^{-4}z,w) = 0,\\
&- q^{-2(4b+6c+1)}Q_{b,c+1}(x,y,z,w) - wzq^{-2(4b+6c+1)}Q_{b+1,c}(x,y,z,w)\\
&+ wx(y-1)zq^{-2(b+2c)}Q_{b,c}(x,q^{-2}y,z,w) + (w-1)(x-1)Q_{b,c}(q^{-4}x,y,z,q^{-2}w)\\
&+ w(x-1)(z-1)q^{-2b}Q_{b,c}(q^{-4}x,q^{2}y,q^{-4}z,w) = 0.
\end{aligned}
$$

The recursion relations (5.88) and (5.89) can be derived by combining the two equations. The normalization condition $K^{0000}_{0000} = 1$ also matches $Q_{0,0}(x,y,z,w) = 1$. □

The power ϕ_K in (5.91) is invariant under the exchange $(a,b,c,d) \leftrightarrow (i,j,k,l)$. Therefore (5.75) implies another general formula:

$$
K^{abcd}_{ijkl} = \delta^{a+b+c}_{i+j+k}\delta^{b+2c+d}_{j+2k+l}\, q^{\phi_K-\varphi_{j,k}} \begin{Bmatrix} l \\ b,d \end{Bmatrix}_{q^2} \begin{Bmatrix} i \\ a,c \end{Bmatrix}_{q^4} Q_{j,k}(q^{4a},q^{2b},q^{4c},q^{2d}). \tag{5.93}
$$

The formulas (5.90) and (5.93) with the recursive definition of $Q_{b,c}(x,y,z,w)$ by (5.88) and (5.89) are no less convenient for computer programming than the explicit formulas in Theorem 5.9 and the forthcoming Theorem 5.15.

Example 5.14 According to (5.90), Example 5.10 is expressed by various special values of $Q_{1,0}(x,y,z,w) = wxy^2z - w - xy + 1$ as

$$K^{3102}_{1300} = \frac{Q_{1,0}(q^4,q^6,1,1)}{q^4(1-q^2)}, \quad K^{3102}_{2110} = \frac{Q_{1,0}(q^8,q^2,q^4,1)}{1-q^2},$$
$$K^{3102}_{2201} = \frac{Q_{1,0}(q^8,q^4,1,q^2)}{1-q^2}, \quad K^{3102}_{3011} = \frac{q^6 Q_{1,0}(q^{12},1,q^4,q^2)}{1-q^2},$$
$$K^{3102}_{3102} = \frac{q^6 Q_{1,0}(q^{12},q^2,1,q^4)}{1-q^2}, \quad K^{3102}_{4003} = \frac{q^{14} Q_{1,0}(q^{16},1,1,q^6)}{1-q^2}.$$

On the other hand, according to (5.93), they are also expressed in terms of $Q_{b,c}(q^{12},q^2,1,q^4)$ with various b, c. For instance, one has

$$K^{3102}_{2110} = \frac{q^{-10}Q_{1,1}(q^{12},q^2,1,q^4)}{(1-q^2)^2(1-q^4)(1-q^{12})}, \quad K^{3102}_{3011} = \frac{q^4 Q_{0,1}(q^{12},q^2,1,q^4)}{(1-q^2)(1-q^4)}.$$

To present a closed formula for the polynomial $Q_{b,c}(x,y,z,w)$, we introduce

$$\mathscr{S}_{b,c} = \{(r,s,t,u) \in \mathbb{Z}^4_{\geq 0} \mid \min(u-t, 2r-s, b-s+2t-u, c-r+s-t) \geq 0\}, \tag{5.94}$$

which is a finite subset of $\{(r,s,t,u) \in \mathbb{Z}^4_{\geq 0} \mid s/2 \leq r \leq b+c,\ t \leq u \leq b+2c\}$.

Theorem 5.15 *Let $\varphi_{b,c}$ be as in (5.92). The following formula is valid:*

$$Q_{b,c}(x,y,z,w) = q^{\varphi_{b,c}} \sum_{(r,s,t,u)\in\mathscr{S}_{b,c}} (-1)^{r+u} q^{\phi_Q - \psi_{r,s}} C^{b,c}_{r,s,t,u} x^r y^s z^t w^u, \tag{5.95}$$

$$C^{b,c}_{r,s,t,u} = \frac{(-1)^s q^{\psi_{r,s}} \left\{ {b,u-t \atop b+2t-s-u, 2r-s} \right\}_{q^2}}{(q^4)_r (q^4)_{u-t} (q^4)_{c-r+s-t}} \sum_{(\alpha,\beta,\gamma)\in\mathbb{Z}^3_{\geq 0}} (-1)^{\beta+\gamma} q^{\phi_C}\, \Xi_{\alpha,\beta,\gamma}, \tag{5.96}$$

$$\Xi_{\alpha,\beta,\gamma} = \left\{ {b-s+t-\alpha, 2r-s+\beta \atop \alpha,\beta,\gamma,u-t-\alpha,t-\beta,b-s-\alpha+\beta,s-\beta-\gamma} \right\}_{q^2} \left\{ {c+s-r-\beta, c+\gamma \atop c-r+\gamma} \right\}_{q^4}, \tag{5.97}$$

$$\phi_Q = (s-2t+u)^2 + 2r(r+2t+1) - (2b-1)(s+u) - 4c(r+t), \tag{5.98}$$

$$\phi_C = \alpha(\alpha+1+2t) + \beta(\beta-1-2\alpha+2b-4r) + \gamma(\gamma-1-4r), \tag{5.99}$$

$$\psi_{r,s} = s(4r-s+1). \tag{5.100}$$

In view of the support property of the symbols in (5.97), (see (3.65)), the sum (5.96) is limited to those α, β, γ such that all the lower entries in (5.97) are non-negative, which also ensures that all the upper entries are so. Thus it ranges over a finite subset of $\{(\alpha,\beta,\gamma) \in \mathbb{Z}^3_{\geq 0} \mid \alpha \leq u-t,\ \beta \leq t,\ \gamma \leq s\}$. The dependence of $\Xi_{\alpha,\beta,\gamma}$ on (b,c,r,s,t,u) has been suppressed in the notation for simplicity.

The proof of Theorem 5.15 is lengthy, hence omitted here. The details can be found in [88]. We quote a few additional results therein:

$$
\begin{aligned}
&Q_{b,c}(x,y,z,w)|_{q=0} = (-1)^c (xy^2)^{b+c-1}(zw)^{b+2c-1} Q_{1,0}(x,y,z,w),\\
&q^{-\varphi_{b,c}} Q_{b,c}(x,y,z,w)|_{q=0} = 1 - xy^{\delta^0_c} - wz^{\delta^0_b} + xwy^{\delta^0_c+\delta^1_{b+c}} z^{\delta^0_b+\delta^1_b\delta^0_c},\\
&Q_{b,c}(x,1,1,w) = (-1)^c x^{b+c} w^{b+2c} (x^{-1};q^4)_{b+c} (w^{-1};q^2)_{b+2c},\\
&Q_{b,c}(x,y,1,1) = (-1)^c (xy^2)^{b+c} (y^{-1};q^2)_{b+2c},\\
&Q_{b,c}(1,1,z,w) = (-1)^c (zw)^{b+2c} (z^{-1};q^4)_{b+c},
\end{aligned}
\tag{5.101}
$$

where $(b,c) \neq (0,0)$ in the first two relations. From them we see $Q_{b,c}(x,y,z,w) \in \mathbb{Z}[q^2,x,y,z,w]$ and that $\varphi_{b,c}$ in (5.92) gives the *exact* degree of $Q_{b,c}(x,y,z,w)$ as a polynomial in q. It is an interesting open problem whether $Q_{b,c}(x,y,z,w)$ admits a formula in terms of appropriately truncated generalized q-hypergeometric type series analogous to (3.75).

5.6 Solution to the 3D Reflection Equation

In Sects. 5.3 and 5.4, we have obtained three kinds of normalized intertwiners for the $A_q(C_n)$ modules corresponding to (5.28), (5.29) and (5.30). They are given by P in (3.23), $\Phi = RP_{13}$ and $\Psi = KP_{14}P_{23}$, respectively.

Theorem 5.16 *The intertwiners R and K satisfy the 3D reflection equation (4.3) in* $\mathrm{End}\big(\overset{1}{\mathcal{F}}_{q^2} \otimes \overset{2}{\mathcal{F}}_q \otimes \overset{3}{\mathcal{F}}_{q^2} \otimes \overset{4}{\mathcal{F}}_q \otimes \overset{5}{\mathcal{F}}_q \otimes \overset{6}{\mathcal{F}}_q \otimes \overset{7}{\mathcal{F}}_{q^2} \otimes \overset{8}{\mathcal{F}}_q \otimes \overset{9}{\mathcal{F}}_q\big)$.

Proof Consider $A_q(C_3)$ and let π_1, π_2, π_3 be the fundamental representations given in (5.19)–(5.22) which are also displayed in Example 5.4. The Weyl group $W(C_3)$ is generated by simple reflections s_1, s_2, s_3 with the relations

$$
s_i^2 = 1, \quad s_1 s_3 = s_3 s_1, \quad s_1 s_2 s_1 = s_2 s_1 s_2, \quad s_2 s_3 s_2 s_3 = s_3 s_2 s_3 s_2. \tag{5.102}
$$

According to Theorem 3.3, the equivalence of the tensor product representations $\pi_{13} \simeq \pi_{31}$, $\pi_{121} \simeq \pi_{212}$ and $\pi_{2323} \simeq \pi_{3232}$ are valid. Their intertwining relations are given by

$$
P \circ \pi_{13} = \pi_{31} \circ P, \quad \Phi \circ \pi_{121} = \pi_{212} \circ \Phi, \quad \Psi \circ \pi_{2323} = \pi_{3232} \circ \Psi. \tag{5.103}
$$

See Remark 3.4, (5.31) and (5.34).

Let $w_0 \in W(C_3)$ be the longest element. As a linear transformation on the root lattice, we know $w_0 = -1$. Pick the two reduced expressions, say,

$$
w_0 = s_1 s_2 s_3 s_1 s_2 s_1 s_3 s_2 s_3 = s_3 s_2 s_3 s_1 s_2 s_1 s_3 s_2 s_1, \tag{5.104}
$$

which are related by the reverse ordering. According to Theorem 3.3, we have the equivalence of the two irreducible representations of $A_q(C_3)$:

$$\pi_{123121323} \simeq \pi_{323121321}. \tag{5.105}$$

Following the transformations of the reduced expressions by the Coxeter relations (5.102), one can construct the intertwiner for (5.105) in two ways. In terms of the indices, they look as follows:

$$\begin{array}{llll}
123\underline{121}323 & \Phi_{456} & 12\underline{31}2\underline{13}23 & P_{34}P_{67} \\
12321\underline{2323} & \Psi_{6789} & \underline{121}323123 & \Phi_{123} \\
1232\underline{13}232 & P_{56} & 21\underline{2323}123 & \Psi_{3456} \\
1\underline{2323}1232 & \Psi_{2345} & 21323\underline{212}3 & \Phi^{-1}_{678} \\
1323\underline{212}32 & \Phi^{-1}_{567} & 2\underline{13}2\underline{31}2\underline{13} & P_{23}P_{56}P_{89} \\
\underline{13}2\underline{31}2\underline{13}2 & P_{12}P_{45}P_{78} & 23\underline{121}3231 & \Phi_{345} \\
3\underline{121}32312 & \Phi_{234} & 2321\underline{2323}1 & \Psi_{5678} \\
321\underline{2323}12 & \Psi_{4567} & 232\underline{13}2321 & P_{45} \\
321323\underline{212} & \Phi^{-1}_{789} & \underline{2323}12321 & \Psi_{1234} \\
32\underline{13}2\underline{31}21 & P_{34}P_{67} & 323\underline{212}321 & \Phi^{-1}_{456} \\
323121321 & & 323121321 &
\end{array} \tag{5.106}$$

The underlines[4] indicate the components to which the intertwiners given on the right are to be applied as in (3.93). Since the intertwiner for (5.105) is unique up to normalization and Φ and Ψ are normalized as (3.29) and (5.35), the following equality is valid:

$$\begin{aligned}
&P_{34}P_{67}\Phi^{-1}_{789}\Psi_{4567}\Phi_{234}P_{12}P_{45}P_{78}\Phi^{-1}_{567}\Psi_{2345}P_{56}\Psi_{6789}\Phi_{456} \\
&= \Phi^{-1}_{456}\Psi_{1234}P_{45}\Psi_{5678}\Phi_{345}P_{23}P_{56}P_{89}\Phi^{-1}_{678}\Psi_{3456}\Phi_{123}P_{34}P_{67}.
\end{aligned}$$

Substitute $\Psi_{ijkl} = K_{ijkl}P_{il}P_{jk}$(5.36) and $\Phi_{ijk} = R_{ijk}P_{ik}$(3.30) and use $\Psi^{-1}_{ijk} = P_{ik}R^{-1}_{ijk} = R_{ijk}P_{ik}$due to Proposition 3.7. The result reads as

$$\begin{aligned}
&P_{34}P_{67}R_{789}P_{79}K_{4567}P_{47}P_{56}R_{234}P_{24}P_{12}P_{45}P_{78} \\
&\quad \times R_{567}P_{57}K_{2345}P_{25}P_{34}P_{56}K_{6789}P_{69}P_{78}R_{456}P_{46} \\
&= R_{456}P_{46}K_{1234}P_{14}P_{23}P_{45}K_{5678}P_{58}P_{67}R_{345}P_{35} \\
&\quad \times P_{23}P_{56}P_{89}R_{678}P_{68}K_{3456}P_{36}P_{45}R_{123}P_{13}P_{34}P_{67}.
\end{aligned}$$

[4] Their pattern exactly corresponds to (4.19) and (4.20).

Sending all the P_{ij}'s to the right we find

$$R_{689}K_{3579}R_{249}R_{258}K_{1478}K_{1236}R_{456}\sigma = R_{456}K_{1236}K_{1478}R_{258}R_{249}K_{3579}R_{689}\sigma',$$

where $\sigma = P_{14}P_{67}P_{79}P_{47}P_{56}P_{24}P_{12}P_{45}P_{78}P_{57}P_{25}P_{34}P_{56}P_{69}P_{78}P_{46}$ and $\sigma' = P_{46}P_{14}P_{23}P_{45}P_{58}P_{67}P_{35}P_{23}P_{56}P_{89}P_{68}P_{36}P_{45}P_{13}P_{34}P_{67}$. Since σ and σ' are both equal to the reverse ordering of the 9-fold tensor product (the longest element in the context of the symmetric group $\mathfrak{S}_9$), the 3D reflection equation (4.3) follows. □

There are 42 reduced expressions for the longest element w_0. Starting from any one of them, one can derive similar equations to the 3D reflection equation (4.3). They are shown to be equivalent by using $R_{ijk} = R_{ijk}^{-1} = P_{ik}R_{ijk}P_{ik}$ and $K = K^{-1}$ reflecting the fact that any reduced expression is transformed to any other by the Coxeter relation [119].

Let $\mathbf{h}$ be the number operator (3.14) either on $\mathcal{F}_q$ and $\mathcal{F}_{q^2}$. As usual we let $\mathbf{h}_i$ denote the one acting only on the ith tensor component from the left.

Remark 5.17 The operators $(-1)^{\mathbf{h}_2}R_{123}$ and $(-1)^{\mathbf{h}_2}K_{1234}$ also satisfy the 3D reflection equation. In view of Remarks 3.13 and 5.11, they yield a positive solution of the 3D reflection equation in the regime $q > 1$.

Proof From Remark 3.14 we know that $(-1)^{\mathbf{h}_2}R_{123}$ also satisfies the tetrahedron equation. As for the 3D reflection equation, we multiply the LHS of (4.3) by $(-1)^{\mathbf{h}_2+\mathbf{h}_5+\mathbf{h}_8}$ and use (3.49) to get

$$\begin{aligned}
&(-1)^{\mathbf{h}_2+\mathbf{h}_5+\mathbf{h}_8}R_{689}K_{3579}R_{249}R_{258}K_{1478}K_{1236}R_{456}\\
&= (-1)^{\mathbf{h}_8}R_{689}(-1)^{\mathbf{h}_5}K_{3579}(-1)^{\mathbf{h}_4}(-1)^{\mathbf{h}_2+\mathbf{h}_4}R_{249}R_{258}K_{1478}K_{1236}R_{456}\\
&= (-1)^{\mathbf{h}_8}R_{689}(-1)^{\mathbf{h}_5}K_{3579}(-1)^{\mathbf{h}_4}R_{249}(-1)^{\mathbf{h}_5}(-1)^{\mathbf{h}_2+\mathbf{h}_5}(-1)^{\mathbf{h}_4}R_{258}K_{1478}K_{1236}R_{456}\\
&= (-1)^{\mathbf{h}_8}R_{689}(-1)^{\mathbf{h}_5}K_{3579}(-1)^{\mathbf{h}_4}R_{249}(-1)^{\mathbf{h}_5}R_{258}(-1)^{\mathbf{h}_4}K_{1478}(-1)^{\mathbf{h}_2}K_{1236}(-1)^{\mathbf{h}_5}R_{456}.
\end{aligned}$$

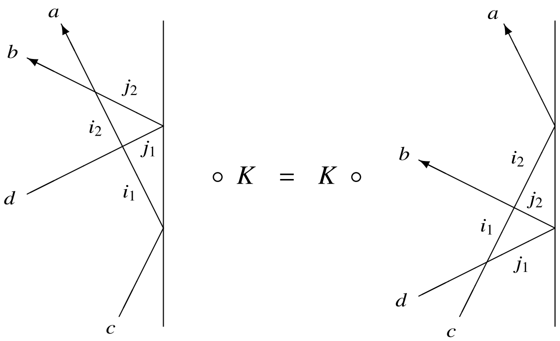
The RHS is similar. □

5.7 Solution to the Quantized Reflection Equation

Recall the quantized reflection equation introduced in (4.12):

$$\sum_{i_1,i_2,j_1,j_2}(L^{ab}_{i_2j_2}\otimes G^{j_2}_{j_1}\otimes L^{j_1i_2}_{d\,i_1}\otimes G^{i_1}_{c})K = \sum_{i_1,i_2,j_1,j_2}K(L^{j_1i_1}_{d\,c}\otimes G^{j_2}_{j_1}\otimes L^{i_2b}_{i_1j_2}\otimes G^{a}_{i_2}). \tag{5.107}$$

Graphically it looks as follows:



(5.108)

In the setting of Sect. 4.4, choose $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1, \mathcal{F} = \mathcal{F}_q$ and $\mathcal{F}' = \mathcal{F}_{q^2}$. Thus we have

$$L^{ab}_{ij} \in \mathrm{End}(\mathcal{F}_{q^2}),\ \ G^k_i \in \mathrm{End}(\mathcal{F}_q),\ \ K \in \mathrm{End}(\mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q). \tag{5.109}$$

Let us take L^{ab}_{ij} to be the Osc_{q^2}-valued six-vertex model. They are obtained by replacing $q, \mathbf{a}^{\pm}, \mathbf{k}$ by $q^2, \mathbf{A}^{\pm}, \mathbf{K}$ respectively in Fig. 3.2 as follows (Fig. 5.1).

As for G^k_j, we specify it along the diagram (a) in (4.11) as in Fig. 5.2.

In formulas these definitions of L^{ab}_{ij} and G^k_j are summarized as

$$\begin{aligned} L = {} & E_{00} \otimes E_{00} \otimes 1 + E_{11} \otimes E_{11} \otimes 1 - E_{11} \otimes E_{00} \otimes \mathbf{K} \\ & - q^2 E_{00} \otimes E_{11} \otimes \mathbf{K} + E_{10} \otimes E_{01} \otimes \mathbf{A}^+ + E_{01} \otimes E_{10} \otimes \mathbf{A}^-, \end{aligned} \tag{5.110}$$

$$G = E_{00} \otimes \mathbf{a}^+ + E_{10} \otimes \mathbf{k} - q E_{01} \otimes \mathbf{k} + E_{11} \otimes \mathbf{a}^-. \tag{5.111}$$

Theorem 5.18 *The 3D K characterized in Sect. 5.4 satisfies the quantized reflection equation (5.107) for L and G given in (5.110) and (5.111).*

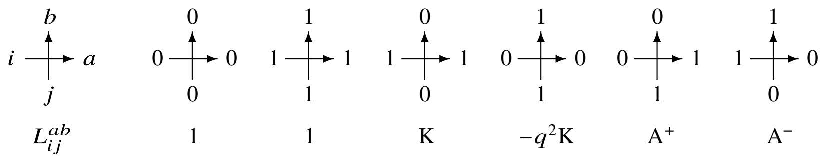


Fig. 5.1 3D L entering the quantized reflection equation (5.107) is taken to be the Osc_{q^2}-valued six-vertex model. It corresponds to replacing q with q^2 in Fig. 3.2. For the notations $\mathbf{a}^{\pm}, \mathbf{k} \in \mathrm{Osc}_\mathrm{q}$ and $\mathbf{A}^{\pm}, \mathbf{K} \in \mathrm{Osc}_{\mathrm{q}^2}$, see (3.13) and (5.16)

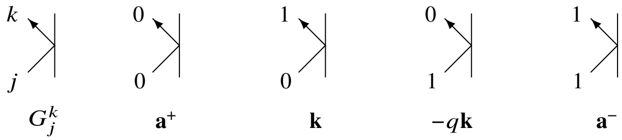


Fig. 5.2 The operator G^k_j entering (5.107)

Proof We show that the quantized reflection equation (5.107) for K coincides with the intertwining relation (5.37). Let us write the former equation as

$$\mathcal{M}^{ab}_{cd} K = K \tilde{\mathcal{M}}^{ab}_{cd}, \tag{5.112}$$

$$\mathcal{M}^{ab}_{cd} = \sum_{i_1,i_2,j_1,j_2} L^{ab}_{i_2 j_2} \otimes G^{j_2}_{j_1} \otimes L^{j_1 i_2}_{d\, i_1} \otimes G^{i_1}_{c}, \tag{5.113}$$

$$\tilde{\mathcal{M}}^{ab}_{cd} = \sum_{i_1,i_2,j_1,j_2} L^{j_1 i_1}_{d\, c} \otimes G^{j_2}_{j_1} \otimes L^{i_2 b}_{i_1 j_2} \otimes G^{a}_{i_2}. \tag{5.114}$$

Then one can directly check

$$\mathcal{M}^{ab}_{cd} = \xi_i^{-1}\xi_j \pi_{2121}(\Delta(t_{ij})), \quad \tilde{\mathcal{M}}^{ab}_{cd} = \xi_i^{-1}\xi_j \pi_{2121}(\tilde{\Delta}(t_{ij})), \tag{5.115}$$

$$i = \gamma_{ab}, \ \ j = \gamma_{cd}, \quad (\gamma_{11}, \gamma_{01}, \gamma_{10}, \gamma_{00}) = (1, 2, 3, 4), \tag{5.116}$$

$$(\xi_1, \xi_2, \xi_3, \xi_4) = (1, \mu_1, -q^{-2}\mu_1\mu_2, q^{-2}\mu_1^2\mu_2), \tag{5.117}$$

where the concrete form (5.117) does not matter for the proof. The 16 choices of the external lines $a, b, c, d \in \{0, 1\}$ are in one-to-one correspondence with the 16 generators t_{ij} with $i, j \in \{1, 2, 3, 4\}$. □

Remark 5.19 As an equation on the 3D K, the quantized reflection equation (5.107) is invariant under the simultaneous changes $L^{ab}_{ij} \to \alpha^{a-j} L^{ab}_{ij}$ and $G^k_j \to \beta^{k-j} G^k_j$ by parameters α and β, endowing the two sides with a common overall factor $\alpha^{d-b}\beta^{a+b-c-d}$. Thus the gauge of L and G in such a sense can be chosen arbitrarily. A similar fact holds also in the quantized Yang–Baxter equation in Remark 3.23.

In Sect. 6.4, Theorem 5.18 will be applied to the proof of the 3D reflection equation of type B. In Chaps. 15 and 16, it will also be applied to a matrix product construction of reflection K matrices in 2D.

5.8 Further Aspects of 3D K

5.8.1 Boundary Vector

Let us introduce boundary vectors

$$|\eta_s\rangle = \sum_{m\geq 0} \frac{|sm\rangle}{(q^{s^2})_m} \in \mathcal{F}_q \quad (s=1,2), \tag{5.118}$$

$$|\chi_s\rangle = \sum_{m\geq 0} \frac{|sm\rangle}{(q^{2s^2})_m} \in \mathcal{F}_{q^2} \quad (s=1,2), \tag{5.119}$$

which are actually elements of a completion of the Fock spaces. The former is the same as (3.132) and the latter is obtained from it just by replacing q with q^2. Set

$$|\Xi_{r,k}\rangle = |\chi_r\rangle \otimes |\eta_k\rangle \otimes |\chi_r\rangle \otimes |\eta_k\rangle \qquad ((r,k) = (1,1),(1,2),(2,2)). \tag{5.120}$$

We also call them boundary vectors. Below we write $\pi_{2121}(\Delta(f))$ and $\pi_{2121}(\tilde{\Delta}(f))$ for $f \in A_q(C_2)$ simply as $\Delta(f)$ and $\tilde{\Delta}(f)$ to save space.

Lemma 5.20

$$1\otimes \mathbf{k}^2 \otimes \mathbf{K}^2 \otimes 1 = \Delta(\mu_1^{-4}\mu_2^{-2}t_{14}^2 - q^{-3}t_{42}t_{13}), \tag{5.121}$$

$$1\otimes \mathbf{k} \otimes \mathbf{K} \otimes \mathbf{k} = \Delta(-\mu_1^{-2}\mu_2^{-1}t_{14}) = \Delta(q^{-4}\mu_1^2\mu_2 t_{41}), \tag{5.122}$$

$$\mathbf{K}\otimes \mathbf{k}^2 \otimes \mathbf{K} \otimes 1 = \Delta(\mu_1^{-2}\mu_2^{-2}(q^{-1}t_{23}t_{14} - t_{24}t_{13})), \tag{5.123}$$

$$1\otimes \mathbf{k} \otimes \mathbf{K} \otimes \mathbf{a}^+ = \Delta(-q^{-3}\mu_1\mu_2 t_{42}), \tag{5.124}$$

$$1\otimes \mathbf{k} \otimes \mathbf{K} \otimes \mathbf{a}^- = \Delta(\mu_1^{-1}\mu_2^{-1}t_{13}), \tag{5.125}$$

$$\mathbf{A}^+\otimes \mathbf{k}^2 \otimes \mathbf{K} \otimes 1 = \Delta(-q^{-5}\mu_1^2\mu_2 t_{33}t_{41} - \mu_1^{-2}\mu_2^{-1}t_{34}t_{13}), \tag{5.126}$$

$$\mathbf{A}^-\otimes \mathbf{k}^2 \otimes \mathbf{K} \otimes 1 = \Delta(q\mu_1^{-2}\mu_2^{-1}t_{22}t_{14} + q^{-4}\mu_1^2\mu_2 t_{21}t_{42}), \tag{5.127}$$

$$1\otimes \mathbf{k}\,\mathbf{a}^+ \otimes \mathbf{K} \otimes 1 = \Delta(q\mu_1^{-1}\mu_2^{-1}t_{44}t_{13} + q^{-4}\mu_1^3\mu_2 t_{43}t_{41}), \tag{5.128}$$

$$1\otimes \mathbf{k}\,\mathbf{a}^- \otimes \mathbf{K} \otimes 1 = \Delta(\mu_1\mu_2(-q^{-3}t_{42}t_{11} + q^{-4}t_{41}t_{12})), \tag{5.129}$$

$$1\otimes \mathbf{k}^2 \otimes \mathbf{K}\mathbf{A}^+ \otimes 1 = \Delta(\mu_1^2\mu_2(-q^{-1}t_{44}t_{41} - q^{-2}t_{43}t_{42})), \tag{5.130}$$

$$1\otimes \mathbf{k}^2 \otimes \mathbf{K}\mathbf{A}^- \otimes 1 = \Delta(-q^{-5}\mu_1^2\mu_2 t_{41}t_{11} + q^{-2}\mu_1^{-2}\mu_2^{-1}t_{12}t_{13}). \tag{5.131}$$

Proof A direct calculation using (5.24). □

From Lemma 3.27, we know that the vector $|\eta_1\rangle$ (resp. $|\chi_1\rangle$) is characterized up to normalization by any one of the following three conditions in the left (resp. right) column:

$$(\mathbf{a}^+ - 1 + \mathbf{k})|\eta_1\rangle = 0, \qquad (\mathbf{A}^+ - 1 + \mathbf{K})|\chi_1\rangle = 0, \tag{5.132}$$

$$(\mathbf{a}^- - 1 - q\mathbf{k})|\eta_1\rangle = 0, \qquad (\mathbf{A}^- - 1 - q^2\mathbf{K})|\chi_1\rangle = 0, \tag{5.133}$$

$$(\mathbf{a}^+ - \mathbf{a}^- + (1+q)\mathbf{k})|\eta_1\rangle = 0, \qquad (\mathbf{A}^+ - \mathbf{A}^- + (1+q^2)\mathbf{K})|\chi_1\rangle = 0. \tag{5.134}$$

Up to normalization, the vectors $|\eta_2\rangle$ and $|\chi_2\rangle$ are characterized by

$$(\mathbf{a}^+ - \mathbf{a}^-)|\eta_2\rangle = 0, \qquad (\mathbf{A}^+ - \mathbf{A}^-)|\chi_2\rangle = 0. \tag{5.135}$$

Proposition 5.21 *The 3D K and the boundary vectors satisfy the following relations:*

$$K|\Xi_{r,k}\rangle = |\Xi_{r,k}\rangle \qquad ((r,k) = (1,1), (1,2), (2,2)). \tag{5.136}$$

Proof From (5.132)–(5.135), It suffices to show

$$((\mathbf{A}^+ - \mathbf{A}^-) \otimes \mathbf{k}^2 \otimes \mathbf{K} \otimes 1)K|\Xi_{2,2}\rangle = 0, \tag{5.137}$$

$$(1 \otimes \mathbf{k}\,(\mathbf{a}^+ - \mathbf{a}^-) \otimes \mathbf{K} \otimes 1)K|\Xi_{2,2}\rangle = 0, \tag{5.138}$$

$$(1 \otimes \mathbf{k}^2 \otimes \mathbf{K}\,(\mathbf{A}^+ - \mathbf{A}^-) \otimes 1)K|\Xi_{2,2}\rangle = 0, \tag{5.139}$$

$$(1 \otimes \mathbf{k} \otimes \mathbf{K} \otimes (\mathbf{a}^+ - \mathbf{a}^-))K|\Xi_{2,2}\rangle = 0, \tag{5.140}$$

$$((\mathbf{A}^+ - \mathbf{A}^- + (1+q^2)\mathbf{K}) \otimes \mathbf{k}^2 \otimes \mathbf{K} \otimes 1)K|\Xi_{1,2}\rangle = 0, \tag{5.141}$$

$$(1 \otimes \mathbf{k}\,(\mathbf{a}^+ - \mathbf{a}^-) \otimes \mathbf{K} \otimes 1)K|\Xi_{1,2}\rangle = 0, \tag{5.142}$$

$$(1 \otimes \mathbf{k}^2 \otimes \mathbf{K}\,(\mathbf{A}^+ - \mathbf{A}^- + (1+q^2)\mathbf{K}) \otimes 1)K|\Xi_{1,2}\rangle = 0, \tag{5.143}$$

$$(1 \otimes \mathbf{k} \otimes \mathbf{K} \otimes (\mathbf{a}^+ - \mathbf{a}^-))K|\Xi_{1,2}\rangle = 0, \tag{5.144}$$

$$((\mathbf{A}^+ - \mathbf{A}^- + (1+q^2)\mathbf{K}) \otimes \mathbf{k}^2 \otimes \mathbf{K} \otimes 1)K|\Xi_{1,1}\rangle = 0, \tag{5.145}$$

$$(1 \otimes \mathbf{k}\,(\mathbf{a}^+ - \mathbf{a}^- + (1+q)\mathbf{k}) \otimes \mathbf{K} \otimes 1)K|\Xi_{1,1}\rangle = 0, \tag{5.146}$$

$$(1 \otimes \mathbf{k}^2 \otimes \mathbf{K}\,(\mathbf{A}^+ - \mathbf{A}^- + (1+q^2)\mathbf{K}) \otimes 1)K|\Xi_{1,1}\rangle = 0, \tag{5.147}$$

$$(1 \otimes \mathbf{k} \otimes \mathbf{K} \otimes (\mathbf{a}^+ - \mathbf{a}^- + (1+q)\mathbf{k}))K|\Xi_{1,1}\rangle = 0. \tag{5.148}$$

As an illustration, (5.137) is verified as

$$((\mathbf{A}^+ - \mathbf{A}^-) \otimes \mathbf{k}^2 \otimes \mathbf{K} \otimes 1)K|\Xi_{2,2}\rangle = \Delta(Y)K|\Xi_{2,2}\rangle = K\tilde{\Delta}(Y)|\Xi_{2,2}\rangle = 0, \tag{5.149}$$

where $Y = -q^{-5}\mu_1^2\mu_2 t_{33}t_{41} - \mu_1^{-2}\mu_2^{-1}t_{34}t_{13} - q\mu_1^{-2}\mu_2^{-1}t_{22}t_{14} - q^{-4}\mu_1^2\mu_2 t_{21}t_{42}$. Here the first equality is due to (5.126) and (5.127), the second equality is due to (5.37), and the last equality is checked directly. The other relations in (5.137)–(5.148) can be shown similarly. Namely, one can always find a polynomial $Y = Y(\{t_{ij}\})$ which is a linear combination of those appearing in Lemma 5.20 such that the relation in question is expressed and shown as

$$\Delta(Y)K|\Xi_{r,k}\rangle = K\tilde{\Delta}(Y)|\Xi_{r,k}\rangle = 0$$

by applying (5.132)–(5.135) in the last step. The only exception is (5.146) involving $X = 1 \otimes \mathbf{k}^2 \otimes \mathbf{K} \otimes 1$ which is *not* contained in Lemma 5.20. In fact, from (5.128) and (5.129), LHS of (5.146) is expressed as

$$(\Delta(Z) + (1+q)X)\, K|\Xi_{1,1}\rangle, \tag{5.150}$$

where $Z = q\mu_1^{-1}\mu_2^{-1}t_{44}t_{13} + q^{-4}\mu_1^3\mu_2 t_{43}t_{41} + \mu_1\mu_2(q^{-3}t_{42}t_{11} - q^{-4}t_{41}t_{12})$. To treat this, we rely on (5.147) which can be proved independently as explained previously. It then shows that the third component of $|\Xi_{1,1}\rangle$ is proportional to $|\chi_1\rangle$. Therefore from (5.132) we know that it also satisfies

$$(1 \otimes \mathbf{k}^2 \otimes \mathbf{K}(\mathbf{A}^+ - 1 + \mathbf{K}) \otimes 1)K|\Xi_{1,1}\rangle = 0. \tag{5.151}$$

This leads to

$$\begin{aligned} XK|\Xi_{1,1}\rangle &= (1 \otimes \mathbf{k}^2 \otimes \mathbf{K}(\mathbf{A}^+ + \mathbf{K}) \otimes 1)K|\Xi_{1,1}\rangle \\ &= \Delta(-\mu_1^2\mu_2(q^{-1}t_{44}t_{41} + q^{-2}t_{43}t_{42}) + \mu_1^{-4}\mu_2^{-2}t_{14}^2 - q^{-3}t_{42}t_{13})K|\Xi_{1,1}\rangle \end{aligned}$$

due to (5.121) and (5.130). Substituting this into (5.150) and applying (5.37) one can check that it indeed vanishes. □

5.8.2 *Combinatorial and Birational Counterparts*

Lemma 5.22 *Matrix elements K^{abcd}_{ijkl} of the 3D K are polynomials in q with integer coefficients. Their constant term is given by*

$$\begin{aligned} &K^{abcd}_{ijkl}|_{q=0} = \delta_i^{a'}\delta_j^{b'}\delta_k^{c'}\delta_l^{d'}, \\ &a' = x + a + b - d, \quad b' = c - x + d - \min(a, c + x), \\ &c' = \min(a, c + x), \quad d' = b + (c + x - a)_+, \quad x = (c - a + (d - b)_+)_+, \end{aligned} \tag{5.152}$$

where the symbol $(x)_+$ is defined in (3.66).

The proof is similar to but more cumbersome than that for Lemma 3.29. The details are available in [93, Appendix C].

Lemma 5.22 and $K = K^{-1}$ tell us that 3D K at $q = 0$ maps a monomial to another monomial as $|a\rangle \otimes |b\rangle \otimes |c\rangle \otimes |d\rangle \mapsto |a'\rangle \otimes |b'\rangle \otimes |c'\rangle \otimes |d'\rangle$. Motivated by this fact, we define the *combinatorial* 3D K to be a map on $(\mathbb{Z}_{\geq 0})^4$ given by

$$K_{\text{combinatorial}} : (a, b, c, d) \mapsto (a', b', c', d') \tag{5.153}$$

Type C

	(211202341)	
R_{456} ↙		↘ R_{689}
(211020341)		(211205314)
K_{1236} ↓		↓ K_{3579}
(301021341)		(213205118)
K_{1478} ↓		↓ R_{249}
(301021341)		(223105119)
R_{258} ↓		↓ R_{258}
(321001361)		(213115109)
R_{249} ↓		↓ K_{1478}
(311101360)		(313015119)
K_{3579} ↓		↓ K_{1236}
(313101164)		(313015119)
R_{689} ↘		↙ R_{456}
	(313106119)	

Type B

	(211202341)	
R^B_{456} ↙		↘ R^B_{689}
(211020341)		(211205314)
K^B_{1236} ↓		↓ K^B_{3579}
(401021341)		(213205116)
K^B_{1478} ↓		↓ R^B_{249}
(301021431)		(223105117)
R^B_{258} ↓		↓ R^B_{258}
(321001451)		(213115107)
R^B_{249} ↓		↓ K^B_{1478}
(311101450)		(413015117)
K^B_{3579} ↓		↓ K^B_{1236}
(314101153)		(314014117)
R^B_{689} ↘		↙ R^B_{456}
	(314105117)	

Fig. 5.3 Left: Type C. The maps R and K denote $R_{\text{combinatorial}}$ in (3.150) and $K_{\text{combinatorial}}$ in (5.153) for type C. The second arrow by K_{1236} in the LHS is due to $K_{\text{combinatorial}} : (2, 1, 1, 0) \mapsto (3, 0, 1, 1)$, which can be seen in Example 5.10. Right: The Type B case which will be treated in Sect. 6.5 is shown for comparison

in terms of a', b', c', d' in (5.152). It has the conserved quantities $a + b + c$ and $b + 2c + d$ corresponding to the weight conservation (5.65). Setting $q = 0$ in Theorem 5.16 we obtain:

Corollary 5.23 *The combinatorial 3D K (5.153) is an involution on* $(\mathbb{Z}_{\geq 0})^4$. *It satisfies the 3D reflection equation on* $(\mathbb{Z}_{\geq 0})^9$ *together with the combinatorial 3D R in (3.150).*

Example 5.24 Examples of the 3D reflection equation in the combinatorial setting (Fig. 5.3).

Let us proceed to the third 3D K. We introduce $2n$-by-$2n$ upper triangular matrices

$$X_i(x) = 1 + xE_{i,i+1} - xE_{2n-i,2n-i+1} \quad (1 \leq i < n), \tag{5.154}$$

$$X_n(x) = 1 + 2xE_{n,n+1}, \tag{5.155}$$

where x is a parameter and $E_{i,j}$ is a matrix unit. The matrix $X_i(x)$ is a generator of the unipotent subgroup of Sp$(2n)$.[5] It satisfies $X_i(x)^{-1} = X_i(-x)$ and $X_i(a)X_j(b) = X_j(b)X_i(a)$ for $|i - j| > 1$. Given parameters a, b, c, d, one can easily check that each of the matrix equations

[5] For instance, the matrices (5.158) for $n = 2$ satisfy ${}^tX_i(z)\Omega X_i(z) = \Omega$ with $\Omega = (C \text{ in } (5.7))|_{q\to 1}$.

$$X_i(a)X_j(b)X_i(c) = X_j(\tilde{c})X_i(\tilde{b})X_j(\tilde{a}) \;\; (|i-j|=1, i, j < n), \tag{5.156}$$
$$X_{n-1}(a)X_n(b)X_{n-1}(c)X_n(d) = X_n(d')X_{n-1}(c')X_n(b')X_{n-1}(a') \tag{5.157}$$

has the unique solution. For (5.156) it is given by (3.151). For (5.157) it essentially reduces to the $n=2$ case:

$$X_1(z) = \begin{pmatrix} 1 & z & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & -z \\ & & & 1 \end{pmatrix}, \qquad X_2(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ & 1 & 2z & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}. \tag{5.158}$$

The solution is given by

$$\begin{aligned} & a' = \frac{abc}{A}, \quad b' = \frac{A^2}{B}, \quad c' = \frac{B}{A}, \quad d' = \frac{bc^2d}{B}, \\ & A = ab + ad + cd, \quad B = a^2b + a^2d + 2acd + c^2d. \end{aligned} \tag{5.159}$$

Based on this fact we define a map

$$K_{\text{birational}} : (a, b, c, d) \mapsto (a', b', c', d') \tag{5.160}$$

using (5.159) and call it the *birational* 3D K. It is easy to see that $K_{\text{birational}}$ is an involution having the two conserved quantities

$$abc, \quad bc^2d. \tag{5.161}$$

Their ultradiscretization (3.152) reproduces the weight conservation (5.65).

By considering Sp(6) and comparing the two ways to achieve the birational maps $(a_1, \ldots, a_9) \mapsto (\tilde{a}_9, \ldots, \tilde{a}_1)$ defined by

$$\begin{aligned} & X_1(a_1)X_2(a_2)X_3(a_3)X_1(a_4)X_2(a_5)X_1(a_6)X_3(a_7)X_2(a_8)X_3(a_9) \\ & = X_3(\tilde{a}_9)X_2(\tilde{a}_8)X_3(\tilde{a}_7)X_1(\tilde{a}_6)X_2(\tilde{a}_5)X_1(\tilde{a}_4)X_3(\tilde{a}_3)X_2(\tilde{a}_2)X_1(\tilde{a}_1), \end{aligned}$$

we obtain:

Proposition 5.25 *The birational 3D R (3.151) and the birational 3D K (5.160) satisfy the 3D reflection equation on the ring of rational functions of nine variables.*

Let us denote the 3D K detailed in Sects. 5.4 and 5.5 by K_{quantum}. Then the triad of the 3D K's and their relation is summarized in the same manner as (3.158) for the 3D R:

$$K_{\text{quantum}} \overset{q \to 0}{\longrightarrow} K_{\text{combinatorial}} \overset{\text{UD}}{\longleftarrow} K_{\text{birational}}. \tag{5.162}$$

$K_{\text{combinatorial}}$ and $K_{\text{birational}}$ (and R^t below) are set-theoretical solutions to the 3D reflection equation.

Remark 5.26 Define a family of maps K^t depending on a parameter t by

$$K^t : (a, b, c, d) \mapsto \left(\frac{abc}{A_t}, \frac{A_t^2}{B_t}, \frac{B_t}{A_t}, \frac{bc^2d}{B_t} \right), \tag{5.163}$$

$$A_t = A + tabcd, \quad B_t = B + tabcd(a + c) \tag{5.164}$$

by using A and B given in (5.159). The birational 3D K in (5.160) corresponds to $t = 0$. Then $K^t = (K^t)^{-1}$ holds. Moreover, K^t satisfies the 3D reflection equation with another one parameter family of birational 3D R's in (3.159) as

$$R^s_{689} K^t_{3579} R^s_{249} R^s_{258} K^t_{1478} K^t_{1236} R^s_{456} = R^s_{456} K^t_{1236} K^t_{1478} R^s_{258} R^s_{249} K^t_{3579} R^s_{689}, \tag{5.165}$$

where s and t can be chosen independently.

5.9 Bibliographical Notes and Comments

This chapter is an extended exposition of [93, Chap. 3]. The RTT realization in Sect. 5.1 is taken from [127]. The 3D K in Sect. 5.5 was the first non-trivial solution to the 3D reflection equation.[6] The explicit formula in Theorem 5.13 is due to [88], which is an analogue of Theorem 3.18 for 3D R. The solution of the quantized reflection equation (Theorem 5.18) was obtained in [105, Sect. 2.2]. The property of the boundary vector (Theorem 5.21) was conjectured in [105, Eq. (78)] and proved in [106, Appendix B].

For the combinatorial and birational 3D K in Sect. 5.8.2, comments similar to those for the 3D R (Sect. 3.7) apply. The t-deformed birational 3D K in Remark 5.26 has appeared in a different context in [110, Remark 5.1]. It originates in the folding type embedding of the Coxeter group $B_2 \hookrightarrow A_3$, which corresponds to the $m = 4$ case of (9.2). Realizing the transformation corresponding to the quartic Coxeter relation of B_2 in the image A_3 as the product of four t-deformed birational 3D R (3.159), one can deduce K^t[152]. See also the last paragraph in Sect. 9.3.

[6] It was reported in a talk in *The XXIX International Colloquium on Group-Theoretical Methods in Physics* (20–26 August 2012, Tianjin, China), where both authors of [62] were in the audience.

Chapter 6
3D K From Quantized Coordinate Ring of Type B

Abstract For the quantized coordinate ring $A_q(B_n)$, fundamental representations of the generators associated with the spin representation of B_n are presented. Reflecting the equivalence of the spin representation of B_2 and the vector representation of C_2, the equivalence $A_q(B_n) \simeq A_q(C_n)$ holds for $n = 2$ but not for $n \geq 3$. In particular $A_q(B_3)$ leads to another solution to the 3D reflection equation different from Chap. 5. The RTT relation for the fundamental representations are proved by making use of the tetrahedron equation of type $MMLL = LLMM$ (Theorem 3.25) and a matrix product formula of the quantum R matrix for the spin representation (Chap. 12).

6.1 Quantized Coordinate Ring $A_q(B_n)$

Like $A_q(A_{n-1})$ and $A_q(C_n)$ treated in the preceding chapters, the quantized coordinate ring $A_q(B_n)$ $(n \geq 2)$ we consider in this chapter is the $\mathfrak{g} = B_n$ case of the Hopf algebra $A_q(\mathfrak{g})$ defined in Sect. 10.2 in a universal manner. On general grounds, $A_q(B_n)$ has generators $t_{\mathbf{ab}}$ associated with the spin representation $V(\varpi_n)$ of $U_q(B_n)$.[1] Here the indices $\mathbf{a}$, $\mathbf{b}$ range over

$$\{0, 1\}^n = \{\mathbf{a} = (a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in \{0, 1\}\}, \tag{6.1}$$

which is a natural labeling set of the base of $V(\varpi_n)$. A feature that distinguishes it from the A_{n-1} and C_n cases is that the complete set of defining relations among the $2^n \times 2^n$ generators $T = (t_{\mathbf{ab}})$ have not been identified explicitly in the literature. They include the RTT relation and the ρTT relation at least:

[1] See the explanations around (10.22). $V(\varpi_n)$ denotes the irreducible $U_q(B_n)$ module whose highest weight is the nth fundamental weight ϖ_n. $A_q(B_n)$ here is different from (in a sense "finer" than) $\mathrm{Fun}(\mathrm{SO}_q(2n+1))$ in [127] based on $(2n+1)^2$ generators associated with the vector representation.

A. Kuniba, *Quantum Groups in Three-Dimensional Integrability*, Theoretical and Mathematical Physics, https://doi.org/10.1007/978-981-19-3262-5_6

$$\sum_{\mathbf{l},\mathbf{m}} R^{\mathbf{ab}}_{\mathbf{lm}} t_{\mathbf{lc}} t_{\mathbf{md}} = \sum_{\mathbf{l},\mathbf{m}} t_{\mathbf{bm}} t_{\mathbf{al}} R^{\mathbf{lm}}_{\mathbf{cd}}, \tag{6.2}$$

$$\sum_{\mathbf{b}} \rho_{\mathbf{b}} t_{\mathbf{ab}} t_{\mathbf{l}'\mathbf{b}'} = \sum_{\mathbf{c}} \rho_{\mathbf{c}} t_{\mathbf{ca}} t_{\mathbf{c}'\mathbf{l}'} = \rho_{\mathbf{a}} \delta_{\mathbf{al}}, \tag{6.3}$$

where $\mathbf{a}'$ is defined by

$$\mathbf{a}' = (1 - a_1, \dots, 1 - a_n). \tag{6.4}$$

The RTT relation is known to be valid from the general argument leading to (10.15). The relation (6.3) originates in the fact that $V(\varpi_n) \otimes V(\varpi_n) \supset V(0)$, which is also the case for $A_q(C_n)$ as in (5.12). The structure constants $R^{\mathbf{ab}}_{\mathbf{ij}}$ and $\rho_{\mathbf{a}}$ are related by (5.12), and given as

$$R^{\mathbf{ab}}_{\mathbf{ij}} = \lim_{x\to\infty} x^{-2n} R(x)^{\mathbf{ab}}_{\mathbf{ij}}, \tag{6.5}$$

$$\rho_{\mathbf{a}} = q^{-\frac{n^2}{2}} \prod_{k=1}^{n} \left((-1)^k q^{2k-1}\right)^{a_{n+1-k}}. \tag{6.6}$$

In (6.5), $R(x)^{\mathbf{ab}}_{\mathbf{ij}}$ is an element of the quantum R matrix of the spin representation. See (6.61) and the explanation around it for a precise description. In (6.5), one is picking the coefficient of the highest order power of x from it as in (3.4) and (5.6). From $\rho_{\mathbf{a}}\rho_{\mathbf{a}'} = (-1)^{n(n+1)/2}$, (6.6) corresponds to $\varepsilon = (-1)^{n(n+1)/2}$ in (5.10).

Remark 6.1 Under the equivalence $U_q(C_2) \simeq U_q(B_2)$, the vector representation of the former corresponds to the spin representation of the latter. Reflecting this fact, $A_q(B_2)$ here is isomorphic to $A_q(C_2)$ in Chap. 5 via the rescaling of generators explained in Remark 5.1. Concretely, the indices $1, 2, 3, 4$ for $A_q(C_2)$ correspond to $(0,0), (0,1), (1,0), (1,1)$ for $A_q(B_2)$, and the generators are identified by (5.13) with $(g_1, g_2, g_3, g_4) = (i, 1, 1, i)$ satisfying (5.14).

6.2 Fundamental Representations

Let $\mathrm{Osc}_q = \langle \mathbf{a}^+, \mathbf{a}^-, \mathbf{k}, \mathbf{k}^{-1} \rangle$ be the q-oscillator algebra (3.12) and $\mathrm{Osc}_{q^2} = \langle \mathbf{A}^+, \mathbf{A}^-, \mathbf{K}, \mathbf{K}^{-1} \rangle$ be the q^2-oscillator algebra (5.15). As before, they are identified with elements of $\mathrm{End}(\mathcal{F}_{q_i})$. The embedding in Theorem 3.3 enables one to write down the fundamental representations

$$\pi_i : A_q(B_n) \to \mathrm{End}(\mathcal{F}_{q_i}) \qquad (1 \le i \le n), \tag{6.7}$$

$$q_1 = \cdots = q_{n-1} = q^2, \;\; q_n = q \tag{6.8}$$

containing a non-zero parameter μ_i. Note the difference of (6.8) from (5.18).

For $1 \le i \le n-1$, let the image of the generators $(t_{\mathbf{ab}})_{\mathbf{a},\mathbf{b}\in\{0,1\}^n}$ by π_i be as follows:

$$\begin{pmatrix} t_{\alpha 00\tilde{\alpha},\alpha 00\tilde{\alpha}} & t_{\alpha 00\tilde{\alpha},\alpha 01\tilde{\alpha}} & t_{\alpha 00\tilde{\alpha},\alpha 10\tilde{\alpha}} & t_{\alpha 00\tilde{\alpha},\alpha 11\tilde{\alpha}} \\ t_{\alpha 01\tilde{\alpha},\alpha 00\tilde{\alpha}} & t_{\alpha 01\tilde{\alpha},\alpha 01\tilde{\alpha}} & t_{\alpha 01\tilde{\alpha},\alpha 10\tilde{\alpha}} & t_{\alpha 01\tilde{\alpha},\alpha 11\tilde{\alpha}} \\ t_{\alpha 10\tilde{\alpha},\alpha 00\tilde{\alpha}} & t_{\alpha 10\tilde{\alpha},\alpha 01\tilde{\alpha}} & t_{\alpha 10\tilde{\alpha},\alpha 10\tilde{\alpha}} & t_{\alpha 10\tilde{\alpha},\alpha 11\tilde{\alpha}} \\ t_{\alpha 11\tilde{\alpha},\alpha 00\tilde{\alpha}} & t_{\alpha 11\tilde{\alpha},\alpha 01\tilde{\alpha}} & t_{\alpha 11\tilde{\alpha},\alpha 10\tilde{\alpha}} & t_{\alpha 11\tilde{\alpha},\alpha 11\tilde{\alpha}} \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathbf{A}^- & \mu_i \mathbf{K} & 0 \\ 0 & -q^2\mu_i^{-1}\mathbf{K} & \mathbf{A}^+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.9}$$

$$\text{otherwise } t_{\mathbf{a},\mathbf{b}} \mapsto 0, \tag{6.10}$$

where $\alpha \in \{0,1\}^{i-1}$ and $\tilde{\alpha} \in \{0,1\}^{n-i-1}$ are arbitrary in (6.9). For π_n, the image of the generators is specified as

$$\begin{pmatrix} t_{\alpha 0,\alpha 0} & t_{\alpha 0,\alpha 1} \\ t_{\alpha 1,\alpha 0} & t_{\alpha 1,\alpha 1} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{a}^- & \mu_n \mathbf{k} \\ -q\mu_n^{-1}\mathbf{k} & \mathbf{a}^+ \end{pmatrix}, \tag{6.11}$$

$$\text{otherwise } t_{\mathbf{a},\mathbf{b}} \mapsto 0, \tag{6.12}$$

where $\alpha \in \{0,1\}^{n-1}$ is arbitrary in (6.11).

Example 6.2 For $A_q(B_2)$, let $T = (t_{\mathbf{ab}})$ be the array with row $\mathbf{a}$ and column $\mathbf{b}$ ordered as $(0,0), (0,1), (1,0), (1,1)$ from the top let. Then its image reads as

$$\pi_1(T) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathbf{A}^- & \mu_1\mathbf{K} & 0 \\ 0 & -q^2\mu_1^{-1}\mathbf{K} & \mathbf{A}^+ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \pi_2(T) = \begin{pmatrix} \mathbf{a}^- & \mu_2\mathbf{k} & 0 & 0 \\ -q\mu_2^{-1}\mathbf{k} & \mathbf{a}^+ & 0 & 0 \\ 0 & 0 & \mathbf{a}^- & \mu_2\mathbf{k} \\ 0 & 0 & -q\mu_2^{-1}\mathbf{k} & \mathbf{a}^+ \end{pmatrix}. \tag{6.13}$$

Remark 6.3 Denote the fundamental representations of $A_q(B_2)$ in Example 6.2 by $\pi_1^{B_2}$ and $\pi_2^{B_2}$. Similarly, denote the fundamental representations of $A_q(C_2)$ in (5.24) by $\pi_1^{C_2}$ and $\pi_2^{C_2}$. Then $\pi_i^{B_2}$ coincides with $\pi_{3-i}^{C_2}$ via the adjustment explained in Remark 6.1 with a suitable redefinition of μ_i parameters.

From Remark 5.1, the parameters $\mu_1, \dots, \mu_n$ are removed by switching to the rescaled generators $\tilde{t}_{\mathbf{ab}}$ in (5.13) with $g_{\mathbf{a}} = \prod_{1\le k\le n} \mu_k^{a_1+\cdots+a_k-k/2}$ satisfying (5.14) with $g_{\mathbf{a}} g_{\mathbf{a}'} = 1$. Thus we set $\mu_1 = \cdots = \mu_n = 1$ in the rest of the chapter without loss of generality.

Example 6.4 Let $T = (t_{\mathbf{a},\mathbf{b}})$ be the 8-by-8 matrix of generators of $A_q(B_3)$, where the row index $\mathbf{a}$ and the column index $\mathbf{b}$ are ordered from the top left corner as 000, 001, 010, 011, 100, 101, 110, 111. Then their image by the fundamental representations π_1, π_2, π_3 according to (6.9)–(6.12) reads as follows:

$$\pi_1(T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{A}^- & 0 & \mathbf{K} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{A}^- & 0 & \mathbf{K} & 0 & 0 \\ 0 & 0 & -q^2\mathbf{K} & 0 & \mathbf{A}^+ & 0 & 0 & 0 \\ 0 & 0 & 0 & -q^2\mathbf{K} & 0 & \mathbf{A}^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.14}$$

$$\pi_2(T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{A}^- & \mathbf{K} & 0 & 0 & 0 & 0 & 0 \\ 0 & -q^2\mathbf{K} & \mathbf{A}^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{A}^- & \mathbf{K} & 0 \\ 0 & 0 & 0 & 0 & 0 & -q^2\mathbf{K} & \mathbf{A}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.15}$$

$$\pi_3(T) = \begin{pmatrix} \mathbf{a}^- & \mathbf{k} & 0 & 0 & 0 & 0 & 0 & 0 \\ -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{a}^- & \mathbf{k} & 0 & 0 & 0 & 0 \\ 0 & 0 & -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{a}^- & \mathbf{k} & 0 & 0 \\ 0 & 0 & 0 & 0 & -q\mathbf{k} & \mathbf{a}^+ & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{a}^- & \mathbf{k} \\ 0 & 0 & 0 & 0 & 0 & 0 & -q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}. \tag{6.16}$$

Proposition 6.5 *The image of the generators by the maps $\pi_1, \dots, \pi_n$ in (6.9)–(6.12) satisfies the RTT relation (6.2) and the ρTT relation (6.3).*

We will present an intriguing proof in Sect. 6.6 making use of the tetrahedron equation of type $MMLL = LLMM$ (3.122), where the MM part yields the structure constant and the LL part the generators.

Let us turn to the tensor products of the fundamental representations. We write $\pi_{i_1} \otimes \cdots \otimes \pi_{i_l}$ as $\pi_{i_1,\dots,i_l}$ for short. The Weyl group $W(B_n)$ is the same as $W(C_n)$ explained in (5.26) and (5.27). Then Theorem 3.3 asserts the same equivalence as (5.28)–(5.30):

$$\pi_{i,j} \simeq \pi_{j,i} \qquad (|i-j| \geq 2), \tag{6.17}$$

$$\pi_{i,i+1,i} \simeq \pi_{i+1,i,i+1} \qquad (1 \leq i \leq n-2), \tag{6.18}$$

$$\pi_{n-1,n,n-1,n} \simeq \pi_{n,n-1,n,n-1}. \tag{6.19}$$

6.3 Intertwiners

By Remark 3.4, the intertwiner responsible for the isomorphism (6.17) is just the exchange of components P defined in (3.23). See the explanation around (3.24).

Next we consider the intertwiner for (6.18), which corresponds to the cubic Coxeter relation. It is an element $\Phi^B \in \mathrm{End}(\mathcal{F}_{q^2}^{\otimes 3})$ characterized by

$$\Phi^B \circ \pi_{i,i+1,i}(\Delta(f)) = \pi_{i+1,i,i+1}(\Delta(f)) \circ \Phi^B \quad (1 \le i < n,\ \forall f \in A_q(B_n)), \tag{6.20}$$

$$\Phi^B(|0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle. \tag{6.21}$$

The latter just fixes the normalization. Set $R^B = \Phi^B P_{13}$ as in (3.30). Then the Eq. (6.20) is identical, as a set, with (3.38)–(3.46) for the 3D R with q replaced by q^2. Therefore, the intertwiner for (6.18) is provided by $\Phi^B = R^B P_{13}$ with $R^B = R|_{q \to q^2}$, where R in the RHS is the 3D R in Chap. 3. As before, R^B will also be called the intertwiner. We know that R^B satisfies the tetrahedron equation of type $RRRR = RRRR$ (2.6).

Finally, we consider the intertwiner for the equivalence (6.19), which corresponds to the quartic Coxeter relation. Due to the nested structure of the representations (6.9)–(6.12) with respect to rank n, the problem reduces to $\pi_{1212} \simeq \pi_{2121}$ for $A_q(B_2)$. Thus we consider the linear map

$$\Psi^B : \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \to \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \tag{6.22}$$

characterized by

$$\pi_{2121}(\Delta(f)) \circ \Psi^B = \Psi^B \circ \pi_{1212}(\Delta(f)) \quad (\forall f \in A_q(B_2)), \tag{6.23}$$

$$\Psi^B(|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle, \tag{6.24}$$

where the latter specifies the normalization. Set

$$K^B = \Psi^B P_{14} P_{23} \in \mathrm{End}(\mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2}), \tag{6.25}$$

where $P_{14}P_{23}$ reverses the order of the 4-fold tensor product. Note a slight difference from the 3D K of $A_q(C_2)$ in (5.36).

Theorem 6.6 *The intertwiner K^B is given by $K^B = P_{14}P_{23}KP_{14}P_{23}$, where K in the RHS is the 3D K for $A_q(C_2)$ in Chap. 5.*

Proof From Remark 6.3, the Eq. (6.23) is equivalent to $\pi^C_{1212}(\Delta(f)) \circ \Psi^B = \Psi^B \circ \pi^C_{2121}(\Delta(f))$. Comparing this with the type C case $\pi^C_{2121}(\Delta(f)) \circ \Psi = \Psi \circ \pi^C_{1212}(\Delta(f))$ in (5.34) and from the unique existence of Ψ^B, we have $\Psi^B = \Psi^{-1}$ taking the normalization into account. Thus we find $K^B P_{14}P_{23} = \Psi^B = \Psi^{-1} \overset{(5.36)}{=} (KP_{14}P_{23})^{-1} \overset{(5.72)}{=} P_{14}P_{23}K$. □

Let us summarize the relation of intertwiners that originate in the cubic and the quartic Coxeter relations exhibiting the types B and C as superscripts.

$$\begin{aligned} &\text{Cubic} && \text{Quartic} \\ &\Phi^B = \Phi^C|_{q\to q^2}, && \Psi^B = (\Psi^C)^{-1}, && (6.26) \\ &R^B = R^C|_{q\to q^2}, && K^B = P_{14}P_{23}K^C P_{14}P_{23}. && (6.27) \end{aligned}$$

The last result implies

$$K^B(|i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle) = \sum_{a,b,c,d} K^{dcba}_{lkji}|a\rangle \otimes |b\rangle \otimes |c\rangle \otimes |d\rangle \tag{6.28}$$

in $\mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2}$ in terms of the matrix elements K^{abcd}_{ijkl} of $K = K^C$ in (5.57). We note that (5.72) also implies

$$K^B = (K^B)^{-1}. \tag{6.29}$$

6.4 3D Reflection Equation

To be explicit, we set

$$S = R^B = R|_{q\to q^2}, \qquad K_{4321} = K^B_{1234} = P_{14}P_{23}K_{1234}P_{23}P_{14} \tag{6.30}$$

according to (6.27).

Theorem 6.7 *The intertwiners R^B and K^B satisfy the 3D reflection equation (4.3), which is presented in terms of the above S and K as*

$$S_{689}K_{9753}S_{249}S_{258}K_{8741}K_{6321}S_{456} = S_{456}K_{6321}K_{8741}S_{258}S_{249}K_{9753}S_{689}. \tag{6.31}$$

It is an equality of linear operators on

$$\overset{1}{\mathcal{F}_q} \otimes \overset{2}{\mathcal{F}_{q^2}} \otimes \overset{3}{\mathcal{F}_q} \otimes \overset{4}{\mathcal{F}_{q^2}} \otimes \overset{5}{\mathcal{F}_{q^2}} \otimes \overset{6}{\mathcal{F}_{q^2}} \otimes \overset{7}{\mathcal{F}_q} \otimes \overset{8}{\mathcal{F}_{q^2}} \otimes \overset{9}{\mathcal{F}_{q^2}}. \tag{6.32}$$

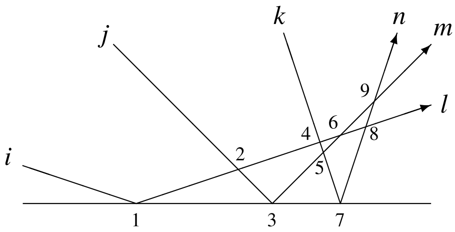
Proof Since $W(B_3) \simeq W(C_3)$, the same proof as the one for Theorem 5.16 remains valid. □

Note that the replacement $q \to q^2$ in (6.27) is done only for R. Therefore, the solution (R^B, K^B) is *not* reducible to (R, K) for type C in Chap. 5.

In the reminder of this section we present another proof of Theorem 6.7 based on the quantized reflection equation introduced in Sect. 4.4. Define L by (5.110), G by (5.111) and $K = K^C$ to be the 3D K for type C in Chap. 5. Then Theorem 5.18 shows that L, G and K satisfy the quantized reflection equation (4.12) with $\mathcal{F}' = \mathcal{F}_{q^2}$ and $\mathcal{F} = \mathcal{F}_q$. We know $K = K^{-1}$ by (5.72). Thus J in (4.14) and (4.15) coincides with K^B in (6.27). From Theorem 3.21, (3.59) and (3.60), we see that R^B in (6.27) and the above L satisfy the quantized Yang–Baxter equations $(2.19)|_{R \to R^B}$ and $(2.20)|_{S \to R^B}$. In this way we have a concrete realization of all the operators appearing in (4.19) and (4.20) in which $R = S = R^B$ and $J = K^B$. Thus the argument leading to (4.23) proves Theorem 6.7 provided that the operators (4.18) act irreducibly on the space (4.22), which is (6.32) in the present setting.

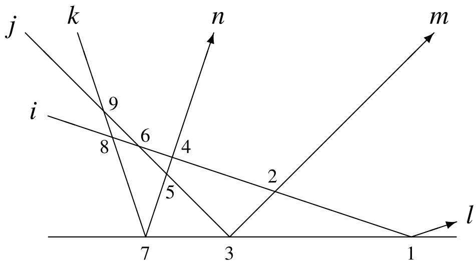
The last point of the irreducibility is established by identifying the quantized three-body reflection amplitude with the representation of $A_q(B_3)$ corresponding to the longest element of $W(B_3)$. To state it precisely, we set

$$\mathcal{M}_{ijk}^{lmn} =$$



(6.33)

$$\tilde{\mathcal{M}}_{ijk}^{lmn} =$$



(6.34)

They stand for the quantized three-body reflection amplitudes

$$\mathcal{M}_{ijk}^{lmn}, \tilde{\mathcal{M}}_{ijk}^{lmn} \in \mathrm{End}(\overset{1}{\mathcal{F}}_q \otimes \overset{2}{\mathcal{F}}_{q^2} \otimes \overset{3}{\mathcal{F}}_q \otimes \overset{4}{\mathcal{F}}_{q^2} \otimes \overset{5}{\mathcal{F}}_{q^2} \otimes \overset{6}{\mathcal{F}}_{q^2} \otimes \overset{7}{\mathcal{F}}_q \otimes \overset{8}{\mathcal{F}}_{q^2} \otimes \overset{9}{\mathcal{F}}_{q^2}). \tag{6.35}$$

They allow us to express the first and the last operators in (4.19) and (4.20) as

$$\overset{9}{L}_{bc}\overset{8}{L}_{ac}\overset{7}{G}_c\overset{6}{L}_{ab}\overset{5}{L}_{cb}\overset{4}{L}_{ca}\overset{3}{G}_b\overset{2}{L}_{ba}\overset{1}{G}_a = \sum \overset{a}{E}_{li} \otimes \overset{b}{E}_{mj} \otimes \overset{c}{E}_{nk} \otimes \mathcal{M}_{ijk}^{lmn}, \tag{6.36}$$

$$\overset{1}{G}_a\overset{2}{L}_{ab}\overset{3}{G}_b\overset{4}{L}_{ac}\overset{5}{L}_{bc}\overset{6}{L}_{ba}\overset{7}{G}_c\overset{8}{L}_{ca}\overset{9}{L}_{cb} = \sum \overset{a}{E}_{li} \otimes \overset{b}{E}_{mj} \otimes \overset{c}{E}_{nk} \otimes \tilde{\mathcal{M}}_{ijk}^{lmn}, \tag{6.37}$$

where the sums extend over $i, j, k, l, m, n \in \{0, 1\}$ and E_{ij} is the matrix unit on V.

Proposition 6.8 *The quantized three-body reflection amplitudes are identified with the representation of $A_q(B_3)$ corresponding to the longest element of $W(B_3)$ as follows:*

$$\mathcal{M}^{lmn}_{ijk} = (-q)^{i+j+k-l-m-n}\pi_{323121321}(\Delta(t_{\mathbf{a},\mathbf{b}})), \tag{6.38}$$

$$\tilde{\mathcal{M}}^{lmn}_{ijk} = (-q)^{i+j+k-l-m-n}\pi_{323121321}(\tilde{\Delta}(t_{\mathbf{a},\mathbf{b}})), \tag{6.39}$$

$$\mathbf{a} = (1-k, 1-j, 1-i), \quad \mathbf{b} = (1-n, 1-m, 1-l). \tag{6.40}$$

This can be verified directly. From this proposition and Theorem 3.3, it follows that $\mathcal{M}^{lmn}_{ijk}$ and $\tilde{\mathcal{M}}^{lmn}_{ijk}$ act irreducibly on (6.32). Therefore the argument in (4.21)–(4.23) proves that (R^B, K^B) satisfies the 3D reflection equation (4.3). We note that the reduced word 323121321 has been encoded in (6.33) as the sequence of "heights" of the points $1, 2, \dots, 9$, where the bottom level is set to be 3.

Example 6.9

$$\mathcal{M}^{000}_{011} = q^2\mathbf{a}^+ \otimes \mathbf{K} \otimes \mathbf{k} \otimes \mathbf{K} \otimes \mathbf{K} \otimes 1 \otimes \mathbf{k} \otimes 1 \otimes 1, \tag{6.41}$$

$$\begin{aligned}\mathcal{M}^{111}_{110} &= q^4\mathbf{a}^- \otimes 1 \otimes \mathbf{a}^- \otimes \mathbf{K} \otimes \mathbf{K} \otimes 1 \otimes \mathbf{k} \otimes 1 \otimes 1\\ &\quad - q^5\mathbf{k} \otimes \mathbf{A}^- \otimes \mathbf{k} \otimes \mathbf{K} \otimes \mathbf{K} \otimes 1 \otimes \mathbf{k} \otimes 1 \otimes 1,\end{aligned} \tag{6.42}$$

$$\begin{aligned}\tilde{\mathcal{M}}^{010}_{000} &= -q^2\mathbf{a}^+ \otimes \mathbf{A}^- \otimes \mathbf{a}^+ \otimes \mathbf{A}^+ \otimes \mathbf{K} \otimes 1 \otimes \mathbf{k} \otimes 1 \otimes 1\\ &\quad - q^2\mathbf{a}^+ \otimes \mathbf{K} \otimes \mathbf{a}^- \otimes 1 \otimes \mathbf{A}^+ \otimes 1 \otimes \mathbf{k} \otimes 1 \otimes 1\\ &\quad - q^2\mathbf{a}^+ \otimes \mathbf{K} \otimes \mathbf{k} \otimes 1 \otimes 1 \otimes 1 \otimes \mathbf{a}^+ \otimes 1 \otimes 1\\ &\quad + q^3\mathbf{k} \otimes 1 \otimes \mathbf{k} \otimes \mathbf{A}^+ \otimes \mathbf{K} \otimes 1 \otimes \mathbf{k} \otimes 1 \otimes 1.\end{aligned} \tag{6.43}$$

On the other hand, for instance, we have

$$\begin{aligned}&\pi_{323121321}(\Delta(t_{001,111}))\\ &= \pi_{323121321}(t_{001,001} \otimes t_{001,010} \otimes t_{010,011} \otimes t_{011,101} \otimes t_{101,110}\\ &\qquad \otimes t_{110,110} \otimes t_{110,111} \otimes t_{111,111} \otimes t_{111,111} + \cdots)\\ &= \mathbf{a}^+ \otimes \mathbf{K} \otimes \mathbf{k} \otimes \mathbf{K} \otimes \mathbf{K} \otimes 1 \otimes \mathbf{k} \otimes 1 \otimes 1,\end{aligned} \tag{6.44}$$

where all the other $+\cdots$ terms are vanishing upon evaluation by $\pi_{323121321}$. Thus we see $\mathcal{M}^{000}_{011} = q^2\pi_{323121321}(\Delta(t_{001,111}))$.

One can similarly define the n-body quantized reflection amplitudes generalizing (6.33) and (6.34) by arranging the n reflecting arrows. They are linear operators on $\mathcal{F}_q^{\otimes n} \otimes \mathcal{F}_{q^2}^{\otimes n(n-1)}$. Thus the total number of the components is n^2, which is equal to the length of the longest element of $W(B_n)$. The formulas (6.38)–(6.40) suggest a natural extension to general n. It is an interesting problem to establish a result like Proposition 6.8 for general n hopefully more intrinsically.

6.5 Combinatorial and Birational Counterparts

In view of (6.27) it is natural to set

$$K^B_{\text{combinatorial}} = P_{14}P_{23}K_{\text{combinatorial}}P_{14}P_{23}, \tag{6.45}$$

$$K^B_{\text{birational}} = P_{14}P_{23}K_{\text{birational}}P_{14}P_{23} \tag{6.46}$$

in terms of (5.153) and (5.160) for type C. On the other hand, we simply set $R^B_{\text{combinatorial}} = R_{\text{combinatorial}}$ in terms of (3.150) and $R^B_{\text{birational}} = R_{\text{birational}}$ in terms of (3.151). These definitions lead to another triad of the 3D R analogous to (5.162):

$$K^B_{\text{quantum}} \overset{q\to 0}{\longrightarrow} K^B_{\text{combinatorial}} \overset{\text{UD}}{\longleftarrow} K^B_{\text{birational}}. \tag{6.47}$$

The 3D reflection equation also remains valid both at combinatorial and birational level. However, it should be stressed that the equation defines *different* transformations for type B and C. See Example 5.24 for comparison in the combinatorial case.

In the rest of the section, we mention a slight variant of the upper triangular matrices relevant to $K^B_{\text{birational}}$ adapted to the natural (rather than spin) representation of B_n. Define the $2n+1$ by $2n+1$ upper triangular matrices

$$Y_i(x) = 1 + xE_{i,i+1} - xE_{2n+1-i,2n-i+2} \quad (1 \le i < n), \tag{6.48}$$

$$Y_n(x) = 1 + xE_{n,n+1} - xE_{n+1,n+2} - \frac{x^2}{2}E_{n,n+2}, \tag{6.49}$$

where x is a parameter and $E_{i,j}$ is a matrix unit. The matrix $Y_i(x)$ is a generator of the unipotent subgroup of $\mathrm{SO}(2n+1)$. It satisfies $Y_i(x)^{-1} = Y_i(-x)$ and $Y_i(a)Y_j(b) = Y_j(b)Y_i(a)$ for $|i-j| > 1$. Given parameters a, b, c, d, each of the matrix equations

$$Y_i(a)Y_j(b)Y_i(c) = Y_j(\tilde{c})Y_i(\tilde{b})Y_j(\tilde{a}) \quad (|i-j| = 1, i, j < n), \tag{6.50}$$

$$Y_{n-1}(a)Y_n(b)Y_{n-1}(c)Y_n(d) = Y_n(d'')Y_{n-1}(c'')Y_n(b'')Y_{n-1}(a'') \tag{6.51}$$

has the unique solution. For (6.50) it is given by (3.151). The second equation (6.51) determines $K^B_{\text{birational}} : (a, b, c, d) \mapsto (a'', b'', c'', d'')$. It essentially reduces to the $n = 2$ case:

$$Y_1(z) = \begin{pmatrix} 1 & z & 0 & 0 & 0 \\ & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 0 \\ & & & 1 & -z \\ & & & & 1 \end{pmatrix}, \qquad Y_2(z) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ & 1 & z & -z^2/2 & 0 \\ & & 1 & -z & 0 \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}.$$

In accordance with (6.46), the solution is given in terms of a', b', c', d' in (5.159) as

$$\begin{aligned}(a'', b'', c'', d'') &= (d', c', b', a')|_{(a,b,c,d)\to(d,c,b,a)}\\ &= \frac{ab^2c}{\mathcal{B}}, \frac{\mathcal{B}}{\mathcal{A}}, \frac{\mathcal{A}^2}{\mathcal{B}}, \frac{bcd}{\mathcal{A}},\\ \mathcal{A} &= ab + ad + cd, \quad \mathcal{B} = ab^2 + 2abd + ad^2 + cd^2.\end{aligned} \tag{6.52}$$

6.6 Proof of Proposition 6.5

6.6.1 Matrix Product Formula of the Structure Function

Let us collect the necessary definitions and facts for the proof of Proposition 6.5. Following (2.24) and (2.25) we set

$$L = \sum_{a,b,i,j} E_{ai} \otimes E_{bj} \otimes L^{ab}_{ij}, \qquad M = \sum_{a,b,i,j} E_{ai} \otimes E_{bj} \otimes M^{ab}_{ij}, \tag{6.53}$$

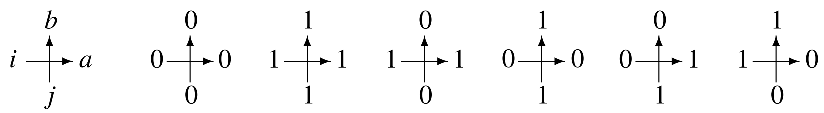
where the sums are taken over $a, b, i, j \in \{0, 1\}$ and E_{ij} is a matrix unit on $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$. The operators L^{ab}_{ij} and M^{ab}_{ij} are defined by

$$L^{ab}_{ij} = M^{ab}_{ij} = 0 \quad \text{if} \;\; a + b \neq i + j, \tag{6.54}$$

$$L^{00}_{00} = L^{11}_{11} = 1, \quad L^{10}_{10} = -q^2\mathbf{K}, \quad L^{01}_{01} = \mathbf{K}, \quad L^{10}_{01} = \mathbf{A}^+, \quad L^{01}_{10} = \mathbf{A}^-, \tag{6.55}$$

$$M^{00}_{00} = M^{11}_{11} = 1, \; M^{10}_{10} = q\tilde{\mathbf{K}}, \quad M^{01}_{01} = q\tilde{\mathbf{K}}, \quad M^{10}_{01} = \mathbf{A}^+, \quad M^{01}_{10} = \mathbf{A}^-. \tag{6.56}$$

Pictorially, the non-zero cases look like



L^{ab}_{ij}	1	1	$-q^2\mathbf{K}$	$\mathbf{K}$	$\mathbf{A}^+$	$\mathbf{A}^-$
M^{ab}_{ij}	1	1	$q\tilde{\mathbf{K}}$	$q\tilde{\mathbf{K}}$	$\mathbf{A}^+$	$\mathbf{A}^-$

(6.57)

The operators $\mathbf{A}^{\pm}$, $\mathbf{K}$ are q^2-oscillators in (5.15) and (5.16), and $\tilde{\mathbf{K}}$ is given by

$$\tilde{\mathbf{K}}|m\rangle = (-q^2)^m|m\rangle. \tag{6.58}$$

Thus we have $L, M \in \mathrm{End}(V \otimes V \otimes \mathcal{F}_{q^2})$. These definitions are related to the earlier ones as

$$L^{ab}_{ij} = L(1)^{ab}_{ij}|_{q\to q^2,\mu\to -q^2}, \qquad M^{ab}_{ij} = M(1)^{ab}_{ij}|_{q\to q^2,\nu\to q}, \tag{6.59}$$

where the RHSs are those in (3.120) and (3.121).

We will further need the boundary vectors

$$\langle\tilde{\eta}_2| = \sum_{m\geq 0}\frac{\langle 2m|}{(q^8;q^8)_m}, \qquad |\tilde{\eta}_1\rangle = \sum_{m\geq 0}\frac{|m\rangle}{(-q^2;-q^2)_m}, \tag{6.60}$$

whose pairing is specified by $\langle m|m'\rangle = (q^4;q^4)_m\delta_{m,m'}$. These are obtained from the earlier ones in (3.132)–(3.133) by replacing q with $-q^2$.

Now the function $R^{\mathbf{ab}}_{\mathbf{ij}}(x)$ appearing in (6.5) is given by the matrix product formula

$$R(x)^{\mathbf{ab}}_{\mathbf{ij}} = (\text{scalar}) \times \langle\tilde{\eta}_2|x^{\mathbf{h}}M^{a_1b_1}_{i_1j_1}\cdots M^{a_nb_n}_{i_nj_n}|\tilde{\eta}_1\rangle, \tag{6.61}$$

where $\mathbf{h}$ is the number operator (3.14) and $\mathbf{a} = (a_1,\ldots,a_n)$, etc. according to (6.1). The scalar can be chosen so that $R(x)^{\mathbf{ab}}_{\mathbf{ij}}$ becomes a polynomial in x^2 with maximal degree exactly n in the following sense[2]:

$$\max\left\{\deg_{x^2}\left(R(x)^{\mathbf{ab}}_{\mathbf{ij}}\right) \mid \mathbf{a},\mathbf{b},\mathbf{i},\mathbf{j}\in\{0,1\}^n\right\} = n. \tag{6.62}$$

Otherwise the normalization is not essential, being required only to validate (5.11) which does not influence (6.2) and (6.3).

Up to normalization and gauge, $R(x)^{\mathbf{ab}}_{\mathbf{ij}}$ is an element of the quantum R matrix of the spin representation of $B^{(1)}_n$ with spectral parameter x.[3] This fact and the related results will be explained in detail in Chapter 12.

What we need here is the tetrahedron equation of type $MMLL = LLMM$:

$$M_{126}M_{346}L_{135}L_{245} = L_{245}L_{135}M_{346}M_{126}. \tag{6.63}$$

This is a corollary of Theorem 3.25 and (6.59). It is an identity in $\mathrm{End}(\overset{1}{V}\otimes\overset{2}{V}\otimes\overset{3}{V}\otimes\overset{4}{V}\otimes\overset{5}{\mathcal{F}}_{q^2}\otimes\overset{6}{\mathcal{F}}_{q^2})$.

Another necessary fact is a property of the boundary vector

$$(\mathbf{A}^+ - 1 + \tilde{\mathbf{K}})|\tilde{\eta}_1\rangle = 0, \qquad (\mathbf{A}^- - 1 + q^2\tilde{\mathbf{K}})|\tilde{\eta}_1\rangle = 0. \tag{6.64}$$

This is a corollary of (3.134) and (3.135) and the origin of $|\tilde{\eta}_1\rangle$ mentioned after (6.60).

---

[2] It is guaranteed from the fact that the spectral decomposition of the R matrix consists of $(n+1)$ eigenvalues. One can also check the claim in the example of $S^{2,1}(z)$ for $n=2$ in Sect. 12.4.

[3] Equation (6.61) is related to (12.9) as $R(x)^{\mathbf{ab}}_{\mathbf{ij}} = (\text{scalar})\ S^{2,1}(x)^{\mathbf{ab}}_{\mathbf{ij}}|_{q\to -q^2,\alpha\to q}$, where α originates in (12.1). Therefore from Theorem 12.2, it is an element of the quantum R matrix of the spin representation of $U_{q^{-2}}(B^{(1)}_n)$.

6.6.2 RTT Relation

This subsection is devoted to the proof of the RTT relation (6.2). According the remark after Remark 6.3, we take the parameter μ_i in π_i to be 1.

Lemma 6.10 *The image of the generator* $t_{\mathbf{ab}} \in A_q(B_n)$ *by* π_i *in (6.9)–(6.12) with* $\mu_i = 1$ *(also denoted by* $t_{\mathbf{ab}}$ *for simplicity) satisfies the* RTT *relation including the spectral parameter:*

$$\sum_{\mathbf{l},\mathbf{m}} R(x)^{\mathbf{ab}}_{\mathbf{lm}}\, t_{\mathbf{lc}}\, t_{\mathbf{md}} = \sum_{\mathbf{l},\mathbf{m}} t_{\mathbf{bm}}\, t_{\mathbf{al}}\, R(x)^{\mathbf{lm}}_{\mathbf{cd}} \tag{6.65}$$

In view of (6.5), the defining RTT relation of $A_q(B_n)$ in (6.2) follows from this lemma by picking the highest order terms in x since π_i's are independent of it.

Proof First we treat π_i with $1 \le i \le n-1$. Then it is easy to see

$$t_{\mathbf{ab}} = \theta(a_k = b_k \text{ for } k \neq i, i+1) L^{a_i a_{i+1}}_{b_{i+1} b_i}, \tag{6.66}$$

where $\theta(\text{true}) = 1$ and $\theta(\text{false}) = 0$. Compare $(6.9)|_{\mu_i=1} - (6.10)$ with (6.55). Thus $\mathbf{l}$ and $\mathbf{m}$ in the LHS (resp. RHS) of (6.65) are restricted to those $(l_k, m_k) = (c_k, d_k)$ (resp. $(l_k, m_k) = (a_k, b_k)$) for $k \neq i, i+1$.

Let us write down the $\mathrm{End}(\overset{5}{\mathcal{F}}_{q^2} \otimes \overset{6}{\mathcal{F}}_{q^2})$ component of the tetrahedron equation (6.63) corresponding to the transition

$$v_{c_{i+1}} \otimes v_{d_{i+1}} \otimes v_{c_i} \otimes v_{d_i} \to v_{a_i} \otimes v_{b_i} \otimes v_{a_{i+1}} \otimes v_{b_{i+1}}.$$

The result reads as

$$\sum M^{a_i b_i}_{l_i m_i} M^{a_{i+1} b_{i+1}}_{l_{i+1} m_{i+1}} L^{l_i l_{i+1}}_{c_{i+1} c_i} L^{m_i m_{i+1}}_{d_{i+1} d_i} = \sum L^{b_i b_{i+1}}_{m_{i+1} m_i} L^{a_i a_{i+1}}_{l_{i+1} l_i} M^{l_i m_i}_{c_i d_i} M^{l_{i+1} m_{i+1}}_{c_{i+1} d_{i+1}}, \tag{6.67}$$

where the sums are taken over $l_i, m_i, l_{i+1}, m_{i+1} \in \{0, 1\}$ on both sides. The operators $L^{\bullet\bullet}_{\bullet\bullet}$ and $M^{\bullet\bullet}_{\bullet\bullet}$ act on a different components $\overset{5}{\mathcal{F}}_{q^2}$ and $\overset{6}{\mathcal{F}}_{q^2}$, respectively. One can check that (6.67) agrees with (2.30).

Putting (6.67) in a sandwich in the space $\overset{6}{\mathcal{F}}_{q^2}$ as

$$\langle \tilde{\eta}_2 | x^{\mathbf{h}} M^{a_1 b_1}_{c_1 d_1} \cdots M^{a_{i-1} b_{i-1}}_{c_{i-1} d_{i-1}} (\cdots) M^{a_{i+2} b_{i+2}}_{c_{i+2} d_{i+2}} \cdots M^{a_n b_n}_{c_n d_n} | \tilde{\eta}_1 \rangle \tag{6.68}$$

and applying (6.66) and (6.61), we obtain (6.65).

The essence of this derivation is that the matrix product structure (6.61) makes things *local* with respect to i, and the local structure is exactly the tetrahedron equation of type $MMLL = LLMM$. Up to this point, the boundary vectors $\langle \tilde{\eta}_2 | x^{\mathbf{h}}$ and $|\tilde{\eta}_1\rangle$ have not played any role.

Next we consider π_n. From (6.11) and (6.12) we only have to concern the last factor $M^{a_n b_n}_{i_n j_n}$ in the matrix product formula (6.61) and the effect of the boundary vector $|\tilde{\eta}_1\rangle$. Moreover, the common indices α in (6.11) can be suppressed. Therefore it suffices to check

$$\sum_{l_n, m_n} \mathrm{t}_{l_n c_n} \mathrm{t}_{m_n d_n} \otimes M^{a_n b_n}_{l_n m_n} |\tilde{\eta}_1\rangle = \sum_{l_n, m_n} \mathrm{t}_{b_n m_n} \mathrm{t}_{a_n l_n} \otimes M^{l_n m_n}_{c_n d_n} |\tilde{\eta}_1\rangle \tag{6.69}$$

for the temporarily defined symbols

$$\begin{pmatrix} \mathrm{t}_{00} & \mathrm{t}_{01} \\ \mathrm{t}_{10} & \mathrm{t}_{11} \end{pmatrix} := \begin{pmatrix} t_{\alpha 0, \alpha 0} & t_{\alpha 0, \alpha 1} \\ t_{\alpha 1, \alpha 0} & t_{\alpha 1, \alpha 1} \end{pmatrix} = \begin{pmatrix} \mathbf{a}^- & \mathbf{k} \\ -q\mathbf{k} & \mathbf{a}^+ \end{pmatrix}, \tag{6.70}$$

which is independent of $\alpha \in \{0, 1\}^{n-1}$. We have set $\mu_n = 1$ in (6.11). The relation (6.69) represents 16 identities in $\mathrm{End}(\mathcal{F}_q) \otimes \mathcal{F}_{q^2}$ corresponding to the choices $(a_n, b_n, c_n, d_n) \in \{0, 1\}^4$. It is elementary to verify them case by case. Here we shall only illustrate the two instructive examples. The other cases are similar.

Case $(a_n, b_n, c_n, d_n) = (0, 1, 0, 0)$. The difference LHS – RHS is

$$\begin{aligned} &\mathrm{t}_{00}\mathrm{t}_{10} \otimes M^{01}_{01}|\tilde{\eta}_1\rangle + \mathrm{t}_{10}\mathrm{t}_{00} \otimes M^{01}_{10}|\tilde{\eta}_1\rangle - \mathrm{t}_{10}\mathrm{t}_{00} \otimes M^{00}_{00}|\tilde{\eta}_1\rangle \\ &= \mathbf{a}^-(-q\mathbf{k}) \otimes q\tilde{\mathbf{K}}|\tilde{\eta}_1\rangle + (-q\mathbf{k})\mathbf{a}^- \otimes \mathbf{A}^-|\tilde{\eta}_1\rangle - (-q\mathbf{k}\mathbf{a}^-) \otimes |\tilde{\eta}_1\rangle \\ &= q\mathbf{k}\mathbf{a}^- \otimes (-q^2\tilde{\mathbf{K}} - \mathbf{A}^- + 1)|\tilde{\eta}_1\rangle = 0, \end{aligned}$$

where the last equality is due to (6.64).

Case $(a_n, b_n, c_n, d_n) = (0, 1, 0, 1)$. The difference LHS – RHS is

$$\begin{aligned} &\mathrm{t}_{00}\mathrm{t}_{11} \otimes M^{01}_{01}|\tilde{\eta}_1\rangle + \mathrm{t}_{10}\mathrm{t}_{01} \otimes M^{01}_{10}|\tilde{\eta}_1\rangle - \mathrm{t}_{11}\mathrm{t}_{00} \otimes M^{01}_{01}|\tilde{\eta}_1\rangle - \mathrm{t}_{10}\mathrm{t}_{01} \otimes M^{10}_{01}|\tilde{\eta}_1\rangle \\ &= (\mathbf{a}^-\mathbf{a}^+ - \mathbf{a}^+\mathbf{a}^-) \otimes q\tilde{\mathbf{K}}|\tilde{\eta}_1\rangle - q\mathbf{k}^2 \otimes (\mathbf{A}^- - \mathbf{A}^+)|\tilde{\eta}_1\rangle \\ &= (1 - q^2)\mathbf{k}^2 \otimes q\tilde{\mathbf{K}}|\tilde{\eta}_1\rangle - q\mathbf{k}^2 \otimes (1 - q^2)\tilde{\mathbf{K}}|\tilde{\eta}_1\rangle = 0, \end{aligned}$$

where the second equality is due to (3.12) and (6.64). □

6.6.3 ρTT Relations

The remaining task for the proof of Proposition 6.5 is to show:

Lemma 6.11 *The image of the generator $t_{\mathbf{ab}} \in A_q(B_n)$ by π_i in (6.9)–(6.12) with $\mu_i = 1$ satisfies (6.3).*

Proof We show

$$\sum_{\mathbf{b}} \rho_{\mathbf{b}}\, t_{\mathbf{ab}}\, t_{\mathbf{l'b'}} = \rho_{\mathbf{a}}\, \delta_{\mathbf{bl}}. \tag{6.71}$$

Another relation in (6.3) can be verified similarly. See (6.4) for the notation $\mathbf{a}'$.

First we treat π_i with $1 \le i \le n-1$. From (6.9)–(6.10), the relation (6.71) is trivially valid as $0 = 0$ unless $a_k = l_k$ for $k \neq i, i+1$. So we focus on this situation which can be classified into the four cases $(a_i, a_{i+1}) \in \{0,1\}^2$. The case $a_i = a_{i+1}$ is trivially valid as $0 = 0$ or $\rho_{\mathbf{a}}\, t_{\mathbf{aa}}\, t_{\mathbf{a'a'}} = \rho_{\mathbf{a}}$. A similar fact holds also for (l_i, l_{i+1}). Thus the only non-trivial case is

$$\sum_{b_1,b_2} r_{b_1b_2} \mathrm{t}_{a_1a_2,b_1b_2} \mathrm{t}_{l'_1l'_2,b'_1b'_2} = r_{a_1a_2} \delta_{a_1l_1} \delta_{a_2l_2} \tag{6.72}$$

for $(a_1, a_2), (l_1, l_2) \in \{(0,1), (1,0)\}$, where $b'_1 = 1 - b_1$, etc. The sum extends over $(b_1, b_2) = (0,1), (1,0)$ only. Here we have written a_i, a_{i+1} as a_1, a_2, etc. for simplicity and introduced the temporary notations

$$\begin{pmatrix} \mathrm{t}_{01,01} & \mathrm{t}_{01,10} \\ \mathrm{t}_{10,01} & \mathrm{t}_{10,10} \end{pmatrix} := \begin{pmatrix} \mathbf{A}^- & \mathbf{K} \\ -q^2\mathbf{K} & \mathbf{A}^+ \end{pmatrix}, \tag{6.73}$$

$$r_{01} = 1, \quad r_{10} = -q^2. \tag{6.74}$$

The former is taken from (6.9) and the latter reflects $\rho_{\dots 10\dots}/\rho_{\dots 01\dots} = -q^2$ according to (6.6). Now (6.72) reads as

$$\begin{aligned} &\mathrm{t}_{01,01}\mathrm{t}_{10,10} - q^2\mathrm{t}_{01,10}\mathrm{t}_{10,01} = 1, \quad \mathrm{t}_{01,01}\mathrm{t}_{01,10} - q^2\mathrm{t}_{01,10}\mathrm{t}_{01,01} = 0, \\ &\mathrm{t}_{10,01}\mathrm{t}_{10,10} - q^2\mathrm{t}_{10,10}\mathrm{t}_{10,01} = 0, \quad \mathrm{t}_{10,01}\mathrm{t}_{01,10} - q^2\mathrm{t}_{10,10}\mathrm{t}_{01,01} = -q^2. \end{aligned} \tag{6.75}$$

These relations can be checked directly by means of (5.15). For instance, the first one reduces to $\mathbf{A}^-\mathbf{A}^+ + q^4\mathbf{K}^2 = 1$.

Next we consider π_n. From (6.11)–(6.12), the relation (6.71) is trivially valid as $0 = 0$ unless $a_j = l_j$ for $j \neq n$. So we focus on this situation which consists of the four cases $(a_n, l_n) \in \{0,1\}^2$. In terms of the symbols in (6.70), they read as

$$\begin{aligned} &\mathrm{t}_{00}\mathrm{t}_{11} - q\mathrm{t}_{01}\mathrm{t}_{10} = 1, \quad \mathrm{t}_{00}\mathrm{t}_{01} - q\mathrm{t}_{01}\mathrm{t}_{00} = 0, \\ &\mathrm{t}_{10}\mathrm{t}_{11} - q\mathrm{t}_{11}\mathrm{t}_{10} = 0, \quad \mathrm{t}_{10}\mathrm{t}_{01} - q\mathrm{t}_{11}\mathrm{t}_{00} = -q, \end{aligned} \tag{6.76}$$

where the coefficient $-q$ reflects $\rho_{\dots 1}/\rho_{\dots 0} = -q$ from (6.6). Comparison of (6.73) and (6.70) shows that (6.76) is exactly reduced to (6.75) by replacing q with q^2. □

This completes a proof of Proposition 6.5. We did not use the property of the boundary vector $\langle\tilde{\eta}_2|$ directly. However, its effect is working implicitly via the fact that the normalization (6.62) is possible. We note that the representation π_i satisfies further relations obtained by taking the coefficients of the non-highest powers of x in (6.65).

6.7 Bibliographical Notes and Comments

This chapter is an extended exposition of [93, Sect. 4] and [94]. As mentioned in the beginning of Sect. 6.1, the algebra $A_q(B_n)$ in this chapter is different from $\mathrm{Fun}(\mathrm{SO}_q(2n+1))$ in [127, Definition 11] which is defined with $(2n+1)^2$ generators based on the vector representation. For $n=2$, an explicit embedding $A_q(B_2)(\simeq A_q(C_2)) \hookrightarrow \mathrm{Fun}(\mathrm{SO}_{q^2}(5))$ is shown in [94, Theorem 2.1]. The concrete forms of the fundamental representations (6.9)–(6.12), Proposition 6.5 and its proof based on the tetrahedron equation $MMLL = LLMM$ are presented for the first time in Sect. 6.6.

Chapter 7
Intertwiners for Quantized Coordinate Ring $A_q(F_4)$

Abstract We study the intertwiners of $A_q(F_4)$ based on the subalgebras $A_q(A_2)$, $A_q(B_2)$ and $A_q(C_2)$ without going into explicit forms of the defining relations among generators and the fundamental representations. A natural F_4 analogue of the tetrahedron/3D reflection equation is presented which contains fifty operators on each side. As suggested by the relevant Dynkin diagrams, it consists of a mixture of B_3 and C_3 structures, and is attributed to a composition of the 3D reflection equations $SKSSKKS = SKKSSKS$ of type B and $RKRRKKR = RKKRRKR$ of type C twelve times for each.

7.1 Fundamental Representations

Let $A_q(F_4)$ be the quantized coordinate ring of F_4. We list the relevant Dynkin diagrams in Fig. 7.1.

Let π_i be the fundamental representation attached to the vertex i of the F_4 Dynkin diagram in Fig. 7.1 in the sense of Theorem 3.3. It is realized in terms of Osc_{q_i} [1] as

$$\pi_i : \; A_q(F_4) \to \mathrm{End}(\mathcal{F}_{q_i}) \quad \text{with} \quad (q_1, q_2, q_3, q_4) = (q, q, q^2, q^2).$$

The Weyl group $W(F_4)$ is generated by simple reflections s_i obeying the Coxeter relations

$$\begin{aligned} &s_i^2 = 1, \quad s_1s_3 = s_3s_1, \quad s_1s_4 = s_4s_1, \quad s_2s_4 = s_4s_2, \\ &s_1s_2s_1 = s_2s_1s_2, \quad s_2s_3s_2s_3 = s_3s_2s_3s_2, \quad s_3s_4s_3 = s_4s_3s_4. \end{aligned} \tag{7.1}$$

[1] For the definition, see (3.12)–(3.13) and (5.15)–(5.16).

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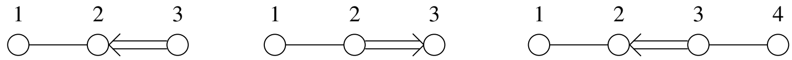

Fig. 7.1 Dynkin diagrams of C_3 (left), B_3 (center) and F_4 (right). Enumeration of vertices for F_4 agrees with [27] which is opposite to [67]

According to Theorem 3.3, the following equivalence between the tensor product representations $\pi_{j_1,\dots,j_k} := \pi_{j_1} \otimes \cdots \otimes \pi_{j_k}$ is valid:

$$\pi_{13} \simeq \pi_{31}, \quad \pi_{14} \simeq \pi_{41}, \quad \pi_{24} \simeq \pi_{42}, \tag{7.2}$$

$$\pi_{121} \simeq \pi_{212}, \quad \pi_{2323} \simeq \pi_{3232}, \quad \pi_{343} \simeq \pi_{434}. \tag{7.3}$$

7.2 Intertwiners

Due to Remark 3.4, the intertwiners for (7.2) is given by the exchange of components P in (3.23). As for (7.3), we introduce three kinds of intertwiners Φ, Ψ, Υ, which are characterized by the intertwining relations ($f \in A_q(F_4)$)

$$\Phi \circ \pi_{121}(\Delta(f)) = \pi_{212}(\Delta(f)) \circ \Phi, \tag{7.4}$$

$$\Psi \circ \pi_{2323}(\Delta(f)) = \pi_{3232}(\Delta(f)) \circ \Psi, \tag{7.5}$$

$$\Upsilon \circ \pi_{343}(\Delta(f)) = \pi_{434}(\Delta(f)) \circ \Upsilon \tag{7.6}$$

and by the same normalization condition as (3.29) for Φ, Υ and (5.35) for Ψ. As before we set

$$\mathrm{R} = \Phi P_{13} \in \mathrm{End}(\mathcal{F}_q \otimes \mathcal{F}_q \otimes \mathcal{F}_q), \tag{7.7}$$

$$\mathrm{K} = \Psi P_{14} P_{23} \in \mathrm{End}(\mathcal{F}_{q^2} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_q), \tag{7.8}$$

$$\mathrm{S} = \Upsilon P_{13} \in \mathrm{End}(\mathcal{F}_{q^2} \otimes \mathcal{F}_{q^2} \otimes \mathcal{F}_{q^2}), \tag{7.9}$$

where we use upright font to temporarily distinguish them from many definitions in the preceding chapters for types A, B and C. P_{ij} is the exchanger of the ith and the jth components from the left as usual.

From Fig. 7.1, we see that $A_q(F_4)$ has the subalgebras $A_q(B_3)$ and $A_q(C_3)$. Accordingly, we consider the subgroups of $W(F_4)$ realized as $W(B_3) = \langle s_2, s_3, s_4 \rangle$ and $W(C_3) = \langle s_1, s_2, s_3 \rangle$. They have the longest elements

$$s_3 s_2 s_3 s_1 s_2 s_1 s_3 s_2 s_1 \in W(C_3) \subset W(F_4), \tag{7.10}$$

$$s_2 s_3 s_2 s_4 s_3 s_4 s_2 s_3 s_4 \in W(B_3) \subset W(F_4). \tag{7.11}$$

Then applying the same argument as Theorems 5.16 and 6.7, we obtain the 3D reflection equations

$$\mathrm{R}_{689}\mathrm{K}_{3579}\mathrm{R}_{249}\mathrm{R}_{258}\mathrm{K}_{1478}\mathrm{K}_{1236}\mathrm{R}_{456} = \mathrm{R}_{456}\mathrm{K}_{1236}\mathrm{K}_{1478}\mathrm{R}_{258}\mathrm{R}_{249}\mathrm{K}_{3579}\mathrm{R}_{689},$$
$$\in \mathrm{End}\big(\overset{1}{\mathcal{F}}_{q^2} \otimes \overset{2}{\mathcal{F}}_q \otimes \overset{3}{\mathcal{F}}_{q^2} \otimes \overset{4}{\mathcal{F}}_q \otimes \overset{5}{\mathcal{F}}_q \otimes \overset{6}{\mathcal{F}}_q \otimes \overset{7}{\mathcal{F}}_{q^2} \otimes \overset{8}{\mathcal{F}}_q \otimes \overset{9}{\mathcal{F}}_q\big), \tag{7.12}$$
$$\mathrm{S}_{689}\mathrm{K}_{9753}\mathrm{S}_{249}\mathrm{S}_{258}\mathrm{K}_{8741}\mathrm{K}_{6321}\mathrm{S}_{456} = \mathrm{S}_{456}\mathrm{K}_{6321}\mathrm{K}_{8741}\mathrm{S}_{258}\mathrm{S}_{249}\mathrm{K}_{9753}\mathrm{S}_{689} \tag{7.13}$$
$$\in \mathrm{End}\big(\overset{1}{\mathcal{F}}_q \otimes \overset{2}{\mathcal{F}}_{q^2} \otimes \overset{3}{\mathcal{F}}_q \otimes \overset{4}{\mathcal{F}}_{q^2} \otimes \overset{5}{\mathcal{F}}_{q^2} \otimes \overset{6}{\mathcal{F}}_{q^2} \otimes \overset{7}{\mathcal{F}}_q \otimes \overset{8}{\mathcal{F}}_{q^2} \otimes \overset{9}{\mathcal{F}}_{q^2}\big).$$

From the Dynkin diagram in Fig. 7.1, $A_q(F_4)$ also has the subalgebras isomorphic to $A_q(A_2)$ corresponding to the vertices $1, 2$ and $A_{q^2}(A_2)$ corresponding to the vertices $3, 4$. Therefore we know that R and S in (7.7) and (7.9) satisfy the tetrahedron equations $RRRR = RRRR$ and $SSSS = SSSS$, although they are not derived in the same manner as the preceding chapters because of $W(A_3) \not\subset W(F_4)$.

7.3 F_4 Analogue of the Tetrahedron/3D Reflection Equations

Let us derive a consistency condition mixing R, K, S, which can be viewed as a natural F_4 analogue of the tetrahedron/3D reflection equations. The procedure is parallel with Theorem 5.16. It is formulated for each reduced word of the longest element in the Weyl group $w_0 = -1 \in W(F_4)$. The length of w_0 is 24 and there are 2144892 reduced expressions for it. Below we demonstrate the derivation along the example

$$\mathbf{w}_0 = s_{i_1} \cdots s_{i_{24}} = s_4 s_3 s_4 s_2 s_3 s_4 s_2 s_3 s_2 s_1 s_2 s_3 s_4 s_2 s_3 s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_2 s_1, \tag{7.14}$$

where we use the bold font to distinguish it as a word rather than an element of $W(F_4)$. The reversed word $\tilde{\mathbf{w}}_0 = s_{i_{24}} \cdots s_{i_1}$ is another reduced expression of $w_0^{-1} = w_0$ which is "most distant" from $\mathbf{w}_0$. From Theorem 3.3, the tensor product representations corresponding to $\mathbf{w}_0$ and $\tilde{\mathbf{w}}_0$ are equivalent, i.e,

$$\pi_{434234232123423123412321} \simeq \pi_{123214321324321232432434}. \tag{7.15}$$

The intertwiner for this is constructed by composing the basic ones (7.4)–(7.6) along the transformation $\mathbf{w}_0 \to \tilde{\mathbf{w}}_0$ by means of the Coxeter relations (7.1). One way to achieve this is shown below.

$$\begin{array}{rll}
\mathbf{w}_0: & \underline{434}23\underline{42}3212342312\underline{341}2321 & P_{6,7}P_{18,19}P_{19,20}\Upsilon^{-1}_{1,2,3} \\
& 34\underline{3232}432123423\underline{121}342321 & \Psi^{-1}_{3,4,5,6}\Phi_{16,17,18} \\
& 3\underline{42}32\underline{343212}3423212342321 & P_{2,3}P_{10,11}P_{9,10}P_{8,9}\Upsilon_{6,7,8} \\
& 3243\underline{24}3212\underline{434}232123\underline{42}321 & P_{5,6}P_{20,21}\Upsilon^{-1}_{11,12,13} \\
& 32\underline{434}2321234\underline{32321}2324321 & \Upsilon^{-1}_{3,4,5}P_{16,17}\Psi^{-1}_{13,14,15,16} \\
& 3234\underline{32321}23\underline{42}321\underline{3232}4321 & P_{8,9}\Psi^{-1}_{5,6,7,8}P_{12,13}\Psi^{-1}_{17,18,19,20} \\
& 323\underline{42}321\underline{3232}432123\underline{234321} & P_{4,5}\Psi^{-1}_{9,10,11,12}P_{19,20}P_{23,24}P_{22,23}\Upsilon_{20,21,22} \\
& \underline{3232}4321232\underline{343212}3423214 & \Psi^{-1}_{1,2,3,4}P_{16,17}P_{15,16}P_{14,15}\Upsilon_{12,13,14} \\
& 232\underline{343212}3\underline{24}3212\underline{434}23214 & \Upsilon^{-1}_{17,18,19}P_{11,12}P_{8,9}P_{7,8}P_{6,7}\Upsilon_{4,5,6} \\
& 23243212\underline{434}232123\underline{432321}4 & \Upsilon^{-1}_{9,10,11}P_{18,19}P_{22,23}\Psi^{-1}_{19,20,21,22} \\
& 23243\underline{2123432321}232432134 & P_{9,10}P_{8,9}\Phi^{-1}_{6,7,8}P_{14,15}\Psi^{-1}_{11,12,13,14} \\
& 2324\underline{31}23412321\underline{3232}432134 & P_{5,6}\Psi^{-1}_{15,16,17,18} \\
& 232\underline{41}3234123\underline{2123}2\underline{34321}34 & P_{4,5}P_{15,16}\Phi^{-1}_{13,14,15}P_{21,22}P_{20,21}\Upsilon_{18,19,20} \\
& 23214323412\underline{31}23124321434 & P_{12,13} \\
& 232143234\underline{121}323124321434 & \Phi_{10,11,12} \\
& 23214323\underline{4212323}124321434 & P_{10,11}P_{9,10}\Psi_{12,13,14,15} \\
& 23214\underline{32321}4323\underline{21243}21434 & P_{9,10}\Psi^{-1}_{6,7,8,9}P_{18,19}P_{17,18}\Phi^{-1}_{15,16,17} \\
& 2321\underline{42}321\underline{3432}3\underline{124}3\underline{121}434 & P_{5,6}P_{12,13}\Upsilon_{10,11,12}P_{15,16}P_{16,17}\Phi_{19,20,21} \\
& 23\underline{212}432\underline{143}2\underline{4341}23212434 & \Phi^{-1}_{3,4,5}P_{10,11}P_{9,10}P_{15,16}\Upsilon^{-1}_{13,14,15} \\
& 2\underline{31}2143\underline{24}312\underline{34132321}2434 & P_{2,3}P_{8,9}P_{13,14}P_{14,15}P_{19,20}\Psi^{-1}_{16,17,18,19} \\
& 21321\underline{434}23\underline{121}3\underline{42}32132434 & \Upsilon^{-1}_{6,7,8}\Phi_{11,12,13}P_{15,16} \\
& 2132\underline{13432321}232432132434 & P_{6,7}P_{5,6}P_{11,12}\Psi^{-1}_{8,9,10,11} \\
& 21323412321\underline{3232}432132434 & \Psi^{-1}_{12,13,14,15} \\
& 213234123\underline{2123}2\underline{34321}32434 & P_{12,13}\Phi^{-1}_{10,11,12}P_{18,19}P_{17,18}\Upsilon_{15,16,17} \\
& 21323412\underline{31}23124321432434 & P_{9,10} \\
& 213234\underline{121}323124321432434 & \Phi_{7,8,9} \\
& 21323\underline{4212323}124321432434 & P_{7,8}P_{6,7}\Psi_{9,10,11,12} \\
& 21\underline{32321}4323\underline{21243}21432434 & P_{6,7}\Psi^{-1}_{3,4,5,6}P_{15,16}P_{14,15}\Phi^{-1}_{12,13,14} \\
& \underline{2123}21\underline{3432}3\underline{124}3\underline{121}432434 & P_{3,4}\Phi^{-1}_{1,2,3}P_{9,10}\Upsilon_{7,8,9}P_{12,13}P_{13,14}\Phi_{16,17,18} \\
& 123\underline{1214}32\underline{4341}23212432434 & P_{6,7}\Phi_{4,5,6}P_{12,13}\Upsilon^{-1}_{10,11,12} \\
& 123214\underline{23234}1\underline{32321}2432434 & P_{10,11}\Psi_{7,8,9,10}P_{16,17}\Psi^{-1}_{13,14,15,16} \\
& 12321432\underline{342123}2132432434 & P_{9,10}P_{10,11}P_{13,14}\Phi^{-1}_{11,12,13} \\
& 1232143213\underline{42}3\underline{121}32432434 & P_{11,12}\Phi_{14,15,16} \\
\tilde{\mathbf{w}}_0: & 123214321324321232432434. &
\end{array} \tag{7.16}$$

The underlines indicate the tensor components on which the operator written on the right-hand side act non-trivially. Analogous procedures have appeared in (3.93) and (5.106).

Let us write (7.16) schematically as

$$\mathbf{w}_0 \xrightarrow{O_1} \mathbf{w}_1 \xrightarrow{O_2} \cdots \xrightarrow{O_{N-1}} \mathbf{w}_{N-1} \xrightarrow{O_N} \mathbf{w}_N = \tilde{\mathbf{w}}_0, \tag{7.17}$$

where $O_m \in \{P_{k,k+1}, \Phi^{\pm 1}_{k,k+1,k+2}, \Psi^{\pm 1}_{k,k+1,k+2,k+3}, \Upsilon^{\pm 1}_{k,k+1,k+2,k+3}\}$ and $N = 126$. For instance, $O_1 = \Upsilon^{-1}_{1,2,3}$ and $O_{126} = P_{11,12}$. Considering the inverse procedure reversing the length 24 arrays at every stage, one finds another route going from $\mathbf{w}_0$ to $\tilde{\mathbf{w}}_0$ as

$$\mathbf{w}_0 = \tilde{\mathbf{w}}_N \xrightarrow{\tilde{O}_N^{-1}} \tilde{\mathbf{w}}_{N-1} \xrightarrow{\tilde{O}_{N-1}^{-1}} \cdots \xrightarrow{\tilde{O}_2^{-1}} \tilde{\mathbf{w}}_1 \xrightarrow{\tilde{O}_1^{-1}} \tilde{\mathbf{w}}_0, \tag{7.18}$$

where $\tilde{\mathbf{w}}_r$ denotes the reverse word of $\mathbf{w}_r$. The operators $\tilde{O}_m$ is chosen according to O_m as

$$\begin{aligned} O_m &: P_{k,k+1}, \quad \Phi^{\pm 1}_{k,k+1,k+2}, \quad \Upsilon^{\pm 1}_{k,k+1,k+2}, \quad \Psi^{\pm 1}_{k,k+1,k+2,k+3}, \\ \tilde{O}_m &: P_{j+1,j+2}, \quad \Phi^{\pm 1}_{j,j+1,j+2}, \quad \Upsilon^{\pm 1}_{j,j+1,j+2}, \quad \Psi^{\mp 1}_{j-1,j,j+1,j+2} \end{aligned} \tag{7.19}$$

with $j + k = 23$. The reason for exceptionally inverting Ψ is that the reverse ordering of 2323 into 3232 interchanges the role of two sides in (7.5). From the uniqueness of the intertwiner for the equivalence (7.15) with the normalization mentioned after (7.6), we get a non-trivial identity

$$O_N \cdots O_2 O_1 = \tilde{O}_1^{-1} \tilde{O}_2^{-1} \ldots \tilde{O}_N^{-1} \in \mathrm{Hom}\big(\mathcal{F}_{q_{i_1}} \otimes \cdots \otimes \mathcal{F}_{q_{i_{24}}}, \mathcal{F}_{q_{i_{24}}} \otimes \cdots \otimes \mathcal{F}_{q_{i_1}}\big), \tag{7.20}$$

where the i_k indices are those appearing in the reduced word (7.14). It is expressed by R, K, S and P by substituting (7.7)–(7.9). Then one can send all the P_{ij}'s to the right separating each side into an R, K, S part and a P part. The P part yields the permutation corresponding to the longest element in the symmetric group $\mathfrak{S}_{24}$ in both sides. Thus it can be cancelled, leaving an identity in $\mathrm{End}\big(\mathcal{F}_{q_{i_{24}}} \otimes \cdots \otimes \mathcal{F}_{q_{i_1}}\big)$ containing the R, K, S part only. Explicitly it has the form

$$
\begin{aligned}
&\mathrm{R}_{14,15,16}\mathrm{R}^{-1}_{16,11,9}\mathrm{K}_{7,8,10,16}\mathrm{K}^{-1}_{17,15,13,9}\mathrm{R}_{4,5,16}\mathrm{S}^{-1}_{17,12,7}\mathrm{R}^{-1}_{16,2,1}\mathrm{S}_{6,10,17}\mathrm{R}_{9,14,18}\\
&\times \mathrm{K}^{-1}_{17,5,3,1}\mathrm{R}^{-1}_{18,15,11}\mathrm{K}_{6,8,12,18}\mathrm{R}_{1,4,18}\mathrm{R}^{-1}_{15,8,1}\mathrm{S}_{7,13,19}\mathrm{K}^{-1}_{19,11,6,1}\mathrm{K}^{-1}_{19,15,12,4}\mathrm{S}^{-1}_{19,10,3}\\
&\times \mathrm{R}_{4,8,11}\mathrm{K}^{-1}_{20,14,7,1}\mathrm{R}^{-1}_{18,5,2}\mathrm{S}^{-1}_{20,13,6}\mathrm{S}_{3,12,20}\mathrm{R}_{1,9,21}\mathrm{K}^{-1}_{20,15,10,2}\mathrm{R}^{-1}_{21,14,4}\mathrm{K}_{3,8,13,21}\\
&\times \mathrm{R}_{2,11,21}\mathrm{R}^{-1}_{14,8,2}\mathrm{S}_{6,7,22}\mathrm{K}^{-1}_{22,4,3,2}\mathrm{R}^{-1}_{21,15,5}\mathrm{K}^{-1}_{22,14,13,11}\mathrm{S}^{-1}_{22,12,10}\mathrm{K}^{-1}_{23,9,6,2}\mathrm{S}^{-1}_{23,7,3}\\
&\times \mathrm{S}_{19,20,22}\mathrm{K}^{-1}_{22,18,17,16}\mathrm{S}_{10,13,23}\mathrm{K}^{-1}_{23,14,12,5}\mathrm{S}_{3,6,24}\mathrm{K}^{-1}_{23,21,19,16}\mathrm{K}^{-1}_{24,9,7,4}\mathrm{S}^{-1}_{23,20,17}\\
&\times \mathrm{K}^{-1}_{24,11,10,5}\mathrm{S}^{-1}_{24,13,12}\mathrm{S}_{17,19,24}\mathrm{K}^{-1}_{24,21,20,18}\mathrm{R}_{5,8,9}\mathrm{S}^{-1}_{24,23,22}\\
&= \mathrm{S}_{22,23,24}\mathrm{R}^{-1}_{9,8,5}\mathrm{K}^{-1}_{24,21,20,18}\mathrm{S}^{-1}_{24,19,17}\mathrm{S}_{12,13,24}\mathrm{K}^{-1}_{24,11,10,5}\\
&\times \mathrm{S}_{17,20,23}\mathrm{K}^{-1}_{24,9,7,4}\mathrm{K}^{-1}_{23,21,19,16}\mathrm{S}^{-1}_{24,6,3}\mathrm{K}^{-1}_{23,14,12,5}\mathrm{S}^{-1}_{23,13,10}\mathrm{K}^{-1}_{22,18,17,16}\mathrm{S}^{-1}_{22,20,19}\\
&\times \mathrm{S}_{3,7,23}\mathrm{K}^{-1}_{23,9,6,2}\mathrm{S}_{10,12,22}\mathrm{K}^{-1}_{22,14,13,11}\mathrm{R}_{5,15,21}\mathrm{K}^{-1}_{22,4,3,2}\mathrm{S}^{-1}_{22,7,6}\mathrm{R}_{2,8,14}\mathrm{R}^{-1}_{21,11,2}\\
&\times \mathrm{K}_{3,8,13,21}\mathrm{R}_{4,14,21}\mathrm{K}^{-1}_{20,15,10,2}\mathrm{R}^{-1}_{21,9,1}\mathrm{S}^{-1}_{20,12,3}\mathrm{S}_{6,13,20}\mathrm{R}_{2,5,18}\mathrm{K}^{-1}_{20,14,7,1}\mathrm{R}^{-1}_{11,8,4}\\
&\times \mathrm{S}_{3,10,19}\mathrm{K}^{-1}_{19,15,12,4}\mathrm{K}^{-1}_{19,11,6,1}\mathrm{S}^{-1}_{19,13,7}\mathrm{R}_{1,8,15}\mathrm{R}^{-1}_{18,4,1}\mathrm{K}_{6,8,12,18}\mathrm{R}_{11,15,18}\mathrm{K}^{-1}_{17,5,3,1}\\
&\times \mathrm{R}^{-1}_{18,14,9}\mathrm{S}^{-1}_{17,10,6}\mathrm{R}_{1,2,16}\mathrm{S}_{7,12,17}\mathrm{R}^{-1}_{16,5,4}\mathrm{K}^{-1}_{17,15,13,9}\mathrm{K}_{7,8,10,16}\mathrm{R}_{9,11,16}\mathrm{R}^{-1}_{16,15,14}.
\end{aligned}
\tag{7.21}
$$

The indices specify the components in $\mathcal{F}_{q_{i_{24}}} \otimes \cdots \otimes \mathcal{F}_{q_{i_1}}$ counted from the left on which the operators act non-trivially. We have set $\mathrm{R}_{k,j,i} = P_{i,k}\mathrm{R}_{i,j,k}P_{i,k}$, $\mathrm{S}_{k,j,i} = P_{i,k}\mathrm{S}_{i,j,k}P_{i,k}$ and $\mathrm{K}_{l,k,j,i} = P_{i,l}P_{j,k}\mathrm{K}_{i,j,k,l}P_{i,l}P_{j,k}$ as usual. This is an F_4 analogue of the tetrahedron/3D reflection equations. The two sides are related by reverse ordering of the operators and the exchange $\mathrm{R}_{i,j,k} \leftrightarrow \mathrm{R}^{-1}_{k,j,i}$, $\mathrm{S}_{i,j,k} \leftrightarrow \mathrm{S}^{-1}_{k,j,i}$. In both sides, each of R, R^{-1}, S, S^{-1} appear 8 times, whereas the numbers of K and K^{-1} are 3 and 15, leading to 50 operators in total. To summarize, we have shown:

Theorem 7.1 *The intertwiners* R, K, S *of* $A_q(F_4)$ *in (7.7)–(7.9) satisfy the* F_4 *analogue of the tetrahedron/3D reflection equations. It is associated with each reduced expression of the longest element* $w_0 \in W(F_4)$ *whose concrete form for the choice (7.14) is given by (7.21).*

Any reduced expression $\mathbf{w}$ of w_0 is transformed to $\mathbf{w}_0$ in (7.14) by the Coxeter relations (7.1) [119]. It follows that the consistency condition arising from the reverse ordering $\mathbf{w} \to \tilde{\mathbf{w}}$ is equivalent to the one for $\mathbf{w}_0 \to \tilde{\mathbf{w}}_0$ by a conjugation. In this sense, (7.21) can be regarded as a canonical form as well as one particular presentation among other numerous guises. See Remark 7.4 for a related observation.

7.4 Reduction to 3D Reflection Equations

The intertwiners are reducible to the rank 2 cases. From the Dynkin diagrams in Fig. 7.1 one can identify them, with appropriate choice of basis, with those appearing in earlier chapters as

$$\mathrm{R} = R \text{ in (3.30)–(3.32)}, \tag{7.22}$$

$$\mathrm{K} = K \text{ in (5.36)–(5.38)}, \tag{7.23}$$

$$\mathrm{S} = S := (R \text{ in } (3.30)–(3.32))\,|_{q\to q^2}\,. \tag{7.24}$$

The last one, S, also appeared in (6.26).

Recall that R and K satisfy $R_{i,j,k} = R^{-1}_{i,j,k} = R_{k,j,i}$ and $K = K^{-1}$. See (3.59), (3.60) and (5.72). Thus S in (7.24) also satisfies $S_{i,j,k} = S^{-1}_{i,j,k} = S_{k,j,i}$. By substituting (7.22)–(7.24) into (7.21) and applying these properties, it is simplified into

$$\begin{aligned}
&R_{14,15,16}R_{9,11,16}K_{7,8,10,16}K_{17,15,13,9}R_{4,5,16}S_{7,12,17}R_{1,2,16}S_{6,10,17}R_{9,14,18}\\
&\times K_{17,5,3,1}R_{11,15,18}K_{6,8,12,18}R_{1,4,18}R_{1,8,15}S_{7,13,19}K_{19,11,6,1}K_{19,15,12,4}S_{3,10,19}\\
&\times R_{4,8,11}K_{20,14,7,1}R_{2,5,18}S_{6,13,20}S_{3,12,20}R_{1,9,21}K_{20,15,10,2}R_{4,14,21}K_{3,8,13,21}\\
&\times R_{2,11,21}R_{2,8,14}S_{6,7,22}K_{22,4,3,2}R_{5,15,21}K_{22,14,13,11}S_{10,12,22}K_{23,9,6,2}S_{3,7,23}\\
&\times S_{19,20,22}K_{22,18,17,16}S_{10,13,23}K_{23,14,12,5}S_{3,6,24}K_{23,21,19,16}K_{24,9,7,4}S_{17,20,23}\\
&\times K_{24,11,10,5}S_{12,13,24}S_{17,19,24}K_{24,21,20,18}R_{5,8,9}S_{22,23,24}\\
=\;&\text{product in reverse order,}
\end{aligned} \tag{7.25}$$

where the reverse ordering does not change the internal indices of $K_{i,j,k,l}$ into $K_{l,k,j,i}$. The 50 operators appearing here have distinct set of indices.

Let us call (7.25) the F_4 compatibility equation or F_4 compatibility for short. In this terminology, the 3D reflection equations (7.13) and (7.12) are the B_3 compatibility and the C_3 compatibility, respectively.[2] From the fact that $A_q(B_3), A_q(C_3) \subset A_q(F_4)$, it is natural to expect that the F_4 compatibility equation is attributed to a composition of the B_3 and C_3 compatibility equations. This is indeed the case. The following theorem and the proof contain finer information.

Theorem 7.2 *The F_4 compatibility equation (7.25) is reduced to a composition of the B_3 compatibility (7.13) and the C_3 compatibility (7.12) twelve times for each.*

Proof Let X_0 denote the expression in the LHS of (7.25) which consists of 16 R's, 16 S's and 18 K's. It can be transformed to the RHS along

$$X_0 \to Y_0 \to X_1 \to Y_1 \to \cdots \to X_{24} \to Y_{24} = \text{reverse ordering of} X_0.$$

Here, rewriting $X_i \to Y_i$ only uses trivial commutativity of operators having totally distinct indices. On the other hand, the step $Y_i \to X_{i+1}$ indicates an application of a 3D reflection equation, which reverses seven consecutive factors somewhere in the length 50 array Y_i. Let us label the 50 operators in X_0 with $1, 2, \ldots, 50$ by saying that $X_0 = 1 \cdot 2 \cdot \cdots \cdot 50$. Thus for instance, $1 = R_{14,15,16}, 2 = R_{9,11,16}, 3 = K_{7,8,10,16}$ and $50 = S_{22,23,24}$. To save space, we specify a length 50 array in two rows. Thus X_0 is expressed as $\left(\substack{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25\\26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50}\right)$. The intermediate forms $Y_0, Y_1, \ldots, Y_{23}$ are listed below in such a notation.[3]

---

[2] The tetrahedron equation $RRRR = RRRR$ is the A_3 compatibility. R, S, K are identified with R, S, K in the sequel.

$$\begin{pmatrix} 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 \\ 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 39, 40, 41, 43, 45, 46, 37, 38, 42, 44, 47, 48, 50, 49 \end{pmatrix}$$
$$\begin{pmatrix} 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 \\ 26, 27, 28, 29, 30, 31, 32, 35, 36, 41, 43, 33, 34, 39, 40, 45, 46, 50, 48, 47, 44, 42, 38, 37, 49 \end{pmatrix}$$
$$\begin{pmatrix} 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 \\ 26, 27, 28, 29, 32, 30, 31, 35, 36, 41, 43, 50, 46, 45, 40, 49, 39, 34, 33, 48, 47, 44, 42, 38, 37 \end{pmatrix}$$
$$\begin{pmatrix} 50, 1, 2, 3, 4, 5, 6, 15, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 19, 18, 20, 21, 22, 23, 24 \\ 25, 26, 27, 28, 29, 43, 41, 36, 35, 31, 30, 46, 32, 45, 40, 49, 39, 34, 33, 48, 47, 44, 42, 38, 37 \end{pmatrix}$$
$$\begin{pmatrix} 50, 1, 2, 3, 5, 19, 4, 6, 15, 7, 8, 17, 10, 16, 14, 9, 13, 20, 24, 26, 43, 18, 12, 22, 23 \\ 27, 41, 46, 36, 11, 21, 25, 28, 32, 45, 48, 29, 35, 40, 49, 47, 39, 44, 42, 31, 30, 34, 33, 38, 37 \end{pmatrix}$$
$$\begin{pmatrix} 50, 1, 2, 3, 5, 19, 4, 6, 15, 7, 8, 17, 10, 16, 14, 9, 13, 20, 24, 26, 43, 18, 12, 22, 23 \\ 27, 41, 46, 48, 36, 45, 47, 32, 28, 25, 21, 29, 35, 40, 49, 39, 44, 42, 31, 30, 34, 11, 33, 38, 37 \end{pmatrix}$$
$$\begin{pmatrix} 50, 1, 2, 3, 5, 19, 4, 6, 15, 7, 8, 17, 10, 16, 14, 9, 13, 20, 24, 26, 43, 48, 46, 18, 41 \\ 27, 23, 22, 36, 45, 47, 32, 28, 25, 12, 21, 29, 35, 40, 49, 39, 44, 42, 31, 30, 34, 11, 33, 38, 37 \end{pmatrix}$$
$$\begin{pmatrix} 50, 48, 1, 2, 3, 5, 19, 4, 6, 15, 17, 43, 46, 7, 8, 10, 16, 14, 26, 24, 20, 13, 18, 41, 45 \\ 27, 23, 22, 36, 47, 32, 28, 25, 9, 12, 21, 29, 35, 40, 49, 39, 44, 42, 31, 30, 34, 11, 33, 38, 37 \end{pmatrix}$$
$$\begin{pmatrix} 50, 48, 1, 2, 3, 5, 19, 4, 6, 15, 17, 43, 26, 46, 7, 8, 10, 16, 18, 41, 45, 47, 14, 24, 27 \\ 32, 49, 20, 23, 36, 40, 28, 22, 25, 35, 39, 29, 13, 21, 44, 42, 31, 12, 30, 34, 9, 11, 33, 38, 37 \end{pmatrix}$$
$$\begin{pmatrix} 50, 48, 1, 2, 3, 5, 19, 4, 6, 15, 17, 43, 46, 47, 26, 45, 41, 18, 7, 16, 10, 14, 24, 27, 32 \\ 49, 20, 23, 36, 40, 28, 8, 22, 25, 35, 39, 44, 29, 13, 21, 42, 31, 12, 30, 34, 9, 11, 33, 38, 37 \end{pmatrix}$$
$$\begin{pmatrix} 50, 48, 47, 46, 1, 2, 3, 5, 19, 43, 45, 41, 17, 15, 6, 26, 18, 7, 16, 4, 10, 14, 24, 27, 32 \\ 49, 20, 23, 36, 40, 28, 8, 22, 25, 35, 39, 44, 29, 13, 21, 42, 31, 12, 30, 34, 9, 11, 33, 38, 37 \end{pmatrix}$$
$$\begin{pmatrix} 50, 48, 47, 46, 1, 2, 3, 5, 19, 43, 45, 49, 41, 17, 15, 6, 26, 18, 7, 16, 32, 27, 24, 14, 10 \\ 20, 23, 36, 40, 28, 4, 8, 22, 25, 35, 39, 44, 29, 13, 21, 42, 31, 12, 30, 34, 9, 11, 33, 38, 37 \end{pmatrix}$$
$$\begin{pmatrix} 50, 48, 47, 46, 1, 2, 3, 5, 19, 43, 45, 49, 41, 17, 15, 26, 18, 7, 16, 32, 27, 24, 14, 6, 10 \\ 20, 23, 36, 40, 44, 39, 28, 35, 42, 25, 22, 29, 13, 21, 31, 8, 12, 30, 34, 4, 9, 11, 33, 38, 37 \end{pmatrix}$$
$$\begin{pmatrix} 50, 48, 47, 44, 46, 1, 2, 3, 5, 19, 43, 45, 49, 41, 17, 15, 26, 18, 7, 16, 32, 27, 24, 14, 40 \\ 36, 23, 20, 10, 6, 39, 28, 35, 42, 25, 22, 29, 13, 21, 31, 8, 12, 30, 34, 4, 9, 11, 33, 38, 37 \end{pmatrix}$$
$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 43, 41, 19, 1, 5, 17, 3, 15, 26, 32, 18, 40, 27, 36, 39, 2, 7, 16 \\ 24, 28, 35, 42, 14, 23, 20, 25, 22, 29, 10, 13, 21, 31, 6, 8, 12, 30, 34, 4, 9, 11, 33, 38, 37 \end{pmatrix}$$
$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 43, 41, 19, 1, 5, 17, 26, 32, 40, 3, 15, 18, 27, 36, 39, 42, 35, 28 \\ 24, 16, 7, 14, 23, 20, 25, 22, 29, 10, 13, 21, 31, 6, 8, 12, 30, 34, 2, 4, 9, 11, 33, 38, 37 \end{pmatrix}$$
$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 43, 41, 19, 1, 5, 17, 26, 32, 40, 42, 39, 36, 27, 18, 15, 35, 28, 24 \\ 16, 3, 7, 14, 23, 20, 25, 22, 29, 10, 13, 21, 31, 6, 8, 12, 30, 34, 2, 4, 9, 11, 33, 38, 37 \end{pmatrix}$$
$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 42, 40, 39, 43, 41, 36, 19, 32, 26, 35, 27, 28, 24, 17, 18, 23, 15, 16 \\ 5, 1, 3, 7, 14, 20, 25, 29, 22, 10, 13, 21, 31, 6, 8, 12, 30, 34, 2, 4, 9, 11, 33, 38, 37 \end{pmatrix}$$
$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 42, 40, 39, 43, 41, 36, 19, 32, 26, 35, 27, 28, 24, 17, 18, 23, 15, 16 \\ 5, 29, 25, 20, 14, 7, 3, 22, 10, 13, 21, 31, 6, 8, 12, 30, 34, 1, 2, 4, 9, 11, 33, 38, 37 \end{pmatrix}$$
$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 42, 40, 39, 43, 41, 36, 19, 32, 26, 35, 27, 28, 24, 17, 18, 23, 29, 25 \\ 15, 16, 20, 14, 5, 7, 22, 10, 13, 21, 31, 3, 6, 8, 12, 30, 34, 38, 33, 37, 11, 9, 4, 2, 1 \end{pmatrix}$$

[3] Thus the first one Y_0 already differs from X_0 slightly.

$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 42, 40, 39, 43, 41, 36, 19, 32, 26, 35, 27, 28, 24, 17, 18, 23, 29, 25 \\ 15, 16, 20, 22, 14, 5, 7, 10, 13, 21, 31, 38, 34, 30, 33, 37, 12, 11, 8, 6, 3, 9, 4, 2, 1 \end{pmatrix}$$
$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 42, 38, 40, 39, 43, 41, 36, 19, 32, 26, 35, 27, 28, 17, 18, 23, 29, 25 \\ 31, 34, 24, 15, 16, 20, 22, 30, 33, 37, 21, 14, 13, 12, 11, 9, 10, 8, 7, 5, 6, 3, 4, 2, 1 \end{pmatrix}$$
$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 44, 42, 38, 40, 39, 43, 41, 36, 19, 32, 26, 35, 27, 28, 29, 17, 18, 23, 25 \\ 31, 34, 37, 33, 30, 22, 24, 20, 16, 15, 21, 14, 13, 12, 11, 9, 10, 8, 7, 5, 6, 3, 4, 2, 1 \end{pmatrix}$$
$$\begin{pmatrix} 50, 49, 48, 47, 46, 45, 43, 41, 44, 42, 40, 39, 38, 37, 34, 36, 35, 32, 19, 26, 27, 28, 29, 31, 33 \\ 25, 23, 18, 30, 22, 24, 20, 17, 16, 15, 21, 14, 13, 12, 11, 9, 10, 8, 7, 5, 6, 3, 4, 2, 1 \end{pmatrix}$$

The blue (resp. red) neighboring seven numbers specify the place and operators to which the B_3 (resp. C_3) compatibility equation is applied. For example, X_1 is obtained from Y_0 by replacing $37 \cdot 38 \cdot 42 \cdot 44 \cdot 47 \cdot 48 \cdot 50 = S_{19,20,22} K_{22,18,17,16} K_{23,21,19,16} S_{17,20,23} S_{17,19,24} K_{24,21,20,18} S_{22,23,24}$ with the reverse ordered form $S_{22,23,24} K_{24,21,20,18} S_{17,19,24} S_{17,20,23} K_{23,21,19,16} K_{22,18,17,16} S_{19,20,22} = 50 \cdot 48 \cdot 47 \cdot 44 \cdot 42 \cdot 38 \cdot 37$ by (7.13). The numbers of red and blue sequences are both twelve. □

Theorem 7.2 shows that (R, S, K) is a solution to the F_4 compatibility equation. We remark that the tetrahedron equations $RRRR = RRRR$ and $SSSS = SSSS$ have not been used. R and S act as catalysts for the main reactions which are 3D reflection equations (7.12) and (7.13) involving K. Similar features are valid also for the combinatorial and the birational versions.

Example 7.3 The two sides of the F_4 compatibility equation (7.25) applied to a monomial $|101101001011200101010102\rangle$[4] produce

$$\begin{aligned} &|530000010101000303110233\rangle \\ &+ q(7\,|\ldots\rangle) + q^2(16\,|\ldots\rangle) + q^3(61\,|\ldots\rangle) + q^4(109\,|\ldots\rangle) + q^5(390\,|\ldots\rangle) \mod q^6\mathbb{Z}[q], \end{aligned}$$

where the first line gives the image of the combinatorial version ($q = 0$) and the second line shows the number of monomials appearing in each order of q.

By closely inspecting the definitions (7.4)–(7.9), the identification (7.22)–(7.24) and using the properties mentioned after it, one can show:

Remark 7.4 Given a reduced expression of the longest element $\mathbf{w}_0 = s_{i_1} \cdots s_{i_{24}} \in W(F_4)$, set $\mathbf{w}'_0 = s_{5-i_1} \cdots s_{5-i_{24}}$, which defines another reduced expression. Suppose the F_4 compatibility equation for $\mathbf{w}_0$ is $Z_1 \cdots Z_{50} = Z_{50} \cdots Z_1$ where Z_r is one of R_{ijk}, S_{ijk} and K_{ijkl} for some $i, j, k, l \in \{1, \ldots, 24\}$. Then the F_4 compatibility equation for $\mathbf{w}'_0$ takes the form $Z'_1 \cdots Z'_{50} = Z'_{50} \cdots Z'_1$, where $R'_{ijk} = S_{ijk}$, $S'_{ijk} = R_{ijk}$ and $K'_{ijkl} = K_{lkji}$.

[4] $|m_1 \ldots m_{24}\rangle$ is a shorthand for $|m_1\rangle \otimes \cdots \otimes |m_{24}\rangle$.

7.5 Bibliographical Notes and Comments

This chapter is an extended exposition of [93, Sect. 4], where many statements were given on a conjectural basis. An F_4 compatibility equation first appeared in [93, Eq. (48)]. It is related to (7.25) via the transformation mentioned in Remark 7.4. According to [126, Theorem 2.17], for any element of a Coxeter group, the loops in its reduced expression (rex) graph are (in a suitable sense) generated by the loops in the reduced expression graph of the longest element in any finite parabolic subgroup of rank 3. Theorem 7.2 is consistent with it, giving finer information distinguishing B_3 and C_3 structures within F_4.

Chapter 8
Intertwiner for Quantized Coordinate Ring $A_q(G_2)$

Abstract The quantized coordinate ring $A_q(G_2)$ is formulated in terms of generators and relations. Fundamental representations are presented explicitly. Basic properties of the intertwiner associated with the order six Coxeter relation are derived. Since there is no "G_3", we do not have a compatibility condition analogous to the tetrahedron or 3D reflection equations. Nevertheless, the intertwining relation still admits a reformulation as what we call the quantized G_2 reflection equation. This fact will be utilized to construct matrix product solutions to the G_2 reflection equation in Chap. 17.

8.1 Introduction

By G_2 reflection equation we mean

$$\begin{aligned} &R_{12}(x)X_{132}(xy)R_{23}(x^2y^3)X_{213}(xy^2)R_{31}(xy^3)X_{321}(y) \\ &= X_{231}(y)R_{13}(xy^3)X_{123}(xy^2)R_{32}(x^2y^3)X_{312}(xy)R_{21}(x), \end{aligned} \tag{8.1}$$

which contains the spectral parameters x and y. As we will explain in Chap. 17, it is a natural G_2 analogue of the Yang–Baxter and the reflection equations which are associated with A_2 and B_2, C_2, respectively. If the spectral parameters are suppressed, it reduces to the constant version $R_{12}X_{132}R_{23}X_{213}R_{31}X_{321} = X_{231}R_{13}X_{123}R_{32}X_{312}R_{21}$.

In this chapter we consider the quantized G_2 reflection equation

$$(L_{12}J_{132}L_{23}J_{213}L_{31}J_{321})F = F(J_{231}L_{13}J_{123}L_{32}J_{312}L_{21}), \tag{8.2}$$

which is a generalization of the constant G_2 reflection equation to a conjugacy equivalence by F. We present a solution of (8.2) connected to the quantized coordinate ring $A_q(G_2)$, where F arises as the characteristic intertwiner. In Chap. 17, it will be utilized to generate infinitely many solutions to (8.1) in matrix product forms. These features are quite parallel with types A and B, C.

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8.2 Quantized Coordinate Ring $A_q(G_2)$

The quantized coordinate ring $A_q(G_2)$ is a Hopf algebra generated by 49 generators $T = (t_{ij})_{1 \le i,j \le 7}$ satisfying (i) and (ii) given below.

(i) The RTT relation

$$\sum_{m,p} R^{ij}_{mp} t_{mk} t_{pl} = \sum_{m,p} t_{jp} t_{im} R^{mp}_{kl}, \tag{8.3}$$

where there are 112 non-zero R^{ij}_{kl} listed in (8.91).

(ii) Additional relations

$$g^{ij} = \sum_{k,l} t_{jl} t_{ik} g^{kl}, \qquad \sum_{k} f^{ij}_{\ k} t_{km} = \sum_{k,l} t_{jl} t_{ik} f^{kl}_{\ m}, \tag{8.4}$$

where g^{ij} and $f^{ij}_{\ k}$ are specified in (8.85)–(8.86).

The coproduct and the counit are defined by the same formula as (3.6) and (3.8), respectively. The antipode is given by

$$S(t_{ij}) = \sum_{kl} g^{li} g_{kj} t_{kl}, \tag{8.5}$$

where g_{kj} is determined from (8.87) and explicitly given by (8.88).

Remark 8.1 Similarly to Remark 5.1, the Hopf algebra $A_q(G_2)$ is equivalently presented in terms of generators and structure constants rescaled by parameters $g_1, \dots, g_7$. Explicitly, $\tilde{t}_{ij}$ and $\tilde{R}^{ij}_{kl}$ are taken as (5.13), and the other structure constants are rescaled as $\tilde{g}^{ij} = g_i g_j g^{ij}$, $\tilde{g}_{ij} = (g_i g_j)^{-1} g_{ij}$ and $\tilde{f}^{ij}_k = g_i g_j g_k^{-1} f^{ij}_k$.

8.3 Fundamental Representations

We set $(q_1, q_2) = (q, q^3)$.[1] Let $\mathrm{Osc}_q = \langle\, \mathbf{a}^+,\ \mathbf{a}^-,\ \mathbf{k},\ \mathbf{k}^{-1} \rangle$ be the q-oscillator as in (3.12) and (3.13). In this chapter we also use q^3-oscillator Osc_{q^3} and denote it by $\langle\, \mathbf{A}^+,\ \mathbf{A}^-,\ \mathbf{K},\ \mathbf{K}^{-1} \rangle$. Thus (5.15) is replaced by

$$\mathbf{K}\ \mathbf{A}^+ = q^3\ \mathbf{A}^+\ \mathbf{K}, \quad \mathbf{K}\ \mathbf{A}^- = q^{-3}\ \mathbf{A}^-\ \mathbf{K}, \quad \mathbf{A}^-\ \mathbf{A}^+ = \mathbf{1} - q^6\ \mathbf{K}^2, \quad \mathbf{A}^+\ \mathbf{A}^- = \mathbf{1} -\ \mathbf{K}^2, \tag{8.6}$$

which has an irreducible representation on the Fock space $\mathcal{F}_{q^3} = \bigoplus_{m \ge 0} \mathbb{C}(q^{\frac{1}{2}})|m\rangle$[2]:

[1] This is opposite to the labeling of vertices of the Dynkin diagram in [129, Eq. (27)].

[2] We include $q^{\frac{1}{2}}$ in the coefficient field in view of (8.13).

$$\mathbf{K}|m\rangle = q^{3m}|m\rangle, \quad \mathbf{A}^+|m\rangle = |m+1\rangle, \quad \mathbf{A}^-|m\rangle = (1-q^{6m})|m-1\rangle. \tag{8.7}$$

Let $T = (t_{ij})_{1\le i,j\le 7}$ be the 7×7 table of the generators of $A_q(G_2)$. The fundamental representations $\pi_i : A_q(G_2) \to \mathrm{End}(\mathcal{F}_{q_i})$ $(i = 1, 2)$ are given by

$$\pi_1(T) = \begin{pmatrix} \mathbf{a}^- & \mu_1\,\mathbf{k} & 0 & 0 & 0 & 0 & 0 \\ -q\mu_1^{-1}\,\mathbf{k} & \mathbf{a}^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\,\mathbf{a}^-)^2 & \rho\mu_1\,\mathbf{k}\;\mathbf{a}^- & (\mu_1\,\mathbf{k})^2 & 0 & 0 \\ 0 & 0 & -q\mu_1^{-1}\,\mathbf{a}^-\;\mathbf{k} & \mathbf{a}^-\;\mathbf{a}^+ - \,\mathbf{k}^2 & \mu_1\,\mathbf{k}\;\mathbf{a}^+ & 0 & 0 \\ 0 & 0 & (q\mu_1^{-1}\,\mathbf{k})^2 & -\rho\mu_1^{-1}\,\mathbf{k}\;\mathbf{a}^+ & (\,\mathbf{a}^+)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{a}^- & \mu_1\,\mathbf{k} \\ 0 & 0 & 0 & 0 & 0 & -q\mu_1^{-1}\,\mathbf{k} & \mathbf{a}^+ \end{pmatrix}, \tag{8.8}$$

$$\pi_2(T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{A}^- & \mu_2\,\mathbf{K} & 0 & 0 & 0 & 0 \\ 0 & -q^3\mu_2^{-1}\,\mathbf{K}^{-1} & \mathbf{A}^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{A}^- & \mu_2\,\mathbf{K} & 0 \\ 0 & 0 & 0 & 0 & -q^3\mu_2^{-1}\,\mathbf{K}^{-1} & \mathbf{A}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8.9}$$

Here and in the rest of this chapter we use the notation

$$\rho = q + q^{-1}. \tag{8.10}$$

Remark 8.2 Nonzero parameters μ_1, μ_2 in the fundamental representations (8.8), (8.9) can be removed by the rescaling in Remark 8.1 with $g_1^{-1} = g_7 = \mu_1^2\mu_2$, $g_2^{-1} = g_6 = \mu_1\mu_2$, $g_3^{-1} = g_5 = \mu_1$ and $g_4 = 1$.

From Remark 8.2, in the rest of this chapter we take $\mu_1 = q^{\frac{1}{2}}$ and $\mu_2 = q^{\frac{3}{2}}$. The choice does not influence the forthcoming intertwining relations (8.15), (8.18) and the intertwiners. Moreover, it has the advantage of simplifying $\pi_i(T)$ into[3]

[3] The (5, 4) element $-r\,\mathbf{k}\;\mathbf{a}^+$ in π_1 of [85, Eq. (40)] needs to be replaced by $-r\,\mathbf{a}^+\,\mathbf{k}$ in the notation therein.

$$\pi_1(T) = \begin{pmatrix} \mathbf{a}^- & \hat{\mathbf{k}} & 0 & 0 & 0 & 0 & 0 \\ -\hat{\mathbf{k}} & \mathbf{a}^+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\mathbf{a}^-)^2 & \rho\,\hat{\mathbf{k}}\,\mathbf{a}^- & \hat{\mathbf{k}}^2 & 0 & 0 \\ 0 & 0 & -\mathbf{a}^-\,\hat{\mathbf{k}} & \mathbf{s} & \hat{\mathbf{k}}\,\mathbf{a}^+ & 0 & 0 \\ 0 & 0 & \hat{\mathbf{k}}^2 & -\rho\,\mathbf{a}^+\,\hat{\mathbf{k}} & (\mathbf{a}^+)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{a}^- & \hat{\mathbf{k}} \\ 0 & 0 & 0 & 0 & 0 & -\hat{\mathbf{k}} & \mathbf{a}^+ \end{pmatrix}, \tag{8.11}$$

$$\pi_2(T) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{A}^- & \hat{\mathbf{K}} & 0 & 0 & 0 & 0 \\ 0 & -\hat{\mathbf{K}} & \mathbf{A}^+ & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{A}^- & \hat{\mathbf{K}} & 0 \\ 0 & 0 & 0 & 0 & -\hat{\mathbf{K}} & \mathbf{A}^+ & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8.12}$$

Here $\hat{\mathbf{k}}$ and $\hat{\mathbf{K}}$ are Osc_{q_i} operators including the zero point energy

$$\hat{\mathbf{k}} = q^{\frac{1}{2}}\,\mathbf{k}, \quad \hat{\mathbf{K}} = q^{\frac{3}{2}}\,\mathbf{K}, \tag{8.13}$$

where $\mathbf{k}$, $\mathbf{K}$ are specified in (3.13) and (8.7). The operator $\mathbf{s}$ at the center of (8.11) is defined by

$$\mathbf{s} = \mathbf{a}^-\,\mathbf{a}^+ - \mathbf{k}^2 = 1 - \rho\,\hat{\mathbf{k}}^2. \tag{8.14}$$

8.4 Intertwiner

The Weyl group $W(G_2) = \langle s_1, s_2 \rangle$ is the Coxeter system with the relations

$$s_1^2 = s_2^2 = 1, \quad s_2s_1s_2s_1s_2s_1 = s_1s_2s_1s_2s_1s_2.$$

According to Theorem 3.3 we have the isomorphism $\pi_{212121} \simeq \pi_{121212}$. Let $\Xi : \mathcal{F}_q \otimes \mathcal{F}_{q^3} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^3} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^3} \to \mathcal{F}_{q^3} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^3} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^3} \otimes \mathcal{F}_q$ be the corresponding intertwiner. Thus it is characterized by

$$\pi_{212121}(\Delta(f)) \circ \Xi = \Xi \circ \pi_{121212}(\Delta(f)) \quad (\forall f \in A_q(G_2)), \tag{8.15}$$
$$\Xi(|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle, \tag{8.16}$$

where the latter just fixes the normalization. We introduce the 3D F[4] by

[4] No 2D counterpart is known despite the name.

$$F = \Xi\, P_{16}P_{25}P_{34} \in \mathrm{End}(\mathcal{F}_{q^3} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^3} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^3} \otimes \mathcal{F}_q), \tag{8.17}$$

where $P_{16}P_{25}P_{34}$ reverses the order of the components in the six-fold tensor product. Then the above relations are cast into

$$\pi_{212121}(\Delta(f)) \circ F = F \circ \pi_{212121}(\tilde{\Delta}(f)) \quad (\forall f \in A_q(G_2)), \tag{8.18}$$

$$F(|0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle) = |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle \otimes |0\rangle, \tag{8.19}$$

where $\tilde{\Delta}(f) = P_{16}P_{25}P_{34}\Delta(f)P_{16}P_{25}P_{34}$. F will also be referred to as the intertwiner.

Example 8.3 The intertwining relation (8.18) is reduced to 49 equations for the generators $f = t_{ij}$ $(1 \le i, j \le 7)$. Although (8.11) and (8.12) are pretty sparse, some equations become lengthy including typically 16 terms on one side or both. So we do not display them all but present a few examples. To save space, the tensor product symbol $\otimes$ is denoted by $\cdot$.

$f = t_{11}:$

$$[F, 1\cdot\, \mathbf{a}^-\cdot 1\cdot\, \mathbf{a}^-\cdot 1\cdot\, \mathbf{a}^- - 1\cdot\, \mathbf{a}^-\cdot 1\cdot\, \hat{\mathbf{k}}\cdot\, \mathbf{A}^-\cdot\, \hat{\mathbf{k}} - 1\cdot\, \hat{\mathbf{k}}\cdot\, \mathbf{A}^-\cdot\, \mathbf{a}^+\cdot\, \mathbf{A}^-\cdot\, \hat{\mathbf{k}}$$
$$- 1\cdot\, \hat{\mathbf{k}}\cdot\, \mathbf{A}^-\cdot\, \hat{\mathbf{k}}\cdot 1\cdot\, \mathbf{a}^- + 1\cdot\, \hat{\mathbf{k}}\cdot\, \hat{\mathbf{K}}\cdot (\mathbf{a}^-)^2\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}] = 0,$$

$f = t_{15}:$

$$(1\cdot\, \mathbf{a}^-\cdot 1\cdot\, \hat{\mathbf{k}}\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}^2 + 1\cdot\, \hat{\mathbf{k}}\cdot\, \mathbf{A}^-\cdot\, \mathbf{a}^+\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}^2 + 1\cdot\, \hat{\mathbf{k}}\cdot\, \hat{\mathbf{K}}\cdot (\mathbf{a}^-)^2\cdot\, \mathbf{A}^+\cdot\, \hat{\mathbf{k}}^2$$
$$+ \rho 1\cdot\, \hat{\mathbf{k}}\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}\ \mathbf{a}^-\cdot 1\cdot\, \hat{\mathbf{k}}\ \mathbf{a}^+ + 1\cdot\, \hat{\mathbf{k}}\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}^2\cdot\, \mathbf{A}^-\cdot (\mathbf{a}^+)^2)F$$
$$= F(\mathbf{A}^-\cdot (\mathbf{a}^+)^2\cdot\, \mathbf{A}^-\cdot\, \hat{\mathbf{k}}^2\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}} + \rho\, \mathbf{A}^-\cdot\, \hat{\mathbf{k}}\ \mathbf{a}^+\cdot 1\cdot\, \hat{\mathbf{k}}\ \mathbf{a}^-\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}$$
$$+ \mathbf{A}^-\cdot\, \hat{\mathbf{k}}^2\cdot\, \mathbf{A}^+\cdot (\mathbf{a}^-)^2\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}} + \mathbf{A}^-\cdot\, \hat{\mathbf{k}}^2\cdot\, \hat{\mathbf{K}}\cdot\, \mathbf{a}^+\cdot\, \mathbf{A}^-\cdot\, \hat{\mathbf{k}}$$
$$+ \mathbf{A}^-\cdot\, \hat{\mathbf{k}}^2\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}\cdot 1\cdot\, \mathbf{a}^- - \hat{\mathbf{K}}\cdot\, \mathbf{a}^-\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}^2\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}),$$

$f = t_{16}:$

$$(1\cdot\, \hat{\mathbf{k}}\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}^2\cdot\, \hat{\mathbf{K}}\cdot\, \mathbf{a}^-)F$$
$$= F(\mathbf{A}^+\cdot\, \mathbf{a}^-\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}^2\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}} + \hat{\mathbf{K}}\cdot (\mathbf{a}^+)^2\cdot\, \mathbf{A}^-\cdot\, \hat{\mathbf{k}}^2\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}} + \rho\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}\ \mathbf{a}^+\cdot 1\cdot\, \hat{\mathbf{k}}\ \mathbf{a}^-\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}$$
$$+ \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}^2\cdot\, \mathbf{A}^+\cdot (\mathbf{a}^-)^2\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}} + \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}^2\cdot\, \hat{\mathbf{K}}\cdot\, \mathbf{a}^+\cdot\, \mathbf{A}^-\cdot\, \hat{\mathbf{k}} + \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}^2\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}\cdot 1\cdot\, \mathbf{a}^-),$$

$f = t_{26}:$

$$(\mathbf{A}^-\cdot\, \mathbf{a}^+\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}^2\cdot\, \hat{\mathbf{K}}\cdot\, \mathbf{a}^- + \hat{\mathbf{K}}\cdot (\mathbf{a}^-)^2\cdot\, \mathbf{A}^+\cdot\, \hat{\mathbf{k}}^2\cdot\, \hat{\mathbf{K}}\cdot\, \mathbf{a}^- + \rho\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}\ \mathbf{a}^-\cdot 1\cdot\, \hat{\mathbf{k}}\ \mathbf{a}^+\cdot\, \hat{\mathbf{K}}\cdot\, \mathbf{a}^-$$
$$+ \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}^2\cdot\, \mathbf{A}^-\cdot (\mathbf{a}^+)^2\cdot\, \hat{\mathbf{K}}\cdot\, \mathbf{a}^- + \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}^2\cdot\, \hat{\mathbf{K}}\cdot\, \mathbf{a}^-\cdot\, \mathbf{A}^+\cdot\, \mathbf{a}^- - \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}^2\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}\cdot 1\cdot\, \hat{\mathbf{k}})F$$
$$= F(\mathbf{A}^+\cdot\, \mathbf{a}^-\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}^2\cdot\, \hat{\mathbf{K}}\cdot\, \mathbf{a}^+ + \hat{\mathbf{K}}\cdot (\mathbf{a}^+)^2\cdot\, \mathbf{A}^-\cdot\, \hat{\mathbf{k}}^2\cdot\, \hat{\mathbf{K}}\cdot\, \mathbf{a}^+ + \rho\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}\ \mathbf{a}^+\cdot 1\cdot\, \hat{\mathbf{k}}\ \mathbf{a}^-\cdot\, \hat{\mathbf{K}}\cdot\, \mathbf{a}^+$$
$$+ \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}^2\cdot\, \mathbf{A}^+\cdot (\mathbf{a}^-)^2\cdot\, \hat{\mathbf{K}}\cdot\, \mathbf{a}^+ + \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}^2\cdot\, \hat{\mathbf{K}}\cdot\, \mathbf{a}^+\cdot\, \mathbf{A}^-\cdot\, \mathbf{a}^+ - \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}^2\cdot\, \hat{\mathbf{K}}\cdot\, \hat{\mathbf{k}}\cdot 1\cdot\, \hat{\mathbf{k}}).$$

Set

$$\mathcal{D}_G = D_{q^3} \otimes D_q \otimes D_{q^3} \otimes D_q \otimes D_{q^3} \otimes D_q, \tag{8.20}$$

where D_q is defined by (3.15).

Proposition 8.4

$$[F, z^{\mathbf{h}} \otimes z^{\mathbf{h}} \otimes z^{2\mathbf{h}} \otimes z^{\mathbf{h}} \otimes z^{\mathbf{h}} \otimes 1] = [F, 1 \otimes z^{\mathbf{h}} \otimes z^{3\mathbf{h}} \otimes z^{2\mathbf{h}} \otimes z^{3\mathbf{h}} \otimes z^{\mathbf{h}}] = 0, \tag{8.21}$$

$$F = F^{-1}, \tag{8.22}$$

$${}^tF = \mathcal{D}_G F \mathcal{D}_G^{-1}, \tag{8.23}$$

where z is a generic parameter.

Proof The proof is similar to Propositions 3.7 and 5.8. The weight conservation (8.21) follows from

$$\begin{aligned} &\pi_{212121}(\Delta(t_{17})) = \pi_{212121}(\tilde{\Delta}(t_{17})) = \pi_{212121}(\Delta(t_{71})) = \pi_{212121}(\tilde{\Delta}(t_{71})) \\ &= 1 \otimes \hat{\mathbf{k}} \otimes \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}^2 \otimes \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}, \end{aligned} \tag{8.24}$$

$$\begin{aligned} &\pi_{212121}(\Delta(t_{27}t_{16} - q^{-1}t_{26}t_{17})) = \pi_{212121}(\tilde{\Delta}(t_{27}t_{16} - q^{-1}t_{26}t_{17})) \\ &= \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}^3 \otimes \hat{\mathbf{K}}^2 \otimes \hat{\mathbf{k}}^3 \otimes \hat{\mathbf{K}} \otimes 1. \end{aligned} \tag{8.25}$$

The involution property (8.22) follows from

$$\pi_{212121}(\Delta(t_{ij})) = \xi_i^{-1}\xi_j \pi_{212121}(\tilde{\Delta}(t_{ji})), \tag{8.26}$$

where $\xi_i = (-1)^i(\rho/q)^{\delta_{i4}}$. The transpose (8.23) is derived from

$${}^t\pi_1(t_{ij}) = \kappa_i^{-1}\kappa_j D_q \pi_1(t_{j'i'}) D_q^{-1}, \quad {}^t\pi_2(t_{ij}) = \kappa_i^{-1}\kappa_j D_{q^3} \pi_2(t_{j'i'}) D_{q^3}^{-1}, \tag{8.27}$$

where $i' = 8 - i$ and $\kappa_i = (\rho/q)^{\delta_{i4}}$. □

The intertwiner F is regarded as a matrix with elements F_{ijklmn}^{abcdef} as

$$\begin{aligned} &F\,(|i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle \otimes |m\rangle \otimes |n\rangle) \\ &= \sum_{a,b,c,d,e,f} F_{ijklmn}^{abcdef} |a\rangle \otimes |b\rangle \otimes |c\rangle \otimes |d\rangle \otimes |e\rangle \otimes |f\rangle. \end{aligned} \tag{8.28}$$

Then Proposition 8.4 implies

$$F_{ijklmn}^{abcdef} = 0 \;\text{ unless }\; \begin{pmatrix} a+b+2c+d+e \\ b+3c+2d+3e+f \end{pmatrix} = \begin{pmatrix} i+j+2k+l+m \\ j+3k+2l+3m+n \end{pmatrix}, \tag{8.29}$$

$$F_{ijklmn}^{abcdef} = \frac{(q^6)_i(q^2)_j(q^6)_k(q^2)_l(q^6)_m(q^2)_n}{(q^6)_a(q^2)_b(q^6)_c(q^2)_d(q^6)_e(q^2)_f} F_{abcdef}^{ijklmn}. \tag{8.30}$$

From (3.16), $(3.16)|_{q\to q^3}$, (8.28) and (8.30), we also have

$$
\begin{aligned}
&(\langle i| \otimes \langle j| \otimes \langle k| \otimes \langle l| \otimes \langle m| \otimes \langle n|)F \\
&= \sum_{a,b,c,d,e,f} F_{ijklmn}^{abcdef} \langle a| \otimes \langle b| \otimes \langle c| \otimes \langle d| \otimes \langle e| \otimes \langle f|.
\end{aligned} \tag{8.31}
$$

Example 8.5 The following is the list of all the non-zero F_{100102}^{abcdef}.

$$
\begin{aligned}
F_{100102}^{000200} &= -q^3(1-q^4)(1-q^6)(1-q^2-q^6), \\
F_{100102}^{001001} &= -q^2(1-q^4)^2(1-q^6)(1+q^4), \\
F_{100102}^{010010} &= (1-q^4)(1-q^6)(1-q^2+q^{10}), \\
F_{100102}^{200004} &= q^{11}, \\
F_{100102}^{020002} &= q^3(1-q^6)(1-q^4-q^6-2q^8-q^{10}-q^{12}), \\
F_{100102}^{100011} &= q^2(1-q^4)(1-q^6+q^{14}), \\
F_{100102}^{100102} &= q^4(1+q^2-2q^6-q^8-q^{10}+q^{14}+q^{16}+q^{18}), \\
F_{100102}^{110003} &= q^6(1+q^2)(1-q^8-q^{12}), \\
F_{100102}^{010101} &= q(1+q^2)(1-q^6)(1-q^2-q^4-q^6+q^{10}+q^{12}+q^{14}).
\end{aligned}
$$

Although a tedious algorithm can be formulated for calculating F_{ijklmn}^{abcdef} for any given indices by combining the intertwining relation (8.18), an explicit general formula is yet to be constructed.

8.5 Quantized G_2 Reflection Equation

The quantized G_2 reflection equation is given by (8.2). In this section we take L to be the 3D L in the preceding chapters (based on Osc_{q^3}) and set up J elaborately. It enables us to identify (8.2) with the intertwining relation (8.18). In other words we obtain a solution to the quantized G_2 reflection equation in terms of 3D F associated with the representation theory of $A_q(G_2)$. It will be an input to Chap. 17, where an infinite family of solutions to the G_2 reflection equation with a spectral parameter will be constructed by a matrix product method. We begin by recalling the 3D L and then proceed to the J which we call the quantized G_2 scattering operator.

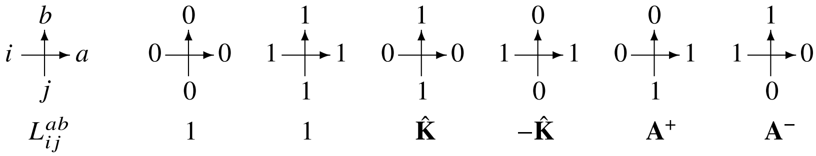
8.5.1 3D L

Set $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1 \simeq \mathbb{C}^2$. Let L be the Osc_{q^3}-valued 3D L:

$$L(v_i \otimes v_j \otimes |m\rangle) = \sum_{a,b \in \{0,1\}} v_a \otimes v_b \otimes L^{ab}_{ij}|m\rangle, \tag{8.32}$$

$$L = \left(L^{ab}_{ij}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \hat{\mathbf{K}} & \mathbf{A}^- & 0 \\ 0 & \mathbf{A}^+ & -\hat{\mathbf{K}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{End}(V \otimes V \otimes \mathcal{F}_{q^3}). \tag{8.33}$$

We attach a diagram to each component $L^{ab}_{ij} \in \mathrm{End}(\mathcal{F}_{q^3})$ as follows:



(8.34)

This is the same as Fig. 5.1 by the gauge transformation $L^{ab}_{ij} \to q^{i-b} L^{ab}_{ij}$ followed by the replacement $q \to q^{\frac{3}{2}}$. Explicitly, we have

$$\begin{aligned}
L(v_0 \otimes v_0 \otimes |m\rangle) &= v_0 \otimes v_0 \otimes |m\rangle, \quad L(v_1 \otimes v_1 \otimes |m\rangle) = v_1 \otimes v_1 \otimes |m\rangle, \\
L(v_0 \otimes v_1 \otimes |m\rangle) &= v_0 \otimes v_1 \otimes \hat{\mathbf{K}}|m\rangle + v_1 \otimes v_0 \otimes \mathbf{A}^+|m\rangle \\
&= q^{3m+\frac{3}{2}} v_0 \otimes v_1 \otimes |m\rangle + v_1 \otimes v_0 \otimes |m+1\rangle, \\
L(v_1 \otimes v_0 \otimes |m\rangle) &= v_0 \otimes v_1 \otimes \mathbf{A}^-|m\rangle - v_1 \otimes v_0 \otimes \hat{\mathbf{K}}|m\rangle \\
&= (1 - q^{6m}) v_0 \otimes v_1 \otimes |m-1\rangle - q^{3m+\frac{3}{2}} v_1 \otimes v_0 \otimes |m\rangle.
\end{aligned}$$

The weight conservation

$$L^{ab}_{ij} = 0, \quad \text{unless} \ \ a + b = i + j, \tag{8.35}$$

$$\mathbf{h}\, L^{ab}_{ij} = L^{ab}_{ij}(\mathbf{h} + j - b), \tag{8.36}$$

holds, where $\mathbf{h}$ is defined in (3.14).

8.5.2 Quantized G_2 Scattering Operator J

Now we introduce a new operator $J \in \mathrm{End}(V \otimes V \otimes V \otimes \mathcal{F}_q)$ of the form

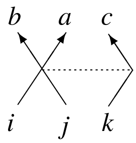
$$J(v_i \otimes v_j \otimes v_k \otimes |m\rangle) = \sum_{a,b,c \in \{0,1\}} v_a \otimes v_b \otimes v_c \otimes J_{ijk}^{abc}|m\rangle, \tag{8.37}$$

$$J_{ijk}^{abc} \in \mathrm{End}(\mathcal{F}_q). \tag{8.38}$$

It encodes a special three-body event characteristic to the G_2 root system. A detailed explanation of it including the geometric interpretation will be given in Chap. 17. Here we just define each component J_{ijk}^{abc} algebraically with graphical representation.

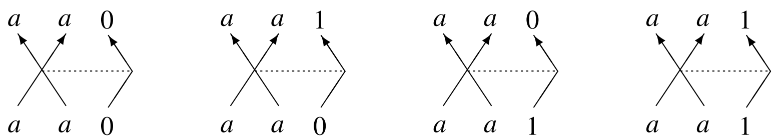
The quantized amplitude J_{ijk}^{abc} is depicted by a diagram in which the two-particle collision and the boundary reflection happen *simultaneously* at the instance indicated by the dotted line.[5]

$J_{ijk}^{abc} =$

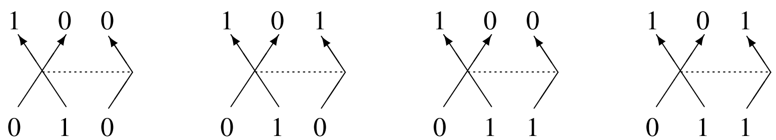


(8.39)

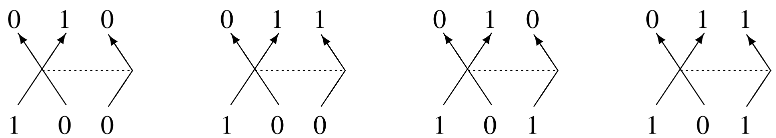
In terms of this graphical representation, $J_{ijk}^{abc} \in \mathrm{End}(\mathcal{F}_q)$ is specified as follows:



$\mathbf{a}^+$ $\qquad$ $\hat{\mathbf{k}}$ $\qquad$ $-\hat{\mathbf{k}}$ $\qquad$ $\mathbf{a}^-$ (8.40)

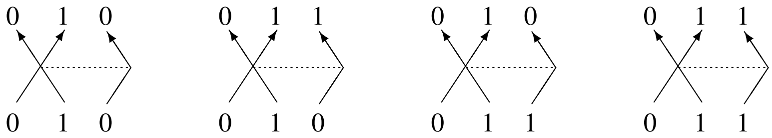


$u_1\hat{\mathbf{k}}\,\mathbf{a}^+$ $\qquad$ $\hat{\mathbf{k}}^2$ $\qquad$ $\rho^{-1}u_1u_3\mathbf{s}$ $\qquad$ $u_3\hat{\mathbf{k}}\,\mathbf{a}^-$ (8.41)

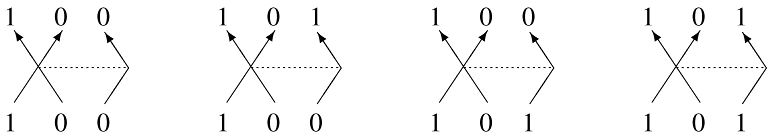


$-u_2\,\mathbf{a}^+\hat{\mathbf{k}}$ $\qquad$ $\rho^{-1}u_2u_4\mathbf{s}$ $\qquad$ $\hat{\mathbf{k}}^2$ $\qquad$ $-u_4\mathbf{a}^-\hat{\mathbf{k}}$ (8.42)

[5] For simplicity the vertical boundary line reflecting k to c is omitted here. It will be depicted in Figs. 17.1 and 17.2 as horizontal lines in the 90°-rotated diagrams.



$$(\mathbf{a}^+)^2 \qquad u_4\hat{\mathbf{k}}\,\mathbf{a}^+ \qquad -u_3\,\mathbf{a}^+\hat{\mathbf{k}} \qquad \rho^{-1}u_3u_4\mathbf{s} \tag{8.43}$$



$$\rho^{-1}u_1u_2\mathbf{s} \qquad u_2\hat{\mathbf{k}}\,\mathbf{a}^- \qquad -u_1\mathbf{a}^-\hat{\mathbf{k}} \qquad (\mathbf{a}^-)^2 \tag{8.44}$$

In (8.40), $a = 0, 1$. The parameters u_1, u_2, u_3, u_4 are arbitrary as long as

$$u_1u_2 + u_3u_4 = \rho \tag{8.45}$$

is satisfied. See (8.10) and (8.14) for the definition of ρ and $\mathbf{s} \in \mathrm{End}(\mathcal{F}_q)$. All the J^{abc}_{ijk}'s not contained in the above list are zero. The weight conservation properties

$$J^{abc}_{ijk} = 0 \;\text{ unless }\; a + b = i + j, \tag{8.46}$$
$$\mathbf{h}\,J^{abc}_{ijk} = J^{abc}_{ijk}(\mathbf{h} + 1 + j - k - b - c) \tag{8.47}$$

are valid, which are analogous to (8.35) and (8.36). For instance, we have

$$\begin{aligned}
&J(v_1 \otimes v_0 \otimes v_0 \otimes |m\rangle) \\
&= v_1 \otimes v_0 \otimes v_0 \otimes J^{100}_{100}|m\rangle + v_1 \otimes v_0 \otimes v_1 \otimes J^{101}_{100}|m\rangle \\
&+ v_0 \otimes v_1 \otimes v_0 \otimes J^{010}_{100}|m\rangle + v_0 \otimes v_1 \otimes v_1 \otimes J^{011}_{100}|m\rangle \\
&= -u_2q^{m+\frac{1}{2}}v_1 \otimes v_0 \otimes v_0 \otimes |m+1\rangle + \rho^{-1}u_2u_4(1-\rho q^{2m+1})v_1 \otimes v_0 \otimes v_1 \otimes |m\rangle \\
&+ \rho^{-1}u_1u_2(1-\rho q^{2m+1})v_0 \otimes v_1 \otimes v_0 \otimes |m\rangle + u_2q^{m-\frac{1}{2}}(1-q^{2m})v_0 \otimes v_1 \otimes v_1 \otimes |m-1\rangle.
\end{aligned}$$

The operator J almost splits into a product of two-particle scattering and one-particle boundary reflection as

$$J^{abc}_{ijk} = \alpha\mathcal{L}^{ab}_{ij}\,G^c_k + \beta\delta_{i+k,1}\delta_{b+c,1}\mathrm{id} \tag{8.48}$$

for some constants α, β. Here $\mathcal{L}$ denotes (8.33)$|_{\mathbf{A}^{\pm}\to\mathbf{a}^{\pm},\, \hat{\mathbf{K}}\to\hat{\mathbf{k}}}$ and G_k^c is (15.4)$|_{\beta=q^{\frac{1}{2}}}$, which is a slight gauge change from Fig. 5.2. Due to the presence of the second term, it is not a direct product, which is indicated by the dotted line in the diagram.

8.5.3 Quantized G_2 Reflection Equation

Theorem 8.6 *The intertwiner $F \in \mathrm{End}(\mathcal{F}_{q^3} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^3} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^3} \otimes \mathcal{F}_q)$ defined by (8.18) and (8.19), the 3D L $L \in \mathrm{End}(V \otimes V \otimes \mathcal{F}_{q^3})$ in Sect. 8.5.1 and the quantized G_2 scattering operator $J \in \mathrm{End}(V \otimes V \otimes V \otimes \mathcal{F}_q)$ in Sect. 8.5.2 satisfy the quantized G_2 reflection equation*

$$(L_{12}J_{132}L_{23}J_{213}L_{31}J_{321})F = F(J_{231}L_{13}J_{123}L_{32}J_{312}L_{21}). \tag{8.49}$$

Theorem 8.6 is a corollary of Proposition 8.8 below. In this solution, the quantized G_2 reflection equation (8.49) becomes an equality of linear operators on $\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V} \otimes \overset{4}{\mathcal{F}}_{q^3} \otimes \overset{5}{\mathcal{F}}_q \otimes \overset{6}{\mathcal{F}}_{q^3} \otimes \overset{7}{\mathcal{F}}_q \otimes \overset{8}{\mathcal{F}}_{q^3} \otimes \overset{9}{\mathcal{F}}_q$, where the superscripts are temporary labels for the explanation. If they are all exhibited, Eq. (8.49) reads as

$$L_{124}J_{1325}L_{236}J_{2137}L_{318}J_{3219}F_{456789} = F_{456789}J_{2319}L_{138}J_{1237}L_{326}J_{3125}L_{214}. \tag{8.50}$$

To explain the notation in (8.50), let us write L (8.32) and J (8.37) symbolically as

$$L = \sum_{a,b,i,j} E_{ai} \otimes E_{bj} \otimes L_{ij}^{ab} = \sum_l \mathcal{L}_l^{(1)} \otimes \mathcal{L}_l^{(2)} \otimes \mathcal{L}_l^{(3)}, \tag{8.51}$$

$$J = \sum_{a,b,c,i,j,k} E_{ai} \otimes E_{bj} \otimes E_{ck} \otimes J_{ijk}^{abc} = \sum_l \mathcal{J}_l^{(1)} \otimes \mathcal{J}_l^{(2)} \otimes \mathcal{J}_l^{(3)} \otimes \mathcal{J}_l^{(4)}. \tag{8.52}$$

Then

$$L_{ij4} = \sum \mathcal{L}_l^{(i)} \otimes \mathcal{L}_l^{(j)} \otimes 1 \otimes \mathcal{L}_l^{(3)} \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \quad (\{i,j\} = \{1,2\}),$$
$$L_{ij6} = \sum 1 \otimes \mathcal{L}_l^{(i-1)} \otimes \mathcal{L}_l^{(j-1)} \otimes 1 \otimes 1 \otimes \mathcal{L}_l^{(3)} \otimes 1 \otimes 1 \otimes 1 \quad (\{i,j\} = \{2,3\}),$$
$$L_{138} = \sum \mathcal{L}_l^{(1)} \otimes 1 \otimes \mathcal{L}_l^{(2)} \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \mathcal{L}_l^{(3)} \otimes 1,$$
$$L_{318} = \sum \mathcal{L}_l^{(2)} \otimes 1 \otimes \mathcal{L}_l^{(1)} \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \mathcal{L}_l^{(3)} \otimes 1,$$
$$J_{ijk5} = \sum \mathcal{J}_l^{(i)} \otimes \mathcal{J}_l^{(j)} \otimes \mathcal{J}_l^{(k)} \otimes 1 \otimes \mathcal{J}_l^{(4)} \otimes 1 \otimes 1 \otimes 1 \otimes 1 \quad (\{i,j,k\} = \{1,2,3\}),$$
$$J_{ijk7} = \sum \mathcal{J}_l^{(i)} \otimes \mathcal{J}_l^{(j)} \otimes \mathcal{J}_l^{(k)} \otimes 1 \otimes 1 \otimes 1 \otimes \mathcal{J}_l^{(4)} \otimes 1 \otimes 1 \quad (\{i,j,k\} = \{1,2,3\}),$$
$$J_{ijk9} = \sum \mathcal{J}_l^{(i)} \otimes \mathcal{J}_l^{(j)} \otimes \mathcal{J}_l^{(k)} \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes \mathcal{J}_l^{(4)} \quad (\{i,j,k\} = \{1,2,3\}).$$

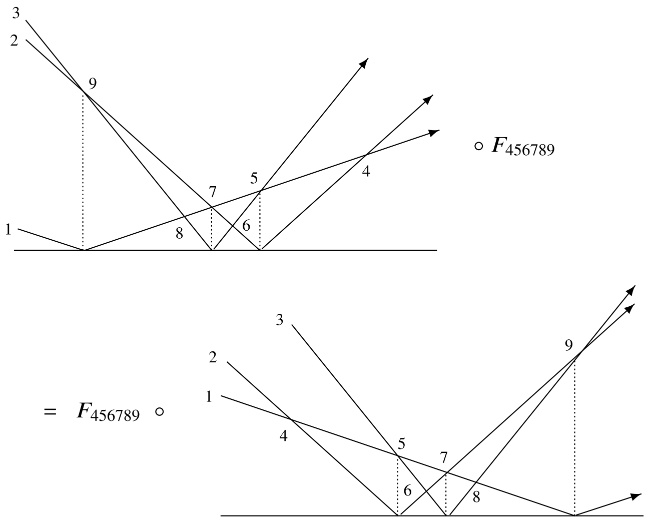


Fig. 8.1 A graphical representation of the quantized G_2 reflection equation (8.50), which admits a natural 3D interpretation

Practically, one can realize these operators by putting L^{ab}_{ij} and J^{abc}_{ijk} at appropriate tensor components with a suitable permutations of the indices. The quantized G_2 reflection equation (8.49) or equivalently (8.50) is depicted in Fig. 8.1 below.

Here the indices 1,2,3 label the lines (arrows) which are being reflected, whereas 4,5,6,7,8,9 are attached to the scattering/reflection events. The latter group of indices are associated with the Fock spaces, and the q-oscillators are acting on them in the direction perpendicular to this planar diagram. If one introduces such q-oscillator arrows going from the back to the front of the diagram, the operator F_{456789} in the LHS (resp. RHS) corresponds to a vertex where the six arrows going toward (resp. coming from) 4, 5, 6, 7, 8, 9 intersect.

The component of (8.50) corresponding to the transition $v_i \otimes v_j \otimes v_k \mapsto v_a \otimes v_b \otimes v_c$ in $\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V}$ is given by

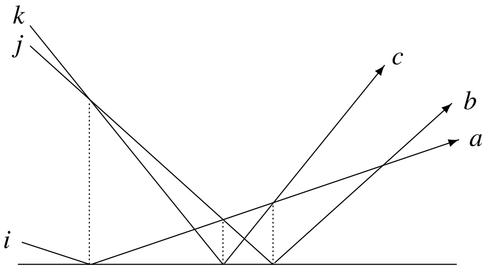
$$\mathcal{M}^{abc}_{ijk} F = F \tilde{\mathcal{M}}^{abc}_{ijk} \quad (a, b, c, i, j, k \in \{0, 1\}), \tag{8.53}$$

$$\mathcal{M}^{abc}_{ijk} = \sum L^{a,b}_{\alpha_1,\alpha_2} \otimes J^{\alpha_1,c,\alpha_2}_{\beta_1,\beta_3,\beta_2} \otimes L^{\beta_2,\beta_3}_{\gamma_1,\gamma_2} \otimes J^{\gamma_1,\beta_1,\gamma_2}_{\lambda_2,\mu_1,\mu_2} \otimes L^{\mu_2,\mu_1}_{\lambda_3,\lambda_1} \otimes J^{\lambda_3,\lambda_2,\lambda_1}_{k,j,i} \tag{8.54}$$

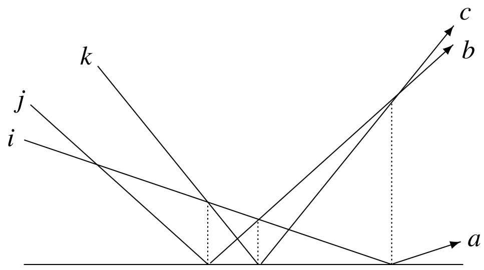
$$\tilde{\mathcal{M}}^{abc}_{ijk} = \sum L^{\alpha_2,\alpha_1}_{j,i} \otimes J^{\beta_3,\beta_1,\beta_2}_{k,\alpha_1,\alpha_2} \otimes L^{\gamma_2,\gamma_1}_{\beta_3,\beta_2} \otimes J^{\mu_1,\lambda_2,\mu_2}_{\beta_1,\gamma_1,\gamma_2} \otimes L^{\lambda_1,\lambda_3}_{\mu_1,\mu_2} \otimes J^{b,c,a}_{\lambda_2,\lambda_3,\lambda_1}, \tag{8.55}$$

with the sums taken over $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2 \in \{0, 1\}$. The quantities $\mathcal{M}^{abc}_{ijk}, \tilde{\mathcal{M}}^{abc}_{ijk}$ are to be called the quantized three-body G_2 amplitudes. They are expressed as sums over diagrams with specified external edges as

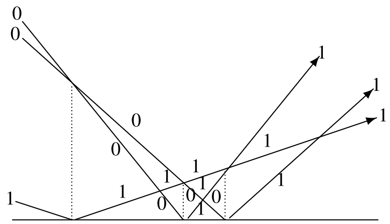
$\mathcal{M}^{abc}_{ijk} =$



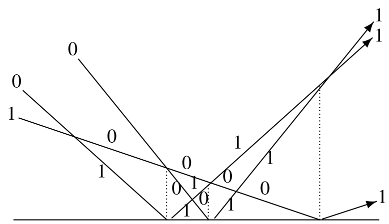
$\tilde{\mathcal{M}}^{abc}_{ijk} =$



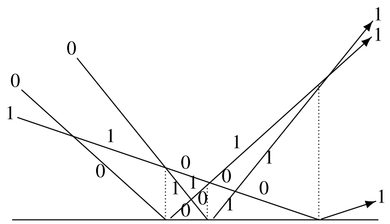
Example 8.7 From the diagrams (8.34) and (8.40)–(8.44), those contributing to $\mathcal{M}^{111}_{100}$, $\tilde{\mathcal{M}}^{111}_{100}$ and the associated quantized amplitudes are given as follows:



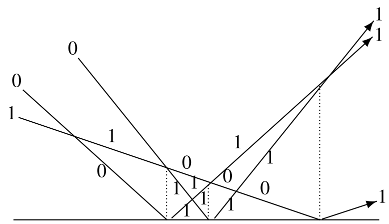
$1 \otimes \hat{\mathbf{k}} \otimes \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}^2 \otimes \hat{\mathbf{K}} \otimes \mathbf{a}^-$



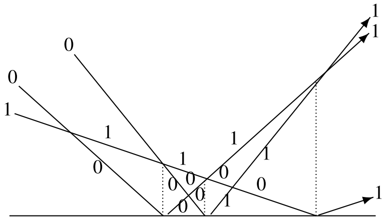
$\mathbf{A}^+ \otimes \mathbf{a}^- \otimes \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}^2 \otimes \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}$



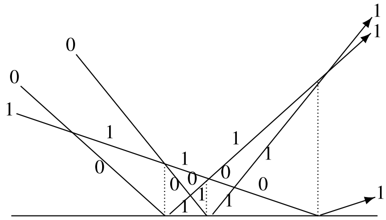
$\hat{\mathbf{K}} \otimes (\mathbf{a}^+)^2 \otimes \mathbf{A}^- \otimes \hat{\mathbf{k}}^2 \otimes \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}$



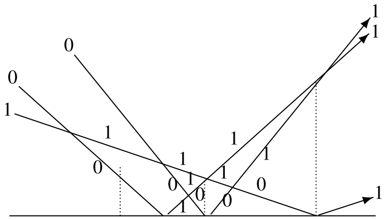
$u_3 u_4\, \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}\, \mathbf{a}^+ \otimes 1 \otimes \hat{\mathbf{k}}\, \mathbf{a}^- \otimes \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}$



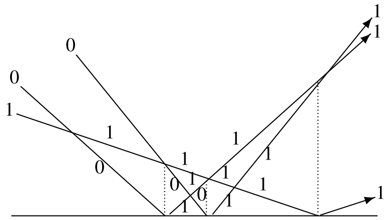
$u_1 u_2\, \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}\, \mathbf{a}^+ \otimes 1 \otimes \hat{\mathbf{k}}\, \mathbf{a}^- \otimes \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}$



$\hat{\mathbf{K}} \otimes \hat{\mathbf{k}}^2 \otimes \mathbf{A}^+ \otimes (\mathbf{a}^-)^2 \otimes \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}$



$\hat{\mathbf{K}} \otimes \hat{\mathbf{k}}^2 \otimes \hat{\mathbf{K}} \otimes \mathbf{a}^+ \otimes \mathbf{A}^- \otimes \hat{\mathbf{k}}$



$\hat{\mathbf{K}} \otimes \hat{\mathbf{k}}^2 \otimes \hat{\mathbf{K}} \otimes \hat{\mathbf{k}} \otimes 1 \otimes \mathbf{a}^-$

Thus we get

$$\mathcal{M}^{111}_{100} = 1 \otimes \hat{\mathbf{k}} \otimes \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}^2 \otimes \hat{\mathbf{K}} \otimes \mathbf{a}^-, \tag{8.56}$$

$$\begin{aligned}\tilde{\mathcal{M}}^{111}_{100} = &\ \mathbf{A}^+ \otimes \mathbf{a}^- \otimes \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}^2 \otimes \hat{\mathbf{K}} \otimes \hat{\mathbf{k}} + \hat{\mathbf{K}} \otimes (\mathbf{a}^+)^2 \otimes \mathbf{A}^- \otimes \hat{\mathbf{k}}^2 \otimes \hat{\mathbf{K}} \otimes \hat{\mathbf{k}} \\ &+ \rho\, \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}\, \mathbf{a}^+ \otimes 1 \otimes \hat{\mathbf{k}}\, \mathbf{a}^- \otimes \hat{\mathbf{K}} \otimes \hat{\mathbf{k}} + \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}^2 \otimes \mathbf{A}^+ \otimes (\mathbf{a}^-)^2 \otimes \hat{\mathbf{K}} \otimes \hat{\mathbf{k}} \\ &+ \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}^2 \otimes \hat{\mathbf{K}} \otimes \mathbf{a}^+ \otimes \mathbf{A}^- \otimes \hat{\mathbf{k}} + \hat{\mathbf{K}} \otimes \hat{\mathbf{k}}^2 \otimes \hat{\mathbf{K}} \otimes \hat{\mathbf{k}} \otimes 1 \otimes \mathbf{a}^-,\end{aligned} \tag{8.57}$$

where (8.45) has been used to combine the fourth and the fifth diagrams into a single term with coefficient ρ.

Notice that the quantized G_2 reflection equation (8.53) with (8.56) and (8.57) coincides with the $f = t_{16}$ case of Example 8.3. This happens generally. In fact a direct calculation shows:

Proposition 8.8 *The quantized three-body G_2 amplitudes are identified with the representations of $A_q(G_2)$ as*

$$\mathcal{M}^{abc}_{ijk} = \lambda_{abc} \nu_{ijk} \rho^{-\delta_{\beta 4}} \pi_{212121}(\Delta(t_{\alpha,\beta})), \tag{8.58}$$

$$\tilde{\mathcal{M}}^{abc}_{ijk} = \lambda_{abc} \nu_{ijk} \rho^{-\delta_{\beta 4}} \pi_{212121}(\tilde{\Delta}(t_{\alpha,\beta})), \tag{8.59}$$

where $\alpha = \kappa_{abc}$, $\beta = \kappa_{ijk}$, and $\kappa_{ijk}, \lambda_{ijk}, \nu_{ijk}$ are defined by the following table:

ijk	111	011	101	001	110	010	100	000
κ_{ijk}	1	2	3	4	5	6	7	8
λ_{ijk}	1	1	1	u_1	u_4	1	1	1
ν_{ijk}	1	1	1	u_2	u_3	1	1	1

In particular, 2^6 quantized G_2 reflection equations (8.53) are equivalent to the intertwining relations (8.18).

8.6 Further Aspects of F

8.6.1 Boundary Vector

Let us introduce the boundary vectors

$$\langle \eta_1| = \sum_{m\geq 0} \frac{\langle m|}{(q)_m}, \qquad |\eta_1\rangle = \sum_{m\geq 0} \frac{|m\rangle}{(q)_m}, \tag{8.60}$$

$$\langle \xi| = \sum_{m\geq 0} \frac{\langle m|}{(q^3)_m}, \qquad |\xi\rangle = \sum_{m\geq 0} \frac{|m\rangle}{(q^3)_m}, \tag{8.61}$$

which are actually elements of a completion of the Fock space. The vectors $\langle \eta_1|$ and $|\eta_1\rangle$ are the same as in (3.132) and (3.133). The vectors $\langle \xi|$ and $|\xi\rangle$ are simply related to them by the replacement $q \to q^3$.

Conjecture 8.9 *The intertwiner F has an eigenvector as follows:*

$$F\big(|\xi\rangle \otimes |\eta_1\rangle \otimes |\xi\rangle \otimes |\eta_1\rangle \otimes |\xi\rangle \otimes |\eta_1\rangle\big) = |\xi\rangle \otimes |\eta_1\rangle \otimes |\xi\rangle \otimes |\eta_1\rangle \otimes |\xi\rangle \otimes |\eta_1\rangle. \tag{8.62}$$

It implies the dual relation

$$(\langle \xi| \otimes \langle \eta_1| \otimes \langle \xi| \otimes \langle \eta_1| \otimes \langle \xi| \otimes \langle \eta_1|)F = \langle \xi| \otimes \langle \eta_1| \otimes \langle \xi| \otimes \langle \eta_1| \otimes \langle \xi| \otimes \langle \eta_1|. \tag{8.63}$$

The conjecture will be utilized in Sect. 17.4.

8.6.2 Combinatorial and Birational Counterparts

Introduce the 7×7 upper triangular matrices

$$W_1(x) = \begin{pmatrix} 1&0&0&0&0&0&0\\ 0&1&x&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&-x&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{pmatrix}, \quad W_2(x) = \begin{pmatrix} 1&x&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&x&-\frac{x^2}{2}&0&0\\ 0&0&0&1&-x&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&-x\\ 0&0&0&0&0&0&1 \end{pmatrix}. \tag{8.64}$$

These are generators of a unipotent subgroup of the Lie group of G_2. Given parameters a, b, c, d, e, f, consider the equation

$$\begin{aligned} &W_2(a)W_1(b)W_2(c)W_1(d)W_2(e)W_1(f)\\ &= W_1(f')W_2(e')W_1(d')W_2(c')W_1(b')W_2(a') \end{aligned} \tag{8.65}$$

for a', b', c', d', e', f'. It has the unique solution

$$a' = \frac{abc^2de}{A}, \quad b' = \frac{A^3}{D}, \quad c' = \frac{D}{AB}, \quad d' = \frac{B^3}{CD}, \quad e' = \frac{C}{B}, \quad f' = \frac{bc^3d^2e^3f}{C}, \tag{8.66}$$

$$A = abc^2d + abf(c+e)^2 + de^2f(a+c), \tag{8.67}$$

$$B = a^2b^2c^3d + a^2b^2f(c+e)^3 + abde^2f(3ac+2ae+2c^2+2ce) + d^2e^3f(a+c)^2, \tag{8.68}$$

$$C = a^3b^2c^3d + a^3b^2f(c+e)^3 + a^2bde^2f(3ac+2ae+3c^2+3ce) + d^2e^3f(a+c)^3, \tag{8.69}$$

$$\begin{aligned} D = {} & a^2b^2c^3d\big(abc^3d + 2abf(c+e)^3 + de^2f(3ac+2ae+3c^2+3ce)\big)\\ & + f^2(ab(c+e)^2 + de^2(a+c))^3. \end{aligned} \tag{8.70}$$

Define the *birational* 3D F by

$$F_{\mathrm{birational}} : (a, b, c, d, e, f) \mapsto (a', b', c', d', e', f'). \tag{8.71}$$

It is easy to see that $F_{\mathrm{birational}}$ is an involution having the two conserved quantities

$$abc^2de, \quad bc^3d^2e^3f. \tag{8.72}$$

Let $a'', b'', c'', d'', e'', f''$ be the piecewise linear functions obtained by applying the tropical variable change (or UD) of (3.152) to the totally positive rational functions a', b', c', d', e', f' in (8.66). For instance, we have

$$a'' = \max(e, d+e-f, c+d-f, 2c+d-e-f, b+2c-e-f, a+b+c-e-f). \tag{8.73}$$

We define the *combinatorial* 3D F to be the piecewise linear map on $(\mathbb{Z}_{\geq 0})^6$ by

$$F_{\mathrm{combinatorial}} : (a, b, c, d, e, f) \mapsto (a'', b'', c'', d'', e'', f''). \tag{8.74}$$

By construction it satisfies $F_{\text{combinatorial}} = F^{-1}_{\text{combinatorial}}$ and preserves $a+b+2c+d+e$ and $b+3c+2d+3e+f$ corresponding to (8.72) in agreement with the weight conservation (8.29).

Example 8.10 $F_{\text{combinatorial}} : (1,0,0,1,0,2) \mapsto (0,1,0,0,1,0)$. This captures Example 8.5 at $q=0$.

Let us denote the intertwiner F defined in Sect. 8.4 by F_{quantum}. Then the triad of the 3D F's and their relation is summarized in the same manner as (3.158), (5.162) and (6.47):

$$F_{\text{quantum}} \overset{q\to 0}{\longrightarrow} F_{\text{combinatorial}} \overset{\text{UD}}{\longleftarrow} F_{\text{birational}}. \tag{8.75}$$

8.7 Data on Relevant Quantum R Matrix

We list the structure constants in the defining relations of $A_q(G_2)$. They originate in the 7-dimensional representation of $U_q(G_2) = \langle k_i^{\pm 1}, e_i, f_i\rangle_{i=1,2}$ and the quantum R matrix acting on its tensor square. In our convention, α_1 (resp. α_2) is the short (resp. long) simple root and the Cartan matrix (a_{ij}) is $a_{11} = a_{22} = 2, a_{12} = -3, a_{21} = -1$. Defining relations of the generators are

$$k_i k_j = k_j k_i, \quad k_i k_i^{-1} = k_i^{-1} k_i = 1, \tag{8.76}$$

$$k_i e_j k_i^{-1} = q_i^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q_i^{-a_{ij}} f_j, \quad [e_i, f_j] = \delta_{ij}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \tag{8.77}$$

and the q-Serre relations (10.1), where $(q_1, q_2) = (q, q^3)$ as in the previous sections. Explicitly, they read as

$$\begin{aligned} &e_1^4 e_2 - (q^3+q+q^{-1}+q^{-3})e_1^3 e_2 e_1 + (q^4+q^2+2+q^{-2}+q^{-4})e_1^2 e_2 e_1^2 \\ &\quad - (q^3+q+q^{-1}+q^{-3})e_1 e_2 e_1^3 + e_2 e_1^4 = 0, \\ &e_2^2 e_1 - (q^3+q^{-3})e_2 e_1 e_2 + e_1 e_2^2 = 0, \end{aligned} \tag{8.78}$$

which are quoted from (10.81).

Let ϖ_1, ϖ_2 be the fundamental weights, which are related to the simple roots as $\varpi_1 = 2\alpha_1 + \alpha_2$, $\varpi_2 = 3\alpha_1 + 2\alpha_2$. We are concerned with the irreducible $U_q(G_2)$ modules with highest weights $2\varpi_1, \varpi_2, \varpi_1$ and 0 whose dimensions are 27, 14, 7 and 1, respectively. In this section we refer to them by the dimensions. The representation 7 has bases $v_1, \ldots, v_7$ having weights $2\alpha_1+\alpha_2, \alpha_1+\alpha_2, \alpha_1, 0, -\alpha_1, -(\alpha_1+\alpha_2), -(2\alpha_1+\alpha_2)$. With respect to them the generators are represented as

$$e_1 = E_{12} + (q + q^{-1})E_{34} + E_{45} + E_{67}, \quad e_2 = E_{23} + E_{56}, \tag{8.79}$$

$$f_1 = E_{21} + E_{43} + (q + q^{-1})E_{54} + E_{76}, \quad f_2 = E_{32} + E_{65}, \tag{8.80}$$

$$k_1 = \mathrm{diag}(q, q^{-1}, q^2, 1, q^{-2}, q, q^{-1}), \quad k_2 = \mathrm{diag}(1, q^3, q^{-3}, 1, q^3, q^{-3}, 1). \tag{8.81}$$

The following tensor product decomposition is valid:

$$7 \otimes 7 = 27 \oplus 14 \oplus 7 \oplus 1. \tag{8.82}$$

Let $v_i^{(\lambda)}$ $(i = 1, \ldots, \lambda)$ be a weight base of the representation $\lambda = 1, 7, 14, 27$ appearing in the RHS. We employ the comultiplication Δ in (10.2). Then the weight bases are given as follows (we write $v[j, k] = v_j \otimes v_k$):

$$\begin{aligned}
v_1^{(27)} &= v[1,1], \quad v_2^{(27)} = qv[1,2] + v[2,1], \quad v_3^{(27)} = qv[1,3] + v[3,1],\\
v_4^{(27)} &= v[2,2], \quad v_5^{(27)} = q^3v[2,3] + v[3,2],\\
v_6^{(27)} &= q^2v[1,4] + v[2,3] + qv[3,2] + v[4,1], \quad v_7^{(27)} = v[3,3],\\
v_8^{(27)} &= q^2v[2,4] + v[4,2], \quad v_9^{(27)} = q^3v[1,5] + q^2v[2,4] + q^2v[4,2] + v[5,1],\\
v_{10}^{(27)} &= qv[2,5] + v[5,2], \quad v_{11}^{(27)} = q^2v[3,4] + v[4,3],\\
v_{12}^{(27)} &= q^3v[1,6] + q^2v[3,4] + q^2v[4,3] + v[6,1],\\
v_{13}^{(27)} &= q^4v[2,6] + qv[3,5] + q^3v[5,3] + v[6,2],\\
v_{14}^{(27)} &= q^4v[3,5] + q^3v[4,4] + v[5,3],\\
v_{15}^{(27)} &= q^4v[1,7] + q^3v[2,6] + q^2v[3,5] + q^3v[4,4] + q^2v[5,3] + qv[6,2] + v[7,1],\\
v_i^{(27)} &= v_{28-i}^{(27)}|_{v[j,k]\to v[8-k,8-j]} \ (16 \le i \le 27).
\end{aligned} \tag{8.83}$$

$$\begin{aligned}
v_1^{(14)} &= v[1,2] - qv[2,1], \quad v_2^{(14)} = v[1,3] - qv[3,1],\\
v_3^{(14)} &= q^2v[1,4] + v[2,3] - q^3v[3,2] - q^2v[4,1],\\
v_4^{(14)} &= -qv[1,5] - v[2,4] + q^2v[4,2] + v[5,1],\\
v_5^{(14)} &= -qv[1,6] - v[3,4] + q^2v[4,3] + v[6,1], \quad v_6^{(14)} = -v[2,5] + qv[5,2],\\
v_7^{(14)} &= -q^2v[1,7] - qv[2,6] - v[3,5] - q^2v[3,5] - qv[4,4]\\
&\quad + q^3v[4,4] + v[5,3] + q^2v[5,3] + qv[6,2] + v[7,1],\\
v_8^{(14)} &= q^3v[2,6] + v[3,5] - q^4v[5,3] - qv[6,2],\\
v_i^{(14)} &= v_{15-i}^{(14)}|_{v[j,k]\to v[8-k,8-j]} \ (9 \le i \le 14).
\end{aligned} \tag{8.84}$$

$$\begin{aligned}
v_1^{(7)} &= q^{-3}v[1,4] - q^{-3}(1+q^2)v[2,3] + (1+q^2)v[3,2] - q^3 v[4,1],\\
v_2^{(7)} &= q^{-4}(1+q^2)v[1,5] - q^{-1}v[2,4] + qv[4,2] - q(1+q^2)v[5,1],\\
v_3^{(7)} &= q^{-4}(1+q^2)v[1,6] - q^{-1}v[3,4] + qv[4,3] - q(1+q^2)v[6,1],\\
v_4^{(7)} &= (1+q^{-2})(q^{-2}v[1,7] + q^{-3}v[2,6] - v[3,5] + v[5,3] - q^3 v[6,2] - q^2 v[7,1])\\
&\quad - q^{-1}(1-q^2)v[4,4],\\
v_i^{(7)} &= v_{8-i}^{(7)}|_{v[j,k]\to v[8-k,8-j]} \ (5 \le i \le 7),\\
v_1^{(1)} &= q^{-5}v[1,7] - q^{-4}v[2,6] + q^{-1}v[3,5] + qv[5,3] - q^4 v[6,2] + q^5 v[7,1]\\
&\quad - \frac{q^2 v[4,4]}{1+q^2}.
\end{aligned} \tag{8.85}$$

Using the above data for the Clebsh–Gordan coefficients, the structure constants g^{ij}, g_{ij} and f_k^{ij} are defined by

$$v_1^{(1)} = \sum_{ij} g^{ij} v_j \otimes v_i, \quad v_k^{(7)} = \sum_{ij} f_k^{ij} v_j \otimes v_i, \tag{8.86}$$

$$\sum_k g^{ik} g_{jk} = \delta_{ij}. \tag{8.87}$$

In particular, the constant g_{ij} necessary for the antipode (8.5) is given by

$$(g_{8-i,i})_{i=1,\dots,7} = (q^5, -q^4, q, -(q+q^{-1}), q^{-1}, -q^{-4}, q^{-5}), \quad \text{otherwise } g_{ij} = 0. \tag{8.88}$$

We note that the normalization of $v_i^{(\lambda)}$ influences g^{ij}, g_{ij}, f_k^{ij}, but the relation (8.4) and the antipode (8.5) remain invariant.

Let $\mathcal{P}^{(\lambda)}$ be the orthonormal projector from $7 \otimes 7$ onto λ in the decomposition (8.82). The quantum R matrix $R(x)$ satisfying the Yang–Baxter equation $R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x)$ is given by

$$\begin{aligned}
R(z) = P([qz^{-1}][q^4z^{-1}][q^6z^{-1}]\mathcal{P}^{(27)} + [qz][q^4z^{-1}][q^6z^{-1}]\mathcal{P}^{(14)}\\
+ [qz^{-1}][q^4z][q^6z^{-1}]\mathcal{P}^{(7)} + [qz][q^4z^{-1}][q^6z]\mathcal{P}^{(1)}),
\end{aligned} \tag{8.89}$$

where $[z] = z - z^{-1}$ and $P(v_i \otimes v_j) = v_j \otimes v_i$. The structure constants R_{kl}^{ij} are extracted from it as

$$\begin{aligned}
\sum_{ijkl} R_{kl}^{ij} E_{ik} \otimes E_{jl} &= -q^9 \lim_{z\to\infty} z^{-3} R(z)\\
&= P(q^{-2}\mathcal{P}^{(27)} - \mathcal{P}^{(14)} - q^6\mathcal{P}^{(7)} + q^{12}\mathcal{P}^{(1)}).
\end{aligned} \tag{8.90}$$

The normalization of $R(z)$ influences R^{ij}_{kl}, but the relation (8.3) is independent of it. There are 112 non-zero R^{ij}_{kl} given by

$$
\begin{aligned}
&R^{14}_{14} = R^{24}_{24} = R^{34}_{34} = R^{41}_{41} = R^{42}_{42} = R^{43}_{43} = R^{44}_{44} \\
&\quad = R^{45}_{45} = R^{46}_{46} = R^{47}_{47} = R^{54}_{54} = R^{64}_{64} = R^{74}_{74} = 1, \\
&R^{12}_{21} = R^{13}_{31} = R^{25}_{52} = R^{36}_{63} = R^{57}_{75} = R^{67}_{76} = -1 + q^{-2}, \\
&R^{11}_{11} = R^{22}_{22} = R^{33}_{33} = R^{55}_{55} = R^{66}_{66} = R^{77}_{77} = q^{-2}, \\
&R^{12}_{12} = R^{13}_{13} = R^{21}_{21} = R^{25}_{25} = R^{31}_{31} = R^{36}_{36} = R^{52}_{52} \\
&\quad = R^{57}_{57} = R^{63}_{63} = R^{67}_{67} = R^{75}_{75} = R^{76}_{76} = q^{-1}, \\
&R^{15}_{15} = R^{16}_{16} = R^{23}_{23} = R^{27}_{27} = R^{32}_{32} = R^{37}_{37} = R^{51}_{51} \\
&\quad = R^{56}_{56} = R^{61}_{61} = R^{65}_{65} = R^{72}_{72} = R^{73}_{73} = q, \\
&R^{17}_{17} = R^{26}_{26} = R^{35}_{35} = R^{53}_{53} = R^{62}_{62} = R^{71}_{71} = q^{2}, \\
&R^{14}_{23} = R^{42}_{51} = R^{43}_{61} = R^{47}_{56} = R^{54}_{72} = R^{64}_{73} = 1 - q^{2}, \\
&R^{15}_{24} = R^{16}_{34} = R^{24}_{42} = R^{27}_{45} = R^{32}_{41} = R^{34}_{43} \\
&\quad = R^{37}_{46} = R^{45}_{54} = R^{46}_{64} = R^{65}_{74} = q^{-2} - q^{2}, \\
&R^{35}_{44} = q^{-3} + q^{-1} - q - q^{3}, \quad R^{17}_{26} = R^{44}_{53} = R^{62}_{71} = q - q^{3}, \\
&R^{23}_{32} = R^{56}_{65} = q^{-2} - q^{4}, \quad R^{15}_{42} = R^{16}_{43} = R^{27}_{54} = R^{37}_{64} = -1 + q^{4}, \\
&R^{24}_{51} = R^{34}_{61} = R^{45}_{72} = R^{46}_{73} = -q^{2} + q^{4}, \\
&R^{35}_{53} = -1 + q^{-2} - q^{2} + q^{4}, \quad R^{26}_{35} = R^{53}_{62} = q^{-1} - q^{5}, \\
&R^{23}_{41} = R^{56}_{74} = -q + q^{5}, \quad R^{14}_{32} = R^{47}_{65} = -q^{3} + q^{5}, \\
&R^{15}_{51} = R^{16}_{61} = R^{27}_{72} = R^{37}_{73} = -1 + q^{-2} + q^{2} - q^{6}, \\
&R^{14}_{41} = R^{47}_{74} = -1 + q^{-2} + q^{4} - q^{6}, \\
&R^{17}_{35} = R^{44}_{62} = R^{53}_{71} = -q^{4} + q^{6}, \quad R^{26}_{44} = -1 - q^{2} + q^{4} + q^{6}, \\
&R^{17}_{44} = q^{-3} - q + q^{3} - q^{7}, \quad R^{26}_{53} = R^{35}_{62} = q^{5} - q^{7}, \\
&R^{44}_{71} = q - 2q^{3} + 2q^{5} - q^{7}, \quad R^{17}_{53} = R^{35}_{71} = -1 + q^{4} - q^{6} + q^{8}, \\
&R^{26}_{62} = -1 + q^{-2} - q^{8} + q^{10}, \quad R^{17}_{62} = R^{26}_{71} = q^{3} - q^{7} + q^{9} - q^{11}, \\
&R^{17}_{71} = -1 + q^{-2} + q^{2} - q^{4} - q^{6} + q^{8} - q^{10} + q^{12}.
\end{aligned}
\tag{8.91}
$$

8.8 Bibliographical Notes and Comments

The definition of $A_q(G_2)$ in terms of generators and relations in this chapter is quoted from [129], although the labeling of indices of the Cartan matrix (see [129, Eq. (27)]) is the opposite there. The data R^{ij}_{kl}, g^{ij} and $f^{\,ij}_{\;k}$ here are the same as [129, Eq. (33)], and [129, Eqs. (30) and (31)]. The relations [129, Eqs. (20) and (22)] are equivalent to (8.4) if the RTT relations are imposed. See the explanation after [129, Definition 7].

The fundamental representations (8.8) and (8.9) were given and the basic property of the intertwiner F in Proposition 8.4 were stated without a proof in [102, Sect. 4.4]. Sect. 8.5 is based on [85], where the quantized G_2 reflection equation was introduced and Theorem 8.6 was obtained. Conjecture 8.6.1 on the boundary vector is taken from [85, Eq. (77)].

As for the combinatorial and birational F in Sect. 8.6.2, comments similar to those for the 3D R in Sect. 3.7 are applicable. The birational map $F_{\text{birational}}$ (8.71) has effectively appeared in [21, Theorem 3.1] in terms of variables t_i, p_i, π_i. They are related to those in (8.66)–(8.71) as

$$(p_6, \ldots, p_1) = F_{\text{birational}}(t_1, \ldots, t_6), \tag{8.92}$$

$$(\pi_1, \pi_2, \pi_3, \pi_4) = (A, B, C, D)|_{(a,b,c,d,e,f)\to(t_1,\ldots,t_6)}. \tag{8.93}$$

Chapter 9
Comments on Tetrahedron-Type Equation for Non-crystallographic Coxeter Groups

Abstract This short chapter is a supplement recalling some basic facts on non-crystallographic finite Coxeter groups and raising questions concerning a possible tetrahedron-type equation.

9.1 Finite Coxeter Groups

The list of finite Coxeter groups[1] is given by [59]:

$$\begin{aligned} &A_n\,(n \geq 1),\ B_n\,(n \geq 2),\ D_n\,(n \geq 4),\ E_6,\ E_7,\ E_8,\ F_4,\ G_2, \\ &H_2,\ H_3,\ H_4,\ I_2(m)\,(m \geq 3). \end{aligned} \tag{9.1}$$

The indices are called ranks. The alphabetically last one $I_2(m)$ is the dihedral group which is the order $2m$ group of symmetry of a regular m-gon consisting of orthogonal transformations. It has overlap with the other rank 2 members for $m = 3, 4, 6$. See Fig. 9.2. Rank n Coxeter groups have a presentation in terms of generators $s_1, \dots, s_n$ obeying the relations $(s_i s_j)^{m_{ij}} = 1$ with $m_{ii} = 1$ and $m_{ij} = m_{ji} \in \{2, 3, \dots\} \cup \{\infty\}$ for $i \neq j$, where $m_{ij} = \infty$ is to be understood as no relation. The data $\{m_{ij}\}$ is customarily encoded in the Coxeter graph. Its vertex set is $\{1, 2, \dots, n\}$, and the two vertices i and j are connected by an unlabeled edge if $m_{ij} = 3$ and by an edge labeled with m_{ij} if $4 \leq m_{ij} < \infty$. The case $\forall m_{ij} \in \{2, 3, 4, 6\}$ is called crystallographic, and has a realization as the Weyl group of the corresponding Lie algebras. Thus those on the second line in (9.1), except $m = 3, 4, 6$, are the non-crystallographic finite Coxeter groups (Fig. 9.1).

[1] In this chapter, symbols like A_n are used to mean Coxeter groups instead of Lie algebras, unlike elsewhere in the book.

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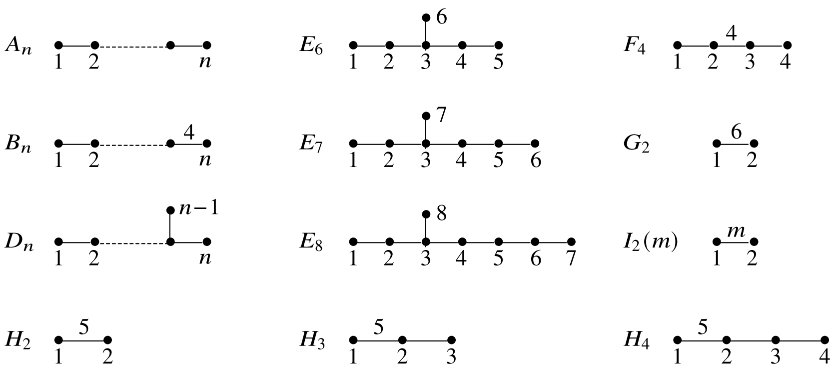
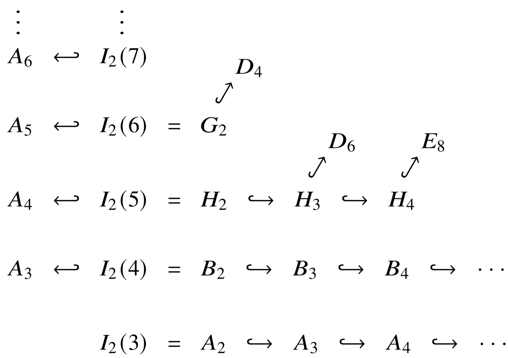


Fig. 9.1 Coxeter graphs of (9.1). Unlike the Dynkin diagrams, there is no arrow and C_n has been merged into B_n

The dihedral groups $I_2(m)$ and H_2, H_3, H_4 admit various embeddings as shown in Fig. 9.2.

Fig. 9.2 Various embeddings concerning non-crystallographic Coxeter groups



The embedding of type $X_n \hookrightarrow X_{n+1}$ just means that X_n is a parabolic subgroup of X_{n+1}. Denoting the generators in the image by t_i's, the other cases are given as follows [134]:

$$I_2(m) \hookrightarrow A_{m-1} : s_1 \mapsto \prod_{\substack{1 \le j \le m-1 \\ j:\text{odd}}} t_j, \quad s_2 \mapsto \prod_{\substack{1 \le j \le m-1 \\ j:\text{even}}} t_j, \tag{9.2}$$

$$G_2 \hookrightarrow D_4 : s_1 \mapsto t_1 t_3 t_4, \quad s_2 \mapsto t_2, \tag{9.3}$$

$$H_3 \hookrightarrow D_6 : s_1 \mapsto t_3 t_5, \quad s_2 \mapsto t_2 t_4, \quad s_3 \mapsto t_1 t_6, \tag{9.4}$$

$$H_4 \hookrightarrow E_8 : s_1 \mapsto t_4 t_8, \quad s_2 \mapsto t_3 t_5, \quad s_3 \mapsto t_2 t_6, \quad s_4 \mapsto t_1 t_7. \tag{9.5}$$

The embedding $B_2 \hookrightarrow A_3$ is a folding by the order 2 diagram automorphism, and has the generalization to $B_n \hookrightarrow A_{2n-1}$ $(n \geq 2)$ as $s_i \mapsto t_i t_{2n-i}$ $(1 \leq i < n)$ and $s_n \mapsto t_n$.

9.2 Tetrahedron-Type Equation for the Coxeter Group H_3

For any element w of a Coxeter group, one can consider a reduced expression (rex) graph. The vertices are reduced expressions of w and the two are connected by an edge if and only if they are transformed by a single application of the Coxeter relation $(s_i s_j)^{m_{ij}} = 1$ $(i \neq j)$. According to [126, Theorem 2.17], any non-trivial loop in a rex graph is generated from the loops in the rex graph of the longest element in the parabolic subgroups of rank 3. See also [44, Sect. 1.4.3]. In this sense, rank 3 cases are essential. In fact, we have seen that the A_3 and B_3 cases led to the tetrahedron and the 3D reflection equations[2] in earlier chapters, respectively. The remaining case is H_3, which we shall consider in what follows.

The Coxeter group H_3 is known as the symmetry of the icosahedron or equivalently the dual dodecahedron [59]. The relations of the generators s_1, s_2, s_3 read as $s_1^2 = s_2^2 = s_3^2 = 1$ and

$$s_1 s_3 = s_3 s_1, \quad s_2 s_3 s_2 = s_3 s_2 s_3, \quad s_1 s_2 s_1 s_2 s_1 = s_2 s_1 s_2 s_1 s_2. \tag{9.6}$$

Unlike the case of crystallographic Coxeter groups, the approach by a quantized coordinate ring is not available. However, one can formulate a compatibility equation formally by an argument similar to those for the crystallographic cases. We attach operators to the transformations in (9.6), denoted by only indices, as follows:

$$P = P^{-1} : 13 \to 31, \ 31 \to 13, \tag{9.7}$$

$$\Phi : 232 \to 323, \quad \Phi_{ijk} = R_{ijk} P_{ik}, \tag{9.8}$$

$$\Omega : 21212 \to 12121, \quad \Omega_{ijklm} = Y_{ijklm} P_{im} P_{jl}, \tag{9.9}$$

where, as before, the lower indices $i, j, k, \ldots$ of the operators specify the components that they act on non-trivially. The operators Ω and Y are the characteristic ones which are expected to come from H_2.

A reduced expression of the longest element of H_3 is

$$s_1 s_2 s_1 s_2 s_1 s_3 s_2 s_1 s_2 s_1 s_3 s_2 s_1 s_2 s_3, \tag{9.10}$$

which has the length 15. Now the process analogous to (3.93), (5.106) and (7.16) reads as

[2] We have actually encountered a fine difference between B_3 and C_3 versions originating in the relevant quantized coordinate rings.

$$
\begin{array}{ll}
1212\underline{13}212132123 & P_{5,6} \\
12123\underline{12121}32123 & \Omega^{-1}_{6,7,8,9,10} \\
121\underline{232}121\underline{232}123 & \Phi_{4,5,6}\Phi_{10,11,12} \\
12\underline{13}2\underline{31}2\underline{13}2\underline{31}23 & P_{3,4}P_{6,7}P_{9,10}P_{12,13} \\
123121\underline{323}121\underline{323} & \Phi^{-1}_{7,8,9}\Phi^{-1}_{13,14,15} \\
12312123\underline{21212}32 & \Omega_{9,10,11,12,13} \\
1231212\underline{31}212\underline{13}2 & P_{8,9}P_{13,14} \\
123\underline{12121}3212312 & \Omega^{-1}_{4,5,6,7,8} \\
1\underline{232}121\underline{232}12312 & \Phi_{2,3,4}\Phi_{8,9,10} \\
132\underline{31}2\underline{13}2\underline{31}2312 & P_{4,5}P_{7,8}P_{10,11} \\
\underline{13}21\underline{323}121\underline{323}12 & P_{12}\Phi^{-1}_{567}\Phi^{-1}_{11,12,13} \\
312123\underline{21212}3212 & \Omega^{-1}_{7,8,9,10,11} \\
31212\underline{31}212\underline{13}212 & P_{6,7}P_{11,12} \\
3\underline{12121}321231212 & \Omega^{-1}_{2,3,4,5,6} \\
32121\underline{232}1231212 & \Phi_{6,7,8} \\
3212\underline{13}2\underline{31}231212 & P_{5,6}P_{8,9} \\
32123121\underline{323}1212 & \Phi^{-1}_{9,10,11} \\
3212312123\underline{21212} & \Omega_{11,12,13,14,15} \\
321231212312121. &
\end{array}
\tag{9.11}
$$

It reverses the initial reduced word. There is another route achieving the reverse ordering which is related to (9.11), similarly to (7.17) and (7.18). Equating the two ways, substituting (9.7), (9.9) and assuming that $P_{i,j}$ just exchanges the indices as $P_{4,7}Y_{1,3,4,9} = Y_{1,3,7,9}P_{4,7}$ etc., we get the H_3 analogue of the tetrahedron equation:

$$
\begin{aligned}
&Y_{11,12,13,14,15}R^{-1}_{15,10,9}R_{5,7,15}Y^{-1}_{15,6,4,3,2}Y_{2,5,8,10,14}R^{-1}_{14,7,3}R^{-1}_{13,9,2}R_{1,6,14} \\
&\times R_{3,8,13}Y^{-1}_{13,10,7,4,1}Y_{1,3,5,9,12}R^{-1}_{12,8,4}R^{-1}_{11,2,1}R_{6,10,12}R_{4,5,11}Y^{-1}_{11,9,8,7,6} \\
&= Y_{6,7,8,9,11}R^{-1}_{11,5,4}R^{-1}_{12,10,6}R_{1,2,11}R_{4,8,12}Y^{-1}_{12,9,5,3,1}Y_{1,4,7,10,13}R^{-1}_{13,8,3} \\
&\times R^{-1}_{14,6,1}R_{2,9,13}R_{3,7,14}Y^{-1}_{14,10,8,5,2}Y_{2,3,4,6,15}R^{-1}_{15,7,5}R_{9,10,15}Y^{-1}_{15,14,13,12,11}.
\end{aligned}
\tag{9.12}
$$

There are 6 $Y^{\pm 1}$'s and 10 $R^{\pm 1}$'s on each side. If $Y^{-1}_{ijklm} = Y_{ijklm} = Y_{mlkji}$ and $R^{-1}_{ijk} = R_{ijk} = R_{kji}$ are valid, the above equation reduces to

$$
\begin{aligned}
&Y_{11,12,13,14,15}R_{9,10,15}R_{5,7,15}Y_{2,3,4,6,15}Y_{2,5,8,10,14}R_{3,7,14}R_{2,9,13}R_{1,6,14} \\
&\times R_{3,8,13}Y_{1,4,7,10,13}Y_{1,3,5,9,12}R_{4,8,12}R_{1,2,11}R_{6,10,12}R_{4,5,11}Y_{6,7,8,9,11} \\
&= \text{product in reverse order.}
\end{aligned}
\tag{9.13}
$$

A diagrammatic representation of the reduced version (9.13) of the H_3 compatibility equation is available in [44, Eq. (4.9)].

9.3 Discussion on the Quintic Coxeter Relation

The operator Y has been introduced formally in (9.9) in association with the quintic Coxeter relation. It is natural to seek it in the parabolic subgroup $H_2 \subset H_3$. In this section, we study a composition of the birational 3D R (Sect. 3.6.2) corresponding to the transformation of $s_1 s_2 s_1 s_2 s_1$ into $s_2 s_1 s_2 s_1 s_2$ in H_2 under the embedding $H_2 \hookrightarrow A_4$.

The embedding is the $m = 5$ case of (9.2), which reads as $s_1 \mapsto t_1 t_3$, $s_2 \mapsto t_2 t_4$. One way to realize $s_1 s_2 s_1 s_2 s_1 = s_2 s_1 s_2 s_1 s_2$ in the image is the following transformation of the reduced expression of the longest element of A_4:

$$
\begin{array}{ll}
\underline{13}2\underline{41}32\underline{41}3 & P_{1,2}P_{4,5}P_{8,9} \\
3\underline{121}432143 & \Phi_{2,3,4} \\
321\underline{24}32143 & P_{4,5} \\
3214\underline{232}143 & \Phi_{5,6,7} \\
321432\underline{31}43 & P_{7,8} \\
3214321\underline{343} & \Phi_{8,9,10} \\
32143\underline{214}34 & P_{6,7}P_{7,8} \\
321\underline{434}2134 & \Phi_{4,5,6} \\
32\underline{13}432134 & P_{3,4} \\
\underline{323}1432134 & \Phi_{1,2,3} \\
2321432\underline{13}4 & P_{8,9} \\
23214\underline{323}14 & \Phi_{6,7,8} \\
2321\underline{42}3214 & P_{5,6} \\
23\underline{212}43214 & \Phi_{3,4,5} \\
2312\underline{143}214 & P_{6,7}P_{5,6} \\
231\underline{24}3\underline{121}4 & P_{4,5}\Phi_{7,8,9} \\
2314\underline{232}124 & \Phi_{5,6,7} \\
2\underline{31}432\underline{31}24 & P_{7,8}P_{2,3} \\
21\underline{343}21324 & \Phi_{3,4,5} \\
2\underline{14}3\underline{42}1324 & P_{2,3}P_{5,6} \\
2413241324. &
\end{array}
\tag{9.14}
$$

As before, we have assigned an operator to each step, where P_{ij} is the transposition and $\Phi_{ijk} = R_{ijk}P_{ik}$ with $R_{ijk} = R^{\lambda}_{ijk}$ being the λ-deformed biraitonal 3D R (3.159).[3] The composition of the operators in (9.14) is rearranged as $\tilde{Y}\sigma$, where σ is a product of P_{ij}'s giving the reverse ordering permutation in $\mathfrak{S}_{10}$, and $\tilde{Y}$ has the form

$$\tilde{Y} = R_{2,4,6}R_{2,5,8}R_{2,7,9}R_{3,8,9}R_{3,5,7}R_{1,6,9}R_{1,4,7}R_{1,3,10}R_{4,5,10}R_{6,8,10}. \tag{9.15}$$

This is a totally positive involutive rational map of 10 variables $(x_1, \dots, x_{10})$. Set $(x'_1, \dots, x'_{10}) = \tilde{Y}((x_1, \dots, x_{10}))$. Then examples of simplest components are

$$x'_2 = \frac{x_2x_4x_5x_7}{x_2x_4x_5 + x_2x_4x_9 + x_2x_8x_9 + x_6x_8x_9 + \lambda x_2x_4x_9(x_5x_7 + x_5x_8 + x_6x_8)}, \tag{9.16}$$

$$x'_{10} = x'_2|_{x_1\leftrightarrow x_9, x_2\leftrightarrow x_{10}, x_3\leftrightarrow x_7, x_4\leftrightarrow x_8}. \tag{9.17}$$

One can directly check:

Proposition 9.1 *The map $\tilde{Y}$ preserves the following:*

$$x_2x_4x_5x_7, \quad x_3x_5x_8x_{10}, \quad x_1x_3x_4x_5x_6x_8, \quad x_4x_5x_6x_7x_8x_9, \tag{9.18}$$

$$\{(x_1, \dots, x_{10}) \mid x_7 = x_3, x_8 = x_4, x_9 = x_1, x_{10} = x_2\}. \tag{9.19}$$

One can get totally positive involutive maps of 5 variables by restricting the 6-dimensional space (9.19) by a conserved quantity. For instance, imposing $a = x_2x_4x_5x_7$ in the space (9.19) leads to the map $(x_1, x_2, x_3, x_4, x_6) \mapsto (x''_1, x''_2, x''_3, x''_4, x''_6)$ defined by

$$\begin{aligned}&(x''_1, x''_2, x''_3, x''_4, \frac{a}{x''_2x''_3x''_4}, x''_6, x''_3, x''_4, x''_1, x''_2)\\ &= \tilde{Y}((x_1, x_2, x_3, x_4, \frac{a}{x_2x_3x_4}, x_6, x_3, x_4, x_1, x_2))\end{aligned} \tag{9.20}$$

depending on the parameter a. However, there is no canonical way of doing such a reduction, and construction of a solution to the H_3 compatibility equation (9.12) or (9.13) remains as a challenge.

These features, especially the discrepancy of (9.19) from the desired dimension 5, stem from the fact that H_2 viewed as a subgroup A_4 is *not* an invariant of the diagram automorphism. In contrast, for the embedding $B_2 \hookrightarrow A_3$ respecting the diagram automorphism, the composition of the birational 3D R's corresponding to the length 6 longest element of A_3 admits a natural restriction to the 4-dimensional subspace

[3] $\Phi^{-1} = \Phi$ has been taken into account due to $P^{-1} = P$, $R^{-1} = R$, $R_{ijk} = R_{kji}$.

matching the 3D K [152] and reproduces [110, Remark 5.1]. Another example of such an embedding is $G_2 \hookrightarrow D_4$, which allows one to construct a λ-deformation of the birational 3D F (8.74).[4]

[4] Private communication with the author of [152].

Chapter 10
Connection to PBW Bases of Nilpotent Subalgebra of U_q

Abstract For a finite-dimensional simple Lie algebra $\mathfrak{g}$, let $U_q^+(\mathfrak{g})$ be the positive part of the quantized universal enveloping algebra $U_q(\mathfrak{g})$ with respect to the triangular decomposition. It has the Poincaré–Birkhoff–Witt (PBW) base labeled with the longest element of the Weyl group W of $\mathfrak{g}$. Let $A_q(\mathfrak{g})$ be the quantized coordinate ring of $\mathfrak{g}$. In this chapter, the intertwiner of the irreducible $A_q(\mathfrak{g})$ modules labeled with two different reduced expressions of W is identified with the transition matrix of the corresponding PBW bases of $U_q^+(\mathfrak{g})$. It leads to an alternative proof of the tetrahedron and 3D reflection equations within $U_q^+(\mathfrak{g})$. The boundary vectors in Sects. 3.6.1, 5.8.1 and 8.6.1 give rise to invariants of an anti-algebra involution in $U_q^+(\mathfrak{g})$ in an infinite product form.

10.1 Quantized Universal Enveloping Algebra $U_q(\mathfrak{g})$

10.1.1 Definition

In this chapter $\mathfrak{g}$ stands for a finite-dimensional simple Lie algebra. Its simple roots, simple coroots, fundamental weights are denoted by $\{\alpha_i\}_{i\in I}$,$\{h_i\}_{i\in I}$, $\{\varpi_i\}_{i\in I}$, where I is the index set of the Dynkin diagram of $\mathfrak{g}$. The weight lattice is $P = \oplus_{i\in I}\mathbb{Z}\varpi_i$ and the Cartan matrix $(a_{ij})_{i,j\in I}$ is given by $a_{ij} = \langle h_i, \alpha_j\rangle = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$.

The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is an associative algebra over $\mathbb{Q}(q)$ generated by $\{e_i, f_i, k_i^{\pm 1} \mid i \in I\}$ satisfying the relations:

$$\begin{aligned}
&k_ik_j = k_jk_i, \quad k_ik_i^{-1} = k_i^{-1}k_i = 1,\\
&k_ie_jk_i^{-1} = q_i^{\langle h_i,\alpha_j\rangle}e_j, \quad k_if_jk_i^{-1} = q_i^{-\langle h_i,\alpha_j\rangle}f_j, \quad [e_i, f_j] = \delta_{ij}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}},\\
&\sum_{r=0}^{1-a_{ij}}(-1)^r e_i^{(r)}e_je_i^{(1-a_{ij}-r)} = \sum_{r=0}^{1-a_{ij}}(-1)^r f_i^{(r)}f_jf_i^{(1-a_{ij}-r)} = 0 \quad (i \neq j).
\end{aligned} \tag{10.1}$$

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Here we use the following notations: $q_i = q^{(\alpha_i,\alpha_i)/2}$, $[m]_i = (q_i^m - q_i^{-m})/(q_i - q_i^{-1})$, $[n]_i! = \prod_{m=1}^n [m]_i$, $e_i^{(n)} = e_i^n/[n]_i!$, $f_i^{(n)} = f_i^n/[n]_i!$. We normalize the simple roots so that $q_i = q$ when α_i is a short root. The relation (10.1) is called q-Serre relation. The algebra $U_q(\mathfrak{g})$ is a Hopf algebra. For the comultiplication (or coproduct), we adopt the following[1]:

$$\Delta(k_i) = k_i \otimes k_i, \quad \Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i. \tag{10.2}$$

10.1.2 PBW Basis

Let W be the Weyl group of $\mathfrak{g}$. It is generated by simple reflections $\{s_i \mid i \in I\}$ obeying the relations: $s_i^2 = 1$, $(s_i s_j)^{m_{ij}} = 1$ $(i \neq j)$, where $m_{ij} = 2, 3, 4, 6$ for $\langle h_i, \alpha_j \rangle \langle h_j, \alpha_i \rangle = 0, 1, 2, 3$, respectively. Let w_0 be the longest element of W and fix a reduced expression $w_0 = s_{i_1} s_{i_2} \cdots s_{i_l}$. Then every positive root occurs exactly once in

$$\beta_1 = \alpha_{i_1}, \ \beta_2 = s_{i_1}(\alpha_{i_2}), \dots, \ \beta_l = s_{i_1} s_{i_2} \cdots s_{i_{l-1}}(\alpha_{i_l}). \tag{10.3}$$

Correspondingly, define elements $e_{\beta_r} \in U_q(\mathfrak{g})$ $(r = 1, \dots, l)$ by

$$e_{\beta_r} = T_{i_1} T_{i_2} \cdots T_{i_{r-1}}(e_{i_r}). \tag{10.4}$$

Here T_i is the action of the braid group on $U_q(\mathfrak{g})$. It is an algebra automorphism and is given on the generators $\{e_j\}$ by

$$T_i(e_i) = -k_i f_i, \quad T_i(e_j) = \sum_{r=0}^{-a_{ij}} (-1)^r q_i^r e_i^{(r)} e_j e_i^{(-a_{ij}-r)} \quad (i \neq j). \tag{10.5}$$

Let $U_q^+(\mathfrak{g})$ be a subalgebra of $U_q(\mathfrak{g})$ generated by $\{e_i \mid i \in I\}$. The only relation among them is the q-Serre relation (10.1) for e_i's. It is known that $e_{\beta_r} \in U_q^+(\mathfrak{g})$ holds for any r. $U_q^+(\mathfrak{g})$ has the PBW basis. It depends on the reduced expression $s_{i_1} s_{i_2} \cdots s_{i_l}$ of w_0. Set $\mathbf{i} = (i_1, i_2, \dots, i_l)$ and define for $A = (a_1, a_2, \dots, a_l) \in (\mathbb{Z}_{\geq 0})^l$

$$E_{\mathbf{i}}^A = e_{\beta_1}^{(a_1)} e_{\beta_2}^{(a_2)} \cdots e_{\beta_l}^{(a_l)}. \tag{10.6}$$

Then $\{E_{\mathbf{i}}^A \mid A \in (\mathbb{Z}_{\geq 0})^l\}$ forms a basis of $U_q^+(\mathfrak{g})$. We warn that the notations e_{i_r} with $i_r \in I$ and e_{β_r} with a positive root β_r should be distinguished properly from the context. In particular $e_{\beta_r}^{(a_r)} = (e_{\beta_r})^{a_r} / \prod_{m=1}^{a_r} \frac{p_r^m - p_r^{-m}}{p_r - p_r^{-1}}$ with $p_r = q^{(\beta_r,\beta_r)/2}$.

[1] This convention will be kept throughout the book.

10.2 Quantized Coordinate Ring $A_q(\mathfrak{g})$

10.2.1 Definition

Let us give the definition of the quantized coordinate ring $A_q(\mathfrak{g})$.[2] The relation to the concrete realization by generators and relations in earlier chapters will be explained later.

Let $O_{\text{int}}(\mathfrak{g})$ be the category of integrable left $U_q(\mathfrak{g})$ modules M such that, for any $v \in M$, there exists $l \geq 0$ satisfying $e_{i_1} \cdots e_{i_l} v = 0$ for any $i_1, \dots, i_l \in I$. Then $O_{\text{int}}(\mathfrak{g})$ is semisimple and any simple object is isomorphic to the irreducible module $V(\lambda)$ with dominant integral highest weight λ. Similarly, we can consider the category $O_{\text{int}}(\mathfrak{g}^{\text{op}})$ of integrable right $U_q(\mathfrak{g})$ modules M^r such that, for any $u \in M^r$, there exists $l \geq 0$ satisfying $u f_{i_1} \cdots f_{i_l} = 0$ for any $i_1, \dots, i_l \in I$. The superscript op signifies "opposite". $O_{\text{int}}(\mathfrak{g}^{\text{op}})$ is also semisimple and any simple object is isomorphic to the irreducible module $V^r(\lambda)$ with dominant integral highest weight λ. Let v_λ (resp. u_λ) be a highest weight vector of $V(\lambda)$ (resp. $V^r(\lambda)$). Then there exists a unique bilinear form (,)

$$V^r(\lambda) \otimes V(\lambda) \to \mathbb{Q}(q)$$

satisfying

$$(u_\lambda, v_\lambda) = 1 \quad \text{and}$$
$$(ug, v) = (u, gv) \quad \text{for } u \in V^r(\lambda),\ v \in V(\lambda),\ g \in U_q(\mathfrak{g}).$$

Let $U_q(\mathfrak{g})^*$ be $\text{Hom}_{\mathbb{Q}(q)}(U_q(\mathfrak{g}), \mathbb{Q}(q))$ and $\langle\ ,\ \rangle$ be the canonical pairing between $U_q(\mathfrak{g})^*$ and $U_q(\mathfrak{g})$. The comultiplication Δ of $U_q(\mathfrak{g})$ induces a multiplication of $U_q(\mathfrak{g})^*$ by

$$\langle \varphi\varphi', g \rangle = \langle \varphi \otimes \varphi', \Delta(g) \rangle \qquad \text{for } g \in U_q(\mathfrak{g}), \tag{10.7}$$

thereby giving $U_q(\mathfrak{g})^*$ the structure of $\mathbb{Q}(q)$-algebra. It also has a $U_q(\mathfrak{g})$ bimodule structure by

$$\langle x\varphi y, g \rangle = \langle \varphi, ygx \rangle \qquad \text{for } x, y, g \in U_q(\mathfrak{g}). \tag{10.8}$$

We define the subalgebra $A_q(\mathfrak{g})$ of $U_q(\mathfrak{g})^*$ by

$$A_q(\mathfrak{g}) = \{\varphi \in U_q(\mathfrak{g})^*;\ U_q(\mathfrak{g})\varphi \text{ belongs to } O_{\text{int}}(\mathfrak{g}) \text{ and } \varphi U_q(\mathfrak{g}) \text{ belongs to } O_{\text{int}}(\mathfrak{g}^{\text{op}})\},$$

and call it the quantized coordinate ring.

The following theorem is the q-analogue of the Peter–Weyl theorem.

---

[2] The definition and Theorem 10.1 are valid for any symmetrizable Kac–Moody algebra.

Theorem 10.1 *As a $U_q(\mathfrak{g})$ bimodule, $A_q(\mathfrak{g})$ is isomorphic to $\bigoplus_\lambda V^r(\lambda) \otimes V(\lambda)$, where λ runs over all dominant integral weights, by the homomorphisms*

$$\Psi_\lambda : V^r(\lambda) \otimes V(\lambda) \to A_q(\mathfrak{g})$$

given by

$$\langle \Psi_\lambda(u \otimes v), g \rangle = (u, gv)$$

for $u \in V^r(\lambda)$, $v \in V(\lambda)$, and $g \in U_q(\mathfrak{g})$.[3]

In our case of a finite-dimensional simple Lie algebra $\mathfrak{g}$, $A_q(\mathfrak{g})$ turns out to be a Hopf algebra. See for example [66, Chap. 9]. Its comultiplication is also denoted by Δ.

Let $\mathcal{R}$ be the universal R matrix for $U_q(\mathfrak{g})$. For its explicit formula see [29, p. 273] for example. For our purpose it is enough to know that

$$\mathcal{R} \in q^{(\mathrm{wt}\,\cdot,\mathrm{wt}\,\cdot)} \bigoplus_{\beta \in Q^+} (U_q^+)_\beta \otimes (U_q^-)_{-\beta}, \tag{10.9}$$

where $q^{(\mathrm{wt}\,\cdot,\mathrm{wt}\,\cdot)}$ is an operator acting on the tenor product $v_\lambda \otimes v_\mu$ of weight vectors v_λ, v_μ of weight λ, μ by $q^{(\mathrm{wt}\,\cdot,\mathrm{wt}\,\cdot)}(v_\lambda \otimes v_\mu) = q^{(\lambda,\mu)} v_\lambda \otimes v_\mu$, $Q_+ = \bigoplus_i \mathbb{Z}_{\geq 0}\alpha_i$, and $(U_q^\pm)_{\pm\beta}$ is the subspace of $U_q^\pm(\mathfrak{g})$ spanned by root vectors corresponding to $\pm\beta$.

Fix λ, let $\{u_j^\lambda\}$ and $\{v_i^\lambda\}$ be bases of $V^r(\lambda)$ and $V(\lambda)$ such that $(u_i^\lambda, v_j^\lambda) = \delta_{ij}$. Set

$$\varphi_{ij}^\lambda = \Psi_\lambda(u_i^\lambda \otimes v_j^\lambda) \in A_q(\mathfrak{g}). \tag{10.10}$$

Let R be the so-called constant R matrix for $V(\lambda) \otimes V(\mu)$. Denoting the homomorphism $U_q(\mathfrak{g}) \to \mathrm{End}(V(\lambda))$ by ρ_λ, it is given as

$$R \propto (\rho_\lambda \otimes \rho_\mu)(P\mathcal{R}), \tag{10.11}$$

where P stands for the exchange of the first and second components. The scalar multiple is determined appropriately depending on $\mathfrak{g}$. The reason we apply P is to fit the so-called RTT relation in (10.15). The dependence of R on λ and μ has been suppressed in the notation. R satisfies

$$R\Delta(g) = \Delta^{\mathrm{op}}(g)R \qquad \text{for any } g \in U_q(\mathfrak{g}), \tag{10.12}$$

where $\Delta^{\mathrm{op}} = P \circ \Delta \circ P$. Define matrix elements R_{kl}^{ij} by

$$R(v_k^\lambda \otimes v_l^\mu) = \sum_{i,j} R_{kl}^{ij} v_i^\lambda \otimes v_j^\mu. \tag{10.13}$$

[3] Of course this Ψ_λ has nothing to do with the intertwiners in (5.33), (6.22) and (7.5).

Define the right action of R on $V^r(\lambda) \otimes V^r(\mu)$ in such a way that $((u_i^\lambda \otimes u_j^\mu)R, v_k^\lambda \otimes v_l^\mu) = (u_i^\lambda \otimes u_j^\mu, R(v_k^\lambda \otimes v_l^\mu))$ holds. Then we have

$$(u_i^\lambda \otimes u_j^\mu)R = \sum_{k,l} R_{kl}^{ij} u_k^\lambda \otimes u_l^\mu. \tag{10.14}$$

Now for any $x \in U_q(\mathfrak{g})$, we have

$$\begin{aligned}
&\sum_{m,p} R_{mp}^{ij} \langle \varphi_{mk}^\lambda \varphi_{pl}^\mu, x\rangle = \sum_{m,p} R_{mp}^{ij} \langle \varphi_{mk}^\lambda \otimes \varphi_{pl}^\mu, \Delta(x)\rangle \\
&= \sum_{m,p} R_{mp}^{ij} \langle \Psi_\lambda(u_m^\lambda \otimes v_k^\lambda) \otimes \Psi_\mu(u_p^\mu \otimes v_l^\mu), \Delta(x)\rangle \\
&= \sum_{m,p} R_{mp}^{ij} (u_m^\lambda \otimes u_p^\mu, \Delta(x)(v_k^\lambda \otimes v_l^\mu)) = ((u_i^\lambda \otimes u_j^\mu)R, \Delta(x)(v_k^\lambda \otimes v_l^\mu)) \\
&= (u_i^\lambda \otimes u_j^\mu, R\Delta(x)(v_k^\lambda \otimes v_l^\mu)) = (u_i^\lambda \otimes u_j^\mu, \Delta^{\mathrm{op}}(x)R(v_k^\lambda \otimes v_l^\mu)) \\
&= \sum_{m,p} (u_i^\lambda \otimes u_j^\mu, \Delta^{\mathrm{op}}(x)(v_m^\lambda \otimes v_p^\mu))R_{kl}^{mp} = \sum_{m,p} (u_j^\mu \otimes u_i^\lambda, \Delta(x)(v_p^\mu \otimes v_m^\lambda))R_{kl}^{mp} \\
&= \sum_{m,p} \langle \varphi_{jp}^\mu \otimes \varphi_{im}^\lambda, \Delta(x)\rangle R_{kl}^{mp} = \sum_{m,p} \langle \varphi_{jp}^\mu \varphi_{im}^\lambda, x\rangle R_{kl}^{mp}.
\end{aligned}$$

Thus we get

$$\sum_{m,p} R_{mp}^{ij} \varphi_{mk}^\lambda \varphi_{pl}^\mu = \sum_{m,p} \varphi_{jp}^\mu \varphi_{im}^\lambda R_{kl}^{mp} \in A_q(\mathfrak{g}). \tag{10.15}$$

We call such a relation an RTT relation. It forms a large family containing conventional ones as the special case where $\lambda = \mu = \varpi_r$ for some specific fundamental weight ϖ_r.

Example 10.2 Consider the simplest case $\mathfrak{g} = A_1$ with $\lambda = \mu = \varpi_1$. We write $u_i^{\varpi_1}, v_i^{\varpi_1}$ simply as u_i, v_i $(i = 1, 2)$. The $U_q(sl_2)$ module structure is

$$f_1 v_1 = v_2,\ f_1 v_2 = 0,\ e_1 v_1 = 0,\ e_1 v_2 = v_1,\ k_1 v_1 = q v_1,\ k_1 v_2 = q^{-1} v_2, \tag{10.16}$$

$$u_1 f_1 = 0,\ u_2 f_1 = u_1,\ u_1 e_1 = u_2,\ u_2 e_1 = 0,\ u_1 k_1 = q u_1,\ u_2 k_1 = q^{-1} u_2. \tag{10.17}$$

The R matrix (3.3) acts as

$$R(v_1 \otimes v_1) = qv_1 \otimes v_1, \quad R(v_1 \otimes v_2) = v_1 \otimes v_2 + (q - q^{-1})v_2 \otimes v_1, \tag{10.18}$$

$$R(v_2 \otimes v_1) = v_2 \otimes v_1, \quad R(v_2 \otimes v_2) = qv_2 \otimes v_2, \tag{10.19}$$

$$(u_1 \otimes u_1)R = qu_1 \otimes u_1, \quad (u_2 \otimes u_1)R = u_2 \otimes u_1 + (q - q^{-1})u_1 \otimes u_2, \tag{10.20}$$

$$(u_1 \otimes u_2)R = u_1 \otimes u_2, \quad (u_2 \otimes u_2)R = qu_2 \otimes u_2. \tag{10.21}$$

Set $t_{ij} = \Psi_{\omega_1}(u_i \otimes v_j) \in A_q(A_1)$. Then we have

$$\begin{aligned}
\langle t_{11}t_{22}, x\rangle &= \langle \Psi_{\omega_1}(u_1 \otimes v_1) \otimes \Psi_{\omega_1}(u_2 \otimes v_2), \Delta(x)\rangle = (u_1 \otimes u_2, \Delta(x)(v_1 \otimes v_2)) \\
&= ((u_1 \otimes u_2)R, \Delta(x)(v_1 \otimes v_2)) = (u_1 \otimes u_2, \Delta^{\mathrm{op}}(x)R(v_1 \otimes v_2)) \\
&= (u_1 \otimes u_2, \Delta^{\mathrm{op}}(x)(v_1 \otimes v_2 + (q - q^{-1})v_2 \otimes v_1)) \\
&= (u_2 \otimes u_1, \Delta(x)(v_2 \otimes v_1 + (q - q^{-1})v_1 \otimes v_2)) \\
&= \langle \Psi_{\omega_1}(u_2 \otimes v_2) \otimes \Psi_{\omega_1}(u_1 \otimes v_1) \\
&\quad + (q - q^{-1})\Psi_{\omega_1}(u_2 \otimes v_1) \otimes \Psi_{\omega_1}(u_1 \otimes v_2), \Delta(x)\rangle \\
&= \langle t_{22} \otimes t_{11} + (q - q^{-1})t_{21} \otimes t_{12}, \Delta(x)\rangle \\
&= \langle t_{22}t_{11} + (q - q^{-1})t_{21}t_{12}, x\rangle,
\end{aligned}$$

which reproduces the relation $[t_{11}, t_{22}] = (q - q^{-1})t_{21}t_{12}$ in (3.9). Similarly, we have

$$\begin{aligned}
\langle t_{11}t_{22} - qt_{12}t_{21}, x\rangle &= \langle t_{11} \otimes t_{22} - qt_{12} \otimes t_{21}, \Delta(x)\rangle \\
&= (u_1 \otimes u_2, \Delta(x)(v_1 \otimes v_2)) - q(u_1 \otimes u_2, \Delta(x)(v_2 \otimes v_1)) \\
&= (u_1 \otimes u_2, \Delta(x)(v_1 \otimes v_2 - qv_2 \otimes v_1)).
\end{aligned}$$

Suppose $x = e_1^l k_1^m f_1^n \in U_q(sl_2)\,(l, m, n \in \mathbb{Z}_{\geq 0})$ without loss of generality. Since $v_1^0 := v_1 \otimes v_2 - qv_2 \otimes v_1$ is a $U_q(sl_2)$-singlet annihilated either by $\Delta(e_1)$ and $\Delta(f_1)$, one has $\Delta(x)v_1^0 = \delta_{l0}\delta_{n0}v_1^0$. Thus the RHS of the above calculation is equal to $\delta_{l0}\delta_{n0}(u_1 \otimes u_2, v_1^0) = \delta_{l0}\delta_{n0} = \langle 1, x\rangle$. This yields $t_{11}t_{22} - qt_{12}t_{21} = 1$ in (3.9).

Let us mention the relation to the formulation of $A_q(\mathfrak{g})$ in earlier chapters using specific generators and relations. Suppose ϖ_l is a fundamental weight such that any $V(\lambda)$ is included in the tensor power $V(\varpi_l)^{\otimes m}$ for some m.[4] Denoting the base of $V^r(\varpi_l)$ and $V(\varpi_l)$ by u_i and v_i, set

$$t_{ij} = \Psi_{\varpi_l}(u_i \otimes v_j) \in A_q(\mathfrak{g}). \tag{10.22}$$

[4] For example, in type B, it is the *spin* representation that qualifies this postulate rather than the vector representation. For type D, the argument in the text needs a slight modification since the two kinds of spin representations $V(\varpi_{n-1})$ and $V(\varpi_n)$ are necessary, but it does not influence the results in the chapter.

We know that t_{ij} satisfies the RTT relation (10.15) whose structure constant is the constant R matrix for $\lambda = \mu = \varpi_l$. Any vectors $u \in V^r(\lambda)$ and $v \in V(\lambda)$ are expressed as linear combinations $u = \sum C_{i_1,\ldots,i_m} u_{i_1} \otimes \cdots \otimes u_{im}$ and $v = \sum D_{j_1,\ldots,j_m} v_{j_1} \otimes \cdots \otimes v_{j_m}$. Theorem 10.1 shows that an arbitrary element of $A_q(\mathfrak{g})$ is constructed as $\Psi_\lambda(u \otimes v)$. A calculation similar to Example 10.2 leads to $\Psi_\lambda(u \otimes v) = \sum C_{i_1,\ldots,i_m} D_{j_1,\ldots,j_m} t_{i_1 j_1} \cdots t_{i_m j_m}$, which says that t_{ij}'s are certainly generators. They satisfy RTT and additional relations reflecting a fine structure of the Grothendieck ring of $\mathfrak{g}$ like $V(\varpi_l)^{\otimes m} \supset V(0)$ and $V(\varpi_l)^{\otimes m} \supset V(\varpi_l)$, etc. Our individual treatment in the earlier chapters corresponds to the choice $l = 1$ for A_{n-1}, C_n, G_2 and $l = n$ for B_n.[5]

10.2.2 Right Quotient Ring $A_q(\mathfrak{g})_S$

Here we prepare the necessary ingredients for the proof of Theorem 10.6. The point is to assure the well definedness of the division in (10.39).

Recall that $w_0 \in W$ is the longest element of the Weyl group. For any $l \in I$, let $v_{w_0\varpi_l} \in V(\varpi_l)$ be a lowest weight vector. Similarly, let $u_{\varpi_l} \in V^r(\varpi_l)$ be a highest weight vector. The following element will play a key role:

$$\sigma_l = \Psi_{\varpi_l}(u_{\varpi_l} \otimes v_{w_0\varpi_l}) \in A_q(\mathfrak{g}). \tag{10.23}$$

Example 10.3 For $\mathfrak{g} = A_1$ treated in Example 10.2, one has $\sigma_1 = \Psi_{\omega_1}(u_1 \otimes v_2) = t_{12}$.

Proposition 10.4 *The commutativity $\sigma_r\sigma_s = \sigma_s\sigma_r$ holds for any $r, s \in I$.*

Proof From (10.9) and (10.11) we have

$$(u_{\varpi_r} \otimes u_{\varpi_s})R = q^{(\varpi_r,\varpi_s)} u_{\varpi_r} \otimes u_{\varpi_s}, \tag{10.24}$$

$$R(v_{w_0\varpi_r} \otimes v_{w_0\varpi_s}) = q^{(\varpi_r,\varpi_s)} v_{w_0\varpi_r} \otimes v_{w_0\varpi_s}, \tag{10.25}$$

where $(w_0\varpi_r, w_0\varpi_s) = (\varpi_r, \varpi_s)$ has been used. Consider the RTT relation (10.15) with $\lambda = \varpi_r$, $\mu = \varpi_s$, and take the indices i, j, k, l so as to specify the following bases:

$$u^\lambda_i = u_{\varpi_r}, \quad u^\mu_j = u_{\varpi_s}, \quad v^\lambda_k = v_{w_0\varpi_r}, \quad v^\mu_l = v_{w_0\varpi_s}. \tag{10.26}$$

Then (10.24) and (10.25) indicate $R^{ij}_{mp} = q^{(\varpi_r,\varpi_s)}\delta^i_m\delta^i_p$ and $R^{mp}_{kl} = q^{(\varpi_r,\varpi_s)}\delta^m_k\delta^p_l$. Thus the RTT relation (10.15) reduces to

$$\varphi^{\varpi_r}_{ik}\varphi^{\varpi_s}_{jl} = \varphi^{\varpi_s}_{jl}\varphi^{\varpi_r}_{ik}. \tag{10.27}$$

[5] As for F_4 we did not present specific generators and relations.

The proof is finished by noting $\varphi_{ik}^{\varpi_r} = \sigma_r$ and $\varphi_{jl}^{\varpi_s} = \sigma_s$ by comparing (10.10) and (10.23). □

Since $A_q(\mathfrak{g})$ is a right $U_q(\mathfrak{g})$ module, we have an element $\sigma_i e_i \in A_q(\mathfrak{g})$. Later in Sect. 10.3.2, we will need the division $(\sigma_i e_i)/\sigma_i$ for $i \in I$. The following localization is known to be possible making sense of it.

Theorem 10.5 *Let n be the rank of $\mathfrak{g}$. For the multiplicatively closed subset $\mathcal{S} = \{\sigma_1^{m_1} \cdots \sigma_n^{m_n} \mid m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}\} \subset A_q(\mathfrak{g})$, the right quotient ring $A_q(\mathfrak{g})_{\mathcal{S}}$ exists.*

Elements of $A_q(\mathfrak{g})_{\mathcal{S}}$ are expressed in the form r/s with $r \in A_q(\mathfrak{g})$ and $s \in \mathcal{S}$. Theorem 10.5 guarantees the well-defined ring structure, namely, the addition and the multiplication of r_1/s_1 and r_2/s_2 in $A_q(\mathfrak{g})_{\mathcal{S}}$ as

$$r_1/s_1 + r_2/s_2 = (r_1 u + r_2 u')/(s_1 u), \qquad (r_1/s_1)(r_2/s_2) = (r_1 v')/(s_2 v), \quad (10.28)$$

where u, u', v, v' are so chosen that $s_1 u = s_2 u'$ $(u \in \mathcal{S}, u' \in A_q(\mathfrak{g}))$, $r_2 v = s_1 v'$ $(v \in \mathcal{S}, v' \in A_q(\mathfrak{g}))$.

10.3 Main Theorem

In this section we fix two reduced words $\mathbf{i} = (i_1, \ldots, i_l)$, $\mathbf{j} = (j_1, \ldots, j_l)$ of the longest element $w_0 = s_{i_1} \cdots s_{i_l} = s_{j_1} \cdots s_{j_l} \in W$.

10.3.1 Definitions of γ_B^A and Φ_B^A

In the $U_q(\mathfrak{g})$ side, we defined the PBW bases $E_{\mathbf{i}}^A$, $E_{\mathbf{j}}^B$ of $U_q^+(\mathfrak{g})$ in Sect. 10.1.2. We define their transition coefficient γ_B^A by

$$E_{\mathbf{i}}^A = \sum_B \gamma_B^A E_{\mathbf{j}}^B.$$

In the $A_q(\mathfrak{g})$ side, we have the intertwiner $\Phi : \mathcal{F}_{q_{i_1}} \otimes \cdots \otimes \mathcal{F}_{q_{i_l}} \to \mathcal{F}_{q_{j_1}} \otimes \cdots \otimes \mathcal{F}_{q_{j_l}}$ satisfying

$$\pi_{\mathbf{j}}(g) \circ \Phi = \Phi \circ \pi_{\mathbf{i}}(g) \qquad (\forall g \in A_q(\mathfrak{g})). \quad (10.29)$$

We take the parameters μ_i as in (3.21) and (5.19) to be 1. The intertwiner Φ is normalized by $\Phi(|0\rangle \otimes \cdots \otimes |0\rangle) = |0\rangle \otimes \cdots \otimes |0\rangle$. Under these conditions a matrix element Φ_B^A of Φ is uniquely specified by

$$\Phi|B\rangle = \sum_A \Phi_B^A |A\rangle,$$

where $A = (a_1, \ldots, a_l) \in (\mathbb{Z}_{\geq 0})^l$ and $|A\rangle = |a_1\rangle \otimes \cdots \otimes |a_l\rangle \in \mathcal{F}_{q_{j_1}} \otimes \cdots \otimes \mathcal{F}_{q_{j_l}}$ and similarly for $|B\rangle \in \mathcal{F}_{q_{i_1}} \otimes \cdots \otimes \mathcal{F}_{q_{i_l}}$. The main result of this chapter is

Theorem 10.6

$$\gamma_B^A = \Phi_B^A.$$

For any pair $(\mathbf{i}, \mathbf{j})$, from $\mathbf{i}$ one can reach $\mathbf{j}$ by applying Coxeter relations (for indices of the simple reflections). In view of the uniqueness of γ and Φ and the fact that the braid group action T_i is an algebra homomorphism, the proof of this theorem reduces to establishing the same equality for the rank 2 case $\mathfrak{g} = A_2, C_2$ and G_2.[6] This will be done in the sequel.

10.3.2 Proof of Theorem 10.6 for Rank 2 Cases

In the rank 2 cases, there are two reduced expressions $s_{i_1} \cdots s_{i_l}$ for the longest element of the Weyl group. Denote the associated sequences $\mathbf{i} = (i_1, \ldots, i_l)$ by $\mathbf{1}, \mathbf{2}$ and set $\mathbf{1}' = \mathbf{2}, \mathbf{2}' = \mathbf{1}$. Concretely, we take them as

$$A_2 : \mathbf{1} = (1, 2, 1), \qquad \mathbf{2} = (2, 1, 2), \qquad (q_1, q_2) = (q, q), \tag{10.30}$$

$$C_2 : \mathbf{1} = (1, 2, 1, 2), \qquad \mathbf{2} = (2, 1, 2, 1), \qquad (q_1, q_2) = (q, q^2), \tag{10.31}$$

$$G_2 : \mathbf{1} = (1, 2, 1, 2, 1, 2), \qquad \mathbf{2} = (2, 1, 2, 1, 2, 1), \qquad (q_1, q_2) = (q, q^3), \tag{10.32}$$

where q_i defined after (10.1) is also recalled. In order to simplify the formulas in Sect. 10.4, we use the PBW bases and the Fock states in yet another normalization as follows:

$$\tilde{E}_{\mathbf{i}}^A := ([a_1]_{i_1}! \cdots [a_l]_{i_l}!) E_{\mathbf{i}}^A = e_{\beta_1}^{a_1} \cdots e_{\beta_l}^{a_l}, \tag{10.33}$$

$$|A\rangle\rangle := d_{i_1,a_1} \cdots d_{i_l,a_l} |A\rangle, \quad d_{i,a} = q_i^{-a(a-1)/2} \lambda_i^a, \quad \lambda_i = (1 - q_i^2)^{-1}, \tag{10.34}$$

where $A = (a_1, \ldots, a_l)$. See after (10.1) for the symbol $[a]_i!$. The root vector e_{β_r} is defined in (10.4). Accordingly, we introduce the matrix elements $\tilde{\gamma}_B^A$ and $\tilde{\Phi}_B^A$ by

$$\tilde{E}_{\mathbf{i}}^A = \sum_B \tilde{\gamma}_B^A \tilde{E}_{\mathbf{i}'}^B, \quad \Phi|B\rangle\rangle = \sum_A \tilde{\Phi}_B^A |A\rangle\rangle, \quad (\mathbf{i} = \mathbf{1}, \mathbf{2}). \tag{10.35}$$

It follows that $\gamma_B^A = \tilde{\gamma}_B^A \prod_{k=1}^l ([b_k]_{i_k}!/[a_k]_{i_k}!)$ and $\Phi_B^A = \tilde{\Phi}_B^A \prod_{k=1}^l (d_{i_k,a_k}/d_{i_k,b_k})$ for $B = (b_1, \ldots, b_l)$. On the other hand, we know $\Phi_B^A = \Phi_A^B \prod_{k=1}^l ((q_{i_k}^2)_{b_k}/(q_{i_k}^2)_{a_k})$ from

[6] The B_2 case reduces to C_2 by the interchange of indices $1 \leftrightarrow 2 \in I$.

(3.63), (5.75) and (8.30). Due to the identity $(q_i^2)_m d_{i,m} = [m]_i!$, the assertion $\gamma^A_B = \Phi^A_B$ of Theorem 10.6 is equivalent to

$$\tilde{\gamma}^A_B = \tilde{\Phi}^B_A. \tag{10.36}$$

Let $\rho_{\mathbf{i}}(x) = (\rho_{\mathbf{i}}(x)_{AB})$ be the matrix for the left multiplication of $x \in U_q^+(\mathfrak{g})$:

$$x \cdot \tilde{E}^A_{\mathbf{i}} = \sum_B \tilde{E}^B_{\mathbf{i}} \rho_{\mathbf{i}}(x)_{BA}. \tag{10.37}$$

Let further $\pi_{\mathbf{i}}(g) = (\pi_{\mathbf{i}}(g)_{AB})$ be the representation matrix of $g \in A_q(\mathfrak{g})$:

$$\pi_{\mathbf{i}}(g)|A\rangle\rangle = \sum_B |B\rangle\rangle \pi_{\mathbf{i}}(g)_{BA}. \tag{10.38}$$

The following element in the right quotient ring $A_q(\mathfrak{g})_{\mathcal{S}}$ (see Theorem 10.5) will play a key role in our proof:

$$\xi_i = \lambda_i(\sigma_i e_i)/\sigma_i \quad (i = 1, 2). \tag{10.39}$$

We recall that the general definition of σ_i is (10.23). Its concrete form in the rank 2 case will be given in Lemmas 10.10, 10.12 and 10.14. In Sect. 10.4 we will check the following statement case by case. It says that the "conjugation" of e_i by σ_i on $A_q(\mathfrak{g})$ modules $(\sigma_i e_i)/\sigma_i$ corresponds to $(1 - q_i^2)e_i$ in $U_q^+(\mathfrak{g})$.

Proposition 10.7 *For $\mathfrak{g}$ of rank 2, $\pi_{\mathbf{i}}(\sigma_i)$ is invertible and the following equality is valid:*

$$\rho_{\mathbf{i}}(e_i)_{AB} = \pi_{\mathbf{i}}(\xi_i)_{AB} \quad (i = 1, 2), \tag{10.40}$$

where the RHS means $\lambda_i \pi_{\mathbf{i}}(\sigma_i e_i)\pi_{\mathbf{i}}(\sigma_i)^{-1}$.

Proof of Theorem 10.6 for rank 2 case. We write both sides of (10.40) as M^i_{AB} and the term for $\mathbf{i}'$ instead of $\mathbf{i}$ as M'^i_{AB}. From

$$\sum_{B,C} \tilde{E}^C_{\mathbf{i}'} M'^i_{CB} \tilde{\gamma}^A_B = e_i \sum_B \tilde{E}^B_{\mathbf{i}'} \tilde{\gamma}^A_B = e_i \tilde{E}^A_{\mathbf{i}} = \sum_B \tilde{E}^B_{\mathbf{i}} M^i_{BA} = \sum_{B,C} \tilde{E}^C_{\mathbf{i}'} \tilde{\gamma}^B_C M^i_{BA}$$

we have $\sum_B M'^i_{CB} \tilde{\gamma}^A_B = \sum_B \tilde{\gamma}^B_C M^i_{BA}$. On the other hand, the actions of the two sides of (10.29) with $g = \xi_i$ and $\mathbf{j} = \mathbf{i}'$ are calculated as

$$\pi_{\mathbf{i}'}(\xi_i) \circ \Phi|A\rangle\rangle = \pi_{\mathbf{i}'}(\xi_i) \sum_B |B\rangle\rangle \tilde{\Phi}^B_A = \sum_{B,C} |C\rangle\rangle M'^i_{CB} \tilde{\Phi}^B_A$$

and

$$\Phi \circ \pi_{\mathbf{i}}(\xi_i)|A\rangle\rangle = \Phi \sum_B |B\rangle\rangle M^i_{BA} = \sum_{B,C} |C\rangle\rangle \tilde{\Phi}^C_B M^i_{BA}.$$

Hence $\sum_B M'^i_{CB}\tilde{\Phi}^B_A = \sum_B \tilde{\Phi}^C_B M^i_{BA}$. Thus $\tilde{\gamma}^A_B$ and $\tilde{\Phi}^B_A$ satisfy the same relation. Moreover, the maps $\pi_{\mathbf{i}}$ and $\rho_{\mathbf{i}}$ are both homomorphisms, i.e. $\pi_{\mathbf{i}}(gh) = \pi_{\mathbf{i}}(g)\pi_{\mathbf{i}}(h)$ and $\rho_{\mathbf{i}}(xy) = \rho_{\mathbf{i}}(x)\rho_{\mathbf{i}}(y)$. We know that Φ is the intertwiner of the irreducible $A_q(\mathfrak{g})$ modules and (10.36) obviously holds as $1 = 1$ at $A = B = (0, \dots, 0)$. Thus it is valid for arbitrary A and B. $\square$

Remark 10.8 The equality (10.40) is valid for any $\mathfrak{g}$.

10.4 Proof of Proposition 10.7

Here we present the explicit formulas of (10.37) with $x = e_i$ and (10.38) with $g = \sigma_i, \sigma_i e_i$ that allow one to check Proposition 10.7. In each case, there are two $\mathbf{i}$-sequences, $\mathbf{1}$ and $\mathbf{2} = \mathbf{1}'$ corresponding to the two reduced words. Define

$$\chi = \text{the anti-algebra involution of } U_q^+(\mathfrak{g}) \text{ such that } \chi(e_i) = e_i. \tag{10.41}$$

Then both $E^A_{\mathbf{i}}$ in (10.6) and $\tilde{E}^A_{\mathbf{i}}$ in (10.33) satisfy

$$\chi(E^A_{\mathbf{i}}) = E^{A^\vee}_{\mathbf{i}'}, \qquad \chi(\tilde{E}^A_{\mathbf{i}}) = \tilde{E}^{A^\vee}_{\mathbf{i}'}, \tag{10.42}$$

where $A^\vee = (a_l, \dots, a_2, a_1)$ denotes the reversal of $A = (a_1, a_2, \dots, a_l)$. Applying χ to (10.37) with $x = e_i$ yields the right multiplication formula $\tilde{E}^{A^\vee}_{\mathbf{i}'} \cdot e_i = \sum_B \tilde{E}^{B^\vee}_{\mathbf{i}'} \rho_{\mathbf{i}}(e_i)_{BA}$ for the $\mathbf{i}'$-sequence. In view of this fact, we shall present the left and right multiplication formulas for $\mathbf{i} = \mathbf{2}$ only.

As for (10.38) with $g = \xi_i$ in (10.39), explicit formulas for $\sigma_i, \sigma_i e_i \in A_q(\mathfrak{g})$ and their image by both representations $\pi_{\mathbf{1}}$ and $\pi_{\mathbf{2}}$ will be given. We include an exposition on how to use these data to check (10.40) along the simplest A_2 case. The C_2 and G_2 cases are similar.

Following (10.34), we write $|m\rangle\!\rangle := d_{i,m}|m\rangle \in \mathcal{F}_{q_i}$ for each component. From the choice (10.30)–(10.32), the action of the q_i-oscillator on $\mathcal{F}_{q_i}$ $(i = 1, 2)$ takes the form

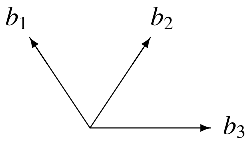
$$\begin{aligned} &\mathbf{a}^+|m\rangle\!\rangle = \lambda_1^{-1}q_1^m|m+1\rangle\!\rangle, \quad \mathbf{a}^-|m\rangle\!\rangle = [m]_1|m-1\rangle\!\rangle, \quad \mathbf{k}|m\rangle\!\rangle = q_1^m|m\rangle\!\rangle, \\ &\mathbf{A}^+|m\rangle\!\rangle = \lambda_2^{-1}q_2^m|m+1\rangle\!\rangle, \quad \mathbf{A}^-|m\rangle\!\rangle = [m]_2|m-1\rangle\!\rangle, \quad \mathbf{K}|m\rangle\!\rangle = q_2^m|m\rangle\!\rangle. \end{aligned} \tag{10.43}$$

See (10.34) and (3.13). We also use the shorthand

$$\langle m\rangle = q^m - q^{-m}. \tag{10.44}$$

10.4.1 Explicit Formulas for A_2

Consider $\mathfrak{g} = A_2$.



The q-Serre relations are

$$e_1^2 e_2 - [2]_1 e_1 e_2 e_1 + e_2 e_1^2 = 0, \quad e_2^2 e_1 - [2]_1 e_2 e_1 e_2 + e_1 e_2^2 = 0, \tag{10.45}$$

where $[m]_1 = \langle m \rangle / \langle 1 \rangle$. For simplicity we write the positive root vectors $e_{\beta_{i_r}}$ in (10.4) with $(i_1, i_2, i_3) = \mathbf{2}$ (10.30) as

$$b_1 = e_{\beta_1} = e_2, \quad b_2 = e_{\beta_2} = e_1 e_2 - q e_2 e_1, \quad b_3 = e_{\beta_3} = e_1. \tag{10.46}$$

The corresponding positive roots are $(\beta_1, \beta_2, \beta_3) = (\alpha_2, \alpha_1 + \alpha_2, \alpha_1)$. In particular, $b_2 = T_2(e_1)$. Their commutation relations are

$$b_2 b_1 = q^{-1} b_1 b_2, \quad b_3 b_1 = b_2 + q b_1 b_3, \quad b_3 b_2 = q^{-1} b_2 b_3. \tag{10.47}$$

Lemma 10.9 *For $\tilde{E}_{\mathbf{2}}^{a,b,c} = b_1^a b_2^b b_3^c$, we have*

$$\begin{aligned}
\tilde{E}_{\mathbf{2}}^{a,b,c} \cdot e_1 &= \tilde{E}_{\mathbf{2}}^{a,b,c+1}, \\
\tilde{E}_{\mathbf{2}}^{a,b,c} \cdot e_2 &= q^{c-b} \tilde{E}_{\mathbf{2}}^{a+1,b,c} + [c]_1 \tilde{E}_{\mathbf{2}}^{a,b+1,c-1}, \\
e_1 \cdot \tilde{E}_{\mathbf{2}}^{a,b,c} &= q^{a-b} \tilde{E}_{\mathbf{2}}^{a,b,c+1} + [a]_1 \tilde{E}_{\mathbf{2}}^{a-1,b+1,c}, \\
e_2 \cdot \tilde{E}_{\mathbf{2}}^{a,b,c} &= \tilde{E}_{\mathbf{2}}^{a+1,b,c}.
\end{aligned}$$

Proof By induction, we have

$$\begin{aligned}
b_3 b_1^n &= q^n b_1^n b_3 + [n]_1 b_1^{n-1} b_2, \quad b_3 b_2^n = q^{-n} b_2^n b_3, \\
b_3^n b_1 &= q^n b_1 b_3^n + [n]_1 b_2 b_3^{n-1}, \quad b_2^n b_1 = q^{-n} b_1 b_2^n.
\end{aligned}$$

The lemma is a direct consequence of these formulas. □

Set $\tilde{E}_{\mathbf{1}}^{a,b,c} = \chi(\tilde{E}_{\mathbf{2}}^{c,b,a}) = \chi(b_3^a)\chi(b_2^b)\chi(b_1^c) = b_3^a b_2'^b b_1^c$, where $b_2' := \chi(b_2) = e_2 e_1 - q e_1 e_2$. By applying χ to the first two relations in Lemma 10.9, we get

$$e_1 \cdot \tilde{E}_{\mathbf{1}}^{a,b,c} = \tilde{E}_{\mathbf{1}}^{a+1,b,c}, \qquad e_2 \cdot \tilde{E}_{\mathbf{1}}^{a,b,c} = q^{a-b} E_{\mathbf{1}}^{a,b,c+1} + [a]_1 \tilde{E}_{\mathbf{1}}^{a-1,b+1,c}. \tag{10.48}$$

Thus we find $\rho_{\mathbf{i}'}(e_i) = \rho_{\mathbf{i}}(e_{3-i})$. This property is only valid for A_2 and not in C_2 and G_2.

Let u_i $(i = 1, 2, 3)$ be the bases of the right $U_q(A_2)$ module $V^r(\varpi_1)$ such that $u_j = u_1 e_1 \cdots e_{j-1} e_j$. Similarly, let v_i $(i = 1, 2, 3)$ be the bases of the left $U_q(A_2)$ module $V(\varpi_1)$ such that $v_j = f_j f_{j-1} \cdots f_1 v_1$.

$$\begin{array}{cccc}
k_1 & k_2 & V^r(\varpi_1) & V(\varpi_1) \\
q & 1 & u_1 & v_1 \\
 & & \downarrow e_1 & f_1 \downarrow \\
q^{-1} & q & u_2 & v_2 \\
 & & \downarrow e_2 & f_2 \downarrow \\
1 & q^{-1} & u_3 & v_3
\end{array}$$

The left two columns specify the weights for example as $u_2 k_1 = q^{-1} u_2, k_1 v_1 = q v_1$. For the coproduct (10.2), the bases of $V^r(\varpi_2)$ and $V(\varpi_2)$ are similarly given as

$$\begin{array}{cccc}
k_1 & k_2 & V^r(\varpi_2) & V(\varpi_2) \\
1 & q & u_1 \otimes u_2 - q u_2 \otimes u_1 & v_1 \otimes v_2 - q v_2 \otimes v_1 \\
 & & \downarrow e_2 & f_2 \downarrow \\
q & q^{-1} & u_1 \otimes u_3 - q u_3 \otimes u_1 & v_1 \otimes v_3 - q v_3 \otimes v_1 \\
 & & \downarrow e_1 & f_1 \downarrow \\
q^{-1} & 1 & u_2 \otimes u_3 - q u_3 \otimes u_2 & v_2 \otimes v_3 - q v_3 \otimes v_2
\end{array}$$

Here $g = k_i, e_i, f_i$ are to be understood as $\Delta(g)$ in (10.2).

Following (10.22) with $l = 1$ we set

$$t_{ij} = \Psi_{\varpi_1}(u_i \otimes v_j) \tag{10.49}$$

for $1 \le i, j \le 3$. They satisfy the relations (3.5) and (3.2) of the earlier definition of $A_q(A_2)$. The formula (10.23) reads

$$\sigma_1 = \Psi_{\varpi_1}(u_1 \otimes v_3), \tag{10.50}$$

$$\sigma_2 = \frac{1}{1+q^2} \Psi_{\varpi_2}\big((u_1 \otimes u_2 - q u_2 \otimes u_1) \otimes (v_2 \otimes v_3 - q v_3 \otimes v_2)\big), \tag{10.51}$$

where $(1+q^2)^{-1}$ is the normalization factor.[7] Thus we see $\sigma_1 = t_{13}$. On the other hand, from

[7] The normalization of σ_i actually does not matter since only $\sigma_i e_i / \sigma_i$ will be used.

$$
\begin{aligned}
\langle \sigma_2, x\rangle &= \frac{(u_1 \otimes u_2 - q u_2 \otimes u_1,\, \Delta(x)(v_2 \otimes v_3 - q v_3 \otimes v_2))}{1+q^2} \\
&= \frac{\langle t_{12} \otimes t_{23} - q t_{13} \otimes t_{22} - q t_{22} \otimes t_{13} + q^2 t_{23} \otimes t_{12},\, \Delta(x)\rangle}{1+q^2} \\
&= \frac{\langle t_{12}t_{23} - q t_{13}t_{22} - q t_{22}t_{13} + q^2 t_{23}t_{12},\, x\rangle}{1+q^2} \quad (\forall x \in U_q(A_2)),
\end{aligned}
$$

we find $\sigma_2 = (1+q^2)^{-1}(t_{12}t_{23} - q t_{13}t_{22} - q t_{22}t_{13} + q^2 t_{23}t_{12})$.[8] Using the relations $[t_{12}, t_{23}] = (q - q^{-1})t_{22}t_{13}$ and $[t_{22}, t_{13}] = 0$ from (3.5), this is simplified into $\sigma_2 = t_{12}t_{23} - q t_{22}t_{13}$, which is the (3, 1)-quantum minor of $(t_{ij})_{1\le i,j\le 3}$.

Let us turn to $\sigma_i e_i$. First we note

$$\langle t_{ij}k_r, x\rangle = (u_i k_r, x v_j) = q^{\delta_{ir}-\delta_{i,r+1}}(u_i, x v_j) = q^{\delta_{ir}-\delta_{i,r+1}}\langle t_{ij}, x\rangle, \tag{10.52}$$

$$\langle t_{ij}e_r, x\rangle = (u_i e_r, x v_j) = \delta_{ir}(u_{i+1}, x v_j) = \delta_{ir}\langle t_{i+1,j}, x\rangle. \tag{10.53}$$

They imply

$$t_{ij}k_r = q^{\delta_{ir}-\delta_{i,r+1}} t_{ij}, \qquad t_{ij}e_r = \delta_{ir} t_{i+1,j}. \tag{10.54}$$

Using this and the coproduct Δ in (10.2), we see

$$
\begin{aligned}
\langle \sigma_1 e_1, x\rangle &= \langle t_{13}e_1, x\rangle = \langle t_{23}, x\rangle, \\
\langle \sigma_2 e_2, x\rangle &= \langle (t_{12} \otimes t_{23} - q t_{22} \otimes t_{13})\Delta(e_2), \Delta(x)\rangle \\
&= \langle t_{12}k_2 \otimes t_{23}e_2 - q t_{22}e_2 \otimes t_{13}, \Delta(x)\rangle \\
&= \langle t_{12} \otimes t_{33} - q t_{32} \otimes t_{13}, \Delta(x)\rangle = \langle t_{12}t_{33} - q t_{32}t_{13}, x\rangle.
\end{aligned}
$$

In these calculations, one should distinctively recognize that $t_{13}e_1$ for instance is an action of $e_1 \in U_q(A_2)$ on $t_{13} \in A_q(A_2)$ viewed as an element of a right $U_q(A_2)$ module, whereas $t_{12}t_{33}$ is just a multiplication within $A_q(A_2)$. To summarize, we have shown:

Lemma 10.10 *For $A_q(A_2)$, the following relations are valid:*

$$\sigma_1 = t_{13}, \quad \sigma_2 = t_{12}t_{23} - q t_{22}t_{13}, \quad \sigma_1 e_1 = t_{23}, \quad \sigma_2 e_2 = t_{12}t_{33} - q t_{32}t_{13}. \tag{10.55}$$

From (3.35) and Lemma 10.10, we find

$$\pi_{\mathbf{1}}(\sigma_1) = \mathbf{k}_1\mathbf{k}_2, \quad \pi_{\mathbf{1}}(\sigma_1 e_1) = \mathbf{a}_1^+\mathbf{k}_2, \quad \pi_{\mathbf{1}}(\sigma_2) = \mathbf{k}_2\mathbf{k}_3, \quad \pi_{\mathbf{1}}(\sigma_2 e_2) = \mathbf{a}_1^-\mathbf{a}_2^+\mathbf{k}_3 + \mathbf{k}_1\mathbf{a}_3^+,$$

where a notation like $\mathbf{k}_1\mathbf{a}_3^+ = \mathbf{k} \otimes 1 \otimes \mathbf{a}^+$ has been used. Since $\mathbf{k} \in \mathrm{End}(\mathcal{F}_q)$ is invertible, so is $\pi_{\mathbf{i}}(\sigma_i)$ and we may write

[8] The calculation is displayed to illustrate how this could be concluded directly from (10.51) and the definition (10.23).

$$\pi_{\mathbf{1}}(\xi_1) = \lambda_1 \mathbf{a}_1^+ \mathbf{k}_1^{-1}, \quad \pi_{\mathbf{1}}(\xi_2) = \lambda_2(\mathbf{a}_1^- \mathbf{a}_2^+ \mathbf{k}_2^{-1} + \mathbf{k}_1 \mathbf{k}_2^{-1} \mathbf{a}_3^+ \mathbf{k}_3^{-1}),$$

where $\lambda_1 = \lambda_2 = (1-q^2)^{-1}$. Thus (10.43) leads to

$$\pi_{\mathbf{1}}(\xi_1)|a,b,c\rangle\rangle = |a+1,b,c\rangle\rangle, \tag{10.56}$$
$$\pi_{\mathbf{1}}(\xi_2)|a,b,c\rangle\rangle = [a]_1|a-1,b+1,c\rangle\rangle + q^{a-b}|a,b,c+1\rangle\rangle. \tag{10.57}$$

These formulas agree with (10.48) proving (10.40) for $\mathbf{i} = \mathbf{1}$. The other case $\mathbf{i} = \mathbf{2}$ also holds due to the symmetry $\pi_{\mathbf{2}}(\xi_i) = \pi_{\mathbf{1}}(\xi_{3-i})$. Thus Proposition 10.7 is established for A_2.

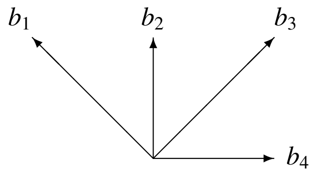
In terms of the 3DR in Chap. 3, Theorem 10.6 implies

$$E_{\mathbf{i}}^{a,b,c} = \sum_{i,j,k} R_{ijk}^{abc}\, E_{\mathbf{i}'}^{k,j,i}. \tag{10.58}$$

This is valid either for $(\mathbf{i}, \mathbf{i}') = (\mathbf{1}, \mathbf{2})$ or $(\mathbf{2}, \mathbf{1})$ thanks to (3.62). The weight conservation (3.48) assures the equality of weights of the two sides.

10.4.2 Explicit Formulas for C_2

Consider $\mathfrak{g} = C_2$.



The q-Serre relations are

$$\begin{aligned} &e_1^3 e_2 - [3]_1 e_1^2 e_2 e_1 + [3]_1 e_1 e_2 e_1^2 - e_2 e_1^3 = 0, \\ &e_2^2 e_1 - [2]_2 e_2 e_1 e_2 + e_1 e_2^2 = 0, \end{aligned} \tag{10.59}$$

where $[m]_1 = \langle m \rangle / \langle 1 \rangle$ and $[m]_2 = \langle 2m \rangle / \langle 2 \rangle$. For simplicity we write the positive root vectors $e_{\beta_{i_r}}$ in (10.4) with $(i_1, \ldots, i_4) = \mathbf{2}$ (10.31) as

$$\begin{aligned} &b_1 = e_{\beta_1} = e_2, \quad b_2 = e_{\beta_2} = e_1 e_2 - q^2 e_2 e_1, \\ &b_3 = e_{\beta_3} = \frac{1}{[2]_1}(e_1 b_2 - b_2 e_1), \quad b_4 = e_{\beta_4} = e_1. \end{aligned} \tag{10.60}$$

Their commutation relations are

$$b_2b_1 = q^{-2}b_1b_2, \qquad b_3b_1 = -q^{-1}\langle 1\rangle[2]_1^{-1}b_2^2 + b_1b_3, \tag{10.61}$$
$$b_4b_1 = b_2 + q^2b_1b_4, \quad b_3b_2 = q^{-2}b_2b_3, \tag{10.62}$$
$$b_4b_2 = [2]_1b_3 + b_2b_4, \quad b_4b_3 = q^{-2}b_3b_4. \tag{10.63}$$

Lemma 10.11 *For $\tilde{E}_{\mathbf{2}}^{a,b,c,d} = b_1^ab_2^bb_3^cb_4^d$, we have*

$$\begin{aligned}
\tilde{E}_{\mathbf{2}}^{a,b,c,d} \cdot e_1 &= \tilde{E}_{\mathbf{2}}^{a,b,c,d+1},\\
\tilde{E}_{\mathbf{2}}^{a,b,c,d} \cdot e_2 &= [d]_1q^{d-2c-1}\tilde{E}_{\mathbf{2}}^{a,b+1,c,d-1} + q^{2(d-b)}\tilde{E}_{\mathbf{2}}^{a+1,b,c,d}\\
&\quad - \langle 1\rangle q^{2d-2c+1}[c]_2[2]_1^{-1}\tilde{E}_{\mathbf{2}}^{a,b+2,c-1,d} + [d-1]_1[d]_1\tilde{E}_{\mathbf{2}}^{a,b,c+1,d-2},\\
e_1 \cdot \tilde{E}_{\mathbf{2}}^{a,b,c,d} &= [2]_1[b]_1q^{2a-b+1}\tilde{E}_{\mathbf{2}}^{a,b-1,c+1,d} + q^{2a-2c}\tilde{E}_{\mathbf{2}}^{a,b,c,d+1} + [a]_2\tilde{E}_{\mathbf{2}}^{a-1,b+1,c,d},\\
e_2 \cdot \tilde{E}_{\mathbf{2}}^{a,b,c,d} &= \tilde{E}_{\mathbf{2}}^{a+1,b,c,d}.
\end{aligned}$$

Proof By induction, we have

$$\begin{aligned}
b_4b_1^n &= b_1^nb_4q^{2n} + [n]_2b_1^{n-1}, b_2,\\
b_4b_2^n &= [2]_1[n]_1b_2^{n-1}b_3q^{-n+1} + b_2^nb_4,\\
b_4b_3^n &= q^{-2n}b_3^nb_4,\\
b_4^nb_1 &= [n]_1b_2b_4^{n-1}q^{n-1} + b_1b_4^nq^{2n} + [n-1]_1[n]_1b_3b_4^{n-2},\\
b_3^nb_1 &= -q^{1-2n}\langle 1\rangle[n]_2[2]_1^{-1}b_2^2b_3^{n-1} + b_1b_3^n,\\
b_3^nb_2 &= q^{-2n}b_2b_3^n,\\
b_2^nb_1 &= q^{-2n}b_1b_2^n.
\end{aligned}$$

The lemma is a direct consequence of these formulas. □

Set $\tilde{E}_{\mathbf{1}}^{a,b,c,d} = \chi(\tilde{E}_{\mathbf{2}}^{d,c,b,a})$. The left multiplication formula for this basis is deduced from the above lemma by applying χ.

Let u_i and v_i $(i = 1, 2, 3, 4)$ be bases of $V^r(\varpi_1)$ and $V(\varpi_1)$ such that $u_j = u_1e_1\cdots e_{j-1}e_j$ and $v_j = f_jf_{j-1}\cdots f_1v_1$, where $e_3 = e_1$, $f_3 = f_1$ just temporarily.

k_1	k_2	$V^r(\varpi_1)$	$V(\varpi_1)$
q	1	u_1	v_1
		$\downarrow e_1$	$f_1 \downarrow$
q^{-1}	q	u_2	v_2
		$\downarrow e_2$	$f_2 \downarrow$
q	q^{-1}	u_3	v_3
		$\downarrow e_1$	$f_1 \downarrow$
q^{-1}	1	u_4	v_4

The left two columns specify the weights as in the A_2 case. For the coproduct (10.2), the bases of $V(\varpi_2)$ and $V^r(\varpi_2)$ are similarly given as

k_1	k_2	$V^r(\varpi_2)$	$V(\varpi_2)$
1	q	$u_1 \otimes u_2 - qu_2 \otimes u_1$	$v_1 \otimes v_2 - qv_2 \otimes v_1$
		$\downarrow e_2$	$f_2 \downarrow$
q^2	q^{-1}	$u_1 \otimes u_3 - qu_3 \otimes u_1$	$v_1 \otimes v_3 - qv_3 \otimes v_1$
		$\downarrow e_1$	$f_1 \downarrow$
1	1	$u_2 \otimes u_3 + qu_1 \otimes u_4$ $-qu_4 \otimes u_1 - q^2 u_3 \otimes u_2$	$v_2 \otimes v_3 + qv_1 \otimes v_4$ $-qv_4 \otimes v_1 - q^2 v_3 \otimes v_2$
		$\downarrow e_1$	$f_1 \downarrow$
q^{-2}	q	$u_2 \otimes u_4 - qu_4 \otimes u_2$	$v_2 \otimes v_4 - qv_4 \otimes v_2$
		$\downarrow e_2$	$f_2 \downarrow$
1	q^{-1}	$u_3 \otimes u_4 - qu_4 \otimes u_3$	$v_3 \otimes v_4 - qv_4 \otimes v_3$

Arrows here indicate the images only up to overall normalization.

We adopt the definition of t_{ij} in (10.22) with $l = 1$ for $1 \le i, j \le 4$. Then t_{ij}'s satisfy the relations (5.1), (5.2) of the earlier definition of $A_q(C_2)$. The formula (10.23) reads as

$$\sigma_1 = \Psi_{\varpi_1}(u_1 \otimes v_4), \tag{10.64}$$

$$\sigma_2 = \frac{1}{1+q^2}\Psi_{\varpi_2}\big((u_1 \otimes u_2 - qu_2 \otimes u_1) \otimes (v_3 \otimes v_4 - qv_4 \otimes v_3)\big). \tag{10.65}$$

By a calculation similar to $A_q(A_2)$ using the commutation relations

$$[t_{24}, t_{13}] = (q - q^{-1})t_{23}t_{14}, \quad [t_{14}, t_{23}] = 0, \tag{10.66}$$

we get:

Lemma 10.12 *For $A_q(C_2)$, the following relations are valid:*

$$\sigma_1 = t_{14}, \quad \sigma_2 = t_{13}t_{24} - qt_{23}t_{14}, \quad \sigma_1 e_1 = t_{24}, \quad \sigma_2 e_2 = t_{13}t_{34} - qt_{33}t_{14}. \tag{10.67}$$

Images of the generators t_{ij} by the representations $\pi_{\mathbf{1}}$ and $\pi_{\mathbf{2}}$ in (10.31) are available in Sect. 5.4 as $\pi_{\mathbf{1}}(t_{ij}) = P_{14}P_{23}\pi_{2121}(\tilde{\Delta}(t_{ij}))P_{14}P_{23}$ and $\pi_{\mathbf{2}}(t_{ij}) = \pi_{2121}(\Delta(t_{ij}))$, where the conjugation by $P_{14}P_{23}$ reverses the order of the four-fold tensor product. See (5.39) and (5.40). From (5.37), the relations (5.41)–(5.56) are displaying the concrete form of $\pi_{\mathbf{2}}(t_{ij})K = K(P_{14}P_{23}\pi_{\mathbf{1}}(t_{ij})P_{14}P_{23})$. For convenience, we pick those generators appearing in Lemma 10.12:

$$\pi_{\mathbf{1}}(t_{13}) = \mathbf{a}_1^- \mathbf{k}_3 \mathbf{K}_4 + \mathbf{k}_1 \mathbf{A}_2^- \mathbf{a}_3^+ \mathbf{K}_4 + \mathbf{k}_1 \mathbf{K}_2 \mathbf{a}_3^- \mathbf{A}_4^+, \tag{10.68}$$
$$\pi_{\mathbf{1}}(t_{14}) = -\mathbf{k}_1 \mathbf{K}_2 \mathbf{k}_3, \tag{10.69}$$
$$\pi_{\mathbf{1}}(t_{23}) = \mathbf{a}_1^+ \mathbf{A}_2^- \mathbf{a}_3^+ \mathbf{K}_4 + \mathbf{a}_1^+ \mathbf{K}_2 \mathbf{a}_3^- \mathbf{A}_4^+ - q \mathbf{k}_1 \mathbf{k}_3 \mathbf{K}_4, \tag{10.70}$$
$$\pi_{\mathbf{1}}(t_{24}) = -\mathbf{a}_1^+ \mathbf{K}_2 \mathbf{k}_3, \tag{10.71}$$
$$\pi_{\mathbf{1}}(t_{33}) = \mathbf{a}_1^- \mathbf{A}_2^+ \mathbf{a}_3^- \mathbf{A}_4^+ - q^2 \mathbf{a}_1^- \mathbf{K}_2 \mathbf{a}_3^+ \mathbf{K}_4 - q \mathbf{k}_1 \mathbf{k}_3 \mathbf{A}_4^+, \tag{10.72}$$
$$\pi_{\mathbf{1}}(t_{34}) = -\mathbf{a}_1^- \mathbf{A}_2^+ \mathbf{k}_3 - \mathbf{k}_1 \mathbf{a}_3^+, \tag{10.73}$$
$$\pi_{\mathbf{2}}(t_{13}) = \mathbf{k}_2 \mathbf{K}_3 \mathbf{a}_4^-, \tag{10.74}$$
$$\pi_{\mathbf{2}}(t_{14}) = -\mathbf{k}_2 \mathbf{K}_3 \mathbf{k}_4, \tag{10.75}$$
$$\pi_{\mathbf{2}}(t_{23}) = \mathbf{A}_1^- \mathbf{a}_2^+ \mathbf{K}_3 \mathbf{a}_4^- + \mathbf{K}_1 \mathbf{a}_2^- \mathbf{A}_3^+ \mathbf{a}_4^- - q \mathbf{K}_1 \mathbf{k}_2 \mathbf{k}_4, \tag{10.76}$$
$$\pi_{\mathbf{2}}(t_{24}) = -\mathbf{A}_1^- \mathbf{a}_2^+ \mathbf{K}_3 \mathbf{k}_4 - \mathbf{K}_1 \mathbf{a}_2^- \mathbf{A}_3^+ \mathbf{k}_4 - \mathbf{K}_1 \mathbf{k}_2 \mathbf{a}_4^+, \tag{10.77}$$
$$\pi_{\mathbf{2}}(t_{33}) = \mathbf{A}_1^+ \mathbf{a}_2^- \mathbf{A}_3^+ \mathbf{a}_4^- - q \mathbf{A}_1^+ \mathbf{k}_2 \mathbf{k}_4 - q^2 \mathbf{K}_1 \mathbf{a}_2^+ \mathbf{K}_3 \mathbf{a}_4^-, \tag{10.78}$$
$$\pi_{\mathbf{2}}(t_{34}) = -\mathbf{A}_1^+ \mathbf{a}_2^- \mathbf{A}_3^+ \mathbf{k}_4 - \mathbf{A}_1^+ \mathbf{k}_2 \mathbf{a}_4^+ + q^2 \mathbf{K}_1 \mathbf{a}_2^+ \mathbf{K}_3 \mathbf{k}_4. \tag{10.79}$$

From this and Lemma 10.12 we get

$$\begin{aligned}
\pi_{\mathbf{1}}(\sigma_1) &= -\mathbf{k}_1 \mathbf{K}_2 \mathbf{k}_3, \\
\pi_{\mathbf{1}}(\sigma_1 e_1) &= -\mathbf{a}_1^+ \mathbf{K}_2 \mathbf{k}_3, \\
\pi_{\mathbf{1}}(\sigma_2) &= -\mathbf{K}_2 {\mathbf{k}_3}^2 \mathbf{K}_4, \\
\pi_{\mathbf{1}}(\sigma_2 e_2) &= -\mathbf{a}_1^{-2} \mathbf{A}_2^+ {\mathbf{k}_3}^2 \mathbf{K}_4 - [2]_1 \mathbf{a}_1^- \mathbf{k}_1 \mathbf{a}_3^+ \mathbf{k}_3 \mathbf{K}_4 - {\mathbf{k}_1}^2 \mathbf{A}_2^- \mathbf{a}_3^{+2} \mathbf{K}_4 - \mathbf{A}_4^+ {\mathbf{k}_1}^2 \mathbf{K}_2, \\
\lambda_1^{-1} \pi_{\mathbf{1}}(\xi_1) &= \mathbf{a}_1^+ {\mathbf{k}_1}^{-1}, \\
\lambda_2^{-1} \pi_{\mathbf{1}}(\xi_2) &= \mathbf{a}_1^{-2} \mathbf{A}_2^+ {\mathbf{K}_2}^{-1} + {\mathbf{k}_1}^2 \mathbf{A}_2^- {\mathbf{K}_2}^{-1} \mathbf{a}_3^{+2} {\mathbf{k}_3}^{-2} + [2]_1 \mathbf{a}_1^- \mathbf{k}_1 {\mathbf{K}_2}^{-1} \mathbf{a}_3^+ {\mathbf{k}_3}^{-1} \\
&\quad + {\mathbf{k}_1}^2 {\mathbf{k}_3}^{-2} \mathbf{A}_4^+ {\mathbf{K}_4}^{-1}, \\
\pi_{\mathbf{2}}(\sigma_1) &= -\mathbf{k}_2 \mathbf{K}_3 \mathbf{k}_4, \\
\pi_{\mathbf{2}}(\sigma_1 e_1) &= -\mathbf{K}_1 \mathbf{k}_2 \mathbf{a}_4^+ - \mathbf{K}_1 \mathbf{a}_2^- \mathbf{A}_3^+ \mathbf{k}_4 - \mathbf{A}_1^- \mathbf{a}_2^+ \mathbf{K}_3 \mathbf{k}_4, \\
\pi_{\mathbf{2}}(\sigma_2) &= -\mathbf{K}_1 \mathbf{k}_2^2 \mathbf{K}_3, \\
\pi_{\mathbf{2}}(\sigma_2 e_2) &= -\mathbf{A}_1^+ \mathbf{k}_2^2 \mathbf{K}_3, \\
\lambda_1^{-1} \pi_{\mathbf{2}}(\xi_1) &= \mathbf{A}_1^- \mathbf{a}_2^+ \mathbf{k}_2^{-1} + \mathbf{K}_1 \mathbf{a}_2^- \mathbf{k}_2^{-1} \mathbf{A}_3^+ \mathbf{K}_3^{-1} + \mathbf{K}_1 \mathbf{K}_3^{-1} \mathbf{a}_4^+ \mathbf{k}_4^{-1}, \\
\lambda_2^{-1} \pi_{\mathbf{2}}(\xi_2) &= \mathbf{A}_1^+ {\mathbf{K}_1}^{-1}.
\end{aligned}$$

Note that $\pi_{\mathbf{i}}(\sigma_i)$ is invertible. Comparing these formulas with Lemma 10.11 by using (10.43), the equality (10.40) is directly checked. Thus Proposition 10.7 is established for C_2.

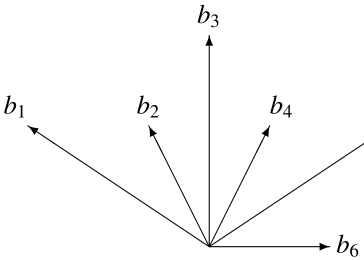
In terms of the 3D K in Chap. 5, Theorem 10.6 implies

$$E_{\mathbf{2}}^{a,b,c,d} = \sum_{i,j,k,l} K_{ijkl}^{abcd}\, E_{\mathbf{1}}^{l,k,j,i}. \tag{10.80}$$

The weight conservation (5.65) assures the equality of weights of the two sides.

10.4.3 Explicit Formulas for G_2

Consider $\mathfrak{g} = G_2$.



The q-Serre relations are

$$e_1^4 e_2 - [4]_1 e_1^3 e_2 e_1 + [4]_1 [3]_1 / [2]_1^{-1} e_1^2 e_2 e_1^2 - [4]_1 e_1 e_2 e_1^3 + e_2 e_1^4 = 0,$$
$$e_2^2 e_1 - [2]_2 e_2 e_1 e_2 + e_1 e_2^2 = 0, \tag{10.81}$$

where $[m]_1 = \langle m \rangle / \langle 1 \rangle$ and $[m]_2 = \langle 3m \rangle / \langle 3 \rangle$. For simplicity we write the positive root vectors $e_{\beta_{i_r}}$ in (10.4) with $(i_1, \ldots, i_6) = \mathbf{2}$ (10.32) as

$$b_1 = e_{\beta_1} = e_2, \quad b_2 = e_{\beta_2} = e_1 e_2 - q^3 e_2 e_1,$$
$$b_4 = e_{\beta_3} = \frac{1}{[2]_1}(e_1 b_2 - q b_2 e_1), \quad b_5 = e_{\beta_4} = \frac{1}{[3]_1}(e_1 b_4 - q^{-1} b_4 e_1), \tag{10.82}$$
$$b_3 = e_{\beta_5} = \frac{1}{[3]_1}(b_4 b_2 - q^{-1} b_2 b_4), \quad b_6 = e_{\beta_6} = e_1.$$

Their commutation relations are

$$b_2 b_1 = b_1 b_2 q^{-3}, \qquad b_3 b_1 = \langle 1 \rangle^2 b_2^3 q^{-3} [3]_1^{-1} + b_1 b_3 q^{-3}, \tag{10.83}$$
$$b_4 b_1 = b_1 b_4 - b_2^2 \langle 1 \rangle q^{-1}, \tag{10.84}$$
$$b_5 b_1 = b_1 b_5 q^3 - b_2 b_4 \langle 1 \rangle q^{-1} - (q^4 + q^2 - 1) b_3 q^{-3}, \tag{10.85}$$
$$b_6 b_1 = b_1 b_6 q^3 + b_2, \qquad b_3 b_2 = b_2 b_3 q^{-3}, \tag{10.86}$$
$$b_4 b_2 = b_2 b_4 q^{-1} + b_3 [3]_1, \qquad b_5 b_2 = b_2 b_5 - b_4^2 \langle 1 \rangle q^{-1}, \tag{10.87}$$
$$b_6 b_2 = q b_2 b_6 + b_4 [2]_1, \qquad b_4 b_3 = b_3 b_4 q^{-3}, \tag{10.88}$$
$$b_5 b_3 = \langle 1 \rangle^2 b_4^3 q^{-3} [3]_1^{-1} + b_3 b_5 q^{-3}, \tag{10.89}$$
$$b_6 b_3 = b_3 b_6 - b_4^2 \langle 1 \rangle q^{-1}, \qquad b_5 b_4 = b_4 b_5 q^{-3}, \tag{10.90}$$
$$b_6 b_4 = [3]_1 b_5 + b_4 b_6 q^{-1}, \qquad b_6 b_5 = b_5 b_6 q^{-3}. \tag{10.91}$$

Lemma 10.13 *For* $\tilde{E}_{\mathbf{2}}^{a,b,c,d,e,f} = b_1^a b_2^b b_3^c b_4^d b_5^e b_6^f$, *we have*

$$\begin{aligned}
\tilde{E}_{\mathbf{2}}^{a,b,c,d,e,f} \cdot e_1 =& \tilde{E}_{\mathbf{2}}^{a,b,c,d,e,f+1}.\\
\tilde{E}_{\mathbf{2}}^{a,b,c,d,e,f} \cdot e_2 =& -\langle 1\rangle[e]_2 q^{-3c-d+3f-1}\tilde{E}_{\mathbf{2}}^{a,b+1,c,d+1,e-1,f}\\
&+\langle 1\rangle^2[e-1]_2[e]_2[3]_1^{-1}q^{-3e+3f+3}\tilde{E}_{\mathbf{2}}^{a,b,c,d+3,e-2,f}\\
&-\langle 3\rangle[d-1]_1[d]_1 q^{-3c-2d+3e+3f+1}\tilde{E}_{\mathbf{2}}^{a,b+1,c+1,d-2,e,f}\\
&-\langle 1\rangle[d]_1 q^{-6c-d+3(e+f)}\tilde{E}_{\mathbf{2}}^{a,b+2,c,d-1,e,f}\\
&+[f-1]_1[f]_1 q^{-3e+f-2}\tilde{E}_{\mathbf{2}}^{a,b,c,d+1,e,f-2}\\
&+[3]_1[d]_1[f]_1 q^{2f-2d}\tilde{E}_{\mathbf{2}}^{a,b,c+1,d-1,e,f-1}\\
&+[f]_1 q^{-3c-d+2f-2}\tilde{E}_{\mathbf{2}}^{a,b+1,c,d,e,f-1}\\
&+q^{-3(b+c-e-f)}\tilde{E}_{\mathbf{2}}^{a+1,b,c,d,e,f}\\
&+\langle 1\rangle^2[c]_2[3]_1^{-1}q^{3(-2c+e+f+1)}\tilde{E}_{\mathbf{2}}^{a,b+3,c-1,d,e,f}\\
&-\langle 3\rangle[d-2]_1[d-1]_1[d]_1 q^{3(-d+e+f+2)}\tilde{E}_{\mathbf{2}}^{a,b,c+2,d-3,e,f}\\
&-\langle 1\rangle[e]_2[f]_1 q^{-3e+2f}\tilde{E}_{\mathbf{2}}^{a,b,c,d+2,e-1,f-1}\\
&-[e]_2 q^{-3d+3f}(q^{2d+1}[3]_1-[2]_2)\tilde{E}_{\mathbf{2}}^{a,b,c+1,d,e-1,f}\\
&+[f-2]_1[f-1]_1[f]_1\tilde{E}_{\mathbf{2}}^{a,b,c,d,e+1,f-3}.\\
e_1 \cdot \tilde{E}_{\mathbf{2}}^{a,b,c,d,e,f} =& -\langle 1\rangle[c]_2 q^{3a+b-3c+2}\tilde{E}_{\mathbf{2}}^{a,b,c-1,d+2,e,f}\\
&+[3]_1[b-1]_1[b]_1 q^{3a-b+2}\tilde{E}_{\mathbf{2}}^{a,b-2,c+1,d,e,f}\\
&+[3]_1[d]_1 q^{3a+b-2d+2}\tilde{E}_{\mathbf{2}}^{a,b,c,d-1,e+1,f}\\
&+q^{3a+b-d-3e}\tilde{E}_{\mathbf{2}}^{a,b,c,d,e,f+1}\\
&+[2]_1[b]_1 q^{3(a-c)}\tilde{E}_{\mathbf{2}}^{a,b-1,c,d+1,e,f}\\
&+[a]_2\tilde{E}_{\mathbf{2}}^{a-1,b+1,c,d,e,f}.\\
e_2 \cdot \tilde{E}_{\mathbf{2}}^{a,b,c,d,e,f} =& \tilde{E}_{\mathbf{2}}^{a+1,b,c,d,e,f}.
\end{aligned}$$

Proof By induction, we have

$$\begin{aligned}
b_6 b_1^n &= q^{3n} b_1^n b_6 + [n]_2 b_1^{n-1} b_2,\\
b_6 b_2^n &= [3]_1 q^{2-n}[n-1]_1[n]_1 b_2^{n-2} b_3 + q^n b_2^n b_6 + [2]_1[n]_1 b_2^{n-1} b_4,\\
b_4 b_3^n &= q^{-3n} b_3^n b_4,\\
b_6 b_3^n &= b_3^n b_6 - \langle 1\rangle q^{2-3n}[n]_2 b_3^{n-1} b_4 b_4,\\
b_6 b_4^n &= [3]_1 q^{2-2n}[n]_1 b_4^{n-1} b_5 + q^{-n} b_4^n b_6,\\
b_6 b_5^n &= q^{-3n} b_5^n b_6,
\end{aligned}$$

and

$$
\begin{aligned}
b_6^n b_1 &= q^{n-2}[n-1]_1[n]_1 b_4 b_6^{n-2} + q^{3n} b_1 b_6^n \\
&\quad + q^{2(n-1)}[n]_1 b_2 b_6^{n-1} + [n-2]_1[n-1]_1[n]_1 b_5 b_6^{n-3}, \\
b_5^n b_1 &= \langle 1\rangle^2 q^{-3(n-1)}[n-1]_2[n]_2[3]_1^{-1} b_4^3 b_5^{n-2} + q^{3n} b_1 b_5^n \\
&\quad - q^{-3}(q^4+q^2-1)[n]_2 b_3 b_5^{n-1} - q^{-1}\langle 1\rangle [n]_2 b_2 b_4 b_5^{n-1}, \\
b_5^n b_2 &= b_2 b_5^n - \langle 1\rangle q^{2-3n}[n]_2 b_4 b_4 b_5^{n-1}, \\
b_5^n b_4 &= q^{-3n} b_4 b_5^n, \\
b_4^n b_1 &= -\langle 3\rangle q^{6-3n}[n-2]_1[n-1]_1[n]_1 b_3^2 b_4^{n-3} - \langle 1\rangle q^{-n}[n]_1 b_2^2 b_4^{n-1} \\
&\quad - \langle 3\rangle q^{1-2n}[n-1]_1[n]_1 b_2 b_3 b_4^{n-2} + b_1 b_4^n, \\
b_4^n b_2 &= [3]_1 q^{2-2n}[n]_1 b_3 b_4^{n-1} + q^{-n} b_2 b_4^n, \\
b_4^n b_3 &= q^{-3n} b_3 b_4^n, \\
b_3^n b_1 &= q^{-3n} b_1 b_3^n + \langle 1\rangle^2 q^{3-6n}[n]_2[3]_1^{-1} b_2^3 b_3^{n-1}, \\
b_3^n b_2 &= q^{-3n} b_2 b_3^n, \\
b_2^n b_1 &= q^{-3n} b_1 b_2^n.
\end{aligned}
$$

The lemma is a direct consequence of these formulas. □

Let v_i ($i = 1, \dots, 7$) be the basis of $V(\varpi_1)$ for which the representation matrix is given by (8.79)–(8.81). Its highest and lowest weight vectors are v_1 and v_7, respectively. Let $u_i \in V^r(\varpi_1)$ be the dual base of v_i.

The representation $V(\varpi_2)$ is the adjoint representation with dimension 14. Its lowest weight vector is $v_{14}^{(14)}$ in (8.84), which is $v_6 \otimes v_7 - q v_7 \otimes v_6$ in the notation here. The highest weight vector of $V^r(\varpi_2)$ is $u_1 \otimes u_2 - q u_2 \otimes u_1$. From these facts we have

$$\sigma_1 = \Psi_{\varpi_1}(u_1 \otimes v_7), \tag{10.92}$$

$$\sigma_2 = \frac{1}{1+q^2}\Psi_{\varpi_2}\big((u_1 \otimes u_2 - q u_2 \otimes u_1) \otimes (v_6 \otimes v_7 - q v_7 \otimes v_6)\big). \tag{10.93}$$

We define t_{ij} by the formula (10.22) with $l = 1$ for $1 \le i, j \le 7$. They satisfy the relations (8.3) and (8.4) of the earlier definition of $A_q(G_2)$. By a calculation similar to $A_q(A_2)$ using the commutation relations

$$[t_{16}, t_{27}] = (q - q^{-1}) t_{26} t_{17}, \quad [t_{17}, t_{26}] = 0, \tag{10.94}$$

we get[9]

[9] σ_2 and $\sigma_2 e_2$ in [102, Eq. (42)] are $(-q)$ times those in Lemma 10.14.

Lemma 10.14 *For $A_q(G_2)$, the following relations are valid:*

$$\sigma_1 = t_{17}, \quad \sigma_2 = t_{16}t_{27} - q t_{27}t_{16}, \quad \sigma_1 e_1 = t_{27}, \quad \sigma_2 e_2 = t_{16}t_{37} - q t_{36}t_{17}. \tag{10.95}$$

Images of the generators t_{ij} by the representations $\pi_{\mathbf{1}}$ and $\pi_{\mathbf{2}}$ in (10.31) are available from (8.11) and (8.12). For convenience, we present explicit formulas for those appearing in Lemma 10.14:

$$\begin{aligned}
\pi_{\mathbf{1}}(t_{16}) &= \mathbf{a}_1^-\mathbf{k}_3\mathbf{K}_4\mathbf{k}_5^2\mathbf{K}_6 + \mathbf{k}_1\mathbf{A}_2^-\mathbf{a}_3^+\mathbf{K}_4\mathbf{k}_5^2\mathbf{K}_6 + \mathbf{k}_1\mathbf{K}_2\mathbf{a}_3^{-2}\mathbf{A}_4^+\mathbf{k}_5^2\mathbf{K}_6 \\
&\quad + [2]_1\mathbf{k}_1\mathbf{K}_2\mathbf{a}_3^-\mathbf{k}_3\mathbf{a}_5^+\mathbf{k}_5\mathbf{K}_6 + \mathbf{k}_1\mathbf{K}_2\mathbf{k}_3^2\mathbf{A}_4^-\mathbf{a}_5^{+2}\mathbf{K}_6 + \mathbf{k}_1\mathbf{K}_2\mathbf{k}_3^2\mathbf{K}_4\mathbf{a}_5^-\mathbf{A}_6^+, \\
\pi_{\mathbf{1}}(t_{17}) &= \mathbf{k}_1\mathbf{K}_2\mathbf{k}_3^2\mathbf{K}_4\mathbf{k}_5, \\
\pi_{\mathbf{1}}(t_{27}) &= \mathbf{a}_1^+\mathbf{K}_2\mathbf{k}_3^2\mathbf{K}_4\mathbf{k}_5, \\
\pi_{\mathbf{1}}(t_{36}) &= \mathbf{a}_1^{-2}\mathbf{A}_2^+\mathbf{a}_2^{-2}\mathbf{A}_4^+\mathbf{k}_5^2\mathbf{K}_6 + [2]_1\mathbf{a}_1^{-2}\mathbf{A}_2^+\mathbf{a}_3^-\mathbf{k}_3\mathbf{a}_5^+\mathbf{k}_5\mathbf{K}_6 + \mathbf{a}_1^{-2}\mathbf{A}_2^+\mathbf{k}_3^2\mathbf{A}_4^-\mathbf{a}_5^{+2}\mathbf{K}_6 \\
&\quad + \mathbf{a}_1^{-2}\mathbf{A}_2^+\mathbf{k}_3^2\mathbf{K}_4\mathbf{a}_5^-\mathbf{A}_6^+ - q^3\mathbf{a}_1^{-2}\mathbf{K}_2\mathbf{a}_3^+\mathbf{K}_4\mathbf{k}_5^2\mathbf{K}_6 + [2]_1\mathbf{a}_1^-\mathbf{k}_1\mathbf{a}_5^+\mathbf{k}_5\mathbf{K}_6 \\
&\quad - [2]_1\mathbf{a}_1^-\mathbf{k}_1\mathbf{a}_3^-\mathbf{k}_3\mathbf{A}_4^+\mathbf{k}_5^2\mathbf{K}_6 + [2]_1\mathbf{a}_1^-\mathbf{k}_1\mathbf{a}_3^+\mathbf{k}_3\mathbf{A}_4^-\mathbf{a}_5^{+2}\mathbf{K}_6 + [2]_1\mathbf{a}_1^-\mathbf{k}_1\mathbf{a}_3^+\mathbf{k}_3\mathbf{K}_4\mathbf{a}_5^-\mathbf{A}_6^+ \\
&\quad - q[2]_1^2\mathbf{a}_1^-\mathbf{k}_1\mathbf{k}_3^2\mathbf{a}_5^+\mathbf{k}_5\mathbf{K}_6 + \mathbf{k}_1^2\mathbf{A}_2^-\mathbf{a}_3^{+2}\mathbf{A}_4^-\mathbf{a}_5^{+2}\mathbf{K}_6 + \mathbf{k}_1^2\mathbf{A}_2^-\mathbf{a}_3^{+2}\mathbf{K}_4\mathbf{a}_5^-\mathbf{A}_6^+ \\
&\quad - q^2[2]_1\mathbf{k}_1^2\mathbf{A}_2^-\mathbf{a}_3^+\mathbf{k}_3\mathbf{a}_5^+\mathbf{k}_5\mathbf{K}_6 + q^2\mathbf{k}_1^2\mathbf{A}_2^-\mathbf{k}_3^2\mathbf{A}_4^+\mathbf{k}_5^2\mathbf{K}_6 + \mathbf{k}_1^2\mathbf{K}_2\mathbf{a}_3^-\mathbf{A}_4^+\mathbf{a}_5^-\mathbf{A}_6^+ \\
&\quad - q^3\mathbf{k}_1^2\mathbf{K}_2\mathbf{a}_3^-\mathbf{K}_4\mathbf{a}_5^{+2}\mathbf{K}_6 - q\mathbf{k}_1^2\mathbf{K}_2\mathbf{k}_3\mathbf{k}_5\mathbf{A}_6^+, \\
\pi_{\mathbf{1}}(t_{37}) &= \mathbf{a}_1^{-2}\mathbf{A}_2^+\mathbf{k}_3^2\mathbf{K}_4\mathbf{k}_5 + [2]_1\mathbf{a}_1^-\mathbf{k}_1\mathbf{a}_3^+\mathbf{k}_3\mathbf{K}_4\mathbf{k}_5 + \mathbf{k}_1^2\mathbf{A}_2^-\mathbf{a}_3^{+2}\mathbf{K}_4\mathbf{k}_5 \\
&\quad + \mathbf{k}_1^2\mathbf{K}_2\mathbf{a}_3^-\mathbf{A}_4^+\mathbf{k}_5 + \mathbf{k}_1^2\mathbf{K}_2\mathbf{k}_3\mathbf{a}_5^+, \\
\pi_{\mathbf{2}}(t_{16}) &= \mathbf{k}_2\mathbf{K}_3\mathbf{k}_4^2\mathbf{K}_5\mathbf{a}_6^-, \\
\pi_{\mathbf{2}}(t_{17}) &= \mathbf{k}_2\mathbf{K}_3\mathbf{k}_4^2\mathbf{K}_5\mathbf{k}_6, \\
\pi_{\mathbf{2}}(t_{27}) &= \mathbf{A}_1^-\mathbf{a}_2^+\mathbf{K}_3\mathbf{k}_4^2\mathbf{K}_5\mathbf{k}_6 + \mathbf{K}_1\mathbf{a}_2^{-2}\mathbf{A}_3^+\mathbf{k}_4^2\mathbf{K}_5\mathbf{k}_6 + [2]_1\mathbf{K}_1\mathbf{a}_2^-\mathbf{k}_2\mathbf{a}_4^+\mathbf{k}_4\mathbf{K}_5\mathbf{k}_6 \\
&\quad + \mathbf{K}_1\mathbf{k}_2^2\mathbf{A}_3^-\mathbf{a}_4^{+2}\mathbf{K}_5\mathbf{k}_6 + \mathbf{K}_1\mathbf{k}_2^2\mathbf{K}_3\mathbf{a}_4^-\mathbf{A}_5^+\mathbf{k}_6 + \mathbf{K}_1\mathbf{k}_2^2\mathbf{K}_3\mathbf{k}_4\mathbf{a}_6^+, \\
\pi_{\mathbf{2}}(t_{36}) &= \mathbf{A}_1^+\mathbf{a}_2^{-2}\mathbf{A}_3^+\mathbf{k}_4^2\mathbf{K}_5\mathbf{a}_6^- + [2]_1\mathbf{A}_1^+\mathbf{a}_2^-\mathbf{k}_2\mathbf{a}_4^+\mathbf{k}_4\mathbf{K}_5\mathbf{a}_6^- + \mathbf{A}_1^+\mathbf{k}_2^2\mathbf{A}_3^-\mathbf{a}_4^{+2}\mathbf{K}_5\mathbf{a}_6^- \\
&\quad + \mathbf{A}_1^+\mathbf{k}_2^2\mathbf{K}_3\mathbf{a}_4^-\mathbf{A}_5^+\mathbf{a}_6^- - q\mathbf{A}_1^+\mathbf{k}_2^2\mathbf{K}_3\mathbf{k}_4\mathbf{k}_6 - q^3\mathbf{K}_1\mathbf{a}_2^+\mathbf{K}_3\mathbf{k}_4^2\mathbf{K}_5\mathbf{a}_6^-, \\
\pi_{\mathbf{2}}(t_{37}) &= \mathbf{A}_1^+\mathbf{a}_2^{-2}\mathbf{A}_3^+\mathbf{k}_4^2\mathbf{K}_5\mathbf{k}_6 + [2]_1\mathbf{A}_1^+\mathbf{a}_2^-\mathbf{k}_2\mathbf{a}_4^+\mathbf{k}_4\mathbf{K}_5\mathbf{k}_6 + \mathbf{A}_1^+\mathbf{k}_2^2\mathbf{A}_3^-\mathbf{a}_4^{+2}\mathbf{K}_5\mathbf{k}_6 \\
&\quad + \mathbf{A}_1^+\mathbf{k}_2^2\mathbf{K}_3\mathbf{a}_4^-\mathbf{A}_5^+\mathbf{k}_6 + \mathbf{A}_1^+\mathbf{k}_2^2\mathbf{K}_3\mathbf{k}_4\mathbf{a}_6^+ - q^3\mathbf{K}_1\mathbf{a}_2^+\mathbf{K}_3\mathbf{k}_4^2\mathbf{K}_5\mathbf{k}_6.
\end{aligned}$$

From this and Lemma 10.14 we get

$$
\begin{aligned}
\pi_{\mathbf{1}}(\sigma_1) &= \mathbf{k}_1\mathbf{K}_2\mathbf{k}_3^2\mathbf{K}_4\mathbf{k}_5,\\
\pi_{\mathbf{1}}(\sigma_2) &= \mathbf{K}_2\mathbf{k}_3^3\mathbf{K}_4^2\mathbf{k}_5^3\mathbf{K}_6,\\
\pi_{\mathbf{1}}(\sigma_1 e_1) &= \mathbf{a}_1^+\mathbf{K}_2\mathbf{k}_3^2\mathbf{K}_4\mathbf{k}_5,\\
\pi_{\mathbf{1}}(\sigma_2 e_2) &= \mathbf{k}_1^3\mathbf{K}_2^2\mathbf{k}_3^3\mathbf{K}_4\mathbf{A}_6^+ + [2]_2\mathbf{k}_1^3\mathbf{A}_2^-\mathbf{K}_2\mathbf{A}_4^+\mathbf{K}_4\mathbf{k}_5^3\mathbf{K}_6 + \mathbf{a}_1^{-3}\mathbf{A}_2^+\mathbf{k}_3^3\mathbf{K}_4^2\mathbf{k}_5^3\mathbf{K}_6\\
&\quad + [3]_1\mathbf{a}_1^{-2}\mathbf{k}_1\mathbf{a}_3^+\mathbf{k}_3^2\mathbf{K}_4^2\mathbf{k}_5^3\mathbf{K}_6 + [3]_1\mathbf{a}_1^-\mathbf{k}_1^2\mathbf{K}_2\mathbf{k}_3^2\mathbf{K}_4\mathbf{a}_5^+\mathbf{k}_5^2\mathbf{K}_6\\
&\quad - q[3]_1\mathbf{k}_1^3\mathbf{A}_2^-\mathbf{K}_2\mathbf{k}_3^2\mathbf{A}_4^+\mathbf{K}_4\mathbf{k}_5^3\mathbf{K}_6 + [3]_1\mathbf{k}_1^3\mathbf{K}_2^2\mathbf{a}_3^-\mathbf{k}_3^2\mathbf{a}_5^{+2}\mathbf{k}_5\mathbf{K}_6\\
&\quad + \mathbf{k}_1^3\mathbf{K}_2^2\mathbf{k}_3^3\mathbf{A}_4^-\mathbf{a}_5^{+3}\mathbf{K}_6 + [3]_1\mathbf{a}_1^-\mathbf{k}_1^2\mathbf{A}_2^-\mathbf{a}_3^{+2}\mathbf{k}_3\mathbf{K}_4^2\mathbf{k}_5^3\mathbf{K}_6\\
&\quad + [3]_1\mathbf{a}_1^-\mathbf{k}_1^2\mathbf{K}_2\mathbf{a}_3^-\mathbf{k}_3\mathbf{A}_4^+\mathbf{K}_4\mathbf{k}_5^3\mathbf{K}_6 + \mathbf{k}_1^3\mathbf{A}_2^{-2}\mathbf{a}_3^{+3}\mathbf{K}_4^2\mathbf{k}_5^3\mathbf{K}_6\\
&\quad + [3]_1\mathbf{k}_1^3\mathbf{A}_2^-\mathbf{K}_2\mathbf{a}_3^+\mathbf{k}_3\mathbf{K}_4\mathbf{a}_5^+\mathbf{k}_5^2\mathbf{K}_6 + \mathbf{k}_1^3\mathbf{K}_2^2\mathbf{a}_3^{-3}\mathbf{A}_4^{+2}\mathbf{k}_5^3\mathbf{K}_6\\
&\quad + [3]_1\mathbf{k}_1^3\mathbf{K}_2^2\mathbf{a}_3^{-2}\mathbf{k}_3\mathbf{A}_4^+\mathbf{a}_5^+\mathbf{k}_5^2\mathbf{K}_6,\\
\lambda_1^{-1}\pi_{\mathbf{1}}(\xi_1) &= \mathbf{a}_1^+\mathbf{k}_1^{-1},\\
\lambda_2^{-1}\pi_{\mathbf{1}}(\xi_2) &= \mathbf{a}_1^{-3}\mathbf{A}_2^+\mathbf{K}_2^{-1} + [2]_2\mathbf{k}_1^3\mathbf{A}_2^-\mathbf{k}_3^{-3}\mathbf{A}_4^+\mathbf{K}_4^{-1} - q[3]_1\mathbf{k}_1^3\mathbf{A}_2^-\mathbf{k}_3^{-1}\mathbf{A}_4^+\mathbf{K}_4^{-1}\\
&\quad + [3]_1\mathbf{a}_1^{-2}\mathbf{k}_1\mathbf{K}_2^{-1}\mathbf{a}_3^+\mathbf{k}_3^{-1} + [3]_1\mathbf{a}_1^-\mathbf{k}_1^2\mathbf{a}_3^-\mathbf{k}_3^{-2}\mathbf{A}_4^+\mathbf{K}_4^{-1}\\
&\quad + [3]_1\mathbf{a}_1^-\mathbf{k}_1^2\mathbf{k}_3^{-1}\mathbf{K}_4^{-1}\mathbf{a}_5^+\mathbf{k}_5^{-1} + \mathbf{k}_1^3\mathbf{K}_2\mathbf{K}_4^{-1}\mathbf{k}_5^{-3}\mathbf{A}_6^+\mathbf{K}_6^{-1}\\
&\quad + [3]_1\mathbf{a}_1^-\mathbf{k}_1^2\mathbf{A}_2^-\mathbf{K}_2^{-1}\mathbf{a}_3^{+2}\mathbf{k}_3^{-2} + [3]_1\mathbf{k}_1^3\mathbf{A}_2^-\mathbf{a}_3^+\mathbf{k}_3^{-2}\mathbf{K}_4^{-1}\mathbf{a}_5^+\mathbf{k}_5^{-1}\\
&\quad + \mathbf{k}_1^3\mathbf{A}_2^{-2}\mathbf{K}_2^{-1}\mathbf{a}_3^{+3}\mathbf{k}_3^{-3} + [3]_1\mathbf{k}_1^3\mathbf{K}_2\mathbf{a}_3^-\mathbf{k}_3^{-1}\mathbf{K}_4^{-2}\mathbf{a}_5^{+2}\mathbf{k}_5^{-2}\\
&\quad + \mathbf{k}_1^3\mathbf{K}_2\mathbf{A}_4^-\mathbf{K}_4^{-2}\mathbf{a}_5^{+3}\mathbf{k}_5^{-3} + \mathbf{k}_1^3\mathbf{K}_2\mathbf{a}_3^{-3}\mathbf{k}_3^{-3}\mathbf{A}_4^{+2}\mathbf{K}_4^{-2}\\
&\quad + [3]_1\mathbf{k}_1^3\mathbf{K}_2\mathbf{a}_3^{-2}\mathbf{k}_3^{-2}\mathbf{A}_4^+\mathbf{K}_4^{-2}\mathbf{a}_5^+\mathbf{k}_5^{-1},
\end{aligned}
$$

$$
\begin{aligned}
\pi_{\mathbf{2}}(\sigma_1) &= \mathbf{k}_2\mathbf{K}_3\mathbf{k}_4^2\mathbf{K}_5\mathbf{k}_6,\\
\pi_{\mathbf{2}}(\sigma_2) &= \mathbf{K}_1\mathbf{k}_2^3\mathbf{K}_3^2\mathbf{k}_4^3\mathbf{K}_5,\\
\pi_{\mathbf{2}}(\sigma_1 e_1) &= \mathbf{K}_1\mathbf{k}_2^2\mathbf{K}_3\mathbf{k}_4\mathbf{a}_6^+ + \mathbf{A}_1^-\mathbf{a}_2^+\mathbf{K}_3\mathbf{k}_4^2\mathbf{K}_5\mathbf{k}_6 + \mathbf{K}_1\mathbf{k}_2^2\mathbf{K}_3\mathbf{a}_4^-\mathbf{A}_5^+\mathbf{k}_6\\
&\quad + \mathbf{K}_1\mathbf{a}_2^{-2}\mathbf{A}_3^+\mathbf{k}_4^2\mathbf{K}_5\mathbf{k}_6 + [2]_1\mathbf{K}_1\mathbf{a}_2^-\mathbf{k}_2\mathbf{a}_4^+\mathbf{k}_4\mathbf{K}_5\mathbf{k}_6 + \mathbf{K}_1\mathbf{k}_2^2\mathbf{A}_3^-\mathbf{a}_4^{+2}\mathbf{K}_5\mathbf{k}_6,\\
\pi_{\mathbf{2}}(\sigma_2 e_2) &= \mathbf{A}_1^+\mathbf{k}_2^3\mathbf{K}_3^2\mathbf{k}_4^3\mathbf{K}_5,\\
\lambda_1^{-1}\pi_{\mathbf{2}}(\xi_1) &= \mathbf{A}_1^-\mathbf{a}_2^+\mathbf{k}_2^{-1} + [2]_1\mathbf{K}_1\mathbf{a}_2^-\mathbf{K}_3^{-1}\mathbf{a}_4^+\mathbf{k}_4^{-1} + \mathbf{K}_1\mathbf{a}_2^{-2}\mathbf{k}_2^{-1}\mathbf{A}_3^+\mathbf{K}_3^{-1}\\
&\quad + \mathbf{K}_1\mathbf{k}_2\mathbf{a}_4^-\mathbf{k}_4^{-2}\mathbf{A}_5^+\mathbf{K}_5^{-1} + \mathbf{K}_1\mathbf{k}_2\mathbf{k}_4^{-1}\mathbf{K}_5^{-1}\mathbf{a}_6^+\mathbf{k}_6^{-1} + \mathbf{K}_1\mathbf{k}_2\mathbf{A}_3^-\mathbf{K}_3^{-1}\mathbf{a}_4^{+2}\mathbf{k}_4^{-2},\\
\lambda_2^{-1}\pi_{\mathbf{2}}(\xi_2) &= \mathbf{A}_1^+\mathbf{K}_1^{-1}.
\end{aligned}
$$

Note that $\pi_{\mathbf{i}}(\sigma_i)$ is invertible. Comparing these formulas with Lemma 10.13 by using (10.43), the equality (10.40) is directly checked. Thus Proposition 10.7 is established for G_2.

In terms of the intertwiner F in Chap. 8, Theorem 10.6 implies

$$
E_{\mathbf{2}}^{a,b,c,d,e,f} = \sum_{i,j,k,l,m,n} F_{ijklmn}^{abcdef}\, E_{\mathbf{1}}^{n,m,l,k,j,i}. \tag{10.96}
$$

The weight conservation (8.29) assures the equality of weights of the two sides.

10.5 Tetrahedron and 3D Reflection Equations from PBW Bases

The relation (10.58) serves as an auxiliary linear system by which the tetrahedron equation (2.6) is established as the non-linear consistency condition. To see this, consider a PBW basis (10.6) of $U_q^+(A_3)$ having the form $E^{a,b,c,d,e}_{1,2,3,1,2,1}$. In addition to $E^{\dots ab \dots}_{\dots 13 \dots} = E^{\dots ba \dots}_{\dots 31 \dots}$, we may apply (10.58) as

$$E^{\dots abc \dots}_{\dots 121 \dots} = \sum R^{abc}_{ijk} E^{\dots kji \dots}_{\dots 212 \dots}, \quad E^{\dots abc \dots}_{\dots 212 \dots} = \sum R^{abc}_{ijk} E^{\dots kji \dots}_{\dots 121 \dots} \tag{10.97}$$

reflecting the $U_q^+(A_2)$ subalgebra structure. Then we have

$$\begin{aligned}
E^{a,b,c,d,e,f}_{1,2,3,1,2,1} &= E^{a,b,d,c,e,f}_{1,2,1,3,2,1} = \sum R^{abd}_{a_1 b_1 d_1} E^{d_1,b_1,a_1,c,e,f}_{2,1,2,3,2,1} \\
&= \sum R^{abd}_{a_1 b_1 d_1} R^{a_1 ce}_{a_2 c_1 e_1} E^{d_1,b_1,e_1,c_1,a_2,f}_{2,1,3,2,3,1} \\
&= \sum R^{abd}_{a_1 b_1 d_1} R^{a_1 ce}_{a_2 c_1 e_1} E^{d_1,e_1,b_1,c_1,f,a_2}_{2,3,1,2,1,3} \\
&= \sum R^{abd}_{a_1 b_1 d_1} R^{a_1 ce}_{a_2 c_1 e_1} R^{b_1 c_1 f}_{b_2 c_2 f_1} E^{d_1,e_1,f_1,c_2,b_2,a_2}_{2,3,2,1,2,3} \\
&= \sum R^{abd}_{a_1 b_1 d_1} R^{a_1 ce}_{a_2 c_1 e_1} R^{b_1 c_1 f}_{b_2 c_2 f_1} R^{d_1 e_1 f_1}_{d_2 e_2 f_2} E^{f_2,e_2,d_2,c_2,b_2,a_2}_{3,2,3,1,2,3}.
\end{aligned}$$

There is another route going from $E^{a,b,c,d,e,f}_{1,2,3,1,2,1}$ to $E^{f_2,e_2,d_2,c_2,b_2,a_2}_{3,2,3,1,2,3}$ as

$$\begin{aligned}
E^{a,b,c,d,e,f}_{1,2,3,1,2,1} &= \sum R^{def}_{d_1 e_1 f_1} E^{a,b,c,f_1,e_1,d_1}_{1,2,3,2,1,2} \\
&= \sum R^{def}_{d_1 e_1 f_1} R^{bcf_1}_{b_1 c_1 f_2} E^{a,f_2,c_1,b_1,e_1,d_1}_{1,3,2,3,1,2} \\
&= \sum R^{def}_{d_1 e_1 f_1} R^{bcf_1}_{b_1 c_1 f_2} E^{f_2,a,c_1,e_1,b_1,d_1}_{3,1,2,1,3,2} \\
&= \sum R^{def}_{d_1 e_1 f_1} R^{bcf_1}_{b_1 c_1 f_2} R^{ac_1 e_1}_{a_1 c_2 e_2} E^{f_2,e_2,c_2,a_1,b_1,d_1}_{3,2,1,2,3,2} \\
&= \sum R^{def}_{d_1 e_1 f_1} R^{bcf_1}_{b_1 c_1 f_2} R^{ac_1 e_1}_{a_1 c_2 e_2} R^{a_1 b_1 d_1}_{a_2 b_2 d_2} E^{f_2,e_2,c_2,d_2,b_2,a_2}_{3,2,1,3,2,3} \\
&= \sum R^{def}_{d_1 e_1 f_1} R^{bcf_1}_{b_1 c_1 f_2} R^{ac_1 e_1}_{a_1 c_2 e_2} R^{a_1 b_1 d_1}_{a_2 b_2 d_2} E^{f_2,e_2,d_2,c_2,b_2,a_2}_{3,2,3,1,2,3}.
\end{aligned}$$

Comparison of them leads to

$$\sum R^{abd}_{a_1 b_1 d_1} R^{a_1 ce}_{a_2 c_1 e_1} R^{b_1 c_1 f}_{b_2 c_2 f_1} R^{d_1 e_1 f_1}_{d_2 e_2 f_2} = \sum R^{def}_{d_1 e_1 f_1} R^{bcf_1}_{b_1 c_1 f_2} R^{ac_1 e_1}_{a_1 c_2 e_2} R^{a_1 b_1 d_1}_{a_2 b_2 d_2} \tag{10.98}$$

for arbitrary a, b, c, d, e, f and $a_2, b_2, c_2, d_2, e_2, f_2$. The sums are over $a_1, b_1, c_1, d_1, e_1, f_1 \in \mathbb{Z}_{\geq 0}$ on both sides. They are finite sums due to the weight conservation (3.48). The identity (10.98) reproduces the tetrahedron equation (2.9).

A similar proof of the 3D reflection equation (4.3) is possible based on (10.80). We now start from a PBW basis (10.6) of $U_q^+(C_3)$ having the form $E^{a,b,c,d,e,f,g,h,i}_{3,2,3,1,2,1,3,2,1}$ and apply (10.97) and $E^{\dots abcd\dots}_{\dots 3232\dots} = \sum K^{abcd}_{ijkl} E^{\dots lkji\dots}_{\dots 2323\dots}$. The two routes are as follows:

$$
\begin{aligned}
&E^{a,b,c,d,e,f,g,h,i}_{3,2,3,1,2,1,3,2,1} = E^{a,b,d,c,e,g,f,h,i}_{3,2,1,3,2,3,1,2,1} = \sum R^{fhi}_{f_1h_1i_1} E^{a,b,d,c,e,g,i_1,h_1,f_1}_{3,2,1,3,2,3,2,1,2}\\
&= \sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} E^{a,b,d,i_2,g_1,e_1,c_1,h_1,f_1}_{3,2,1,2,3,2,3,1,2}\\
&= \sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} R^{bdi_2}_{b_1d_1i_3} E^{a,i_3,d_1,b_1,g_1,e_1,c_1,h_1,f_1}_{3,1,2,1,3,2,3,1,2}\\
&= \sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} R^{bdi_2}_{b_1d_1i_3} E^{i_3,a,d_1,g_1,b_1,e_1,h_1,c_1,f_1}_{1,3,2,3,1,2,1,3,2}\\
&= \sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} R^{bdi_2}_{b_1d_1i_3} R^{b_1e_1h_1}_{b_2e_2h_2} E^{i_3,a,d_1,g_1,h_2,e_2,b_2,c_1,f_1}_{1,3,2,3,2,1,2,3,2}\\
&= \sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} R^{bdi_2}_{b_1d_1i_3} R^{b_1e_1h_1}_{b_2e_2h_2} K^{ad_1g_1h_2}_{a_1d_2g_2h_3} E^{i_3,h_3,g_2,d_2,a_1,e_2,b_2,c_1,f_1}_{1,2,3,2,3,1,2,3,2}\\
&= \sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} R^{bdi_2}_{b_1d_1i_3} R^{b_1e_1h_1}_{b_2e_2h_2} K^{ad_1g_1h_2}_{a_1d_2g_2h_3} E^{i_3,h_3,g_2,d_2,e_2,a_1,b_2,c_1,f_1}_{1,2,3,2,1,3,2,3,2}\\
&= \sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} R^{bdi_2}_{b_1d_1i_3} R^{b_1e_1h_1}_{b_2e_2h_2} K^{ad_1g_1h_2}_{a_1d_2g_2h_3} K^{a_1b_2c_1f_1}_{a_2b_3c_2f_2} E^{i_3,h_3,g_2,d_2,e_2,f_2,c_2,b_3,a_2}_{1,2,3,2,1,2,3,2,3}\\
&= \sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} R^{bdi_2}_{b_1d_1i_3} R^{b_1e_1h_1}_{b_2e_2h_2} K^{ad_1g_1h_2}_{a_1d_2g_2h_3} K^{a_1b_2c_1f_1}_{a_2b_3c_2f_2} R^{d_2e_2f_2}_{d_3e_3f_3} E^{i_3,h_3,g_2,f_3,e_3,d_3,c_2,b_3,a_2}_{1,2,3,1,2,1,3,2,3}
\end{aligned}
$$

and

$$
\begin{aligned}
&E^{a,b,c,d,e,f,g,h,i}_{3,2,3,1,2,1,3,2,1} = \sum R^{def}_{d_1e_1f_1} E^{a,b,c,f_1,e_1,d_1,g,h,i}_{3,2,3,2,1,2,3,2,1}\\
&= \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} E^{f_2,c_1,b_1,a_1,e_1,d_1,g,h,i}_{2,3,2,3,1,2,3,2,1}\\
&= \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} E^{f_2,c_1,b_1,e_1,a_1,d_1,g,h,i}_{2,3,2,1,3,2,3,2,1}\\
&= \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} K^{a_1d_1gh}_{a_2d_2g_1h_1} E^{f_2,c_1,b_1,e_1,h_1,g_1,d_2,a_2,i}_{2,3,2,1,2,3,2,3,1}\\
&= \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} K^{a_1d_1gh}_{a_2d_2g_1h_1} R^{b_1e_1h_1}_{b_2e_2h_2} E^{f_2,c_1,h_2,e_2,b_2,g_1,d_2,a_2,i}_{2,3,1,2,1,3,2,3,1}\\
&= \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} K^{a_1d_1gh}_{a_2d_2g_1h_1} R^{b_1e_1h_1}_{b_2e_2h_2} E^{f_2,h_2,c_1,e_2,g_1,b_2,d_2,i,a_2}_{2,1,3,2,3,1,2,1,3}\\
&= \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} K^{a_1d_1gh}_{a_2d_2g_1h_1} R^{b_1e_1h_1}_{b_2e_2h_2} R^{b_2d_2i}_{b_3d_3i_1} E^{f_2,h_2,c_1,e_2,g_1,i_1,d_3,b_3,a_2}_{2,1,3,2,3,2,1,2,3}\\
&= \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} K^{a_1d_1gh}_{a_2d_2g_1h_1} R^{b_1e_1h_1}_{b_2e_2h_2} R^{b_2d_2i}_{b_3d_3i_1} K^{c_1e_2g_1i_1}_{c_2e_3g_2i_2} E^{f_2,h_2,i_2,g_2,e_3,c_2,d_3,b_3,a_2}_{2,1,2,3,2,3,1,2,3}\\
&= \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} K^{a_1d_1gh}_{a_2d_2g_1h_1} R^{b_1e_1h_1}_{b_2e_2h_2} R^{b_2d_2i}_{b_3d_3i_1} K^{c_1e_2g_1i_1}_{c_2e_3g_2i_2} R^{f_2h_2i_2}_{f_3h_3i_3} E^{i_3,h_3,f_3,g_2,e_3,c_2,d_3,b_3,a_2}_{1,2,1,3,2,3,1,2,3}\\
&= \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} K^{a_1d_1gh}_{a_2d_2g_1h_1} R^{b_1e_1h_1}_{b_2e_2h_2} R^{b_2d_2i}_{b_3d_3i_1} K^{c_1e_2g_1i_1}_{c_2e_3g_2i_2} R^{f_2h_2i_2}_{f_3h_3i_3} E^{i_3,h_3,g_2,f_3,e_3,d_3,c_2,b_3,a_2}_{1,2,3,1,2,1,3,2,3}.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
&\sum R^{fhi}_{f_1h_1i_1} K^{cegi_1}_{c_1e_1g_1i_2} R^{bdi_2}_{b_1d_1i_3} R^{b_1e_1h_1}_{b_2e_2h_2} K^{ad_1g_1h_2}_{a_1d_2g_2h_3} K^{a_1b_2c_1f_1}_{a_2b_3c_2f_2} R^{d_2e_2f_2}_{d_3e_3f_3}\\
&= \sum R^{def}_{d_1e_1f_1} K^{abcf_1}_{a_1b_1c_1f_2} K^{a_1d_1gh}_{a_2d_2g_1h_1} R^{b_1e_1h_1}_{b_2e_2h_2} R^{b_2d_2i}_{b_3d_3i_1} K^{c_1e_2g_1i_1}_{c_2e_3g_2i_2} R^{f_2h_2i_2}_{f_3h_3i_3}
\end{aligned}
\tag{10.99}
$$

for any $a, b, c, d, e, f, g, h, i$ and $a_2, b_3, c_2, d_3, e_3, f_3, g_2, h_3, i_3$. The sums are over $a_1, b_1, b_2, c_1, d_1, d_2, e_1, e_2, f_1, f_2, g_1, h_1, h_2, i_1, i_2 \in \mathbb{Z}_{\geq 0}$ on both sides. They are finite sums due to the weight conservation (3.48) and (5.65). The identity (10.99) reproduces the 3D reflection equation (4.5). By a parallel argument for $U_q^+(B_3)$, the 3D reflection equation of type B (6.31) can also be derived.

10.6 χ-Invariants

Theorem 10.6 implies non-trivial identities in (a completion of) $U_q^+(\mathfrak{g})$. They are stated as invariance of some infinite products under the anti-involution χ introduced in (10.42). Here we illustrate the derivation along $\mathfrak{g} = A_2$ and present the results for C_2 and G_2. The point is to translate the boundary vectors in Sects. 3.6.1, 5.8.1 and 8.6.1 in terms of the PBW basis.

Let us write the boundary vectors (3.132) as

$$|\eta_s\rangle = \sum_{m\geq 0} \eta_{s,m}|m\rangle \quad (s = 1, 2). \tag{10.100}$$

By comparing the coefficient of $|a\rangle \otimes |b\rangle \otimes |c\rangle$ on the two sides of (3.143) using (3.47), we get

$$\sum_{i,j,k} \eta_{s,i}\eta_{s,j}\eta_{s,k}R^{abc}_{ijk} = \eta_{s,a}\eta_{s,b}\eta_{s,c}. \tag{10.101}$$

In view of (3.63), this is equivalent to

$$\sum_{a,b,c} \hat{\eta}_{s,a}\hat{\eta}_{s,b}\hat{\eta}_{s,c}R^{abc}_{ijk} = \hat{\eta}_{s,i}\hat{\eta}_{s,j}\hat{\eta}_{s,k}, \qquad \hat{\eta}_{s,a} = (q^2)_a\eta_{s,a}. \tag{10.102}$$

Multiply this by $E_{\mathbf{2}}^{k,j,i}$ and sum over $i, j, k \in \mathbb{Z}_{\geq 0}$. From (10.46) and (10.6), the RHS gives

$$\sum_{i,j,k} E_{\mathbf{2}}^{k,j,i}\hat{\eta}_{s,i}\hat{\eta}_{s,j}\hat{\eta}_{s,k} = \Big(\sum_k \hat{\eta}_{s,k}\frac{(b_1)^k}{[k]_1!}\Big)\Big(\sum_j \hat{\eta}_{s,j}\frac{(b_2)^j}{[j]_1!}\Big)\Big(\sum_i \hat{\eta}_{s,i}\frac{(b_3)^i}{[i]_1!}\Big). \tag{10.103}$$

As for the LHS, we have

$$\begin{aligned}\sum_{a,b,c}\Big(\sum_{i,j,k} R^{abc}_{ijk}E_{\mathbf{2}}^{k,j,i}\Big)\hat{\eta}_{s,a}\hat{\eta}_{s,b}\hat{\eta}_{s,c} &= \sum_{a,b,c} E_{\mathbf{1}}^{a,b,c}\hat{\eta}_{s,a}\hat{\eta}_{s,b}\hat{\eta}_{s,c}\\ &= \chi\Big(\sum_{a,b,c} E_{\mathbf{2}}^{c,b,a}\hat{\eta}_{s,a}\hat{\eta}_{s,b}\hat{\eta}_{s,c}\Big).\end{aligned} \tag{10.104}$$

The first equality is due to (10.58) which is the A_2 case of the main theorem of this chapter. The second equality is (10.42). The quantity within χ in (10.104) is equal to (10.103). Thus we find that (10.103) is χ-invariant. To describe the result neatly we introduce a quantum-dilogarithm-type infinite product:

$$\Theta_q(z) = \sum_m \frac{q^{m(m-1)/2}z^m}{(q)_m} = (-z; q)_\infty. \tag{10.105}$$

Then a direct calculation using (3.132) yields

$$\sum_m \hat{\eta}_{s,m} \frac{z^m}{[m]_1!} = \begin{cases} \Theta_q((1-q^2)z) & s=1, \\ \Theta_{q^4}(q(1-q^2)^2 z^2) & s=2. \end{cases} \tag{10.106}$$

Thus we get a corollary of Theorem 10.6 and Proposition 3.28.

Corollary 10.15 *Set $c_i = (1-q^2)b_i, c'_i = \chi(c_i) \in U_q^+(A_2)$ $(i=1,2,3)$ using b_i in (10.46) and the anti-algebra involution χ in (10.41). Then the following equalities are valid:*

$$\Theta_q(c_1)\Theta_q(c_2)\Theta_q(c_3) = \Theta_q(c'_3)\Theta_q(c'_2)\Theta_q(c'_1), \tag{10.107}$$

$$\Theta_{q^4}(qc_1^2)\Theta_{q^4}(qc_2^2)\Theta_{q^4}(qc_3^2) = \Theta_{q^4}(q{c'_3}^2)\Theta_{q^4}(q{c'_2}^2)\Theta_{q^4}(q{c'_1}^2). \tag{10.108}$$

Remark 10.16 By the rescaling $e_1 \to xe_1, e_2 \to ye_2$ with parameters x, y, the identity (10.107) is seemingly generalized to

$$\Theta_q(xc_1)\Theta_q(xyc_2)\Theta_q(yc_3) = \Theta_q(yc'_3)\Theta_q(xyc'_2)\Theta_q(xc'_1)$$

containing x, y in the same manner as spectral parameters in the Yang–Baxter equation. The same holds for (10.108). Similar remarks apply to the C_2 and G_2 cases in the sequel where the parameters arranged along the positive roots fit the spectral parameters in the reflection and the G_2 reflection equations.

The product (10.107) is expanded as

$$\begin{aligned}
&\Theta_q(c_1)\Theta_q(c_2)\Theta_q(c_3) \\
&= 1 + (1+q)(e_1+e_2) + q(1+q)(e_1^2+e_2^2) + (1+q)(e_1e_2+e_2e_1) \\
&+ (1+q)^2(e_1e_2e_1+e_2e_1e_2) + \frac{q^3(1-q^2)^2(e_1^3+e_2^3)}{(1-q)(1-q^3)} + \frac{q^6(1-q^2)^3(e_1^4+e_2^4)}{(1-q)(1-q^3)(1-q^4)} \\
&+ \frac{q^2(1-q^2)^2(e_1e_2e_1^2+e_1^2e_2e_1+e_2e_1e_2^2+e_2^2e_1e_2)}{(1-q)(1-q^3)} \\
&+ \frac{q(1-q^2)^2\big(q(e_1^2e_2^2+e_2^2e_1^2)+(1+q)^2e_1e_2^2e_1-q(1+q^2)e_2e_1^2e_2\big)}{(1-q)(1-q^4)} + \cdots,
\end{aligned} \tag{10.109}$$

where the q-Serre relation (10.45) has been used to make it manifestly invariant under χ. Similarly, (10.108) is expanded as

$$\begin{aligned}
&\Theta_{q^4}(qc_1^2)\Theta_{q^4}(qc_2^2)\Theta_{q^4}(qc_3^2) \\
&= 1 + \frac{q(1-q^2)^2(e_1^2+e_2^2)}{1-q^4} + \frac{q^6(1-q^2)^4(e_1^4+e_2^4)}{(1-q^4)(1-q^8)} \\
&+ \frac{q^2(1-q^2)^3(e_1^2e_2^2+e_2^2e_1^2-(1+q^2)e_2e_1^2e_2)}{(1-q^4)^2} + \cdots.
\end{aligned} \tag{10.110}$$

For C_2, the relevant results are (10.80) and Proposition 5.21 concerning the boundary vectors in (5.118)–(5.120). There are three identities corresponding to the choices of (r, k) in (5.136).

Corollary 10.17 *Set* $c_i = (1-q^4)b_i$ $(i = 1, 3)$, $c_i = (1-q^2)b_i$ $(i = 2, 4)$ *and* $c'_i = \chi(c_i) \in U_q^+(C_2)$ $(i = 1, 2, 3, 4)$ *using* b_i *in (10.60) and the anti-algebra involution* χ *in (10.41). Then the following equalities are valid:*

$$\Theta_{q^2}(c_1)\Theta_q(c_2)\Theta_{q^2}(c_3)\Theta_q(c_4) = \Theta_q(c'_4)\Theta_{q^2}(c'_3)\Theta_q(c'_2)\Theta_{q^2}(c'_1), \tag{10.111}$$

$$\Theta_{q^2}(c_1)\Theta_{q^4}(qc_2^2)\Theta_{q^2}(c_3)\Theta_{q^4}(qc_4^2) = \Theta_{q^4}(qc_4'^2)\Theta_{q^2}(c'_3)\Theta_{q^4}(qc_2'^2)\Theta_{q^2}(c'_1), \tag{10.112}$$

$$\Theta_{q^8}(q^2c_1^2)\Theta_{q^4}(qc_2^2)\Theta_{q^8}(q^2c_3^2)\Theta_{q^4}(qc_4^2) = \Theta_{q^4}(qc_4'^2)\Theta_{q^8}(q^2c_3'^2)\Theta_{q^4}(qc_2'^2)\Theta_{q^8}(q^2c_1'^2). \tag{10.113}$$

For G_2, the relevant result is Conjecture 8.9 for the boundary vector (8.61) and (10.96).

Corollary 10.18 *Set* $c_i = (1-q^6)b_i$ $(i = 1, 3, 5)$, $c_i = (1-q^2)b_i$ $(i = 2, 4, 6)$ *and* $c'_i = \chi(c_i) \in U_q^+(G_2)$ $(i = 1, \ldots, 6)$ *using* b_i *in (10.82) and the anti-algebra involution* χ *in (10.41). If Conjecture 8.9 holds, the following equality is valid:*

$$\begin{aligned} &\Theta_{q^3}(c_1)\Theta_q(c_2)\Theta_{q^3}(c_3)\Theta_q(c_4)\Theta_{q^3}(c_5)\Theta_q(c_6) \\ &= \Theta_q(c'_6)\Theta_{q^3}(c'_5)\Theta_q(c'_4)\Theta_{q^3}(c'_3)\Theta_q(c'_2)\Theta_{q^3}(c'_1). \end{aligned} \tag{10.114}$$

10.7 Bibliographical Notes and Comments

This chapter is an extended exposition of [102]. The braid group action (10.5) is introduced in [111]. The formulation of quantized coordinate ring in this chapter follows [76, 139]. See also [43] and [29, Chap. 7]. For quantum cluster algebra structure of quantized coordinate rings, see [52].

The Peter–Weyl-type Theorem 10.1 is taken from [76, Proposition 7.2.2]. Proposition 10.4 is a special case of [66, Corollary 9.1.4]. In [149, Theorem 7], $U_q^+(\mathfrak{g})$ has been identified with an explicit subalgebra of $A_q(\mathfrak{g})_{\mathcal{S}}$. A proof of Theorem 10.5 adapted to the present setting has been given in [102, Sect. 3.2]. The main result, Theorem 10.6, is due to [102, Theorem 5]. The case $\mathfrak{g} = A_2$ was obtained earlier in the pioneering work [131]. Remark 10.8 is due to [141], where a unified conceptual proof of Theorem 10.6 has been attained. See also [128] for yet another proof using the representation theory of q-boson algebra and the Drinfeld pairing of $U_q(\mathfrak{g})$. The multiplication rule on the PBW bases like Lemmas 10.9, 10.11 and 10.13 plays an important role also in the study of the positive principal series representations and modular double [61]. For type C_2, one can adjust the definition of $E_{\mathbf{i}}^A$ in (10.6) with that in [148] by setting $v = q^{-1}$. Some of the results like Lemma 10.13 have also been obtained in [147]. An analogue of Sect. 10.5 for quantum superalgebras has been argued in [151].

Chapter 11
Trace Reductions of $RLLL = LLLR$

Abstract From this chapter onwards, we turn to applications of the 3D structures. The tetrahedron equation $RLLL = LLLR$ can be composed n times in various directions. By tracing out a part of the spaces appropriately, it reduces to the Yang–Baxter equation among the remaining objects. The prescription, which we call the trace reduction, generates infinitely many solutions to the Yang–Baxter equation labeled with n in a matrix product form. In this chapter we demonstrate the method using the 3D R and 3D L in Sect. 3.5.2. The resulting solutions to the Yang–Baxter equation are trigonometric. They are identified with the quantum R matrices for the symmetric and the anti-symmetric tensor representations of $U_{\pm q^{-1}}(A^{(1)}_{n-1})$. The cyclicity of the trace is reflected in the Dynkin diagram of the affine Lie algebra $A^{(1)}_{n-1}$. The matrix product formula of the quantum R matrices naturally leads to an interpretation of the commuting transfer matrix of $m \times n$ layer either as the one for a $U_{q^{-1}}(A^{(1)}_{n-1})$ vertex model with size m or a $U_{q^{-1}}(A^{(1)}_{m-1})$ vertex model with size n.

11.1 Introduction

This chapter consists of two main parts. The first part, Sects. 11.1–11.4, is devoted to a *construction* of infinitely many solutions to the Yang–Baxter equation by the method we call the trace reduction. The basic ingredients are the tetrahedron equation $RLLL = LLLR$ and its solution associated with $A_q(A_{n-1})$ in Chap. 3. The second part, Sect. 11.5, provides the *characterization* of them as the quantum R matrices in the standard framework of $U_p(A^{(1)}_{n-1})$ and its finite-dimensional representations.

There are two essential parameters, q from the quantized coordinate rings A_q and p from the quantum affine algebras U_p. They should be adjusted properly depending on the direction along which the trace reduction is performed. All these features are common in the rest of the book except the last chapter.

Let us prepare notations that will also be used in the subsequent chapters. For n component arrays of 0 and 1, the following notations will be used:

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$$\mathfrak{s} = \mathfrak{s}^{(n)} = \{0,1\}^n = \{\mathbf{a} = (a_1, \dots, a_n) = a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n\}, \tag{11.1}$$

$$\mathfrak{s} = \mathfrak{s}_0 \cup \mathfrak{s}_1 \cup \dots \cup \mathfrak{s}_n, \qquad \mathfrak{s} = \mathfrak{s}_+ \cup \mathfrak{s}_-, \tag{11.2}$$

$$\mathfrak{s}_k = \mathfrak{s}_k^{(n)} = \{\mathbf{a} \in \mathfrak{s} \mid |\mathbf{a}| = k\}, \qquad \mathfrak{s}_\pm = \{\mathbf{a} \in \mathfrak{s} \mid (-1)^{|\mathbf{a}|} = \pm 1\}, \tag{11.3}$$

$$|\mathbf{a}| = a_1 + \dots + a_n, \qquad \mathbf{a}^\vee = (a_n, \dots, a_1), \tag{11.4}$$

where the last one is the reverse ordering of $\mathbf{a} = (a_1, \dots, a_n)$. For the n-fold tensor product of $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$, we set

$$\mathbf{V} = \mathbf{V}^{(n)} = V^{\otimes n} = \mathbf{V}_0 \oplus \mathbf{V}_1 \oplus \dots \oplus \mathbf{V}_n, \qquad \mathbf{V} = \mathbf{V}_+ \oplus \mathbf{V}_-, \tag{11.5}$$

$$\mathbf{V} = \bigoplus_{\mathbf{a}\in\mathfrak{s}} \mathbb{C}\, v_{\mathbf{a}}, \quad \mathbf{V}_k = \mathbf{V}_k^{(n)} = \bigoplus_{\mathbf{a}\in\mathfrak{s}_k^{(n)}} \mathbb{C}\, v_{\mathbf{a}}, \quad \mathbf{V}_\pm = \bigoplus_{\mathbf{a}\in\mathfrak{s}_\pm} \mathbb{C}\, v_{\mathbf{a}}, \tag{11.6}$$

$$v_{\mathbf{a}} = v_{a_1} \otimes \dots \otimes v_{a_n} \quad \text{for} \quad \mathbf{a} = (a_1, \dots, a_n). \tag{11.7}$$

For n-arrays of $\mathbb{Z}_{\ge 0}$ and n-fold tensor product of $\mathcal{F}_q$, we still use (11.4) and similar notations as follows:

$$B = B^{(n)} = (\mathbb{Z}_{\ge 0})^n = \{\mathbf{a} = (a_1, \dots, a_n)\}, \tag{11.8}$$

$$B = \bigcup_{k\ge 0} B_k, \qquad B = B_+ \cup B_-, \tag{11.9}$$

$$B_k = B_k^{(n)} = \{\mathbf{a} \in B^{(n)} \mid |\mathbf{a}| = k\}, \quad B_\pm = \{\mathbf{a} \in B \mid (-1)^{|\mathbf{a}|} = \pm 1\}, \tag{11.10}$$

$$\mathbf{W} = \mathbf{W}^{(n)} = \mathcal{F}_q^{\otimes n} = \bigoplus_{k\ge 0} \mathbf{W}_k, \qquad \mathbf{W} = \mathbf{W}_+ \oplus \mathbf{W}_-, \tag{11.11}$$

$$\mathbf{W} = \bigoplus_{\mathbf{a}\in B} \mathbb{C}\,|\mathbf{a}\rangle, \quad \mathbf{W}_k = \mathbf{W}_k^{(n)} = \bigoplus_{\mathbf{a}\in B_k^{(n)}} \mathbb{C}\,|\mathbf{a}\rangle, \quad \mathbf{W}_\pm = \bigoplus_{\mathbf{a}\in B_\pm} \mathbb{C}\,|\mathbf{a}\rangle, \tag{11.12}$$

$$|\mathbf{a}\rangle = |a_1\rangle \otimes \dots \otimes |a_n\rangle \quad \text{for} \quad \mathbf{a} = (a_1, \dots, a_n). \tag{11.13}$$

Except for Sects. 11.6, 12.3 and 13.8, the integer n is fixed and the superscript "(n)" will be suppressed.

In Sect. 3.5.2, we have presented a solution to the quantized Yang–Baxter equation. We recall it below for readers' convenience. The 3D L is depicted as in Fig. 11.1.

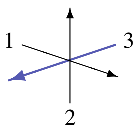
$L_{123} =$ 

Fig. 11.1 Diagram for the 3D L, where 1, 2, 3 are labels of the arrows. Black arrows carry V and the blue one carries $\mathcal{F}_q$

It is a linear operator $L = \sum E_{ai} \otimes E_{bj} \otimes L^{ab}_{ij} \in \mathrm{End}(\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{\mathcal{F}}_q)$ with

$$\begin{pmatrix} L^{00}_{00} & L^{00}_{01} & L^{00}_{10} & L^{00}_{11} \\ L^{01}_{00} & L^{01}_{01} & L^{01}_{10} & L^{01}_{11} \\ L^{10}_{00} & L^{10}_{01} & L^{10}_{10} & L^{10}_{11} \\ L^{11}_{00} & L^{11}_{01} & L^{11}_{10} & L^{11}_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -q\alpha^{-1}\mathbf{k} & \mathbf{a}^- & 0 \\ 0 & \mathbf{a}^+ & \alpha\mathbf{k} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{11.14}$$

where $\mathbf{a}^\pm$, $\mathbf{k}$ are q-oscillators in (3.12)–(3.13) . This is the same as Fig. 3.2 up to an extra gauge parameter α. We let the number operator $\mathbf{h}$ (3.14) on $\mathcal{F}_q$ also act on V by $\mathbf{h}v_i = iv_i$ $(i = 0, 1)$ and set $\mathbf{h}_1 = \mathbf{h} \otimes 1 \otimes 1$, $\mathbf{h}_2 = 1 \otimes \mathbf{h} \otimes 1$ and $\mathbf{h}_3 = 1 \otimes 1 \otimes \mathbf{h}$. The 3D L satisfies

$$[x^{\mathbf{h}_1+\mathbf{h}_2} y^{\mathbf{h}_2+\mathbf{h}_3}, L_{123}] = 0, \tag{11.15}$$

$$\overline{L^{ab}_{ij}} = L^{ij}_{ab} = L^{ba}_{ij}|_{\alpha \to -q\alpha^{-1}}. \tag{11.16}$$

The weight conservation (11.15) with arbitrary x, y is equivalent to (3.102) and (3.104). $\overline{L^{ab}_{ij}}$ is defined by the rule mentioned after (3.18).

From Theorem 3.21 and Remark 3.23, we know that L satisfies the quantized Yang–Baxter equation, i.e. the tetrahedron equation of type $RLLL = LLLR$

$$L_{124}L_{135}L_{236}R_{456} = R_{456}L_{236}L_{135}L_{124}, \tag{11.17}$$

where R denotes the 3D R detailed in Chapter 3. This is an identity in $\mathrm{End}(\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V} \otimes \overset{4}{\mathcal{F}}_q \otimes \overset{5}{\mathcal{F}}_q \otimes \overset{6}{\mathcal{F}}_q)$.

11.2 Trace Reduction Over the Third Component of L

Consider n copies of (11.17) in which the spaces labeled with 1, 2, 3 are replaced by $1_i, 2_i, 3_i$ with $i = 1, 2, \dots, n$:

$$(L_{1_i2_i4}L_{1_i3_i5}L_{2_i3_i6})\, R_{456} = R_{456}\, (L_{2_i3_i6}L_{1_i3_i5}L_{1_i2_i4}).$$

Sending R_{456} to the left by applying this relation repeatedly, we get

$$\begin{aligned}(L_{1_12_14}L_{1_13_15}L_{2_13_16}) \cdots (L_{1_n2_n4}L_{1_n3_n5}L_{2_n3_n6})\, R_{456} \\ = R_{456}\, (L_{2_13_16}L_{1_13_15}L_{1_12_14}) \cdots (L_{2_n3_n6}L_{1_n3_n5}L_{1_n2_n4}).\end{aligned} \tag{11.18}$$

One can rearrange this without changing the order of operators sharing common labels, hence by using the trivial commutativity, as

$$\begin{aligned}&(L_{1_1 2_1 4}\cdots L_{1_n 2_n 4})(L_{1_1 3_1 5}\cdots L_{1_n 3_n 5})(L_{2_1 3_1 6}\cdots L_{2_n 3_n 6})R_{456}\\&\quad = R_{456}(L_{2_1 3_1 6}\cdots L_{2_n 3_n 6})(L_{1_1 3_1 5}\cdots L_{1_n 3_n 5})(L_{1_1 2_1 4}\cdots L_{1_n 2_n 4}).\end{aligned} \tag{11.19}$$

The weight conservation (3.49) of the 3D R may be stated as

$$R_{456}\, x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5} y^{\mathbf{h}_6} = x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5} y^{\mathbf{h}_6} R_{456} \tag{11.20}$$

for arbitrary parameters x and y. See (3.14) for the definition of $\mathbf{h}$. Multiplying this by (11.19) from the left and applying $R^2 = 1$ from (3.60), we get

$$\begin{aligned}&R_{456}\, x^{\mathbf{h}_4}(L_{1_1 2_1 4}\cdots L_{1_n 2_n 4})(xy)^{\mathbf{h}_5}(L_{1_1 3_1 5}\cdots L_{1_n 3_n 5})y^{\mathbf{h}_6}(L_{2_1 3_1 6}\cdots L_{2_n 3_n 6})R_{456}\\&\quad = y^{\mathbf{h}_6}(L_{2_1 3_1 6}\cdots L_{2_n 3_n 6})(xy)^{\mathbf{h}_5}(L_{1_1 3_1 5}\cdots L_{1_n 3_n 5})x^{\mathbf{h}_4}(L_{1_1 2_1 4}\cdots L_{1_n 2_n 4}).\end{aligned} \tag{11.21}$$

This relation will also be utilized in the boundary vector reduction in Sect. 12.1 (Fig. 11.2).

Take the trace of (11.21) over $\overset{4}{\mathcal{F}}_q \otimes \overset{5}{\mathcal{F}}_q \otimes \overset{6}{\mathcal{F}}_q$ using the cyclicity of the trace and $R^2 = 1$. The result reads as

$$\begin{aligned}&\mathrm{Tr}_4\big(x^{\mathbf{h}_4} L_{1_1 2_1 4}\cdots L_{1_n 2_n 4}\big)\mathrm{Tr}_5\big((xy)^{\mathbf{h}_5} L_{1_1 3_1 5}\cdots L_{1_n 3_n 5}\big)\mathrm{Tr}_6\big(y^{\mathbf{h}_6} L_{2_1 3_1 6}\cdots L_{2_n 3_n 6}\big)\\&\quad = \mathrm{Tr}_6\big(y^{\mathbf{h}_6} L_{2_1 3_1 6}\cdots L_{2_n 3_n 6}\big)\mathrm{Tr}_5\big((xy)^{\mathbf{h}_5} L_{1_1 3_1 5}\cdots L_{1_n 3_n 5}\big)\mathrm{Tr}_4\big(x^{\mathbf{h}_4} L_{1_1 2_1 4}\cdots L_{1_n 2_n 4}\big).\end{aligned} \tag{11.22}$$

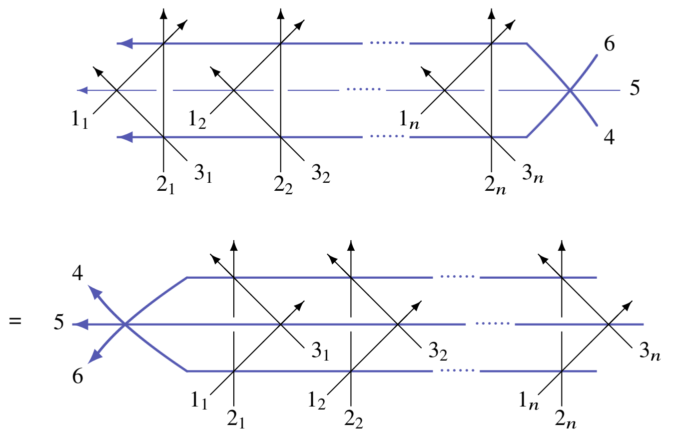


Fig. 11.2 A graphical representation of (11.18) and (11.19). It is a concatenation of Fig. 2.3 which corresponds to the basic $RLLL = LLLR$ relation. Black and blue arrows carry V and $\mathcal{F}_q$, respectively

Let us denote the operators appearing here by

$$\begin{aligned}
S^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{2}}(z) &= \varrho^{\mathrm{tr}_3}(z)\mathrm{Tr}_4(z^{\mathbf{h}_4} L_{1_1 2_1 4} \cdots L_{1_n 2_n 4}) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}} \otimes \overset{\mathbf{2}}{\mathbf{V}}),\\
S^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{3}}(z) &= \varrho^{\mathrm{tr}_3}(z)\mathrm{Tr}_5(z^{\mathbf{h}_5} L_{1_1 3_1 5} \cdots L_{1_n 3_n 5}) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}} \otimes \overset{\mathbf{3}}{\mathbf{V}}),\\
S^{\mathrm{tr}_3}_{\mathbf{2},\mathbf{3}}(z) &= \varrho^{\mathrm{tr}_3}(z)\mathrm{Tr}_6(z^{\mathbf{h}_6} L_{2_1 3_1 6} \cdots L_{2_n 3_n 6}) \in \mathrm{End}(\overset{\mathbf{2}}{\mathbf{V}} \otimes \overset{\mathbf{3}}{\mathbf{V}}).
\end{aligned} \tag{11.23}$$

The superscript tr_3 indicates that the trace is taken over the *3rd* (rightmost) component of L, whereas Tr_j in the RHSs signifies the *label* j of a space. A similar convention will be employed in subsequent sections.

Those appearing in (11.23) are the same operators acting on different copies of $\mathbf{V} \otimes \mathbf{V}$ specified as $\overset{\mathbf{1}}{\mathbf{V}} = \overset{1_1}{V} \otimes \cdots \otimes \overset{1_n}{V}$, $\overset{\mathbf{2}}{\mathbf{V}} = \overset{2_1}{V} \otimes \cdots \otimes \overset{2_n}{V}$ and $\overset{\mathbf{3}}{\mathbf{V}} = \overset{3_1}{V} \otimes \cdots \otimes \overset{3_n}{V}$. The normalization factor $\varrho^{\mathrm{tr}_3}(z)$ will be specified later in (11.33). Now the relation (11.22) is stated as the Yang–Baxter equation:

$$S^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{2}}(x) S^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{3}}(xy) S^{\mathrm{tr}_3}_{\mathbf{2},\mathbf{3}}(y) = S^{\mathrm{tr}_3}_{\mathbf{2},\mathbf{3}}(y) S^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{3}}(xy) S^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{2}}(x). \tag{11.24}$$

When arguing a single S, we will often suppress the labels **1**, **2** etc. Set

$$S^{\mathrm{tr}_3}(z)(v_{\mathbf{i}} \otimes v_{\mathbf{j}}) = \sum_{\mathbf{a},\mathbf{b} \in \mathfrak{s}} S^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} v_{\mathbf{a}} \otimes v_{\mathbf{b}}. \tag{11.25}$$

Then the construction (11.23) implies the matrix product formula

$$S^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \varrho^{\mathrm{tr}_3}(z)\mathrm{Tr}\big(z^{\mathbf{h}} L^{a_1 b_1}_{i_1 j_1} \cdots L^{a_n b_n}_{i_n j_n}\big) \tag{11.26}$$

in terms of the components of the 3D L in (11.14) (Fig. 11.3).

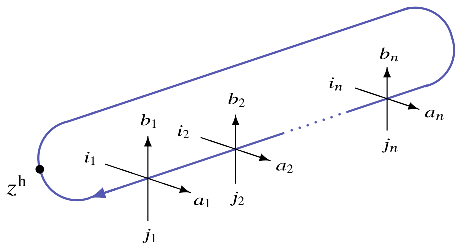


Fig. 11.3 Matrix product construction by the trace reduction (11.26) is depicted as a concatenation of Fig. 11.1 along the blue arrow carrying $\mathcal{F}_q$. It is a BBQ stick with n-fold X-shaped sausages. It is closed cyclically reflecting the trace

By definition, the trace is given by $\mathrm{Tr}(X) = \sum_{m\geq 0} \frac{\langle m|X|m\rangle}{\langle m|m\rangle} = \sum_{m\geq 0} \frac{\langle m|X|m\rangle}{(q^2)_m}$. See (3.12)–(3.17). Then (11.26) is evaluated by using the commutation relations of q-oscillators and the following formula[1]:

$$\mathrm{Tr}(z^{\mathbf{h}}\mathbf{k}^r(\mathbf{a}^+)^s(\mathbf{a}^-)^{s'}) = \delta_{s,s'}\frac{(zq^r)^s(q^2;q^2)_s}{(zq^r;q^2)_{s+1}}. \tag{11.27}$$

From (3.104), (11.15), (11.16) and (3.18), it is easy to see

$$S^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = 0 \text{ unless } \mathbf{a}+\mathbf{b} = \mathbf{i}+\mathbf{j} \text{ and } |\mathbf{a}| = |\mathbf{i}|, |\mathbf{b}| = |\mathbf{j}|, \tag{11.28}$$

$$S^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = S^{\mathrm{tr}_3}(z)^{\mathbf{i}^\vee \mathbf{j}^\vee}_{\mathbf{a}^\vee \mathbf{b}^\vee}, \tag{11.29}$$

$$S^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = z^{j_1-b_1}S^{\mathrm{tr}_3}(z)^{\sigma(\mathbf{a})\sigma(\mathbf{b})}_{\sigma(\mathbf{i})\sigma(\mathbf{j})}, \tag{11.30}$$

where $\sigma(\mathbf{a}) = (a_2, \ldots, a_n, a_1)$ is a cyclic shift. The property (11.28) implies the decomposition

$$S^{\mathrm{tr}_3}(z) = \bigoplus_{0\leq l,m\leq n} S^{\mathrm{tr}_3}_{l,m}(z), \qquad S^{\mathrm{tr}_3}_{l,m}(z) \in \mathrm{End}(\mathbf{V}_l \otimes \mathbf{V}_m). \tag{11.31}$$

The Yang–Baxter equation (11.24) is valid in each subspace $\mathbf{V}_k \otimes \mathbf{V}_l \otimes \mathbf{V}_m$ of $\overset{1}{\mathbf{V}} \otimes \overset{2}{\mathbf{V}} \otimes \overset{3}{\mathbf{V}}$. The scalar $\varrho^{\mathrm{tr}_3}(z)$ in (11.26) may be specified depending on the summands in (11.31). We take them as

$$S^{\mathrm{tr}_3}_{l,m}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \varrho^{\mathrm{tr}_3}_{l,m}(z)\mathrm{Tr}\big(z^{\mathbf{h}}L^{a_1b_1}_{i_1j_1}\cdots L^{a_nb_n}_{i_nj_n}\big) \qquad (\mathbf{a},\mathbf{i} \in \mathfrak{s}_l,\ \mathbf{b},\mathbf{j} \in \mathfrak{s}_m), \tag{11.32}$$

$$\varrho^{\mathrm{tr}_3}_{l,m}(z) = (-q)^{-(m-l)_+}\alpha^{m-l}(1 - zq^{|m-l|}), \tag{11.33}$$

where $(x)_+ = \max(x, 0)$ as in (3.66). In this normalization we have

$$S^{\mathrm{tr}_3}_{l,m}(z)(v_{\mathbf{e}_1+\cdots+\mathbf{e}_l} \otimes v_{\mathbf{e}_1+\cdots+\mathbf{e}_m}) = v_{\mathbf{e}_1+\cdots+\mathbf{e}_l} \otimes v_{\mathbf{e}_1+\cdots+\mathbf{e}_m}. \tag{11.34}$$

General elements $S^{\mathrm{tr}_3}_{l,m}(z)^{\mathbf{ab}}_{\mathbf{ij}}$ become rational functions of q and z which are *independent* of the parameter α in (11.14). The latter fact follows from the condition on the number of operators $\#L^{10}_{10} - \#L^{01}_{01} = l - m$ in (11.26) which is necessary for the trace to survive. By combining (3.18), (11.16) and $\varrho_{l,m}(z) = \varrho_{m,l}(z)|_{\alpha\to -q\alpha^{-1}}$, we also have

$$S^{\mathrm{tr}_3}_{l,m}(z)^{\mathbf{ab}}_{\mathbf{ij}} = S^{\mathrm{tr}_3}_{m,l}(z)^{\mathbf{b}^\vee \mathbf{a}^\vee}_{\mathbf{j}^\vee \mathbf{i}^\vee}. \tag{11.35}$$

[1] In the formula [106, Eq. (59)] which should correspond to $(11.27)|_{q\to q^2}$, the factor $(zq^{2r})^s$ is missing.

Example 11.1 Consider the simplest case $n = 2, l = m = 1$ of $S^{\mathrm{tr}_3}_{l,m}(z)$. We have

$$S^{\mathrm{tr}_3}_{1,1}(z)(v_{01} \otimes v_{10}) = S^{\mathrm{tr}_3}_{1,1}(z)^{01,10}_{01,10}\, v_{01} \otimes v_{10} + S^{\mathrm{tr}_3}_{1,1}(z)^{10,01}_{01,10}\, v_{10} \otimes v_{01},$$

and the coefficients are given by the matrix product formula

$$S^{\mathrm{tr}_3}_{1,1}(z)^{01,10}_{01,10} = \varrho^{\mathrm{tr}_3}_{1,1}(z)\mathrm{Tr}(z^{\mathbf{h}} L^{01}_{01} L^{10}_{10}) = (1-z)\mathrm{Tr}(-qz^{\mathbf{h}}\mathbf{k}^2) = -\frac{q(1-z)}{1-q^2 z},$$
$$S^{\mathrm{tr}_3}_{1,1}(z)^{10,01}_{01,10} = \varrho^{\mathrm{tr}_3}_{1,1}(z)\mathrm{Tr}(z^{\mathbf{h}} L^{10}_{01} L^{01}_{10}) = (1-z)\mathrm{Tr}(z^{\mathbf{h}}\mathbf{a}^+\mathbf{a}^-) = \frac{(1-q^2)z}{1-q^2 z},$$

where (11.14), (11.27) and (11.33) are used. Similar calculations show that $S^{\mathrm{tr}_3}_{l,m}(z)$ is a map such that

$$\begin{aligned}
v_{ij} \otimes v_{ij} &\mapsto v_{ij} \otimes v_{ij} \quad (i, j \in \{0, 1\}),\\
v_{01} \otimes v_{10} &\mapsto -\frac{q(1-z)v_{01} \otimes v_{10}}{1-q^2 z} + \frac{(1-q^2)z v_{10} \otimes v_{01}}{1-q^2 z},\\
v_{10} \otimes v_{01} &\mapsto \frac{(1-q^2)v_{01} \otimes v_{10}}{1-q^2 z} - \frac{q(1-z)v_{10} \otimes v_{01}}{1-q^2 z},
\end{aligned}$$

which is known as a six-vertex model R matrix [10, Chaps. 8 and 9].

Example 11.2 We consider general $n \geq 2$. Elements of $S^{\mathrm{tr}_3}_{m,1}(z)$ are given by

$$S^{\mathrm{tr}_3}_{m,1}(z)^{\mathbf{a}\,\mathbf{e}_b}_{\mathbf{i}\,\mathbf{e}_j} = \begin{cases} 1 & j = b, a_j = 1,\\ -\frac{q(1-q^{m-1}z)}{1-q^{m+1}z} & j = b, a_j = 0,\\ \frac{z(1-q^2)}{1-q^{m+1}z} q^{m-i_{j+1}-i_{j+2}-\cdots-i_b} & j < b,\\ \frac{1-q^2}{1-q^{m+1}z} q^{i_{b+1}+i_{b+2}+\cdots+i_j} & j > b, \end{cases} \tag{11.36}$$

where $\mathbf{a}, \mathbf{i} \in \mathfrak{s}_m$ and $\mathbf{a} + \mathbf{e}_b = \mathbf{i} + \mathbf{e}_j$ are assumed. Similarly, elements of $S^{\mathrm{tr}_3}_{1,m}(z)$ are given by

$$S^{\mathrm{tr}_3}_{1,m}(z)^{\mathbf{e}_a\,\mathbf{b}}_{\mathbf{e}_i\,\mathbf{j}} = \begin{cases} 1 & i = a, j_a = 1,\\ -\frac{q(1-q^{m-1}z)}{1-q^{m+1}z} & i = a, j_a = 0,\\ \frac{z(1-q^2)}{1-q^{m+1}z} q^{m-j_a-j_{a+1}-\cdots-j_{i-1}} & i > a,\\ \frac{1-q^2}{1-q^{m+1}z} q^{j_i+i_{i+1}+\cdots+j_{a-1}} & i < a, \end{cases}$$

where $\mathbf{b}, \mathbf{j} \in \mathfrak{s}_m$ and $\mathbf{e}_a + \mathbf{b} = \mathbf{e}_i + \mathbf{j}$ are assumed. The case $n = 2$ reduces to Example 11.1.

11.3 Trace Reduction Over the First Component of L

The previous section was concerned with the reduction over q-oscillator Fock spaces. It is also possible to make reductions over $V \simeq \mathbb{C}^2$, which is the subject of this and the next sections. We restrict ourselves to the choice $\alpha = 1$ in (11.14).[2]

Consider n copies of the tetrahedron equation (11.17) in which the spaces 3, 5, 6 are replaced by $3_i, 5_i, 6_i$ with $i = 1, \dots, n$:

$$R_{45_i6_i} L_{23_i6_i} L_{13_i5_i} L_{124} = L_{124} L_{13_i5_i} L_{23_i6_i} R_{45_i6_i}.$$

Sending L_{124} to the left by applying this repeatedly, we get

$$\begin{aligned}&(R_{45_16_1} L_{23_16_1} L_{13_15_1}) \cdots (R_{45_n6_n} L_{23_n6_n} L_{13_n5_n}) L_{124}\\ &\quad = L_{124}(L_{13_15_1} L_{23_16_1} R_{45_16_1}) \cdots (L_{13_n5_n} L_{23_n6_n} R_{45_n6_n}),\end{aligned} \tag{11.37}$$

which can be rearranged as (Fig. 11.4)

$$\begin{aligned}&(R_{45_16_1} \cdots R_{45_n6_n})(L_{23_16_1} \cdots L_{23_n6_n})(L_{13_15_1} \cdots L_{13_n5_n}) L_{124}\\ &\quad = L_{124}(L_{13_15_1} \cdots L_{13_n5_n})(L_{23_16_1} \cdots L_{23_n6_n})(R_{45_16_1} \cdots R_{45_n6_n}).\end{aligned} \tag{11.38}$$

From Remark 3.24 we know that L_{124} is invertible. Multiply $x^{\mathbf{h}_1}(xy)^{\mathbf{h}_2} y^{\mathbf{h}_4} L_{124}^{-1}$ from the left by (11.38) and take the trace over $\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{4}{\mathcal{F}}_q$. Using the weight conservation (11.15) we get the Yang–Baxter equation.

$$R^{\mathrm{tr}_1}_{\mathbf{5},\mathbf{6}}(y) S^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{6}}(xy) S^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{5}}(x) = S^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{5}}(x) S^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{6}}(xy) R^{\mathrm{tr}_1}_{\mathbf{5},\mathbf{6}}(y) \in \mathrm{End}(\overset{\mathbf{3}}{\mathbf{V}} \otimes \overset{\mathbf{5}}{\mathbf{W}} \otimes \overset{\mathbf{6}}{\mathbf{W}}), \tag{11.39}$$

where $\overset{\mathbf{3}}{\mathbf{V}} = \overset{3_1}{V} \otimes \cdots \otimes \overset{3_n}{V}$, $\overset{\mathbf{5}}{\mathbf{W}} = \overset{5_1}{\mathcal{F}_q} \otimes \cdots \otimes \overset{5_n}{\mathcal{F}_q}$ and $\overset{\mathbf{6}}{\mathbf{W}} = \overset{6_1}{\mathcal{F}_q} \otimes \cdots \otimes \overset{6_n}{\mathcal{F}_q}$. The superscript tr_1 signifies that the trace is taken over the 1st (leftmost) component of the 3D R and 3D L as

$$R^{\mathrm{tr}_1}_{\mathbf{5},\mathbf{6}}(z) = \mathrm{Tr}_4(z^{\mathbf{h}_4} R_{45_16_1} \cdots R_{45_n6_n}) \in \mathrm{End}(\overset{\mathbf{5}}{\mathbf{W}} \otimes \overset{\mathbf{6}}{\mathbf{W}}), \tag{11.40}$$

$$S^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{5}}(z) = \mathrm{Tr}_1(z^{\mathbf{h}_1} L_{13_15_1} \cdots L_{13_n5_n}) \in \mathrm{End}(\overset{\mathbf{3}}{\mathbf{V}} \otimes \overset{\mathbf{5}}{\mathbf{W}}), \tag{11.41}$$

$$S^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{6}}(z) = \mathrm{Tr}_2(z^{\mathbf{h}_2} L_{23_16_1} \cdots L_{23_n6_n}) \in \mathrm{End}(\overset{\mathbf{3}}{\mathbf{V}} \otimes \overset{\mathbf{6}}{\mathbf{W}}). \tag{11.42}$$

The operator (11.40) reappears in (13.22) and will be studied in Sect. 13.3. At this point we only mention that it satisfies the same selection rule as (11.45) since the 3D R obeys the same weight conservation (3.49) as the 3D L in (11.15). The other

[2] Other choices of α invalidate Theorem 11.5.

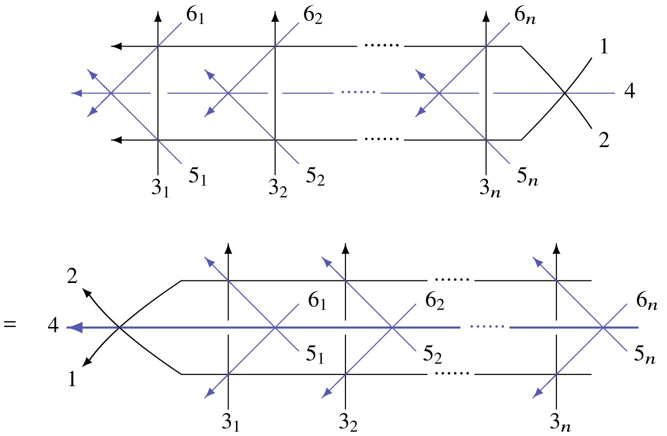


Fig. 11.4 A graphical representation of (11.37) and (11.38)

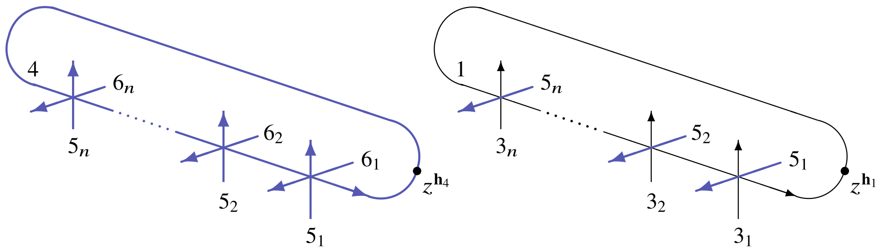


Fig. 11.5 Matrix product construction by the trace reduction (11.40) (left) and (11.41) (right). They are concatenations of the diagrams in (2.3) and (2.14) which are closed cyclically reflecting the trace. The diagram for (11.42) is the same as the one for (11.41) up to the labels of arrows

operators (11.41) and (11.42) involve the trace over V which consists of only two terms. We will often suppress the labels **3**, **5** etc. (Fig. 11.5).

The operator $S^{\mathrm{tr}_1}(z)$ acts on the basis as

$$S^{\mathrm{tr}_1}(z)(v_{\mathbf{i}} \otimes |\mathbf{j}\rangle) = \sum_{\mathbf{a}\in\mathfrak{s},\mathbf{b}\in B} S^{\mathrm{tr}_1}(z)^{\mathbf{ab}}_{\mathbf{ij}}\, v_{\mathbf{a}} \otimes |\mathbf{b}\rangle, \tag{11.43}$$

$$S^{\mathrm{tr}_1}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \sum_{r_0,\dots,r_{n-1}=0,1} z^{r_0} L^{r_0 a_1 b_1}_{r_1 i_1 j_1} L^{r_1 a_2 b_2}_{r_2 i_2 j_2} \cdots L^{r_{n-1} a_n b_n}_{r_0\, i_n\, j_n}, \tag{11.44}$$

where the elements $L^{r'ab}_{rij}$ are defined in (3.105). From the weight conservation of the 3D L (11.15), we have

$$S^{\mathrm{tr}_1}(z)^{\mathbf{ab}}_{\mathbf{ij}} = 0 \text{ unless } \mathbf{a}+\mathbf{b} = \mathbf{i}+\mathbf{j} \text{ and } |\mathbf{a}| = |\mathbf{i}|, |\mathbf{b}| = |\mathbf{j}|. \tag{11.45}$$

It implies the decomposition

$$S^{\mathrm{tr}_1}(z) = \bigoplus_{0\le l\le n, m\ge 0} S^{\mathrm{tr}_1}_{l,m}(z), \qquad S^{\mathrm{tr}_1}_{l,m}(z) \in \mathrm{End}(\mathbf{V}_l \otimes \mathbf{W}_m). \tag{11.46}$$

The sum (11.43) corresponding to the component $S^{\mathrm{tr}_1}_{l,m}(z)$ is restricted to $\mathbf{a} \in \mathfrak{s}_l$, $\mathbf{b} \in B_m$. The Yang–Baxter relation (11.39) holds on each subspace $\mathbf{V}_k \otimes \mathbf{W}_l \otimes \mathbf{W}_m$.

11.4 Trace Reduction Over the Second Component of L

Similarly to Sect. 11.3, we assume $\alpha = 1$ in this section.[3]

Consider n copies of the tetrahedron equation (11.17) in which the spaces $1, 4, 5$ are replaced by $1_i, 4_i, 5_i$ with $i = 1, \ldots, n$:

$$R_{4_i 5_i 6} L_{1_i 2 4_i} L_{1_i 3 5_i} L_{236} = L_{236} L_{1_i 3 5_i} L_{1_i 2 4_i} R_{4_i 5_i 6}.$$

Here we have relocated R by using $R = R^{-1}$ (3.60). Sending L_{236} to the left by applying this repeatedly, we get

$$\begin{aligned}&(R_{4_1 5_1 6} L_{1_1 2 4_1} L_{1_1 3 5_1}) \cdots (R_{4_n 5_n 6} L_{1_n 2 4_n} L_{1_n 3 5_n}) L_{236} \\ &\quad = L_{236} (L_{1_1 3 5_1} L_{1_1 2 4_1} R_{4_1 5_1 6}) \cdots (L_{1_n 3 5_n} L_{1_n 2 4_n} R_{4_n 5_n 6}),\end{aligned} \tag{11.47}$$

which can be rearranged as

$$\begin{aligned}&(R_{4_1 5_1 6} \cdots R_{4_n 5_n 6})(L_{1_1 2 4_1} \cdots L_{1_n 2 4_n})(L_{1_1 3 5_1} \cdots L_{1_n 3 5_n}) L_{236} \\ &\quad = L_{236} (L_{1_1 3 5_1} \cdots L_{1_n 3 5_n})(L_{1_1 2 4_1} \cdots L_{1_n 2 4_n})(R_{4_1 5_1 6} \cdots R_{4_n 5_n 6}).\end{aligned} \tag{11.48}$$

From Remark 3.24 we know that L_{236} is invertible. Multiply $x^{\mathbf{h}_2}(xy)^{\mathbf{h}_3} y^{\mathbf{h}_6} L^{-1}_{236}$ from the left by (11.48) and take the trace over $\overset{2}{V} \otimes \overset{3}{V} \otimes \overset{6}{\mathcal{F}}_q$. Using the weight conservation (11.15) we get the Yang–Baxter equation (Fig. 11.6).

$$R^{\mathrm{tr}_3}_{\mathbf{4},\mathbf{5}}(y) S^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{4}}(x) S^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{5}}(xy) = S^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{5}}(xy) S^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{4}}(x) R^{\mathrm{tr}_3}_{\mathbf{4},\mathbf{5}}(y) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}} \otimes \overset{\mathbf{4}}{\mathbf{W}} \otimes \overset{\mathbf{5}}{\mathbf{W}}), \tag{11.49}$$

where $\overset{\mathbf{1}}{\mathbf{V}} = \overset{1_1}{V} \otimes \cdots \otimes \overset{1_n}{V}$, $\overset{\mathbf{4}}{\mathbf{W}} = \overset{4_1}{W} \otimes \cdots \otimes \overset{4_n}{W}$ and $\overset{\mathbf{5}}{\mathbf{W}}$ was defined after (11.39). The superscript tr_2 signifies that the trace is taken over the second (middle) component as

[3] Other choice of α invalidates Theorem 11.6.

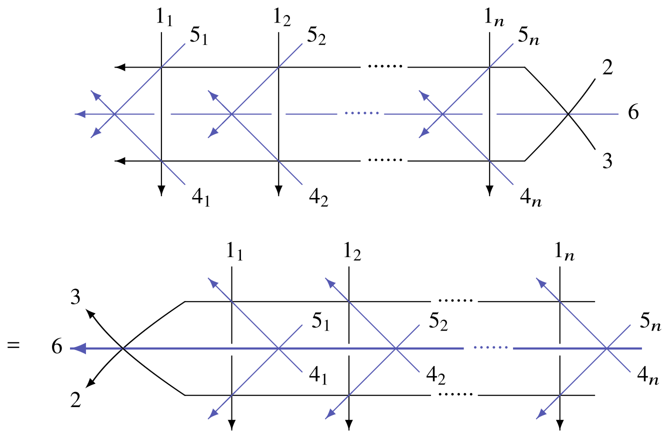


Fig. 11.6 A graphical representation of (11.47) and (11.48)

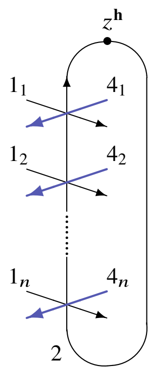


Fig. 11.7 A graphical representation of (11.50). The one for (11.51) just corresponds to a relabeling of the arrows

$$S^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{4}}(z) = \mathrm{Tr}_2(z^{\mathbf{h}_2} L_{1_1 2 4_1} \cdots L_{1_n 2 4_n}) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}} \otimes \overset{\mathbf{4}}{\mathbf{W}}), \tag{11.50}$$

$$S^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{5}}(z) = \mathrm{Tr}_3(z^{\mathbf{h}_3} L_{1_1 3 5_1} \cdots L_{1_n 3 5_n}) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}} \otimes \overset{\mathbf{5}}{\mathbf{W}}). \tag{11.51}$$

This matrix product construction is depicted as Fig. 11.7.

The operator $S^{\mathrm{tr}_2}(z)$ acts on the basis as

$$S^{\mathrm{tr}_2}(z)(v_{\mathbf{i}} \otimes |\mathbf{j}\rangle) = \sum_{\mathbf{a}\in\mathfrak{s},\mathbf{b}\in B} S^{\mathrm{tr}_2}(z)^{\mathbf{ab}}_{\mathbf{ij}}\, v_{\mathbf{a}} \otimes |\mathbf{b}\rangle, \tag{11.52}$$

$$S^{\mathrm{tr}_2}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \sum_{r_0,\dots,r_{n-1}=0,1} z^{r_0} L^{a_1 r_0 b_1}_{i_1 r_1 j_1} L^{a_2 r_1 b_2}_{i_2 r_2 j_2} \cdots L^{a_n r_{n-1} b_n}_{i_n r_0 j_n}, \tag{11.53}$$

where the elements $L^{ar'b}_{irj}$ are defined in (3.105). From the weight conservation of the 3D L (11.15), we have

$$S^{\mathrm{tr}_2}(z)^{\mathbf{ab}}_{\mathbf{ij}} = 0 \text{ unless } \mathbf{a} - \mathbf{b} = \mathbf{i} - \mathbf{j} \text{ and } |\mathbf{a}| = |\mathbf{i}|, |\mathbf{b}| = |\mathbf{j}|. \tag{11.54}$$

It implies the decomposition

$$S^{\mathrm{tr}_2}(z) = \bigoplus_{0 \le l \le n, m \ge 0} S^{\mathrm{tr}_2}_{l,m}(z), \qquad S^{\mathrm{tr}_2}_{l,m}(z) \in \mathrm{End}(\mathbf{V}_l \otimes \mathbf{W}_m). \tag{11.55}$$

The sum (11.52) corresponding to the component $S^{\mathrm{tr}_2}_{l,m}(z)$ is restricted to $\mathbf{a} \in \mathfrak{s}_l$, $\mathbf{b} \in B_m$. The Yang–Baxter relation (11.49) holds on each subspace $\mathbf{V}_k \otimes \mathbf{W}_l \otimes \mathbf{W}_m$.

11.5 Identification with Quantum R Matrices of $A^{(1)}_{n-1}$

Let $U_p(A^{(1)}_{n-1})$ be the quantized universal enveloping algebra of the affine Kac–Moody algebra $A^{(1)}_{n-1}$ corresponding to the Dynkin diagram in Fig. 11.8.

The algebra $U_p(A^{(1)}_{n-1})$ has the generators $e_i, f_i, k^{\pm 1}_i$ $(i \in \mathbb{Z}_n)$ obeying the relations

$$\begin{aligned}
&k_i k_j = k_j k_i, \quad k_i k_i^{-1} = k_i^{-1} k_i = 1,\\
&k_i e_j k_i^{-1} = p^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = p^{-a_{ij}} f_j, \quad [e_i, f_j] = \delta_{ij} \frac{k_i - k_i^{-1}}{p - p^{-1}},\\
&\sum_{r=0}^{1-a_{ij}} (-1)^r e_i^{(r)} e_j e_i^{(1-a_{ij}-r)} = \sum_{r=0}^{1-a_{ij}} (-1)^r f_i^{(r)} f_j f_i^{(1-a_{ij}-r)} = 0 \quad (i \ne j),
\end{aligned} \tag{11.56}$$

where $e^{(m)}_i = e^m_i/[m]_p!$, $f^{(m)}_i = f^m_i/[m]_p!$ with $[m]_p! = \prod_{j=1}^m [j]_p$ and[4]

$$[m]_p = \frac{p^m - p^{-m}}{p - p^{-1}}. \tag{11.57}$$

The indices should be understood as elements of $\mathbb{Z}_n$. Thus the element of the Cartan matrix $a_{ij} = 2\delta_{i,j} - \delta_{i,i+1} - \delta_{i,j-1}$ is $a_{0,n-1} = -1$ for example. Note that $A^{(1)}_1$ is exceptional in that $a_{00} = a_{11} = -a_{01} = -a_{10} = 2$. The algebra $U_p(A_{n-1})$ defined in Sect. 10.1.1 is the subalgebra of $U_p(A^{(1)}_{n-1})$ with generators restricted to $e_i, f_i, k^{\pm 1}_i$ $(0 < i < n)$.

We adopt the coproduct Δ in (10.2) and its opposite Δ^{op} mentioned after (10.12) for e_i, f_i, k_i for all $i \in \mathbb{Z}_n$:

[4] The symbol $[m]_i$ defined after (10.1) is $[m]_{q_i}$ here.

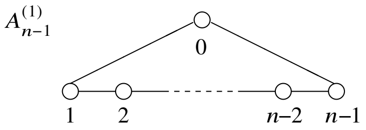


Fig. 11.8 Dynkin diagram of $A^{(1)}_{n-1}$ ($n \geq 3$) with enumeration of vertices. (For $n = 2$, See [67])

$$\Delta(k_i) = k_i \otimes k_i, \quad \Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes k_i^{-1}, \tag{11.58}$$

$$\Delta^{\mathrm{op}}(k_i) = k_i \otimes k_i, \quad \Delta^{\mathrm{op}}(e_i) = 1 \otimes e_i + e_i \otimes k_i, \quad \Delta^{\mathrm{op}}(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i. \tag{11.59}$$

11.5.1 $S^{\mathrm{tr}_3}(z)$

Let $\pi_{\varpi_k,x} : U_p(A^{(1)}_{n-1}) \to \mathrm{End}(\mathbf{V}_k)$ $(0 \leq k \leq n)$ be a p analogue of the degree k anti-symmetric tensor representation with spectral parameter x:

$$e_i v_{\mathbf{m}} = x^{\delta_{i0}} v_{\mathbf{m}+\mathbf{e}_i-\mathbf{e}_{i+1}}, \quad f_i v_{\mathbf{m}} = x^{-\delta_{i0}} v_{\mathbf{m}-\mathbf{e}_i+\mathbf{e}_{i+1}}, \quad k_i v_{\mathbf{m}} = p^{m_i - m_{i+1}} v_{\mathbf{m}}, \tag{11.60}$$

where $\mathbf{m} \in \mathfrak{s}_k$ and $i \in \mathbb{Z}_n$.[5] See Sect. 11.1 for the notations $\mathbf{V}_k$, $v_{\mathbf{m}}$, $\mathfrak{s}_k$ and $\mathbf{e}_i$. In the RHSs, the vector $v_{\mathbf{m}}$ should be understood as 0 if $\mathbf{m} \notin \mathfrak{s}_k$. In the LHSs, e_i for example actually means $\pi_{\varpi_k,x}(e_i)$. A similar simplified notation will be employed in what follows.

As a representation of the classical subalgebra $U_p(A_{n-1}) \subset U_p(A^{(1)}_{n-1})$ without $e_0, f_0, k_0^{\pm}$, the space $\mathbf{V}_k$ is isomorphic to the irreducible highest weight module $V(\varpi_k)$ with highest weight ϖ_k in the notation of Sect. 10.1.1. The highest weight vector is $v_{\mathbf{e}_1+\cdots+\mathbf{e}_k}$.

Let $\Delta_{x,y} = (\pi_{\varpi_l,x} \otimes \pi_{\varpi_m,y}) \circ \Delta$ and $\Delta^{\mathrm{op}}_{x,y} = (\pi_{\varpi_l,x} \otimes \pi_{\varpi_m,y}) \circ \Delta^{\mathrm{op}}$ be the tensor product representations. Let $\mathcal{R}_{\varpi_l,\varpi_m}(z) \in \mathrm{End}(\mathbf{V}_l \otimes \mathbf{V}_m)$ be a quantum R matrix of $U_p(A^{(1)}_{n-1})$ which is characterized, up to normalization, by the commutativity

$$\mathcal{R}_{\varpi_l,\varpi_m}(\tfrac{x}{y})\Delta_{x,y}(g) = \Delta^{\mathrm{op}}_{x,y}(g)\mathcal{R}_{\varpi_l,\varpi_m}(\tfrac{x}{y}) \quad (\forall g \in U_p(A^{(1)}_{n-1})). \tag{11.61}$$

Here we have taken into account the obvious fact that $\mathcal{R}_{\varpi_l,\varpi_m}$ depends only on the ratio x/y. The relation (11.61) is a generalization of $(10.12)|_{q\to p}$ including the latter as the classical part $g \in U_p(A_{n-1})$.

[5] It is a Kirillov–Reshetikhin module $W^{(k)}_1$ up to specification of the spectral parameter.

Theorem 11.3 *Up to normalization, $S^{\mathrm{tr}_3}_{l,m}(z)$ by the matrix product construction (11.31)–(11.32) based on the 3D L (11.14) for arbitrary α coincides with the quantum R matrix as*

$$S^{\mathrm{tr}_3}_{l,m}(z) = \mathcal{R}_{\varpi_l,\varpi_m}(z^{-1}) \quad \text{at} \;\; p = -q^{-1}. \tag{11.62}$$

Proof It suffices to check

$$S^{\mathrm{tr}_3}(\tfrac{y}{x})(e_r \otimes 1 + k_r \otimes e_r) = (1 \otimes e_r + e_r \otimes k_r)S^{\mathrm{tr}_3}(\tfrac{y}{x}), \tag{11.63}$$
$$S^{\mathrm{tr}_3}(\tfrac{y}{x})(1 \otimes f_r + f_r \otimes k_r^{-1}) = (f_r \otimes 1 + k_r^{-1} \otimes f_r)S^{\mathrm{tr}_3}(\tfrac{y}{x}), \tag{11.64}$$
$$S^{\mathrm{tr}_3}(\tfrac{y}{x})(k_r \otimes k_r) = (k_r \otimes k_r)S^{\mathrm{tr}_3}(\tfrac{y}{x}) \tag{11.65}$$

under the image of $\pi_{\varpi_l,x} \otimes \pi_{\varpi_m,y}$ for $r \in \mathbb{Z}_n$. The matrix product structure (11.26) allows us to focus on the rth and the $(r+1)$th components in the representation (11.60). Thus we write $v_{\mathbf{m}}$ simply as $v_{m_r,m_{r+1}}$.

Let us look at the result of application of (11.63) on $v_{\mathbf{i}} \otimes v_{\mathbf{j}}$. The LHS/$\varrho^{\mathrm{tr}_3}(\frac{y}{x})$ leads to

$$\begin{aligned}
&\varrho^{\mathrm{tr}_3}(\tfrac{y}{x})^{-1}S^{\mathrm{tr}_3}(\tfrac{y}{x})(e_r \otimes 1 + k_r \otimes e_r)(v_{i_r,i_{r+1}} \otimes v_{j_r,j_{r+1}}) \\
&= \varrho^{\mathrm{tr}_3}(\tfrac{y}{x})^{-1}S^{\mathrm{tr}_3}(\tfrac{y}{x})(x^{\delta_{r,0}}v_{i_r+1,i_{r+1}-1} \otimes v_{j_r,j_{r+1}} + y^{\delta_{r,0}}p^{i_r-i_{r+1}}v_{i_r,i_{r+1}} \otimes v_{j_r+1,j_{r+1}-1}) \\
&= \sum \mathrm{Tr}(\cdots L^{a_r,b_r}_{i_r+1,j_r}(\tfrac{y}{x})^{\delta_{r,0}\mathbf{h}}L^{a_{r+1},b_{r+1}}_{i_{r+1}-1,j_{r+1}}\cdots)x^{\delta_{r,0}}v_{a_r,a_{r+1}} \otimes v_{b_r,b_{r+1}} \\
&+ \sum \mathrm{Tr}(\cdots L^{a_r,b_r}_{i_r,j_r+1}(\tfrac{y}{x})^{\delta_{r,0}\mathbf{h}}L^{a_{r+1},b_{r+1}}_{i_{r+1},j_{r+1}-1}\cdots)y^{\delta_{r,0}}p^{i_r-i_{r+1}}v_{a_r,a_{r+1}} \otimes v_{b_r,b_{r+1}},
\end{aligned}$$

where $\delta_{r,0} = 1$ at $r = n \in \mathbb{Z}_n$. The RHS/$\varrho^{\mathrm{tr}_3}(\frac{y}{x})$ leads to

$$\begin{aligned}
&(1 \otimes e_r + e_r \otimes k_r)\sum \mathrm{Tr}(\cdots L^{a_r,b_r}_{i_r,j_r}(\tfrac{y}{x})^{\delta_{r,0}\mathbf{h}}L^{a_{r+1},b_{r+1}}_{i_{r+1},j_{r+1}}\cdots)v_{a_r,a_{r+1}} \otimes v_{b_r,b_{r+1}} \\
&= \sum \mathrm{Tr}(\cdots L^{a_r,b_r}_{i_r,j_r}(\tfrac{y}{x})^{\delta_{r,0}\mathbf{h}}L^{a_{r+1},b_{r+1}}_{i_{r+1},j_{r+1}}\cdots) \\
&\quad (y^{\delta_{r,0}}v_{a_r,a_{r+1}} \otimes v_{b_r+1,b_{r+1}-1} + x^{\delta_{r,0}}p^{b_r-b_{r+1}}v_{a_r+1,a_{r+1}-1} \otimes v_{b_r,b_{r+1}}).
\end{aligned}$$

The factors $\cdots$ in the traces are common, therefore the two sides are equal if the coefficients of $v_{a_r,a_{r+1}} \otimes v_{b_r,b_{r+1}}$ agree. This yields

$$\begin{aligned}
&L^{a_r,b_r}_{i_r+1,j_r}(\tfrac{y}{x})^{\delta_{r,0}\mathbf{h}}L^{a_{r+1},b_{r+1}}_{i_{r+1}-1,j_{r+1}}x^{\delta_{r,0}} + L^{a_r,b_r}_{i_r,j_r+1}(\tfrac{y}{x})^{\delta_{r,0}\mathbf{h}}L^{a_{r+1},b_{r+1}}_{i_{r+1},j_{r+1}-1}y^{\delta_{r,0}}p^{i_r-i_{r+1}} \\
&= L^{a_r,b_r-1}_{i_r,j_r}(\tfrac{y}{x})^{\delta_{r,0}\mathbf{h}}L^{a_{r+1},b_{r+1}+1}_{i_{r+1},j_{r+1}}y^{\delta_{r,0}} + L^{a_r-1,b_r}_{i_r,j_r}(\tfrac{y}{x})^{\delta_{r,0}\mathbf{h}}L^{a_{r+1}+1,b_{r+1}}_{i_{r+1},j_{r+1}}x^{\delta_{r,0}}p^{b_r-b_{r+1}}.
\end{aligned}$$

Thanks to (3.104), the dependence on x and y becomes a common overall factor even for $r = 0$, leaving a quadratic relation

$$L^{a,b}_{i+1,j}L^{a',b'}_{i'-1,j'} + L^{a,b}_{i,j+1}L^{a',b'}_{i',j'-1}p^{i-i'} = L^{a,b-1}_{i,j}L^{a',b'+1}_{i',j'} + L^{a-1,b}_{i,j}L^{a'+1,b'}_{i',j'}p^{b-b'}, \tag{11.66}$$

where $a_r, a_{r+1}, b_r, b_{r+1}, i_r, i_{r+1}, j_r, j_{r+1}$ are denoted by $a, a', b, b', i, i', j, j'$ for simplicity. Let us consider the case $(a, a', b, b', i, i', j, j') = (0, 0, 1, 1, 0, 1, 0, 1)$ for example. Since $L^{ab}_{ij} = 0$ unless $a, b, i, j \in \{0, 1\}$, the RHS is zero. From (11.14), the LHS is

$$\begin{aligned} L^{01}_{10}L^{01}_{01} + L^{01}_{01}L^{01}_{10}p^{-1} &= \mathbf{a}^-(-q\alpha^{-1}\mathbf{k}) + (-q\alpha^{-1}\mathbf{k})\mathbf{a}^- p^{-1} \\ &= \mathbf{a}^-(-q\alpha^{-1}\mathbf{k})(1 + q^{-1}p^{-1}), \end{aligned}$$

which certainly vanishes at $p = -q^{-1}$. Similarly, one can check that all the 2^8 equations (11.66) are valid if and only if $p = -q^{-1}$. The relation (11.64) can be verified in the same manner. To check (11.65) is much simpler. □

Remark 11.4 The trace reduction is depicted by a circle in Fig. 11.3. It corresponds to the Dynkin diagram of $A^{(1)}_{n-1}$ $(n \geq 3)$ in Fig. 11.8. Further intriguing correspondence of this kind will be observed in Remarks 12.3 and 14.3.

11.5.2 $S^{\mathrm{tr}_1}(z)$

Here we consider $S^{\mathrm{tr}_1}(z)$, therefore $\alpha = 1$ is assumed in (11.14) as stated in the beginning of Sect. 11.3.

Let $\pi_{k\varpi_1,x} : U_p(A^{(1)}_{n-1}) \to \mathrm{End}(\mathbf{W}_k)$ $(k \geq 0)$ be the p-analogue of the degree k symmetric tensor representation with parameter x:

$$\begin{aligned} e_i|\mathbf{m}\rangle &= x^{\delta_{i0}} p^{m_i - m_{i+1}+1}[m_{i+1}]_p|\mathbf{m} + \mathbf{e}_i - \mathbf{e}_{i+1}\rangle, \\ f_i|\mathbf{m}\rangle &= x^{-\delta_{i0}} p^{m_{i+1} - m_i+1}[m_i]_p|\mathbf{m} - \mathbf{e}_i + \mathbf{e}_{i+1}\rangle, \\ k_i|\mathbf{m}\rangle &= p^{m_i - m_{i+1}}|\mathbf{m}\rangle, \end{aligned} \qquad (11.67)$$

where $\mathbf{m} \in B_k$ and $i \in \mathbb{Z}_n$.[6] See Sect. 11.1 for the notations $\mathbf{W}_k$, $|\mathbf{m}\rangle$, B_k, $\mathbf{e}_i$ and (11.57) for $[m]_p$. In the RHSs, the vector $|\mathbf{m}\rangle$ should be understood as 0 if $\mathbf{m} \notin B_k$. As a representation of the classical subalgebra $U_p(A_{n-1}) \subset U_p(A^{(1)}_{n-1})$ without $e_0, f_0, k^{\pm}_0$, the space $\mathbf{W}_k$ is isomorphic to the irreducible highest weight module $V(k\varpi_1)$ with highest weight $k\varpi_1$ in the notation of Sect. 10.1.1. The highest weight vector is $|k\mathbf{e}_1\rangle$.

Let $\Delta_{x,y} = (\pi_{\varpi_l,x} \otimes \pi_{m\varpi_1,y}) \circ \Delta$ and $\Delta^{\mathrm{op}}_{x,y} = (\pi_{\varpi_l,x} \otimes \pi_{m\varpi_1,y}) \circ \Delta^{\mathrm{op}}$ be the tensor product representations, where Δ and Δ^{op} are specified in (11.58) and (11.59).

Let $\mathcal{R}_{\varpi_l,m\varpi_1}(z) \in \mathrm{End}(\mathbf{V}_l \otimes \mathbf{W}_m)$ be a quantum R matrix of $U_p(A^{(1)}_{n-1})$ which is characterized, up to normalization, by the commutativity

$$\mathcal{R}_{\varpi_l,m\varpi_1}(\tfrac{x}{y})\Delta_{x,y}(g) = \Delta^{\mathrm{op}}_{x,y}(g)\mathcal{R}_{\varpi_l,m\varpi_1}(\tfrac{x}{y}) \quad (\forall g \in U_p(A^{(1)}_{n-1})). \qquad (11.68)$$

[6] It is a Kirillov–Reshetikhin module $W^{(1)}_k$ up to specification of the spectral parameter.

Theorem 11.5 *Up to normalization, $S^{\mathrm{tr}_1}_{l,m}(z)$ by the matrix product construction (11.41)–(11.46) based on the 3D L (11.14) with $\alpha = 1$ coincides with the quantum R matrix as*

$$S^{\mathrm{tr}_1}_{l,m}(z) = \mathcal{R}_{\varpi_l, m\varpi_1}(z) \quad \text{at} \;\; p = q^{-1}. \tag{11.69}$$

Proof It suffices to check

$$S^{\mathrm{tr}_1}(\tfrac{x}{y})(e_r \otimes 1 + k_r \otimes e_r) = (1 \otimes e_r + e_r \otimes k_r)S^{\mathrm{tr}_3}(\tfrac{x}{y}), \tag{11.70}$$

$$S^{\mathrm{tr}_1}(\tfrac{x}{y})(1 \otimes f_r + f_r \otimes k_r^{-1}) = (f_r \otimes 1 + k_r^{-1} \otimes f_r)S^{\mathrm{tr}_3}(\tfrac{x}{y}), \tag{11.71}$$

$$S^{\mathrm{tr}_1}(\tfrac{x}{y})(k_r \otimes k_r) = (k_r \otimes k_r)S^{\mathrm{tr}_3}(\tfrac{x}{y}) \tag{11.72}$$

under the image of $\pi_{\varpi_l, x} \otimes \pi_{m\varpi_1, y}$ for $r \in \mathbb{Z}_n$. The matrix product structure (11.44) allows us to focus on the rth and the $(r+1)$th components in the representations (11.60) and (11.67). Thus we write $v_{\mathbf{j}} \otimes |\mathbf{m}\rangle$ simply as $v_{j_r, j_{r+1}} \otimes |m_r, m_{r+1}\rangle$. Then from (11.44), the action of $S^{\mathrm{tr}_1}(z)$ is described as

$$\begin{aligned} &S^{\mathrm{tr}_1}(z)(v_{j_r, j_{r+1}} \otimes |m_r, m_{r+1}\rangle) \\ &= \sum_{\mathbf{i}, \mathbf{b} \in \mathfrak{s}, \mathbf{l} \in B} (\cdots L^{i_{r-1}, b_r, l_r}_{i_r, j_r, m_r} z^{i_r \delta_{r,0}} L^{i_r, b_{r+1}, l_{r+1}}_{i_{r+1}, j_{r+1}, m_{r+1}} \cdots) v_{b_r, b_{r+1}} \otimes |l_r, l_{r+1}\rangle. \end{aligned} \tag{11.73}$$

See Sect. 11.1 for the definition of $\mathfrak{s}$ and B. Let us compare the coefficients of the transition $v_{j_r, j_{r+1}} \otimes |m_r, m_{r+1}\rangle \mapsto v_{b_r, b_{r+1}} \otimes |l_r, l_{r+1}\rangle$ by the two sides of (11.70). Omitting the common factors denoted by $\cdots$ in (11.73), we are to show

$$\begin{aligned} &\sum_{i=0,1} (\tfrac{x}{y})^{i\delta_{r,0}} (x^{\delta_{r,0}} L^{a,b,l}_{i,j+1,m} L^{i,b',l'}_{a',j'-1,m'} + y^{\delta_{r,0}} p^{j-j'+m-m'+1}[m']_p L^{a,b,l}_{i,j,m+1} L^{i,b',l'}_{a',j',m'-1}) \\ &= \sum_{i=0,1} (\tfrac{x}{y})^{i\delta_{r,0}} (y^{\delta_{r,0}} p^{l-l'-1}[l'+1]_p L^{a,b,l-1}_{i,j,m} L^{i,b',l'+1}_{a',j',m'} + x^{\delta_{r,0}} p^{l-l'} L^{a,b-1,l}_{i,j,m} L^{i,b'+1,l'}_{a',j',m'}), \end{aligned} \tag{11.74}$$

where $i_{r-1}, i_r, i_{r+1}, b_r, b_{r+1}, j_r, j_{r+1}, l_r, l_{r+1}, m_r, m_{r+1}$ are denoted by a, i,a',b, b', j, j', l, l', m, m', respectively for simplicity. This splits into three equations for the coefficients of $x^{\delta_{r,0}}$, $y^{\delta_{r,0}}$ and $(\frac{x^2}{y})^{\delta_{r,0}}$. Explicitly they read as

$$\begin{aligned} &L^{a,b,l}_{0,j+1,m} L^{0,b',l'}_{a',j'-1,m'} + p^{j-j'+m-m'+1}[m']_p L^{a,b,l}_{1,j,m+1} L^{1,b',l'}_{a',j',m'-1} \\ &\quad = p^{l-l'-1}[l'+1]_p L^{a,b,l-1}_{1,j,m} L^{1,b',l'+1}_{a',j',m'} + p^{l-l'} L^{a,b-1,l}_{0,j,m} L^{0,b'+1,l'}_{a',j',m'}, \end{aligned} \tag{11.75}$$

$$p^{j-j'+m-m'+1}[m']_p L^{a,b,l}_{0,j,m+1} L^{0,b',l'}_{a',j',m'-1} = p^{l-l'-1}[l'+1]_p L^{a,b,l-1}_{0,j,m} L^{0,b',l'+1}_{a',j',m'}, \tag{11.76}$$

$$L^{a,b,l}_{1,j+1,m} L^{1,b',l'}_{a',j'-1,m'} = p^{l-l'} L^{a,b-1,l}_{1,j,m} L^{1,b'+1,l'}_{a',j',m'}. \tag{11.77}$$

A non-trivial case is $(a, b, i, a', b', j') = (1, 1, 0, 1, 0, 1)$ and $(l, m) = (l', m')$ of (11.77), which reads as $q^{l'} = p^{l-l'} q^{l}$ by (3.105). This enforces $p = q^{-1}$. Under this identification, the other relations are confirmed straightforwardly. The relations (11.71) and (11.72) can be checked similarly. □

11.5.3 $S^{\mathrm{tr}_2}(z)$

Here we consider $S^{\mathrm{tr}_2}(z)$, therefore $\alpha = 1$ assumed in (11.14) as stated in the beginning of Sect. 11.4.

Let $\pi'_{\varpi_k,x} : U_p(A^{(1)}_{n-1}) \to \mathrm{End}(\mathbf{V}_k)$ $(0 \le k \le n)$ be a representation with spectral parameter x:

$$e_i v_{\mathbf{m}} = x^{\delta_{i0}} v_{\mathbf{m}-\mathbf{e}_i+\mathbf{e}_{i+1}}, \quad f_i v_{\mathbf{m}} = x^{-\delta_{i0}} v_{\mathbf{m}+\mathbf{e}_i-\mathbf{e}_{i+1}}, \quad k_i v_{\mathbf{m}} = p^{m_{i+1}-m_i} v_{\mathbf{m}}, \tag{11.78}$$

where $\mathbf{m} \in \mathfrak{s}_k$ and $i \in \mathbb{Z}_n$. In the RHSs, the vector $v_{\mathbf{m}}$ should be understood as 0 if $\mathbf{m} \notin \mathfrak{s}_k$. Note the difference from $\pi_{\varpi_k,x}$ in (11.60). As a representation of the classical subalgebra $U_p(A_{n-1}) \subset U_p(A^{(1)}_{n-1})$ without $e_0, f_0, k_0^{\pm}$, the representation $\pi'_{\varpi_k,x}$ is equivalent to $\pi_{\varpi_{n-k},x}$. In fact, the relation

$$\pi'_{\varpi_k,x}(g) = \iota \circ \pi_{\varpi_{n-k},x}(g) \circ \iota \quad (g \in U_p(A^{(1)}_{n-1})), \tag{11.79}$$

$$\iota(v_{\mathbf{m}}) = v_{\mathbf{m}'}, \quad \mathbf{m}' = (1 - m_1, \dots, 1 - m_n), \tag{11.80}$$

holds for $\mathbf{m} = (m_1, \dots, m_n)$.[7]

Let $\Delta_{x,y} = (\pi'_{\varpi_l,x} \otimes \pi_{m\varpi_1,y}) \circ \Delta$ and $\Delta^{\mathrm{op}}_{x,y} = (\pi'_{\varpi_l,x} \otimes \pi_{m\varpi_1,y}) \circ \Delta^{\mathrm{op}}$ be the tensor product representations, where Δ and Δ^{op} are defined in (11.58) and (11.59).

Let $\mathcal{R}'_{\varpi_l,m\varpi_1}(z) \in \mathrm{End}(\mathbf{V}_l \otimes \mathbf{W}_m)$ be a quantum R matrix of $U_p(A^{(1)}_{n-1})$ which is characterized, up to normalization, by the commutativity

$$\mathcal{R}'_{\varpi_l,m\varpi_1}(\tfrac{x}{y}) \Delta_{x,y}(g) = \Delta^{\mathrm{op}}_{x,y}(g) \mathcal{R}'_{\varpi_l,m\varpi_1}(\tfrac{x}{y}) \quad (\forall g \in U_p(A^{(1)}_{n-1})). \tag{11.81}$$

It is simply related to the quantum R matrix $\mathcal{R}_{\varpi_{n-l},m\varpi_1}(z)$ defined as the intertwiner of $(\pi_{\varpi_{n-l},x} \otimes \pi_{m\varpi_1,y}) \circ \Delta$ and $(\pi_{\varpi_{n-l},x} \otimes \pi_{m\varpi_1,y}) \circ \Delta^{\mathrm{op}}$ by the involution ι as

$$\mathcal{R}'_{\varpi_l,m\varpi_1}(z) = (\iota \otimes 1) \mathcal{R}_{\varpi_{n-l},m\varpi_1}(z) (\iota \otimes 1) \tag{11.82}$$

up to normalization.

[7] The notation $\mathbf{m}'$ was also used earlier in (6.4).

Theorem 11.6 *Up to normalization, $S^{\mathrm{tr}_2}_{l,m}(z)$ by the matrix product construction (11.50)–(11.55) based on the 3D L (11.14) with $\alpha = 1$ coincides with the quantum R matrix as*

$$S^{\mathrm{tr}_2}_{l,m}(z) = \mathcal{R}'_{\varpi_l, m\varpi_1}(z^{-1}) \quad \text{at} \;\; p = q^{-1}. \tag{11.83}$$

We omit the proof since it is quite parallel with that for Theorem 11.5.

11.6 Commuting Layer Transfer Matrices and Duality

Let $m, n \geq 2$ and consider the composition of $m \times n$ 3D L's as in Fig. 11.9.

At the intersection of 1_i and 2_j, we have the 3D L $L_{1_i,2_j,3_{ij}}$ as in Fig. 11.1, where the label 3_{ij} corresponds to the vertical blue arrows carrying $\mathcal{F}_q$. We take the parameters μ_i and ν_j as

$$\mu_i = xu_i \; (i = 1, \ldots, m), \qquad \nu_j = yw_j \; (j = 1, \ldots, n). \tag{11.84}$$

Tracing out the horizontal degrees of freedom leaves us with a linear operator acting vertically along the blue arrows. We write the resulting *layer* transfer matrix as

$$T(x, y) = T(x, y|\mathbf{u}, \mathbf{w}) \in \mathrm{End}(\mathcal{F}_q^{\otimes mn}), \tag{11.85}$$

$$\mathbf{u} = (u_1, \ldots, u_m), \quad \mathbf{w} = (w_1, \ldots, w_n). \tag{11.86}$$

Figure 11.9 shows its action on the basis $\bigotimes_{1 \leq i \leq m, 1 \leq j \leq n} |l_{ij}\rangle \in \mathcal{F}_q^{\otimes mn}$.

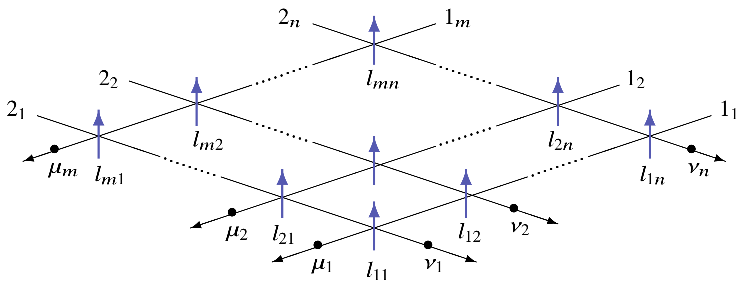


Fig. 11.9 Graphical representation of the layer transfer matrix $T(x, y)$. There are m horizontal arrows $1_1, \ldots, 1_m$ carrying $V \simeq \mathbb{C}^2$ and being traced out, which corresponds to the periodic boundary condition. The mark • with μ_i signifies an operator $\mu_i^{\mathbf{h}}$ attached to 1_i. Similarly, there are also n horizontal arrows $2_1, \ldots, 2_n$ which are to be traced out including the operator $\nu_j^{\mathbf{h}}$. At the intersection of 1_i and 2_j, there is a q-oscillator Fock space $\mathcal{F}_q$ depicted with a blue arrow

We exhibit the n-dependence in the notations prepared in Sect. 11.1 as $\mathfrak{s}^{(n)}, \mathbf{V}^{(n)}, \mathbf{V}_k^{(n)}$, etc. In what follows, $\mathbf{u}^H$ for $\mathbf{u} \in \mathbb{C}^m$ should be understood as the linear diagonal operator $u_1^{\mathbf{h}_1} \cdots u_m^{\mathbf{h}_m}$, i.e.

$$\mathbf{u}^H : v_{\mathbf{a}} \mapsto u_1^{a_1} \cdots u_m^{a_m} v_{\mathbf{a}} \quad \text{for} \quad \mathbf{a} = (a_1, \ldots, a_m) \in \mathfrak{s}^{(m)} \tag{11.87}$$

and similarly for $x\mathbf{u} \in \mathbb{C}^m$ and $\mathbf{w}, y\mathbf{w} \in \mathbb{C}^n$.[8]

Viewing Fig. 11.9 from the SW, or taking the traces over $1_1, \ldots, 1_m$ first, we find that it represents the trace of the product of $(y\mathbf{w})^H$ and $S^{\mathrm{tr}_1}(\mu_1), \ldots, S^{\mathrm{tr}_1}(\mu_m)$:

$$\begin{aligned} T(x,y) &= \mathrm{Tr}_{\mathbf{V}^{(n)}}\big((y\mathbf{w})^H S^{\mathrm{tr}_1}(xu_1) \cdots S^{\mathrm{tr}_1}(xu_m)\big) \\ &= \sum_{k=0}^{n} y^k \mathrm{Tr}_{\mathbf{V}_k^{(n)}}\big(\mathbf{w}^H S^{\mathrm{tr}_1}(xu_1) \cdots S^{\mathrm{tr}_1}(xu_m)\big) \in \mathrm{End}\big((\mathbf{W}^{(n)})^{\otimes m}\big), \end{aligned} \tag{11.88}$$

where $S^{\mathrm{tr}_1}(xu_i) \in \mathrm{End}(\overset{\mathbf{2}}{\mathbf{V}}{}^{(n)} \otimes \mathbf{W}^{(n)})$ is a quantum R matrix of $U_{q^{-1}}(A_{n-1}^{(1)})$ due to Theorem 11.5 and (11.46). The product is taken with respect to $\overset{\mathbf{2}}{\mathbf{V}}{}^{(n)} = \overset{2_1}{V} \otimes \cdots \otimes \overset{2_n}{V}$ in Fig. 11.9, which corresponds to the first (left) component of S^{tr_1}'s.

Alternatively, Fig. 11.9 viewed from the SE or first taking the traces over $2_1, \ldots, 2_n$ gives rise to the trace of the product of $(x\mathbf{u})^H$ and $S^{\mathrm{tr}_2}(\nu_1), \ldots, S^{\mathrm{tr}_2}(\nu_n)$, namely,

$$\begin{aligned} T(x,y) &= \mathrm{Tr}_{\mathbf{V}^{(m)}}\big((x\mathbf{u})^H S^{\mathrm{tr}_2}(yw_1) \cdots S^{\mathrm{tr}_2}(yw_n)\big) \\ &= \sum_{k=0}^{m} x^k \mathrm{Tr}_{\mathbf{V}_k^{(m)}}\big(\mathbf{u}^H S^{\mathrm{tr}_2}(yw_1) \cdots S^{\mathrm{tr}_2}(yw_n)\big) \in \mathrm{End}\big((\mathbf{W}^{(m)})^{\otimes n}\big), \end{aligned} \tag{11.89}$$

where $S^{\mathrm{tr}_2}(yw_j) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}}{}^{(m)} \otimes \mathbf{W}^{(m)})$ is a quantum R matrix of $U_{q^{-1}}(A_{m-1}^{(1)})$ due to Theorem 11.6 and (11.55). The product is taken with respect to $\overset{\mathbf{1}}{\mathbf{V}}{}^{(m)} = \overset{1_1}{V} \otimes \cdots \otimes \overset{1_m}{V}$ in Fig. 11.9, which corresponds to the first (left) component of S^{tr_2}'s.

The identifications (11.88) and (11.89) correspond to the two complementary pictures $\mathcal{F}_q^{\otimes mn} = (\mathbf{W}^{(n)})^{\otimes m} = (\mathbf{W}^{(m)})^{\otimes n}$. In either case, $S^{\mathrm{tr}_1}(z)$ and $S^{\mathrm{tr}_2}(z)$ satisfy the Yang–Baxter equations,[9] which imply the two-parameter commutativity

$$[T(x,y), T(x',y')] = 0 \tag{11.90}$$

for *fixed* $\mathbf{u}$ and $\mathbf{w}$.

Due to the weight conservations (11.45) and (11.54), the space $\mathcal{F}_q^{\otimes mn}$ decomposes into many invariant subspaces under $T(x,y)$ as

[8] For H we do not exhibit the number of components m, n as $H^{(m)}$ or $H^{(n)}$ for simplicity.

[9] They are not (11.39) or (11.49) but those valid in $\mathrm{End}(\mathbf{V} \otimes \mathbf{V} \otimes \mathbf{W})$, which we omit here.

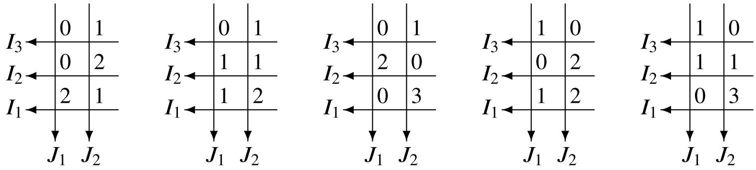


Fig. 11.10 Five base vectors of $\mathcal{F}_q^{\otimes 6}$ in the sector $I = (3, 2, 1)$ and $J = (2, 4)$ are shown as the configurations of $\{l_{ij} \mid i = 1, 2, 3, j = 1, 2\}$ of Fig. 11.9 viewed from the top

$$\mathcal{F}_q^{\otimes mn} = \bigoplus_{I,J} \mathbf{F}_{I,J}, \tag{11.91}$$

$$\mathbf{F}_{I,J} = \bigoplus_{l_{ij} \geq 0} \mathbb{C}|l_{11}\rangle \otimes \cdots \otimes |l_{mn}\rangle, \tag{11.92}$$

where the sum (11.91) is taken over $I = (I_1, \ldots, I_m) \in B^{(m)}$ and $J = (J_1, \ldots, J_n) \in B^{(n)}$ with only one constraint

$$I_1 + \cdots + I_m = J_1 + \cdots + J_n, \tag{11.93}$$

which is the total number of q-oscillator excitations. The sum (11.92) extends over $l_{ij} \in \mathbb{Z}_{\geq 0}$ under the $m + n$ constraints

$$l_{i1} + \cdots + l_{in} = I_i \quad (i = 1, \ldots, m), \quad l_{1j} + \cdots + l_{mj} = J_j \quad (j = 1, \ldots, n). \tag{11.94}$$

Example 11.7 When $(m, n) = (3, 2)$, $I = (3, 2, 1)$ and $J = (2, 4)$, we have $\dim \mathbf{F}_{I,j} = 5$ whose bases are given as follows:

According to (11.91), we have the decompositions (Fig. 11.10).

$$\mathcal{F}_q^{\otimes mn} = \bigoplus_{I_1,\ldots,I_m \geq 0} \mathbf{W}_{I_1}^{(n)} \otimes \cdots \otimes \mathbf{W}_{I_m}^{(n)} = \bigoplus_{J_1,\ldots,J_n \geq 0} \mathbf{W}_{J_1}^{(m)} \otimes \cdots \otimes \mathbf{W}_{J_n}^{(m)}. \tag{11.95}$$

Correspondingly, each summand in (11.88) and (11.89) is further decomposed as

$$\begin{aligned} &\mathrm{Tr}_{\mathbf{V}_k^{(n)}} \left(\mathbf{w}^H S^{\mathrm{tr}_1}(xu_1) \cdots S^{\mathrm{tr}_1}(xu_m) \right) \\ &= \bigoplus_{I_1,\ldots,I_m \geq 0} \mathrm{Tr}_{\mathbf{V}_k^{(n)}} \left(\mathbf{w}^H S^{\mathrm{tr}_1}_{k,I_1}(xu_1) \cdots S^{\mathrm{tr}_1}_{k,I_m}(xu_m) \right), \end{aligned} \tag{11.96}$$

$$\begin{aligned} &\mathrm{Tr}_{\mathbf{V}_k^{(m)}} \left(\mathbf{u}^H S^{\mathrm{tr}_2}(yw_1) \cdots S^{\mathrm{tr}_2}(yw_n) \right) \\ &= \bigoplus_{J_1,\ldots,J_n \geq 0} \mathrm{Tr}_{\mathbf{V}_k^{(m)}} \left(\mathbf{u}^H S^{\mathrm{tr}_2}_{k,J_1}(yw_1) \cdots S^{\mathrm{tr}_2}_{k,J_n}(yw_n) \right). \end{aligned} \tag{11.97}$$

In the terminology of the quantum inverse scattering method, each summand in the RHS of (11.96) is a row transfer matrix of the $U_{q^{-1}}(A^{(1)}_{n-1})$ vertex model of size m whose auxiliary space is $\mathbf{V}^{(n)}_k$ and the quantum space is $\mathbf{W}^{(n)}_{l_1} \otimes \cdots \otimes \mathbf{W}^{(n)}_{l_m}$ having the spectral parameter x with inhomogeneity $u_1, \ldots, u_m$ and the "horizontal" boundary electric/magnetic field $\mathbf{w}$.[10] It forms a commuting family with respect to x provided that the other parameters are fixed. In the dual picture (11.97), the role of these data is interchanged as $m \leftrightarrow n$, $x \leftrightarrow y$, $\mathbf{u} \leftrightarrow \mathbf{w}$. This is an example of duality between rank and size, spectral inhomogeneity and field, which follows directly from the 3D picture and the matrix product constructions. It is highly non-trivial from the viewpoint of the Bethe ansatz.

Remark 11.8 Consider the cube of size $l \times m \times n$ formed by concatenating Fig. 11.9 vertically for l times. One can formulate two further versions of the duality corresponding to the interchanges $l \leftrightarrow m$ and $l \leftrightarrow n$.

11.7 Bibliographical Notes and Comments

The idea of generating infinitely many solutions to the Yang–Baxter equation from trace reductions of n-concatenation of the tetrahedron equations was implemented in [18]. The present chapter presents the proofs, the precise U_p module structures (11.60), (11.67) and (11.78) and the relation $p = \pm q^{-1}$ of the two essential parameters q from Osc_q and p from U_p such that the solutions are characterized as the quantum R matrices in the standard framework of the quantum group theory [63]. See also [64, p. 540]. The matrix product formula (11.26) has an application to the multispecies totally asymmetric simple exclusion process [89]. See Remark 18.6. The matrix product construction of $S^{\mathrm{tr}_1}(z)$ in (11.41) has appeared as $\mathbf{L_{Vb}}$ in [18, Eq. (39)]. The literary and Australian nomenclature "BBQ stick" for the diagram in Fig. 11.3 is due to the second author of [18]. In Fig. 12.1 in the next chapter, we will encounter more ordinary BBQ sticks which are not cyclically closed but have two ends. The duality in Sect. 11.6 was formulated in [18, Sect. 6]. It implies the non-trivial equality of the Bethe ansatz eigenvalue formulas based on $U_p(A^{(1)}_{n-1})$ and $U_p(A^{(1)}_{m-1})$. The layer transfer matrices in this chapter correspond to the periodic boundary condition. There are other boundary conditions leading to the layer transfer matrices obeying a rich family of *bilinear identities* which contain the commutativity as a special case. Such a topic will be treated in Sect. 18.4.

[10] This terminology originates in the fact that the simplest case of $n = 2, k = \forall l_i = 1$ known as the six-vertex model is regarded as a ferroelectric model in a field [10, Sect. 8.12] and the relevant XXZ spin chain is a model of magnetism.

Chapter 12
Boundary Vector Reductions of $RLLL = LLLR$

Abstract This chapter presents yet another reduction of an n-concatenation of the tetrahedron equation $RLLL = LLLR$ different from the previous chapter. We eliminate the 3D R not by taking the trace but by evaluation with respect to the boundary vectors using Proposition 3.28. We call it the boundary vector reduction. In contrast to the trace reduction that led to the quantum R matrices of $U_{\pm q^{-1}}(A^{(1)}_{n-1})$ (Chap. 11), it leads to the quantum R matrices for the spin representations of $U_{-q^{-1}}(B^{(1)}_n)$, $U_{-q^{-1}}(D^{(1)}_n)$ and $U_{-q^{-1}}(D^{(2)}_{n+1})$. These algebras have Dynkin diagrams with double outward arrows or double branches. It turns out that the two kinds of the boundary vectors correspond to the two choices of the end shape of the relevant Dynkin diagrams. For simplicity, we treat the reduction with respect to the q-oscillator Fock space only.

12.1 Boundary Vector Reductions

We retain the notations $\mathfrak{s}$, $\mathfrak{s}_\pm$, V, $\mathbf{V}$, $\mathbf{V}_\pm$, $v_{\mathbf{a}}$ etc. and 3D L in Sect. 11.1:

$$\begin{pmatrix} L^{00}_{00} & L^{00}_{01} & L^{00}_{10} & L^{00}_{11} \\ L^{01}_{00} & L^{01}_{01} & L^{01}_{10} & L^{01}_{11} \\ L^{10}_{00} & L^{10}_{01} & L^{10}_{10} & L^{10}_{11} \\ L^{11}_{00} & L^{11}_{01} & L^{11}_{10} & L^{11}_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -q\alpha^{-1}\mathbf{k} & \mathbf{a}^- & 0 \\ 0 & \mathbf{a}^+ & \alpha\mathbf{k} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{12.1}$$

In (11.21) we have obtained an n-concatenation of the tetrahedron equation $RLLL = LLLR$ as

$$\begin{aligned} &R_{456}\, x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5} y^{\mathbf{h}_6} (L_{1_1 2_1 4} \cdots L_{1_n 2_n 4})(L_{1_1 3_1 5} \cdots L_{1_n 3_n 5})(L_{2_1 3_1 6} \cdots L_{2_n 3_n 6}) R_{456} \\ &\quad = x^{\mathbf{h}_4}(xy)^{\mathbf{h}_5} y^{\mathbf{h}_6} (L_{2_1 3_1 6} \cdots L_{2_n 3_n 6})(L_{1_1 3_1 5} \cdots L_{1_n 3_n 5})(L_{1_1 2_1 4} \cdots L_{1_n 2_n 4}). \end{aligned} \tag{12.2}$$

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Recall the boundary vectors (3.132) and (3.133) given as[1]

$$\langle \eta_r| = \sum_{m\ge 0} \frac{\langle rm|}{(q^{r^2})_m}, \qquad |\eta_r\rangle = \sum_{m\ge 0} \frac{|rm\rangle}{(q^{r^2})_m} \qquad (r = 1, 2). \tag{12.3}$$

Sandwich (12.2) between them as

$$(\overset{4}{\langle \eta_r|} \otimes \overset{5}{\langle \eta_r|} \otimes \overset{6}{\langle \eta_r|})(\cdots)(\overset{4}{|\eta_{r'}\rangle} \otimes \overset{5}{|\eta_{r'}\rangle} \otimes \overset{6}{|\eta_{r'}\rangle}) \qquad (r, r' = 1, 2). \tag{12.4}$$

Thanks to Proposition 3.28, the two R_{456}'s disappear, leading to

$$\begin{aligned} &\langle \overset{4}{\eta_r}|x^{\mathbf{h}_4} L_{1_1 2_1 4} \cdots L_{1_n 2_n 4}|\overset{4}{\eta_{r'}}\rangle \langle \overset{5}{\eta_r}|(xy)^{\mathbf{h}_5} L_{1_1 3_1 5} \cdots L_{1_n 3_n 5}|\overset{5}{\eta_{r'}}\rangle \times \\ &\quad \times \langle \overset{6}{\eta_r}|y^{\mathbf{h}_6} L_{2_1 3_1 6} \cdots L_{2_n 3_n 6}|\overset{6}{\eta_{r'}}\rangle \\ &= \langle \overset{6}{\eta_r}|y^{\mathbf{h}_6} L_{2_1 3_1 6} \cdots L_{2_n 3_n 6}|\overset{6}{\eta_{r'}}\rangle \langle \overset{5}{\eta_r}|(xy)^{\mathbf{h}_5} L_{1_1 3_1 5} \cdots L_{1_n 3_n 5}|\overset{5}{\eta_{r'}}\rangle \times \\ &\quad \times \langle \overset{4}{\eta_r}|x^{\mathbf{h}_4} L_{1_1 2_1 4} \cdots L_{1_n 2_n 4}|\overset{4}{\eta_{r'}}\rangle. \end{aligned} \tag{12.5}$$

Let us denote the operators appearing here by

$$\begin{aligned} S^{r,r'}_{\mathbf{1},\mathbf{2}}(z) &= \varrho^{r,r'}(z) \langle \overset{4}{\eta_r}|z^{\mathbf{h}_4} L_{1_1 2_1 4} \cdots L_{1_n 2_n 4}|\overset{4}{\eta_{r'}}\rangle \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}} \otimes \overset{\mathbf{2}}{\mathbf{V}}), \\ S^{r,r'}_{\mathbf{1},\mathbf{3}}(z) &= \varrho^{r,r'}(z) \langle \overset{5}{\eta_r}|z^{\mathbf{h}_5} L_{1_1 3_1 5} \cdots L_{1_n 3_n 5}|\overset{5}{\eta_{r'}}\rangle \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}} \otimes \overset{\mathbf{3}}{\mathbf{V}}), \\ S^{r,r'}_{\mathbf{2},\mathbf{3}}(z) &= \varrho^{r,r'}(z) \langle \overset{6}{\eta_r}|z^{\mathbf{h}_6} L_{2_1 3_1 6} \cdots L_{2_n 3_n 6}|\overset{6}{\eta_{r'}}\rangle \in \mathrm{End}(\overset{\mathbf{2}}{\mathbf{V}} \otimes \overset{\mathbf{3}}{\mathbf{V}}), \end{aligned} \tag{12.6}$$

where $r, r' = 1, 2$. The normalization factor $\varrho^{r,r'}(z)$ will be specified in (12.15). They are the same operators acting on different copies of $\overset{\mathbf{1}}{\mathbf{V}}, \overset{\mathbf{2}}{\mathbf{V}}, \overset{\mathbf{3}}{\mathbf{V}}$ of $\mathbf{V}$ in (11.5) and (11.6). Now the relation (12.5) is stated as the Yang–Baxter equation:

$$S^{r,r'}_{\mathbf{1},\mathbf{2}}(x) S^{r,r'}_{\mathbf{1},\mathbf{3}}(xy) S^{r,r'}_{\mathbf{2},\mathbf{3}}(y) = S^{r,r'}_{\mathbf{2},\mathbf{3}}(y) S^{r,r'}_{\mathbf{1},\mathbf{3}}(xy) S^{r,r'}_{\mathbf{1},\mathbf{2}}(x) \quad (r, r' = 1, 2). \tag{12.7}$$

Suppressing the labels **1**, **2** etc., we set

$$S^{r,r'}(z)(v_{\mathbf{i}} \otimes v_{\mathbf{j}}) = \sum_{\mathbf{a},\mathbf{b} \in \mathfrak{s}} S^{r,r'}(z)^{\mathbf{ab}}_{\mathbf{ij}} v_{\mathbf{a}} \otimes v_{\mathbf{b}}. \tag{12.8}$$

[1] There is no decent meaning of r^2 in this fitting formula which makes sense only for $r = 1, 2$.

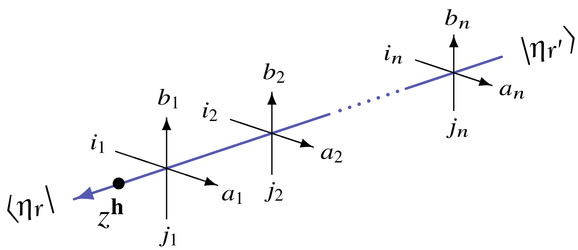

Fig. 12.1 The boundary vector reduction. The matrix product formula (12.9) is depicted as a concatenation of Fig. 11.1 along the blue arrow carrying $\mathcal{F}_q$ sandwiched by the boundary vectors $\langle\eta_r|$ and $|\eta_{r'}\rangle$ in (12.3). It is a BBQ stick with X-shaped sausages and extra caps at the two ends. The dual pairing is defined by (3.16)

Then the construction (12.6) implies the matrix product formula

$$S^{r,r'}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \varrho^{r,r'}(z)\langle\eta_r|z^{\mathbf{h}}L^{a_1b_1}_{i_1j_1}\cdots L^{a_nb_n}_{i_nj_n}|\eta_{r'}\rangle \quad (r,r'=1,2) \tag{12.9}$$

in terms of the components of the 3D L in (12.1) (Fig. 12.1).

From the q-oscillator relations (3.12) and the dual pairing rule (3.16), calculation of the quantities $\langle\eta_r|(\cdots)|\eta_{r'}\rangle$ is reduced to the following:

$$\begin{aligned}
&\langle\eta_r|z^{\mathbf{h}}(\mathbf{a}^{\pm})^j\mathbf{k}^m w^{\mathbf{h}}|\eta_{r'}\rangle = \langle\eta_{r'}|w^{\mathbf{h}}\mathbf{k}^m(\mathbf{a}^{\mp})^j z^{\mathbf{h}}|\eta_r\rangle \quad (r,r'=1,2),\\
&\langle\eta_1|z^{\mathbf{h}}(\mathbf{a}^{+})^j\mathbf{k}^m w^{\mathbf{h}}|\eta_1\rangle = z^j(-q;q)_j\frac{(-q^{j+m+1}zw;q)_\infty}{(q^m zw;q)_\infty},\\
&\langle\eta_1|z^{\mathbf{h}}(\mathbf{a}^{-})^j\mathbf{k}^m w^{\mathbf{h}}|\eta_2\rangle = z^{-j}\sum_{i=0}^{j}(-1)^i q^{\frac{1}{2}i(i+1-2j)}\frac{(q)_j(-q^{2i+2m+1}z^2w^2;q^2)_\infty}{(q)_i(q)_{j-i}(q^{2i+2m}z^2w^2;q^2)_\infty},\\
&\langle\eta_1|z^{\mathbf{h}}(\mathbf{a}^{+})^j\mathbf{k}^m w^{\mathbf{h}}|\eta_2\rangle = z^{j}\sum_{i=0}^{j} q^{\frac{1}{2}i(i+1)}\frac{(q)_j(-q^{2i+2m+1}z^2w^2;q^2)_\infty}{(q)_i(q)_{j-i}(q^{2i+2m}z^2w^2;q^2)_\infty},\\
&\langle\eta_2|z^{\mathbf{h}}(\mathbf{a}^{+})^j\mathbf{k}^m w^{\mathbf{h}}|\eta_2\rangle = \theta(j\in 2\mathbb{Z})\,z^j(q^2;q^4)_{j/2}\frac{(q^{2j+2m+2}z^2w^2;q^4)_\infty}{(q^{2m}z^2w^2;q^4)_\infty}.
\end{aligned} \tag{12.10}$$

See (3.65) for the notation. The symbol θ is defined after (6.66). These formulas are easily derived from the elementary identity (3.82). From (12.9) we see

$$S^{r,r'}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \alpha^{|\mathbf{a}|-|\mathbf{j}|}\left(S^{r,r'}(z)^{\mathbf{ab}}_{\mathbf{ij}}|_{\alpha=1}\right), \tag{12.11}$$

$$S^{r,r'}(z)^{\mathbf{ab}}_{\mathbf{ij}} = 0 \ \text{ unless } \ \mathbf{a}+\mathbf{b}=\mathbf{i}+\mathbf{j}, \tag{12.12}$$

$$S^{2,2}(z)^{\mathbf{ab}}_{\mathbf{ij}} = 0 \ \text{ unless } \ |\mathbf{a}|\equiv|\mathbf{i}| \text{ and } |\mathbf{b}|\equiv|\mathbf{j}| \mod 2. \tag{12.13}$$

The α-dependence (12.11) is a direct consequence of (12.1), the weight conservation (12.12) follows from (11.15) and the parity constraint (12.13) is due to the fact that the boundary vectors $\langle\eta_2|, |\eta_2\rangle$ in (12.3) contain "even modes" only. It leads to the decomposition

$$S^{2,2}(z) = \bigoplus_{\sigma,\sigma'=+,-} S^{\sigma,\sigma'}(z), \qquad S^{\sigma,\sigma'}(z) \in \mathrm{End}(\mathbf{V}_\sigma \otimes \mathbf{V}_{\sigma'}). \tag{12.14}$$

When $(r, r') = (2, 2)$, the Yang–Baxter equation (12.7) is valid in each subspace $\mathbf{V}_\sigma \otimes \mathbf{V}_{\sigma'} \otimes \mathbf{V}_{\sigma''}$ of $\mathbf{V} \otimes \mathbf{V} \otimes \mathbf{V}$. The scalar $\varrho^{2,2}(z)$ in (12.9) may be chosen as $\varrho^{\sigma,\sigma'}(z)$ depending on the summands in (12.14). We take them as

$$\begin{aligned} &\varrho^{r,r'}(z) = \frac{(z^{\max(r,r')}; q^{rr'})_\infty}{(-z^{\max(r,r')}q; q^{rr'})_\infty} \quad ((r, r') = (1, 1), (1, 2), (2, 1)),\\ &\varrho^{\pm,\pm}(z) = \frac{(z^2; q^4)_\infty}{(z^2q^2; q^4)_\infty}, \qquad \varrho^{\pm,\mp}(z) = \frac{(z^2q^2; q^4)_\infty}{(z^2q^4; q^4)_\infty}. \end{aligned} \tag{12.15}$$

Then $S^{r,r'}(z)^{\mathbf{ab}}_{\mathbf{ij}}$ becomes a rational function of q and z^r normalized as

$$S(z)(v_{\mathbf{a}} \otimes v_{\mathbf{a}}) = v_{\mathbf{a}} \otimes v_{\mathbf{a}} \quad (\mathbf{a} \in \mathfrak{s},\ S = S^{1,1}, S^{1,2}, S^{2,1}, S^{+,+}), \tag{12.16}$$

$$S^{-,-}(z)(v_{\mathbf{e}_1} \otimes v_{\mathbf{e}_1}) = v_{\mathbf{e}_1} \otimes v_{\mathbf{e}_1}, \tag{12.17}$$

$$S^{+,-}(z)(v_{\mathbf{0}} \otimes v_{\mathbf{e}_1}) = -q\alpha^{-1} v_{\mathbf{0}} \otimes v_{\mathbf{e}_1}, \quad S^{-,+}(z)(v_{\mathbf{e}_1} \otimes v_{\mathbf{0}}) = \alpha v_{\mathbf{e}_1} \otimes v_{\mathbf{0}}. \tag{12.18}$$

From (3.18), (11.15) and (11.16), we also have

$$S^{r,r'}(z)^{\mathbf{ab}}_{\mathbf{ij}} = z^{|\mathbf{j}|-|\mathbf{b}|} S^{r',r}(z)^{\mathbf{i}^\vee \mathbf{j}^\vee}_{\mathbf{a}^\vee \mathbf{b}^\vee} = S^{r,r'}(z)^{\mathbf{j}\,\mathbf{i}}_{\mathbf{b}\,\mathbf{a}}|_{\alpha \to -q\alpha^{-1}}. \tag{12.19}$$

Example 12.1 We consider the simplest case $n = 1$. $S^{r,r'}(z)$ with $(r, r') = (1, 1)$, $(1, 2)$, $(2, 1)$ are given as follows:

$$\begin{aligned} &v_i \otimes v_i \mapsto v_i \otimes v_i \quad (i = 0, 1),\\ &v_0 \otimes v_1 \mapsto -\frac{q(1 - z^s)v_0 \otimes v_1}{\alpha(1 + qz^s)} + \frac{(1 + q)z^r v_1 \otimes v_0}{1 + qz^s},\\ &v_1 \otimes v_0 \mapsto \frac{(1 + q)z^{r'-1} v_0 \otimes v_1}{1 + qz^s} + \frac{\alpha(1 - z^s)v_1 \otimes v_0}{1 + qz^s}, \end{aligned}$$

where $s = \max(r, r')$. $S^{2,2}(z)$ with $n = 1$ reads as

$$v_i \otimes v_i \mapsto v_i \otimes v_i \ \ (i = 0, 1), \quad v_0 \otimes v_1 \mapsto -q\alpha^{-1} v_0 \otimes v_1, \quad v_1 \otimes v_0 \mapsto \alpha v_1 \otimes v_0.$$

Examples of the case $n = 2$ are available in Sect. 12.4.

12.2 Identification with Quantum R Matrices of $B_n^{(1)}$, $D_n^{(1)}$, $D_{n+1}^{(2)}$

12.2.1 Quantum Affine Algebra $U_p(\mathfrak{g}^{r,r'})$

We will be concerned with the affine Kac–Moody algebras[2]

$$\mathfrak{g}^{1,1} = D_{n+1}^{(2)}, \quad \mathfrak{g}^{2,1} = B_n^{(1)}, \quad \mathfrak{g}^{1,2} = \tilde{B}_n^{(1)}, \quad \mathfrak{g}^{2,2} = D_n^{(1)}, \tag{12.20}$$

where the notation $\mathfrak{g}^{r,r'}$ will turn out to fit to $S^{r,r'}(z)$ in the previous section.

Let $U_p = U_p(D_{n+1}^{(2)})$ $(n \geq 2)$, $U_p(B_n^{(1)})$ $(n \geq 3)$, $U_p(\tilde{B}_n^{(1)})$ $(n \geq 3)$, $U_p(D_n^{(1)})$ $(n \geq 3)$ be the quantum affine algebras. They are Hopf algebras generated by $\{e_i, f_i, k_i^{\pm 1} \mid i \in \{0, 1, \ldots, n\}\}$ satisfying the relations (10.1) with q replaced by p (and the index set I there understood as $\{0, 1, \ldots, n\}$). Beside the commutativity of $k_i^{\pm 1}$ and the p-Serre relations, they include

$$k_i e_j k_i^{-1} = p_i^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = p_i^{-a_{ij}} f_j, \quad [e_i, f_j] = \delta_{i,j} \frac{k_i - k_i^{-1}}{p_i - p_i^{-1}}, \tag{12.21}$$

where the constants p_i $(0 \leq i \leq n)$ are taken as[3]

$$p_i = p \quad \text{except for} \quad p_0 = p^{r/2}, \; p_n = p^{r'/2}. \tag{12.22}$$

Thus the actual exceptions are $p_0 = p_n = p^{1/2}$ for $D_{n+1}^{(2)}$, $p_n = p^{1/2}$ for $B_n^{(1)}$ and $p_0 = p^{1/2}$ for $\tilde{B}_n^{(1)}$.

The affine Lie algebra $\tilde{B}_n^{(1)}$ is just $B_n^{(1)}$ with different enumeration of the vertices as shown in Fig. 12.2. We keep it for uniformity of the description. The Cartan matrix $(a_{ij})_{0 \leq i,j \leq n}$ is determined from the Dynkin diagrams of the relevant affine Lie algebras according to the convention of [67]. Thus for instance in $U_p(D_{n+1}^{(2)})$, one has $a_{01} = -2, a_{10} = -1$ and $k_0 e_0 = p e_0 k_0$, $k_0 e_1 = p^{-1} e_1 k_0$, $k_1 e_0 = p^{-1} e_0 k_1$ and $k_1 e_1 = p^2 e_1 k_1$. Forgetting the 0th node in the Dynkin diagrams yields the classical subalgebras $U_p(B_n) \subset U_p(D_{n+1}^{(2)})$, $U_p(B_n) \subset U_p(B_n^{(1)})$, $U_p(D_n) \subset U_p(\tilde{B}_n^{(1)})$ and $U_p(D_n) \subset U_p(D_n^{(1)})$.

[2] Some symbols including $\mathfrak{g}^{r,r'}$ here and Sect. 14.2.1 are apparently the same, but they should be understood as redefined in each place.

[3] This normalization agrees with (14.19). The normalization mentioned after (10.1) for $U_q(\mathfrak{g})$ with non-affine $\mathfrak{g}$ has not been adopted here.

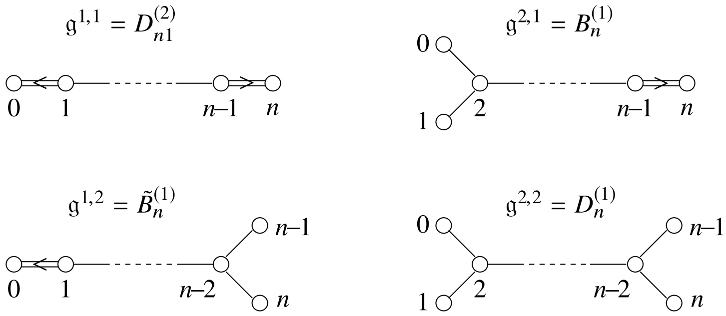

Fig. 12.2 Dynkin diagrams of (12.20) with enumeration of vertices

12.2.2 Spin Representations of $U_p(\mathfrak{g}^{r,r'})$

Let $\pi_{\varpi_n,x} : U_p(\mathfrak{g}^{r,r'}) \to \mathrm{End}(\mathbf{V})$ be the representations

$$e_0 v_{\mathbf{m}} = x v_{\mathbf{m}-\mathbf{e}_1}, \quad f_0 v_{\mathbf{m}} = x^{-1} v_{\mathbf{m}+\mathbf{e}_1}, \quad k_0 v_{\mathbf{m}} = p^{\frac{1}{2}-m_1} v_{\mathbf{m}} \quad (r = 1), \tag{12.23}$$

$$e_0 v_{\mathbf{m}} = x^2 v_{\mathbf{m}-\mathbf{e}_1-\mathbf{e}_2}, \quad f_0 v_{\mathbf{m}} = x^{-2} v_{\mathbf{m}+\mathbf{e}_1+\mathbf{e}_2}, \quad k_0 v_{\mathbf{m}} = p^{1-m_1-m_2} v_{\mathbf{m}} \quad (r = 2), \tag{12.24}$$

$$e_i v_{\mathbf{m}} = v_{\mathbf{m}+\mathbf{e}_i-\mathbf{e}_{i+1}}, \quad f_i v_{\mathbf{m}} = v_{\mathbf{m}-\mathbf{e}_i+\mathbf{e}_{i+1}}, \quad k_i v_{\mathbf{m}} = p^{m_i-m_{i+1}} v_{\mathbf{m}} \quad (0 < i < n), \tag{12.25}$$

$$e_n v_{\mathbf{m}} = v_{\mathbf{m}+\mathbf{e}_n}, \quad f_n v_{\mathbf{m}} = v_{\mathbf{m}-\mathbf{e}_n}, \quad k_n v_{\mathbf{m}} = p^{m_n-\frac{1}{2}} v_{\mathbf{m}} \quad (r' = 1), \tag{12.26}$$

$$e_n v_{\mathbf{m}} = v_{\mathbf{m}+\mathbf{e}_{n-1}+\mathbf{e}_n}, \quad f_n v_{\mathbf{m}} = v_{\mathbf{m}-\mathbf{e}_{n-1}-\mathbf{e}_n}, \quad k_n v_{\mathbf{m}} = p^{m_n+m_{n-1}-1} v_{\mathbf{m}} \quad (r' = 2), \tag{12.27}$$

where $\mathbf{m} \in \mathfrak{s}$. See Sect. 11.1 for the definitions of $\mathbf{V}$, $v_{\mathbf{m}}$, $\mathfrak{s}$ and $\mathbf{e}_i$. In the LHSs, e_i for example actually means $\pi_{\varpi_n,x}(e_i)$. In the RHSs, the vector $v_{\mathbf{m}}$ should be understood as 0 if $\mathbf{m} \notin \mathfrak{s}$. The choice of $x^{\pm r}$ rather than $x^{\pm 1}$ in (12.23) and (12.24) is the option leading to a uniform description of the results in Theorem 12.2.

The algebras $U_p(\mathfrak{g}^{1,1})$ and $U_p(\mathfrak{g}^{2,1})$ have a common classical subalgebra $U_p(B_n)$ without $e_0, f_0, k_0^{\pm}$. As a $U_p(B_n)$ module, $\mathbf{V}$ is already irreducible and is isomorphic to the highest weight module $V(\varpi_n)$ in the notation of Sect. 10.1.1 with highest weight vector $v_{\mathbf{e}_1+\cdots+\mathbf{e}_n}$. It is called the spin representation.[4]

The algebras $U_p(\mathfrak{g}^{1,2})$ and $U_p(\mathfrak{g}^{2,2})$ have a common classical subalgebra $U_p(D_n)$ without $e_0, f_0, k_0^{\pm}$. As a $U_p(D_n)$ module, $\mathbf{V}$ decomposes into two irreducible

[4] As a $U_p(\mathfrak{g}^{1,1})$ or $U_p(\mathfrak{g}^{2,1})$ module, it is a Kirillov–Reshetikhin module $W_1^{(n)}$ up to specification of the spectral parameter.

components $\mathbf{V}_+$ and $\mathbf{V}_-$ in (11.6). The space $\mathbf{V}_{(-1)^n}$ (resp. $\mathbf{V}_{(-1)^{n-1}}$) is isomorphic to the highest weight module $V(\varpi_n)$ (resp. $V(\varpi_{n-1})$) in the notation of Sect. 10.1.1 whose highest weight vector is $v_{\mathbf{e}_1+\cdots+\mathbf{e}_n}$ (resp. $v_{\mathbf{e}_1+\cdots+\mathbf{e}_{n-1}}$). Both $V(\varpi_n)$ and $V(\varpi_{n-1})$ are called spin representations. As a $U_p(\mathfrak{g}^{1,2})$ module, $\mathbf{V}$ is irreducible. As a $U_p(\mathfrak{g}^{2,2})$ module, each $\mathbf{V}_\pm$ remains individually irreducible since the parity of $|\mathbf{m}| = m_1 + \cdots + m_n$ in $v_\mathbf{m}$ is preserved.[5] We will simply refer to $\pi_{\varpi_n,x}$ as the spin representation of $U_p(\mathfrak{g}^{r,r'})$.

12.2.3 $S^{r,r'}(z)$ as Quantum R Matrices for Spin Representations

Let $\Delta_{x,y} = (\pi_{\varpi_n,x} \otimes \pi_{\varpi_n,y}) \circ \Delta$ and $\Delta^{\mathrm{op}}_{x,y} = (\pi_{\varpi_n,x} \otimes \pi_{\varpi_n,y}) \circ \Delta^{\mathrm{op}}$ be the tensor product of the spin representations, where Δ and Δ^{op} are the coproduct (11.58) and its opposite (11.59). For $(r, r') \neq (2, 2)$, let $\mathcal{R}^{r,r'}(z) \in \mathrm{End}(\mathbf{V} \otimes \mathbf{V})$ be a quantum R matrix of $U_p(\mathfrak{g}^{r,r'})$ which is characterized, up to normalization, by the commutativity

$$\mathcal{R}^{r,r'}(\tfrac{x}{y})\Delta_{x,y}(g) = \Delta^{\mathrm{op}}_{x,y}(g)\mathcal{R}^{r,r'}(\tfrac{x}{y}) \quad (\forall g \in U_p(\mathfrak{g}^{r,r'})). \tag{12.28}$$

For $(r, r') = (2, 2)$, we set

$$\mathcal{R}^{2,2}(z) = \mathcal{R}^{+,+}(z) \oplus \mathcal{R}^{+,-}(z) \oplus \mathcal{R}^{-,+}(z) \oplus \mathcal{R}^{-,-}(z) \in \mathrm{End}(\mathbf{V} \otimes \mathbf{V}), \tag{12.29}$$

where $\mathcal{R}^{\varepsilon,\varepsilon'}(z) \in \mathrm{End}(\mathbf{V}_\varepsilon \otimes \mathbf{V}_{\varepsilon'})$ $(\varepsilon, \varepsilon' = \pm)$ is a quantum R matrix of $U_p(\mathfrak{g}^{2,2})$ which is characterized, up to normalization, by the commutativity

$$\mathcal{R}^{\varepsilon,\varepsilon'}(\tfrac{x}{y})\Delta_{x,y}(g) = \Delta^{\mathrm{op}}_{x,y}(g)\mathcal{R}^{\varepsilon,\varepsilon'}(\tfrac{x}{y}) \quad (\forall g \in U_p(\mathfrak{g}^{2,2})). \tag{12.30}$$

We have taken the obvious fact that the R matrices depend only on the ratio x/y into account. The relations (12.28) and (12.30) are generalizations of (10.12) including the latter as the classical part. The main result in this chapter is the following.

Theorem 12.2 *Up to normalization, $S^{r,r'}(z)$ by the matrix product construction (12.8)–(12.9) based on the 3D L (12.1) with $\alpha = p^{-1/2}$ coincides with the quantum R matrix of $U_p(\mathfrak{g}^{r,r'})$ as*

$$S^{r,r'}(z) = \mathcal{R}^{r,r'}(z^{-1}) \quad \text{at } q = -p^{-1}. \tag{12.31}$$

[5] They are Kirillov–Reshetikhin modules $W_1^{(n)}$ and $W_1^{(n-1)}$ up to specification of the spectral parameters.

Proof It suffices to check

$$S^{r,r'}(\tfrac{y}{x})(e_s \otimes 1 + k_s \otimes e_s) = (1 \otimes e_s + e_s \otimes k_s)S^{r,r'}(\tfrac{y}{x}), \tag{12.32}$$

$$S^{r,r'}(\tfrac{y}{x})(1 \otimes f_s + f_s \otimes k_s^{-1}) = (f_s \otimes 1 + k_s^{-1} \otimes f_s)S^{r,r'}(\tfrac{y}{x}), \tag{12.33}$$

$$S^{r,r'}(\tfrac{y}{x})(k_s \otimes k_s) = (k_s \otimes k_s)S^{r,r'}(\tfrac{y}{x}) \tag{12.34}$$

under the image by $\pi_{\varpi_n,x} \otimes \pi_{\varpi_n,y}$ for $0 \le s \le n$. For $0 < s < n$, the formula (12.25) coincides with (11.60) for $i \neq 0$. Therefore it is indeed valid if $q = -p^{-1}$ thanks to the proof of Theorem 11.3. Let us illustrate the proof of (12.32) for $s = n$ using the properties of the boundary vectors. The other relations can be treated similarly.

First we consider the case $r' = 1$. Then up to the normalization factor $\varrho^{r,1}(\tfrac{y}{x})$, the vector $S^{r,1}(\tfrac{y}{x})(e_n \otimes 1 + k_n \otimes e_n)(v_{\mathbf{i}} \otimes v_{\mathbf{j}})$ is calculated by using (12.26) as

$$\begin{aligned}&\langle \eta_r|\left(\tfrac{y}{x}\right)^{\mathbf{h}} L_{1_1 2_1} \cdots L_{1_n 2_n}|\eta_1\rangle(v_{\mathbf{i}+\mathbf{e}_n} \otimes v_{\mathbf{j}} + p^{i_n - \frac{1}{2}} v_{\mathbf{i}} \otimes v_{\mathbf{j}+\mathbf{e}_n}) \\ &= \sum_{\mathbf{a},\mathbf{b} \in \mathfrak{s}} \langle \eta_r|X(L^{a_n,b_n}_{i_n+1,j_n} + p^{i_n-\frac{1}{2}} L^{a_n,b_n}_{i_n,j_n+1})|\eta_1\rangle v_{\mathbf{a}} \otimes v_{\mathbf{b}},\end{aligned} \tag{12.35}$$

where $X = \left(\tfrac{y}{x}\right)^{\mathbf{h}} L^{a_1,b_1}_{i_1,j_1} \cdots L^{a_{n-1},b_{n-1}}_{i_{n-1},j_{n-1}}$. Similarly, $(1 \otimes e_n + e_n \otimes k_n)S^{r,1}(\tfrac{y}{x})(v_{\mathbf{i}} \otimes v_{\mathbf{j}})$ yields

$$\begin{aligned}&(1 \otimes e_n + e_n \otimes k_n) \sum_{\mathbf{a},\mathbf{b} \in \mathfrak{s}} \langle \eta_r|X L^{a_n,b_n}_{i_n,j_n}|\eta_1\rangle(v_{\mathbf{a}} \otimes v_{\mathbf{b}}) \\ &= \sum_{\mathbf{a},\mathbf{b} \in \mathfrak{s}} \langle \eta_r|X L^{a_n,b_n}_{i_n,j_n}|\eta_1\rangle(v_{\mathbf{a}} \otimes v_{\mathbf{b}+\mathbf{e}_n} + p^{b_n - \frac{1}{2}} v_{\mathbf{a}+\mathbf{e}_n} \otimes v_{\mathbf{b}}).\end{aligned} \tag{12.36}$$

From the comparison of the coefficient of $v_{\mathbf{a}} \otimes v_{\mathbf{b}}$, it suffices to show

$$(L^{a,b}_{i+1,j} + p^{i-\frac{1}{2}} L^{a,b}_{i,j+1} - L^{a.b-1}_{i,j} - p^{b-\frac{1}{2}} L^{a-1,b}_{i,j})|\eta_1\rangle = 0, \tag{12.37}$$

where a_n, b_n, i_n, j_n are denoted by a, b, i, j. As an example for $(a, b, i, j) = (1, 1, 0, 1)$, it reads, from (11.14), as

$$0 = (L^{11}_{11} - L^{10}_{01} - p^{\frac{1}{2}} L^{01}_{01})|\eta_1\rangle = (1 - \mathbf{a}^+ - p^{\frac{1}{2}}(-q\alpha^{-1}\mathbf{k}))|\eta_1\rangle. \tag{12.38}$$

From $q = -p^{-1}$ this is indeed valid at $\alpha = p^{-1/2}$ due to the property (3.134) of the boundary vector $|\eta_1\rangle$. With the choice $(q, \alpha) = (-p^{-1}, p^{-1/2})$, all the other relations in (12.37) can be similarly checked by also using (3.135).

Next we consider the case $r' = 2$. The main difference from the $r' = 1$ case is that (12.27) concerns the "two boundary sites" $n - 1$ and n. Thus a similar argument comparing the coefficients of $v_{\ldots,a_{n-1},a_n} \otimes v_{\ldots,b_{n-1},b_n}$ leads to a quadratic relation

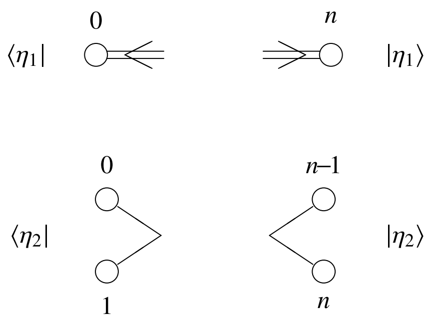

Fig. 12.3 Correspondence between the boundary vectors and the end shape of the Dynkin diagrams. These data are relevant to the LHS and the RHS of (12.31) in Theorem 12.2, respectively

$$(L^{a,b}_{i+1,j}L^{a',b'}_{i'+1,j'} + p^{i+i'-1}L^{a,b}_{i,j+1}L^{a',b'}_{i',j'+1} - L^{a,b-1}_{i,j}L^{a',b'-1}_{i',j'} - p^{b+b'-1}L^{a-1,b}_{i,j}L^{a'-1,b'}_{i',j'})|\eta_2\rangle = 0. \tag{12.39}$$

Consider the LHS for $(a, a', b, b', i, i', j, j') = (1, 1, 1, 1, 0, 0, 1, 1)$ for example:

$$(L^{11}_{11}L^{11}_{11} - L^{10}_{01}L^{10}_{01} - pL^{01}_{01}L^{01}_{01})|\eta_2\rangle = (1 - (\mathbf{a}^+)^2 - p(-q\alpha^{-1}\mathbf{k})^2)|\eta_2\rangle. \tag{12.40}$$

From $(q, \alpha) = (-p^{-1}, p^{-1/2})$, this is evaluated as

$$(1 - (\mathbf{a}^+)^2 - \mathbf{k}^2)|\eta_2\rangle \overset{(3.137)}{=} (1 - \mathbf{a}^+\mathbf{a}^- - \mathbf{k}^2)|\eta_2\rangle \overset{(3.12)}{=} 0. \tag{12.41}$$

All the other relations in (12.39) can be checked similarly by using (3.137) and (3.12). □

Remark 12.3 Theorem 12.2 suggests the following correspondence between the boundary vectors $\langle\eta_r|$, $|\eta_{r'}\rangle$ in (12.3) and the end shape of the Dynkin diagrams in Fig. 12.2: (Fig. 12.3)

A similar correspondence is observed in Remark 11.4 and 14.3.

12.3 Commuting Layer Transfer Matrix

This section is a continuation from Sect. 11.6 from which we will borrow some terminology. Given parameters $\mathbf{u} = (u_1, \ldots, u_m)$ and $\mathbf{w} = (w_1, \ldots, w_n)$, consider the row transfer matrix of the vertex model associated with the spin representation of $U_p(\mathfrak{g}^{r,r'})$:

$$T(x|\mathbf{u}, \mathbf{w}) = \mathrm{Tr}_{\mathbf{1}}\big(\mathbf{w}^H S^{r,r'}_{\mathbf{1},\mathbf{2}_1}(xu_1)\cdots S^{r,r'}_{\mathbf{1},\mathbf{2}_m}(xu_m)\big) \in \mathrm{End}(\overset{\mathbf{2}_1}{\mathbf{V}} \otimes \cdots \otimes \overset{\mathbf{2}_m}{\mathbf{V}}). \tag{12.42}$$

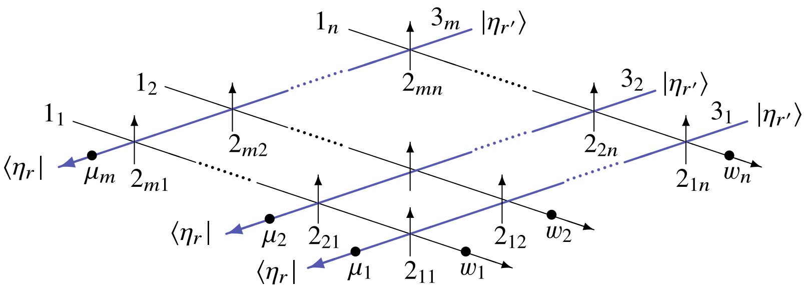

Fig. 12.4 A layer transfer matrix interpretation of the row transfer matrix of the $U_p(\mathfrak{g}^{r,r'})$ vertex model associated with the spin representation. There are n black horizontal arrows $1_1, \dots, 1_n$ carrying $V \simeq \mathbb{C}^2$ which are being traced out corresponding to the periodic boundary condition. There are also m blue horizontal arrows $3_1, \dots, 3_m$ carrying $\mathcal{F}_q$ which are to be evaluated between the boundary vectors. The mark • with z signifies an operator $z^{\mathbf{h}}$. At the intersection of 1_i and 3_j, there is a vertical black arrow 2_{ij} carrying V, which corresponds to a 3D L $L_{1_i,2_{ij},3_j}$. The parameter μ_i is taken as $\mu_i = xu_i$ as in (11.84)

To each $S^{r,r'}(xu_i)$, labels have been attached indicating the spaces it acts. The label $\mathbf{1} = (1_1, \dots, 1_n)$ is the one for the auxiliary space $\overset{\mathbf{1}}{\mathbf{V}} = \overset{1_1}{V} \otimes \cdots \otimes \overset{1_n}{V}$ and $\mathbf{2}_j$ is the one for the jth component $\overset{\mathbf{2}_j}{\mathbf{V}} = \overset{2_{j1}}{V} \otimes \cdots \otimes \overset{2_{jn}}{V}$ in the quantum space $\overset{\mathbf{2}_1}{\mathbf{V}} \otimes \cdots \otimes \overset{\mathbf{2}_m}{\mathbf{V}}$. For the symbol $\mathbf{w}^H$, see (11.87). The parameters x, $\mathbf{u}$ and $\mathbf{w}$ are spectral parameters, their inhomogeneity and the boundary field. From the Yang–Baxter equation (12.7) and the weight conservation (12.12), it forms a commuting family:

$$[T(x|\mathbf{u}, \mathbf{w}), T(x'|\mathbf{u}, \mathbf{w})] = 0. \tag{12.43}$$

Theorem 12.2 endows $T(x|\mathbf{u}, \mathbf{w})$ with an interpretation as a layer transfer matrix of a 3D lattice model with a special boundary condition explained below.

The formula (12.42) corresponds to looking at Fig. 12.4 from the SW, or evaluating $\langle\eta_r|(\cdots)|\eta_{r'}\rangle$ first. On the other hand, one can look at it from the SE or first take the trace over $1_1, \dots, 1_n$. From (11.41), it leads to an alternative interpretation:

$$T(x|\mathbf{u}, \mathbf{w}) = \langle\eta_r|^{\otimes m}(x\mathbf{u})^H S^{\mathrm{tr}_1}_{\mathbf{2}_1,\mathbf{3}}(w_1) \cdots S^{\mathrm{tr}_1}_{\mathbf{2}_n,\mathbf{3}}(w_n)|\eta_{r'}\rangle^{\otimes m} \in \mathrm{End}(V^{\otimes mn}). \tag{12.44}$$

Here $\mathbf{3} = (3_1, \dots, 3_m)$ is the label of the auxiliary space $\overset{\mathbf{3}}{\mathbf{W}} = \overset{3_1}{\mathcal{F}}_q \otimes \cdots \otimes \overset{3_m}{\mathcal{F}}_q$ along which the product of S^{tr_1} is taken and $\langle\eta_r|^{\otimes m}(\cdots)|\eta_{r'}\rangle^{\otimes m}$ is evaluated. The label $\mathbf{2}_j = (2_{1j}, \dots, 2_{mj})$ signifies the jth component $\overset{\mathbf{2}_j}{\mathbf{V}} = \overset{2_{1j}}{V} \otimes \cdots \overset{2_{mj}}{V}$ of the quantum space $\overset{\mathbf{2}_1}{\mathbf{V}} \otimes \cdots \otimes \overset{\mathbf{2}_n}{\mathbf{V}} \simeq V^{\otimes mn}$.

The operator (12.44) arises from the dual pairing between $\mathbf{W}^{(m)} = \mathcal{F}_q^{\otimes m} = \bigoplus_{k \geq 0} \mathbf{W}_k^{(m)}$ in (11.11) and its dual. From the weight conservation (11.45) and the decomposition (11.46), it is expanded as

$$T(x|\mathbf{u}, \mathbf{w}) = \bigoplus_{k\geq 0,\, \mathbf{l}=(l_1,\dots,l_n)\in\{0,1,\dots,m\}^n} x^k T_{k,\mathbf{l}}(\mathbf{u}, \mathbf{w}), \tag{12.45}$$

$$T_{k,\mathbf{l}}(\mathbf{u}, \mathbf{w}) = \langle \eta^m_{r,k}|\mathbf{u}^H S^{\mathrm{tr}_1}_{l_1,k}(w_1)\cdots S^{\mathrm{tr}_1}_{l_n,k}(w_n)|\eta^m_{r',k}\rangle \in \mathrm{End}(\mathbf{V}^{(m)}_{l_1}\otimes\cdots\otimes\mathbf{V}^{(m)}_{l_n}), \tag{12.46}$$

where the vector $|\eta^m_{r',k}\rangle$ is the projection of $|\eta_{r'}\rangle^{\otimes m}$ onto $\mathbf{W}^{(m)}_k$ in (11.11). The vector $\langle\eta^m_{r,k}|$ is the dual of $|\eta^m_{r,k}\rangle$. From (12.3) they are explicitly given as *finite* sums:

$$|\eta^m_{r',k}\rangle = \sum_{(d_1,\dots,d_m)\in B^{(m)}_k} \frac{|r'd_1\rangle\otimes\cdots\otimes|r'd_m\rangle}{(q^{r'^2})_{d_1}\cdots(q^{r'^2})_{d_m}}, \tag{12.47}$$

$$\langle\eta^m_{r,k}| = \sum_{(d_1,\dots,d_m)\in B^{(m)}_k} \frac{\langle rd_1|\otimes\cdots\otimes\langle rd_m|}{(q^{r^2})_{d_1}\cdots(q^{r^2})_{d_m}}. \tag{12.48}$$

See (11.10) for the notation $B^{(m)}_k$. Now the commutativity (12.43) implies

$$[T_{k,\mathbf{l}}(\mathbf{u}, \mathbf{w}), T_{k',\mathbf{l}}(\mathbf{u}, \mathbf{w})] = 0 \quad (k, k' \in \mathbb{Z}_{\geq 0}). \tag{12.49}$$

In 2D terminology, the 3D picture in Fig. 12.4 and Theorem 11.5, 12.2 show the equivalence of the spectral problem for row transfer matrices of the vertex models associated with the spin representations of $U_{-q^{-1}}(B^{(1)}_n)$, $U_{-q^{-1}}(D^{(1)}_n)$, $U_{-q^{-1}}(D^{(2)}_{n+1})$ on length m system with the periodic boundary condition and the $U_{q^{-1}}(A^{(1)}_{m-1})$ vertex model associated with the (anti-symmetric tensor rep.) $\otimes$ (symmetric tensor rep.) on a length n system with a special boundary condition.

12.4 Examples of $S^{1,1}(z)$, $S^{2,1}(z)$, $S^{2,2}(z)$ for $n = 2$

Let us present explicit formulas of $S^{r,r'}(z)$ for $n = 2$.

$S^{1,1}(z)$ is given as follows:

$$v_{ij}\otimes v_{ij} \mapsto v_{ij}\otimes v_{ij} \quad (i, j \in \{0, 1\}),$$

$$v_{00}\otimes v_{01} \mapsto \frac{q(z-1)v_{00}\otimes v_{01}}{\alpha(qz+1)} + \frac{(q+1)zv_{01}\otimes v_{00}}{qz+1},$$

$$v_{00}\otimes v_{10} \mapsto \frac{q(z-1)v_{00}\otimes v_{10}}{\alpha(qz+1)} + \frac{(q+1)zv_{10}\otimes v_{00}}{qz+1},$$

$$v_{00}\otimes v_{11} \mapsto \frac{(z-1)(qz-1)q^2v_{00}\otimes v_{11}}{\alpha^2(qz+1)\left(zq^2+1\right)} + \frac{q^2(q+1)(z-1)zv_{01}\otimes v_{10}}{\alpha(qz+1)\left(zq^2+1\right)}$$
$$+ \frac{q(q+1)(z-1)zv_{10}\otimes v_{01}}{\alpha(qz+1)\left(zq^2+1\right)} + \frac{(q+1)\left(q^2+1\right)z^2v_{11}\otimes v_{00}}{(qz+1)\left(zq^2+1\right)},$$

$$v_{01} \otimes v_{00} \mapsto \frac{(q+1)v_{00} \otimes v_{01}}{qz+1} - \frac{\alpha(z-1)v_{01} \otimes v_{00}}{qz+1},$$

$$v_{01} \otimes v_{10} \mapsto \frac{q(q+1)(z-1)v_{00} \otimes v_{11}}{\alpha(qz+1)\left(zq^2+1\right)} - \frac{q(z-1)(qz-1)v_{01} \otimes v_{10}}{(qz+1)\left(zq^2+1\right)}$$
$$+ \frac{(q+1)z\left(zq^2-zq+q+1\right)v_{10} \otimes v_{01}}{(qz+1)\left(zq^2+1\right)} - \frac{(q+1)\alpha(z-1)zv_{11} \otimes v_{00}}{(qz+1)\left(zq^2+1\right)},$$

$$v_{01} \otimes v_{11} \mapsto \frac{q(z-1)v_{01} \otimes v_{11}}{\alpha(qz+1)} + \frac{(q+1)zv_{11} \otimes v_{01}}{qz+1},$$

$$v_{10} \otimes v_{00} \mapsto \frac{(q+1)v_{00} \otimes v_{10}}{qz+1} - \frac{\alpha(z-1)v_{10} \otimes v_{00}}{qz+1},$$

$$v_{10} \otimes v_{01} \mapsto \frac{q^2(q+1)(z-1)v_{00} \otimes v_{11}}{\alpha(qz+1)\left(zq^2+1\right)} - \frac{q(z-1)(qz-1)v_{10} \otimes v_{01}}{(qz+1)\left(zq^2+1\right)}$$
$$- \frac{q(q+1)\alpha(z-1)zv_{11} \otimes v_{00}}{(qz+1)\left(zq^2+1\right)} + \frac{(q+1)\left(zq^2+zq-q+1\right)v_{01} \otimes v_{10}}{(qz+1)\left(zq^2+1\right)},$$

$$v_{10} \otimes v_{11} \mapsto \frac{q(z-1)v_{10} \otimes v_{11}}{\alpha(qz+1)} + \frac{(q+1)zv_{11} \otimes v_{10}}{qz+1},$$

$$v_{11} \otimes v_{00} \mapsto \frac{\alpha^2(z-1)(qz-1)v_{11} \otimes v_{00}}{(qz+1)\left(zq^2+1\right)} - \frac{q(q+1)\alpha(z-1)v_{01} \otimes v_{10}}{(qz+1)\left(zq^2+1\right)}$$
$$- \frac{(q+1)\alpha(z-1)v_{10} \otimes v_{01}}{(qz+1)\left(zq^2+1\right)} + \frac{(q+1)\left(q^2+1\right)v_{00} \otimes v_{11}}{(qz+1)\left(zq^2+1\right)},$$

$$v_{11} \otimes v_{01} \mapsto \frac{(q+1)v_{01} \otimes v_{11}}{qz+1} - \frac{\alpha(z-1)v_{11} \otimes v_{01}}{qz+1},$$

$$v_{11} \otimes v_{10} \mapsto \frac{(q+1)v_{10} \otimes v_{11}}{qz+1} - \frac{\alpha(z-1)v_{11} \otimes v_{10}}{qz+1}.$$

$S^{2,1}(z)$ depends on z only via z^2. It is given as follows:

$$v_{ij} \otimes v_{ij} \mapsto v_{ij} \otimes v_{ij} \quad (i, j \in \{0, 1\}),$$

$$v_{00} \otimes v_{01} \mapsto \frac{(q+1)z^2v_{01} \otimes v_{00}}{qz^2+1} + \frac{q(z^2-1)v_{00} \otimes v_{01}}{\alpha\left(qz^2+1\right)},$$

$$v_{00} \otimes v_{10} \mapsto \frac{(q+1)z^2v_{10} \otimes v_{00}}{qz^2+1} + \frac{q(z^2-1)v_{00} \otimes v_{10}}{\alpha\left(qz^2+1\right)},$$

$$v_{00} \otimes v_{11} \mapsto \frac{q^3(q+1)z^2(z^2-1)v_{01} \otimes v_{10}}{\alpha\left(qz^2+1\right)\left(z^2q^3+1\right)} + \frac{q^2(z^2-1)(q^2z^2-1)v_{00} \otimes v_{11}}{\alpha^2\left(qz^2+1\right)\left(z^2q^3+1\right)}$$
$$+ \frac{q^2(q+1)z^2(z^2-1)v_{10} \otimes v_{01}}{\alpha\left(qz^2+1\right)\left(z^2q^3+1\right)} + \frac{(q+1)z^2\left(z^2q^3+z^2q-q+1\right)v_{11} \otimes v_{00}}{\left(qz^2+1\right)\left(z^2q^3+1\right)},$$

$$v_{01} \otimes v_{00} \mapsto \frac{(q+1)v_{00} \otimes v_{01}}{qz^2+1} - \frac{\alpha(z^2-1)v_{01} \otimes v_{00}}{qz^2+1},$$

$$\begin{aligned}
v_{01} \otimes v_{10} \mapsto\ & \frac{(q+1)z^2\left(z^2q^3 - z^2q^2 + q^2 + 1\right) v_{10} \otimes v_{01}}{\left(qz^2+1\right)\left(z^2q^3+1\right)} - \frac{q(q+1)\alpha z^2(z^2-1)v_{11} \otimes v_{00}}{\left(qz^2+1\right)\left(z^2q^3+1\right)} \\
& + \frac{q(q+1)(z^2-1)v_{00} \otimes v_{11}}{\alpha\left(qz^2+1\right)\left(z^2q^3+1\right)} - \frac{q(z^2-1)(q^2z^2-1)v_{01} \otimes v_{10}}{\left(qz^2+1\right)\left(z^2q^3+1\right)}, \\
v_{01} \otimes v_{11} \mapsto\ & \frac{(q+1)v_{11} \otimes v_{01} z^2}{qz^2+1} + \frac{q(z^2-1)v_{01} \otimes v_{11}}{\alpha\left(qz^2+1\right)}, \\
v_{10} \otimes v_{00} \mapsto\ & \frac{(q+1)v_{00} \otimes v_{10}}{qz^2+1} - \frac{\alpha(z^2-1)v_{10} \otimes v_{00}}{qz^2+1}, \\
v_{10} \otimes v_{01} \mapsto\ & \frac{q^2(q+1)(z^2-1)v_{00} \otimes v_{11}}{\alpha\left(qz^2+1\right)\left(z^2q^3+1\right)} - \frac{q^2(q+1)\alpha z^2(z^2-1)v_{11} \otimes v_{00}}{\left(qz^2+1\right)\left(z^2q^3+1\right)} \\
& - \frac{q(z^2-1)(q^2z^2-1)v_{10} \otimes v_{01}}{\left(qz^2+1\right)\left(z^2q^3+1\right)} + \frac{(q+1)\left(z^2q^3 + z^2q - q + 1\right) v_{01} \otimes v_{10}}{\left(qz^2+1\right)\left(z^2q^3+1\right)}, \\
v_{10} \otimes v_{11} \mapsto\ & \frac{(q+1)z^2 v_{11} \otimes v_{10}}{qz^2+1} + \frac{q(z^2-1)v_{10} \otimes v_{11}}{\alpha\left(qz^2+1\right)}, \\
v_{11} \otimes v_{00} \mapsto\ & \frac{\alpha^2(z^2-1)(q^2z^2-1)v_{11} \otimes v_{00}}{\left(qz^2+1\right)\left(z^2q^3+1\right)} - \frac{q(q+1)\alpha(z^2-1)v_{01} \otimes v_{10}}{\left(qz^2+1\right)\left(z^2q^3+1\right)} \\
& - \frac{(q+1)\alpha(z^2-1)v_{10} \otimes v_{01}}{\left(qz^2+1\right)\left(z^2q^3+1\right)} + \frac{(q+1)\left(z^2q^3 - z^2q^2 + q^2 + 1\right) v_{00} \otimes v_{11}}{\left(qz^2+1\right)\left(z^2q^3+1\right)}, \\
v_{11} \otimes v_{01} \mapsto\ & \frac{(q+1)v_{01} \otimes v_{11}}{qz^2+1} - \frac{\alpha(z^2-1)v_{11} \otimes v_{01}}{qz^2+1}, \\
v_{11} \otimes v_{10} \mapsto\ & \frac{(q+1)v_{10} \otimes v_{11}}{qz^2+1} - \frac{\alpha(z^2-1)v_{11} \otimes v_{10}}{qz^2+1}.
\end{aligned}$$

$S^{1,2}(z)$ can be obtained from the above $S^{2,1}(z)$ by applying (12.19).

$S^{2,2}(z)$ depends on z and α only via z^2 and α^2 up to overall $\alpha^{\pm 1}$. It is given as follows:

$$\begin{aligned}
& v_{ij} \otimes v_{ij} \mapsto v_{ij} \otimes v_{ij} \quad (i, j \in \{0, 1\}), \\
& v_{00} \otimes v_{01} \mapsto -q\alpha^{-1} v_{00} \otimes v_{01}, \quad v_{00} \otimes v_{10} \mapsto -q\alpha^{-1} v_{00} \otimes v_{10}, \quad v_{01} \otimes v_{00} \mapsto \alpha v_{01} \otimes v_{00}, \\
& v_{00} \otimes v_{11} \mapsto \frac{q^2(z^2-1)v_{00} \otimes v_{11}}{\alpha^2(q^2z^2-1)} + \frac{(q^2-1)z^2 v_{11} \otimes v_{00}}{q^2z^2-1}, \\
& v_{01} \otimes v_{10} \mapsto \frac{(q^2-1)z^2 v_{10} \otimes v_{01}}{q^2z^2-1} - \frac{q(z^2-1)v_{01} \otimes v_{10}}{q^2z^2-1}, \\
& v_{01} \otimes v_{11} \mapsto -q\alpha^{-1} v_{01} \otimes v_{11}, \quad v_{10} \otimes v_{00} \mapsto \alpha v_{10} \otimes v_{00}, \\
& v_{10} \otimes v_{01} \mapsto \frac{(q^2-1)v_{01} \otimes v_{10}}{q^2z^2-1} - \frac{q(z^2-1)v_{10} \otimes v_{01}}{q^2z^2-1}, \\
& v_{10} \otimes v_{11} \mapsto -q\alpha^{-1} v_{10} \otimes v_{11}, \quad v_{11} \otimes v_{01} \mapsto \alpha v_{11} \otimes v_{01}, \quad v_{11} \otimes v_{10} \mapsto \alpha v_{11} \otimes v_{10}, \\
& v_{11} \otimes v_{00} \mapsto \frac{\alpha^2(z^2-1)v_{11} \otimes v_{00}}{q^2z^2-1} + \frac{(q^2-1)v_{00} \otimes v_{11}}{q^2z^2-1}.
\end{aligned}$$

12.5 Bibliographical Notes and Comments

The content of this chapter, save for Sect. 12.3, is based on [107] where the reduction using the boundary vectors was introduced with a proof of Proposition 3.28. As for the relevant quantum R matrices for the spin representations, a description in terms of spectral decomposition was shown earlier for $U_p(B_n^{(1)})$ and $U_p(D_n^{(1)})$ in [121]. The eigenvalues in the spectral decomposition for the $U_p(D_{n+1}^{(2)})$ case is available in [107, Eq. (6.15)]. The matrix product form (12.9) provides a most handy and programmable formula for these R matrices via (12.31). It indicates a recursive structure of the R matrices with respect to rank n observed in earlier works including [121].

Chapter 13
Trace Reductions of $RRRR = RRRR$

Abstract Like $RLLL = LLLR$, the tetrahedron equation $RRRR = RRRR$ admits various reductions to the Yang–Baxter equation leading to several families of solutions in matrix product forms. In this chapter we focus on the trace reduction as done for $RLLL = LLLR$ in Chap. 11. We identify the solutions with quantum R matrices of $U_q(A^{(1)}_{n-1})$, present their explicit formulas, construct commuting layer transfer matrices, and demonstrate that the birational versions reproduce the distinguished example of set-theoretical solutions to the Yang–Baxter equation known as geometric R.

13.1 Preliminaries

Let $n \geq 2$ be an integer. We retain the notations for the sets $B^{(n)} = (\mathbb{Z}_{\geq 0})^n$, $B^{(n)}_k$, the vector spaces $\mathbf{W}^{(n)} = \mathcal{F}_q^{\otimes n}$ and $\mathbf{W}^{(n)}_k$ having bases $|\mathbf{a}\rangle$ labeled with n-arrays $\mathbf{a} = (a_1, \dots, a_n)$ in (11.8)–(11.13). We will also use $|\mathbf{a}| = a_1 + \cdots + a_n$, $\mathbf{a}^\vee = (a_n, \dots, a_1)$ in (11.4) and the elementary vector $\mathbf{e}_i$ in (11.1). As for the q-oscillator algebra Osc_q and the Fock space $\mathcal{F}_q$, see Sect. 3.2. Except in Sect. 13.8, n is fixed, hence the superscript "(n)" will be suppressed.

In Chap. 3, we have introduced a linear operator $R_{123} \in \mathrm{End}(\overset{1}{\mathcal{F}}_q \otimes \overset{2}{\mathcal{F}}_q \otimes \overset{3}{\mathcal{F}}_q)$ which we called a 3D R.

In Theorem 3.20 it was shown to satisfy the tetrahedron equation

$$R_{124}R_{135}R_{236}R_{456} = R_{456}R_{236}R_{135}R_{124}, \tag{13.1}$$

which is an equality in $\mathrm{End}(\overset{1}{\mathcal{F}}_q \otimes \cdots \otimes \overset{6}{\mathcal{F}}_q)$.

A. Kuniba, *Quantum Groups in Three-Dimensional Integrability*, Theoretical and Mathematical Physics, https://doi.org/10.1007/978-981-19-3262-5_13

13.2 Trace Reduction Over the Third Component of *R*

The following procedure is quite parallel with that in Sect. 11.2. Consider n copies of (13.1) in which the spaces labeled with $1, 2, 3$ are replaced by $1_i, 2_i, 3_i$ with $i = 1, 2, \dots, n$:

$$(R_{1_i 2_i 4} R_{1_i 3_i 5} R_{2_i 3_i 6})\, R_{456} = R_{456}\, (R_{2_i 3_i 6} R_{1_i 3_i 5} R_{1_i 2_i 4}).$$

Sending R_{456} to the left by applying this relation repeatedly, we get

$$\begin{aligned}&(R_{1_1 2_1 4} R_{1_1 3_1 5} R_{2_1 3_1 6}) \cdots (R_{1_n 2_n 4} R_{1_n 3_n 5} R_{2_n 3_n 6})\, R_{456} \\ &\quad = R_{456}\, (R_{2_1 3_1 6} R_{1_1 3_1 5} R_{1_1 2_1 4}) \cdots (R_{2_n 3_n 6} R_{1_n 3_n 5} R_{1_n 2_n 4}).\end{aligned} \tag{13.2}$$

One can rearrange this without changing the order of operators sharing common labels, hence by using the trivial commutativity, as

$$\begin{aligned}&(R_{1_1 2_1 4} \cdots R_{1_n 2_n 4})(R_{1_1 3_1 5} \cdots R_{1_n 3_n 5})(R_{2_1 3_1 6} \cdots R_{2_n 3_n 6}) R_{456} \\ &\quad = R_{456} (R_{2_1 3_1 6} \cdots R_{2_n 3_n 6})(R_{1_1 3_1 5} \cdots R_{1_n 3_n 5})(R_{1_1 2_1 4} \cdots R_{1_n 2_n 4}).\end{aligned} \tag{13.3}$$

The weight conservation (3.49) of the 3D R may be stated as

$$R_{456}\, x^{\mathbf{h}_4} (xy)^{\mathbf{h}_5} y^{\mathbf{h}_6} = x^{\mathbf{h}_4} (xy)^{\mathbf{h}_5} y^{\mathbf{h}_6} R_{456} \tag{13.4}$$

for arbitrary parameters x and y. See (3.14) for the definition of $\mathbf{h}$. Multiplying this by (13.3) from the left and applying $R^2 = 1$ from (3.60), we get

$$\begin{aligned}&R_{456}\, x^{\mathbf{h}_4} (R_{1_1 2_1 4} \cdots R_{1_n 2_n 4})(xy)^{\mathbf{h}_5} (R_{1_1 3_1 5} \cdots R_{1_n 3_n 5}) y^{\mathbf{h}_6} (R_{2_1 3_1 6} \cdots R_{2_n 3_n 6}) R_{456} \\ &\quad = y^{\mathbf{h}_6} (R_{2_1 3_1 6} \cdots R_{2_n 3_n 6})(xy)^{\mathbf{h}_5} (R_{1_1 3_1 5} \cdots R_{1_n 3_n 5}) x^{\mathbf{h}_4} (R_{1_1 2_1 4} \cdots R_{1_n 2_n 4}).\end{aligned} \tag{13.5}$$

This relation will also be utilized in the boundary vector reduction in Chap. 14 (Fig. 13.2).

Take the trace of (13.5) over $\overset{4}{\mathcal{F}}_q \otimes \overset{5}{\mathcal{F}}_q \otimes \overset{6}{\mathcal{F}}_q$ using the cyclicity of trace and $R^2 = 1$. The result reads as

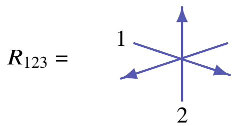

Fig. 13.1 A graphical representation of the 3D R, where 1, 2, 3 are labels of the blue arrows. Each on them carries a q-oscillator Fock space $\mathcal{F}_q$

$$\mathrm{Tr}_4\big(x^{\mathbf{h}_4} R_{1_1 2_1 4} \cdots R_{1_n 2_n 4}\big)\mathrm{Tr}_5\big((xy)^{\mathbf{h}_5} R_{1_1 3_1 5} \cdots R_{1_n 3_n 5}\big)\mathrm{Tr}_6\big(y^{\mathbf{h}_6} R_{2_1 3_1 6} \cdots R_{2_n 3_n 6}\big)$$
$$= \mathrm{Tr}_6\big(y^{\mathbf{h}_6} R_{2_1 3_1 6} \cdots R_{2_n 3_n 6}\big)\mathrm{Tr}_5\big((xy)^{\mathbf{h}_5} R_{1_1 3_1 5} \cdots R_{1_n 3_n 5}\big)\mathrm{Tr}_4\big(x^{\mathbf{h}_4} R_{1_1 2_1 4} \cdots R_{1_n 2_n 4}\big). \tag{13.6}$$

Let us denote the operators appearing here by

$$\begin{aligned} R^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{2}}(z) &= \mathrm{Tr}_4(z^{\mathbf{h}_4} R_{1_1 2_1 4} \cdots R_{1_n 2_n 4}) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{W}} \otimes \overset{\mathbf{2}}{\mathbf{W}}), \\ R^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{3}}(z) &= \mathrm{Tr}_5(z^{\mathbf{h}_5} R_{1_1 3_1 5} \cdots R_{1_n 3_n 5}) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{W}} \otimes \overset{\mathbf{3}}{\mathbf{W}}), \\ R^{\mathrm{tr}_3}_{\mathbf{2},\mathbf{3}}(z) &= \mathrm{Tr}_6(z^{\mathbf{h}_6} R_{2_1 3_1 6} \cdots R_{2_n 3_n 6}) \in \mathrm{End}(\overset{\mathbf{2}}{\mathbf{W}} \otimes \overset{\mathbf{3}}{\mathbf{W}}). \end{aligned} \tag{13.7}$$

The superscript tr_3 indicates that the trace is taken over the *3rd* (rightmost) component of R, whereas Tr_j in RHSs signifies the *label* j of a space. A similar convention will be employed in the subsequent sections.

Those appearing in (13.7) are the same operators acting on different copies of $\mathbf{W}$ specified as $\overset{\mathbf{1}}{\mathbf{W}} = \overset{1_1}{\mathcal{F}_q} \otimes \cdots \otimes \overset{1_n}{\mathcal{F}_q}$, $\overset{\mathbf{2}}{\mathbf{W}} = \overset{2_1}{\mathcal{F}_q} \otimes \cdots \otimes \overset{2_n}{\mathcal{F}_q}$ and $\overset{\mathbf{3}}{\mathbf{W}} = \overset{3_1}{\mathcal{F}_q} \otimes \cdots \otimes \overset{3_n}{\mathcal{F}_q}$. Now the relation (13.6) is stated as the Yang–Baxter equation:

$$R^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{2}}(x) R^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{3}}(xy) R^{\mathrm{tr}_3}_{\mathbf{2},\mathbf{3}}(y) = R^{\mathrm{tr}_3}_{\mathbf{2},\mathbf{3}}(y) R^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{3}}(xy) R^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{2}}(x). \tag{13.8}$$

Suppressing the labels **1**, **2** etc., we set

$$R^{\mathrm{tr}_3}(z)(|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle) = \sum_{\mathbf{a},\mathbf{b}\in B} R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} |\mathbf{a}\rangle \otimes |\mathbf{b}\rangle. \tag{13.9}$$

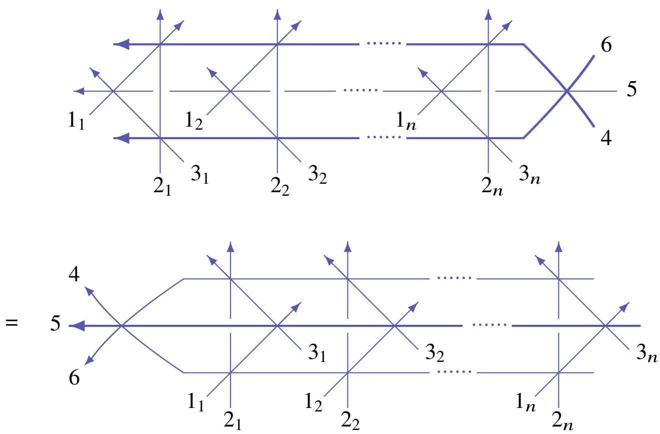

Fig. 13.2 A graphical representation of (13.2) and (13.3). It is a concatenation of Fig. 2.1 which corresponds to the basic $RRRR = RRRR$ relation. Each blue arrow carries $\mathcal{F}_q$

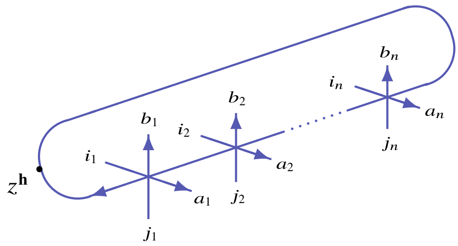

Fig. 13.3 Matrix product construction by the trace reduction (13.10) is depicted as a concatenation of Fig. 13.1 along the blue arrow corresponding to the third component of R. It is closed cyclically reflecting the trace

Then the construction (13.7) implies the matrix product formula

$$R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \mathrm{Tr}\big(z^{\mathbf{h}} R^{a_1 b_1}_{i_1 j_1} \cdots R^{a_n b_n}_{i_n j_n}\big) \tag{13.10}$$

in terms of the operator $R^{ab}_{ij} \in \mathrm{Osc}_q$ introduced in (2.4) and (2.5). In our case of the 3D R, it is explicitly given by (3.69).

By the definition, the trace is given by $\mathrm{Tr}(X) = \sum_{m\geq 0} \frac{\langle m|X|m\rangle}{\langle m|m\rangle} = \sum_{m\geq 0} \frac{\langle m|X|m\rangle}{(q^2)_m}$. See (3.12)–(3.17). Then (13.10) is evaluated by using the commutation relations of q-oscillators (3.12) and the formula (11.27). The matrix product formula (13.10) may also be presented as

$$R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \sum_{c_1,\ldots,c_n \geq 0} z^{c_1} R^{a_1 b_1 c_1}_{i_1 j_1 c_2} R^{a_2 b_2 c_2}_{i_2 j_2 c_3} \cdots R^{a_n b_n c_n}_{i_n j_n c_1} \tag{13.11}$$

in terms of the elements R^{abc}_{ijk} of the 3D R in the sense of (3.47). Explicit formulas of R^{abc}_{ijk} are available in Theorems 3.11, 3.18 and (3.84) (Fig. 13.3).

From the weight conservation (3.48), c_β in (13.11) is reducible to c_1 as

$$c_\beta = c_1 + \sum_{1\leq \alpha < \beta} (b_\alpha - j_\alpha), \tag{13.12}$$

therefore (13.11) is actually a *single* sum over c_1.

From (3.63), (3.48) and (3.70) it is easy to see

$$R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = 0 \text{ unless } \mathbf{a}+\mathbf{b} = \mathbf{i}+\mathbf{j} \text{ and } |\mathbf{a}| = |\mathbf{i}|, |\mathbf{b}| = |\mathbf{j}|, \tag{13.13}$$

$$R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = R^{\mathrm{tr}_3}(z)^{\mathbf{i}^\vee \mathbf{j}^\vee}_{\mathbf{a}^\vee \mathbf{b}^\vee} \prod_{k=1}^{n} \frac{(q^2)_{i_k}(q^2)_{j_k}}{(q^2)_{a_k}(q^2)_{b_k}}, \tag{13.14}$$

$$R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = z^{j_1 - b_1} R^{\mathrm{tr}_3}(z)^{\sigma(\mathbf{a})\sigma(\mathbf{b})}_{\sigma(\mathbf{i})\sigma(\mathbf{j})}, \tag{13.15}$$

where $\sigma(\mathbf{a}) = (a_2, \ldots, a_n, a_1)$ is a cyclic shift. The property (13.13) implies the decomposition

$$R^{\mathrm{tr}_3}(z) = \bigoplus_{l,m\geq 0} R^{\mathrm{tr}_3}_{l,m}(z), \qquad R^{\mathrm{tr}_3}_{l,m}(z) \in \mathrm{End}(\mathbf{W}_l \otimes \mathbf{W}_m). \tag{13.16}$$

The Yang–Baxter equation (13.8) is valid in each finite-dimensional subspace $\mathbf{W}_k \otimes \mathbf{W}_l \otimes \mathbf{W}_m$ of $\overset{1}{\mathbf{W}} \otimes \overset{2}{\mathbf{W}} \otimes \overset{3}{\mathbf{W}}$. In the current normalization we have

$$R^{\mathrm{tr}_3}_{l,m}(z)(|l\mathbf{e}_k\rangle \otimes |m\mathbf{e}_k\rangle) = \Lambda_{l,m}(z,q)\, |l\mathbf{e}_k\rangle \otimes |m\mathbf{e}_k\rangle \tag{13.17}$$

for any $1 \leq k \leq n$, where the factor $\Lambda_{l,m}(z,q)$ is given by

$$\Lambda_{l,m}(z,q) = \sum_{c\geq 0} z^c R^{lmc}_{lmc} = (-1)^m q^{m(l+1)} \frac{(q^{-l-m}z;q^2)_m}{(q^{l-m}z;q^2)_{m+1}}. \tag{13.18}$$

The second equality is shown by means of the general identity like (13.82). General elements $R^{\mathrm{tr}_3}_{l,m}(z)^{\mathbf{ab}}_{\mathbf{ij}}$ also become rational functions of q and z.

Example 13.1 Substituting the formulas in Example 3.17 into (13.10) and evaluating the trace we get

$$R^{\mathrm{tr}_3}_{m,1}(z)^{\mathbf{a}\,\mathbf{e}_b}_{\mathbf{i}\,\mathbf{e}_j} = \begin{cases} (q^{m-a_j}z - q^{a_j+1})/D & j = b, \\ z(1-q^{2a_b+2})q^{m-a_j-a_{j+1}-\cdots-a_b}/D & j < b, \\ (1-q^{2a_b+2})q^{a_{b+1}+a_{b+2}+\cdots+a_{j-1}}/D & j > b, \end{cases}$$

where $D = (1-q^{m-1}z)(1-q^{m+1}z)$, and $\mathbf{a}, \mathbf{i} \in B_m$ and $\mathbf{a}+\mathbf{e}_b = \mathbf{i}+\mathbf{e}_j$ are assumed.

From the remark after (3.71), this should coincide with (11.36) divided by $\varrho^{\mathrm{tr}_3}(z)|_{\alpha=1}$ in (11.33) provided that $\mathbf{a}, \mathbf{i} \in \mathfrak{s}_m$ [1] and $a_j = i_j = 0$ when $j = b$. This can be checked directly.

13.3 Trace Reduction Over the First Component of *R*

The following procedure is quite parallel with that in Sect. 11.3. Consider n copies of the tetrahedron equation (13.1) in which the spaces 3, 5, 6 are replaced by $3_i, 5_i, 6_i$ with $i = 1, \ldots, n$:

$$R_{45_i6_i} R_{23_i6_i} R_{13_i5_i} R_{124} = R_{124} R_{13_i5_i} R_{23_i6_i} R_{45_i6_i}.$$

Sending R_{124} to the left by applying this repeatedly, we get

[1] See (11.3) for the definition of $\mathfrak{s}_m$.

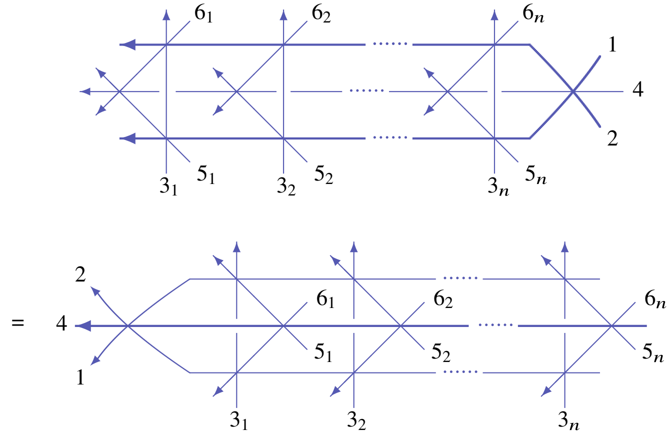

Fig. 13.4 A graphical representation of (13.19) and (13.20)

$$\begin{aligned}&(R_{45_16_1}R_{23_16_1}R_{13_15_1})\cdots(R_{45_n6_n}R_{23_n6_n}R_{13_n5_n})R_{124}\\&\quad = R_{124}(R_{13_15_1}R_{23_16_1}R_{45_16_1})\cdots(R_{13_n5_n}R_{23_n6_n}R_{45_n6_n}),\end{aligned} \tag{13.19}$$

which can be rearranged as (Fig. 13.4)

$$\begin{aligned}&(R_{45_16_1}\cdots R_{45_n6_n})(R_{23_16_1}\cdots R_{23_n6_n})(R_{13_15_1}\cdots R_{13_n5_n})R_{124}\\&\quad = R_{124}(R_{13_15_1}\cdots R_{13_n5_n})(R_{23_16_1}\cdots R_{23_n6_n})(R_{45_16_1}\cdots R_{45_n6_n}).\end{aligned} \tag{13.20}$$

Multiply $x^{\mathbf{h}_1}(xy)^{\mathbf{h}_2}y^{\mathbf{h}_4}R_{124}^{-1}$ from the left by (13.20) and take the trace over $\overset{1}{\mathcal{F}}_q \otimes \overset{2}{\mathcal{F}}_q \otimes \overset{4}{\mathcal{F}}_q$. Using the weight conservation (13.4) we get the Yang–Baxter equation.

$$R^{\mathrm{tr}_1}_{\mathbf{5},\mathbf{6}}(y)R^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{6}}(xy)R^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{5}}(x) = R^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{5}}(x)R^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{6}}(xy)R^{\mathrm{tr}_1}_{\mathbf{5},\mathbf{6}}(y) \in \mathrm{End}(\overset{\mathbf{3}}{\mathbf{W}} \otimes \overset{\mathbf{5}}{\mathbf{W}} \otimes \overset{\mathbf{6}}{\mathbf{W}}), \tag{13.21}$$

where $\overset{\mathbf{3}}{\mathbf{W}} = \overset{3_1}{\mathcal{F}_q} \otimes \cdots \otimes \overset{3_n}{\mathcal{F}_q}$ $\overset{\mathbf{5}}{\mathbf{W}} = \overset{5_1}{\mathcal{F}_q} \otimes \cdots \otimes \overset{5_n}{\mathcal{F}_q}$ and $\overset{\mathbf{6}}{\mathbf{W}} = \overset{6_1}{\mathcal{F}_q} \otimes \cdots \otimes \overset{6_n}{\mathcal{F}_q}$. The superscript tr_1 signifies that the trace is taken over the 1st (leftmost) component of the 3D R as

$$R^{\mathrm{tr}_1}_{\mathbf{5},\mathbf{6}}(z) = \mathrm{Tr}_4(z^{\mathbf{h}_4}R_{45_16_1}\cdots R_{45_n6_n}) \in \mathrm{End}(\overset{\mathbf{5}}{\mathbf{W}} \otimes \overset{\mathbf{6}}{\mathbf{W}}), \tag{13.22}$$

$$R^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{5}}(z) = \mathrm{Tr}_1(z^{\mathbf{h}_1}R_{13_15_1}\cdots R_{13_n5_n}) \in \mathrm{End}(\overset{\mathbf{3}}{\mathbf{W}} \otimes \overset{\mathbf{5}}{\mathbf{W}}), \tag{13.23}$$

$$R^{\mathrm{tr}_1}_{\mathbf{3},\mathbf{6}}(z) = \mathrm{Tr}_2(z^{\mathbf{h}_2}R_{23_16_1}\cdots R_{23_n6_n}) \in \mathrm{End}(\overset{\mathbf{3}}{\mathbf{W}} \otimes \overset{\mathbf{6}}{\mathbf{W}}). \tag{13.24}$$

These are the same operators acting on different copies of $\mathbf{W} \otimes \mathbf{W}$. We will often suppress the labels **3**, **5** etc. The expression (13.22) has already appeared in (11.40) and it is depicted as the left diagram in Fig. 11.5. The operator $R^{\mathrm{tr}_1}(z)$ acts on the basis in (11.13) as

$$R^{\mathrm{tr}_1}(z)(|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle) = \sum_{\mathbf{a},\mathbf{b}\in B} R^{\mathrm{tr}_1}(z)^{\mathbf{ab}}_{\mathbf{ij}} |\mathbf{a}\rangle \otimes |\mathbf{b}\rangle, \tag{13.25}$$

$$R^{\mathrm{tr}_1}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \sum_{k_1,\dots,k_n\geq 0} z^{k_1} R^{k_1 a_1 b_1}_{k_2 i_1 j_1} R^{k_2 a_2 b_2}_{k_3 i_2 j_2} \cdots R^{k_n a_n b_n}_{k_1 i_n j_n}. \tag{13.26}$$

Comparing this with (13.11) and using (3.62), we find that $R^{\mathrm{tr}_1}(z)$ is simply related to $R^{\mathrm{tr}_3}(z)$ as

$$R^{\mathrm{tr}_1}(z)^{\mathbf{ab}}_{\mathbf{ij}} = R^{\mathrm{tr}_3}(z)^{\mathbf{ba}}_{\mathbf{ji}} \quad \text{i.e.} \quad R^{\mathrm{tr}_1}(z) = P R^{\mathrm{tr}_3}(z) P, \tag{13.27}$$

where $P(u \otimes v) = v \otimes u$ is the exchange of the components. Consequently, all the properties in (13.14)–(13.17) are valid beside minor changes in (13.15) and (13.17):

$$R^{\mathrm{tr}_1}(z)^{\mathbf{ab}}_{\mathbf{ij}} = 0 \text{ unless } \mathbf{a}+\mathbf{b} = \mathbf{i}+\mathbf{j} \text{ and } |\mathbf{a}| = |\mathbf{i}|, |\mathbf{b}| = |\mathbf{j}|, \tag{13.28}$$

$$R^{\mathrm{tr}_1}(z) = \bigoplus_{l,m\geq 0} R^{\mathrm{tr}_1}_{l,m}(z), \qquad R^{\mathrm{tr}_1}_{l,m}(z) \in \mathrm{End}(\mathbf{W}_l \otimes \mathbf{W}_m), \tag{13.29}$$

$$R^{\mathrm{tr}_1}_{l,m}(z)(|l\mathbf{e}_k\rangle \otimes |m\mathbf{e}_k\rangle) = \Lambda_{m,l}(z,q)\, |l\mathbf{e}_k\rangle \otimes |m\mathbf{e}_k\rangle, \tag{13.30}$$

$$R^{\mathrm{tr}_1}(z)^{\mathbf{ab}}_{\mathbf{ij}} = R^{\mathrm{tr}_1}(z)^{\mathbf{i}^\vee \mathbf{j}^\vee}_{\mathbf{a}^\vee \mathbf{b}^\vee} \prod_{k=1}^{n} \frac{(q^2)_{i_k}(q^2)_{j_k}}{(q^2)_{a_k}(q^2)_{b_k}}, \tag{13.31}$$

$$R^{\mathrm{tr}_1}(z)^{\mathbf{ab}}_{\mathbf{ij}} = z^{b_1 - j_1} R^{\mathrm{tr}_1}(z)^{\sigma(\mathbf{a})\sigma(\mathbf{b})}_{\sigma(\mathbf{i})\sigma(\mathbf{j})}, \tag{13.32}$$

where $\Lambda_{m,l}(z,q)$ in (13.30) is given by $(13.18)_{l\leftrightarrow m}$. The Yang–Baxter equation (13.21) holds in each finite-dimensional subspace $\mathbf{W}_k \otimes \mathbf{W}_l \otimes \mathbf{W}_m$ of $\mathbf{W} \otimes \mathbf{W} \otimes \mathbf{W}$.

13.4 Trace Reduction Over the Second Component of *R*

The following procedure is quite parallel with that in Sect. 11.4. Consider n copies of the tetrahedron equation (13.1) in which the spaces 1, 4, 5 are replaced by $1_i, 4_i, 5_i$ with $i = 1, \dots, n$:

$$R_{4_i 5_i 6} R_{1_i 2 4_i} R_{1_i 3 5_i} R_{236} = R_{236} R_{1_i 3 5_i} R_{1_i 2 4_i} R_{4_i 5_i 6}.$$

Here we have relocated R by using $R = R^{-1}$ (3.60). Sending R_{236} to the left by applying this repeatedly, we get

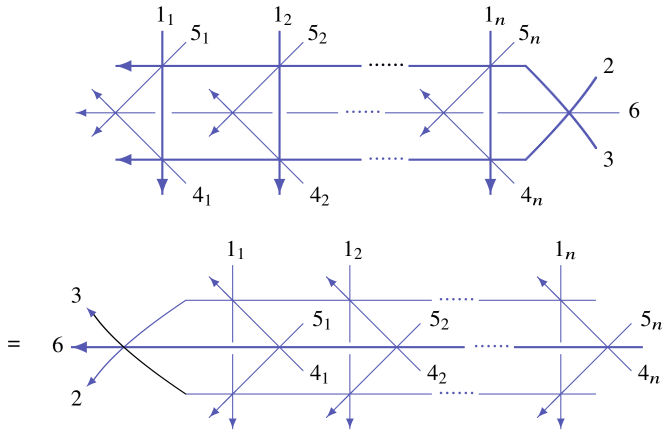

Fig. 13.5 A graphical representation of (13.33) and (13.34)

$$\begin{aligned}(R_{4_15_16}R_{1_124_1}R_{1_135_1})\cdots(R_{4_n5_n6}R_{1_n24_n}R_{1_n35_n})R_{236}\\ = R_{236}(R_{1_135_1}R_{1_124_1}R_{4_15_16})\cdots(R_{1_n35_n}R_{1_n24_n}R_{4_n5_n6}),\end{aligned} \tag{13.33}$$

which can be rearranged as (Fig. 13.5)

$$\begin{aligned}(R_{4_15_16}\cdots R_{4_n5_n6})(R_{1_124_1}\cdots R_{1_n24_n})(R_{1_135_1}\cdots R_{1_n35_n})R_{236}\\ = R_{236}(R_{1_135_1}\cdots R_{1_n35_n})(R_{1_124_1}\cdots R_{1_n24_n})(R_{4_15_16}\cdots R_{4_n5_n6}).\end{aligned} \tag{13.34}$$

Multiply $x^{\mathbf{h}_2}(xy)^{\mathbf{h}_3}y^{\mathbf{h}_6}R_{236}^{-1}$ from the left by (13.34) and take the trace over $\overset{2}{\mathcal{F}}_q \otimes \overset{3}{\mathcal{F}}_q \otimes \overset{6}{\mathcal{F}}_q$. Using the weight conservation (13.4) we get the Yang–Baxter equation.

$$R^{\mathrm{tr}_3}_{\mathbf{4},\mathbf{5}}(y)R^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{4}}(x)R^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{5}}(xy) = R^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{5}}(xy)R^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{4}}(x)R^{\mathrm{tr}_3}_{\mathbf{4},\mathbf{5}}(y) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{W}} \otimes \overset{\mathbf{4}}{\mathbf{W}} \otimes \overset{\mathbf{5}}{\mathbf{W}}), \tag{13.35}$$

where $\overset{\mathbf{1}}{\mathbf{W}} = \overset{1_1}{\mathcal{F}}_q \otimes \cdots \otimes \overset{1_n}{\mathcal{F}}_q$ $\overset{\mathbf{4}}{\mathbf{W}} = \overset{4_1}{\mathcal{F}}_q \otimes \cdots \otimes \overset{4_n}{\mathcal{F}}_q$ and $\overset{\mathbf{5}}{\mathbf{W}} = \overset{5_1}{\mathcal{F}}_q \otimes \cdots \otimes \overset{5_n}{\mathcal{F}}_q$. The superscript tr_2 signifies that the trace is taken over the second (middle) component as (Fig. 13.6)

$$R^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{4}}(z) = \mathrm{Tr}_2(z^{\mathbf{h}_2}R_{1_124_1}\cdots R_{1_n24_n}) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{W}} \otimes \overset{\mathbf{4}}{\mathbf{W}}), \tag{13.36}$$

$$R^{\mathrm{tr}_2}_{\mathbf{1},\mathbf{5}}(z) = \mathrm{Tr}_3(z^{\mathbf{h}_3}R_{1_135_1}\cdots R_{1_n35_n}) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{W}} \otimes \overset{\mathbf{5}}{\mathbf{W}}). \tag{13.37}$$

These are the same operators acting on different copies of $\mathbf{W} \otimes \mathbf{W}$. We will often suppress the labels like $\mathbf{1}, \mathbf{4}$. The operator $R^{\mathrm{tr}_3}(y)$ has already appeared in (11.40).

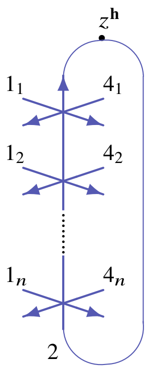

Fig. 13.6 A graphical representation of (13.36). The one for (13.37) just corresponds to a relabeling of the arrows

The operator $R^{\mathrm{tr}_2}(z)$ acts on the basis as

$$R^{\mathrm{tr}_2}(z)(|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle) = \sum_{\mathbf{a},\mathbf{b}\in B} R^{\mathrm{tr}_2}(z)^{\mathbf{ab}}_{\mathbf{ij}} |\mathbf{a}\rangle \otimes |\mathbf{b}\rangle, \tag{13.38}$$

$$R^{\mathrm{tr}_2}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \sum_{k_1,\dots,k_n\geq 0} z^{k_1} R^{a_1k_1b_1}_{i_1k_2j_1} R^{a_2k_2b_2}_{i_2k_3j_2} \cdots R^{a_nk_nb_n}_{i_nk_1j_n}. \tag{13.39}$$

Comparing (13.39) and (13.11) using (3.86) and (3.62), we find

$$R^{\mathrm{tr}_2}(z)^{\mathbf{ab}}_{\mathbf{ij}} = (-q)^{-l+\sum_{k=1}^n k(j_k-b_k)} \left(\prod_{k=1}^n \frac{(q^2)_{j_k}}{(q^2)_{b_k}}\right) R^{\mathrm{tr}_3}((-q)^n z)^{\mathbf{ja}}_{\mathbf{bi}} \tag{13.40}$$

for $\mathbf{a}, \mathbf{i} \in B_l$ and $\mathbf{b}, \mathbf{j} \in B_m$. One can derive properties similar to $R^{\mathrm{tr}_1}(z)$ as follows:

$$R^{\mathrm{tr}_2}(z)^{\mathbf{ab}}_{\mathbf{ij}} = 0 \text{ unless } \mathbf{a}-\mathbf{b} = \mathbf{i}-\mathbf{j} \text{ and } |\mathbf{a}| = |\mathbf{i}|, |\mathbf{b}| = |\mathbf{j}|, \tag{13.41}$$

$$R^{\mathrm{tr}_2}(z) = \bigoplus_{l,m\geq 0} R^{\mathrm{tr}_2}_{l,m}(z), \qquad R^{\mathrm{tr}_2}_{l,m}(z) \in \mathrm{End}(\mathbf{W}_l \otimes \mathbf{W}_m), \tag{13.42}$$

$$R^{\mathrm{tr}_2}_{l,m}(z)(|l\mathbf{e}_1\rangle \otimes |m\mathbf{e}_2\rangle) = \frac{|l\mathbf{e}_1\rangle \otimes |m\mathbf{e}_2\rangle}{1+(-1)^{n+1}q^{l+m+n}z}, \tag{13.43}$$

$$R^{\mathrm{tr}_2}(z)^{\mathbf{ab}}_{\mathbf{ij}} = R^{\mathrm{tr}_2}(z)^{\mathbf{ba}}_{\mathbf{ji}}, \tag{13.44}$$

$$R^{\mathrm{tr}_2}(z)^{\mathbf{ab}}_{\mathbf{ij}} = z^{j_1-b_1} R^{\mathrm{tr}_2}(z)^{\sigma(\mathbf{a})\sigma(\mathbf{b})}_{\sigma(\mathbf{i})\sigma(\mathbf{j})}, \tag{13.45}$$

$$R^{\mathrm{tr}_2}(z)^{\mathbf{ab}}_{\mathbf{ij}} = R^{\mathrm{tr}_2}(z)^{\mathbf{i}^\vee \mathbf{j}^\vee}_{\mathbf{a}^\vee \mathbf{b}^\vee} \prod_{k=1}^n \frac{(q^2)_{i_k}(q^2)_{j_k}}{(q^2)_{a_k}(q^2)_{b_k}}. \tag{13.46}$$

13.5 Explicit Formulas of $R^{\mathrm{tr}_1}(z)$, $R^{\mathrm{tr}_2}(z)$, $R^{\mathrm{tr}_3}(z)$

The main result of this section is the explicit formulas in Theorem 13.3 which are derived from the matrix product construction by a direct calculation. The detail of the proof will not be used elsewhere and can be skipped. It is included in the light of the fact that the relevant quantum R matrices (Theorems 13.10, 13.11 and 13.12) are very fundamental examples associated with higher rank type A quantum groups with higher "spin" representations.

13.5.1 Function $A(z)^{\mathbf{ab}}_{\mathbf{ij}}$

For integer arrays $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k) \in \mathbb{Z}^k$ of *any* length k, we use the notation

$$|\boldsymbol{\alpha}| = \sum_{1 \le i \le k} \alpha_i, \quad \overline{\boldsymbol{\alpha}} = (\alpha_1, \dots, \alpha_{k-1}), \tag{13.47}$$

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = \sum_{1 \le i < j \le k} \alpha_i \beta_j, \quad (\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{1 \le i \le k} \alpha_i \beta_i, \tag{13.48}$$

where $|\boldsymbol{\alpha}|$ appeared also in (11.4) for $\boldsymbol{\alpha} \in \{0, 1\}^n$.

For parameters λ, μ and arrays $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k)$, $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k) \in \mathbb{Z}^k_{\ge 0}$ of *any* length k, define

$$\Phi_q(\boldsymbol{\gamma}|\boldsymbol{\beta}; \lambda, \mu) = q^{\langle \boldsymbol{\beta} - \boldsymbol{\gamma}, \boldsymbol{\gamma} \rangle} \left(\frac{\mu}{\lambda}\right)^{|\boldsymbol{\gamma}|} \overline{\Phi}_q(\boldsymbol{\gamma}|\boldsymbol{\beta}; \lambda, \mu), \tag{13.49}$$

$$\overline{\Phi}_q(\boldsymbol{\gamma}|\boldsymbol{\beta}; \lambda, \mu) = \frac{(\lambda; q)_{|\boldsymbol{\gamma}|} (\frac{\mu}{\lambda}; q)_{|\boldsymbol{\beta}| - |\boldsymbol{\gamma}|}}{(\mu; q)_{|\boldsymbol{\beta}|}} \prod_{i=1}^{k} \binom{\beta_i}{\gamma_i}_q. \tag{13.50}$$

From the definition of the q-binomial in (3.65), $\overline{\Phi}_q(\boldsymbol{\gamma}|\boldsymbol{\beta}; \lambda, \mu) = 0$ unless $\gamma_i \le \beta_i$ for all $1 \le i \le k$. We will write this condition as $\boldsymbol{\gamma} \le \boldsymbol{\beta}$.

Given n component arrays $\mathbf{a}, \mathbf{i} \in B_l$ and $\mathbf{b}, \mathbf{j} \in B_m$ (see (11.10) for the definition of B_k), we introduce a quadratic combination of (13.49) as

$$A(z)^{\mathbf{ab}}_{\mathbf{ij}} = q^{\langle \mathbf{i}, \mathbf{j} \rangle - \langle \mathbf{b}, \mathbf{a} \rangle} \times \sum_{\overline{\mathbf{k}}} \Phi_{q^2}(\overline{\mathbf{a}} - \overline{\mathbf{k}} | \overline{\mathbf{a}} + \overline{\mathbf{b}} - \overline{\mathbf{k}}; q^{m-l} z, q^{-l-m} z) \Phi_{q^2}(\overline{\mathbf{k}} | \overline{\mathbf{j}}; q^{-l-m} z^{-1}, q^{-2m}), \tag{13.51}$$

where the sum ranges over $\overline{\mathbf{k}} \in \mathbb{Z}_{\geq 0}^{n-1}$.[2] Due to the remark after (13.50), it is actually confined into the finite set $0 \leq \overline{\mathbf{k}} \leq \min(\overline{\mathbf{b}}, \overline{\mathbf{j}})$ meaning that $0 \leq k_r \leq \min(b_r, j_r)$ for $1 \leq r \leq n-1$. A characteristic feature of the formula (13.51) is that Φ_{q^2} depends on $\mathbf{a} = (a_1, \dots, a_n) \in B_l$ via $\overline{\mathbf{a}} = (a_1, \dots, a_{n-1})$ and l by which the last component is taken into account as $a_n = l - |\overline{\mathbf{a}}|$. Dependence on $\mathbf{b}$ and $\mathbf{j}$ is similar. Substituting (13.49) and (13.50) into (13.51) we get

$$
\begin{aligned}
A(z)_{\mathbf{i}\,\mathbf{j}}^{\mathbf{a}\mathbf{b}} &= (-1)^{b_n - j_n} q^{\varphi} \frac{(q^2)_{j_n}}{(q^2)_{b_n}} \sum_{\overline{\mathbf{k}}} q^{2\langle \overline{\mathbf{j}} - \overline{\mathbf{b}} - \overline{\mathbf{k}}, \overline{\mathbf{k}} \rangle + (l+m)|\overline{\mathbf{k}}|} \prod_{\alpha=1}^{n-1} \binom{a_\alpha + b_\alpha - k_\alpha}{b_\alpha}_{q^2} \binom{j_\alpha}{k_\alpha}_{q^2} \\
&\quad \times z^{|\overline{\mathbf{k}}|} \frac{(q^{m-l} z; q^2)_{|\overline{\mathbf{a}} - \overline{\mathbf{k}}|} (q^{l-m} z; q^2)_{|\overline{\mathbf{j}} - \overline{\mathbf{k}}|} (q^{-l-m} z^{-1}; q^2)_{|\overline{\mathbf{k}}|}}{(q^{-l-m} z; q^2)_{|\overline{\mathbf{a}} + \overline{\mathbf{b}} - \overline{\mathbf{k}}|}}, \qquad (13.52)
\end{aligned}
$$

$$
\varphi = \langle \overline{\mathbf{i}}, \overline{\mathbf{j}} \rangle + \langle \overline{\mathbf{b}}, \overline{\mathbf{a}} \rangle + m a_n + l j_n + (b_n - j_n)(i_n + j_n + 1) - 2ml. \qquad (13.53)
$$

The factor $(q^2)_{j_n}/(q^2)_{b_n}$ here originates in $(q^{-2m})_{|\overline{\mathbf{b}}|}/(q^{-2m})_{|\overline{\mathbf{j}}|}$ contained in (13.51).

Remark 13.2 By an induction on k, it can be shown that

$$
\sum_{\boldsymbol{\gamma} \in (\mathbb{Z}_{\geq 0})^k,\, \boldsymbol{\gamma} \leq \boldsymbol{\beta}} \Phi_q(\boldsymbol{\gamma} | \boldsymbol{\beta}; \lambda, \mu) = 1 \qquad (\forall \boldsymbol{\beta} \in (\mathbb{Z}_{\geq 0})^k). \qquad (13.54)
$$

This property has an application to stochastic models, where it plays the role of the total probability conservation. It can also be derived from Proposition 13.13 and (13.132).

13.5.2 $A(z)_{\mathbf{i}\,\mathbf{j}}^{\mathbf{a}\mathbf{b}}$ as Elements of $R^{\mathrm{tr}_1}(z)$, $R^{\mathrm{tr}_2}(z)$ and $R^{\mathrm{tr}_3}(z)$

Theorem 13.3 *For* $\mathbf{a}, \mathbf{i} \in B_l$, $\mathbf{b}, \mathbf{j} \in B_m$, *the following formulas are valid:*

$$
\Lambda_{l,m}(z,q)^{-1} R^{\mathrm{tr}_3}(z)_{\mathbf{i}\,\mathbf{j}}^{\mathbf{a}\mathbf{b}} = \delta_{\mathbf{i}+\mathbf{j}}^{\mathbf{a}+\mathbf{b}} A(z)_{\mathbf{i}\,\mathbf{j}}^{\mathbf{a}\mathbf{b}}, \qquad (13.55)
$$

$$
\Lambda_{m,l}(z,q)^{-1} R^{\mathrm{tr}_1}(z)_{\mathbf{i}\,\mathbf{j}}^{\mathbf{a}\mathbf{b}} = \delta_{\mathbf{i}+\mathbf{j}}^{\mathbf{a}+\mathbf{b}} A(z)_{\mathbf{j}\,\mathbf{i}}^{\mathbf{b}\mathbf{a}}, \qquad (13.56)
$$

$$
\Lambda_{m,l}((-q)^n z, q)^{-1} R^{\mathrm{tr}_2}(z)_{\mathbf{i}\,\mathbf{j}}^{\mathbf{a}\mathbf{b}} = (-q)^{-l + \sum_{\alpha=1}^{n} \alpha (j_\alpha - b_\alpha)} \left(\prod_{\alpha=1}^{n} \frac{(q^2)_{j_\alpha}}{(q^2)_{b_\alpha}} \right) \delta_{\mathbf{b}+\mathbf{i}}^{\mathbf{a}+\mathbf{j}} A((-q)^n z)_{\mathbf{b}\,\mathbf{i}}^{\mathbf{j}\,\mathbf{a}}, \qquad (13.57)
$$

where $\Lambda_{l,m}(z,q)$ *is defined by (13.18).*

[2] $\overline{\mathbf{k}}$ is just an array of summation variables. We have not introduced an n component array $\mathbf{k}$ which is related to it as in (13.47).

13.5.3 *Proof of Theorem 13.3*

The formulas (13.56) and (13.57) follow from (13.55) by virtue of (13.27) and (13.40). Therefore we concentrate on (13.55) in the sequel. The following lemma is nothing but a quantum group symmetry (13.105) with $R^{\mathrm{tr}_3}(z)$ replaced by the matrix having the elements $A(z)^{\mathbf{ab}}_{\mathbf{ij}}$.

Lemma 13.4 *Suppose $n \geq 3$. For $1 \leq r \leq n-2$, the function $A(z)^{\mathbf{ab}}_{\mathbf{ij}}$ satisfies the relation*

$$\begin{aligned}&[b_{r+1}+1]_{q^2} A(z)^{\mathbf{a},\mathbf{b}-\hat{r}}_{\mathbf{i},\mathbf{j}} + q^{b_r-b_{r+1}}[a_{r+1}+1]_{q^2} A(z)^{\mathbf{a}-\hat{r},\mathbf{b}}_{\mathbf{i},\mathbf{j}} \\ &- [i_{r+1}]_{q^2} A(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i}+\hat{r},\mathbf{j}} - q^{i_r-i_{r+1}}[j_{r+1}]_{q^2} A(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}+\hat{r}} = 0\end{aligned} \tag{13.58}$$

for $\mathbf{a}+\mathbf{b} = \mathbf{i}+\mathbf{j}+\hat{r}$. Here $\hat{r} = \mathbf{e}_r - \mathbf{e}_{r+1}$ with $\mathbf{e}_r$ being an elementary vector in (11.1). The symbol $[m]_{q^2}$ is defined in (11.57).

Proof Let $\overline{\mathbf{k}} = (k_1, \ldots, k_{n-1})$ in (13.52). It turns out that (13.58) holds for the partial sum of (13.52) in which $k_\alpha (\alpha \neq r, r+1)$ and $|\overline{\mathbf{k}}|$ are fixed. Under this constraint $A(z)^{\mathbf{ab}}_{\mathbf{ij}}$ is proportional to

$$q^{\langle \bar{\mathbf{i}}, \bar{\mathbf{j}} \rangle - \langle \overline{\mathbf{b}}, \overline{\mathbf{a}} \rangle} \sum q^{2(j_r-b_r-k_r)k_{r+1}} \prod_{\alpha=r,r+1} \binom{a_\alpha+b_\alpha-k_\alpha}{b_\alpha}_{q^2} \binom{j_\alpha}{k_\alpha}_{q^2} \tag{13.59}$$

up to a common overall factor. The sum here is taken over $k_r, k_{r+1} \geq 0$ under the condition $k_r + k_{r+1} = k$ for any fixed k. There is no dependence on the spectral parameter z owing to the assumption $r \neq 0, n-1$. Substituting this into (13.58) and using $\langle \hat{r}, \mathbf{j} \rangle = j_{r+1}$ and $\langle \mathbf{b}, \hat{r} \rangle = -b_r$, we find that (13.58) follows from

$$\begin{aligned}&q^{-a_2-b_2-1}(1-q^{2b_2+2}) \sum q^{2(j_1-b_1-k_1+1)k_2} \\ &\quad \times \binom{a_1+b_1-k_1-1}{b_1-1}_{q^2} \binom{a_2+b_2-k_2+1}{b_2+1}_{q^2} \binom{j_1}{k_1}_{q^2} \binom{j_2}{k_2}_{q^2} \\ &+ q^{2b_1-b_2-a_2-1}(1-q^{2a_2+2}) \sum q^{2(j_1-b_1-k_1)k_2} \\ &\quad \times \binom{a_1+b_1-k_1-1}{b_1}_{q^2} \binom{a_2+b_2-k_2+1}{b_2}_{q^2} \binom{j_1}{k_1}_{q^2} \binom{j_2}{k_2}_{q^2} \\ &- q^{j_2-i_2}(1-q^{2i_2}) \sum q^{2(j_1-b_1-k_1)k_2} \\ &\quad \times \binom{a_1+b_1-k_1}{b_1}_{q^2} \binom{a_2+b_2-k_2}{b_2}_{q^2} \binom{j_1}{k_1}_{q^2} \binom{j_2}{k_2}_{q^2} \\ &- q^{-i_2-j_2}(1-q^{2j_2}) \sum q^{2(j_1-b_1-k_1+1)k_2} \\ &\quad \times \binom{a_1+b_1-k_1}{b_1}_{q^2} \binom{a_2+b_2-k_2}{b_2}_{q^2} \binom{j_1+1}{k_1}_{q^2} \binom{j_2-1}{k_2}_{q^2} = 0,\end{aligned} \tag{13.60}$$

where we have denoted a_r, a_{r+1} by a_1, a_2 for simplicity and similarly for the other letters. Thus in particular, $a_1 + b_1 = i_1 + j_1 + 1$ and $a_2 + b_2 = i_2 + j_2 - 1$, reflecting the assumption $\mathbf{a} + \mathbf{b} = \mathbf{i} + \mathbf{j} + \hat{r}$.

The sums in (13.60) are taken over $k_1, k_2 \geq 0$ with the constraint $k_1 + k_2 = k$ for any fixed k. Apart from this constraint, the summation variables k_1 and k_2 are coupling via the factor $q^{-2k_1k_2}$. Fortunately this can be decoupled by rewriting the q^2-binomials as

$$\binom{a_\alpha + b_\alpha - k_\alpha}{b_\alpha}_{q^2} \binom{j_\alpha}{k_\alpha}_{q^2} = (-1)^{k_\alpha} q^{-k_\alpha^2 + (2j_\alpha - 2b_\alpha + 1)k_\alpha} \frac{(q^{2b_\alpha+2}; q^2)_{a_\alpha} (q^{-2a_\alpha}; q^2)_{k_\alpha} (q^{-2j_\alpha}; q^2)_{a_\alpha}}{(q^2; q^2)_{a_\alpha} (q^{-2a_\alpha - 2b_\alpha}; q^2)_{k_\alpha} (q^2; q^2)_{k_\alpha}}. \tag{13.61}$$

In fact, this converts the quadratic power of k_1 and k_2 into an overall constant $q^{-k_1^2 - k_2^2 - 2k_1k_2} = q^{-k^2}$ which can be removed. Consequently, each sum in (13.60) is rewritten in the form $\sum_{k_1+k_2=k} (\sum_{k_1 \geq 0} X_{k_1})(\sum_{k_2 \geq 0} Y_{k_2})$ for any fixed k. Thus introducing the generating series $\sum_{k \geq 0} \zeta^k (\cdots)$ decouples it into the product $(\sum_{k_1 \geq 0} \zeta^{k_1} X_{k_1})(\sum_{k_2 \geq 0} \zeta^{k_2} Y_{k_2})$. Each factor here becomes q^2-hypergeometric defined in (3.73). After some calculation one finds that the explicit form is given, up to an overall factor, by the LHS of (13.62) with the variables replaced as $q \to q^2$, $u_\alpha \to q^{-2a_\alpha}$, $v_\alpha \to q^{-2a_\alpha - 2b_\alpha}$, $w_\alpha \to q^{-2j_\alpha}$ for $\alpha = 1, 2$. This also means $q^{-2i_1} = q^2 v_1/w_1$ and $q^{-2i_2} = q^{-2} v_2/w_2$. Therefore the proof is reduced to Lemma 13.5. □

Lemma 13.5 *The q-hypergeometric $\phi\left({a,b \atop c}; \zeta\right) := {}_2\phi_1\left({a,b \atop c}; q, \zeta\right)$ in (3.73) satisfies the quadratic relation involving the six parameters $u_\alpha, v_\alpha, w_\alpha (\alpha = 1, 2)$ in addition to q and ζ:*

$$\begin{aligned} &u_1(1 - u_1^{-1}v_1)(q - v_2)\phi\left({u_1, w_1 \atop qv_1}; q\zeta\right)\phi\left({u_2, w_2 \atop q^{-1}v_2}; u_2^{-1}v_2w_2^{-1}\zeta\right) \\ &+ (1 - u_1)(q - v_2)\phi\left({qu_1, w_1 \atop qv_1}; \zeta\right)\phi\left({q^{-1}u_2, w_2 \atop q^{-1}v_2}; u_2^{-1}v_2w_2^{-1}\zeta\right) \\ &- (1 - v_1)(q - v_2w_2^{-1})\phi\left({u_1, w_1 \atop v_1}; \zeta\right)\phi\left({u_2, w_2 \atop v_2}; u_2^{-1}v_2w_2^{-1}\zeta\right) \\ &- v_2w_2^{-1}(1 - v_1)(1 - w_2)\phi\left({u_1, q^{-1}w_1 \atop v_1}; q\zeta\right)\phi\left({u_2, qw_2 \atop v_2}; u_2^{-1}v_2w_2^{-1}\zeta\right) = 0. \end{aligned} \tag{13.62}$$

Proof First, we apply

$$\phi\left({a, b \atop c}; \zeta\right) = \frac{(c - abz)}{c(1 - z)}\phi\left({a, b \atop c}; q\zeta\right) + \frac{z(a - c)(b - c)}{c(1 - c)(1 - z)}\phi\left({a, b \atop qc}; q\zeta\right) \tag{13.63}$$

to the left ϕ's in the second and the third terms to change their argument from ζ to $q\zeta$ to adjust to the first and the fourth terms. The resulting sum is a linear combination of

$$X = \phi\left(\begin{matrix} u_1, w_1 \\ qv_1 \end{matrix}; q\zeta\right), \quad Y = \phi\left(\begin{matrix} qu_1, w_1 \\ qv_1 \end{matrix}; q\zeta\right), \tag{13.64}$$

$$\phi\left(\begin{matrix} qu_1, w_1 \\ q^2v_1 \end{matrix}; q\zeta\right), \quad \phi\left(\begin{matrix} u_1, w_1 \\ v_1 \end{matrix}; q\zeta\right), \quad \phi\left(\begin{matrix} u_1, q^{-1}w_1 \\ v_1 \end{matrix}; q\zeta\right). \tag{13.65}$$

Second, we express (13.65) in terms of X and Y by means of the contiguous relations:

$$\phi\left(\begin{matrix} qu_1, w_1 \\ q^2v_1 \end{matrix}; q\zeta\right) = -\frac{v_1(1-qv_1)}{u_1(qv_1-w_1)\zeta}X + \frac{(1-qv_1)(v_1-u_1w_1\zeta)}{u_1(qv_1-w_1)\zeta}Y, \tag{13.66}$$

$$\phi\left(\begin{matrix} u_1, w_1 \\ v_1 \end{matrix}; q\zeta\right) = \frac{(u_1-v_1)}{u_1(1-v_1)}X + \frac{(1-u_1)v_1}{u_1(1-v_1)}Y, \tag{13.67}$$

$$\begin{aligned}\phi\left(\begin{matrix} u_1, q^{-1}w_1 \\ v_1 \end{matrix}; q\zeta\right) &= \frac{(u_1-v_1)\big(v_1(q-w_1)-q(1-v_1)w_1\zeta\big)}{qu_1(1-v_1)(qv_1-w_1)\zeta}X \\ &+ \frac{(v_1-u_1w_1\zeta)\big((u_1-v_1)(q-w_1)-q(1-v_1)(qu_1-w_1)\zeta\big)}{qu_1(1-v_1)(qv_1-w_1)\zeta}Y.\end{aligned} \tag{13.68}$$

As the result, the LHS of (13.62) is cast into the form $AX + BY$ where A and B are *linear* combinations of the four right ϕ's all having the argument $u_2^{-1}v_2w_2^{-1}\zeta$. The coefficients of the linear combinations are Laurent polynomials of ζ. Then it is straightforward to check $A = B = 0$ by picking the coefficient of each power of ζ. □

In the remainder of this section, $(\zeta)_m$ always means $(\zeta; q^2)_m$ for any ζ.[3]

Lemma 13.6 *The formula (13.55) is valid provided that* $\mathbf{a} = (a_1, \ldots, a_n)$ *has vanishing components as* $a_2 = \cdots = a_{n-1} = 0$.

Proof Throughout the proof $\mathbf{a}$ should be understood as the special one $\mathbf{a} = (a_1, 0, \ldots, 0, a_n)$. We also keep assuming $\mathbf{a}, \mathbf{i} \in B_l$, $\mathbf{b}, \mathbf{j} \in B_m$ and $\mathbf{a} + \mathbf{b} = \mathbf{i} + \mathbf{j}$ following Theorem 13.3. Then we have the relations like

$$l = a_1 + a_n = i_n + |\bar{\mathbf{i}}|, \quad m = b_n + |\bar{\mathbf{b}}| = j_n + |\bar{\mathbf{j}}|, \tag{13.69}$$

$$a_\alpha + b_\alpha = i_\alpha + j_\alpha \ (\alpha = 1, n), \quad b_\alpha = i_\alpha + j_\alpha \ (\alpha \neq 1, n). \tag{13.70}$$

[3] This is cautioned since the convention (3.65) may wrongly indicate $(q^{-2k})_{k_1} = (q^{-2k}; q^{-2k})_{k_1}$ for example.

Substitute (3.87) into the sum (13.11) for $R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}}$ with $a_2 = \cdots = a_{n-1} = 0$. The result reads as

$$R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = (-1)^m q^{m-(\mathbf{a},\mathbf{j})} \sum_{c_1,k_1,k_n} (-1)^{k_1+k_n} z^{c_1} q^{\varphi_1} \prod_{\alpha=1,n} \binom{a_\alpha + b_\alpha - k_\alpha}{b_\alpha}_{q^2} \binom{j_\alpha}{k_\alpha}_{q^2}, \tag{13.71}$$

$$\varphi_1 = (\mathbf{a}+\mathbf{j}, \mathbf{c}) + \sum_{\alpha=1,n} k_\alpha(k_\alpha - 2c_\alpha - 1), \tag{13.72}$$

$$c_\beta = c_1 + \sum_{1\le\alpha<\beta} (b_\alpha - j_\alpha), \tag{13.73}$$

where the sum (13.71) extends over $c_1 \in \mathbb{Z}_{\ge 0}$ and $k_1, k_n \in \mathbb{Z}_{\ge 0}$. See (13.48) for the definition of $(\mathbf{a}, \mathbf{j})$ and $(\mathbf{a}+\mathbf{j}, \mathbf{c})$. The relation (13.73) is quoted from (13.12). It leads to $(\mathbf{a}+\mathbf{j}, \mathbf{c}) = \langle \mathbf{b}-\mathbf{j}, \mathbf{a}+\mathbf{j}\rangle + (l+m)c_1$ and $c_n = c_1 + |\bar{\mathbf{b}}| - |\bar{\mathbf{j}}| = c_1 + j_n - b_n$ due to (13.69). Thus the sum over c_1 yields

$$R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = (-1)^m q^{\varphi_3} \sum_{k\ge 0} \frac{(-1)^k}{1 - zq^{l+m-2k}} \sum_{k_1\ge 0} q^{\varphi_2} \prod_{\alpha=1,n} \binom{a_\alpha + b_\alpha - k_\alpha}{b_\alpha}_{q^2} \binom{j_\alpha}{k_\alpha}_{q^2}, \tag{13.74}$$

$$\varphi_2 = k_1^2 + (k-k_1)^2 - k + 2(b_n - j_n)(k - k_1), \tag{13.75}$$

$$\varphi_3 = m - (\mathbf{a}, \mathbf{j}) + \langle \mathbf{b}-\mathbf{j}, \mathbf{a}+\mathbf{j}\rangle. \tag{13.76}$$

Here and in what follows, k_n is to be understood as $k_n = k - k_1$. Both sums are actually finite due to the non-vanishing condition of the q^2-binomials.[4] For example, from $k_\alpha \le \min(a_\alpha, j_\alpha)$, k is bounded as $k = k_1 + k_n \le \min(l, m) \le m$ at most.

Rewrite the q^2-binomial factor with $\alpha = n$ as

$$\binom{a_n + b_n - k_n}{b_n}_{q^2} \binom{j_n}{k_n}_{q^2} = \frac{(q^2)_{j_n} (q^{2a_n - 2k_n + 2})_{b_n}}{(q^2)_{b_n} (q^2)_{k_n} (q^2)_{j_n - k_n}}, \tag{13.77}$$

$$\frac{1}{(q^2)_{k_n}} = (-1)^{k_1} q^{k_1(2k-k_1+1)} \frac{(q^{-2k})_{k_1}}{(q^2)_k}, \tag{13.78}$$

$$\frac{1}{(q^2)_{j_n-k_n}} = (-1)^k q^{k(2m-k+1)} \frac{(q^{-2m})_k (q^{2j_n - 2k_n + 2})_{m - j_n - k_1}}{(q^2)_m}. \tag{13.79}$$

Then (13.74) is expressed as

$$R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \frac{(-1)^m q^{\varphi_3} (q^2)_{j_n}}{(q^2)_{b_n} (q^2)_m} \sum_{k=0}^{m} \frac{1}{1 - zq^{l+m-2k}} \frac{(q^{-2m})_k}{(q^2)_k} \mathcal{P}(q^{2k}), \tag{13.80}$$

---

[4] Conditions like $k \ge k_1$ can formally be dispensed with since the negative k_n kills $\binom{j_n}{k_n}_{q^2}$.

$$\mathcal{P}(w) = w^{m+b_n-j_n} \sum_{k_1=0}^{\min(b_1,j_1)} (-1)^{k_1} q^{k_1^2+(2j_n-2b_n+1)k_1} (w^{-1})_{k_1} \times (w^{-1}q^{2a_n+2k_1+2})_{b_n} (w^{-1}q^{2j_n+2k_1+2})_{m-j_n-k_1} \binom{a_1+b_1-k_1}{b_1}_{q^2} \binom{j_1}{k_1}_{q^2}. \tag{13.81}$$

The upper bound $k_1 \le \min(b_1, j_1)$ in (13.81) is necessary and sufficient for the q^2-binomials and $(w^{-1}q^{2j_n+2k_1+2})_{m-j_n-k_1}$ to survive individually since $m - j_n \ge j_1$ because of $\mathbf{j} \in B_m$. Obviously, $\mathcal{P}(w)$ is a polynomial of w with $\deg\mathcal{P}(w) \le m + b_n - j_n$. In Lemma 13.7 we will show $\deg\mathcal{P}(w) \le m$ even if $b_n > j_n$ due to a non-trivial cancellation. Thanks to this fact, the sum in (13.80) is taken either for $b_n \le j_n$ or $b_n > j_n$ as

$$\sum_{k=0}^{m} \frac{1}{1 - zq^{l+m-2k}} \frac{(q^{-2m})_k}{(q^2)_k} \mathcal{P}(q^{2k}) = \frac{(-1)^m q^{-m(m+1)} (q^2)_m}{(zq^{l-m})_{m+1}} \mathcal{P}(zq^{l+m}), \tag{13.82}$$

which is just a partial fraction expansion. Consequently (13.80) gives

$$\Lambda_{l,m}(z,q)^{-1} R^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \frac{(-1)^m q^{\varphi_3 - m(l+m+2)} (q^2)_{j_n}}{(q^2)_{b_n} (zq^{-l-m})_m} \mathcal{P}(zq^{l+m}), \tag{13.83}$$

where we have used $\Lambda_{l,m}(z,q)$ in (13.18). On the other hand, the formula (13.53) of $A^{\mathrm{tr}_3}(z)^{\mathbf{ab}}_{\mathbf{ij}}$ for the special case $a_2 = \cdots = a_{n-1} = 0$ is simplified considerably. In fact the multidimensional sum over $\overline{\mathbf{k}} = (k_1, \ldots, k_{n-1})$ is reduced to the single sum over k_1 entering $\overline{\mathbf{k}} = (k_1, 0, \ldots, 0)$. The result reads as

$$A(z)^{\mathbf{ab}}_{\mathbf{ij}} = (-1)^{b_n - j_n} \frac{q^{\varphi}(q^2)_{j_n}}{(q^2)_{b_n}} \sum_{k_1 \ge 0} (zq^{l+m})^{k_1} \binom{a_1+b_1-k_1}{b_1}_{q^2} \binom{j_1}{k_1}_{q^2} \times \frac{(q^{m-l}z)_{l-a_n-k_1} (q^{l-m}z)_{m-j_n-k_1} (q^{-l-m}z^{-1})_{k_1}}{(q^{-l-m}z)_{l+m-a_n-b_n-k_1}}, \tag{13.84}$$

where φ is defined in (13.53). By using (13.81) and relations like

$$(\mathbf{a}, \mathbf{j}) = lm - (l - a_n) j_n - (m - j_n) a_n - \langle \overline{\mathbf{a}}, \overline{\mathbf{j}} \rangle, \quad \langle \overline{\mathbf{i}}, \overline{\mathbf{j}} \rangle = \langle \overline{\mathbf{a}} + \overline{\mathbf{b}} - \overline{\mathbf{j}}, \overline{\mathbf{j}} \rangle, \tag{13.85}$$

$$\langle \mathbf{b} - \mathbf{j}, \mathbf{a} + \mathbf{j} \rangle = (j_n - b_n)(a_n + j_n) + \langle \overline{\mathbf{b}} - \overline{\mathbf{j}}, \overline{\mathbf{j}} \rangle, \tag{13.86}$$

the two expressions (13.83) and (13.84) can be identified directly. □

Apart from q, the polynomial $\mathcal{P}(w)$ (13.81) depends on m and $a_\alpha, b_\alpha, j_\alpha$ with $\alpha = 1, n$. From (13.69) and (13.70), we have $a_1 + a_n = l \ge i_1 + i_n = a_1 + a_n + b_1 + b_n - j_1 - j_n$ and $m \ge j_1 + j_n$.

Lemma 13.7 *The polynomial* $\mathcal{P}(w)$ *(13.81) satisfies* $\deg\mathcal{P}(w) \le m$.

Proof From the preceding remark we assume

$$b_1 + b_n \le j_1 + j_n \le m, \quad b_n > j_n, \tag{13.87}$$

where the last condition selects the non-trivial case of the claim. Up to an overall factor independent of w, $\mathcal{P}(w)$ is equal to

$$\sum_{k_1 \ge 0} (-1)^{k_1} q^{k_1(k_1-1)} (wq^{-2k_1+2})_{k_1} (xwq^{-2k_1})_{b_n} (wq^{-2m})_{m-j_n-k_1} (yq^{-2k_1})_{b_1} \binom{j_1}{k_1}_{q^2} \tag{13.88}$$

at $x = q^{-2a_n-2b_n}$ and $y = q^{2a_1+2}$. This is further expanded into the powers of x and y as

$$\sum_{r=0}^{b_n} \sum_{s=0}^{b_1} (-1)^{r+s} x^r y^s q^{r(r-1)+s(s-1)} \binom{b_n}{r}_{q^2} \binom{b_1}{s}_{q^2} w^r \mathcal{F}_{r+s}(w), \tag{13.89}$$

$$\mathcal{F}_d(w) = \sum_{k_1=0}^{j_1} (-1)^{k_1} q^{k_1(k_1-1-2d)} (wq^{-2k_1+2})_{k_1} (wq^{-2m})_{m-j_n-k_1} \binom{j_1}{k_1}_{q^2}. \tag{13.90}$$

The variable d has the range $0 \le d = r + s \le b_1 + b_n \le j_1 + j_n$ due to (13.87). Thus it suffices to show $\deg\mathcal{F}_d(w) \le m - d$. The reason we consider this slightly stronger inequality rather than $\deg\mathcal{F}_d(w) \le m - r$ is of course that $\mathcal{F}_d(w)$ depends on d instead of r. It is a non-trivial claim when $j_n < d(\le j_1 + j_n)$.

The w-dependent factors in (13.90) are expanded as

$$\begin{aligned} &(wq^{-2k_1+2})_{k_1} (wq^{-2m})_{m-j_n-k_1} \\ &= \sum_{t=0}^{m-j_n} w^{m-j_n-t} \sum_{\alpha+\beta=t} C_{\alpha,\beta} q^{2(j_n+\beta+1)k_1} \binom{k_1}{\alpha}_{q^2} \binom{m-j_n-k_1}{\beta}_{q^2}, \end{aligned} \tag{13.91}$$

$$\binom{k_1}{\alpha}_{q^2} = \sum_{u=0}^{\alpha} f_u q^{2uk_1}, \quad \binom{m-j_n-k_1}{\beta}_{q^2} = \sum_{v=0}^{\beta} g_v q^{-2vk_1}, \tag{13.92}$$

where $\sum_{\alpha+\beta=t}$ denotes the finite sum over $(\alpha, \beta) \in \{0, 1, \dots, t\}^2$ under the condition $\alpha + \beta = t$. In the following argument, precise forms of the coefficients $C_{\alpha,\beta}$, f_u, g_v do not matter and only the fact that they are independent of k_1 is used. Substituting (13.91) and (13.92) into (13.90) we get

$$\mathcal{F}_d(w) = \sum_{t=0}^{m-j_n} w^{m-j_n-t} \sum_{\alpha+\beta=t} \sum_{u=0}^{\alpha} \sum_{v=0}^{\beta} D_{u,v}^{\alpha,\beta} (q^{2(j_n-d+1+\beta+u-v)}; q^2)_{j_1} \tag{13.93}$$

for some coefficient $D^{\alpha,\beta}_{u,v}$. Thus it is sufficient to show that all the q^2-factorials appearing here are zero for $t = 0, 1, \dots, d - j_n - 1$. It amounts to checking

$$\text{(i) } j_n - d + 1 + \beta + u - v \le 0, \qquad \text{(ii) } j_1 + j_n - d + \beta + u - v \ge 0 \tag{13.94}$$

for all the terms for $t = 0, 1, \dots, d - j_n - 1$. For (i), the most critical case is $v = 0$ and $\beta + u = t = d - j_n - 1$ for which the LHS is exactly 0. Therefore it is satisfied. For (ii), the most critical case is $\beta - v = 0$ and $u = 0$ for which the LHS is $j_1 + j_n - d$. This is indeed non-negative according to the remark after (13.90). □

Proof of Theorem 13.3. Consider the relation (13.58) with $\mathbf{a}$ replaced by $\mathbf{a} + \hat{r}$. The result is a recursion formula which reduces $\mathbf{a} = (a_1, \dots, a_r, a_{r+1}, \dots, a_n)$ in $A(z)^{\mathbf{a}\bullet}_{\bullet\bullet}$ to $\mathbf{a} + \hat{r} = (a_1, \dots, a_r + 1, a_{r+1} - 1, \dots, a_n)$ for $r = n - 2, \dots, 2, 1$. Thus $\mathbf{a}$ can ultimately be reduced to the form $(a_1, 0, \dots, 0, a_n)$. As remarked before Lemma 13.4, the quantum group symmetry (13.105) in Theorem 13.10 shows that $R^{\mathrm{tr}_3}(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}}$ also satisfies the same relation as (13.58). Therefore Lemma 13.4 reduces the proof of Theorem 13.3 to the situation $\mathbf{a} = (a_1, 0, \dots, 0, a_n)$. Since this has been established in Lemma 13.6, the proof is completed. □

13.6 Identification with Quantum *R* Matrices of $A^{(1)}_{n-1}$

Let $U_p(A^{(1)}_{n-1})$ be the quantum affine algebra. We keep the convention specified in the beginning of Sect. 11.5. We take $p = q$ throughout this section, hence the relevant algebra is always $U_q(A^{(1)}_{n-1})$.

Consider the n-fold tensor product $\mathrm{Osc}_q^{\otimes n}$ of q-oscillators and let $\mathbf{a}^+_i, \mathbf{a}^-_i, \mathbf{k}_i, \mathbf{k}^{-1}_i$ be the copy of the generators $\mathbf{a}^+, \mathbf{a}^-, \mathbf{k}, \mathbf{k}^{-1}$ (3.12) corresponding to its ith component. By the definition, generators with different indices are trivially commutative.

Proposition 13.8 *The following maps for $i \in \mathbb{Z}_n$ define algebra homomorphisms $U_q(A^{(1)}_{n-1}) \to \mathrm{Osc}_q^{\otimes n}$ depending on a spectral parameter x:*

$$\rho^{(3)}_x : \; e_i \mapsto \frac{x^{\delta_{i0}} q \mathbf{a}^+_i \mathbf{a}^-_{i+1} \mathbf{k}^{-1}_{i+1}}{1 - q^2}, \quad f_i \mapsto \frac{x^{-\delta_{i0}} q \mathbf{a}^-_i \mathbf{a}^+_{i+1} \mathbf{k}^{-1}_i}{1 - q^2}, \quad k_i \mapsto \mathbf{k}_i \mathbf{k}^{-1}_{i+1}, \tag{13.95}$$

$$\rho^{(1)}_x : \; e_i \mapsto \frac{x^{\delta_{i0}} q \mathbf{a}^-_i \mathbf{a}^+_{i+1} \mathbf{k}^{-1}_i}{1 - q^2} \quad f_i \mapsto \frac{x^{-\delta_{i0}} q \mathbf{a}^+_i \mathbf{a}^-_{i+1} \mathbf{k}^{-1}_{i+1}}{1 - q^2}, \quad k_i \mapsto \mathbf{k}^{-1}_i \mathbf{k}_{i+1}. \tag{13.96}$$

Proof The relations (11.56) with $p = q$ are directly checked by using (3.12). □

The maps $\rho^{(1)}_x$ and $\rho^{(3)}_x$ are interchanged via the algebra automorphism $e_i \leftrightarrow f_i, k_i \leftrightarrow k^{-1}_i$ up to the spectral parameter.

By (3.13) one can further let $\mathrm{Osc}_q^{\otimes n}$ act on $\mathbf{W} = \mathcal{F}_q^{\otimes n} = \bigoplus_{\mathbf{a} \in B} \mathbb{C}|\mathbf{a}\rangle$ in (11.11). Since (13.95) and (13.96) preserve $|\mathbf{a}|$ in (11.4), the representation space can be restricted to $\mathbf{W}_k$ (11.12) for any $k \in \mathbb{Z}_{\geq 0}$. Let us denote the resulting representations by

$$\tilde{\pi}_{k\varpi_1,x}: \ U_q(A^{(1)}_{n-1}) \xrightarrow{\rho^{(3)}_x} \mathrm{Osc}_q^{\otimes n}[x, x^{-1}] \to \mathrm{End}(\mathbf{W}_k), \tag{13.97}$$

$$\tilde{\pi}_{k\varpi_{n-1},x}: \ U_q(A^{(1)}_{n-1}) \xrightarrow{\rho^{(1)}_x} \mathrm{Osc}_q^{\otimes n}[x, x^{-1}] \to \mathrm{End}(\mathbf{W}_k), \tag{13.98}$$

where the second arrow is given by (3.13) for each component. Explicitly they are given by

$$\tilde{\pi}_{k\varpi_1,x}: \quad \begin{aligned} e_i|\mathbf{m}\rangle &= x^{\delta_{i0}}[m_{i+1}]_q|\mathbf{m}+\mathbf{e}_i-\mathbf{e}_{i+1}\rangle, \\ f_i|\mathbf{m}\rangle &= x^{-\delta_{i0}}[m_i]_q|\mathbf{m}-\mathbf{e}_i+\mathbf{e}_{i+1}\rangle, \\ k_i|\mathbf{m}\rangle &= q^{m_i-m_{i+1}}|\mathbf{m}\rangle, \end{aligned} \tag{13.99}$$

$$\tilde{\pi}_{k\varpi_{n-1},x}: \quad \begin{aligned} e_i|\mathbf{m}\rangle &= x^{\delta_{i0}}[m_i]_q|\mathbf{m}-\mathbf{e}_i+\mathbf{e}_{i+1}\rangle, \\ f_i|\mathbf{m}\rangle &= x^{-\delta_{i0}}[m_{i+1}]_q|\mathbf{m}+\mathbf{e}_i-\mathbf{e}_{i+1}\rangle, \\ k_i|\mathbf{m}\rangle &= q^{m_{i+1}-m_i}|\mathbf{m}\rangle \end{aligned} \tag{13.100}$$

for $\mathbf{m} \in B_k$ and $i \in \mathbb{Z}_n$.[5] As a representation of the classical part $U_q(A_{n-1})$ without $e_0, f_0, k_0^{\pm 1}$, $\tilde{\pi}_{k\varpi_1,x}$ (resp. $\tilde{\pi}_{k\varpi_{n-1},x}$) is the irreducible highest weight representation with the highest weight vector $|k\mathbf{e}_1\rangle$ (resp. $|k\mathbf{e}_n\rangle$) with highest weight $k\varpi_1$ (resp. $k\varpi_{n-1}$). They are q-analogues of the k-fold symmetric tensor of the vector and the anti-vector representations.

Remark 13.9 The representations $\tilde{\pi}_{k\varpi_1,x}$ in (13.99), (13.95) and the earlier one $\pi_{k\varpi_1,x}$ in (11.67) with $p = q$ are equivalent. In fact, by an automorphism

$$\mathbf{a}^+_j \mapsto \mathbf{a}^+_j\mathbf{k}_j, \quad \mathbf{a}^-_j \mapsto \mathbf{k}_j^{-1}\mathbf{a}^-_j, \quad \mathbf{k}_j \mapsto \mathbf{k}_j \tag{13.101}$$

of Osc_q induced by the conjugation $\mathbf{a}^{\pm}_j \mapsto q^{\mathbf{h}_j(\mathbf{h}_j-1)/2}\mathbf{a}^{\pm}_j q^{-\mathbf{h}_j(\mathbf{h}_j-1)/2}$, we get another algebra homomorphism $U_q(A^{(1)}_{n-1}) \to \mathrm{Osc}_q^{\otimes n}$ as

$$\rho^{(3)'}_x : \ e_i \mapsto \frac{x^{\delta_{i0}}q^2\mathbf{a}^+_i\mathbf{a}^-_{i+1}\mathbf{k}_i\mathbf{k}^{-2}_{i+1}}{1-q^2}, \quad f_i \mapsto \frac{x^{-\delta_{i0}}q^2\mathbf{a}^+_{i+1}\mathbf{a}^-_i\mathbf{k}^{-2}_i\mathbf{k}_{i+1}}{1-q^2}, \quad k_i \mapsto \mathbf{k}_i\mathbf{k}^{-1}_{i+1}. \tag{13.102}$$

Employing this $\rho^{(3)'}_x$ in (13.97) instead of $\rho^{(3)}_x$ yields $(11.67)|_{p=q}$.

[5] The definition of $[m]_q$ is in (11.57).

13.6.1 $R^{\mathrm{tr}_3}(z)$

Let $\tilde{\pi}_{k\varpi_1,x} : U_q(A^{(1)}_{n-1}) \to \mathrm{End}(\mathbf{W}_k)$ be the representation (13.99). Let $\Delta_{x,y} = (\tilde{\pi}_{l\varpi_1,x} \otimes \tilde{\pi}_{m\varpi_1,y}) \circ \Delta$ and $\Delta^{\mathrm{op}}_{x,y} = (\tilde{\pi}_{l\varpi_1,x} \otimes \tilde{\pi}_{m\varpi_1,y}) \circ \Delta^{\mathrm{op}}$ be the tensor product representations, where the coproducts Δ and Δ^{op} are specified in (11.58) and (11.59).

Let $\mathcal{R}_{l\varpi_1,m\varpi_1}(z) \in \mathrm{End}(\mathbf{W}_l \otimes \mathbf{W}_m)$ be the quantum R matrix of $U_q(A^{(1)}_{n-1})$ which is characterized, up to normalization, by the commutativity

$$\mathcal{R}_{l\varpi_1,m\varpi_1}(\tfrac{x}{y})\Delta_{x,y}(g) = \Delta^{\mathrm{op}}_{x,y}(g)\mathcal{R}_{l\varpi_1,m\varpi_1}(\tfrac{x}{y}) \quad (\forall g \in U_q(A^{(1)}_{n-1})), \tag{13.103}$$

where we have taken into account the obvious fact that $\mathcal{R}_{l\varpi_1,m\varpi_1}$ depends only on the ratio x/y. The relation (13.103) is a generalization of $(10.12)|_{q\to p}$ including the latter as the classical part $g \in U_q(A_{n-1})$.

Theorem 13.10 *Up to normalization, $R^{\mathrm{tr}_3}_{l,m}(z)$ by the matrix product construction (13.9)–(13.11) based on the 3D R coincides with the quantum R matrix of $U_q(A^{(1)}_{n-1})$ as*

$$R^{\mathrm{tr}_3}_{l,m}(z) = \mathcal{R}_{l\varpi_1,m\varpi_1}(z^{-1}). \tag{13.104}$$

Proof It suffices to check

$$R^{\mathrm{tr}_3}(\tfrac{y}{x})(e_r \otimes 1 + k_r \otimes e_r) = (1 \otimes e_r + e_r \otimes k_r)R^{\mathrm{tr}_3}(\tfrac{y}{x}), \tag{13.105}$$

$$R^{\mathrm{tr}_3}(\tfrac{y}{x})(1 \otimes f_r + f_r \otimes k_r^{-1}) = (f_r \otimes 1 + k_r^{-1} \otimes f_r)R^{\mathrm{tr}_3}(\tfrac{y}{x}), \tag{13.106}$$

$$R^{\mathrm{tr}_3}(\tfrac{y}{x})(k_r \otimes k_r) = (k_r \otimes k_r)R^{\mathrm{tr}_3}(\tfrac{y}{x}) \tag{13.107}$$

under the image by $\tilde{\pi}_{l\varpi_1,x} \otimes \tilde{\pi}_{m\varpi_1,y}$. Actually, they can be shown by using (13.95) instead of (13.99), which means that the commutativity holds already in $\mathrm{Osc}_q^{\otimes n} \otimes \mathrm{Osc}_q^{\otimes n}$ without taking the image in $\mathrm{End}(\mathbf{W}_l \otimes \mathbf{W}_m)$. Due to the $\mathbb{Z}_n$ symmetry of (13.95) and (13.7) up to the spectral parameter, it suffices to check this for $r = 0$.[6] The relevant part of (13.11) is $R^{a_n b_n c_n}_{i_n j_n c_1} z^{c_1} R^{a_1 b_1 c_1}_{i_1 j_1 c_2}$, which we regard as an element of the product $R_{123} z^{\mathbf{h}_3} R_{1'2'3}$ of 3D R. The indices here are labels of the corresponding spaces as in Fig. 13.7.

In terms of the labels, the image by (13.95) reads as

$$\begin{aligned}
e_0 \otimes 1 &= xd\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_{1'}^{-1}, & 1 \otimes e_0 &= yd\mathbf{a}_2^+\mathbf{a}_{2'}^-\mathbf{k}_{2'}^{-1},\\
f_0 \otimes 1 &= x^{-1}d\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_1^{-1}, & 1 \otimes f_0 &= y^{-1}d\mathbf{a}_{2'}^+\mathbf{a}_2^-\mathbf{k}_2^{-1},\\
k_0 \otimes 1 &= \mathbf{k}_1\mathbf{k}_{1'}^{-1}, & 1 \otimes k_0 &= \mathbf{k}_2\mathbf{k}_{2'}^{-1},
\end{aligned} \tag{13.108}$$

[6] The case $r \neq 0$ corresponds to the special case $x = y = 1$.

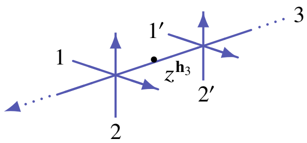

Fig. 13.7 The part of the matrix product construction (13.11) relevant to the commutation relations with e_0, f_0, k_0

where $d = q(1-q^2)^{-1}$. Then (13.105)–(13.107) are attributed to

$$
\begin{aligned}
& Rz^{\mathbf{h}_3}R'(x\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_{1'}^{-1} + y\mathbf{k}_1\mathbf{k}_{1'}^{-1}\mathbf{a}_2^+\mathbf{a}_{2'}^-\mathbf{k}_{2'}^{-1}) \\
&\quad = (y\mathbf{a}_2^+\mathbf{a}_{2'}^-\mathbf{k}_{2'}^{-1} + x\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_{1'}^{-1}\mathbf{k}_2\mathbf{k}_{2'}^{-1})Rz^{\mathbf{h}_3}R', && (13.109) \\
& Rz^{\mathbf{h}_3}R'(y^{-1}\mathbf{a}_{2'}^+\mathbf{a}_2^-\mathbf{k}_2^{-1} + x^{-1}\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_1^{-1}\mathbf{k}_2^{-1}\mathbf{k}_{2'}) \\
&\quad = (x^{-1}\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_1^{-1} + y^{-1}\mathbf{k}_1^{-1}\mathbf{k}_{1'}\mathbf{a}_{2'}^+\mathbf{a}_2^-\mathbf{k}_2^{-1})Rz^{\mathbf{h}_3}R', && (13.110) \\
& Rz^{\mathbf{h}_3}R'\mathbf{k}_1\mathbf{k}_{1'}^{-1}\mathbf{k}_2\mathbf{k}_{2'}^{-1} = \mathbf{k}_1\mathbf{k}_{1'}^{-1}\mathbf{k}_2\mathbf{k}_{2'}^{-1}Rz^{\mathbf{h}_3}R', && (13.111)
\end{aligned}
$$

where $z = yx^{-1}$ and we have set $R = R_{123}$ and $R' = R_{1'2'3}$ for short. To show these relations we invoke the intertwining relations (3.127)–(3.131),[7] i.e.

$$
\begin{aligned}
R\,\mathbf{k}_2\mathbf{a}_1^+ &= (\mathbf{k}_3\mathbf{a}_1^+ + \mathbf{k}_1\mathbf{a}_2^+\mathbf{a}_3^-)R, & R\,\mathbf{k}_2\mathbf{a}_1^- &= (\mathbf{k}_3\mathbf{a}_1^- + \mathbf{k}_1\mathbf{a}_2^-\mathbf{a}_3^+)R, && (13.112) \\
R\,\mathbf{a}_2^+ &= (\mathbf{a}_1^+\mathbf{a}_3^+ - q\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^+)R, & R\,\mathbf{a}_2^- &= (\mathbf{a}_1^-\mathbf{a}_3^- - q\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^-)R, && (13.113) \\
R\,\mathbf{k}_2\mathbf{a}_3^+ &= (\mathbf{k}_1\mathbf{a}_3^+ + \mathbf{k}_3\mathbf{a}_1^-\mathbf{a}_2^+)R, & R\,\mathbf{k}_2\mathbf{a}_3^- &= (\mathbf{k}_1\mathbf{a}_3^- + \mathbf{k}_3\mathbf{a}_1^+\mathbf{a}_2^-)R, && (13.114) \\
R\,\mathbf{k}_1\mathbf{k}_2 &= \mathbf{k}_1\mathbf{k}_2R, & R\,\mathbf{k}_2\mathbf{k}_3 &= \mathbf{k}_2\mathbf{k}_3R && (13.115)
\end{aligned}
$$

and their copy where R and the indices $1, 2$ are replaced with R' and $1', 2'$. The relation (13.111) follows from (13.115) immediately. By multiplying $\mathbf{k}_{1'}\mathbf{k}_{2'}$ from the right by (13.109) and $\mathbf{k}_1\mathbf{k}_2$ from the left to (13.110) and using the commutativity with R and R' by (13.115), they are slightly simplified into

$$
\begin{aligned}
& Rz^{\mathbf{h}_3}R'(x\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_{2'} + y\mathbf{k}_1\mathbf{a}_2^+\mathbf{a}_{2'}^-) = (y\mathbf{a}_2^+\mathbf{a}_{2'}^-\mathbf{k}_{1'} + x\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_2)Rz^{\mathbf{h}_3}R', && (13.116) \\
& Rz^{\mathbf{h}_3}R'(y^{-1}\mathbf{a}_{2'}^+\mathbf{a}_2^-\mathbf{k}_1 + x^{-1}\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_{2'}) = (x^{-1}\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_2 + y^{-1}\mathbf{k}_{1'}\mathbf{a}_{2'}^+\mathbf{a}_2^-)Rz^{\mathbf{h}_3}R'. && (13.117)
\end{aligned}
$$

To get (13.117) we have used $\mathbf{k}_j\mathbf{a}_j^{\pm} = q^{\pm 1}\mathbf{a}_j^{\pm}\mathbf{k}_j$. All the terms appearing here can be brought to the form $Rz^{\mathbf{h}_3}(\cdots)R'$ by means of $z^{\mathbf{h}_3}\mathbf{a}^{\pm} = \mathbf{a}^{\pm}z^{\mathbf{h}_3 \pm 1}$, $R = R^{-1}$, (13.112)–(13.115) and the corresponding relations for R'. Explicitly, we have the following for (13.116):

---

[7] The relation (3.130) can be dispensed with.

$$
\begin{aligned}
Rz^{\mathbf{h}_3}R'x\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_{2'} &= xRz^{\mathbf{h}_3}\mathbf{a}_1^+(\underline{\mathbf{k}_3\mathbf{a}_{1'}^-} + \underline{\mathbf{k}_{1'}\mathbf{a}_{2'}^-\mathbf{a}_3^+})R',\\
Rz^{\mathbf{h}_3}R'y\mathbf{k}_1\mathbf{a}_2^+\mathbf{a}_{2'}^- &= yRz^{\mathbf{h}_3}\mathbf{k}_1\mathbf{a}_2^+(\underline{\mathbf{a}_{1'}^-\mathbf{a}_3^-} - q\underline{\mathbf{k}_{1'}\mathbf{k}_3\mathbf{a}_{2'}^-})R',\\
y\mathbf{a}_2^+\mathbf{a}_{2'}^-\mathbf{k}_{1'}Rz^{\mathbf{h}_3}R' &= yR(\mathbf{a}_1^+\mathbf{a}_3^+ - q\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^+)z^{\mathbf{h}_3}\mathbf{a}_{2'}^-\mathbf{k}_{1'}R'\\
&= yRz^{\mathbf{h}_3}(z^{-1}\underline{\mathbf{a}_1^+\mathbf{a}_3^+} - q\underline{\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^+})\mathbf{a}_{2'}^-\mathbf{k}_{1'}R',\\
x\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_2Rz^{\mathbf{h}_3}R' &= xR(\mathbf{k}_3\mathbf{a}_1^+ + \mathbf{k}_1\mathbf{a}_2^+\mathbf{a}_3^-)z^{\mathbf{h}_3}\mathbf{a}_{1'}^-R'\\
&= xRz^{\mathbf{h}_3}(\underline{\mathbf{k}_3\mathbf{a}_1^+} + z\underline{\mathbf{k}_1\mathbf{a}_2^+\mathbf{a}_3^-})\mathbf{a}_{1'}^-R'.
\end{aligned}
$$

As shown by the underlines, (13.116) is indeed valid at $z = yx^{-1}$. A similar calculation casts the four terms in (13.117) into

$$
\begin{aligned}
Rz^{\mathbf{h}_3}R'y^{-1}\mathbf{a}_{2'}^+\mathbf{a}_2^-\mathbf{k}_1 &= y^{-1}Rz^{\mathbf{h}_3}\mathbf{a}_2^-\mathbf{k}_1(\underline{\mathbf{a}_{1'}^+\mathbf{a}_3^+} - q\underline{\mathbf{k}_{1'}\mathbf{k}_3\mathbf{a}_{2'}^+})R',\\
Rz^{\mathbf{h}_3}R'x^{-1}\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_{2'} &= x^{-1}Rz^{\mathbf{h}_3}\mathbf{a}_1^-(\underline{\mathbf{k}_3\mathbf{a}_{1'}^+} + \underline{\mathbf{k}_{1'}\mathbf{a}_{2'}^+\mathbf{a}_3^-})R',\\
x^{-1}\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_2Rz^{\mathbf{h}_3}R' &= x^{-1}Rz^{\mathbf{h}_3}(\underline{\mathbf{k}_3\mathbf{a}_1^-} + z^{-1}\underline{\mathbf{k}_1\mathbf{a}_2^-\mathbf{a}_3^+})\mathbf{a}_{1'}^+R',\\
y^{-1}\mathbf{k}_{1'}\mathbf{a}_{2'}^+\mathbf{a}_2^-Rz^{\mathbf{h}_3}R' &= y^{-1}Rz^{\mathbf{h}_3}(z\underline{\mathbf{a}_1^-\mathbf{a}_3^-} - q\underline{\mathbf{k}_1\mathbf{k}_3\mathbf{a}_2^-})\mathbf{k}_{1'}\mathbf{a}_{2'}^+R',
\end{aligned}
$$

which are again valid at $z = yx^{-1}$. □

13.6.2 $R^{\mathrm{tr}_1}(z)$

Let $\tilde{\pi}_{k\varpi_{n-1},x} : U_q(A_{n-1}^{(1)}) \to \mathrm{End}(\mathbf{W}_k)$ be the representation (13.100). Let $\Delta_{x,y} = (\tilde{\pi}_{l\varpi_{n-1},x} \otimes \tilde{\pi}_{m\varpi_{n-1},y}) \circ \Delta$ and $\Delta^{\mathrm{op}}_{x,y} = (\tilde{\pi}_{l\varpi_{n-1},x} \otimes \tilde{\pi}_{m\varpi_{n-1},y}) \circ \Delta^{\mathrm{op}}$ be the tensor product representations, where the coproducts Δ and Δ^{op} are specified in (11.58) and (11.59).

Let $\mathcal{R}_{l\varpi_{n-1},m\varpi_{n-1}}(z) \in \mathrm{End}(\mathbf{W}_l \otimes \mathbf{W}_m)$ be the quantum R matrix of $U_q(A_{n-1}^{(1)})$ which is characterized, up to normalization, by the commutativity

$$
\mathcal{R}_{l\varpi_{n-1},m\varpi_{n-1}}(\tfrac{x}{y})\Delta_{x,y}(g) = \Delta^{\mathrm{op}}_{x,y}(g)\mathcal{R}_{l\varpi_{n-1},m\varpi_{n-1}}(\tfrac{x}{y}) \quad (\forall g \in U_q(A_{n-1}^{(1)})), \tag{13.118}
$$

where we have taken into account the fact that $\mathcal{R}_{l\varpi_{n-1},m\varpi_{n-1}}$ depends only on the ratio x/y.

Theorem 13.11 *Up to normalization, $R^{\mathrm{tr}_1}_{l,m}(z)$ by the matrix product construction (13.25)–(13.26) and (13.29) based on the 3D R coincides with the quantum R matrix of $U_q(A_{n-1}^{(1)})$ as*

$$
R^{\mathrm{tr}_1}_{l,m}(z) = \mathcal{R}_{l\varpi_{n-1},m\varpi_{n-1}}(z^{-1}). \tag{13.119}
$$

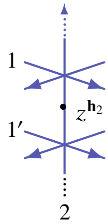

Fig. 13.8 The part of the matrix product construction (13.39) relevant to the commutation relations with e_0, f_0, k_0

Proof This follows from the relation (13.27), Theorem 13.10, the commutativity (13.105)–(13.107) and the fact that $\tilde{\pi}_{k\varpi_1,x}$ (13.99) and $\tilde{\pi}_{k\varpi_{n-1},x^{-1}}$ (13.100) are interchanged via the algebra automorphism $e_i \leftrightarrow f_i, k_i \leftrightarrow k_i^{-1}$. □

13.6.3 $R^{\mathrm{tr}_2}(z)$

Let $\tilde{\pi}_{k\varpi_1,x}$ and $\tilde{\pi}_{k\varpi_{n-1},x}$ be the representations $U_q(A^{(1)}_{n-1}) \to \mathrm{End}(\mathbf{W}_k)$ in (13.99) and (13.100). Let $\Delta_{x,y} = (\tilde{\pi}_{l\varpi_1,x} \otimes \tilde{\pi}_{m\varpi_{n-1},y}) \circ \Delta$ and $\Delta^{\mathrm{op}}_{x,y} = (\tilde{\pi}_{l\varpi_1,x} \otimes \tilde{\pi}_{m\varpi_{n-1},y}) \circ \Delta^{\mathrm{op}}$ be the tensor product representations, where the coproducts Δ and Δ^{op} are specified in (11.58) and (11.59).

Let $\mathcal{R}_{l\varpi_1,m\varpi_{n-1}}(z) \in \mathrm{End}(\mathbf{W}_l \otimes \mathbf{W}_m)$ be the quantum R matrix of $U_q(A^{(1)}_{n-1})$ which is characterized, up to normalization, by the commutativity

$$\mathcal{R}_{l\varpi_1,m\varpi_{n-1}}(\tfrac{x}{y})\Delta_{x,y}(g) = \Delta^{\mathrm{op}}_{x,y}(g)\mathcal{R}_{l\varpi_1,m\varpi_{n-1}}(\tfrac{x}{y}) \quad (\forall g \in U_q(A^{(1)}_{n-1})), \tag{13.120}$$

where we have taken into account the fact that $\mathcal{R}_{l\varpi_1,m\varpi_{n-1}}$ depends only on the ratio x/y.

Theorem 13.12 *Up to normalization, $R^{\mathrm{tr}_2}_{l,m}(z)$ by the matrix product construction (13.38)–(13.39) and (13.42) based on the 3D R coincides with the quantum R matrix of $U_q(A^{(1)}_{n-1})$ as*

$$R^{\mathrm{tr}_2}_{l,m}(z) = \mathcal{R}_{l\varpi_1,m\varpi_{n-1}}(z). \tag{13.121}$$

Proof The proof is similar to the one for Theorem 13.10. So we shall list the corresponding formulas along the labeling in Fig. 13.8 without a detailed explanation.

We are to investigate the commutation relation of $Rz^{\mathbf{h}_2}R' = R_{123}z^{\mathbf{h}_2}R_{1'23'}$ and

$$\begin{aligned}
e_0 \otimes 1 &= xd\mathbf{a}^+_1\mathbf{a}^-_{1'}\mathbf{k}^{-1}_{1'}, & 1 \otimes e_0 &= yd\mathbf{a}^+_{3'}\mathbf{a}^-_3\mathbf{k}^{-1}_3,\\
f_0 \otimes 1 &= x^{-1}d\mathbf{a}^+_{1'}\mathbf{a}^-_1\mathbf{k}^{-1}_1, & 1 \otimes f_0 &= y^{-1}d\mathbf{a}^+_3\mathbf{a}^-_{3'}\mathbf{k}^{-1}_{3'},\\
k_0 \otimes 1 &= \mathbf{k}_1\mathbf{k}^{-1}_{1'}, & 1 \otimes k_0 &= \mathbf{k}^{-1}_3\mathbf{k}_{3'},
\end{aligned} \tag{13.122}$$

where $d = q(1-q^2)^{-1}$. The relation (13.118) with $g = e_0$ becomes, after multiplying $\mathbf{k}_{1'}\mathbf{k}_2\mathbf{k}_3$ from the right,

$$Rz^{\mathbf{h}_2}R'(x\mathbf{k}_2\mathbf{k}_3\mathbf{a}_1^+\mathbf{a}_{1'}^- + y\mathbf{k}_1\mathbf{k}_2\mathbf{a}_{3'}^+\mathbf{a}_3^-) = (y\mathbf{a}_{3'}^+\mathbf{a}_3^-\mathbf{k}_2\mathbf{k}_{1'} + x\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_2\mathbf{k}_{3'})Rz^{\mathbf{h}_2}R'. \tag{13.123}$$

The four terms here are rewritten by means of (13.112)–(13.115) as

$$\begin{aligned}
Rz^{\mathbf{h}_2}R'x\mathbf{k}_2\mathbf{k}_3\mathbf{a}_1^+\mathbf{a}_{1'}^- &= xRz^{\mathbf{h}_2}\mathbf{k}_3\mathbf{a}_1^+(\underline{\mathbf{k}_{3'}\mathbf{a}_{1'}^-} + \underline{\mathbf{k}_{1'}\mathbf{a}_2^-\mathbf{a}_{3'}^+})R',\\
Rz^{\mathbf{h}_2}R'y\mathbf{k}_1\mathbf{k}_2\mathbf{a}_{3'}^+\mathbf{a}_3^- &= yRz^{\mathbf{h}_2}\mathbf{k}_1\mathbf{a}_3^-(\underline{\mathbf{k}_{1'}\mathbf{a}_{3'}^+} + \underline{\mathbf{k}_{3'}\mathbf{a}_{1'}^-\mathbf{a}_2^+})R',\\
y\mathbf{a}_{3'}^+\mathbf{a}_3^-\mathbf{k}_2\mathbf{k}_{1'}Rz^{\mathbf{h}_2}R' &= yRz^{\mathbf{h}_2}(\underline{\mathbf{k}_1\mathbf{a}_3^-} + z\underline{\mathbf{k}_3\mathbf{a}_1^+\mathbf{a}_2^-})\mathbf{a}_{3'}^+\mathbf{k}_{1'}R',\\
x\mathbf{a}_1^+\mathbf{a}_{1'}^-\mathbf{k}_2\mathbf{k}_{3'}Rz^{\mathbf{h}_2}R' &= xRz^{\mathbf{h}_2}(\underline{\mathbf{k}_3\mathbf{a}_1^+} + z^{-1}\underline{\mathbf{k}_1\mathbf{a}_2^+\mathbf{a}_3^-})\mathbf{a}_{1'}^-\mathbf{k}_{3'}R'.
\end{aligned}$$

Thus (13.123) is valid at $z = xy^{-1}$. The relation (13.118) with $g = f_0$ becomes, after multiplying $\mathbf{k}_1\mathbf{k}_2\mathbf{k}_{3'}$ from the left,

$$Rz^{\mathbf{h}_2}R'(y^{-1}\mathbf{k}_1\mathbf{k}_2\mathbf{a}_3^+\mathbf{a}_{3'}^- + x^{-1}\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_2\mathbf{k}_3) = (x^{-1}\mathbf{k}_2\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_{3'} + y^{-1}\mathbf{k}_2\mathbf{k}_{1'}\mathbf{a}_3^+\mathbf{a}_{3'}^-)Rz^{\mathbf{h}_2}R'. \tag{13.124}$$

The four terms here are rewritten by means of (13.112)–(13.115) as

$$\begin{aligned}
Rz^{\mathbf{h}_2}R'y^{-1}\mathbf{k}_1\mathbf{k}_2\mathbf{a}_3^+\mathbf{a}_{3'}^- &= y^{-1}Rz^{\mathbf{h}_2}\mathbf{k}_1\mathbf{a}_3^+(\underline{\mathbf{k}_{1'}\mathbf{a}_{3'}^-} + \underline{\mathbf{k}_{3'}\mathbf{a}_{1'}^+\mathbf{a}_2^-})R',\\
Rz^{\mathbf{h}_2}R'x^{-1}\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_2\mathbf{k}_3 &= x^{-1}Rz^{\mathbf{h}_2}\mathbf{a}_1^-\mathbf{k}_3(\underline{\mathbf{k}_{3'}\mathbf{a}_{1'}^+} + \underline{\mathbf{k}_{1'}\mathbf{a}_2^+\mathbf{a}_{3'}^-})R',\\
x^{-1}\mathbf{k}_2\mathbf{a}_{1'}^+\mathbf{a}_1^-\mathbf{k}_{3'}Rz^{\mathbf{h}_2}R' &= x^{-1}Rz^{\mathbf{h}_2}(\underline{\mathbf{k}_3\mathbf{a}_1^-} + z\underline{\mathbf{k}_1\mathbf{a}_2^-\mathbf{a}_3^+})\mathbf{a}_{1'}^+\mathbf{k}_{3'}R,\\
y^{-1}\mathbf{k}_2\mathbf{k}_{1'}\mathbf{a}_3^+\mathbf{a}_{3'}^-Rz^{\mathbf{h}_2}R' &= y^{-1}Rz^{\mathbf{h}_2}(\underline{\mathbf{k}_1\mathbf{a}_3^+} + z^{-1}\underline{\mathbf{k}_3\mathbf{a}_1^-\mathbf{a}_2^+})\mathbf{k}_{1'}\mathbf{a}_{3'}^-R'.
\end{aligned}$$

Thus (13.124) is valid at $z = xy^{-1}$. □

We note that (13.113) has not been used in the above proof.

13.7 Stochastic *R* Matrix

This section is a small digression on a special gauge of the R matrix. For $l, m \in \mathbb{Z}_{\geq 1}$, we introduce $\mathcal{S}(z) \in \mathrm{End}(\mathbf{W}_l \otimes \mathbf{W}_m)$ by

$$\mathcal{S}(z)(|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle) = \sum_{\mathbf{a}\in B_l, \mathbf{b}\in B_m} \mathcal{S}(z)_{\mathbf{ij}}^{\mathbf{ab}} |\mathbf{a}\rangle \otimes |\mathbf{b}\rangle, \tag{13.125}$$

$$\mathcal{S}(z)_{\mathbf{ij}}^{\mathbf{ab}} = \delta_{\mathbf{i}+\mathbf{j}}^{\mathbf{a}+\mathbf{b}} \mathcal{A}(z)_{\mathbf{ij}}^{\mathbf{ab}}, \tag{13.126}$$

where $\mathcal{A}(z)_{\mathbf{ij}}^{\mathbf{ab}}$ is a slight modification of $A(z)_{\mathbf{ij}}^{\mathbf{ab}}$ (13.51):

$$\mathcal{A}(z)^{\mathbf{ab}}_{\mathbf{ij}} = q^{\langle \mathbf{b},\mathbf{a}\rangle - \langle \mathbf{i},\mathbf{j}\rangle} A(z)^{\mathbf{ab}}_{\mathbf{ij}}$$
$$= \sum_{\overline{\mathbf{k}}} \Phi_{q^2}(\overline{\mathbf{a}} - \overline{\mathbf{k}} | \overline{\mathbf{a}} + \overline{\mathbf{b}} - \overline{\mathbf{k}}; q^{m-l}z, q^{-l-m}z)\Phi_{q^2}(\overline{\mathbf{k}}|\overline{\mathbf{j}}; q^{-l-m}z^{-1}, q^{-2m}). \tag{13.127}$$

From (13.17), (13.55) and Theorem 13.10, $\mathcal{S}(z)$ satisfies

$$\text{Yang–Baxter relation: } \mathcal{S}_{12}(x)\mathcal{S}_{13}(xy)\mathcal{S}_{23}(y) = \mathcal{S}_{23}(y)\mathcal{S}_{13}(xy)\mathcal{S}_{12}(x), \tag{13.128}$$
$$\text{Inversion relation: } \mathcal{S}(z)P\mathcal{S}(z^{-1})P = \mathrm{id}, \tag{13.129}$$
$$\text{Normalization: } \mathcal{S}(z)(|l\mathbf{e}_k\rangle \otimes |m\mathbf{e}_k\rangle) = |l\mathbf{e}_k\rangle \otimes |m\mathbf{e}_k\rangle, \tag{13.130}$$

where $P(u \otimes v) = v \otimes u$ and $k \in \mathbb{Z}_n$ is arbitrary. In fact, it is easy to check that the extra factor $q^{\langle \mathbf{b},\mathbf{a}\rangle - \langle \mathbf{i},\mathbf{j}\rangle}$ in (13.127) does not spoil these properties.[8]

A notable feature of this gauge is the *sum to unity* property:

Proposition 13.13

$$\sum_{\mathbf{a}\in B_l, \mathbf{b}\in B_m} \mathcal{S}(z)^{\mathbf{ab}}_{\mathbf{ij}} = 1 \qquad (\forall(\mathbf{i},\mathbf{j}) \in B_l \times B_m). \tag{13.131}$$

$\mathcal{S}(z)$ has an application to stochastic models where Proposition 13.13 plays the role of the total probability conservation. In such a context, it is called a *stochastic* R matrix.[9]

From (13.49) and (13.50), one sees $\Phi_{q^2}(\boldsymbol{\gamma}|\boldsymbol{\beta}, \lambda = 1, \mu) = \delta_{\boldsymbol{\gamma},0}$. Therefore $\mathcal{S}(z)$ has a factorized special value:

$$\mathcal{S}(z = q^{l-m})^{\mathbf{ab}}_{\mathbf{ij}} = \delta^{\mathbf{a}+\mathbf{b}}_{\mathbf{i}+\mathbf{j}} \, \Phi_{q^2}(\overline{\mathbf{a}}|\overline{\mathbf{j}}; q^{-2l}, q^{-2m}). \tag{13.132}$$

The specialization of (13.131) to (13.132) agrees with (13.54).

13.8 Commuting Layer Transfer Matrices and Duality

This section is parallel with Sect. 11.6. Let $m, n \geq 2$ and consider the composition of $m \times n$ 3D R's as follows:

At the intersection of 1_i and 2_j, we have the 3D R $L_{1_i,2_j,3_{ij}}$ as in Fig. 13.1, where the arrow 3_{ij} corresponds to the vertical arrows carrying $\mathcal{F}_q$. We take the parameters μ_i and ν_j as

$$\mu_i = xu_i \ (i = 1, \ldots, m), \qquad \nu_j = yw_j \ (j = 1, \ldots, n). \tag{13.133}$$

[8] See [87, Proposition 4].

[9] For reasons of convention, the R matrix $R^{\mathrm{tr}_3}_{l,m}(z) = \mathcal{R}_{l\varpi_1, m\varpi_1}(z^{-1})$ in (13.104) of this book is proportional to $R(z)$ in [87, Eq. (6)].

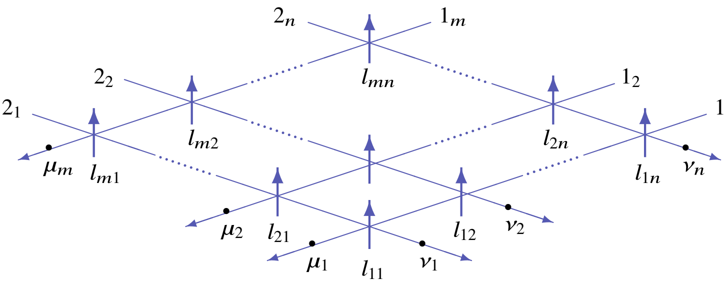

Fig. 13.9 Graphical representation of the layer transfer matrix $T(x, y)$. There are $m + n$ horizontal arrows $1_1, \ldots, 1_m$ and $2_1, \ldots, 2_n$ carrying $\mathcal{F}_q$ and being traced out, which corresponds to the periodic boundary condition. The mark • with μ_i and ν_j signifies an operator $\mu_i^{\mathbf{h}}$ and $\nu_j^{\mathbf{h}}$ attached to 1_i and 2_j, respectively. At the intersection of 1_i and 2_j, there is a q-oscillator Fock space $\mathcal{F}_q$ depicted with a vertical arrow

Tracing out the horizontal degrees of freedom leaves us with a linear operator acting along vertical arrows. We write the resulting *layer* transfer matrix in the third direction as[10]

$$T(x, y) = T(x, y|\mathbf{u}, \mathbf{w}) \in \mathrm{End}(\mathcal{F}_q^{\otimes mn}), \tag{13.134}$$

$$\mathbf{u} = (u_1, \ldots, u_m), \quad \mathbf{w} = (w_1, \ldots, w_n). \tag{13.135}$$

Figure 13.9 shows its action on the basis $\bigotimes_{1 \le i \le m, 1 \le j \le n} |l_{ij}\rangle \in \mathcal{F}_q^{\otimes mn}$.

We exhibit the n-dependence in the notations in Sect. 11.1 as $B^{(n)}, \mathbf{W}^{(n)}, \mathbf{W}_k^{(n)}$, etc. In what follows, $\mathbf{u}^H$ for $\mathbf{u} \in \mathbb{C}^m$ should be understood as the linear diagonal operator $u_1^{\mathbf{h}_1} \cdots u_m^{\mathbf{h}_m}$, i.e.[11]

$$\mathbf{u}^H : |\mathbf{a}\rangle \mapsto u_1^{a_1} \cdots u_m^{a_m} |\mathbf{a}\rangle \quad \text{for } \mathbf{a} = (a_1, \ldots, a_m) \in B^{(m)}. \tag{13.136}$$

Viewing Fig. 13.9 from the SW, or taking the traces over $1_1, \ldots, 1_m$ first, we find that it represents the trace of the product of $(y\mathbf{w})^H$ and $R^{\mathrm{tr}_1}(\mu_1), \ldots, R^{\mathrm{tr}_1}(\mu_m)$:

$$\begin{aligned} T(x, y) &= \mathrm{Tr}_{\mathbf{W}^{(n)}} \left((y\mathbf{w})^H R^{\mathrm{tr}_1}(xu_1) \cdots R^{\mathrm{tr}_1}(xu_m) \right) \\ &= \sum_{k \ge 0} y^k \mathrm{Tr}_{\mathbf{W}_k^{(n)}} \left(\mathbf{w}^H R^{\mathrm{tr}_1}(xu_1) \cdots R^{\mathrm{tr}_1}(xu_m) \right) \in \mathrm{End}\big((\mathbf{W}^{(n)})^{\otimes m} \big), \end{aligned} \tag{13.137}$$

where the matrix product constructed $R^{\mathrm{tr}_1}(xu_i) \in \mathrm{End}(\overset{2}{\mathbf{W}}{}^{(n)} \otimes \mathbf{W}^{(n)})$ is a quantum R matrix of $U_q(A_{n-1}^{(1)})$ due to Theorem 13.11 and (13.29). The product is taken with respect to $\overset{2}{\mathbf{W}}{}^{(n)} = \overset{2_1}{\mathcal{F}}_q \otimes \cdots \otimes \overset{2_n}{\mathcal{F}}_q$, which corresponds to the first (left) component of R^{tr_1}'s.

[10] $T(x, y)$ here is different from the one in (11.85).

[11] For H we do not exhibit the number of components m, n as $H^{(m)}$ or $H^{(n)}$.

Alternatively, Fig. 13.9 viewed from the SE or first taking the traces over $2_1, \ldots, 2_n$ gives rise to the trace of the product of $(x\mathbf{u})^H$ and $R^{\mathrm{tr}_2}(v_1), \ldots, R^{\mathrm{tr}_2}(v_n)$:

$$\begin{aligned} T(x,y) &= \mathrm{Tr}_{\mathbf{W}^{(m)}}\left((x\mathbf{u})^H R^{\mathrm{tr}_2}(yw_1)\cdots R^{\mathrm{tr}_2}(yw_n)\right) \\ &= \sum_{k\geq 0} x^k \mathrm{Tr}_{\mathbf{W}^{(m)}_k}\left(\mathbf{u}^H R^{\mathrm{tr}_2}(yw_1)\cdots R^{\mathrm{tr}_2}(yw_n)\right) \in \mathrm{End}\big((\mathbf{W}^{(m)})^{\otimes n}\big), \end{aligned} \tag{13.138}$$

where the matrix product constructed $R^{\mathrm{tr}_2}(yw_j) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{W}}{}^{(m)} \otimes \mathbf{W}^{(m)})$ is a quantum R matrix of $U_q(A^{(1)}_{m-1})$ due to Theorem 13.12 and (13.42). The product is taken with respect to $\overset{\mathbf{1}}{\mathbf{W}}{}^{(m)} = \overset{1_1}{\mathscr{F}}_q \otimes \cdots \otimes \overset{1_m}{\mathscr{F}}_q$ in Fig. 13.9, which corresponds to the first (left) component of R^{tr_2}'s.

The identifications (13.137) and (13.138) correspond to the two complementary pictures $\mathscr{F}_q^{\otimes mn} = (\mathbf{W}^{(n)})^{\otimes m} = (\mathbf{W}^{(m)})^{\otimes n}$. In either case, $R^{\mathrm{tr}_1}(z)$ and $R^{\mathrm{tr}_2}(z)$ satisfy the Yang–Baxter equations, which implies the two-parameter commutativity

$$[T(x,y), T(x',y')] = 0 \tag{13.139}$$

for fixed $\mathbf{u}$ and $\mathbf{w}$.

Due to the weight conservations (13.28) and (13.41), the layer transfer matrix $T(x,y)$ has many invariant subspaces. The resulting decomposition is again described as (11.91)–(11.95) for another layer transfer matrix $T(x,y)$ considered in Sect. 11.6.

Consequently, each summand in (13.137) and (13.138) is further decomposed as

$$\begin{aligned} &\mathrm{Tr}_{\mathbf{W}^{(n)}_k}\left(\mathbf{w}^H R^{\mathrm{tr}_1}(xu_1)\cdots R^{\mathrm{tr}_1}(xu_m)\right) \\ &= \bigoplus_{I_1,\ldots,I_m\geq 0} \mathrm{Tr}_{\mathbf{W}^{(n)}_k}\left(\mathbf{w}^H R^{\mathrm{tr}_1}_{k,I_1}(xu_1)\cdots R^{\mathrm{tr}_1}_{k,I_m}(xu_m)\right), \end{aligned} \tag{13.140}$$

$$\begin{aligned} &\mathrm{Tr}_{\mathbf{W}^{(m)}_k}\left(\mathbf{u}^H R^{\mathrm{tr}_2}(yw_1)\cdots R^{\mathrm{tr}_2}(yw_n)\right) \\ &= \bigoplus_{J_1,\ldots,J_n\geq 0} \mathrm{Tr}_{\mathbf{W}^{(m)}_k}\left(\mathbf{u}^H R^{\mathrm{tr}_2}_{k,J_1}(yw_1)\cdots R^{\mathrm{tr}_2}_{k,J_n}(yw_n)\right). \end{aligned} \tag{13.141}$$

In the terminology of the quantum inverse scattering method, each summand in the RHS of (13.140) is a row transfer matrix of the $U_q(A^{(1)}_{n-1})$ vertex model of size m whose auxiliary space is $\mathbf{W}^{(n)}_k$ and the quantum space is $\mathbf{W}^{(n)}_{I_1} \otimes \cdots \otimes \mathbf{W}^{(n)}_{I_m}$ having the spectral parameter x with inhomogeneity $u_1, \ldots, u_m$ and the "horizontal" boundary electric/magnetic field $\mathbf{w}$. It forms a commuting family with respect to x provided that the other parameters are fixed. In the dual picture (13.141), the role of these data is interchanged as $m \leftrightarrow n$, $x \leftrightarrow y$, $\mathbf{u} \leftrightarrow \mathbf{w}$. This is another example of duality between rank and size, spectral inhomogeneity and field in addition to the one demonstrated in Sect. 11.6.

Consider the cube of size $l \times m \times n$ formed by concatenating Fig. 13.9 vertically for l times. As in Remark 11.8, one can formulate further two versions of the duality on the layer transfer matrices in the first and the second directions, which correspond to the interchanges $l \leftrightarrow m$ and $l \leftrightarrow n$.

13.9 Geometric R From Trace Reductions of Birational 3D R

We have constructed solutions to the Yang–Baxter equation by the trace reduction of the compositions of the 3D R. They were identified with the quantum R matrices for specific representations of $U_q(A^{(1)}_{n-1})$. Here we present a parallel story for the birational 3D R in Sect. 3.6.2 without going into the detailed proof.

Let us write the birational 3D R $R_{\text{birational}}$ in (3.151) simply as

$$R : (a, b, c) \mapsto \left(\frac{ab}{a+c},\ a+c,\ \frac{bc}{a+c} \right). \tag{13.142}$$

Given arrays of n variables $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ and an extra single variable z_{n+1}, we construct $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$, $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_n)$ and $z_1, \dots, z_n$ by postulating the following relations successively in the order $i = n, n-1, \dots, 1$:

$$R : (x_i, y_i, z_{i+1}) \mapsto (\tilde{x}_i, \tilde{y}_i, z_i). \tag{13.143}$$

See Fig. 13.10.

By the construction, z_1 is expressed as

$$z_1 = \frac{z_{n+1} \prod_{j=1}^{n} y_j}{\prod_{j=1}^{n} x_j + z_{n+1} Q_0(x, y)} \tag{13.144}$$

in terms of $Q_0(x, y)$ which will be given in (13.146). Reflecting the "trace", we impose the periodic boundary condition $z_1 = z_{n+1}$. This determines z_{n+1} hence every

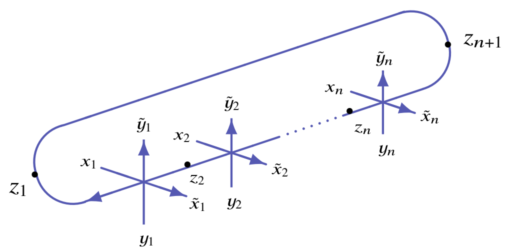

Fig. 13.10 Trace reduction of the birational 3D R along the third component. Each vertex is defined by (13.143) and (13.142). The periodic boundary condition $z_1 = z_{n+1}$ is imposed

z_i in terms of x and y. Explicitly, we get $z_i = (\prod_{k=1}^{n} y_k - \prod_{k=1}^{n} x_k)/Q_{i-1}(x, y)$. Substituting it back to $\tilde{x}$ and $\tilde{y}$, we obtain a map of $2n$ variables

$$\mathcal{R}^{(3)} : (x, y) \mapsto (\tilde{y}, \tilde{x}), \quad \tilde{x}_i = x_i \frac{Q_i(x, y)}{Q_{i-1}(x, y)}, \quad \tilde{y}_i = y_i \frac{Q_{i-1}(x, y)}{Q_i(x, y)}, \tag{13.145}$$

where the superscript (3) signifies that the third component is used for the trace reduction. The function $Q_i(x, y)$ is defined by

$$Q_i(x, y) = \sum_{k=1}^{n} \Big(\prod_{j=1}^{k-1} x_{i+j}\Big)\Big(\prod_{j=k+1}^{n} y_{i+j}\Big). \tag{13.146}$$

The indices of $Q_i, x_i, y_i, \tilde{x}_i, \tilde{y}_i$ are to be understood as belonging to $\mathbb{Z}_n$.

Example 13.14 For $n = 2, 3$, we have

$$n = 2: \quad Q_0(x, y) = x_2 + y_1, \quad Q_1(x, y) = x_1 + y_2, \tag{13.147}$$

$$n = 3: \quad Q_0(x, y) = x_1x_2 + x_1y_3 + y_2y_3, \tag{13.148}$$

$$Q_1(x, y) = x_2x_3 + x_2y_1 + y_1y_3, \tag{13.149}$$

$$Q_2(x, y) = x_1x_3 + x_3y_2 + y_1y_2. \tag{13.150}$$

One can construct similar maps $\mathcal{R}^{(1)}$ and $\mathcal{R}^{(2)}$ by replacing the elementary step (13.143) by

$$R : (z_{i+1}, x_i, y_i) \mapsto (z_i, \tilde{x}_i, \tilde{y}_i), \tag{13.151}$$

$$R : (x_i, z_{i+1}, y_i) \mapsto (\tilde{x}_i, z_i, \tilde{y}_i), \tag{13.152}$$

respectively, and applying them still in the order $i = n, n-1, \dots, 1$. For (13.151), z_1 is given by (13.144) with the interchange $x \leftrightarrow y$ reflecting the symmetry (3.59) of the birational 3D R (13.142). Thus we have

$$\mathcal{R}^{(1)} : (x, y) \mapsto (\tilde{y}, \tilde{x}); \quad \tilde{x}_i = x_i \frac{Q_{i-1}(y, x)}{Q_i(y, x)}, \quad \tilde{y}_i = y_i \frac{Q_i(y, x)}{Q_{i-1}(y, x)}. \tag{13.153}$$

For (13.152), z_i becomes much simpler as $z_i = x_i + y_i$, leading to

$$\mathcal{R}^{(2)} : (x, y) \mapsto (\tilde{y}, \tilde{x}); \quad \tilde{x}_i = x_i \frac{x_{i+1} + y_{i+1}}{x_i + y_i}, \quad \tilde{y}_i = y_i \frac{x_{i+1} + y_{i+1}}{x_i + y_i}. \tag{13.154}$$

We also introduce

$$\mathcal{R}^{\vee(2)} : (x, y) \mapsto (\tilde{y}, \tilde{x}); \quad \tilde{x}_i = x_i \frac{x_{i-1} + y_{i-1}}{x_i + y_i}, \quad \tilde{y}_i = y_i \frac{x_{i-1} + y_{i-1}}{x_i + y_i}. \tag{13.155}$$

It is obtained by the reverse procedure for $\mathcal{R}^{(2)}$ where $R : (x_i, z_i, y_i) \mapsto (\tilde{x}_i, z_{i+1}, \tilde{y}_i)$ is applied in the order $i = 1, 2, \ldots, n$ followed by $z_{n+1} = z_1$. It is related to $\mathcal{R}^{(2)}$ as

$$\mathcal{R}^{(2)} : (x^\vee, y^\vee) \mapsto (u, v) \quad \Leftrightarrow \quad \mathcal{R}^{\vee(2)} : (x, y) \to (u^\vee, v^\vee), \tag{13.156}$$

where $\vee$ denotes the reverse ordering of the n component arrays as in (11.4).

The maps $\mathcal{R}^{(1)}, \mathcal{R}^{(2)}, \mathcal{R}^{\vee(2)}$ and $\mathcal{R}^{(3)}$ are examples of *geometric R* of type A.[12] They satisfy the inversion relations and the Yang–Baxter equations. To describe them uniformly, we introduce a temporary notation

$$\mathcal{R}^{3,3} = \mathcal{R}^{(3)}, \quad \mathcal{R}^{1,3} = \mathcal{R}^{(2)}, \quad \mathcal{R}^{3,1} = \mathcal{R}^{\vee(2)}, \quad \mathcal{R}^{1,1} = \mathcal{R}^{(1)}. \tag{13.157}$$

Then the inversion relations read as

$$\mathcal{R}^{\alpha,\beta}\mathcal{R}^{\beta,\alpha} = \mathrm{id} \tag{13.158}$$

for $\alpha, \beta \in \{1, 3\}$. Thus these geometric R's are birational maps. They form set-theoretical solutions to the eight types of the Yang–Baxter equations

$$\big(1 \otimes \mathcal{R}^{\alpha,\beta}\big)\big(\mathcal{R}^{\alpha,\gamma} \otimes 1\big)\big(1 \otimes \mathcal{R}^{\beta,\gamma}\big) = \big(\mathcal{R}^{\beta,\gamma} \otimes 1\big)\big(1 \otimes \mathcal{R}^{\alpha,\gamma}\big)\big(\mathcal{R}^{\alpha,\beta} \otimes 1\big) \tag{13.159}$$

labeled with $\alpha, \beta, \gamma \in \{1, 3\}$. Here for instance $\big(1 \otimes \mathcal{R}^{\alpha,\beta}\big)(u, x, y) = (u, \tilde{y}, \tilde{x})$ and $\big(\mathcal{R}^{\alpha,\beta} \otimes 1\big)(x, y, u) = (\tilde{y}, \tilde{x}, u)$ in terms of the $\tilde{x}$ and $\tilde{y}$ corresponding to $\mathcal{R}^{\alpha,\beta}$ given by (13.145), (13.153), (13.154) or (13.155). One can bilinearize $\mathcal{R}^{\alpha,\beta}$ in terms of tau functions by incorporating the result in Sect. 3.6.3 into the trace reduction here.

Remark 13.15 The trace reduction considered here admits a two-parameter deformation leading to $\mathcal{R}^{\alpha,\beta}(\lambda, \omega)$. The parameter λ is introduced by replacing the birational 3D R (13.142) with the λ-deformed one in (3.159). The parameter ω is introduced by replacing the periodicity $z_1 = z_{n+1}$ of the auxiliary variable by the *quasi*-periodicity condition $z_1 = \omega z_{n+1}$. Then the inversion relation $\mathcal{R}^{\alpha,\beta}(\lambda, \omega)\mathcal{R}^{\beta,\alpha}(\lambda, \omega) = \mathrm{id}$ persists for any λ and ω. The Yang–Baxter equations remain valid for $\mathcal{R}^{\alpha,\beta}(\lambda, 1)$ and $\mathcal{R}^{\alpha,\beta}(0, \omega)$.

13.10 Bibliographical Notes and Comments

The trace reduction of the 3D R with respect to the first component was considered in [18, Eq. (36)], and the identification with the type A quantum R matrices for symmetric tensor representations was announced in [18, Eq. (54)]. See also [75]. A proof of a similar identification concerning the third component was given in

[12] Some early publications refer to them as "tropical R".

[96, Proposition 17]. This chapter provides a unified treatment of the trace reductions along the three possible directions. They are symbolically expressed, for $n = 3$, as

$$\mathrm{Tr}_{\bullet}\Big(z^{\mathbf{h}_{\bullet}} R_{\bullet\circ\circ} R_{\bullet\circ\circ} R_{\bullet\circ\circ}\Big), \quad \mathrm{Tr}_{\bullet}\Big(z^{\mathbf{h}_{\bullet}} R_{\circ\bullet\circ} R_{\circ\bullet\circ} R_{\circ\bullet\circ}\Big). \quad \mathrm{Tr}_{\bullet}\Big(z^{\mathbf{h}_{\bullet}} R_{\circ\circ\bullet} R_{\circ\circ\bullet} R_{\circ\circ\bullet}\Big).$$

Other variations mixing the components like $\mathrm{Tr}_{\bullet}\Big(z^{\mathbf{h}_{\bullet}} R_{\bullet\circ\circ} R_{\circ\circ\bullet} R_{\bullet\circ\circ}\Big)$ also yield solutions to the Yang–Baxter equation. Their quantum group symmetry has been described in [86] using the appropriate automorphisms of q-oscillator algebra interchanging the creation and the annihilation operators.

Even if the auxiliary Fock space $\bullet$ to take the trace is limited to the third component, there are more significant generalizations mixing the 3D R and 3D L as

$$\mathrm{Tr}\big(z^{\mathbf{h}} \mathcal{R}^{(\epsilon_1)} \cdots \mathcal{R}^{(\epsilon_n)}\big), \qquad (\mathcal{R}^{(0)} = R, \;\; \mathcal{R}^{(1)} = L) \tag{13.160}$$

for $\epsilon_1, \ldots, \epsilon_n = 0, 1$. These 2^n objects are easily seen to satisfy the Yang–Baxter equation by a mixed usage of the tetrahedron equations of type $RRRR = RRRR$ and $RLLL = LLLR$ [95, Theorem 12]. Chapter 11 and the present one correspond to the two special cases without the coexistence of the 3D L and 3D R. In order to characterize them as the intertwiner, one is naturally led to an algebra $\mathcal{U}_A(\epsilon_1, \ldots, \epsilon_n)$ interpolating $\mathcal{U}_A(0, \ldots, 0) = U_{-q^{-1}}(A^{(1)}_{n-1})$ in Theorem 11.3 and $\mathcal{U}_A(1, \ldots, 1) = U_q(A^{(1)}_{n-1})$ in Theorem 13.10 via some quantum superalgebras in between [98]. The algebra $\mathcal{U}_A(\epsilon_1, \ldots, \epsilon_n)$ has been identified as an example of *generalized quantum groups*. This notion emerged in [56] through the classification of pointed Hopf algebras [2, 55] and it has been studied further in [3, 6, 9, 57]. For recent developments related to the content of this book, see [108, 109].

The algebra homomorphism from U_q to q-oscillators as in Proposition 13.8 goes back to [54] for example. The proof of Theorem 13.10 utilizing such a homomorphism is simpler and is due to [97].

The explicit formula $A(z)^{\mathbf{ab}}_{\mathbf{ij}}$ in Theorem 13.3 was presented in [26]. Unfortunately the derivation therein has a gap when $|\mathbf{i}| > |\mathbf{i}'|$ in [26, Eq. (3.15)]. Section 13.5.3 provides the first complete proof of (13.55). It fills the gap effectively by Lemma 13.7, and provides a new insight that the quantum group symmetry is translated into a bilinear identity of q-hypergeometric as in Lemma 13.5.

Section 13.7 is based on [87], where the building block Φ_q (13.49) of the R matrices was extracted which plays the role of local hopping rate of an integrable Markov process of multispecies particles subject to a particular zero-range-type interaction. The case $n = 2$ of Φ_q first appeared in [123]. See also [25, 81, 100] for the subsequent developments.

The 3D lattice model in Sect. 13.8 has been considered in [17]. The layer transfer matrix corresponds to a quantization of the earlier work [68], where the 3D R is replaced by the birational 3D R and the description in terms of geometric R was adopted in accordance with Sect. 13.9. In such a setting, the duality shows up as the $W(A^{(1)}_{m-1} \times A^{(1)}_{n-1})$ symmetry.

One of the earliest appearances of the birational map $\mathcal{R}^{(1)}$ is [150]. The maps $\mathcal{R}^{(3)}$, $\mathcal{R}^{(2)}$, $\mathcal{R}^{\vee(2)}$ and $\mathcal{R}^{(1)}$ in (13.145)–(13.155) are the geometric lifts of R, $R^{\vee}$, ${}^{\vee}R$ and $R^{\vee\vee}$ in [101, Eqs. (2.1)–(2.4)], respectively. $\mathcal{R}^{(3)}$, $\mathcal{R}^{(2)}$ and $\mathcal{R}^{(1)}$ are also contained in the first example of set-theoretical solutions to the reflection equation [101, Appendix A]. Associated with the type A Kirillov–Reshetikhin (KR) module $W_s^{(r)}$ with $1 \leq r \leq n-1$, $s \geq 1$, one has the geometric crystal $\mathcal{B}^{(r)}$. The most general geometric R $R^{r,r'} : \mathcal{B}^{(r)} \times \mathcal{B}^{(r')} \to \mathcal{B}^{(r')} \times \mathcal{B}^{(r)}$ has been constructed in [49]. See also [99]. The four examples in Sect. 13.9 are the special cases of it as $\mathcal{R}^{3,3} = R^{1,1}, \mathcal{R}^{3,1} = R^{1,n-1}, \mathcal{R}^{1,3} = R^{n-1,1}, \mathcal{R}^{1,1} = R^{n-1,n-1}$. Set-theoretical solutions to the Yang–Baxter equation are also called Yang–Baxter maps [145]. Geometric R's form an important class in it having the quantum and combinatorial counterparts which are connected to the KR modules and integrable soliton cellular automata known as (generalized) box–ball systems [60].

Chapter 14
Boundary Vector Reductions of $RRRR = RRRR$

Abstract This chapter presents the boundary vector reduction of an n-concatenation of the tetrahedron equation $RRRR = RRRR$ of the 3D R. It generates infinite families of solutions to the Yang–Baxter equation in matrix product forms. In contrast with the boundary vector reduction starting from $RLLL = LLLR$ (Chap. 12), they turn out to be the quantum R matrices of the q-oscillator representations of $U_q(A^{(2)}_{2n})$, $U_q(C^{(1)}_n)$ and $U_q(D^{(2)}_{n+1})$. These algebras have Dynkin diagrams with double arrows. It turns out that the two kinds of boundary vectors correspond to the two directions of the double arrows. For simplicity, we treat the reduction with respect to the third component only.

14.1 Boundary Vector Reductions

We fix a positive integer n and keep the notations for the sets $B = (\mathbb{Z}_{\geq 0})^n$, the vector spaces $\mathbf{W} = \mathcal{F}_q^{\otimes n}$, $\mathbf{W}_\pm$ with bases $|\mathbf{a}\rangle = |a_1\rangle \otimes \cdots \otimes |a_n\rangle$ labeled by n-arrays $\mathbf{a} = (a_1, \ldots, a_n) \in B$ in (11.8)–(11.13). We will also use $|\mathbf{a}| = a_1 + \cdots + a_n$ and $\mathbf{a}^\vee = (a_n, \ldots, a_1)$ in (11.4) and the elementary vector $\mathbf{e}_i$ in (11.1). As for the q-oscillator algebra Osc_q and the Fock space $\mathcal{F}_q$, see Sect. 3.2.

14.1.1 n-Concatenation of the Tetrahedron Equation

In Sect. 13.2, we started from the tetrahedron equation $RRRR = RRRR$ of the 3D R and derived its n-concatenation with respect to the third component (13.5):

$$\begin{aligned} &R_{456}\, x^{\mathbf{h}_4}(R_{1_1 2_1 4} \cdots R_{1_n 2_n 4})(xy)^{\mathbf{h}_5}(R_{1_1 3_1 5} \cdots R_{1_n 3_n 5})y^{\mathbf{h}_6}(R_{2_1 3_1 6} \cdots R_{2_n 3_n 6})R_{456} \\ &\quad = y^{\mathbf{h}_6}(R_{2_1 3_1 6} \cdots R_{2_n 3_n 6})(xy)^{\mathbf{h}_5}(R_{1_1 3_1 5} \cdots R_{1_n 3_n 5})x^{\mathbf{h}_4}(R_{1_1 2_1 4} \cdots R_{1_n 2_n 4}). \end{aligned} \tag{14.1}$$

The indices $1_i, 2_i, 3_i\, (i = 1, \ldots, n)$ and $4, 5, 6$ are labels of the q-oscillator Fock space $\mathcal{F}_q$ on which a 3D R acts non-trivially. The operator $\mathbf{h}$ is defined in (3.14) and x, y are free parameters.

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14.1.2 Boundary Vector Reductions

Recall the boundary vectors introduced in Sect. 3.6.1:

$$\langle \eta_r| = \sum_{m\geq 0} \frac{\langle rm|}{(q^{r^2})_m}, \qquad |\eta_r\rangle = \sum_{m\geq 0} \frac{|rm\rangle}{(q^{r^2})_m} \qquad (r = 1, 2). \tag{14.2}$$

Evaluate (14.1) between $\langle \overset{4}{\eta}_r| \otimes \langle \overset{5}{\eta}_r| \otimes \langle \overset{6}{\eta}_r|$ and $|\overset{4}{\eta}_{r'}\rangle \otimes |\overset{5}{\eta}_{r'}\rangle \otimes |\overset{6}{\eta}_{r'}\rangle$. Thanks to Proposition 3.28, the two R_{456} disappear, leading to

$$\begin{aligned}
&\langle \overset{4}{\eta}_r|x^{\mathbf{h}_4} R_{1_1 2_1 4} \cdots R_{1_n 2_n 4}|\overset{4}{\eta}_{r'}\rangle \langle \overset{5}{\eta}_r|(xy)^{\mathbf{h}_5} R_{1_1 3_1 5} \cdots R_{1_n 3_n 5}|\overset{5}{\eta}_{r'}\rangle \times \\
&\quad \times \langle \overset{6}{\eta}_r|y^{\mathbf{h}_6} R_{2_1 3_1 6} \cdots R_{2_n 3_n 6}|\overset{6}{\eta}_{r'}\rangle \\
&= \langle \overset{6}{\eta}_r|y^{\mathbf{h}_6} R_{2_1 3_1 6} \cdots R_{2_n 3_n 6}|\overset{6}{\eta}_{r'}\rangle \langle \overset{5}{\eta}_r|(xy)^{\mathbf{h}_5} R_{1_1 3_1 5} \cdots R_{1_n 3_n 5}|\overset{5}{\eta}_{r'}\rangle \times \\
&\quad \times \langle \overset{4}{\eta}_r|x^{\mathbf{h}_4} R_{1_1 2_1 4} \cdots R_{1_n 2_n 4}|\overset{4}{\eta}_{r'}\rangle.
\end{aligned} \tag{14.3}$$

Let us denote the operators appearing here by

$$\begin{aligned}
R^{r,r'}_{\mathbf{1},\mathbf{2}}(z) &= \varrho^{r,r'}(z)\langle \overset{4}{\eta}_r|z^{\mathbf{h}_4} R_{1_1 2_1 4} \cdots R_{1_n 2_n 4}|\overset{4}{\eta}_{r'}\rangle \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{W}} \otimes \overset{\mathbf{2}}{\mathbf{W}}), \\
R^{r,r'}_{\mathbf{1},\mathbf{3}}(z) &= \varrho^{r,r'}(z)\langle \overset{5}{\eta}_r|z^{\mathbf{h}_5} R_{1_1 3_1 5} \cdots R_{1_n 3_n 5}|\overset{5}{\eta}_{r'}\rangle \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{W}} \otimes \overset{\mathbf{3}}{\mathbf{W}}), \\
R^{r,r'}_{\mathbf{2},\mathbf{3}}(z) &= \varrho^{r,r'}(z)\langle \overset{6}{\eta}_r|z^{\mathbf{h}_6} R_{2_1 3_1 6} \cdots R_{2_n 3_n 6}|\overset{6}{\eta}_{r'}\rangle \in \mathrm{End}(\overset{\mathbf{2}}{\mathbf{W}} \otimes \overset{\mathbf{3}}{\mathbf{W}}),
\end{aligned} \tag{14.4}$$

where $r, r' = 1, 2$. The normalization factor $\varrho^{r,r'}(z)$ will be specified in (14.13). They are the same operators acting on different copies of $\overset{\mathbf{1}}{\mathbf{W}}, \overset{\mathbf{2}}{\mathbf{W}}, \overset{\mathbf{3}}{\mathbf{W}}$ of $\mathbf{W}$ in (11.11) and (11.12). Now the relation (14.3) is stated as the Yang–Baxter equation:

$$R^{r,r'}_{\mathbf{1},\mathbf{2}}(x) R^{r,r'}_{\mathbf{1},\mathbf{3}}(xy) R^{r,r'}_{\mathbf{2},\mathbf{3}}(y) = R^{r,r'}_{\mathbf{2},\mathbf{3}}(y) R^{r,r'}_{\mathbf{1},\mathbf{3}}(xy) R^{r,r'}_{\mathbf{1},\mathbf{2}}(x) \quad (r, r' = 1, 2). \tag{14.5}$$

We will often suppress the labels $\mathbf{1}, \mathbf{2}$ etc. Then $R^{r,r'}(z) \in \mathrm{End}(\mathbf{W} \otimes \mathbf{W})$ is defined by

$$R^{r,r'}(z)(|\mathbf{i}\rangle \otimes |\mathbf{j}\rangle) = \sum_{\mathbf{a},\mathbf{b}\in B} R^{r,r'}(z)^{\mathbf{ab}}_{\mathbf{ij}} |\mathbf{a}\rangle \otimes |\mathbf{b}\rangle, \tag{14.6}$$

where the elements are given by the matrix product formula

$$R^{r,r'}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \varrho^{r,r'}(z)\langle \eta_r|z^{\mathbf{h}} R^{a_1 b_1}_{i_1 j_1} \cdots R^{a_n b_n}_{i_n j_n}|\eta_{r'}\rangle \quad (r, r' = 1, 2) \tag{14.7}$$

in terms of the components of the 3D R in (3.69) (Fig. 14.1).

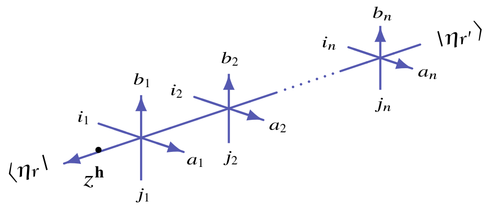

Fig. 14.1 The boundary vector reduction. All the arrows carry the q-oscillator Fock space $\mathcal{F}_q$. The matrix product formula (14.7) is depicted as a concatenation of Fig. 13.1 along the arrow sandwiched by the boundary vectors $\langle\eta_r|$ and $|\eta_{r'}\rangle$ in (14.2). It is a BBQ stick with X-shaped sausages and extra caps at the two ends

The dual pairing is defined by (3.16) and concretely evaluated by (12.10). From the equivalence of (2.4) to (2.2), the formula (14.7) is also expressed as

$$R^{r,r'}(z)^{\mathbf{ab}}_{\mathbf{ij}} = \varrho^{r,r'}(z) \sum_{c_0,\dots,c_n\geq 0} \frac{z^{rc_0}(q^2)_{rc_0}}{(q^{r^2})_{c_0}(q^{r'^2})_{c_n}} \times R^{a_1,b_1,rc_0}_{i_1,j_1,c_1} R^{a_2,b_2,c_1}_{i_2,j_2,c_2} \cdots R^{a_{n-1},b_{n-1},c_{n-2}}_{i_{n-1},j_{n-1},c_{n-1}} R^{a_n,b_n,c_{n-1}}_{i_n,j_n,r'c_n} \tag{14.8}$$

in terms of R^{abc}_{ijk} detailed in Sect. 3.4. The numerator $(q^2)_{rc_0}$ originates in (3.16). From the constraint (3.48), the apparent $(n+1)$-fold sum (14.8) is actually a single sum which in particular imposes $|\mathbf{b}| + rc_0 = |\mathbf{j}| + r'c_n$. By using this fact and (3.63), it is easy to derive an exchange rule under $r \leftrightarrow r'$:

$$R^{r,r'}(z)^{\mathbf{ab}}_{\mathbf{ij}}/\varrho^{r,r'}(z) = z^{|\mathbf{j}|-|\mathbf{b}|}\left(\prod_{l=1}^{n} \frac{(q^2)_{i_l}(q^2)_{j_l}}{(q^2)_{a_l}(q^2)_{b_l}}\right) R^{r',r}(z)^{\mathbf{i}^\vee\mathbf{j}^\vee}_{\mathbf{a}^\vee\mathbf{b}^\vee}/\varrho^{r',r}(z), \tag{14.9}$$

where $\mathbf{a}^\vee$ is the reverse array defined in (11.4).

From (14.7) and (14.8) we also have

$$R^{r,r'}(z)^{\mathbf{ab}}_{\mathbf{ij}} = 0 \ \text{ unless } \ \mathbf{a}+\mathbf{b} = \mathbf{i}+\mathbf{j}, \tag{14.10}$$

$$R^{2,2}(z)^{\mathbf{ab}}_{\mathbf{ij}} = 0 \ \text{ unless } \ |\mathbf{a}| \equiv |\mathbf{i}| \text{ and } |\mathbf{b}| \equiv |\mathbf{j}| \mod 2. \tag{14.11}$$

The weight conservation (14.10) follows either from (3.48) or (3.69), and the parity constraint (14.11) is due to the fact that the boundary vectors $\langle\eta_2|, |\eta_2\rangle$ in (14.2) contain "even modes" only. It leads to the decomposition

$$R^{2,2}(z) = \bigoplus_{\sigma,\sigma'=+,-} R^{\sigma,\sigma'}(z), \qquad R^{\sigma,\sigma'}(z) \in \mathrm{End}(\mathbf{W}_\sigma \otimes \mathbf{W}_{\sigma'}). \tag{14.12}$$

When $(r, r') = (2, 2)$, the Yang–Baxter equation (14.5) is valid in each subspace $\mathbf{W}_\sigma \otimes \mathbf{W}_{\sigma'} \otimes \mathbf{W}_{\sigma''}$ of $\mathbf{W} \otimes \mathbf{W} \otimes \mathbf{W}$. The scalar $\varrho^{2,2}(z)$ in (14.7) may be chosen as $\varrho^{\sigma,\sigma'}(z)$ depending on the summands in (14.12). We take them as

$$\varrho^{r,r'}(z) = \frac{(z^{\max(r,r')}; q^{rr'})_\infty}{(-z^{\max(r,r')}q; q^{rr'})_\infty} \quad ((r, r') = (1, 1), (1, 2), (2, 1)),$$

$$\varrho^{\pm,\pm}(z) = \frac{(z^2; q^4)_\infty}{(z^2q^2; q^4)_\infty}, \qquad \varrho^{\pm,\mp}(z) = \frac{(z^2q^2; q^4)_\infty}{(z^2q^4; q^4)_\infty}. \tag{14.13}$$

This is the same choice as (12.15). It makes all the matrix elements rational in q and z^r. For example,

$$R^{1,1}(z)(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle) = R^{1,2}(z)(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle) = R^{+,+}(z)(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle) = |\mathbf{0}\rangle \otimes |\mathbf{0}\rangle, \tag{14.14}$$

$$R^{+,-}(z)^{\mathbf{0},\mathbf{e}_1}_{\mathbf{0},\mathbf{e}_1} = \frac{-q}{1-z^2}, \quad R^{-,+}(z)^{\mathbf{e}_1,\mathbf{0}}_{\mathbf{e}_1,\mathbf{0}} = \frac{1}{1-z^2}, \quad R^{-,-}(z)^{\mathbf{e}_1,\mathbf{e}_1}_{\mathbf{e}_1,\mathbf{e}_1} = \frac{z^2-q^2}{1-z^2q^2}, \tag{14.15}$$

where $\mathbf{0} = (0, \dots, 0)$. For instance, to derive the last result in (14.15), one looks at the corresponding sum (14.8):

$$\sum_{c_0,\dots,c_n \geq 0} \frac{z^{2c_0}(q^2)_{2c_0}}{(q^4)_{c_0}(q^4)_{c_n}} R^{1,1,2c_0}_{1,1,c_1} R^{0,0,c_1}_{0,0,c_2} \cdots R^{0,0,c_{n-2}}_{0,0,c_{n-1}} R^{0,0,c_{n-1}}_{0,0,2c_n}. \tag{14.16}$$

Due to (3.48) this is a single sum over $k = c_0 = c_n = c_{1/2} = \cdots = c_{n-1}/2$. Moreover, from Example 3.15, the product of R's is equal to $R^{1,1,2k}_{1,1,2k} = 1 - (1+q^2)q^{4k}$. Thus it is calculated as

$$\sum_{k\geq 0} \frac{z^{2k}(q^2)_{2k}}{(q^4)_k^2}\big(1-(1+q^2)q^{4k}\big) = \sum_{k\geq 0} \frac{z^{2k}(q^2; q^4)_k}{(q^4; q^4)_k}\big(1-(1+q^2)q^{4k}\big)$$
$$= \frac{(z^2q^2; q^4)_\infty}{(z^2; q^4)_\infty} - (1+q^2)\frac{(z^2q^6; q^4)_\infty}{(z^2q^4; q^4)_\infty} = \varrho^{-,-}(z)^{-1}\frac{z^2-q^2}{1-z^2q^2}.$$

Although $R^{r,r'}(z)$ acts on the infinite-dimensional space $\mathbf{W} \otimes \mathbf{W}$, the property (14.10) shows that it is locally finite, i.e. the RHS of (14.6) always contains finitely many terms. Therefore the composition of $R^{r,r'}(z)$ is convergent and the Yang–Baxter equation makes sense as an equality of rational functions of q and z.

14.2 Identification with Quantum R Matrices of $A^{(2)}_{2n}$, $C^{(1)}_n$, $D^{(2)}_{n+1}$

14.2.1 Quantum Affine Algebra $U_q(\mathfrak{g}^{r,r'})$.

We will be concerned with the affine Kac–Moody algebras[1]

$$\mathfrak{g}^{1,1} = D^{(2)}_{n+1}, \quad \mathfrak{g}^{1,2} = A^{(2)}_{2n}, \quad \mathfrak{g}^{2,1} = \tilde{A}^{(2)}_{2n}, \quad \mathfrak{g}^{2,2} = C^{(1)}_n, \tag{14.17}$$

where the notation $\mathfrak{g}^{r,r'}$ will turn out to fit to $R^{r,r'}(z)$ in the previous subsection.

Let $U_q = U_q(D^{(2)}_{n+1})$ $(n \geq 2)$, $U_q(B^{(1)}_n)$ $(n \geq 3)$, $U_q(\tilde{B}^{(1)}_n)$ $(n \geq 3)$, $U_q(D^{(1)}_n)$ $(n \geq 3)$ be the quantum affine algebras. They are Hopf algebras generated by $\{e_i, f_i, k_i^{\pm 1} \mid i \in \{0, 1, \dots, n\}\}$ satisfying the relations (10.1). Beside the commutativity of $k_i^{\pm 1}$ and the q-Serre relations, they read as

$$k_i e_j k_i^{-1} = q_i^{a_{ij}} e_j, \quad k_i f_j k_i^{-1} = q_i^{-a_{ij}} f_j, \quad [e_i, f_j] = \delta_{i,j} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \tag{14.18}$$

where the constants q_i $(0 \leq i \leq n)$ are taken as[2]

$$q_i = q \quad \text{except for} \quad q_0 = q^{r^2/2}, \; q_n = q^{r'^2/2}. \tag{14.19}$$

The affine Lie algebra $\tilde{A}^{(2)}_{2n}$ is just $A^{(2)}_{2n}$ with different enumeration of the vertices as shown in Fig. 14.2. We keep it for uniformity of the description. The Cartan matrix $(a_{ij})_{0 \leq i,j \leq n}$ is determined from the Dynkin diagrams of the relevant affine Lie algebras according to the convention of [67]. Thus for instance in $U_q(A^{(2)}_{2n})$, one has $a_{01} = -2, a_{10} = -1$ and $k_0 e_0 = q e_0 k_0$, $k_0 e_1 = q^{-1} e_1 k_0$, $k_1 e_0 = q^{-1} e_0 k_1$ and $k_1 e_1 = q^2 e_1 k_1$.

We retain the coproduct Δ and its opposite Δ^{op} in (11.58) and (11.59):

$$\Delta(k_i) = k_i \otimes k_i, \quad \Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = 1 \otimes f_i + f_i \otimes k_i^{-1}, \tag{14.20}$$

$$\Delta^{\mathrm{op}}(k_i) = k_i \otimes k_i, \quad \Delta^{\mathrm{op}}(e_i) = 1 \otimes e_i + e_i \otimes k_i, \quad \Delta^{\mathrm{op}}(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i. \tag{14.21}$$

[1] Some symbols including $\mathfrak{g}^{r,r'}$ here and Sect. 12.2.1 are apparently the same, but they should be understood as independently redefined.

[2] For $U_q(\mathfrak{g}^{1,1}) = U_q(D^{(2)}_{n+1})$, which is appearing also in Sect. 12.2.1, this normalization and (12.22) coincide.

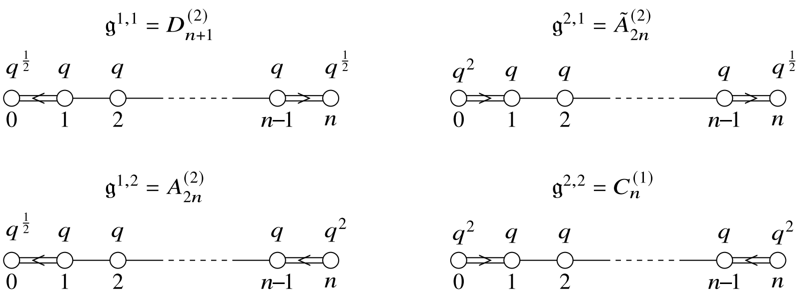

Fig. 14.2 Dynkin diagrams of (14.17) with enumeration of vertices. The data q_i is given above the corresponding vertex i

14.2.2 *q*-Oscillator Representations

Let Osc_q be the q-oscillator algebra (3.12). Instead of $\mathbf{k}$ we use the generator including the zero point energy[3]

$$\hat{\mathbf{k}} = q^{\frac{1}{2}}\mathbf{k}. \tag{14.22}$$

Then Osc_q is an algebra over $\mathbb{C}(q^{\frac{1}{2}})$ generated by $\mathbf{a}^+, \mathbf{a}^-, \hat{\mathbf{k}}$ and $\hat{\mathbf{k}}^{-1}$ obeying the relations

$$\hat{\mathbf{k}}\,\hat{\mathbf{k}}^{-1} = \hat{\mathbf{k}}^{-1}\,\hat{\mathbf{k}} = 1, \quad \hat{\mathbf{k}}\,\mathbf{a}^{\pm} = q^{\pm 1}\mathbf{a}^{\pm}\,\hat{\mathbf{k}}, \quad \mathbf{a}^{\pm}\,\mathbf{a}^{\mp} = 1 - q^{\mp 1}\hat{\mathbf{k}}^2. \tag{14.23}$$

The property of the boundary vectors in (3.134)–(3.141) are rephrased as

$$\mathbf{a}^{\pm}|\eta_1\rangle = (1 \mp q^{\mp\frac{1}{2}}\hat{\mathbf{k}})|\eta_1\rangle, \quad \langle\eta_1|\mathbf{a}^{\pm} = \langle\eta_1|(1 \pm q^{\pm\frac{1}{2}}\hat{\mathbf{k}}), \tag{14.24}$$
$$\mathbf{a}^{+}|\eta_2\rangle = \mathbf{a}^{-}|\eta_2\rangle, \quad \langle\eta_2|\mathbf{a}^{+} = \langle\eta_2|\mathbf{a}^{-}. \tag{14.25}$$

In what follows, an element $\mathbf{a}^+ \otimes 1 \otimes \hat{\mathbf{k}} \otimes \mathbf{a}^- \in \mathrm{Osc}_q^{\otimes 4}$ for example will be denoted by $\mathbf{a}_1^+\hat{\mathbf{k}}_3\mathbf{a}_4^-$ etc. Thus the q-oscillator generators with different indices are commutative. The intertwining relations for the 3D R in (13.112)–(13.115) become "autonomous" in that there is no apparent q as follows:

$$R\,\hat{\mathbf{k}}_2\mathbf{a}_1^+ = (\hat{\mathbf{k}}_3\mathbf{a}_1^+ + \hat{\mathbf{k}}_1\mathbf{a}_2^+\mathbf{a}_3^-)R, \qquad R\,\hat{\mathbf{k}}_2\mathbf{a}_1^- = (\hat{\mathbf{k}}_3\mathbf{a}_1^- + \hat{\mathbf{k}}_1\mathbf{a}_2^-\mathbf{a}_3^+)R, \tag{14.26}$$
$$R\,\mathbf{a}_2^+ = (\mathbf{a}_1^+\mathbf{a}_3^+ - \hat{\mathbf{k}}_1\hat{\mathbf{k}}_3\mathbf{a}_2^+)R, \qquad R\,\mathbf{a}_2^- = (\mathbf{a}_1^-\mathbf{a}_3^- - \hat{\mathbf{k}}_1\hat{\mathbf{k}}_3\mathbf{a}_2^-)R, \tag{14.27}$$
$$R\,\hat{\mathbf{k}}_2\mathbf{a}_3^+ = (\hat{\mathbf{k}}_1\mathbf{a}_3^+ + \hat{\mathbf{k}}_3\mathbf{a}_1^-\mathbf{a}_2^+)R, \qquad R\,\hat{\mathbf{k}}_2\mathbf{a}_3^- = (\hat{\mathbf{k}}_1\mathbf{a}_3^- + \hat{\mathbf{k}}_3\mathbf{a}_1^+\mathbf{a}_2^-)R, \tag{14.28}$$
$$R\,\hat{\mathbf{k}}_1\hat{\mathbf{k}}_2 = \hat{\mathbf{k}}_1\hat{\mathbf{k}}_2R, \qquad R\,\hat{\mathbf{k}}_2\hat{\mathbf{k}}_3 = \hat{\mathbf{k}}_2\hat{\mathbf{k}}_3R. \tag{14.29}$$

[3] It is also used in (8.13).

Set

$$d = \frac{q}{(q-q^{-1})^2}, \quad d_1 = d|_{q\to q^{1/2}}, \quad d_2 = d|_{q\to q^2}. \tag{14.30}$$

Proposition 14.1 *The following map with a parameter x defines an algebra homomorphism $\rho_x : U_q(\mathfrak{g}^{r,r'}) \to \mathrm{Osc}_q^{\otimes n}[x^r, x^{-r}]$. (On the LHS $\rho_x(g)$ is denoted by g for simplicity.)*

$$\begin{aligned}
e_0 &= x^r \mathrm{i}^{r^2}(\mathbf{a}_1^-)^r \hat{\mathbf{k}}_1^{-r}, & f_0 &= x^{-r} d_r (\mathbf{a}_1^+)^r, & k_0 &= (-\mathrm{i}\hat{\mathbf{k}}_1^{-1})^r,\\
e_i &= \mathbf{a}_i^+ \mathbf{a}_{i+1}^- \hat{\mathbf{k}}_{i+1}^{-1}, & f_i &= d\, \mathbf{a}_i^- \mathbf{a}_{i+1}^+ \hat{\mathbf{k}}_i^{-1}, & k_i &= \hat{\mathbf{k}}_i \hat{\mathbf{k}}_{i+1}^{-1},\\
e_n &= (\mathbf{a}_n^+)^{r'}, & f_n &= \mathrm{i}^{r'^2} d_{r'} (\mathbf{a}_n^-)^{r'} \hat{\mathbf{k}}_n^{-r'}, & k_n &= (\mathrm{i}\hat{\mathbf{k}}_n)^{r'},
\end{aligned} \tag{14.31}$$

where $0 < i < n$ and $\mathrm{i} = \sqrt{-1}$.

The proposition can be shown by directly checking (14.18) and the q-Serre relations. If the images of e_i, f_i, k_i for $0 < i < n$ are extended to $i \in \mathbb{Z}_n$, it becomes equivalent to $\rho^{(3)}_{x=1} : U_q(A^{(1)}_{n-1}) \to \mathrm{Osc}_q^{\otimes n}$ in (13.95).

By (3.13) and (14.22) one can further let $\mathrm{Osc}_q^{\otimes n}$ act on $\mathbf{W} = \mathcal{F}_q^{\otimes n} = \bigoplus_{\mathbf{a}\in B} \mathbb{C}|\mathbf{a}\rangle$ in (11.11). For $\mathfrak{g}^{2,2}$, the representation space can be restricted to $\mathbf{W}_+$ or $\mathbf{W}_-$ since (14.31) does not change the parity of $|\mathbf{a}|$ of $|\mathbf{a}\rangle$. Let us denote the resulting representations by

$$\pi_x : U_q(\mathfrak{g}^{r,r'}) \xrightarrow{\rho_x} \mathrm{Osc}_q^{\otimes n}[x^r, x^{-r}] \to \mathrm{End}(\mathbf{W}), \tag{14.32}$$

$$\pi_{\pm,x} : U_q(\mathfrak{g}^{2,2}) \xrightarrow{\rho_x} \mathrm{Osc}_q^{\otimes n}[x^r, x^{-r}] \to \mathrm{End}(\mathbf{W}_\pm), \tag{14.33}$$

where the second arrow is given by (3.13) for each component. By the definition $\pi_x = \pi_{+,x} \oplus \pi_{-,x}$ for $\mathfrak{g}^{2,2}$. We call (14.33) for $\mathfrak{g}^{2,2}$ and (14.32) for the other $\mathfrak{g}^{r,r'}$ the *q-oscillator representations*. They are infinite-dimensional irreducible representations which are singular at $q = 1$ because of the factors d, d_1, d_2 in (14.30).

14.2.3 Quantum Group Symmetry

Let $D \in \mathrm{End}(\mathbf{W})$ be the diagonal operator such that $D|\mathbf{a}\rangle = (\mathrm{i}q^{\frac{1}{2}})^{|\mathbf{a}|}|\mathbf{a}\rangle$. It can be realized as $D = (\mathrm{i}q^{\frac{1}{2}})^{\mathbf{h}_1+\cdots+\mathbf{h}_n}$ where $\mathbf{h}_i$ denotes $\mathbf{h}$ (3.14) acting on the ith component of $\mathbf{W} = \mathcal{F}_q^{\otimes n}$. Introduce a slight gauge transformation of $R^{r,r'}(z)$ as

$$\hat{R}^{r,r'}(z) = (D \otimes 1) R^{r,r'}(z) (1 \otimes D^{-1}). \tag{14.34}$$

By using $[R^{r,r'}(z), D \otimes D] = 0$ which is implied by the weight conservation (14.10), it is easy to see that $\hat{R}^{r,r'}(z)$ also satisfies the Yang–Baxter equation.

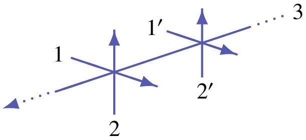

Fig. 14.3 The part of the matrix product formula (14.7) relevant to (14.36) with $0 < i < n$. It is formally the same with Fig. 13.7 except the absence of the operator $z^{\mathbf{h}_3}$

Let $\Delta_{x,y} = (\pi_x \otimes \pi_y) \circ \Delta$ and $\Delta^{\mathrm{op}}_{x,y} = (\pi_x \otimes \pi_y) \circ \Delta^{\mathrm{op}}$ be the tensor product representations, where the coproducts Δ and Δ^{op} are specified in (14.20) and (14.21). For $\mathfrak{g}^{2,2}$, we set $\Delta_{\epsilon_1,x,\epsilon_2,y} = (\pi_{\epsilon_1,x} \otimes \pi_{\epsilon_2,y}) \circ \Delta$ and $\Delta^{\mathrm{op}}_{\epsilon_1,x,\epsilon_2,y} = (\pi_{\epsilon_1,x} \otimes \pi_{\epsilon_2,y}) \circ \Delta^{\mathrm{op}}$ for $\epsilon_1, \epsilon_2 = \pm$. In [96, Prop.12], it has been proved that $\Delta_{x,y}$ for $\mathfrak{g}^{1,1}$, $\mathfrak{g}^{1,2}$, $\mathfrak{g}^{2,1}$ and $\Delta_{\epsilon_1,x,\epsilon_2,y}$ for $\mathfrak{g}^{2,2}$ are irreducible for generic x, y. Therefore $R^{r,r'}(z)$ is characterized, up to normalization, by the following theorem.

Theorem 14.2 *$R^{r,r'}(z)$ given by the matrix product construction (14.6)–(14.8) enjoys the following quantum group symmetry:*

$$\Delta^{\mathrm{op}}_{x,y}(g)\hat{R}^{r,r'}(\tfrac{y}{x}) = \hat{R}^{r,r'}(\tfrac{y}{x})\Delta_{x,y}(g) \qquad (\forall g \in U_q(\mathfrak{g}^{r,r'})). \tag{14.35}$$

For $r = r' = 2$, the same relation holds for $\Delta^{\mathrm{op}}_{\epsilon_1,x,\epsilon_2,y}$ and $\Delta_{\epsilon_1,x,\epsilon_2,y}$ with $\epsilon_1, \epsilon_2 = \pm$.

Proof It suffices to show (14.35) for $g = k_i$, e_i and f_i for $0 \le i \le n$. The case $g = k_i$ is easy. Below we present a proof for $g = f_i$. The case $g = e_i$ can be shown similarly. In terms of $R^{r,r'}(z)$ without hat, the relation (14.35) takes the form

$$(\tilde{f}_i \otimes 1 + k_i^{-1} \otimes f_i)R^{r,r'}(\tfrac{x}{y}) - R^{r,r'}(\tfrac{x}{y})(1 \otimes \tilde{f}_i + f_i \otimes k_i^{-1}) = 0, \tag{14.36}$$

where $\tilde{f}_i = D^{-1} f_i D$ and $\pi_x \otimes \pi_y$ is omitted. There are five cases (i)–(v) to be confirmed. Below we treat them separately.

(i) Case $0 < i < n$. From (14.31) one has $\tilde{f}_i = f_i$. In view of (14.31), the relevant part in $R^{r,r'}(z)$ is the ith and the $(i+1)$th factors in the matrix product formula (14.7). We regard it as an element of the product $R_{123}R_{1'2'3}$ of the 3D R, where the indices $1, 2, 1', 2', 3$ signify the arrows as in Fig. 14.3.
In this label, (14.31) is expressed as

$$f_i \otimes 1 = d\mathbf{a}_1^- \mathbf{a}_{1'}^+ \hat{\mathbf{k}}_1^{-1}, \qquad 1 \otimes f_i = d\mathbf{a}_2^- \mathbf{a}_{2'}^+ \hat{\mathbf{k}}_2^{-1}, \tag{14.37}$$

$$k_i \otimes 1 = \hat{\mathbf{k}}_1 \hat{\mathbf{k}}_{1'}^{-1}, \qquad 1 \otimes k_i = \hat{\mathbf{k}}_2 \hat{\mathbf{k}}_{2'}^{-1}. \tag{14.38}$$

Substituting them into (14.36), we find that it follows from (13.110) with $x = y = z = 1$. Thus it reduces to a special case of Theorem 13.10.

(ii) Case $i = 0$ and $r = 1$. From (14.31) one has $\tilde{f}_0 = -iq^{-\frac{1}{2}} f_0$. The generators f_0 and k_0 interact non-trivially only with the leftmost 3D R in (14.4). Denote it by $R = R_{123}$. In terms of the labels 1, 2, 3, one has

$$f_0 \otimes 1 = x^{-1} d_1 \mathbf{a}_1^+, \qquad 1 \otimes f_0 = y^{-1} d_1 \mathbf{a}_2^+, \tag{14.39}$$

$$k_0 \otimes 1 = -\mathrm{i}\hat{\mathbf{k}}_1^{-1}, \qquad 1 \otimes k_0 = -\mathrm{i}\hat{\mathbf{k}}_2^{-1}. \tag{14.40}$$

Then (14.36) follows from

$$0 = \langle \eta_1 | z^{\mathbf{h}_3} (-\mathrm{i}q^{-\frac{1}{2}} x^{-1} \mathbf{a}_1^+ + \mathrm{i}y^{-1} \hat{\mathbf{k}}_1 \mathbf{a}_2^+) R - \langle \eta_1 | z^{\mathbf{h}_3} R (-\mathrm{i}q^{-\frac{1}{2}} y^{-1} \mathbf{a}_2^+ + \mathrm{i}x^{-1} \mathbf{a}_1^+ \hat{\mathbf{k}}_2) \tag{14.41}$$

with $z = y/x$. Up to an overall factor, the RHS is calculated by means of (14.26) and (14.27) as

$$\begin{aligned} &\langle \eta_1 | z^{\mathbf{h}_3} \left(q^{-\frac{1}{2}} z \mathbf{a}_1^+ R - \hat{\mathbf{k}}_1 \mathbf{a}_2^+ R - q^{-\frac{1}{2}} R \mathbf{a}_2^+ + z R \hat{\mathbf{k}}_2 \mathbf{a}_1^+ \right) \\ &= \langle \eta_1 | z^{\mathbf{h}_3} \left(q^{-\frac{1}{2}} z \mathbf{a}_1^+ - \hat{\mathbf{k}}_1 \mathbf{a}_2^+ - q^{-\frac{1}{2}} (\mathbf{a}_1^+ \mathbf{a}_3^+ - \hat{\mathbf{k}}_1 \hat{\mathbf{k}}_3 \mathbf{a}_2^+) + z (\hat{\mathbf{k}}_3 \mathbf{a}_1^+ + \hat{\mathbf{k}}_1 \mathbf{a}_2^+ \mathbf{a}_3^-) \right) R. \end{aligned} \tag{14.42}$$

Due to $\langle \eta_1 | z^{\mathbf{h}_3} \mathbf{a}_3^{\pm} = z^{\pm 1} \langle \eta_1 | z^{\mathbf{h}_3} (1 \pm q^{\pm \frac{1}{2}} \hat{\mathbf{k}}_3)$ by (14.24), this vanishes.

(iii) Case $i = 0$ and $r = 2$. From (14.31) one has $\tilde{f}_0 = -q^{-1} f_0$. The situation is the same as the previous case (i). Using the same labels 1, 2, 3, one has

$$f_0 \otimes 1 = x^{-2} d_2 (\mathbf{a}_1^+)^2, \qquad 1 \otimes f_0 = y^{-2} d_2 (\mathbf{a}_2^+)^2, \tag{14.43}$$

$$k_0 \otimes 1 = -\hat{\mathbf{k}}_1^{-2}, \qquad 1 \otimes k_0 = -\hat{\mathbf{k}}_2^{-2}. \tag{14.44}$$

Here and in what follows, $\hat{\mathbf{k}}_1^{-2}$ for instance is a shorthand for $(\hat{\mathbf{k}}_1^{-1})^2$. Then (14.36) follows from

$$\begin{aligned} 0 &= \langle \eta_2 | z^{\mathbf{h}_3} \left(q^{-1} z^2 (\mathbf{a}_1^+)^2 R + \hat{\mathbf{k}}_1^2 (\mathbf{a}_2^+)^2 R - q^{-1} R (\mathbf{a}_2^+)^2 - z^2 R (\mathbf{a}_1^+)^2 \hat{\mathbf{k}}_2^2 \right) \\ &= \langle \eta_2 | z^{\mathbf{h}_3} \Big(q^{-1} z^2 (\mathbf{a}_1^+)^2 + \hat{\mathbf{k}}_1^2 (\mathbf{a}_2^+)^2 \\ &\qquad - q^{-1} (\mathbf{a}_1^+ \mathbf{a}_3^+ - \hat{\mathbf{k}}_1 \hat{\mathbf{k}}_3 \mathbf{a}_2^+)^2 - z^2 (\hat{\mathbf{k}}_3 \mathbf{a}_1^+ + \hat{\mathbf{k}}_1 \mathbf{a}_2^+ \mathbf{a}_3^-)^2 \Big) R. \end{aligned}$$

Due to $\langle \eta_2 | z^{\mathbf{h}_3} \mathbf{a}_3^+ = z^2 \langle \eta_2 | z^{\mathbf{h}_3} \mathbf{a}_3^-$ by (14.25), this vanishes.

(iv) Case $i = n$ and $r' = 1$. From (14.31) one has $\tilde{f}_n = \mathrm{i}q^{\frac{1}{2}} f_n$. The generators f_n and k_n interact non-trivially only with the rightmost 3D R in (14.4). Denote it by $R = R_{123}$. In terms of the labels 1, 2, 3, one has

$$f_n \otimes 1 = \mathrm{i} d_1 \mathbf{a}_1^- \hat{\mathbf{k}}_1^{-1}, \qquad 1 \otimes f_n = \mathrm{i} d_1 \mathbf{a}_2^- \hat{\mathbf{k}}_2^{-1}, \tag{14.45}$$

$$k_n \otimes 1 = \mathrm{i}\hat{\mathbf{k}}_1, \qquad 1 \otimes k_n = \mathrm{i}\hat{\mathbf{k}}_2. \tag{14.46}$$

Then (14.36) follows from

$$\begin{aligned}0 &= \left(q^{\frac{1}{2}}\mathbf{a}_1^-\hat{\mathbf{k}}_1^{-1} - \hat{\mathbf{k}}_1^{-1}\mathbf{a}_2^-\hat{\mathbf{k}}_2^{-1}\right)R|\eta_1\rangle - R\left(q^{\frac{1}{2}}\mathbf{a}_2^-\hat{\mathbf{k}}_2^{-1} - \mathbf{a}_1^-\hat{\mathbf{k}}_1^{-1}\hat{\mathbf{k}}_2^{-1}\right)|\eta_1\rangle \\ &= \left(q^{\frac{1}{2}}\hat{\mathbf{k}}_2\mathbf{a}_1^- - \mathbf{a}_2^-\right)R\hat{\mathbf{k}}_1^{-1}\hat{\mathbf{k}}_2^{-1}|\eta_1\rangle - R\left(q^{\frac{1}{2}}\mathbf{a}_2^-\hat{\mathbf{k}}_2^{-1} - \mathbf{a}_1^-\hat{\mathbf{k}}_1^{-1}\hat{\mathbf{k}}_2^{-1}\right)|\eta_1\rangle \\ &= R\Big(\big(q^{\frac{1}{2}}(\hat{\mathbf{k}}_3\mathbf{a}_1^- + \hat{\mathbf{k}}_1\mathbf{a}_2^-\mathbf{a}_3^+) - (\mathbf{a}_1^-\mathbf{a}_3^- - \hat{\mathbf{k}}_1\hat{\mathbf{k}}_3\mathbf{a}_2^-)\big)\hat{\mathbf{k}}_1^{-1}\hat{\mathbf{k}}_2^{-1} \\ &\quad - q^{\frac{1}{2}}\mathbf{a}_2^-\hat{\mathbf{k}}_2^{-1} + \mathbf{a}_1^-\hat{\mathbf{k}}_1^{-1}\hat{\mathbf{k}}_2^{-1}\Big)|\eta_1\rangle,\end{aligned}$$

where $R = R^{-1}$ (3.60) and (14.26)–(14.27) are used. Due to $\mathbf{a}_3^{\pm}|\eta_1\rangle = (1 \mp q^{\mp\frac{1}{2}}\hat{\mathbf{k}}_3)|\eta_1\rangle$ by (14.24), this vanishes.

(v) Case $i = n$ and $r' = 2$. From (14.31) one has $\tilde{f}_n = -qf_n$. The situation is the same as the previous case (i). Using the same labels 1, 2, 3, one has

$$f_n \otimes 1 = d_2(\mathbf{a}_1^-)^2\hat{\mathbf{k}}_1^{-2}, \qquad 1 \otimes f_n = d_2(\mathbf{a}_2^-)^2\hat{\mathbf{k}}_2^{-2}, \tag{14.47}$$

$$k_n \otimes 1 = -\hat{\mathbf{k}}_1^2, \qquad 1 \otimes k_n = -\hat{\mathbf{k}}_2^2. \tag{14.48}$$

Then (14.36) follows from

$$\begin{aligned}0 &= \left(q(\mathbf{a}_1^-)^2\hat{\mathbf{k}}_1^{-2} + \hat{\mathbf{k}}_1^{-2}(\mathbf{a}_2^-)^2\hat{\mathbf{k}}_2^{-2}\right)R|\eta_2\rangle - R\left(q(\mathbf{a}_2^-)^2\hat{\mathbf{k}}_2^{-2} + (\mathbf{a}_1^-)^2\hat{\mathbf{k}}_1^{-2}\hat{\mathbf{k}}_2^{-2}\right)|\eta_2\rangle \\ &= R\Big(\big(q(\hat{\mathbf{k}}_3\mathbf{a}_1^- + \hat{\mathbf{k}}_1\mathbf{a}_2^-\mathbf{a}_3^+)^2 + (\mathbf{a}_1^-\mathbf{a}_3^- - \hat{\mathbf{k}}_1\hat{\mathbf{k}}_3\mathbf{a}_2^-)^2\big)\hat{\mathbf{k}}_1^{-2}\hat{\mathbf{k}}_2^{-2} \\ &\quad - q(\mathbf{a}_2^-)^2\hat{\mathbf{k}}_2^{-2} - (\mathbf{a}_1^-)^2\hat{\mathbf{k}}_1^{-2}\hat{\mathbf{k}}_2^{-2}\Big)|\eta_2\rangle.\end{aligned}$$

This can be shown by utilizing the relations like $(\mathbf{a}_3^+)^2|\eta_2\rangle = \mathbf{a}_3^+\mathbf{a}_3^-|\eta_2\rangle = (1 - q^{-1}\hat{\mathbf{k}}_3^2)|\eta_2\rangle$ originating from $\mathbf{a}_3^+|\eta_2\rangle = \mathbf{a}_3^-|\eta_2\rangle$ in (14.25). □

Remark 14.3 Theorem 14.2 suggests the following correspondence between the boundary vectors $\langle\eta_r|, |\eta_{r'}\rangle$ in (14.2) and the end shape of the Dynkin diagrams in Fig. 14.2: (Fig. 14.4)

A similar correspondence is observed in Remarks 11.4 and 12.3.

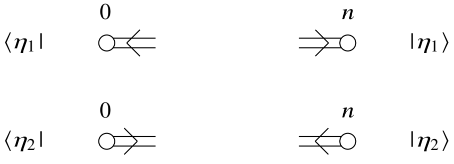

Fig. 14.4 Correspondence between the boundary vectors and the end shape of the Dynkin diagrams in Fig. 14.2 implied by Theorem 14.2

14.3 Bibliographical Notes and Comments

This chapter is mainly based on [96, 97]. The boundary vector reduction was first applied to 3D R in [95], where the sum (14.8) was explicitly evaluated for $n = 1$. For general n, the quantum group symmetry of $R^{r,r'}(z)$ in Theorem 14.2 is due to [96, Theorem 13] and [97, Theorem 4.1]. The former reference contains a proof of the irreducibility of the tensor product of the q-oscillator representations and the spectral decomposition of $R^{r,r'}(z)$.

Similarly to (13.160), the boundary vector reduction also works for the mixed product of 3D R and 3D L, producing the solutions to the Yang–Baxter equation of the form

$$\langle \eta_r | z^{\mathbf{h}} \mathcal{R}^{(\epsilon_1)} \cdots \mathcal{R}^{(\epsilon_n)} | \eta_{r'} \rangle \qquad (\mathcal{R}^{(0)} = R, \;\; \mathcal{R}^{(1)} = L) \tag{14.49}$$

for $\epsilon_1, \ldots, \epsilon_n = 0, 1$. For $r = r' = 1$, they have been characterized as the intertwiner of a generalized quantum group denoted by $\mathcal{U}_B(\epsilon_1, \ldots, \epsilon_n)$ interpolating $\mathcal{U}_B(1, \ldots, 1) = U_{-q^{-1}}(D^{(2)}_{n+1})$ in Theorem 12.2 and $\mathcal{U}_B(0, \ldots, 0) = U_q(D^{(2)}_{n+1})$ in Theorem 14.2 [98, Theorem 4.1].[4]

Remark 14.3 is taken from [96, Remark 14]. It lacks double branches in the end shape of Dynkin diagrams, whereas the similar Remark 12.3 does not cover inward double arrows. It remains a challenge to supplement these missing cases by further devising the boundary vector reductions and to explore the relevant generalized quantum groups.

[4] The irreducibility of the tensor product representations has been established only when $(\epsilon_1, \ldots, \epsilon_n)$ is of type $(1, \ldots, 1, 0, \ldots, 0)$.

Chapter 15
Trace Reduction of $(LGLG)K = K(GLGL)$

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Abstract In this chapter we demonstrate a 3D approach to the reflection equation by the trace reduction. Starting from the quantized reflection equation in Sect. 4.4 and its solution in Theorem 5.18, an infinite family of trigonometric solutions to the usual reflection equation including spectral parameters are constructed. The procedure is parallel with the one applied to the tetrahedron equation in Chap. 11. The resulting K matrices are expressed by the matrix product formula. They are characterized in the quantum group theoretical framework based on the Onsager coideal of $U_p(A^{(1)}_{n-1})$.

15.1 Introduction

This chapter and the next present a 3D approach to the reflection equations in 2D and (1+1)D integrable systems:

$$\mathcal{R}_{12}(xy^{-1})\mathcal{K}_2(x)\mathcal{R}_{21}(xy)\mathcal{K}_1(y) = \mathcal{K}_1(y)\mathcal{R}_{12}(xy)\mathcal{K}_2(x)\mathcal{R}_{21}(xy^{-1}), \tag{15.1}$$

where x, y are spectral parameters and $\mathcal{R}(z)$ is supposed to satisfy the Yang–Baxter equation by itself. If the spectral parameters are suppressed, it reduces to the constant version $\mathcal{R}_{12}\mathcal{K}_2\mathcal{R}_{21}\mathcal{K}_1 = \mathcal{K}_1\mathcal{R}_{12}\mathcal{K}_2\mathcal{R}_{21}$.

In Sect. 5.7, we have obtained a solution to the quantized reflection equation

$$(L_{12}G_2L_{21}G_1)K = K(G_1L_{12}G_2L_{21}), \tag{15.2}$$

which is a generalization (or relaxation) of the constant reflection equation to a conjugacy equivalence by the 3D K. It admits a composition along the auxiliary Fock space for arbitrary n times. The resulting equation allows the trace reduction and the boundary vector reduction which eliminate K and generate infinitely many trigonometric solutions to (15.1) labeled with n. They are dense in that all the elements satisfying the obvious weight constraint are non-vanishing, allow a matrix product formula, and are characterized as the intertwiner of the Onsager coideal of appropriate

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quantum affine algebras. These aspects are quite parallel with Chaps. 11–14 for the Yang–Baxter equation except that special coideal subalgebras are necessary to be selected here. In this chapter and the next we deal with the trace reduction and the boundary reduction, respectively.

Let us summarize the result from Sect. 5.7 for convenience. Set $V = \mathbb{C}v_0 \oplus \mathbb{C}v_1$ and define $L = \sum E_{ai} \otimes E_{bj} \otimes L_{ij}^{ab} \in \mathrm{End}(V \otimes V \otimes \mathcal{F}_{q^2})$ and $G = \sum E_{kj} \otimes G_j^k \in \mathrm{End}(V \otimes \mathcal{F}_q)$ by

$$\begin{pmatrix} L_{00}^{00} & L_{01}^{00} & L_{10}^{00} & L_{11}^{00} \\ L_{00}^{01} & L_{01}^{01} & L_{10}^{01} & L_{11}^{01} \\ L_{00}^{10} & L_{01}^{10} & L_{10}^{10} & L_{11}^{10} \\ L_{00}^{11} & L_{01}^{11} & L_{10}^{11} & L_{11}^{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -q^2\alpha^{-1}\mathbf{K} & \mathbf{A}^- & 0 \\ 0 & \mathbf{A}^+ & \alpha\mathbf{K} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{15.3}$$

$$\begin{pmatrix} G_0^0 & G_1^0 \\ G_0^1 & G_1^1 \end{pmatrix} = \begin{pmatrix} \mathbf{a}^+ & -q\beta^{-1}\mathbf{k} \\ \beta\mathbf{k} & \mathbf{a}^- \end{pmatrix}. \tag{15.4}$$

Here $\mathbf{a}^{\pm}$, $\mathbf{k}$ are q-oscillators in (3.12)–(3.13) and $\mathbf{A}^{\pm}$, $\mathbf{K}$ are q^2-oscillators in (5.15)–(5.16). These L and G are taken from Figs. 5.1 and 5.2, where we have attached the extra gauge parameters α, β in view of Remark 5.19. We remark that

$$(L \text{ in } (15.3)) = (L \text{ in } (11.14))|_{q\to q^2} = (L \text{ in } (12.1))|_{q\to q^2}. \tag{15.5}$$

We also note that

$$z^{\mathbf{h}} G_a^b = z^{1-a-b} G_a^b z^{\mathbf{h}}, \tag{15.6}$$

$$G_a^b = L_{b,1-b}^{1-a,a}\,\big|_{q\to q^{1/2},\alpha\to\beta}\,, \tag{15.7}$$

where $\mathbf{h}$ is defined in (3.14).

From Theorem 5.18 and Remark 5.19, L and G satisfy the quantized reflection equation (4.9)

$$L_{123}G_{24}L_{215}G_{16}K_{3456} = K_{3456}G_{16}L_{125}G_{24}L_{213} \tag{15.8}$$

in $\mathrm{End}(\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{\mathcal{F}}_{q^2} \otimes \overset{4}{\mathcal{F}}_q \otimes \overset{5}{\mathcal{F}}_{q^2} \otimes \overset{6}{\mathcal{F}}_q)$, where we have written $\overset{i}{G}_x$ as G_{xi}, and $L_{215} = P_{12}L_{125}P_{12}$ etc. as usual.

15.2 Concatenation of Quantized Reflection Equation

Consider n copies of (15.8) in which the spaces labeled with 1, 2 are replaced by the copies $1_i, 2_i$ with $i = 1, 2, \ldots, n$:

$$L_{1_i 2_i 3} G_{2_i 4} L_{2_i 1_i 5} G_{1_i 6} K_{3456} = K_{3456} G_{1_i 6} L_{1_i 2_i 5} G_{2_i 4} L_{2_i 1_i 3}. \tag{15.9}$$

Using (15.9) successively, one can send K_{3456} to the left through $L_{1_i 2_i 3} G_{2_i 4} L_{2_i 1_i 5} G_{1_i 6}$ converting it into $G_{1_i 6} L_{1_i 2_i 5} G_{2_i 4} L_{2_i 1_i 3}$ ($i = 1, 2, \ldots, n$) as

$$\begin{aligned}&(L_{1_1 2_1 3} G_{2_1 4} L_{2_1 1_1 5} G_{1_1 6}) \cdots (L_{1_n 2_n 3} G_{2_n 4} L_{2_n 1_n 5} G_{1_n 6}) K_{3456}\\&\quad = K_{3456} (G_{1_1 6} L_{1_1 2_1 5} G_{2_1 4} L_{2_1 1_1 3}) \cdots (G_{1_n 6} L_{1_n 2_n 5} G_{2_n 4} L_{2_n 1_n 3}).\end{aligned}$$

One can rearrange this without changing the order of operators sharing common labels as

$$\begin{aligned}&(L_{1_1 2_1 3} \cdots L_{1_n 2_n 3})(G_{2_1 4} \cdots G_{2_n 4})(L_{2_1 1_1 5} \cdots L_{2_n 1_n 5})(G_{1_1 6} \cdots G_{1_n 6}) K_{3456}\\&\quad = K_{3456} (G_{1_1 6} \cdots G_{1_n 6})(L_{1_1 2_1 5} \cdots L_{1_n 2_n 5})(G_{2_1 4} \cdots G_{2_n 4})(L_{2_1 1_1 3} \cdots L_{2_n 1_n 3}).\end{aligned} \tag{15.10}$$

This relation will serve as the base of the reduction procedure in this and the next chapters.

15.3 Trace Reduction

The weight conservation (5.66) of the 3D K is stated as

$$K_{3456} (xy^{-1})^{\mathbf{h}_3} x^{\mathbf{h}_4} (xy)^{\mathbf{h}_5} y^{\mathbf{h}_6} = (xy^{-1})^{\mathbf{h}_3} x^{\mathbf{h}_4} (xy)^{\mathbf{h}_5} y^{\mathbf{h}_6} K_{3456} \tag{15.11}$$

for arbitrary parameters x and y. See (3.14) for the definition of $\mathbf{h}$. Multiply this by (15.10) from the left and take the trace over $\overset{3}{\mathcal{F}}_{q^2} \otimes \overset{4}{\mathcal{F}}_q \otimes \overset{5}{\mathcal{F}}_{q^2} \otimes \overset{6}{\mathcal{F}}_q$. Using $K^2 = 1$ from (5.72) we obtain

$$\begin{aligned}&\mathrm{Tr}_3\big((xy^{-1})^{\mathbf{h}_3} L_{1_1 2_1 3} \cdots L_{1_n 2_n 3}\big) \mathrm{Tr}_4\big(x^{\mathbf{h}_4} G_{2_1 4} \cdots G_{2_n 4}\big) \times\\&\qquad \times \mathrm{Tr}_5\big((xy)^{\mathbf{h}_5} L_{2_1 1_1 5} \cdots L_{2_n 1_n 5}\big) \mathrm{Tr}_6\big(y^{\mathbf{h}_6} G_{1_1 6} \cdots G_{1_n 6}\big)\\&= \mathrm{Tr}_6\big(y^{\mathbf{h}_6} G_{1_1 6} \cdots G_{1_n 6}\big) \mathrm{Tr}_5\big((xy)^{\mathbf{h}_5} L_{1_1 2_1 5} \cdots L_{1_n 2_n 5}\big) \times\\&\qquad \times \mathrm{Tr}_4\big(x^{\mathbf{h}_4} G_{2_1 4} \cdots G_{2_n 4}\big) \mathrm{Tr}_3\big((xy^{-1})^{\mathbf{h}_3} L_{2_1 1_1 3} \cdots L_{2_n 1_n 3}\big).\end{aligned} \tag{15.12}$$

Here $\mathrm{Tr}_3(\cdots)$ and $\mathrm{Tr}_5(\cdots)$ in the LHS are the R matrices $S^{\mathrm{tr}_3}_{\mathbf{1},\mathbf{2}}(xy^{-1})|_{q\to q^2}$ and $S^{\mathrm{tr}_3}_{\mathbf{2},\mathbf{1}}(xy)|_{q\to q^2}$ in (11.23) up to a scalar multiple. The change $q \to q^2$ is necessary because of (15.5). Similar identifications hold for the RHS. Since they appear frequently, in this chapter we use the notation

$$S^{\mathrm{tr}}(z) := \big(S^{\mathrm{tr}_3}(z) \text{ in } (11.26)\big)\,|_{q\to q^2}. \tag{15.13}$$

The tiny abbreviation of tr_3 to tr is the sign of the replacement $q \to q^2$ in $S(z)$ from Chap. 11. It also balances with the notation $K^{\mathrm{tr}}(z)$ introduced below.

Returning to (15.12), the other factors emerging from G constitute the linear operators

$$K^{\mathrm{tr}}_{\mathbf{1}}(z) = \kappa^{\mathrm{tr}}(z)\mathrm{Tr}_6\big(z^{\mathbf{h}_6} G_{1_1 6}\cdots G_{1_n 6}\big) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}}), \tag{15.14}$$

$$K^{\mathrm{tr}}_{\mathbf{2}}(z) = \kappa^{\mathrm{tr}}(z)\mathrm{Tr}_4\big(z^{\mathbf{h}_4} G_{2_1 4}\cdots G_{2_n 4}\big) \in \mathrm{End}(\overset{\mathbf{2}}{\mathbf{V}}), \tag{15.15}$$

where the traces are taken over $\overset{4}{\mathcal{F}}_q$ and $\overset{6}{\mathcal{F}}_q$, and the scalar $\kappa^{\mathrm{tr}}(z)$ will be specified in (15.22). They are the same linear operators acting on the different copies of $V^{\otimes n}$ given as $\overset{\mathbf{1}}{\mathbf{V}} = \overset{1_1}{V} \otimes \cdots \otimes \overset{1_n}{V}$ and $\overset{\mathbf{2}}{\mathbf{V}} = \overset{2_1}{V} \otimes \cdots \otimes \overset{2_n}{V}$.

The relation (15.12) is the reflection equation in 2D:

$$S^{\mathrm{tr}}_{\mathbf{1},\mathbf{2}}(xy^{-1})K^{\mathrm{tr}}_{\mathbf{2}}(x)S^{\mathrm{tr}}_{\mathbf{2},\mathbf{1}}(xy)K^{\mathrm{tr}}_{\mathbf{1}}(y) = K^{\mathrm{tr}}_{\mathbf{1}}(y)S^{\mathrm{tr}}_{\mathbf{1},\mathbf{2}}(xy)K^{\mathrm{tr}}_{\mathbf{2}}(x)S^{\mathrm{tr}}_{\mathbf{2},\mathbf{1}}(xy^{-1}). \tag{15.16}$$

The construction (15.14)–(15.15) provides the matrix product formula for each element as

$$\begin{aligned} K^{\mathrm{tr}}(z)v_{\mathbf{a}} &= \sum_{\mathbf{b}\in\mathfrak{s}} K^{\mathrm{tr}}(z)^{\mathbf{b}}_{\mathbf{a}} v_{\mathbf{b}} \qquad (\mathbf{a}\in\mathfrak{s}),\\ K^{\mathrm{tr}}(z)^{\mathbf{b}}_{\mathbf{a}} &= \kappa^{\mathrm{tr}}(z)\mathrm{Tr}\big(z^{\mathbf{h}} G^{b_1}_{a_1}\cdots G^{b_n}_{a_n}\big), \end{aligned} \tag{15.17}$$

where the base vector $v_{\mathbf{a}} \in \mathbf{V}$, the labeling set $\mathfrak{s}$ and the relevant notations are defined in (11.1)–(11.7). From (15.4), we see that it depends on the parameter β via G^b_a as the conjugation:

$$K^{\mathrm{tr}}(z) = \beta^{\mathbf{h}_1+\cdots+\mathbf{h}_n}\big(K^{\mathrm{tr}}(z)|_{\beta=1}\big)\beta^{-\mathbf{h}_1-\cdots-\mathbf{h}_n}. \tag{15.18}$$

Comparison of (15.17) and (11.26) using (15.7) also shows that

$$K^{\mathrm{tr}}(z)^{\mathbf{b}}_{\mathbf{a}} = (\mathrm{scalar})S^{\mathrm{tr}_3}(z)^{\mathbf{a}'\,\mathbf{a}}_{\mathbf{b}\,\mathbf{b}'}, \tag{15.19}$$

where $\mathbf{a}' = (1 - a_1, \dots, 1 - a_n)$ as in (6.4). In (15.19), the changes $q \to q^{1/2}$ in (15.7) and $q \to q^2$ in (15.13) are canceled, therefore the RHS is just the one in (11.26) up to a scalar, hence $S^{\mathrm{tr}_3}(z)$ instead of $S^{\mathrm{tr}}(z)$.

Let us derive the "selection rule" or the weight conservation law of $K^{\mathrm{tr}}(z)$. Suppose the number of pairs (0, 0), (0, 1), (1, 0), (1, 1) in the multiset $\{(a_1, b_1), \dots, (a_n, b_n)\}$ is r, s, t, u, respectively in (15.17). Then from (11.4) we have $|\mathbf{a}| = t + u$, $|\mathbf{b}| = s + u$ and $n = r + s + t + u$. Moreover, in order to have a non-vanishing matrix element (15.17), there must be as many creation operators as annihilation operators. From (15.4), this imposes the constraint $r = u$. These relations enforce $|\mathbf{a}| + |\mathbf{b}| = n$. Namely we have the selection rule:

$$K^{\mathrm{tr}}(z)_{\mathbf{a}}^{\mathbf{b}} = 0 \quad \text{unless} \quad |\mathbf{a}| + |\mathbf{b}| = n. \tag{15.20}$$

It implies the direct sum decomposition:

$$K^{\mathrm{tr}}(z) = \bigoplus_{0 \le l \le n} K_l^{\mathrm{tr}}(z), \qquad K_l^{\mathrm{tr}}(z) : \mathbf{V}_l \to \mathbf{V}_{n-l}. \tag{15.21}$$

The space $\mathbf{V}_l$ is the lth fundamental representation of $U_p(A_{n-1}^{(1)})$. Note that the selection rule (15.21) is *not* $\mathbf{V}_l \to \mathbf{V}_l$.

The reflection equation (15.16) holds in a finer manner, i.e. as the identity of linear operators $\mathbf{V}_l \otimes \mathbf{V}_m \to \mathbf{V}_{n-l} \otimes \mathbf{V}_{n-m}$ for each pair $(l, m) \in \{0, 1, \dots, n\}^2$. The scalar in (15.17) can be specified depending on l as $\kappa_l^{\mathrm{tr}}(z)$. We take it as

$$\kappa_l^{\mathrm{tr}}(z) = (-1)^l q^{-\frac{n}{2}}(1 - zq^n). \tag{15.22}$$

This choice leads to the normalization

$$K_l^{\mathrm{tr}}(z) v_{\mathbf{e}_1 + \cdots + \mathbf{e}_l} = (q^{-\frac{1}{2}}\beta)^{n-2l} v_{\mathbf{e}_{l+1} + \cdots + \mathbf{e}_n} + \cdots \quad (0 \le l \le n). \tag{15.23}$$

Example 15.1 We present $K^{\mathrm{tr}}(z)$ (15.17) with $\beta = q^{\frac{1}{2}}$ for $n = 2, 3$. The general β case can be deduced from them by using (15.18). We temporarily write $v_0 \otimes v_1$ as $|01\rangle$ etc.

When $n = 2$, $K^{\mathrm{tr}}(z)|_{\beta = q^{\frac{1}{2}}}$ acts on the basis as

$$|00\rangle \mapsto |11\rangle, \qquad |01\rangle \mapsto -\frac{q^{-1}(-1+q^2)z|01\rangle}{(-1+z)} + |10\rangle,$$
$$|11\rangle \mapsto |00\rangle, \qquad |10\rangle \mapsto |01\rangle - \frac{q^{-1}(-1+q^2)|10\rangle}{(-1+z)}.$$

When $n = 3$, $K^{\mathrm{tr}}(z)|_{\beta = q^{\frac{1}{2}}}$ acts on the basis as

$$
\begin{aligned}
&|000\rangle \mapsto |111\rangle, \qquad |111\rangle \mapsto |000\rangle,\\
&|001\rangle \mapsto -\frac{(-1+q^2)z|011\rangle}{q(-1+qz)} - \frac{(-1+q^2)z|101\rangle}{-1+qz} + |110\rangle,\\
&|010\rangle \mapsto -\frac{(-1+q^2)z|011\rangle}{-1+qz} + |101\rangle - \frac{(-1+q^2)|110\rangle}{q(-1+qz)},\\
&|011\rangle \mapsto -\frac{(-1+q^2)z|001\rangle}{q(-1+qz)} - \frac{(-1+q^2)z|010\rangle}{-1+qz} + |100\rangle,\\
&|100\rangle \mapsto |011\rangle - \frac{(-1+q^2)|101\rangle}{q(-1+qz)} - \frac{(-1+q^2)|110\rangle}{-1+qz},\\
&|101\rangle \mapsto -\frac{(-1+q^2)z|001\rangle}{-1+qz} + |010\rangle - \frac{(-1+q^2)|100\rangle}{q(-1+qz)},\\
&|110\rangle \mapsto |001\rangle - \frac{(-1+q^2)|010\rangle}{q(-1+qz)} - \frac{(-1+q^2)|100\rangle}{-1+qz}.
\end{aligned}
$$

These formulas are consistent with the selection rule (15.21).

15.4 Characterization as the Intertwiner of the Onsager Coideal

Let us establish the characterization of the matrix product constructed K matrix $K^{\mathrm{tr}}(z)$ (15.17) as the intertwiner of the *Onsager coideal* of $U_p = U_p(A^{(1)}_{n-1})$ at $p = -q^{-2}$. We retain the definitions concerning $U_p(A^{(1)}_{n-1})$ in Sect. 11.5. In particular we use the irreducible representations $\pi_{\varpi_k,x} : U_p \to \mathrm{End}(\mathbf{V}_k)$ in (11.60) and $\pi'_{\varpi_k,x} : U_p \to \mathrm{End}(\mathbf{V}_k)$ in (11.78)–(11.80). We further assume

$$p = -q^{-2}, \qquad n \geq 3 \tag{15.24}$$

in the rest of the chapter and allow the coexistence of the letters p and q.

15.4.1 Generalized p-Onsager Algebra $O_p(A^{(1)}_{n-1})$

The algebra $O_p(A^{(1)}_{n-1})$ is generated by $\mathfrak{b}_1, \ldots, \mathfrak{b}_n$ with the relations

$$
\begin{aligned}
&\mathfrak{b}_i\mathfrak{b}_j - \mathfrak{b}_j\mathfrak{b}_i = 0 \quad (a_{ij} = 0),\\
&\mathfrak{b}_i^2\mathfrak{b}_j - (p + p^{-1})\mathfrak{b}_i\mathfrak{b}_j\mathfrak{b}_i + \mathfrak{b}_j\mathfrak{b}_i^2 = \mathfrak{b}_j \quad (a_{ij} = -1),
\end{aligned} \tag{15.25}
$$

where $(a_{ij})_{i,j\in\mathbb{Z}_n}$ is the Cartan matrix given after (11.57). There is an embedding $O_p(A^{(1)}_{n-1}) \hookrightarrow U_p(A^{(1)}_{n-1})$ given by

$$\mathfrak{b}_i \mapsto g_i := e_i + pk_i f_i + \frac{1}{q+q^{-1}}k_i \quad (i \in \mathbb{Z}_n). \tag{15.26}$$

In this context, (15.25) can be viewed as a modified p-Serre relation. Define

$$\mathcal{B}^{\mathrm{tr}} = \text{ the subalgebra of } U_p(A^{(1)}_{n-1}) \text{ generated by } g_1, \ldots, g_n. \tag{15.27}$$

Observe in general that the elements of the form $g'_i = e_i + c_i k_i f_i + d_i k_i \in U_p(A^{(1)}_{n-1})$ with arbitrary coefficients c_i, d_i behave under the coproduct Δ (specified in (11.58)–(11.59)) as

$$\begin{aligned}\Delta g'_i &= e_i \otimes 1 + k_i \otimes e_i + c_i(k_i \otimes k_i)(1 \otimes f_i + f_i \otimes k_i^{-1}) + d_i k_i \otimes k_i \\ &= (e_i + c_i k_i f_i) \otimes 1 + k_i \otimes g'_i \in U_p \otimes 1 + U_p \otimes g'_i.\end{aligned} \tag{15.28}$$

It follows that $\Delta\mathcal{B}^{\mathrm{tr}} \subset U_p \otimes \mathcal{B}^{\mathrm{tr}}$, which implies that the subalgebra $\mathcal{B}^{\mathrm{tr}}$ is a left coideal. The specific choice of the coefficients in (15.26) makes $g_1, \ldots, g_n$ further close among themselves as in (15.25). Having these features in mind we call the coideal subalgebra $\mathcal{B}^{\mathrm{tr}}$ an *Onsager coideal* of type $A^{(1)}_{n-1}$.

Remark 15.2 Let $O_p(A_{n-1})$ be the subalgebra of $O_p(A^{(1)}_{n-1})$ generated by $\mathfrak{b}_1, \ldots, \mathfrak{b}_{n-1}$ without $\mathfrak{b}_n$. Let $T_{q,n}$ denote the Temperley–Lieb algebra [142] generated by $t_1, \ldots, t_{n-1}$ obeying the relations

$$\begin{aligned}&t_i t_j - t_j t_i = 0 \quad (|i-j| \geq 2),\\ &t_i^2 = (q+q^{-1})t_i,\\ &t_i t_j t_i = t_i \quad (|i-j| = 1).\end{aligned} \tag{15.29}$$

Under the relation $p = -q^{\pm 2}$, the map

$$\mathfrak{b}_i \mapsto t_i - \frac{1}{q+q^{-1}} \tag{15.30}$$

yields an isomorphism $O_p(A_{n-1}) \to T_{q,n}$.

15.4.2 $K^{\mathrm{tr}}(z)$ as the Intertwiner of Onsager Coideal

Recall the two representations of $U_p(A^{(1)}_{n-1})$ given as $\pi_{\varpi_k,x}$ in (11.60) and $\pi'_{\varpi_k,x}$ in (11.78), which are simply related by (11.79). We quote $\pi_{\varpi_k,x}$ for convenience.

$$e_i v_{\mathbf{m}} = x^{\delta_{i0}} v_{\mathbf{m}+\mathbf{e}_i-\mathbf{e}_{i+1}}, \quad f_i v_{\mathbf{m}} = x^{-\delta_{i0}} v_{\mathbf{m}-\mathbf{e}_i+\mathbf{e}_{i+1}}, \quad k_i v_{\mathbf{m}} = p^{m_i - m_{i+1}} v_{\mathbf{m}}, \tag{15.31}$$

where the indices belong to $\mathbb{Z}_n$. We exhibit the spectral parameter dependence as

$$\pi_{\varpi_k,x},\ \pi'_{\varpi_k,x}:\ U_p(A^{(1)}_{n-1}) \to \mathrm{End}(\mathbf{V}_{k,x}), \tag{15.32}$$

where $\mathbf{V}_k \simeq \mathbf{V}_{k,x}$ as vector spaces, and x just indicates that the actions of e_i and f_i involve the factors $x^{\delta_{i0}}$ and $x^{-\delta_{i0}}$, respectively. Setting $\mathbf{V}_x = \bigoplus_{0\le k\le n} \mathbf{V}_{k,x}$,[1] we interpret the K matrix in (15.17) and (15.21) as the linear map

$$K^{\mathrm{tr}}(z) : \mathbf{V}_{z^{-1}} \to \mathbf{V}_z. \tag{15.33}$$

Theorem 15.3 *The K matrix $K^{\mathrm{tr}}(z)$ is characterized, up to normalization, as the intertwiner of the Onsager coideal $\mathcal{B}^{\mathrm{tr}} \subset U_p(A^{(1)}_{n-1})$ at $p = -q^{-2}$ (15.24) as*

$$K^{\mathrm{tr}}(z)\pi_{\varpi_k,z^{-1}}(g) = \pi_{\varpi_k,z}(g)K^{\mathrm{tr}}(z) \qquad (\forall g \in \mathcal{B}^{\mathrm{tr}}). \tag{15.34}$$

Proof First we prove (15.34) for $g = g_r$ in (15.26) with $r \in \mathbb{Z}_n$. We write $\gamma = (q+q^{-1})^{-1}$ and $\hat{r} = \mathbf{e}_r - \mathbf{e}_{r+1}$ ($r \in \mathbb{Z}_n$) for short. The action of the two sides on $v_{\mathbf{a}}$ reads as

$$\begin{aligned} K^{\mathrm{tr}}(z)\pi_{\varpi_k,z^{-1}}(g_r)v_{\mathbf{a}} &= K^{\mathrm{tr}}(z)(z^{-\delta_{r0}} v_{\mathbf{a}+\hat{r}} + z^{\delta_{r0}} p^{a_r - a_{r+1}-1} v_{\mathbf{a}-\hat{r}} + \gamma p^{a_r-a_{r+1}} v_{\mathbf{a}}) \\ &= \sum_{\mathbf{b}} (z^{-\delta_{r0}} K^{\mathrm{tr}}(z)^{\mathbf{b}}_{\mathbf{a}+\hat{r}} + z^{\delta_{r0}} p^{a_r-a_{r+1}-1} K^{\mathrm{tr}}(z)^{\mathbf{b}}_{\mathbf{a}-\hat{r}} + \gamma p^{a_r-a_{r+1}} K^{\mathrm{tr}}(z)^{\mathbf{b}}_{\mathbf{a}}) v_{\mathbf{b}}, \\ \pi_{\varpi_k,z}(g_r)K^{\mathrm{tr}}(z)v_{\mathbf{a}} &= \sum_{\mathbf{c}} K^{\mathrm{tr}}(z)^{\mathbf{c}}_{\mathbf{a}} \pi_{\varpi_k,z}(g_r)v_{\mathbf{c}} \\ &= \sum_{\mathbf{c}} K^{\mathrm{tr}}(z)^{\mathbf{c}}_{\mathbf{a}} (z^{\delta_{r0}} v_{\mathbf{c}+\hat{r}} + z^{-\delta_{r0}} p^{c_r-c_{r+1}-1} v_{\mathbf{c}-\hat{r}} + \gamma p^{c_r-c_{r+1}} v_{\mathbf{c}}). \end{aligned}$$

Comparing the coefficients of $v_{\mathbf{b}}$, we are to show

$$\begin{aligned} &z^{-\delta_{r0}} K^{\mathrm{tr}}(z)^{\mathbf{b}}_{\mathbf{a}+\hat{r}} + z^{\delta_{r0}} p^{a_r-a_{r+1}-1} K^{\mathrm{tr}}(z)^{\mathbf{b}}_{\mathbf{a}-\hat{r}} + \gamma p^{a_r-a_{r+1}} K^{\mathrm{tr}}(z)^{\mathbf{b}}_{\mathbf{a}} \\ &= z^{\delta_{r0}} K^{\mathrm{tr}}(z)^{\mathbf{b}-\hat{r}}_{\mathbf{a}} + z^{-\delta_{r0}} p^{b_r-b_{r+1}+1} K^{\mathrm{tr}}(z)^{\mathbf{b}+\hat{r}}_{\mathbf{a}} + \gamma p^{b_r-b_{r+1}} K^{\mathrm{tr}}(z)^{\mathbf{b}}_{\mathbf{a}}. \end{aligned} \tag{15.35}$$

In view of the matrix product formula (15.17), this follows from a relation involving two adjacent operators G^b_a in (15.4). Explicitly, it reads as

$$\begin{aligned} &G^b_{a+1}G^{b'}_{a'-1} + p^{a-a'-1}G^b_{a-1}G^{b'}_{a'+1} + \gamma p^{a-a'} G^b_a G^{b'}_{a'} \\ &= G^{b-1}_a G^{b'+1}_{a'} + p^{b-b'+1} G^{b+1}_a G^{b'-1}_{a'} + \gamma p^{b-b'} G^b_a G^{b'}_{a'}, \end{aligned} \tag{15.36}$$

[1] The temporary notation $\mathbf{V}_z$ with spectral parameters $z = x^{\pm 1}, y^{\pm 1}, \ldots$ is to be distinguished from $\mathbf{V}_k$ with $k \in [0, n]$ in (11.6) by the letters used as the index.

where we have set $(a, a', b, b') = (a_r, a_{r+1}, b_r, b_{r+1})$. The factor $z^{\pm\delta_{r0}}$ has been removed thanks to (15.6). The relation (15.36) can be checked directly. For instance, when $(a, a', b, b') = (0, 0, 0, 1)$, it reads as $\gamma \mathbf{a}^+ \beta \mathbf{k} = \beta \mathbf{k}\, \mathbf{a}^+ + \gamma p^{-1} \mathbf{a}^+ \beta \mathbf{k}$, which is valid due to (15.24) and (3.12). As seen in this example, β in (15.4) always becomes an overall factor for (15.36) reflecting (15.18) and the invariance of $\pi_{\varpi_k, z^{\pm 1}}$ under the conjugation by $\beta^{\mathbf{h}_1 + \cdots + \mathbf{h}_n}$.

We have proved that $K^{\mathrm{tr}}(z)$ satisfies the intertwining relation (15.34). The remaining task is to show that the solution is unique up to normalization. As for this we refer to [104, Sect. 5.2], where the claim has been proved in a more general setting. □

15.4.3 Reflection Equation From Onsager Coideal

In Sect. 15.3 we have already proved the reflection equation (15.16) by using the quantized reflection equation. Here we present an alternative and more conventional proof based on the Onsager coideal $\mathcal{B}^{\mathrm{tr}}$.

We temporarily introduce

$$\mathrm{K}(z) = \iota \circ K^{\mathrm{tr}}(z) : \ \mathbf{V}_{z^{-1}} \to \mathbf{V}_z, \tag{15.37}$$

$$\mathrm{R}(z) = S^{\mathrm{tr}}(z^{-1}), \tag{15.38}$$

$$\mathrm{R}'(z) = (1 \otimes \iota) S^{\mathrm{tr}}(z^{-1})(1 \otimes \iota), \tag{15.39}$$

$$\mathrm{R}''(z) = (\iota \otimes \iota) S^{\mathrm{tr}}(z^{-1})(\iota \otimes \iota), \tag{15.40}$$

where $\iota(v_{\mathbf{m}}) = v_{\mathbf{m}'} = v_{(1-m_1, \ldots, 1-m_n)}$ as defined in (11.80) and (6.4). From (15.21), $\mathrm{K}(z)$ is a direct sum of linear operators $\mathbf{V}_{k, z^{-1}} \to \mathbf{V}_{k,z}$ over $0 \le k \le n$. Since $\pi'_{\varpi_k, x}(g) = \iota \circ \pi_{\varpi_{n-k}, x}(g) \circ \iota$ by (11.79), the intertwining relation (15.34) takes the form

$$\mathrm{K}(z) \pi_{\varpi_k, z^{-1}}(g) = \pi'_{\varpi_{n-k}, z}(g) \mathrm{K}(z) \qquad (\forall g \in \mathcal{B}^{\mathrm{tr}}). \tag{15.41}$$

In (15.38), the reason for $S^{\mathrm{tr}}(z^{-1})$ instead of $S^{\mathrm{tr}}(z)$ is to take the inversion of the spectral parameter in (11.62) into account. Matrix elements, in the same convention as (15.17) and (11.25), are given by

$$\mathrm{K}(z)^{\mathbf{b}}_{\mathbf{a}} = K^{\mathrm{tr}}(z)^{\mathbf{b}'}_{\mathbf{a}}, \quad \mathrm{R}'(z)^{\mathbf{ab}}_{\mathbf{ij}} = S^{\mathrm{tr}}(z^{-1})^{\mathbf{ab}'}_{\mathbf{ij}'}, \quad \mathrm{R}''(z)^{\mathbf{ab}}_{\mathbf{ij}} = S^{\mathrm{tr}}(z^{-1})^{\mathbf{a}'\mathbf{b}'}_{\mathbf{i}'\mathbf{j}'}, \tag{15.42}$$

where $\mathbf{a}' = (1 - a_1, \ldots, 1 - a_n)$ as defined in (6.4).

Following (11.61) we also prepare the tensor product representations[2]

$$\Delta_{x,y} = (\pi_{\varpi_l,x} \otimes \pi_{\varpi_m,y}) \circ \Delta, \tag{15.43}$$

$$\Delta'_{x,y} = (\pi_{\varpi_l,x} \otimes \pi'_{\varpi_m,y}) \circ \Delta = (1 \otimes \iota)\Delta_{x,y}(1 \otimes \iota), \tag{15.44}$$

$${}'\Delta_{x,y} = (\pi'_{\varpi_l,x} \otimes \pi_{\varpi_m,y}) \circ \Delta = (\iota \otimes 1)\Delta_{x,y}(\iota \otimes 1), \tag{15.45}$$

$$\Delta''_{x,y} = (\pi'_{\varpi_l,x} \otimes \pi'_{\varpi_m,y}) \circ \Delta = (\iota \otimes \iota)\Delta_{x,y}(\iota \otimes \iota), \tag{15.46}$$

where the coproduct Δ is specified in (11.58). Let $P(u \otimes v) = v \otimes u$ be the transposition as usual. Then from (11.61), (11.62) and $\Delta^{\mathrm{op}} = P \circ \Delta \circ P$, we know that the commutativity

$$P\mathrm{R}(z)\Delta_{x,y}(g) = \Delta_{y,x}(g)P\mathrm{R}(z), \tag{15.47}$$

$$P\mathrm{R}'(z)\Delta'_{x,y}(g) = {}'\Delta_{y,x}(g)P\mathrm{R}'(z), \tag{15.48}$$

$$P\mathrm{R}''(z)\Delta''_{x,y}(g) = \Delta''_{y,x}(g)P\mathrm{R}''(z) \tag{15.49}$$

hold for $g \in \mathcal{B}^{\mathrm{tr}} \subset U_p(A^{(1)}_{n-1})$ provided $z = x/y$.

Consider the two maps going from the representation space $\mathbf{V}_{x^{-1}} \otimes \mathbf{V}_{y^{-1}}$ of $\Delta_{x^{-1},y^{-1}}$ to $\mathbf{V}_x \otimes \mathbf{V}_y$ of $\Delta''_{x,y}$ constructed as follows:

$$\begin{array}{ccc} & \mathbf{V}_{x^{-1}} \otimes \mathbf{V}_{y^{-1}} & \\ \swarrow {\scriptstyle \mathrm{K}_2(y)} & & {\scriptstyle P\mathrm{R}(y/x)} \searrow \\ \mathbf{V}_{x^{-1}} \otimes \mathbf{V}_y & & \mathbf{V}_{y^{-1}} \otimes \mathbf{V}_{x^{-1}} \\ \downarrow {\scriptstyle P\mathrm{R}'((xy)^{-1})} & & \downarrow {\scriptstyle \mathrm{K}_2(x)} \\ \mathbf{V}_y \otimes \mathbf{V}_{x^{-1}} & & \mathbf{V}_{y^{-1}} \otimes \mathbf{V}_x \\ \downarrow {\scriptstyle \mathrm{K}_2(x)} & & \downarrow {\scriptstyle P\mathrm{R}'((xy)^{-1})} \\ \mathbf{V}_y \otimes \mathbf{V}_x & & \mathbf{V}_x \otimes \mathbf{V}_{y^{-1}} \\ {\scriptstyle P\mathrm{R}''(y/x)} \searrow & & \swarrow {\scriptstyle \mathrm{K}_2(y)} \\ & \mathbf{V}_x \otimes \mathbf{V}_y & \end{array} \tag{15.50}$$

Here $\mathrm{K}_2(z) = 1 \otimes \mathrm{K}(z)$. Note $\Delta(\mathcal{B}^{\mathrm{tr}}) \subset U_p \otimes \mathcal{B}^{\mathrm{tr}}$ and that the K matrices act only on the right component. Therefore for any $g \in \mathcal{B}^{\mathrm{tr}}$, the compositions X and Y corresponding to the LHS and the RHS of (15.50) possess the same intertwining property

[2] The indices l and m do not play a role in the following argument and can be chosen arbitrarily.

$$X\Delta_{x^{-1},y^{-1}}(g) = \Delta''_{x,y}(g)X, \quad X = P\mathrm{R}''(y/x)\mathrm{K}_2(x)P\mathrm{R}'((xy)^{-1})\mathrm{K}_2(y), \tag{15.51}$$

$$Y\Delta_{x^{-1},y^{-1}}(g) = \Delta''_{x,y}(g)Y, \quad Y = \mathrm{K}_2(y)P\mathrm{R}'((xy)^{-1})\mathrm{K}_2(x)P\mathrm{R}(y/x) \tag{15.52}$$

thanks to (15.41) and (15.47)–(15.49).[3] Thus $X = (\mathrm{const})Y$ must hold provided that $\Delta_{x^{-1},y^{-1}}$ is irreducible as a representation of $\mathcal{B}^{\mathrm{tr}}$. The constant here is 1 for any l, m in view of (11.34) and (15.23). The irreducibility has been proved in [104, Sect. 5.2]. Thus we obtain $X = Y$. Then the equality $(\iota \otimes \iota)PXP = (\iota \otimes \iota)PYP$ yields

$$\begin{aligned}
&(\iota \otimes \iota)\mathrm{R}''(y/x)\mathrm{K}_2(x)P\mathrm{R}'((xy)^{-1})\mathrm{K}_2(y)P \\
&= (\iota \otimes \iota)P\mathrm{K}_2(y)P\mathrm{R}'((xy)^{-1})\mathrm{K}_2(x)P\mathrm{R}(y/x)P.
\end{aligned} \tag{15.53}$$

Upon substitution of (15.37)–(15.40), this becomes

$$\begin{aligned}
&S^{\mathrm{tr}}_{\mathbf{1},\mathbf{2}}(xy^{-1})\underline{(\iota \otimes \iota)(1 \otimes \iota)K^{\mathrm{tr}}_{\mathbf{2}}(x)(\iota \otimes 1)}PS^{\mathrm{tr}}_{\mathbf{1},\mathbf{2}}(xy)K^{\mathrm{tr}}_{\mathbf{2}}(y)P \\
&= \underline{(\iota \otimes \iota)P(1 \otimes \iota)K^{\mathrm{tr}}_{\mathbf{2}}(y)P(1 \otimes \iota)}S^{\mathrm{tr}}_{\mathbf{1},\mathbf{2}}(xy)K^{\mathrm{tr}}_{\mathbf{2}}(x)PS^{\mathrm{tr}}_{\mathbf{1},\mathbf{2}}(xy^{-1})P.
\end{aligned} \tag{15.54}$$

The underlined operators on the LHS and the RHS are equal to $K^{\mathrm{tr}}_{\mathbf{2}}(x)$ and $K^{\mathrm{tr}}_{\mathbf{1}}(y)$, respectively. Thus the reflection equation (15.16) is reproduced.

15.5 Further Properties of $K^{\mathrm{tr}}(z)$

15.5.1 Commutativity

Proposition 15.4 *The K matrix $K^{\mathrm{tr}}(z)$ forms a commuting family:*

$$[K^{\mathrm{tr}}(z), K^{\mathrm{tr}}(w)] = 0. \tag{15.55}$$

Proof In view of (15.18) we set $\beta = 1$ without loss of generality. For simplicity we denote $K^{\mathrm{tr}}(z)|_{\beta=1}$ of (15.17) without the prefactor $\kappa^{\mathrm{tr}}(z)$ just by $K(z)$. In order to describe the elements of $K(z)K(w)$, we prepare two copies of q-oscillators and their product:

$$G_i = \begin{pmatrix} \mathbf{a}^+_i & -q\mathbf{k}_i \\ \mathbf{k}_i & \mathbf{a}^-_i \end{pmatrix}, \quad \begin{pmatrix} M^0_0 & M^0_1 \\ M^1_0 & M^1_1 \end{pmatrix} = G_1 \cdot G_2 = \begin{pmatrix} \mathbf{a}^+_1\mathbf{a}^+_2 - q\mathbf{k}_1\mathbf{k}_2 & -q(\mathbf{a}^+_1\mathbf{k}_2 + \mathbf{k}_1\mathbf{a}^-_2) \\ \mathbf{k}_1\mathbf{a}^+_2 + \mathbf{a}^-_1\mathbf{k}_2 & \mathbf{a}^-_1\mathbf{a}^-_2 - q\mathbf{k}_1\mathbf{k}_2 \end{pmatrix}, \tag{15.56}$$

where $i = 1, 2$ and $\cdot$ signifies the usual product as 2-by-2 matrices. We will also use the copies $\mathbf{h}_1, \mathbf{h}_2$ of the number operator $\mathbf{h}$ (3.14). Operators with different indices are commutative as they act on different q-oscillator Fock spaces.

[3] The relation $p = -q^{-1}$ in Theorem 11.3 fits $p = -q^{-2}$ in Theorem 15.3 because of (15.5).

The matrix element of $K(z)K(w)$ is expressed as

$$(K(z)K(w))^{b_1,\ldots,b_n}_{a_1,\ldots,a_n} = \mathrm{Tr}_{12}\left(z^{\mathbf{h}_1} w^{\mathbf{h}_2} M^{b_1}_{a_1} \cdots M^{b_n}_{a_n}\right), \tag{15.57}$$

where the trace extends over the two q-oscillator Fock spaces 1 and 2.

Let r be the exchange operator of the two q-oscillators:

$$r^2 = 1, \quad r\,\mathbf{a}^{\pm}_i = \mathbf{a}^{\pm}_{3-i} r, \quad r\,\mathbf{k}_i = \mathbf{k}_{3-i} r, \quad r\mathbf{h}_i = \mathbf{h}_{3-i} r \qquad (i = 1, 2). \tag{15.58}$$

One can easily check the following relations for any $a, b = 0, 1$:

$$r M^b_a = (-q)^{a-b} M^a_b r, \tag{15.59}$$

$$M^0_1 M^1_0 = M^1_0 M^0_1, \quad M^a_{1-a} M^0_0 = q M^0_0 M^a_{1-a}, \quad M^a_{1-a} M^1_1 = q^{-1} M^1_1 M^a_{1-a}. \tag{15.60}$$

Note that the relation (15.60) is also satisfied by each G_i (15.56) individually. The product $G_1 \cdot G_2$ preserves the relation because it coincides with the coproduct (3.6).

Insert $1 = r^2$ anywhere in the trace (15.57) and let one of the r's encircle the whole array once using (15.58) and (15.59). The result gives

$$\mathrm{Tr}_{12}\left(z^{\mathbf{h}_1} w^{\mathbf{h}_2} M^{b_1}_{a_1} \cdots M^{b_n}_{a_n}\right) = \mathrm{Tr}_{12}\left(w^{\mathbf{h}_1} z^{\mathbf{h}_2} M^{a_1}_{b_1} \cdots M^{a_n}_{b_n}\right)(-q)^{|\mathbf{a}|-|\mathbf{b}|}, \tag{15.61}$$

where the symbol $|\mathbf{a}|$ is defined in (11.4). From (15.20) we know $(K(z)K(w))^{b_1,\ldots,b_n}_{\alpha_1,\ldots,\alpha_n} = 0$ unless $|\mathbf{a}| = |\mathbf{b}|$. Thus the factor $(-q)^{|\mathbf{a}|-|\mathbf{b}|}$ in the above can be removed, leading to

$$(K(z)K(w))^{b_1,\ldots,b_n}_{a_1,\ldots,a_n} = (K(w)K(z))^{a_1,\ldots,a_n}_{b_1,\ldots,b_n}. \tag{15.62}$$

Next consider the expression (15.57) again. Under the assumption $|\mathbf{a}| = |\mathbf{b}|$, the number of M^1_0 and M^0_1 in the trace is equal, which we denote by m. Then by means of (15.60) one can send M^1_0 and M^0_1 to the left to rewrite (15.57) uniquely in the form

$$(K(z)K(w))^{b_1,\ldots,b_n}_{a_1,\ldots,a_n} = q^{\Phi}\,\mathrm{Tr}_{12}\left(z^{\mathbf{h}_1} w^{\mathbf{h}_2} (M^1_0 M^0_1)^m N_1 \cdots N_{n-2m}\right), \tag{15.63}$$

where $N_i = M^0_0$ or M^1_1 are in the original order and Φ is some integer. Starting from $(K(z)K(w))^{a_1,\ldots,a_n}_{b_1,\ldots,b_n}$, the same rewriting procedure leads to the identical expression thanks to (15.60). Thus we find

$$(K(z)K(w))^{b_1,\ldots,b_n}_{a_1,\ldots,a_n} = (K(z)K(w))^{a_1,\ldots,a_n}_{b_1,\ldots,b_n}. \tag{15.64}$$

Combining (15.64) with (15.62) we conclude

$$(K(w)K(z))^{a_1,\ldots,a_n}_{b_1,\ldots,b_n} = (K(z)K(w))^{a_1,\ldots,a_n}_{b_1,\ldots,b_n}, \tag{15.65}$$

which completes the proof. □

The K matrices $K^{k,k'}(z)$ $(k, k' = 1, 2)$ which will be constructed in the next chapter do not satisfy the commutativity (15.55).

15.5.2 $K^{\mathrm{tr}}(z)$ as a Symmetry of XXZ-Type Spin Chain

We have considered a representation of the generalized p-Onsager algebra obtained by the composition

$$\begin{array}{ccccc} O_p(A^{(1)}_{n-1}) & \hookrightarrow & U_p(A^{(1)}_{n-1}) & \to & \mathrm{End}(\mathbf{V}_x) \\ \mathfrak{b}_i & \longmapsto & g_i & \longmapsto & h_i(x) \quad (i \in \mathbb{Z}_n), \end{array} \tag{15.66}$$

where the left arrow is given by (15.26) and the right one is by the direct sum representation $\pi_x = \bigoplus_{0\le k\le n} \pi_{\varpi_k, x}$ of (15.32) and (15.31).

In order to get familiarized with the operator $h_i(x) = \pi_x(g_i)$, we write a base vector $v_{\mathbf{m}} \in \mathbf{V}_x$ labeled with an array $\mathbf{m} = (m_1, \dots, m_n) \in \mathfrak{s}$ (11.1) as $|m_1, \dots, m_n\rangle$,[4] and interpret it as a state of a spin $\frac{1}{2}$ chain on length n periodic lattice. Let $\sigma^x_i, \sigma^y_i, \sigma^z_i$ and $\sigma^{\pm}_i = \frac{1}{2}(\sigma^x_i \pm \mathrm{i}\sigma^y_i)$ $(1 \le i \le n)$ denote the Pauli matrices acting on the ith component of $\mathbf{V} = V^{\otimes n}$ regarding $m_i = 1$ as an up-spin and $m_i = 0$ as a down-spin. Namely,

$$\begin{aligned} \sigma^x_i|\dots,1,\dots\rangle &= |\dots,0,\dots\rangle, & \sigma^x_i|\dots,0,\dots\rangle &= |\dots,1,\dots\rangle, \\ \sigma^y_i|\dots,1,\dots\rangle &= \mathrm{i}|\dots,0,\dots\rangle, & \sigma^y_i|\dots,0,\dots\rangle &= -\mathrm{i}|\dots,1,\dots\rangle, \\ \sigma^z_i|\dots,1,\dots\rangle &= |\dots,1,\dots\rangle, & \sigma^z_i|\dots,0,\dots\rangle &= -|\dots,0,\dots\rangle, \\ \sigma^+_i|\dots,1,\dots\rangle &= 0, & \sigma^+_i|\dots,0,\dots\rangle &= |\dots,1,\dots\rangle, \\ \sigma^-_i|\dots,1,\dots\rangle &= |\dots,0,\dots\rangle, & \sigma^-_i|\dots,0,\dots\rangle &= 0. \end{aligned} \tag{15.67}$$

Then the representation (15.31) is expressed as spin chain operators as

$$e_i = x^{\delta_{i0}}\sigma^+_i\sigma^-_{i+1}, \quad p^2 k_i f_i = x^{-\delta_{i0}}\sigma^-_i\sigma^+_{i+1}, \tag{15.68}$$

$$\frac{1}{q+q^{-1}}k_i = \frac{q+q^{-1}}{4}\sigma^z_i\sigma^z_{i+1} + \frac{q-q^{-1}}{4}(\sigma^z_i - \sigma^z_{i+1}) - \frac{(q-q^{-1})^2}{4(q+q^{-1})}, \tag{15.69}$$

where $p = -q^{-2}$ (15.24) has been taken into account in (15.69). Thus from (15.26), the image $h_i(x)$ in (15.66) takes the form

[4] This is a temporary notation only for (15.67). Elsewhere it is reserved for the base of $\mathbf{W}$ in (11.12)–(11.13).

$$h_i(x) = \frac{x^{\delta_{i0}} + x^{-\delta_{i0}}}{4}(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) - \frac{x^{\delta_{i0}} - x^{-\delta_{i0}}}{4\mathrm{i}}(\sigma_i^x \sigma_{i+1}^y - \sigma_i^y \sigma_{i+1}^x)$$
$$+ \frac{q+q^{-1}}{4}\sigma_i^z \sigma_{i+1}^z + \frac{q-q^{-1}}{4}(\sigma_i^z - \sigma_{i+1}^z) - \frac{(q-q^{-1})^2}{4(q+q^{-1})}. \tag{15.70}$$

This is a typical local Hamiltonian of a spin $\frac{1}{2}$ chain with magnetic field (second term in the second line) and with a Dzyaloshinskii–Moriya interaction term (second term in the first line). The constructions so far shows that they yield a representation of the p-Onsager algebra $O_p(A^{(1)}_{n-1})$.

Now the intertwining relation (15.34) for $g = g_i$ reads as

$$K^{\mathrm{tr}}(z) h_i(z^{-1}) = h_i(z) K^{\mathrm{tr}}(z), \tag{15.71}$$

or equivalently in terms of (15.37) as

$$\mathrm{K}(z) h_i(z^{-1}) = (\iota \circ h_i(z) \circ \iota)\mathrm{K}(z), \tag{15.72}$$

where ι defined in (11.80) is a spin reversal operator in the spin chain context.

One can consider various "Hamiltonians"[5] commuting with $K^{\mathrm{tr}}(z)$ or $\mathrm{K}(z)$ thanks to (15.71) and (15.72). For instance, $h_i = h_i(x)$ is x-independent for $i \neq 0$, therefore $[\sum_{1 \le i \le n-1} c_i h_i, K^{\mathrm{tr}}(z)] = 0$ for any coefficients $c_1, \dots, c_{n-1}$. Another example is $H(z) = \sum_{i \in \mathbb{Z}_n} h_i(z^{-1})$ which eliminates the magnetic field term, leading to

$$H(z) = \sum_{i \in \mathbb{Z}_n} \Big(z^{-\delta_{i0}} \sigma_i^+ \sigma_{i+1}^- + z^{\delta_{i0}} \sigma_i^- \sigma_{i+1}^+ + \frac{q+q^{-1}}{4} \sigma_i^z \sigma_{i+1}^z \Big) - \frac{n(q-q^{-1})^2}{4(q+q^{-1})}. \tag{15.73}$$

It satisfies $H(z^{-1}) = \iota \circ H(z) \circ \iota$, therefore (15.72) leads to

$$[\mathrm{K}(z), H(z)] = 0. \tag{15.74}$$

15.6 Bibliographical Notes and Comments

The reflection equation in 2D or $(1+1)$D has been recognized as a key ingredient in quantum integrable systems in the presence of boundaries since the pioneering works [30, 53, 83, 137]. A variety of solutions and aspects have been explored, for example in [8, 36, 103, 116, 117, 120, 125]. Below we focus on trigonometric solutions.

The 3D approach to the reflection equation was launched in [105], where $K^{\mathrm{tr}}(z)$ in (15.17) (and $K^{k,k'}(z)$ in the next chapter) were constructed from the quantized reflection equation (15.8). Most K matrices known by then were associated with the quantum affine algebra with lowest rank (typically $U_p(A^{(1)}_1)$) with higher spins,

[5] With a suitable choice of q and z so that Hermiticity is fulfilled.

or general rank non-exceptional type but with lowest "spin" (typically the vector representation). The majority of the latter are sparse (or even diagonal) in the sense that $K(z)^{\mathbf{b}}_{\mathbf{a}} = 0$ except for a relatively few choices of **a**, **b**. In contrast to them, $K^{\mathrm{tr}}(z)$ in this chapter is associated with higher rank $U_p(A^{(1)}_{n-1})$ with general degree anti-symmetric tensor representations. Moreover, it is *dense* in that *all* the elements are non-trivial trigonometric functions within each sector (15.21). The most distinctive feature is the matrix product structure (15.17) reflecting the 3D integrability behind the scene. A similar solution associated with general degree *symmetric tensor representations* of $U_p(A^{(1)}_{n-1})$ has been constructed in [103], where the matrix product operators analogous to (15.4) are truncated q-hypergeometric series of q-oscillators. The commutativity (Proposition 15.4) is due to [106, Appendix A]. Construction of the associated double-row-type commuting transfer matrices in the spirit of [137] and especially their spectral problem involving those "dense" K have been left as a future problem.

The idea to characterize the spectral parameter dependent K matrices in terms of coideal subalgebras of quantum affine algebras (Sect. 15.4) was proposed in the context of affine Toda field theory with boundaries. See for example [37, 38], more recently [79, 125] and references therein. It essentially achieves *linearization* of the reflection equation, eliminating the task of proving the original quartic relation $RKRK = KRKR$ "manually". It is a boundary analogue of the classic idea [43, 63] that the cubic Yang–Baxter equation $RRR = RRR$ is attributed to the linear equation $[\check{R}, \Delta(U_p)] = 0$ representing the U_p symmetry. See [64, p. 540]. One may interpret that the full U_p symmetry in the bulk is lost at the boundary but still survives partially as some subalgebra symmetry which should be a coideal to fit the quantum group machinery.

The coideal must be small enough, otherwise the intertwining relation like (15.34) may not allow a solution. Nonetheless it must be also large enough, otherwise the relevant space like $\mathbf{V}_x \otimes \mathbf{V}_y$ in (15.50) may not become irreducible as a module over the coideal. In this way one is led to a fundamental question: what is the right "size" or choice of the coideal in order to make the linearization work legitimately for a given representation? It is still an outstanding problem in general.

Section 15.4 demonstrates that the Onsager coideal is one such example. The generalized p-Onsager algebra associated with general affine Lie algebra has been formulated in [7]. The relation (15.25) with $p = 1$ goes back to [144, Eqs. (11) and (12)]. The early history of the Onsager algebra starting from [122] can be found in [143, Remark 9.1]. A more recent account is available in [79, Sect. 1(1)]. Remark 15.2 on the relation with the Temperley–Lieb algebra [142] is taken from [106, Remark 3.1]. The scheme (15.66) to realize the Onsager algebras in terms of quantum spin chain Hamiltonians like (15.70) has been explored also for other non-exceptional quantum affine algebras [106]. The characteristic aspect is, it accommodates a length n spin $\frac{1}{2}$ chains in a *single* irreducible representation of $U_p(A^{(1)}_{n-1})$ instead of (spin $\frac{1}{2}$ representation)$^{\otimes n}$ of $U_p(A^{(1)}_1)$. This is another manifestation of the rank-size duality (Sects. 11.6, 13.8 and Remark 18.7) in the sense that the Dynkin diagram of the "internal symmetry" $U_p(A^{(1)}_{n-1})$ pops out as the periodic lattice ("external space") on which the spin system is defined.

The Onsager algebra $O_p(A^{(1)}_{n-1})$ has a natural classical part without the generator $\mathfrak{b}_0$. The commutativity in Theorem 15.3 implies the usual commutativity with the classical part. The corresponding spectral decomposition of $K^{\mathrm{tr}}(z)$ has been described in [106, Sect. 4]. A more detailed account will be given of the p-Onsager algebras for other types in Sect. 16.3. This chapter is parallel with the next where most of the results are extended to the $B^{(1)}_n$, $D^{(1)}_n$, $D^{(2)}_{n+1}$ cases by boundary vector reductions.

Chapter 16
Boundary Vector Reductions of $(LGLG)K = K(GLGL)$

Abstract This chapter is a continuation of the 3D approach to the reflection equation from the previous one. We start from the n-concatenation of the quantized reflection equation $(LGLG)K = K(GLGL)$ and perform the boundary vector reduction. The L part gives rise to the quantum R matrices for the spin representations of $\mathfrak{g}^{r,r'} = B_n^{(1)}, D_n^{(1)}, D_n^{(2)}, \tilde{B}_n^{(1)}$, which have been detailed in Chap. 12. The G part generates the companion K matrices that satisfy the reflection equation. They are expressed by a matrix product formula in terms of G and characterized as the intertwiners of various Onsager coideals of the quantum affine algebras $U_p(\mathfrak{g}^{r,r'})$. The final list of the solutions is summarized in Table 16.1.

16.1 Preliminaries

We keep the setting in Sect. 15.1 and continue to work with the solution (L, G, K) to the quantized reflection equation $L_{123}G_{24}L_{215}G_{16}K_{3456} = K_{3456}G_{16}L_{125}G_{24}L_{213}$ summarized there. Thus L and G are given by

$$\begin{pmatrix} L_{00}^{00} & L_{01}^{00} & L_{10}^{00} & L_{11}^{00} \\ L_{00}^{01} & L_{01}^{01} & L_{10}^{01} & L_{11}^{01} \\ L_{00}^{10} & L_{01}^{10} & L_{10}^{10} & L_{11}^{10} \\ L_{00}^{11} & L_{01}^{11} & L_{10}^{11} & L_{11}^{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -q^2\alpha^{-1}\mathbf{K} & \mathbf{A}^- & 0 \\ 0 & \mathbf{A}^+ & \alpha\mathbf{K} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} G_0^0 & G_1^0 \\ G_0^1 & G_1^1 \end{pmatrix} = \begin{pmatrix} \mathbf{a}^+ & -q\beta^{-1}\mathbf{k} \\ \beta\mathbf{k} & \mathbf{a}^- \end{pmatrix}. \tag{16.1}$$

The 3D K has been detailed in Chap. 5. Note that

$$(L \text{ in } (16.1)) = (L \text{ in } (11.14))|_{q\to q^2} = (L \text{ in } (12.1))|_{q\to q^2}. \tag{16.2}$$

Our starting point is the n-concatenation of the quantized reflection equation

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$$(L_{1_1 2_1 3} \cdots L_{1_n 2_n 3})(G_{2_1 4} \cdots G_{2_n 4})(L_{2_1 1_1 5} \cdots L_{2_n 1_n 5})(G_{1_1 6} \cdots G_{1_n 6})K_{3456}$$
$$= K_{3456}(G_{1_1 6} \cdots G_{1_n 6})(L_{1_1 2_1 5} \cdots L_{1_n 2_n 5})(G_{2_1 4} \cdots G_{2_n 4})(L_{2_1 1_1 3} \cdots L_{2_n 1_n 3}) \tag{16.3}$$

and the weight conservation of the 3D K

$$K_{3456}(xy^{-1})^{\mathbf{h}_3} x^{\mathbf{h}_4} (xy)^{\mathbf{h}_5} y^{\mathbf{h}_6} = (xy^{-1})^{\mathbf{h}_3} x^{\mathbf{h}_4} (xy)^{\mathbf{h}_5} y^{\mathbf{h}_6} K_{3456}, \tag{16.4}$$

which are quoted from (15.10) and (15.11). The operator $\mathbf{h}_i$ is the number operator $\mathbf{h}$ (3.14) acting on the ith Fock space.

16.2 Boundary Vector Reduction

Recall the boundary vectors in (5.118) and (5.119):

$$\langle \eta_r | = \sum_{m \geq 0} \frac{\langle rm |}{(q^{r^2})_m}, \quad | \eta_r \rangle = \sum_{m \geq 0} \frac{| rm \rangle}{(q^{r^2})_m}, \tag{16.5}$$

$$\langle \chi_r | = \sum_{m \geq 0} \frac{\langle rm |}{(q^{2r^2})_m}, \quad | \chi_r \rangle = \sum_{m \geq 0} \frac{| rm \rangle}{(q^{2r^2})_m}, \tag{16.6}$$

where $r = 1, 2$. The second line is obtained by setting $q \to q^2$ in the first line. The vectors (16.5) (resp. (16.6)) are elements of a completion of $\mathcal{F}_q^*$ and $\mathcal{F}_q$ (resp. $\mathcal{F}_{q^2}^*$ and $\mathcal{F}_{q^2}$).[1] We invoke Proposition 5.21, which states that they yield particular eigenvectors of the 3D K as

$$(\langle \chi_r | \otimes \langle \eta_k | \otimes \langle \chi_r | \otimes \langle \eta_k |)K = \langle \chi_r | \otimes \langle \eta_k | \otimes \langle \chi_r | \otimes \langle \eta_k |,$$
$$K(|\chi_r\rangle \otimes |\eta_k\rangle \otimes |\chi_r\rangle \otimes |\eta_k\rangle) = |\chi_r\rangle \otimes |\eta_k\rangle \otimes |\chi_r\rangle \otimes |\eta_k\rangle, \tag{16.7}$$

where $1 \leq r \leq k \leq 2$.

Multiply $(xy^{-1})^{\mathbf{h}_3} x^{\mathbf{h}_4} (xy)^{\mathbf{h}_5} y^{\mathbf{h}_6}$ from the left by (16.3) and sandwich the result by the boundary vectors as

$$\langle \overset{3}{\chi_r} | \otimes \langle \overset{4}{\eta_k} | \otimes \langle \overset{5}{\chi_r} | \otimes \langle \overset{6}{\eta_k} | (\cdots) | \overset{3}{\chi_{r'}} \rangle \otimes | \overset{4}{\eta_{k'}} \rangle \otimes | \overset{5}{\chi_{r'}} \rangle \otimes | \overset{6}{\eta_{k'}} \rangle.$$

Thanks to the commutativity (16.4) and the eigen-property (16.7), the 3D K disappears and the result becomes

[1] From (3.16), dual pairing of $\mathcal{F}_{q^2}^*$ and $\mathcal{F}_{q^2}$ should be calculated by $\langle m | m' \rangle = (q^4)_m \delta_{m,m'}$.

$$\langle \overset{3}{\chi}_r|(xy^{-1})^{\mathbf{h}_3} L_{1_1 2_1 3} \cdots L_{1_n 2_n 3}|\overset{3}{\chi}_{r'}\rangle \langle \overset{4}{\eta}_k| x^{\mathbf{h}_4} G_{2_1 4} \cdots G_{2_n 4}|\overset{4}{\eta}_{k'}\rangle \times$$
$$\times \langle \overset{5}{\chi}_r|(xy)^{\mathbf{h}_5} L_{2_1 1_1 5} \cdots L_{2_n 1_n 5}|\overset{5}{\chi}_{r'}\rangle \langle \overset{6}{\eta}_k| y^{\mathbf{h}_6} G_{1_1 6} \cdots G_{1_n 6}|\overset{6}{\eta}_{k'}\rangle$$
$$= \langle \overset{6}{\eta}_k| y^{\mathbf{h}_6} G_{1_1 6} \cdots G_{1_n 6}|\overset{6}{\eta}_{k'}\rangle \langle \overset{5}{\chi}_r|(xy)^{\mathbf{h}_5} L_{1_1 2_1 5} \cdots L_{1_n 2_n 5}|\overset{5}{\chi}_{r'}\rangle \times$$
$$\times \langle \overset{4}{\eta}_k| x^{\mathbf{h}_4} G_{2_1 4} \cdots G_{2_n 4}|\overset{4}{\eta}_{k'}\rangle \langle \overset{3}{\chi}_r|(xy^{-1})^{\mathbf{h}_3} L_{2_1 1_1 3} \cdots L_{2_n 1_n 3}|\overset{3}{\chi}_{r'}\rangle. \qquad (16.8)$$

Up to scalar multiples, the factors $\langle \chi_r|(\cdots)|\chi_{r'}\rangle$ involving L yield $S^{r,r'}(z)|_{q\to q^2}$ in (12.6). In the identification one uses (16.2) and $\langle \chi_r| = \langle \eta_r||_{q\to q^2}$ and $|\chi_r\rangle = |\eta_r\rangle|_{q\to q^2}$ in (16.6). Since they appear frequently, we adopt the convention:

$$S^{r,r'}(z) \text{ in this chapter } = (S^{r,r'}(z) \text{ in (12.8)–(12.9)})_{q\to q^2}. \qquad (16.9)$$

By Theorem 12.2$|_{q\to q^2}$, we know that $S^{r,r'}(z)$ is the quantum R matrix for the spin representation of $U_p(\mathfrak{g}^{r,r'})$ at $p = -q^{-2}$.

Returning to (16.8), the other factors emerging from G have the form

$$K_{\mathbf{1}}^{k,k'}(z) = \kappa^{k,k'}(z)\langle \overset{6}{\eta}_k| z^{\mathbf{h}_6} G_{1_1 6} \cdots G_{1_n 6}|\overset{6}{\eta}_{k'}\rangle \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}}), \qquad (16.10)$$
$$K_{\mathbf{2}}^{k,k'}(z) = \kappa^{k,k'}(z)\langle \overset{4}{\eta}_k| z^{\mathbf{h}_4} G_{2_1 4} \cdots G_{2_n 4}|\overset{4}{\eta}_{k'}\rangle \in \mathrm{End}(\overset{\mathbf{2}}{\mathbf{V}}), \qquad (16.11)$$

where $k, k' = 1, 2$. The scalar $\kappa^{k,k'}(z)$ will be specified in (16.17). They are the same linear operators (16.13) acting on the different copies of $V^{\otimes n}$ given as $\overset{\mathbf{1}}{\mathbf{V}} = \overset{1_1}{V} \otimes \cdots \otimes \overset{1_n}{V}$ and $\overset{\mathbf{2}}{\mathbf{V}} = \overset{2_1}{V} \otimes \cdots \otimes \overset{2_n}{V}$.

In terms of (16.10)–(16.11) and (12.6)$|_{q\to q^2}$, the relation (16.8) is stated as the reflection equation

$$S_{\mathbf{1},\mathbf{2}}^{r,r'}(xy^{-1}) K_{\mathbf{2}}^{k,k'}(x) S_{\mathbf{2},\mathbf{1}}^{r,r'}(xy) K_{\mathbf{1}}^{k,k'}(y) = K_{\mathbf{1}}^{k,k'}(y) S_{\mathbf{1},\mathbf{2}}^{r,r'}(xy) K_{\mathbf{2}}^{k,k'}(x) S_{\mathbf{2},\mathbf{1}}^{r,r'}(xy^{-1}) \qquad (16.12)$$

for $1 \le r \le k \le 2$ and $1 \le r' \le k' \le 2$.

The construction (16.10)–(16.11) yields the matrix product formula for each element as

$$K^{k,k'}(z) v_{\mathbf{a}} = \sum_{\mathbf{b}\in\mathfrak{s}} K^{k,k'}(z)_{\mathbf{a}}^{\mathbf{b}} v_{\mathbf{b}},$$
$$K^{k,k'}(z)_{\mathbf{a}}^{\mathbf{b}} = \kappa^{k,k'}(z)\langle \eta_k| z^{\mathbf{h}} G_{a_1}^{b_1} \cdots G_{a_n}^{b_n}|\eta_{k'}\rangle. \qquad (16.13)$$

See (11.1)–(11.7) for the notations $\mathfrak{s}$, $v_{\mathbf{a}}$, $\mathbf{V}$ etc. From (16.1), we see that it depends on the parameter β in (16.1) as the conjugation:

$$K^{k,k'}(z) = \beta^{\mathbf{h}_1+\cdots+\mathbf{h}_n}\left(K^{k,k'}(z)|_{\beta=1}\right)\beta^{-\mathbf{h}_1-\cdots-\mathbf{h}_n}. \qquad (16.14)$$

From (3.18) and the fact that $\kappa^{k,k'}(z) = \kappa^{k',k}(z)$ in (16.17), it can be shown that

$$K^{k,k'}(z)^{\mathbf{b}}_{\mathbf{a}} = z^{n-|\mathbf{a}|-|\mathbf{b}|} K^{k',k}(z)^{\mathbf{e}_1+\cdots+\mathbf{e}_n-\mathbf{a}^\vee}_{\mathbf{e}_1+\cdots+\mathbf{e}_n-\mathbf{b}^\vee}, \tag{16.15}$$

where $(a_1, \ldots, a_n)^\vee = (a_n, \ldots, a_1)$ is the reverse ordered array as in (11.4). Noting the factor $\theta(j \in 2\mathbb{Z})$ in the last formula in (12.10), one can show

$$K^{2,2}(z)^{\mathbf{b}}_{\mathbf{a}} = 0 \quad \text{unless} \quad |\mathbf{a}| + |\mathbf{b}| \equiv n \mod 2 \tag{16.16}$$

by an argument similar to the one for deriving (15.20). Consequently, the direct sum decomposition

$$K^{2,2}(z) = K^{2,2}_{+}(z) \oplus K^{2,2}_{-}(z), \qquad K^{2,2}_{\sigma}(z) : \mathbf{V}_\sigma \to \mathbf{V}_{\sigma(-1)^n}$$

holds, where $\mathbf{V}_\pm$ was defined in (11.6). As for $K^{k,k'}(z)$ with $(k, k') \neq (2, 2)$, there is no selection rule like (15.20) or (16.16). We choose the scalar $\kappa^{k,k'}(z)$ as

$$\kappa^{k,k'}(z) = q^{-\frac{n}{2}} \frac{((zq^n)^t; q^{kk'})_\infty}{((-q)^s(zq^n)^t; q^{kk'})_\infty}, \qquad s = \min(k, k'), \quad t = \max(k, k'), \tag{16.17}$$

which is the inverse of $q^{\frac{n}{2}} \langle \eta_k | z^{\mathbf{h}} \mathbf{k}^n | \eta_{k'} \rangle$ calculated from (12.10). In this normalization,

$$K^{k,k'}(z) v_{\mathbf{e}_1+\cdots+\mathbf{e}_l} = (-1)^l (q^{-\frac{1}{2}}\beta)^{n-2l} v_{\mathbf{e}_{l+1}+\cdots+\mathbf{e}_n} + \cdots \tag{16.18}$$

for $0 \leq l \leq n, \ 1 \leq k, k' \leq 2$ holds, and general elements are rational functions of $\beta, q^{\frac{1}{2}}$ and z.

As seen from (16.13) and also in Example 16.1 below, the K matrix $K^{k,k'}(z)$ is *dense* in the sense that all the elements are non-zero (for $K^{2,2}$ non-zero within each sector implied by (16.16)).

Example 16.1 We present $K^{k,k'}(z)$ with $\beta = q^{\frac{1}{2}}$ for $n = 1, 2$ and $(k, k') = (1, 1)$, $(1, 2)$, $(2, 2)$. The general β case and $(k, k') = (2, 1)$ can be deduced from them by (16.14) and (16.15). We write $v_0 \otimes v_1$ as $|01\rangle$ etc.

For $n = 1$, $K^{k,k'}(z)|_{\beta=q^{\frac{1}{2}}}$ acts on the basis as

$$K^{1,1}(z) : |0\rangle \mapsto -\frac{q^{-\frac{1}{2}}(1+q)z|0\rangle}{-1+z} + |1\rangle, \qquad |1\rangle \mapsto -|0\rangle - \frac{q^{-\frac{1}{2}}(1+q)|1\rangle}{-1+z},$$

$$K^{1,2}(z) : |0\rangle \mapsto -\frac{q^{-\frac{1}{2}}(1+q)z|0\rangle}{-1+z^2} + |1\rangle, \qquad |1\rangle \mapsto -|0\rangle - \frac{q^{-\frac{1}{2}}(1+q)z|1\rangle}{-1+z^2},$$

$$K^{2,2}(z) : |0\rangle \mapsto |1\rangle, \qquad |1\rangle \mapsto -|0\rangle.$$

For $n = 2$, $K^{1,1}(z)|_{\beta=q^{\frac{1}{2}}}$ acts on the basis as

$$|00\rangle \mapsto \frac{q^{-1}(1+q)(1+q^2)z^2|00\rangle}{(-1+z)(-1+qz)} - \frac{q^{-\frac{1}{2}}(1+q)z|01\rangle}{(-1+qz)} - \frac{q^{\frac{1}{2}}(1+q)z|10\rangle}{-1+qz} + |11\rangle,$$

$$|01\rangle \mapsto \frac{q^{-\frac{1}{2}}(1+q)z|00\rangle}{-1+qz} + \frac{q^{-1}(1+q)z(1+q-qz+q^2z)|01\rangle}{(-1+z)(-1+qz)} - |10\rangle - \frac{q^{-\frac{1}{2}}(1+q)|11\rangle}{-1+qz},$$

$$|10\rangle \mapsto \frac{q^{\frac{1}{2}}(1+q)z|00\rangle}{-1+qz} - |01\rangle + \frac{q^{-1}(1+q)(1-q+qz+q^2z)|10\rangle}{(-1+z)(-1+qz)} - \frac{q^{\frac{1}{2}}(1+q)|11\rangle}{-1+qz},$$

$$|11\rangle \mapsto |00\rangle + \frac{q^{-\frac{1}{2}}(1+q)|01\rangle}{-1+qz} + \frac{q^{\frac{1}{2}}(1+q)|10\rangle}{-1+qz} + \frac{q^{-1}(1+q)(1+q^2)|11\rangle}{(-1+z)(-1+qz)}.$$

$K^{1,2}(z)|_{\beta=q^{\frac{1}{2}}}$ acts on the basis as

$$|00\rangle \mapsto \frac{q^{-1}(1+q)z^2(1+q^2-q^2z^2+q^3z^2)|00\rangle}{(-1+z^2)(-1+q^2z^2)} - \frac{q^{-\frac{1}{2}}(1+q)z|01\rangle}{-1+q^2z^2} - \frac{q^{\frac{1}{2}}(1+q)z|10\rangle}{-1+q^2z^2} + |11\rangle,$$

$$|01\rangle \mapsto \frac{q^{-\frac{1}{2}}(1+q)z|00\rangle}{-1+q^2z^2} + \frac{q^{-1}(1+q)z^2(1+q^2-q^2z^2+q^3z^2)|01\rangle}{(-1+z^2)(-1+q^2z^2)} - |10\rangle - \frac{q^{\frac{1}{2}}(1+q)z|11\rangle}{-1+q^2z^2},$$

$$|10\rangle \mapsto \frac{q^{\frac{1}{2}}(1+q)z|00\rangle}{-1+q^2z^2} - |01\rangle + \frac{q^{-1}(1+q)(1-q+qz^2+q^3z^2)|10\rangle}{(-1+z^2)(-1+q^2z^2)} - \frac{q^{\frac{3}{2}}(1+q)z|11\rangle}{-1+q^2z^2},$$

$$|11\rangle \mapsto |00\rangle + \frac{q^{\frac{1}{2}}(1+q)z|01\rangle}{-1+q^2z^2} + \frac{q^{\frac{3}{2}}(1+q)z|10\rangle}{-1+q^2z^2} + \frac{q^{-1}(1+q)(1-q+qz^2+q^3z^2)|11\rangle}{(-1+z^2)(-1+q^2z^2)}.$$

$K^{2,2}(z)|_{\beta=q^{\frac{1}{2}}}$ acts on the basis as

$$|00\rangle \mapsto \frac{q^{-1}(-1+q^2)z^2|00\rangle}{-1+z^2} + |11\rangle, \qquad |01\rangle \mapsto \frac{q^{-1}(-1+q^2)z^2|01\rangle}{-1+z^2} - |10\rangle,$$
$$|10\rangle \mapsto -|01\rangle + \frac{q^{-1}(-1+q^2)|10\rangle}{-1+z^2}, \qquad |11\rangle \mapsto |00\rangle + \frac{q^{-1}(-1+q^2)|11\rangle}{-1+z^2}.$$

16.3 Characterization as the Intertwiner of the Onsager Coideal

We keep the definitions of the quantum affine algebras $U_p(\mathfrak{g}^{r,r'})$ $(r, r' = 1, 2)$ in Sect. 12.2, where

$$\mathfrak{g}^{1,1} = D^{(2)}_{n+1}, \quad \mathfrak{g}^{2,1} = B^{(1)}_n, \quad \mathfrak{g}^{1,2} = \tilde{B}^{(1)}_n, \quad \mathfrak{g}^{2,2} = D^{(1)}_n \tag{16.19}$$

as in (12.20). We use the spin representation $\pi_{\varpi_n,x} : U_p(\mathfrak{g}^{r,r'}) \to \mathrm{End}(\mathbf{V})$ in (12.23)–(12.27), which we quote here for convenience:

$$e_0 v_{\mathbf{m}} = x v_{\mathbf{m}-\mathbf{e}_1}, \quad f_0 v_{\mathbf{m}} = x^{-1} v_{\mathbf{m}+\mathbf{e}_1}, \quad k_0 v_{\mathbf{m}} = p^{\frac{1}{2}-m_1} v_{\mathbf{m}} \quad (r = 1), \tag{16.20}$$

$$e_0 v_{\mathbf{m}} = x^2 v_{\mathbf{m}-\mathbf{e}_1-\mathbf{e}_2}, \quad f_0 v_{\mathbf{m}} = x^{-2} v_{\mathbf{m}+\mathbf{e}_1+\mathbf{e}_2}, \quad k_0 v_{\mathbf{m}} = p^{1-m_1-m_2} v_{\mathbf{m}} \quad (r = 2), \tag{16.21}$$

$$e_i v_{\mathbf{m}} = v_{\mathbf{m}+\mathbf{e}_i-\mathbf{e}_{i+1}}, \quad f_i v_{\mathbf{m}} = v_{\mathbf{m}-\mathbf{e}_i+\mathbf{e}_{i+1}}, \quad k_i v_{\mathbf{m}} = p^{m_i-m_{i+1}} v_{\mathbf{m}} \quad (0 < i < n), \tag{16.22}$$

$$e_n v_{\mathbf{m}} = v_{\mathbf{m}+\mathbf{e}_n}, \quad f_n v_{\mathbf{m}} = v_{\mathbf{m}-\mathbf{e}_n}, \quad k_n v_{\mathbf{m}} = p^{m_n-\frac{1}{2}} v_{\mathbf{m}} \quad (r' = 1), \tag{16.23}$$

$$e_n v_{\mathbf{m}} = v_{\mathbf{m}+\mathbf{e}_{n-1}+\mathbf{e}_n}, \quad f_n v_{\mathbf{m}} = v_{\mathbf{m}-\mathbf{e}_{n-1}-\mathbf{e}_n}, \quad k_n v_{\mathbf{m}} = p^{m_n+m_{n-1}-1} v_{\mathbf{m}} \quad (r' = 2), \tag{16.24}$$

where $\mathbf{m} \in \mathfrak{s}$. As mentioned before, it is irreducible except for $\mathfrak{g}^{2,2} = D^{(1)}_n$, where $\mathbf{V} = \mathbf{V}_+ \oplus \mathbf{V}_-$ as defined in (11.6) corresponding to the two kinds of spin representations.

According to the remark after (16.9), we will be concerned with $U_p(\mathfrak{g}^{r,r'})$ with $p = -q^{-2}$. In the rest of the chapter we set

$$p^{\frac{1}{2}} = -\mathrm{i}\epsilon q^{-1}, \qquad \epsilon = \pm 1 \tag{16.25}$$

and allow the coexistence of the letters p, q and ϵ.

16.3.1 Generalized p-Onsager Algebra $O_p(\mathfrak{g}^{r,r'})$

For each $\mathfrak{g}^{r,r'}$ in (16.19) we consider the quantum affine algebra $U_p(\mathfrak{g}^{r,r'})$ (12.20) and the Onsager algebra $O_p(\mathfrak{g}^{r,r'})$.

For comparison we write down the p-Serre relations in $U_p(\mathfrak{g}^{r,r'})$ which were not displayed together with (12.21):

$$e_i e_j - e_j e_i = 0 \quad (a_{ij} = 0), \tag{16.26}$$

$$e_i^2 e_j - (p + p^{-1}) e_i e_j e_i + e_j e_i^2 = 0 \quad (a_{ij} = -1), \tag{16.27}$$

$$e_i^3 e_j - (p + 1 + p^{-1}) e_i^2 e_j e_i + (p + 1 + p^{-1}) e_i e_j e_i^2 - e_j e_i^3 = 0 \quad (a_{ij} = -2). \tag{16.28}$$

The same relations are imposed also for f_j's. The data $(a_{ij})_{0 \le i,j \le n}$ is the Cartan matrix of the affine Lie algebra $\mathfrak{g}^{r,r'}$. The Onsager algebra $O_p(\mathfrak{g}^{r,r'})$ is generated by $\mathfrak{b}_0, \ldots, \mathfrak{b}_n$ obeying the modified p-Serre relations:

$$\mathfrak{b}_i \mathfrak{b}_j - \mathfrak{b}_j \mathfrak{b}_i = 0 \quad (a_{ij} = 0), \tag{16.29}$$

$$\mathfrak{b}_i^2 \mathfrak{b}_j - (p + p^{-1}) \mathfrak{b}_i \mathfrak{b}_j \mathfrak{b}_i + \mathfrak{b}_j \mathfrak{b}_i^2 = \mathfrak{b}_j \quad (a_{ij} = -1), \tag{16.30}$$

$$\begin{aligned} &\mathfrak{b}_i^3 \mathfrak{b}_j - (p + 1 + p^{-1}) \mathfrak{b}_i^2 \mathfrak{b}_j \mathfrak{b}_i + (p + 1 + p^{-1}) \mathfrak{b}_i \mathfrak{b}_j \mathfrak{b}_i^2 - \mathfrak{b}_j \mathfrak{b}_i^3 \\ &= (p^{\frac{1}{2}} + p^{-\frac{1}{2}})^2 (\mathfrak{b}_i \mathfrak{b}_j - \mathfrak{b}_j \mathfrak{b}_i) \quad (a_{ij} = -2). \end{aligned} \tag{16.31}$$

Except for (16.31) which are void for the simply-laced case $\mathfrak{g}^{2,2} = D_n^{(1)}$, these relations are formally the same with (15.25) for $O_p(A_{n-1}^{(1)})$.

In terms of commutators $[X, Y] = [X, Y]_1$, $[X, Y]_r = XY - rYX$, the relations (16.29)–(16.31) are written more compactly as

$$[\mathfrak{b}_i, \mathfrak{b}_j] = 0 \quad (a_{ij} = 0), \tag{16.32}$$

$$[\mathfrak{b}_i, [\mathfrak{b}_i, \mathfrak{b}_j]_p]_{p^{-1}} = \mathfrak{b}_j \quad (a_{ij} = -1), \tag{16.33}$$

$$[\mathfrak{b}_i, [\mathfrak{b}_i, [\mathfrak{b}_i, \mathfrak{b}_j]_p]_{p^{-1}}] = (p^{\frac{1}{2}} + p^{-\frac{1}{2}})^2 [\mathfrak{b}_i, \mathfrak{b}_j] \quad (a_{ij} = -2). \tag{16.34}$$

There is an embedding $O_p(\mathfrak{g}^{r,r'}) \hookrightarrow U_p(\mathfrak{g}^{r,r'})$, depending on integer indices k, k' satisfying $1 \le r \le k \le 2$ and $1 \le r' \le k' \le 2$, given by

$$\mathfrak{b}_0 \mapsto g_0 := e_0 + p^{r/2} k_0 f_0 + d^r_k k_0, \tag{16.35}$$

$$\mathfrak{b}_i \mapsto g_i := e_i + p k_i f_i + \frac{1}{q+q^{-1}} k_i \quad (0 < i < n), \tag{16.36}$$

$$\mathfrak{b}_n \mapsto g_n := e_n + p^{r'/2} k_n f_n + d^{r'}_{k'} k_n, \tag{16.37}$$

$$d^1_1 = \epsilon \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{q+q^{-1}}, \quad d^1_2 = 0, \quad d^2_2 = \frac{1}{q+q^{-1}}, \tag{16.38}$$

where $\epsilon = \pm 1$ has been introduced in (16.25). Define

$$\mathcal{B}^{r,r'}_{k,k'} = \text{ the subalgebra of } U_p(\mathfrak{g}^{r,r'}) \text{ generated by } g_0, \dots, g_n \text{ in (16.35)–(16.37).} \tag{16.39}$$

By the remark on (15.28), it becomes a left coideal; $\Delta \mathcal{B}^{r,r'}_{k,k'} \subset U_p(\mathfrak{g}^{r,r'}) \otimes \mathcal{B}^{r,r'}_{k,k'}$. Henceforth $\mathcal{B}^{r,r'}_{k,k'}$ will be referred to as an Onsager coideal.

16.3.2 $K^{k,k'}(z)$ as the Intertwiner of Onsager Coideal

Recall that $\pi_{\varpi_n,x} : U_p(\mathfrak{g}^{r,r'}) \to \mathrm{End}(\mathbf{V})$ denotes the spin representation in (16.20)–(16.24).

Theorem 16.2 *The K matrix (16.13) with $\beta = \mathrm{i}q^{\frac{1}{2}}$ is characterized, up to normalization, as the intertwiner of the Onsager coideal $\mathcal{B}^{r,r'}_{k,k'} \subset U_p(\mathfrak{g}^{r,r'})$ at $p^{\frac{1}{2}} = -\mathrm{i}\epsilon q^{-1}$ (16.25) as*

$$K^{k,k'}(z)\pi_{\varpi_n,z^{-1}}(g) = \pi_{\varpi_n,z}(g) K^{k,k'}(z) \qquad (\forall g \in \mathcal{B}^{r,r'}_{k,k'}), \tag{16.40}$$

where $1 \le r \le k$ and $1 \le r' \le k'$.

Proof We focus on the existence referring to [104, Sect. 5.2] for the uniqueness. There are seven cases in (16.40) to verify:

(i) $g = g_i \ (0 < i < n)$,
(ii) $g = g_0, \ (r,k) = (1,2)$,
(iii) $g = g_0, \ (r,k) = (1,1)$,
(iv) $g = g_0, \ (r,k) = (2,2)$,
(v) $g = g_n, \ (r',k') = (1,2)$,
(vi) $g = g_n, \ (r',k') = (1,1)$,
(vii) $g = g_n, \ (r',k') = (2,2)$.

Thanks to (3.18), the cases (v), (vi) and (vii) are attributed to (ii), (iii), and (iv) at $z = 1$, respectively. Thus we consider (i)–(iv) below. The case (i) reduces to the already shown identity (15.36).

(ii) From (16.35) and (16.38), the Eq. (16.40) reads as

$$K^{2,k'}(z)\pi_{\varpi_n,z^{-1}}(e_0 + p^{\frac{1}{2}} k_0 f_0) = \pi_{\varpi_n,z}(e_0 + p^{\frac{1}{2}} k_0 f_0) K^{2,k'}(z). \tag{16.41}$$

From (16.20), this is translated to the relation of the coefficients for the transition $v_{\mathbf{a}} \to v_{\mathbf{b}}$ ($\mathbf{a}, \mathbf{b} \in \mathfrak{s}$ in (11.1)) as

$$z^{-1}K^{2,k'}(z)^{\mathbf{b}}_{\mathbf{a}-\mathbf{e}_1} + zp^{-a_1}K^{2,k'}(z)^{\mathbf{b}}_{\mathbf{a}+\mathbf{e}_1} = zK^{2,k'}(z)^{\mathbf{b}+\mathbf{e}_1}_{\mathbf{a}} + z^{-1}p^{1-b_1}K^{2,k'}(z)^{\mathbf{b}-\mathbf{e}_1}_{\mathbf{a}}. \tag{16.42}$$

One can drop the factors p^{-a_1} and p^{1-b_1} since the attached terms are non-vanishing only for $\mathbf{a}+\mathbf{e}_1, \mathbf{b}-\mathbf{e}_1 \in \mathfrak{s}$ compelling $a_1 = 0$ and $b_1 = 1$. Then, in view of the matrix product formula (16.13), the relation in question follows from

$$\langle \eta_2 | z^{\mathbf{h}}(z^{-1}G^b_{a-1} + zG^b_{a+1}) = \langle \eta_2 | z^{\mathbf{h}}(zG^{b+1}_a + z^{-1}G^{b-1}_a) \tag{16.43}$$

for $a, b = 0, 1$. From (15.6) this is further reduced to the z-independent relation

$$\langle \eta_2 | (G^b_{a-1} + G^b_{a+1}) = \langle \eta_2 | (G^{b+1}_a + G^{b-1}_a). \tag{16.44}$$

It contains two non-trivial cases

$$0 = \langle \eta_2 | (G^0_0 - G^1_1) = \langle \eta_2 | (\mathbf{a}^+ - \mathbf{a}^-), \tag{16.45}$$

$$0 = \langle \eta_2 | (G^1_0 - G^0_1) = \langle \eta_2 | (\beta + q\beta^{-1})\mathbf{k}, \tag{16.46}$$

where (16.1) is substituted. The first equality holds due to (3.141) and the second does from the assumption $\beta = \mathrm{i}q^{\frac{1}{2}}$ of the theorem.

(iii) By an argument parallel with (ii), the proof reduces to showing

$$\begin{aligned} &z^{-1}K^{1,k'}(z)^{\mathbf{b}}_{\mathbf{a}-\mathbf{e}_1} + zp^{-a_1}K^{1,k'}(z)^{\mathbf{b}}_{\mathbf{a}+\mathbf{e}_1} + d^1_1 p^{\frac{1}{2}-a_1}K^{1,k'}(z)^{\mathbf{b}}_{\mathbf{a}} \\ &= zK^{1,k'}(z)^{\mathbf{b}+\mathbf{e}_1}_{\mathbf{a}} + z^{-1}p^{1-b_1}K^{1,k'}(z)^{\mathbf{b}-\mathbf{e}_1}_{\mathbf{a}} + d^1_1 p^{\frac{1}{2}-b_1}K^{1,k'}(z)^{\mathbf{b}}_{\mathbf{a}}. \end{aligned} \tag{16.47}$$

The matrix product formula (16.13) and (15.6) attribute it to

$$\langle \eta_1 | (G^b_{a-1} + G^b_{a+1} + d^1_1 p^{\frac{1}{2}-a}G^b_a) = \langle \eta_1 | (G^{b+1}_a + G^{b-1}_a + d^1_1 p^{\frac{1}{2}-b}G^b_a) \tag{16.48}$$

for $a, b = 0, 1$, where d^1_1 is specified in (16.38). This can be checked case by case by using $\beta = \mathrm{i}q^{\frac{1}{2}}$, (16.25) and the property of $\langle \eta_1|$ given in (3.138) and (3.139).

(iv) By a parallel argument with respect to the representation (16.21), the proof reduces to showing

$$\begin{aligned} &z^{-2}K^{2,k'}(z)^{\mathbf{b}}_{\mathbf{a}-\mathbf{e}_1-\mathbf{e}_2} + z^2 p^{-a_1-a_2}K^{2,k'}(z)^{\mathbf{b}}_{\mathbf{a}+\mathbf{e}_1+\mathbf{e}_2} + \frac{p^{1-a_1-a_2}}{q+q^{-1}}K^{2,k'}(z)^{\mathbf{b}}_{\mathbf{a}} \\ &= z^2K^{2,k'}(z)^{\mathbf{b}+\mathbf{e}_1+\mathbf{e}_2}_{\mathbf{a}} + z^{-2}p^{2-b_1-b_2}K^{2,k'}(z)^{\mathbf{b}-\mathbf{e}_1-\mathbf{e}_2}_{\mathbf{a}} + \frac{p^{1-b_1-b_2}}{q+q^{-1}}K^{2,k'}(z)^{\mathbf{b}}_{\mathbf{a}}. \end{aligned} \tag{16.49}$$

One may trivialize the coefficients of the middle terms as $p^{-a_1-a_2} = p^{2-b_1-b_2} = 1$ for the non-zero contributions. The matrix product formula (16.13) and (15.6) attribute the resulting relation to

$$\begin{aligned}&\langle\eta_2|(G^b_{a-1}G^{b'}_{a'-1} + G^b_{a+1}G^{b'}_{a'+1} + \frac{p^{1-a-a'}}{q+q^{-1}}G^b_aG^{b'}_{a'})\\&= \langle\eta_2|(G^{b+1}_aG^{b'+1}_{a'} + G^{b-1}_aG^{b'-1}_{a'} + \frac{p^{1-b-b'}}{q+q^{-1}}G^b_aG^{b'}_{a'})\end{aligned} \tag{16.50}$$

for $a, a', b, b' = 0, 1$. We have set $(a, a', b, b') = (a_1, a_2, b_1, b_2)$. This can be checked similarly by using $\beta = \mathrm{i}q^{\frac{1}{2}}$, (16.25) and the property of $\langle\eta_2|$ in (3.141). In particular it involves a maneuver like $\langle\eta_2|(\mathbf{a}^+)^2 = \langle\eta_2|\mathbf{a}^-\mathbf{a}^+ = \langle\eta_2|(1 - q^2\mathbf{k}^2)$, etc. □

One can give an alternative derivation of the reflection equation (16.12) based on the Onsager coideal $\mathcal{B}^{r,r'}_{k,k'}$ by an argument parallel with Sect. 15.4.3.

Let us summarize the solutions to the reflection equation obtained by the 3D approach in Chaps. 15 and 16. There are nine cases in (16.12), where the conditions $1 \le r \le k \le 2$ and $1 \le r' \le k' \le 2$ originate in Proposition 5.21.

Table 16.1 The quantum affine algebra $U_p(\mathfrak{g})$ with $\mathfrak{g} = A^{(1)}_{n-1}$ and $\mathfrak{g}^{r,r'}$ (16.19), the associated R matrices $S^{\mathrm{tr}}(z)$ and $S^{r,r'}(z)$, the associated K matrices $K^{\mathrm{tr}}(z)$ and $K^{k,k'}(z)$. There are a few choices of $K^{k,k'}(z)$ that can be paired with $S^{r,r'}(z)$ to jointly constitute a solution to the reflection equation depending on (r, r')

$\mathfrak{g}$	R matrix	K matrix
$A^{(1)}_{n-1}$	$S^{\mathrm{tr}}(z)$	$K^{\mathrm{tr}}(z)$
$D^{(2)}_{n+1}$	$S^{1,1}(z)$	$K^{1,1}(z)$, $K^{1,2}(z)$, $K^{2,1}(z)$, $K^{2,2}(z)$
$B^{(1)}_n$	$S^{2,1}(z)$	$K^{2,1}(z)$, $K^{2,2}(z)$
$\tilde{B}^{(1)}_n$	$S^{1,2}(z)$	$K^{1,2}(z)$, $K^{2,2}(z)$
$D^{(1)}_n$	$S^{2,2}(z)$	$K^{2,2}(z)$

16.4 Bibliographical Notes and Comments

The boundary vector reduction of the quantized reflection equation was introduced in [105], where the property (16.7) of the boundary vector remained as a conjecture. The first proof of the reflection equation (16.12) was done independently in the quantum group framework based on the Onsager coideal $\mathcal{B}^{r,r'}_{k,k'}$ and the argument like Sect. 15.4.3 [104]. Later the property (16.7) was proved in [106, Appendix B], which completed the 3D approach. Its detail has been reproduced in Proposition 5.21 of this book.

In the 3D approach to the reflection equation, either by the trace reduction (Chap. 15) or the boundary vector reduction (this chapter), the 3D K disappears at an early stage. In fact "reduction" more or less means eliminating it to return to 2D from 3D. However, the 3D K essentially controls the construction behind the scene in the sense that it guides precisely how the local operators L and G should be combined, how the spectral parameters should be arranged and what kind of boundary vectors are acceptable.

Concerning the generalized Onsager algebras, the quartic relation of the form (16.34) with $p^2 = 1$ is often referred to as the Dolan–Grady condition [41]. It is typical for the situation $a_{ij} = -2$, which was utilized to reformulate the original Onsager algebra for $A_1^{(1)}$ [122] by only a few generators. The Onsager algebra $O_p(D_n^{(1)})$ with $p = 1$ was introduced in [34]. It is an interesting open question if there is an analogue of Remark 15.2 for $\mathfrak{g} \neq A_{n-1}^{(1)}$ related to a boundary extension of the Temperley–Lieb algebra like [35].

Generalized Onsager algebras $O_p(\mathfrak{g}^{r,r'})$ have a natural classical part without the generator $\mathfrak{b}_0$. The commutativity in Theorem 16.2 interchanging z and z^{-1} implies the usual commutativity with the classical part. The corresponding spectral decomposition of $K^{r,r'}(z)$ has been described in [106, Sec.10,11].

Chapter 17
Reductions of Quantized G_2 Reflection Equation

Abstract Spectral parameters in the Yang–Baxter and the reflection equations correspond to the positive roots of A_2 and B_2/C_2, respectively. They appear as angles, or relative rapidity, of the world lines of particles that undergo factorized scattering in integrable $(1+1)$D quantum field theories in the bulk and at the boundary. There is an analogous equation associated with G_2, which we call the G_2 *reflection equation* in this book. It describes the three-body scattering related to the geometry of the Desargues–Pappus theorem. In addition to the usual two-body collision in the bulk, it involves the special three-particle event in which a two-body collision takes place at exactly the same instant as the boundary reflection of the third particle. In this chapter we construct infinite families of trigonometric solutions to the G_2 reflection equation by the 3D approach parallel with Chaps. 11–16. We start from the quantized G_2 reflection equation and its solution in Theorem 8.6, and perform the trace and the boundary vector reductions. The resulting solutions to the G_2 reflection equation involve quantum R matrices of $A^{(1)}_{n-1}$ and $D^{(2)}_{n+1}$, and they are coupled with the scattering amplitude of the special three-particle event expressed by a matrix product formula.

17.1 Introduction

Thus far we have presented a 3D approach to the Yang–Baxter and the reflection equations, which are presented in terms of additive spectral parameters as

$$R_{12}(\alpha_1)R_{13}(\alpha_1+\alpha_2)R_{23}(\alpha_2) = R_{23}(\alpha_2)R_{13}(\alpha_1+\alpha_2)R_{12}(\alpha_1), \tag{17.1}$$

$$R_{12}(\alpha_1)K_2(\alpha_1+\alpha_2)R_{21}(\alpha_1+2\alpha_2)K_1(\alpha_2)$$
$$= K_1(\alpha_2)R_{12}(\alpha_1+2\alpha_2)K_2(\alpha_1+\alpha_2)R_{21}(\alpha_1). \tag{17.2}$$

They are spectral parameter dependent versions (sometimes referred to as Yang–Baxterizations) of the cubic and the quartic Coxeter relations for the simple reflections s_1, s_2 of the root systems of A_2 and B_2/C_2:

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$$s_1s_2s_1 = s_2s_1s_2, \qquad \Delta_+ = \{\alpha_1, \alpha_1+\alpha_2, \alpha_2\},$$
$$s_1s_2s_1s_2 = s_2s_1s_2s_1, \qquad \Delta_+ = \{\alpha_1, \alpha_1+\alpha_2, \alpha_1+2\alpha_2, \alpha_2\}.$$

Here α_1, α_2 are the simple roots and Δ_+ denotes the set of positive roots which formally correspond to the spectral parameters. They are so ordered that the kth one from the left is $s_{i_1}\cdots s_{i_{k-1}}(\alpha_{i_k})$ with $i_k = 1$ (k: odd) and $i_k = 2$ (k: even). See (10.3).

In this chapter we consider a natural G_2 analogue of them as

$$\begin{aligned} &R_{12}(\alpha_1)X_{132}(\alpha_1+\alpha_2)R_{23}(2\alpha_1+3\alpha_2)X_{213}(\alpha_1+2\alpha_2)R_{31}(\alpha_1+3\alpha_2)X_{321}(\alpha_2)\\ &= X_{231}(\alpha_2)R_{13}(\alpha_1+3\alpha_2)X_{123}(\alpha_1+2\alpha_2)R_{32}(2\alpha_1+3\alpha_2)X_{312}(\alpha_1+\alpha_2)R_{21}(\alpha_1), \end{aligned} \tag{17.3}$$

which we call the G_2 *reflection equation*. Based on the results on $A_q(G_2)$ in Chap. 8, we construct infinite families of solutions by extending the 3D approach further. The basic ingredient is the quantized G_2 reflection equation (8.2):

$$(L_{12}J_{132}L_{23}J_{213}L_{31}J_{321})F = F(J_{231}L_{13}J_{123}L_{32}J_{312}L_{21}). \tag{17.4}$$

It is a generalization of the constant G_2 reflection equation $R_{12}X_{132}R_{23}X_{213}R_{31}X_{321} = X_{231}R_{13}X_{123}R_{32}X_{312}R_{21}$ to a conjugacy equivalence by the intertwiner F. The contents are parallel with those for the Yang–Baxter and the reflection equations in Chaps. 11–16.

17.2 The G_2 Reflection Equation

Let $\mathbf{V}$ be a vector space and consider the operators

$$R(z) \in \mathrm{End}(\mathbf{V}\otimes\mathbf{V}), \qquad X(z) \in \mathrm{End}(\mathbf{V}\otimes\mathbf{V}\otimes\mathbf{V}) \tag{17.5}$$

depending on the spectral parameter z. We assume that $R(z)$ satisfies the Yang–Baxter equation by itself:

$$R_{12}(x)R_{13}(xy)R_{23}(y) = R_{23}(y)R_{13}(xy)R_{12}(x) \in \mathrm{End}(\mathbf{V}\otimes\mathbf{V}\otimes\mathbf{V}). \tag{17.6}$$

We consider the G_2 reflection equation in $\mathrm{End}(\mathbf{V}\otimes\mathbf{V}\otimes\mathbf{V})$ with multiplicative spectral parameters:[1]

$$\begin{aligned} &R_{12}(x)X_{132}(xy)R_{23}(x^2y^3)X_{213}(xy^2)R_{31}(xy^3)X_{321}(y)\\ &= X_{231}(y)R_{13}(xy^3)X_{123}(xy^2)R_{32}(x^2y^3)X_{312}(xy)R_{21}(x). \end{aligned} \tag{17.7}$$

[1] In the solutions that we will obtain later, $\mathbf{V}$ has the structure $\mathbf{V} = V^{\otimes n}$, hence bold font will be used there for the indices.

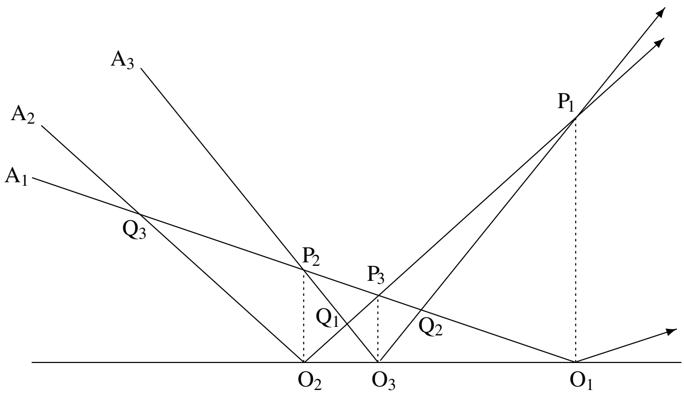


Fig. 17.1 Scattering diagram for the RHS of (17.7)

To clarify the notation, write temporarily as $R(z) = \sum r_l^{(1)} \otimes r_l^{(2)}$ and $X(z) = \sum x_l^{(1)} \otimes x_l^{(2)} \otimes x_l^{(3)}$ in terms of sums over l.[2] Then

$$\begin{aligned} R_{12}(z) &= \sum r_l^{(1)} \otimes r_l^{(2)} \otimes 1, \quad R_{21}(z) = \sum r_l^{(2)} \otimes r_l^{(1)} \otimes 1, \\ R_{13}(z) &= \sum r_l^{(1)} \otimes 1 \otimes r_l^{(2)}, \quad R_{31}(z) = \sum r_l^{(2)} \otimes 1 \otimes r_l^{(1)}, \\ R_{23}(z) &= \sum 1 \otimes r_l^{(1)} \otimes r_l^{(2)}, \quad R_{32}(z) = \sum 1 \otimes r_l^{(2)} \otimes r_l^{(1)}, \\ X_{ijk}(z) &= \sum x_l^{(i)} \otimes x_l^{(j)} \otimes x_l^{(k)}. \end{aligned} \tag{17.8}$$

Let us illustrate the special three-particle scattering diagram corresponding to the G_2 reflection equation. Consider the three particles 1,2,3 coming from A_1,A_2,A_3 and being reflected by the boundary at O_1, O_2, O_3, respectively. See Fig. 17.1. The bottom horizontal line is the boundary which may also be viewed as the time axis. The vertical direction corresponds to the 1D space. Each arrow carries $\mathbf{V}$ which specifies internal degrees of the freedom of a particle. So a three-particle state at a time is described by an element in $\mathbf{V} \otimes \mathbf{V} \otimes \mathbf{V}$.

One can arrange the three particle world lines so that the two-particle scattering P_1, P_2, P_3 happens exactly at the same instant as the boundary reflection O_1, O_2, O_3 of the other particle, respectively. This is non-trivial. For instance, suppose there were only particles 2 and 3. They already determine the reflecting points O_2, O_3 and the intersection P_1 (and Q_1) and its projection O_1 onto the boundary. Let P_2, P_3 be the points on the world lines of particles 3 and 2 whose projection are O_2 and O_3, respectively. In order to be able to draw the world line for the last particle 1, the three points P_2, P_3 and O_1 must be collinear. This is guaranteed by a special case of the Pappus theorem from the fourth century.

[2] Although these expansions do not specify $r_l^{(i)}, x_l^{(i)}$ uniquely, it suffices to make (17.8) unambiguous.

One can state it more symmetrically just by starting from $\mathrm{P}_1, \mathrm{P}_2$ and their projection $\mathrm{O}_1, \mathrm{O}_2$ onto the boundary. Let $\mathrm{P}'_1, \mathrm{P}'_2$ be the mirror image of $\mathrm{P}_1, \mathrm{P}_2$ with respect to the boundary. Then the three intersections $\overline{\mathrm{P}_1\mathrm{O}_2} \cap \overline{\mathrm{O}_1\mathrm{P}_2}$, $\overline{\mathrm{P}_1\mathrm{P}'_2} \cap \overline{\mathrm{P}'_1\mathrm{P}_2}$ and $\overline{\mathrm{O}_1\mathrm{P}'_2} \cap \overline{\mathrm{P}'_1\mathrm{O}_2}$ are collinear; in fact they are P_3, O_3 and the mirror image of P_3.

Let us call the so arranged scattering diagram a *Pappus configuration*. The reflection at O_i with the simultaneous two-particle scattering at P_i will be referred to as a *special three-particle event* ($i = 1, 2, 3$). Up to translation in the horizontal direction and the overall scale a Pappus configuration is parameterized by two real numbers, for instance, by the reflection angles $\angle \mathrm{P}_3\mathrm{O}_2\mathrm{O}_3$ and $\angle \mathrm{P}_3\mathrm{O}_1\mathrm{O}_3$. Set

$$\begin{aligned} &u = \angle \mathrm{P}_3\mathrm{O}_2\mathrm{O}_3, \quad w = \angle \mathrm{P}_2\mathrm{O}_3\mathrm{O}_2, \quad v = \angle \mathrm{P}_3\mathrm{O}_1\mathrm{O}_3,\\ &\theta_1 = \angle \mathrm{A}_2\mathrm{Q}_3\mathrm{A}_1, \quad \theta_2 = \angle \mathrm{A}_3\mathrm{P}_2\mathrm{A}_1, \quad \theta_3 = \angle \mathrm{A}_3\mathrm{Q}_1\mathrm{O}_2,\\ &\theta_4 = \angle \mathrm{A}_1\mathrm{P}_3\mathrm{O}_2, \quad \theta_5 = \angle \mathrm{A}_1\mathrm{Q}_2\mathrm{O}_3, \quad \theta_6 = \angle \mathrm{O}_2\mathrm{P}_1\mathrm{O}_3. \end{aligned} \tag{17.9}$$

Then it is elementary to see

$$\tan w = \tan u + \tan v, \tag{17.10}$$

$$\theta_1 = u - v, \quad \theta_2 = w - v, \quad \theta_3 = u + w, \quad \theta_4 = u + v, \quad \theta_5 = v + w, \quad \theta_6 = w - u. \tag{17.11}$$

We formally consider the infinitesimal angles, hence replace (17.10) by $w = u + v$. By a further substitution $u = \alpha_1 + \alpha_2$ and $v = \alpha_2$, (17.11) becomes

$$\theta_1 = \alpha_1, \;\; \theta_2 = \alpha_1 + \alpha_2, \;\; \theta_3 = 2\alpha_1 + 3\alpha_2, \;\; \theta_4 = \alpha_1 + 2\alpha_2, \;\; \theta_5 = \alpha_1 + 3\alpha_2, \;\; \theta_6 = \alpha_2. \tag{17.12}$$

Regard the symbols α_1, α_2 formally as the simple roots of G_2. They are transformed by the simple reflections s_1, s_2 of the Weyl group $W(G_2)$ as

$$s_1(\alpha_1) = -\alpha_1, \quad s_1(\alpha_2) = \alpha_1 + \alpha_2, \quad s_2(\alpha_1) = \alpha_1 + 3\alpha_2, \quad s_2(\alpha_2) = -\alpha_2. \tag{17.13}$$

Thus we find

$$\theta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}), \qquad (i_1, i_2, i_3, i_4, i_5, i_6) = (1, 2, 1, 2, 1, 2), \tag{17.14}$$

and $\{\theta_1, \ldots, \theta_6\}$ yields the set of the positive roots of G_2.

The RHS of the G_2 reflection equation (17.7) is obtained by attaching $R(\mathrm{e}^{\theta_k})$ to the two particle scattering at Q_i and $G(\mathrm{e}^{\theta_k})$ to the special three particle event at $\mathrm{P}_i\mathrm{O}_i$ if it is the kth event starting from the left in Fig. 17.1. Setting $\mathrm{e}^{\alpha_1} = x$ and $\mathrm{e}^{\alpha_2} = y$, the assignment reads as

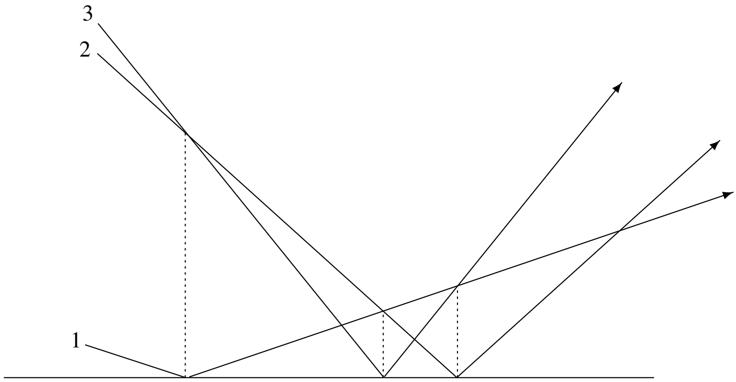


Fig. 17.2 Scattering diagram for the LHS of (17.7)

$$\begin{aligned}
R_{21}(x) &: \text{two-particle scattering at } Q_3,\\
X_{312}(xy) &: \text{special three-particle event at } P_2O_2,\\
R_{32}(x^2y^3) &: \text{two-particle scattering at } Q_1,\\
X_{123}(xy^2) &: \text{special three-particle event at } P_3O_3,\\
R_{13}(xy^3) &: \text{two-particle scattering at } Q_2,\\
X_{231}(y) &: \text{special three-particle event at } P_1O_1.
\end{aligned}$$

The indices for each operator correspond to the ordering of the relevant particles before the process. For instance, just before the special three-particle event at P_2O_2, the incoming particles are 3,1,2 from the top to the bottom, which is encoded in $X_{312}(xy)$. The LHS of the G_2 reflection equation (17.7) represents the Pappus configuration in which the time ordering of the processes are reversed. See Fig. 17.2.

Applications of the G_2 reflection equation to integrable systems are yet to be explored.

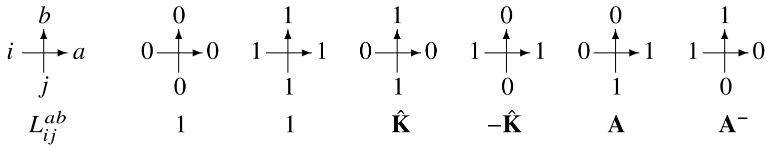
17.3 Quantized G_2 Reflection Equation

Let us recall the quantized G_2 reflection equation and its solution obtained in Sect. 8.5. The quantized G_2 reflection equation (8.50) is

$$L_{124}J_{1325}L_{236}J_{2137}L_{318}J_{3219}F_{456789} = F_{456789}J_{2319}L_{138}J_{1237}L_{326}J_{3125}L_{214}. \tag{17.15}$$

It is an equality of linear operators on $\overset{1}{V} \otimes \overset{2}{V} \otimes \overset{3}{V} \otimes \overset{4}{\mathcal{F}_{q^3}} \otimes \overset{5}{\mathcal{F}_q} \otimes \overset{6}{\mathcal{F}_{q^3}} \otimes \overset{7}{\mathcal{F}_q} \otimes \overset{8}{\mathcal{F}_{q^3}} \otimes \overset{9}{\mathcal{F}_q}$.

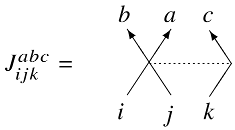
Let us recall L, J and F appearing here. First, $L \in \mathrm{End}(V \otimes V \otimes \mathcal{F}_{q^3})$ is the 3D L in (8.32)–(8.33) depicted as



(17.16)

$\mathbf{A}^{\pm}$ and $\hat{\mathbf{K}}$ are q^3-oscillators (8.7) including the zero point energy as in (8.13). This L is precisely equal to $((11.14)|_{\alpha=q^{1/2}})|_{q\to q^3}$.

Second, $J \in \mathrm{End}(V \otimes V \otimes V \otimes \mathcal{F}_q)$ is the quantized G_2 scattering operator. It is a collection of the operators $J_{ijk}^{abc} \in \mathrm{End}(\mathcal{F}_q)$ expressed by q-oscillators with zero point energy as (8.40)–(8.44). The quantized amplitude J_{ijk}^{abc} is depicted by the diagram which corresponds to the 90° rotation of the special three-particle events in Figs. 17.1 and 17.2:



(17.17)

Finally, $F \in \mathrm{End}(\mathcal{F}_{q^3} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^3} \otimes \mathcal{F}_q \otimes \mathcal{F}_{q^3} \otimes \mathcal{F}_q)$ is the intertwiner of the $A_q(G_2)$ modules detailed in Sect. 8.4.

17.4 Reduction of the Quantized G_2 Reflection Equation

Starting from the quantized G_2 reflection equation (17.15), one can perform two kinds of reductions to construct solutions to the G_2 reflection equation (17.7) in the matrix product form.

17.4.1 Concatenation of Quantized G_2 Reflection Equation

Consider n copies of (17.15) in which the spaces labeled with 1, 2, 3 are replaced by $1_i, 2_i, 3_i$ with $i = 1, 2, \ldots, n$:

$$
\begin{aligned}
&(L_{1_i 2_i 4} J_{1_i 3_i 2_i 5} L_{2_i 3_i 6} J_{2_i 1_i 3_i 7} L_{3_i 1_i 8} J_{3_i 2_i 1_i 9}) F_{456789} \\
&= F_{456789} (J_{2_i 3_i 1_i 9} L_{1_i 3_i 8} J_{1_i 2_i 3_i 7} L_{3_i 2_i 6} J_{3_i 1_i 2_i 5} L_{2_i 1_i 4}). \qquad (17.18)
\end{aligned}
$$

Write this as $Z_i F_{456789} = F_{456789} \tilde{Z}_i$. Then repeated use of it leads to $Z_1 Z_2 \cdots Z_n F_{456789} = F_{456789} \tilde{Z}_1 \tilde{Z}_2 \cdots \tilde{Z}_n$, namely,

$$
\begin{aligned}
&(L_{1_1 2_1 4} J_{1_1 3_1 2_1 5} L_{2_1 3_1 6} J_{2_1 1_1 3_1 7} L_{3_1 1_1 8} J_{3_1 2_1 1_1 9}) \cdots \\
&\qquad \cdots (L_{1_n 2_n 4} J_{1_n 3_n 2_n 5} L_{2_n 3_n 6} J_{2_n 1_n 3_n 7} L_{3_n 1_n 8} J_{3_n 2_n 1_n 9}) F_{456789} \\
&= F_{456789} (J_{2_1 3_1 1_1 9} L_{1_1 3_1 8} J_{1_1 2_1 3_1 7} L_{3_1 2_1 6} J_{3_1 1_1 2_1 5} L_{2_1 1_1 4}) \cdots \\
&\qquad \cdots (J_{2_n 3_n 1_n 9} L_{1_n 3_n 8} J_{1_n 2_n 3_n 7} L_{3_n 2_n 6} J_{3_n 1_n 2_n 5} L_{2_n 1_n 4}). \qquad (17.19)
\end{aligned}
$$

This can be rearranged without changing the order of operators sharing common labels as

$$
\begin{aligned}
&(L_{1_1 2_1 4} \cdots L_{1_n 2_n 4})(J_{1_1 3_1 2_1 5} \cdots J_{1_n 3_n 2_n 5})(L_{2_1 3_1 6} \cdots L_{2_n 3_n 6}) \\
&\quad \times (J_{2_1 1_1 3_1 7} \cdots J_{2_n 1_n 3_n 7})(L_{3_1 1_1 8} \cdots L_{3_n 1_n 8})(J_{3_1 2_1 1_1 9} \cdots J_{3_n 2_n 1_n 9}) F_{456789} \\
&= F_{456789} (J_{2_1 3_1 1_1 9} \cdots J_{2_n 3_n 1_n 9})(L_{1_1 3_1 8} \cdots L_{1_n 3_n 8})(J_{1_1 2_1 3_1 7} \cdots J_{1_n 2_n 3_n 7}) \\
&\quad \times (L_{3_1 2_1 6} \cdots L_{3_n 2_n 6})(J_{3_1 1_1 2_1 5} \cdots J_{3_n 1_n 2_n 5})(L_{2_1 1_1 4} \cdots L_{2_n 1_n 4}). \qquad (17.20)
\end{aligned}
$$

Now we utilize the weight conservation (8.21) of F in the form

$$
\begin{aligned}
&F^{-1}_{456789} x^{\mathbf{h}_4} (xy)^{\mathbf{h}_5} (x^2 y^3)^{\mathbf{h}_6} (xy^2)^{\mathbf{h}_7} (xy^3)^{\mathbf{h}_8} y^{\mathbf{h}_9} \\
&= y^{\mathbf{h}_9} (xy^3)^{\mathbf{h}_8} (xy^2)^{\mathbf{h}_7} (x^2 y^3)^{\mathbf{h}_6} (xy)^{\mathbf{h}_5} x^{\mathbf{h}_4} F^{-1}_{456789}. \qquad (17.21)
\end{aligned}
$$

Multiply it by (17.20) side by side from the left. The result reads as

$$
\begin{aligned}
&F^{-1}_{456789} \big(x^{\mathbf{h}_4} L_{1_1 2_1 4} \cdots L_{1_n 2_n 4}\big)\big((xy)^{\mathbf{h}_5} J_{1_1 3_1 2_1 5} \cdots J_{1_n 3_n 2_n 5}\big) \\
&\quad \times \big((x^2 y^3)^{\mathbf{h}_6} L_{2_1 3_1 6} \cdots L_{2_n 3_n 6}\big)\big((xy^2)^{\mathbf{h}_7} J_{2_1 1_1 3_1 7} \cdots J_{2_n 1_n 3_n 7}\big) \\
&\quad \times \big((xy^3)^{\mathbf{h}_8} L_{3_1 1_1 8} \cdots L_{3_n 1_n 8}\big)\big(y^{\mathbf{h}_9} J_{3_1 2_1 1_1 9} \cdots J_{3_n 2_n 1_n 9}\big) F_{456789} \\
&= \big(y^{\mathbf{h}_9} J_{2_1 3_1 1_1 9} \cdots J_{2_n 3_n 1_n 9}\big)\big((xy^3)^{\mathbf{h}_8} L_{1_1 3_1 8} \cdots L_{1_n 3_n 8}\big) \\
&\quad \times \big((xy^2)^{\mathbf{h}_7} J_{1_1 2_1 3_1 7} \cdots J_{1_n 2_n 3_n 7}\big)\big((x^2 y^3)^{\mathbf{h}_6} L_{3_1 2_1 6} \cdots L_{3_n 2_n 6}\big) \\
&\quad \times \big((xy)^{\mathbf{h}_5} J_{3_1 1_1 2_1 5} \cdots J_{3_n 1_n 2_n 5}\big)\big(x^{\mathbf{h}_4} L_{2_1 1_1 4} \cdots L_{2_n 1_n 4}\big). \qquad (17.22)
\end{aligned}
$$

17.4.2 Trace Reduction

Taking the trace of (17.22) over $\overset{4}{\mathcal{F}}_{q^3} \otimes \overset{5}{\mathcal{F}}_q \otimes \overset{6}{\mathcal{F}}_{q^3} \otimes \overset{7}{\mathcal{F}}_q \otimes \overset{8}{\mathcal{F}}_{q^3} \otimes \overset{9}{\mathcal{F}}_q$, we obtain

$$\begin{aligned}
&\mathrm{Tr}_4\big(x^{\mathbf{h}_4} L_{1_1 2_1 4} \cdots L_{1_n 2_n 4}\big)\mathrm{Tr}_5\big((xy)^{\mathbf{h}_5} J_{1_1 3_1 2_1 5} \cdots J_{1_n 3_n 2_n 5}\big)\\
&\times \mathrm{Tr}_6\big((x^2y^3)^{\mathbf{h}_6} L_{2_1 3_1 6} \cdots L_{2_n 3_n 6}\big)\mathrm{Tr}_7\big((xy^2)^{\mathbf{h}_7} J_{2_1 1_1 3_1 7} \cdots J_{2_n 1_n 3_n 7}\big)\\
&\times \mathrm{Tr}_8\big((xy^3)^{\mathbf{h}_8} L_{3_1 1_1 8} \cdots L_{3_n 1_n 8}\big)\mathrm{Tr}_9\big(y^{\mathbf{h}_9} J_{3_1 2_1 1_1 9} \cdots J_{3_n 2_n 1_n 9}\big)\\
&= \mathrm{Tr}_9\big(y^{\mathbf{h}_9} J_{2_1 3_1 1_1 9} \cdots J_{2_n 3_n 1_n 9}\big)\mathrm{Tr}_8\big((xy^3)^{\mathbf{h}_8} L_{1_1 3_1 8} \cdots L_{1_n 3_n 8}\big)\\
&\times \mathrm{Tr}_7\big((xy^2)^{\mathbf{h}_7} J_{1_1 2_1 3_1 7} \cdots J_{1_n 2_n 3_n 7}\big)\mathrm{Tr}_6\big((x^2y^3)^{\mathbf{h}_6} L_{3_1 2_1 6} \cdots L_{3_n 2_n 6}\big)\\
&\times \mathrm{Tr}_5\big((xy)^{\mathbf{h}_5} J_{3_1 1_1 2_1 5} \cdots J_{3_n 1_n 2_n 5}\big)\mathrm{Tr}_4\big(x^{\mathbf{h}_4} L_{2_1 1_1 4} \cdots L_{2_n 1_n 4}\big).
\end{aligned} \tag{17.23}$$

Here $\mathrm{Tr}_4(\cdots)$, $\mathrm{Tr}_6(\cdots)$, $\mathrm{Tr}_8(\cdots)$ involving the 3D L are identified with

$$S^{\mathrm{tr}}(z) := (S^{\mathrm{tr}_3}(z) \text{ in } (11.26))|_{q \to q^3} \tag{17.24}$$

up to a scalar multiple. The replacement $q \to q^3$ takes into account the comment after (17.16). It satisfies the Yang–Baxter equation (11.24) and is identified with the quantum R matrix of $U_{-q^{-3}}(A^{(1)}_{n-1})$ for the anti-symmetric tensor representations according to (Theorem 11.3)$|_{q \to q^3}$.

The other factors emerging from J have the form

$$X^{\mathrm{tr}}_{\mathbf{123}}(z) = \mathrm{Tr}_a(z^{\mathbf{h}_a} J_{1_1 2_1 3_1 a} \cdots J_{1_n 2_n 3_n a}) \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}} \otimes \overset{\mathbf{2}}{\mathbf{V}} \otimes \overset{\mathbf{3}}{\mathbf{V}}), \tag{17.25}$$

where $\overset{\mathbf{k}}{\mathbf{V}} = \overset{k_1}{V} \otimes \cdots \otimes \overset{k_n}{V} \simeq (\mathbb{C}^2)^{\otimes n}$ for $\mathbf{k} = \mathbf{1}, \mathbf{2}, \mathbf{3}$. The trace is taken over $\overset{a}{\mathcal{F}}_q$ and evaluated by means of (3.12) and (11.27). Now the relation (17.23) is rephrased as

$$\begin{aligned}
&S^{\mathrm{tr}}_{\mathbf{12}}(x) X^{\mathrm{tr}}_{\mathbf{132}}(xy) S^{\mathrm{tr}}_{\mathbf{23}}(x^2y^3) X^{\mathrm{tr}}_{\mathbf{213}}(xy^2) S^{\mathrm{tr}}_{\mathbf{31}}(xy^3) X^{\mathrm{tr}}_{\mathbf{321}}(y)\\
&= X^{\mathrm{tr}}_{\mathbf{231}}(y) S^{\mathrm{tr}}_{\mathbf{13}}(xy^3) X^{\mathrm{tr}}_{\mathbf{123}}(xy^2) S^{\mathrm{tr}}_{\mathbf{32}}(x^2y^3) X^{\mathrm{tr}}_{\mathbf{312}}(xy) S^{\mathrm{tr}}_{\mathbf{21}}(x).
\end{aligned} \tag{17.26}$$

Thus the pair $(S^{\mathrm{tr}}(z), X^{\mathrm{tr}}(z))$ yields a solution to the G_2 reflection equation (17.7) for any $n \geq 1$. Elements of $X^{\mathrm{tr}}(z)$ are rational functions of $q^{1/2}$ and z.

17.4.3 Boundary Vector Reduction

Recall the boundary vectors in (8.60) and (8.61):

$$\langle \eta_1 | = \sum_{m \geq 0} \frac{\langle m |}{(q)_m}, \qquad | \eta_1 \rangle = \sum_{m \geq 0} \frac{| m \rangle}{(q)_m}, \tag{17.27}$$

$$\langle\xi| = \sum_{m\ge 0}\frac{\langle m|}{(q^3)_m},\qquad |\xi\rangle = \sum_{m\ge 0}\frac{|m\rangle}{(q^3)_m}. \tag{17.28}$$

Sandwich the relation (17.22) between $\overset{4}{\langle\xi|}\otimes\overset{5}{\langle\eta_1|}\otimes\overset{6}{\langle\xi|}\otimes\overset{7}{\langle\eta_1|}\otimes\overset{8}{\langle\xi|}\otimes\overset{9}{\langle\eta_1|}$ and $\overset{4}{|\xi\rangle}\otimes\overset{5}{|\eta_1\rangle}\otimes\overset{6}{|\xi\rangle}\otimes\overset{7}{|\eta_1\rangle}\otimes\overset{8}{|\xi\rangle}\otimes\overset{9}{|\eta_1\rangle}$. Assuming Conjecture 8.9 and using $F = F^{-1}$ (8.22), we get

$$\begin{aligned}
&\overset{4}{\langle\xi|}x^{\mathbf{h}_4}L_{1_12_14}\cdots L_{1_n2_n4}\overset{4}{|\xi\rangle}\overset{5}{\langle\eta_1|}(xy)^{\mathbf{h}_5}J_{1_13_12_15}\cdots J_{1_n3_n2_n5}\overset{5}{|\eta_1\rangle}\\
&\quad\times\overset{6}{\langle\xi|}(x^2y^3)^{\mathbf{h}_6}L_{2_13_16}\cdots L_{2_n3_n6}\overset{6}{|\xi\rangle}\overset{7}{\langle\eta_1|}(xy^2)^{\mathbf{h}_7}J_{2_11_13_17}\cdots J_{2_n1_n3_n7}\overset{7}{|\eta_1\rangle}\\
&\quad\times\overset{8}{\langle\xi|}(xy^3)^{\mathbf{h}_8}L_{3_11_18}\cdots L_{3_n1_n8}\overset{8}{|\xi\rangle}\overset{9}{\langle\eta_1|}y^{\mathbf{h}_9}J_{3_12_11_19}\cdots J_{3_n2_n1_n9}\overset{9}{|\eta_1\rangle}\\
&=\overset{9}{\langle\eta_1|}y^{\mathbf{h}_9}J_{2_13_11_19}\cdots J_{2_n3_n1_n9}\overset{9}{|\eta_1\rangle}\overset{8}{\langle\xi|}(xy^3)^{\mathbf{h}_8}L_{1_13_18}\cdots L_{1_n3_n8}\overset{8}{|\xi\rangle}\\
&\quad\times\overset{7}{\langle\eta_1|}(xy^2)^{\mathbf{h}_7}J_{1_12_13_17}\cdots J_{1_n2_n3_n7}\overset{7}{|\eta_1\rangle}\overset{6}{\langle\xi|}(x^2y^3)^{\mathbf{h}_6}L_{3_12_16}\cdots L_{3_n2_n6}\overset{6}{|\xi\rangle}\\
&\quad\times\overset{5}{\langle\eta_1|}(xy)^{\mathbf{h}_5}J_{3_11_12_15}\cdots J_{3_n1_n2_n5}\overset{5}{|\eta_1\rangle}\overset{4}{\langle\xi|}x^{\mathbf{h}_4}L_{2_11_14}\cdots L_{2_n1_n4}\overset{4}{|\xi\rangle}.
\end{aligned}\tag{17.29}$$

The operators arising from $\langle\xi|(\cdots)|\xi\rangle$ involving L are identified, up to a scalar multiple, with

$$S^{\mathrm{bv}}(z) := (S^{1,1}(z)\ \text{in}\ (12.9))|_{q\to q^3}, \tag{17.30}$$

where the superscript "bv" indicates the boundary vector reduction. The relation of the boundary vectors (17.28) = (12.3)$|_{r=1,q\to q^3}$ has also been used for the identification. The result (12.7)$|_{r=r'=1}$ shows that $S^{\mathrm{bv}}(z)$ satisfies the Yang–Baxter equation. It is identified with the quantum R matrix of $U_p(D^{(2)}_{n+1})$ for the spin representation at $p = -q^{-3}$ according to Theorem 12.2.

The other factors emerging from J have the form

$$X^{\mathrm{bv}}_{\mathbf{123}}(z) = \kappa^{\mathrm{bv}}(z)\overset{a}{\langle\eta_1|}z^{\mathbf{h}_a}J_{1_12_13_1a}\cdots J_{1_n2_n3_na}\overset{a}{|\eta_1\rangle} \in \mathrm{End}(\overset{\mathbf{1}}{\mathbf{V}}\otimes\overset{\mathbf{2}}{\mathbf{V}}\otimes\overset{\mathbf{3}}{\mathbf{V}}), \tag{17.31}$$

$$\kappa^{\mathrm{bv}}(z) = \frac{(z;q)_\infty}{(-qz;q)_\infty}, \tag{17.32}$$

where the normalization factor $\kappa^{\mathrm{bv}}(z)$ is introduced to make elements of $X^{\mathrm{bv}}(z)$ rational functions of $q^{1/2}$ and z. Now the relation (17.29) is rephrased as

$$\begin{aligned}
&R^{\mathrm{bv}}_{\mathbf{12}}(x)X^{\mathrm{bv}}_{\mathbf{132}}(xy)R^{\mathrm{bv}}_{\mathbf{23}}(x^2y^3)X^{\mathrm{bv}}_{\mathbf{213}}(xy^2)R^{\mathrm{bv}}_{\mathbf{31}}(xy^3)X^{\mathrm{bv}}_{\mathbf{321}}(y)\\
&=X^{\mathrm{bv}}_{\mathbf{231}}(y)R^{\mathrm{bv}}_{\mathbf{13}}(xy^3)X^{\mathrm{bv}}_{\mathbf{123}}(xy^2)R^{\mathrm{bv}}_{\mathbf{32}}(x^2y^3)X^{\mathrm{bv}}_{\mathbf{312}}(xy)R^{\mathrm{bv}}_{\mathbf{21}}(x).
\end{aligned}\tag{17.33}$$

Thus the pair $(R^{\mathrm{bv}}(z), X^{\mathrm{bv}}(z))$ provides another solution to the G_2 reflection equation (17.7) for any $n \geq 1$ provided that Conjecture 8.9 holds.

17.5 Properties of $X^{\mathrm{tr}}(z)$ and $X^{\mathrm{bv}}(z)$

We use notations like $\mathfrak{s} = \{0, 1\}^n$, $\mathbf{a} = (a_1, \dots, a_n)$, $\mathbf{e}_k$, $|\mathbf{a}| = a_1 + \cdots + a_n$, $v_{\mathbf{a}} \in \mathbf{V}$ and $\mathbf{V}_k \subset \mathbf{V}$ introduced in (11.1)–(11.7). The construction (17.25) and (17.31) imply the matrix product formula for the elements as

$$X(z)(v_{\mathbf{i}} \otimes v_{\mathbf{j}} \otimes v_{\mathbf{k}}) = \sum_{\mathbf{a},\mathbf{b},\mathbf{c} \in \mathfrak{s}} X(z)^{\mathbf{abc}}_{\mathbf{ijk}}\, v_{\mathbf{a}} \otimes v_{\mathbf{b}} \otimes v_{\mathbf{c}} \quad (X = X^{\mathrm{tr}}, X^{\mathrm{bv}}), \tag{17.34}$$

$$X^{\mathrm{tr}}(z)^{\mathbf{abc}}_{\mathbf{ijk}} = \mathrm{Tr}\big(z^{\mathbf{h}} J^{a_1,b_1,c_1}_{i_1,j_1,k_1} \cdots J^{a_n,b_n,c_n}_{i_n,j_n,k_n}\big), \tag{17.35}$$

$$X^{\mathrm{bv}}(z)^{\mathbf{abc}}_{\mathbf{ijk}} = \kappa^{\mathrm{bv}}(z)\langle \eta_1 | z^{\mathbf{h}} J^{a_1,b_1,c_1}_{i_1,j_1,k_1} \cdots J^{a_n,b_n,c_n}_{i_n,j_n,k_n} | \eta_1 \rangle \tag{17.36}$$

in terms of J^{abc}_{ijk} specified in (8.39)–(8.44). They are rational functions of z and $q^{1/2}$.

From (8.46) and (8.47), $X^{\mathrm{tr}}(z)$ has the selection rule

$$X^{\mathrm{tr}}(z)^{\mathbf{abc}}_{\mathbf{ijk}} = 0 \quad \text{unless} \quad \mathbf{a} + \mathbf{b} = \mathbf{i} + \mathbf{j} \in \mathbb{Z}^n \quad \text{and} \quad n + |\mathbf{j}| - |\mathbf{k}| = |\mathbf{b}| + |\mathbf{c}| \tag{17.37}$$

or equivalently the direct sum decomposition:

$$\begin{aligned} X^{\mathrm{tr}}(z) &= \bigoplus_{l,m,k} X^{\mathrm{tr}}(z)_{l,m,k}, \\ X^{\mathrm{tr}}(z)_{l,m,k} &: \mathbf{V}_l \otimes \mathbf{V}_m \otimes \mathbf{V}_k \to \bigoplus_{k'} \mathbf{V}_{l+k+k'-n} \otimes \mathbf{V}_{m-k-k'+n} \otimes \mathbf{V}_{k'}, \end{aligned} \tag{17.38}$$

where the sums extend over $l, m, k, k' \in [0, n]$ such that the indices $l + k + k' - n$ and $m - k - k' + n$ also belong to $[0, n]$.

Similarly, (8.46) leads to the selection rule of $X^{\mathrm{bv}}(z)$ as

$$X^{\mathrm{bv}}(z)^{\mathbf{abc}}_{\mathbf{ijk}} = 0 \quad \text{unless} \quad \mathbf{a} + \mathbf{b} = \mathbf{i} + \mathbf{j} \in \mathbb{Z}^n. \tag{17.39}$$

Example 17.1 We temporarily write $v_{\mathbf{a}}$ as $|\mathbf{a}\rangle$ to magnify the array $\mathbf{a}$. We set $\mathbf{e}_{[1,m]} = \mathbf{e}_1 + \cdots + \mathbf{e}_m$. In particular, $|\mathbf{0}\rangle = |0, \dots, 0\rangle$ and $|\mathbf{1}\rangle = |\mathbf{e}_{[1,n]}\rangle = |1, \dots, 1\rangle$.

$$\begin{aligned} &X^{\mathrm{tr}}(z)(|\mathbf{e}_{[1,l]}\rangle \otimes |\mathbf{e}_{[1,m]}\rangle \otimes |\mathbf{0}\rangle) \\ &= \frac{(q^{\frac{1}{2}})^{m-l+n}}{1 - zq^{m-l+n}} |\mathbf{e}_{[1,l]}\rangle \otimes |\mathbf{e}_{[1,m]}\rangle \otimes |\mathbf{1}\rangle + \cdots \quad (l \leq m), \end{aligned} \tag{17.40}$$

$$X^{\mathrm{tr}}(z)(|\mathbf{e}_{[1,l]}\rangle \otimes |\mathbf{e}_{[1,m]}\rangle \otimes |\mathbf{1}\rangle)$$
$$= \frac{(-q^{\frac{1}{2}})^{l-m+n}}{1-zq^{l-m+n}}|\mathbf{e}_{[1,l]}\rangle \otimes |\mathbf{e}_{[1,m]}\rangle \otimes |\mathbf{0}\rangle + \cdots \quad (l \geq m), \tag{17.41}$$
$$X^{\mathrm{bv}}(z)(|\mathbf{e}_{[1,l]}\rangle \otimes |\mathbf{e}_{[1,m]}\rangle \otimes |\mathbf{0}\rangle)$$
$$= q^{\frac{m-l+n}{2}}\frac{(z;q)_{m-l+n}}{(-qz;q)_{m-l+n}}|\mathbf{e}_{[1,l]}\rangle \otimes |\mathbf{e}_{[1,m]}\rangle \otimes |\mathbf{1}\rangle + \cdots \quad (l \leq m), \tag{17.42}$$
$$X^{\mathrm{bv}}(z)(|\mathbf{e}_{[1,l]}\rangle \otimes |\mathbf{e}_{[1,m]}\rangle \otimes |\mathbf{1}\rangle)$$
$$= (-q^{\frac{1}{2}})^{l-m+n}\frac{(z;q)_{l-m+n}}{(-qz;q)_{l-m+n}}|\mathbf{e}_{[1,l]}\rangle \otimes |\mathbf{e}_{[1,m]}\rangle \otimes |\mathbf{0}\rangle + \cdots \quad (l \geq m). \tag{17.43}$$

Example 17.2 Let us present examples of $X^{\mathrm{tr}}(z)$. We temporarily write $v_{\mathbf{a}} \otimes v_{\mathbf{b}} \otimes v_{\mathbf{c}}$ as $|\mathbf{a}, \mathbf{b}, \mathbf{c}\rangle$ for short. For $n = 1$, $X^{\mathrm{tr}}(z)$ acts on $\mathbf{V}^{\otimes 3} = V^{\otimes 3}$ as

$$|0,0,0\rangle \mapsto \frac{q^{\frac{1}{2}}|0,0,1\rangle}{1-qz}, \quad |0,0,1\rangle \mapsto -\frac{q^{\frac{1}{2}}|0,0,0\rangle}{1-qz}, \quad |0,1,0\rangle \mapsto \frac{q|0,1,1\rangle}{1-q^2z},$$
$$|0,1,1\rangle \mapsto -\frac{u_1u_3(q^2-z)|0,1,0\rangle}{\rho(1-z)(1-q^2z)} - \frac{u_3u_4(q^2-z)|1,0,1\rangle}{\rho(1-z)(1-q^2z)},$$
$$|1,0,0\rangle \mapsto -\frac{u_1u_2(q^2-z)|0,1,0\rangle}{\rho(1-z)(1-q^2z)} - \frac{u_2u_4(q^2-z)|1,0,1\rangle}{\rho(1-z)(1-q^2z)},$$
$$|1,0,1\rangle \mapsto \frac{q|1,0,0\rangle}{1-q^2z}, \quad |1,1,0\rangle \mapsto \frac{q^{\frac{1}{2}}|1,1,1\rangle}{1-qz}, \quad |1,1,1\rangle \mapsto -\frac{q^{\frac{1}{2}}|1,1,0\rangle}{1-qz},$$

where ρ defined in (8.45) and u_1, u_2, u_3, u_4 are to obey (8.10). The two kinds of the denominators $1-qz$ and $1-q^2z$ originate in $J^{001}_{000} = \hat{\mathbf{k}}$ and $J^{011}_{010} = \hat{\mathbf{k}}^2$.

For $n = 2$, it is too lengthy to present all the data. So we give just a few examples:

$$|00,00,00\rangle \mapsto \frac{q|00,00,11\rangle}{1-q^2z}, \quad |00,00,01\rangle \mapsto \frac{(1-q^2)z|00,00,01\rangle}{(1-z)(1-q^2z)} - \frac{q|00,00,10\rangle}{1-q^2z},$$
$$|00,10,11\rangle \mapsto \frac{q^{\frac{3}{2}}u_1u_3(q-z)|00,10,00\rangle}{\rho(1-qz)(1-q^3z)} - \frac{q^{\frac{1}{2}}(1-q^2)u_3z|10,00,01\rangle}{(1-qz)(1-q^3z)}$$
$$+ \frac{q^{\frac{3}{2}}u_3u_4(q-z)|10,00,10\rangle}{\rho(1-qz)(1-q^3z)},$$
$$|10,01,01\rangle \mapsto \frac{u_1^2u_2u_3(q^4+z-2q^2z-2q^4z+q^6z+q^2z^2)|00,11,00\rangle)}{\rho^2(1-z)(1-q^2z)(1-q^4z)}$$
$$+ \frac{u_1u_2u_3u_4(q^4+z-2q^2z-2q^4z+q^6z+q^2z^2)|01,10,01\rangle}{\rho^2(1-z)(1-q^2z)(1-q^4z)}$$
$$- \frac{q(1-q^2)u_2u_3|01,10,10\rangle}{(1-q^2z)(1-q^4z)} - \frac{q(1-q^2)u_2u_3z|10,01,01\rangle}{(1-q^2z)(1-q^4z)}$$

$$+\frac{u_1u_2u_3u_4(q^4+z-2q^2z-2q^4z+q^6z+q^2z^2)|10,01,10\rangle}{\rho^2(1-z)(1-q^2z)(1-q^4z)}$$
$$+\frac{u_2u_3u_4^2(q^4+z-2q^2z-2q^4z+q^6z+q^2z^2)|11,00,11\rangle}{\rho^2(1-z)(1-q^2z)(1-q^4z)}.$$

Example 17.3 $S^{\mathrm{bv}}(z)$ with $n = 1$ is available in Example 12.1 with $r = r' = 1$ and the replacement $q \to q^3$. Let us present examples of $X^{\mathrm{bv}}(z)$ with $n = 1$ using the same notation as Example 17.2. It acts on $\mathbf{V}^{\otimes 3} = V^{\otimes 3}$ as

$$|0,0,0\rangle \mapsto \frac{(1+q)z|0,0,0\rangle}{1+qz}+\frac{q^{\frac{1}{2}}(1-z)|0,0,1\rangle}{1+qz},$$
$$|0,0,1\rangle \mapsto -\frac{q^{\frac{1}{2}}(1-z)|0,0,0\rangle}{1+qz}+\frac{(1+q)|0,0,1\rangle}{1+qz},$$
$$|0,1,1\rangle \mapsto \frac{q^{\frac{3}{2}}(1+q)u_1(1-z)z|0,1,0\rangle}{(1+qz)(1+q^2z)}+\frac{q(1-z)(1-qz)|0,1,1\rangle}{(1+qz)(1+q^2z)}$$
$$+\frac{(1+q)(1+q^2)z^2|1,0,0\rangle}{(1+qz)(1+q^2z)}+\frac{q^{\frac{3}{2}}(1+q)u_4(1-z)z|1,0,1\rangle}{(1+qz)(1+q^2z)},$$
$$|0,1,1\rangle \mapsto \frac{u_3(-q^2+z+2qz+2q^2z+q^3z-qz^2)(u_1|0,1,0\rangle+u_4|1,0,1\rangle)}{\rho(1+qz)(1+q^2z)}$$
$$+\frac{q^{\frac{1}{2}}(1+q)u_3(1-z)(|0,1,1\rangle-z|1,0,0\rangle)}{(1+qz)(1+q^2z)},$$
$$|1,0,0\rangle \mapsto \frac{u_2(-q^2+z+2qz+2q^2z+q^3z-qz^2)(u_1|0,1,0\rangle+u_4|1,0,1\rangle)}{\rho(1+qz)(1+q^2z)}$$
$$+\frac{q^{\frac{1}{2}}(1+q)u_2(1-z)(|0,1,1\rangle-z|1,0,0\rangle)}{(1+qz)(1+q^2z)},$$
$$|1,0,1\rangle \mapsto -\frac{q^{\frac{3}{2}}(1+q)u_1(1-z)|0,1,0\rangle}{(1+qz)(1+q^2z)}+\frac{(1+q)(1+q^2)|0,1,1\rangle}{(1+qz)(1+q^2z)}$$
$$+\frac{q(1-z)(1-qz)|1,0,0\rangle}{(1+qz)(1+q^2z)}-\frac{q^{\frac{3}{2}}(1+q)u_4(1-z)|1,0,1\rangle}{(1+qz)(1+q^2z)},$$
$$|1,1,0\rangle \mapsto \frac{(1+q)z|1,1,0\rangle}{1+qz}+\frac{q^{\frac{1}{2}}(1-z)|1,1,1\rangle}{1+qz},$$
$$|1,1,1\rangle \mapsto -\frac{q^{\frac{1}{2}}(1-z)|1,1,0\rangle}{1+qz}+\frac{(1+q)|1,1,1\rangle}{1+qz}.$$

17.6 Bibliographical Notes and Comments

This chapter is based on [85]. The G_2 reflection equation (17.3) or (17.7) up to spectral parameters was suggested on [30, p. 982], where the Desargues–Pappus geometry of the G_2 scattering diagram was mentioned instead of the equation itself. The equation of the form (17.3) for generic symbols R and X without assuming a tensor product structure of their representation space (i.e. without indices) has appeared as a defining relation of the *root algebra* of type G_2 in [31, Sect. 2].

The reduction procedures in Sect. 17.4 are parallel with earlier chapters. The intertwiner F of $A_q(G_2)$ is eliminated in an early stage but it controls the matrix product construction essentially.

It is an outstanding problem whether the solution $X^{\mathrm{tr}}(z)$ and the conjectural solution $X^{\mathrm{bv}}(z)$ admit a characterization analogous to Theorems 15.3 and 16.2 by some sort of quantum group theoretical structure like coideals.

Chapter 18
Application to Multispecies TASEP

Abstract This chapter is an exposition of a 3D approach to an integrable Markov process called the n species totally asymmetric simple exclusion process (n-TASEP). The main result is a matrix product formula of the stationary probability involving layer transfer matrices of the $q = 0$-oscillator-valued five-vertex model on an $n \times n$ lattice. The stationary condition is translated into their quadratic relations, the so-called Faddeev–Zamolodchikov algebra, which are highly non-local from the viewpoint of the five-vertex model. They are shown to be a far-reaching consequence of the single tetrahedron equation of type $MMLL = LLMM$ in Sect. 2.6 and its solution in Theorem 3.25.

18.1 Introduction

The totally asymmetric simple exclusion process (TASEP) is a continuous-time Markov process of particles obeying a stochastic dynamics governed by a master equation. We consider the n-TASEP on the 1D periodic lattice $\mathbb{Z}_L$, where each site variable assumes $\{0, 1, \dots, n\}$ (Fig. 18.1).

The first basic problem is the determination of the stationary state, which is analogous to the ground state of quantum spin chains. The probability of finding a given particle configuration in the stationary state is called the stationary probability. It is an analogue of the amplitude of a configuration in the ground state for quantum spin chains. In integrable situations, the amplitude should be obtained by the Bethe ansatz, therefore it is *transcendental* in general since the Bethe roots are so. On the other hand, the stationary state is the unique null eigenvector of the Markov matrix, implying that it should be *algebraic* with respect the parameters of the model. These arguments suggest that stationary probabilities of integrable Markov processes should be something between transcendental and algebraic, and it is the place where the matrix product structure emerges naturally.

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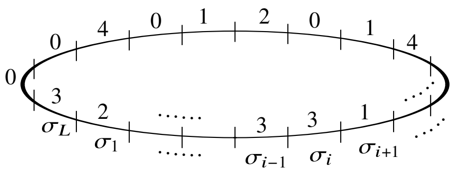


Fig. 18.1 A configuration of particles in n-TASEP ($n \geq 4$)

The n-TASEP considered in this chapter is indeed integrable, being a special case of a more general partially asymmetric simple exclusion process or the $A^{(1)}_n$ vertex model with the standard nested Bethe ansatz solution, e.g. [4]. However, providing a full combinatorial description with the stationary probabilities is another problem, which we are going to address in this chapter by a 3D approach. As we will note in Remark 18.7, it leads to an intriguing duality between $A^{(1)}_n$ and $A^{(1)}_{L-1}$ exchanging the role of internal and external spaces.

18.2 *n*-TASEP

18.2.1 Definition of n-TASEP

Consider the periodic 1D chain with L sites $\mathbb{Z}_L$. Each site $i \in \mathbb{Z}_L$ is populated with a local state $\sigma_i \in \{0, 1, \dots, n\}$. It is interpreted as a species of the particle occupying the site i.[1] We assume $1 \leq n < L$. Consider a stochastic model on $\mathbb{Z}_L$ such that neighboring pairs of local states $(\sigma_i, \sigma_{i+1}) = (\alpha, \beta)$ are interchanged as $\alpha\,\beta \to \beta\,\alpha$ if $\alpha > \beta$ with the uniform transition rate. The space of states is given by

$$(\mathbb{C}^{n+1})^{\otimes L} \simeq \bigoplus_{(\sigma_1,\dots,\sigma_L)\in\{0,\dots,n\}^L} \mathbb{C}|\sigma_1, \dots, \sigma_L\rangle. \tag{18.1}$$

Let $\mathbb{P}(\sigma_1, \dots, \sigma_L; t)$ be the probability of finding the configuration $(\sigma_1, \dots, \sigma_L)$ at time t, and set

$$|P(t)\rangle = \sum_{(\sigma_1,\dots,\sigma_L)\in\{0,\dots,n\}^L} \mathbb{P}(\sigma_1, \dots, \sigma_L; t)|\sigma_1, \dots, \sigma_L\rangle. \tag{18.2}$$

By n-TASEP we mean the stochastic system governed by the continuous-time master equation

$$\frac{d}{dt}|P(t)\rangle = H|P(t)\rangle, \tag{18.3}$$

[1] $\sigma_i = 0$ is may be regarded as an empty site. In such an interpretation, there are n species of particles.

where H is a Markov matrix defined by

$$H = \sum_{i \in \mathbb{Z}_L} h_{i,i+1}, \qquad h|\alpha, \beta\rangle = \begin{cases} |\beta, \alpha\rangle - |\alpha, \beta\rangle & (\alpha > \beta), \\ 0 & (\alpha \leq \beta). \end{cases} \tag{18.4}$$

Here $h_{i,i+1}$ is the local Markov matrix that acts as h on the ith and the $(i+1)$th components and as the identity elsewhere. The master equation (18.3) preserves the total probability.

The Markov matrix H preserves the subspaces, called *sectors*, consisting of the configurations with prescribed *multiplicity* $\mathbf{m} = (m_0, \dots, m_n) \in (\mathbb{Z}_{\geq 0})^{n+1}$ of particles:

$$\mathcal{S}(\mathbf{m}) = \{\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_L) \in \{0, \dots, n\}^L \mid \sum_{1 \leq j \leq L} \delta_{k,\sigma_j} = m_k, \forall k\}. \tag{18.5}$$

The space of states (18.1) is decomposed as $\bigoplus_{\mathbf{m}} \bigoplus_{\boldsymbol{\sigma} \in \mathcal{S}(\mathbf{m})} \mathbb{C}|\boldsymbol{\sigma}\rangle$, where the outer sum ranges over $m_i \in \mathbb{Z}_{\geq 0}$ such that $m_0 + \cdots + m_n = L$. A sector $\bigoplus_{\boldsymbol{\sigma} \in \mathcal{S}(\mathbf{m})} \mathbb{C}|\boldsymbol{\sigma}\rangle$ such that $m_i \geq 1$ for all $0 \leq i \leq n$ is called *basic*. Non-basic sectors are equivalent to a basic sector for n'-TASEP with some $n' < n$ by a suitable relabeling of species. Thus we shall exclusively deal with basic sectors, therefore $n < L$ is assumed as mentioned before. The spectrum of H is known to exhibit a remarkable duality described by a Hasse diagram [4].

18.2.2 Stationary States

In each sector $\bigoplus_{\boldsymbol{\sigma} \in \mathcal{S}(\mathbf{m})} \mathbb{C}|\boldsymbol{\sigma}\rangle$ there is a unique vector $|\bar{P}(\mathbf{m})\rangle$ up to normalization, called the *stationary state*, satisfying $H|\bar{P}(\mathbf{m})\rangle = 0$. The stationary state for 1-TASEP is trivial under the periodic boundary condition in the sense that all the monomials have the same coefficient, i.e. all the configurations are realized with an equal probability.

Example 18.1 Let us present (unnormalized) stationary states in small sectors of 2-TASEP and 3-TASEP in the form

$$|\bar{P}(\mathbf{m})\rangle = |\xi(\mathbf{m})\rangle + C|\xi(\mathbf{m})\rangle + \cdots + C^{L-1}|\xi(\mathbf{m})\rangle \tag{18.6}$$

respecting the translational symmetry $HC = CH$ under the $\mathbb{Z}_L$ cyclic shift $C|\sigma_1, \sigma_2, \dots, \sigma_L\rangle = |\sigma_L, \sigma_1, \dots, \sigma_{L-1}\rangle$. The choice of the vector $|\xi(\mathbf{m})\rangle$ is not unique.

$$
\begin{aligned}
|\xi(1,1,1)\rangle &= 2|012\rangle + |102\rangle,\\
|\xi(2,1,1)\rangle &= 3|0012\rangle + 2|0102\rangle + |1002\rangle,\\
|\xi(1,2,1)\rangle &= 2|0112\rangle + |1012\rangle + |1102\rangle,\\
|\xi(1,1,2)\rangle &= 3|1220\rangle + 2|2120\rangle + |2210\rangle,\\
|\xi(1,2,2)\rangle &= 3|11220\rangle + 2|12120\rangle + |12210\rangle\\
&\quad + 2|21120\rangle + |21210\rangle + |22110\rangle,\\
|\xi(2,1,2)\rangle &= \underline{|00221\rangle} + 2|02021\rangle + 3|02201\rangle\\
&\quad + 3|20021\rangle + \underline{5|20201\rangle} + 6|22001\rangle,\\
|\xi(2,2,1)\rangle &= 3|00112\rangle + 2|01012\rangle + 2|01102\rangle\\
&\quad + |10012\rangle + |10102\rangle + |11002\rangle,\\
|\xi(1,1,1,1)\rangle &= 9|0123\rangle + 3|0213\rangle + 3|1023\rangle\\
&\quad + 5|1203\rangle + 3|2013\rangle + |2103\rangle,\\
|\xi(2,1,1,1)\rangle &= 24|00123\rangle + 6|00213\rangle + 12|01023\rangle + 17|01203\rangle\\
&\quad + 8|02013\rangle + 3|02103\rangle + 4|10023\rangle + 7|10203\rangle\\
&\quad + 9|12003\rangle + 6|20013\rangle + 3|20103\rangle + |21003\rangle,\\
|\xi(1,2,1,1)\rangle &= 12|01123\rangle + 5|01213\rangle + 3|02113\rangle + 4|10123\rangle\\
&\quad + 3|10213\rangle + 4|11023\rangle + 7|11203\rangle + 5|12013\rangle\\
&\quad + 2|12103\rangle + 3|20113\rangle + |21013\rangle + |21103\rangle,\\
|\xi(1,1,2,1)\rangle &= 12|01223\rangle + 5|02123\rangle + 3|02213\rangle + 3|10223\rangle\\
&\quad + 5|12023\rangle + 7|12203\rangle + 4|20123\rangle + 3|20213\rangle\\
&\quad + |21023\rangle + 2|21203\rangle + 4|22013\rangle + |22103\rangle,\\
|\xi(1,1,1,2)\rangle &= 24|12330\rangle + 12|13230\rangle + 4|13320\rangle + 6|21330\rangle\\
&\quad + 8|23130\rangle + 6|23310\rangle + 17|31230\rangle + 7|31320\rangle\\
&\quad + 3|32130\rangle + 3|32310\rangle + 9|33120\rangle + |33210\rangle.
\end{aligned}
$$

The red underlines are put for convenience for Example 18.3. As these coefficients indicate, stationary states are non-trivial for $n \geq 2$. The theme of this chapter is to elucidate a 3D integrability behind them, which will ultimately be related to the tetrahedron equation.

18.2.3 Matrix Product Formula

Consider a stationary state

$$
|\bar{P}(\mathbf{m})\rangle = \sum_{\boldsymbol{\sigma} \in \mathcal{S}(\mathbf{m})} \mathbb{P}(\boldsymbol{\sigma})|\boldsymbol{\sigma}\rangle \tag{18.7}
$$

and postulate that the stationary probability $\mathbb{P}(\boldsymbol{\sigma})$ is expressed in the matrix product form

$$\mathbb{P}(\sigma_1, \ldots, \sigma_L) = \mathrm{Tr}(X_{\sigma_1} \cdots X_{\sigma_L}) \tag{18.8}$$

in terms of some operators $X_0, \ldots, X_n$. Introduce the notations for the matrix elements of the local Markov matrix (18.4) and the associated product of X_i's as

$$h|\alpha, \beta\rangle = \sum_{\gamma,\delta} h^{\gamma,\delta}_{\alpha,\beta}|\gamma, \delta\rangle, \qquad (hXX)_{\alpha,\beta} := \sum_{\gamma,\delta} h^{\alpha,\beta}_{\gamma,\delta} X_\gamma X_\delta, \tag{18.9}$$

where both sums range over $\gamma, \delta \in \{0, 1, \ldots, n\}$. Then we have

$$\begin{aligned} H|\bar{P}(\mathbf{m})\rangle &= \sum_{i\in\mathbb{Z}_L} \sum_{\boldsymbol{\sigma}\in\mathcal{S}(\mathbf{m})} \mathbb{P}(\ldots, \sigma_i, \sigma_{i+1}, \ldots) h_{i,i+1}|\ldots, \sigma_i, \sigma_{i+1}, \ldots\rangle \\ &= \sum_{i\in\mathbb{Z}_L} \sum_{\boldsymbol{\sigma}\in\mathcal{S}(\mathbf{m})} \sum_{\sigma'_i,\sigma'_{i+1}} \mathrm{Tr}(\cdots X_{\sigma_i} X_{\sigma_{i+1}} \cdots) h^{\sigma'_i,\sigma'_{i+1}}_{\sigma_i,\sigma_{i+1}}|\ldots, \sigma'_i, \sigma'_{i+1}, \ldots\rangle \\ &= \sum_{\boldsymbol{\sigma}\in\mathcal{S}(\mathbf{m})} \sum_{i\in\mathbb{Z}_L} \mathrm{Tr}(\cdots (hXX)_{\sigma_i,\sigma_{i+1}} \cdots)|\ldots, \sigma_i, \sigma_{i+1}, \ldots\rangle. \end{aligned} \tag{18.10}$$

Therefore if there is another set of operators $\hat{X}_0, \ldots, \hat{X}_n$ obeying the so-called *hat relation*

$$(hXX)_{\alpha,\beta} = X_\alpha \hat{X}_\beta - \hat{X}_\alpha X_\beta, \tag{18.11}$$

the stationary condition $H|\bar{P}(\mathbf{m})\rangle = 0$ holds thanks to the cyclicity of the trace. Then the trace (18.8), if convergent, must coincide with the actual stationary probability up to overall normalization due to the uniqueness of the stationary state in every sector. Note, on the other hand, that $\hat{X}_i$ satisfying the hat relation with a given X_i is not unique. For instance, $\hat{X}_i \to \hat{X}_i + cX_i$ leaves (18.11) unchanged.

From (18.4) and (18.9), the hat relation (18.11) is given concretely as

$$[X_i, \hat{X}_j] = [\hat{X}_i, X_j] \qquad (0 \le i, j \le n), \tag{18.12}$$

$$X_i X_j = \hat{X}_i X_j - X_i \hat{X}_j \qquad (0 \le j < i \le n). \tag{18.13}$$

Suppose we have the operators $X_0(z), \ldots, X_n(z)$ which depend on a spectral parameter z and satisfy

$$[X_i(x), X_j(y)] = [X_i(y), X_j(x)] \qquad (0 \le i, j \le n), \tag{18.14}$$

$$xX_i(y)X_j(x) = yX_i(x)X_j(y) \qquad (0 \le j < i \le n). \tag{18.15}$$

Then the hat relations (18.12) and (18.13) are realized by setting

$$X_i = X_i(z=1), \qquad \hat{X}_i = \frac{d}{dz}X_i(z)|_{z=1} \qquad (0 \le i \le n). \tag{18.16}$$

The relations (18.14) and (18.15) are rearranged as

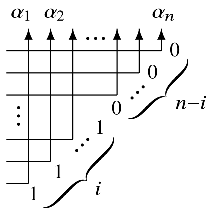
$$X_i(x)X_j(y) = \begin{cases} X_i(y)X_j(x) + (1-\frac{x}{y})X_j(y)X_i(x) & i < j, \\ X_i(y)X_i(x) & i = j, \\ \frac{x}{y}X_i(y)X_j(x) & i > j. \end{cases} \tag{18.17}$$

This exchange rule satisfies the Yang–Baxter relation in the sense that the two ways of rewriting $X_i(x)X_j(y)X_k(z)$ as linear combinations of $X_{k'}(z)X_{j'}(y)X_{i'}(x)$ with $\{i', j', k'\} = \{i, j, k\}$ lead to the identical result. Such a quadratic exchange rule, which is sometimes referred to as Faddeev–Zamolodchikov algebra, is a signal of the integrable structure of the n-TASEP. In this way, seeking the matrix product formula (18.8) of the stationary probability is transformed to the problem of constructing the operator $X_i(z)$ satisfying (18.17).

18.2.4 Matrix Product Operator $X_i(z)$

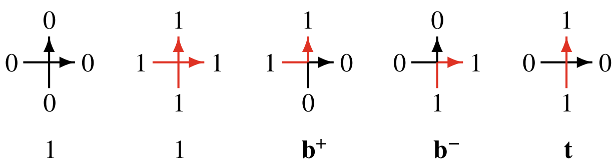
We will show that the following operator $X_i(z)$ fulfills (18.17):

$$X_i(z) = \sum z^{\alpha_1+\cdots+\alpha_n}$$



(18.18)

It represents a configuration sum, i.e. the partition function of the $\mathrm{Osc}_{q=0}$-*valued five-vertex model* on the size n triangular shape region with a prescribed condition along the SE boundary. The $\mathrm{Osc}_{q=0}$-valued five-vertex model has the local "Boltzmann weight" 1, $\mathbf{b}^+$, $\mathbf{b}^-$, $\mathbf{t}$ assigned to each vertex as



(18.19)

Each edge of (18.18) takes 0 or 1 and the sum (except those fixed on the SE boundary) extends over all the configurations such that every vertex is one of the above five types.[2] In (18.19), edges assuming 0 and 1 are colored in black and red respectively. This convention will apply in the rest of the chapter.[3] Given such a configuration, the summand is the *tensor product* of the local Boltzmann weights $1, \mathbf{b}^+, \mathbf{b}^-, \mathbf{t}$. They are linear operators on the Fock space $\mathcal{F} = \bigoplus_{m\geq 0} \mathbb{C}|m\rangle$[4] defined by

$$\mathbf{b}^+|m\rangle = |m+1\rangle, \quad \mathbf{b}^-|m\rangle = |m-1\rangle, \quad \mathbf{t}|m\rangle = \delta_{m,0}|m\rangle, \tag{18.20}$$

which obey the relations

$$\mathbf{t}\,\mathbf{b}^+ = 0, \qquad \mathbf{b}^-\,\mathbf{t} = 0, \qquad \mathbf{b}^+\,\mathbf{b}^- = 1 - \mathbf{t}, \qquad \mathbf{b}^-\,\mathbf{b}^+ = 1. \tag{18.21}$$

The relations (18.20) and (18.21) are identified with the q-oscillator ones (3.13) and (3.12) in the well defined limit

$$\mathbf{b}^\pm = \lim_{q\to 0} \mathbf{a}^\pm, \qquad \mathbf{t} = \lim_{q\to 0} \mathbf{k}, \tag{18.22}$$

where an extra relation $\mathbf{t}^2 = \mathbf{t}$ is acquired. The $\mathrm{Osc}_{q=0}$ operators $\mathbf{b}^\pm, \mathbf{t}$ attached to different vertices act on different copies of $\mathcal{F}$. Thus $X_i(z) \in \mathrm{End}(\mathcal{F}^{\otimes n(n-1)/2})$.

The trace in (18.8) is taken over $\mathcal{F}^{\otimes n(n-1)/2}$, where each component is calculated by $\mathrm{Tr}_{\mathcal{F}}(X) = \sum_{m\geq 0}\langle m|X|m\rangle$ with $\langle m|m'\rangle = \delta_{m,m'}$. See (3.16) and the explanation after Fig. 11.3. Finally, the summands in (18.18) are attached with the overall factor $z^{\alpha_1+\cdots+\alpha_n}$, where $\alpha_i = 0, 1$ is the variable on the ith vertical edge from the left on the top.

The matrix product operator $X_i(z)$ has the form of a *corner transfer matrix* [10, Chap. 13] of the $\mathrm{Osc}_{q=0}$-valued five-vertex model, although it acts along the perpendicular direction to the layer as opposed to the usual 2D setting. Equivalently, one may view it as a layer transfer matrix of the 3D lattice model where the edges perpendicular to the plane (18.18) are assigned with $\mathcal{F}$. The stationary probability (18.8) is then interpreted as a *partition function* of the 3D system of prism shape which is periodic along the third direction.

Remark 18.2 The result (18.8) with X_i defined by (18.16) and (18.18) corresponds to the *integer normalization*

$$\mathbb{P}(\sigma_1, \ldots, \sigma_L) = 1 \quad \text{for} \quad \sigma_1 \geq \cdots \geq \sigma_{\mathrm{L}}.$$

In this normalization $\mathbb{P}(\boldsymbol{\sigma}) \in \mathbb{Z}_{\geq 1}$ holds for all the state $\boldsymbol{\sigma} \in \mathcal{S}(\mathbf{m})$.

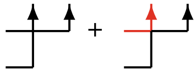
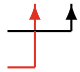
---

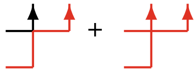
[2] At the SE boundary in (18.18), we do not assign $1, \mathbf{b}^+, \mathbf{b}^-, \mathbf{t}$, and just let arrows make 90° left turns without changing the edge variable. See Examples 18.3 and 18.4.

[3] Although, in some formulas like (18.18), those black edges not on the SE boundary should be understood as taking both 0 or 1.

[4] The ket vector here should not be confused with the TASEP states in Sects. 18.2.1–18.2.3. We take $|-1\rangle = 0,\ 1|m\rangle = |m\rangle$ for granted.

Example 18.3 For $n = 2$ the operators $X_0(z)$, $X_1(z)$, $X_2(z)$ are given by

$X_0(z) =$  $= 1 + z\mathbf{b}^+$, $X_1(z) =$  $= z\mathbf{t}$,

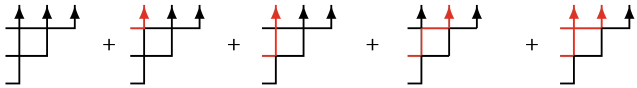
$X_2(z) =$  $= z\mathbf{b}^- + z^2$.

From (18.16) we have $\hat{X}_0 = \mathbf{b}^+$, $\hat{X}_1 = \mathbf{t}$, $\hat{X}_2 = \mathbf{b}^- + 2$. For instance,

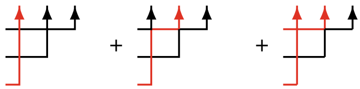
$$\mathbb{P}(00221) = \mathrm{Tr}(X_0X_0X_2X_2X_1) = \mathrm{Tr}\big((1+\mathbf{b}^+)(1+\mathbf{b}^+)(1+\mathbf{b}^-)(1+\mathbf{b}^-)\mathbf{t}\big) = 1,$$
$$\mathbb{P}(20201) = \mathrm{Tr}(X_2X_0X_2X_0X_1) = \mathrm{Tr}\big((1+\mathbf{b}^-)(1+\mathbf{b}^+)(1+\mathbf{b}^-)(1+\mathbf{b}^+)\mathbf{t}\big) = 5,$$

which reproduce the coefficients in the underlined terms in $|\xi(2,1,2)\rangle$ in Example 18.1. As this example indicates, for the convergence of the trace, it is sufficient to have at least one $\mathbf{t}$ for every $\mathcal{F}$ component of $\mathrm{Tr}_{\mathcal{F}^{\otimes n(n-1)/2}}(X_{\sigma_1}\cdots X_{\sigma_L})$.

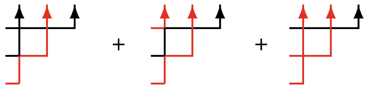
Example 18.4 For $n = 3$, the operators $X_0(z), \ldots, X_3(z)$ are given by

$X_0(z) =$ 

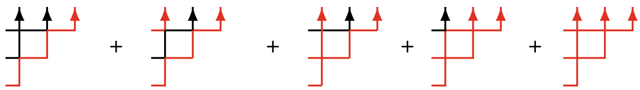
$= 1\otimes 1\otimes 1 + z\mathbf{b}^+\otimes 1\otimes 1 + z\mathbf{t}\otimes\mathbf{b}^+\otimes 1 + z\mathbf{b}^-\otimes\mathbf{b}^+\otimes\mathbf{b}^+ + z^2 1\otimes\mathbf{b}^+\otimes\mathbf{b}^+$,

$X_1(z) =$ 

$= z\mathbf{t}\otimes\mathbf{t}\otimes 1 + z\mathbf{b}^-\otimes\mathbf{t}\otimes\mathbf{b}^+ + z^2 1\otimes\mathbf{t}\otimes\mathbf{b}^+$,

$X_2(z) =$ 

$= z1\otimes\mathbf{b}^-\otimes\mathbf{t} + z^2\mathbf{b}^+\otimes\mathbf{b}^-\otimes\mathbf{t} + z^2\mathbf{t}\otimes 1\otimes\mathbf{t}$,

$X_3(z) =$ 

$= z1\otimes\mathbf{b}^-\otimes\mathbf{b}^- + z^2\mathbf{b}^+\otimes\mathbf{b}^-\otimes\mathbf{b}^- + z^2\mathbf{t}\otimes 1\otimes\mathbf{b}^- + z^2\mathbf{b}^-\otimes 1\otimes 1 + z^3 1\otimes 1\otimes 1$.

Here and in what follows, the components of the tensor product will always be ordered so that they correspond, from left to right, to the vertices from the top to the bottom and from the left to the right.

To summarize so far, we are to show:

Theorem 18.5 *The operators $X_0(z), \ldots, X_n(z)$ defined by (18.18) satisfy the Faddeev–Zamolodchikov algebra relations (18.14) and (18.15).*

From the viewpoint of the five-vertex model, this is a highly non-local property. Our goal in the rest of the chapter is to reveal that Theorem 18.5 is a far-reaching consequence of the *single* local relation which is nothing but the tetrahedron equation.

Remark 18.6 The five vertices in (18.19) are identified with those for the 3D L $\mathcal{L}(z=1)^{ab}_{ij}$ in (18.25) at $q=0$. See (18.22). Therefore each Fock space component of the trace (18.8) takes the form $\mathrm{Tr}_{\mathcal{F}}(\mathcal{L}(1)^{a_1,b_1}_{i_1,j_1}\cdots\mathcal{L}(1)^{a_L,b_L}_{i_L,j_L})$. It coincides with the matrix product formula (11.26) of a quantum R matrix $S^{\mathrm{tr}_3}(z)$ at $z=1, q=0$ up to an overall factor and the conjugation by $(\sigma\otimes\sigma)$.[5] The coincidence leads to a further reformulation of the stationary probability in terms of a composition of the quantum R matrices at $q=0$ [89]. An important consequence of it is the convergence of the trace. In fact, it assures that at least one $\mathbf{t}$ is included in $\mathcal{L}(1)^{a_1,b_1}_{i_1,j_1},\ldots,\mathcal{L}(1)^{a_L,b_L}_{i_L,j_L}$ for every Fock space $\mathcal{F}$ provided that we are in a basic sector defined after (18.5).

Remark 18.7 Another notable feature of the observation in Remark 18.6 is that the relevant quantum affine algebra becomes $U_p(A^{(1)}_{L-1})$ rather than $U_p(A^{(1)}_n)$. Thus, dealing with n-TASEP on the periodic lattice $\mathbb{Z}_L$ eventually leads to the size n system (18.18) with "symmetry algebra" of rank $L-1$. It is another manifestation of the duality mentioned in the second last paragraph of Sect. 15.6.

18.3 3D L, M Operators and the Tetrahedron Equation

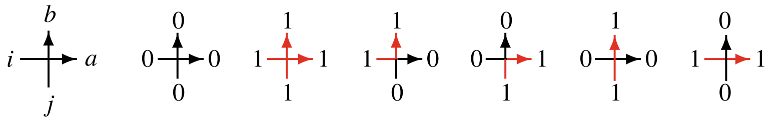
We invoke the results in Sect. 3.5.3. Let $V=\mathbb{C}v_0\oplus\mathbb{C}v_1$ and $L(z), M(z)\in\mathrm{End}(V\otimes V\otimes\mathcal{F}_q)$ be the 3D L and M operators defined in (3.118)–(3.121). They contain the parameters μ and ν, respectively. In this chapter, for reasons of convention, we will work with

$$\mathcal{L}(z)=(\sigma\otimes\sigma\otimes 1)L(z)|_{\mu=1}(\sigma\otimes\sigma\otimes 1)=\sum_{a,b,i,j}E_{ai}\otimes E_{bj}\otimes\mathcal{L}(z)^{ab}_{ij}, \quad (18.23)$$

$$\mathcal{M}(z)=(\sigma\otimes\sigma\otimes 1)M(z)|_{\nu=1}(\sigma\otimes\sigma\otimes 1)=\sum_{a,b,i,j}E_{ai}\otimes E_{bj}\otimes\mathcal{M}(z)^{ab}_{ij}, \quad (18.24)$$

where $\sigma(v_k)=v_{1-k}$ and the other notations are parallel with (3.118) and (3.119). From (3.120), their non-zero matrix elements are given as follows:

[5] σ is defined after (18.24), which just interchanges the indices 0 and 1.



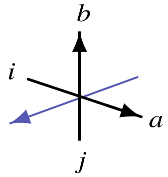
$\mathcal{L}(z)^{ab}_{ij}$	1	1	$z\mathbf{a}^+$	$z^{-1}\mathbf{a}^-$	$\mathbf{k}$	$-q\mathbf{k}$
$\mathcal{M}(z)^{ab}_{ij}$	1	1	$z\mathbf{a}^+$	$z^{-1}\mathbf{a}^-$	$\tilde{\mathbf{k}}$	$q\tilde{\mathbf{k}}$

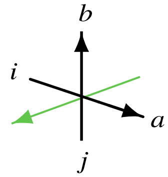
(18.25)

Here $\mathbf{a}^{\pm}$, $\mathbf{k}$ are q-oscillators (3.13) and $\tilde{\mathbf{k}} = \mathbf{k}|_{q\to -q}$ as defined in (3.121). The operators $\mathcal{L}(z)$ and $\mathcal{M}(z)$ are simply related by $\mathcal{M}(z) = \mathcal{L}(z)|_{q\to -q}$. From Theorem 3.25 they also satisfy the tetrahedron equation

$$\begin{aligned}&\mathcal{M}_{126}(z_{12})\mathcal{M}_{346}(z_{34})\mathcal{L}_{135}(z_{13})\mathcal{L}_{245}(z_{24})\\&= \mathcal{L}_{245}(z_{24})\mathcal{L}_{135}(z_{13})\mathcal{M}_{346}(z_{34})\mathcal{M}_{126}(z_{12}),\end{aligned} \tag{18.26}$$

where $z_{ij} = z_i/z_j$. In terms of the 3D diagram representation (cf. Sect. 2.6) as

$\mathcal{L}(z)^{ab}_{ij} =$ 

$\mathcal{M}(z)^{ab}_{ij} =$ 

(18.27)

the tetrahedron equation is expressed as

Let us introduce the dual of V by

$$V^* = \mathbb{C}v_0^* \oplus \mathbb{C}v_1^*, \quad \langle v_i^*, v_j\rangle = \delta_{ij}. \tag{18.28}$$

We let $\mathcal{M}(z)$ act on $V^* \otimes V^* \otimes \mathcal{F}_q^*$ from the right as

$$(v_a^* \otimes v_b^* \otimes \langle \xi|)\mathcal{M}(z) = \sum_{i,j=0,1} v_i^* \otimes v_j^* \otimes \langle \xi|\mathcal{M}(z)^{ab}_{ij}. \tag{18.29}$$

Set

$$|\chi(z)\rangle = \sum_{m\geq 0}\frac{z^m}{(-q;-q)_m}|m\rangle, \qquad \langle \chi(z)| = \sum_{m\geq 0}\frac{z^m}{(-q;-q)_m}\langle m|. \tag{18.30}$$

Proposition 18.8 *The vectors*

$$v_0 \otimes v_0 \otimes |\xi\rangle, \quad v_1 \otimes v_1 \otimes |\xi\rangle, \quad (\mu v_1 \otimes v_0 + \nu v_0 \otimes v_1)\otimes |\chi(\tfrac{\mu z}{\nu})\rangle, \tag{18.31}$$

$$v_0^* \otimes v_0^* \otimes \langle \xi|, \quad v_1^* \otimes v_1^* \otimes \langle \xi|, \quad (\mu v_1^* \otimes v_0^* + \nu v_0^* \otimes v_1^*)\otimes \langle \chi(\tfrac{\mu}{\nu z})| \tag{18.32}$$

are right and left eigenvectors of $\mathcal{M}(z)$ *with eigenvalue 1 for any* $|\xi\rangle \in \mathcal{F}_q$, $\langle\xi| \in \mathcal{F}_q^*$, *and* $\mu, \nu(\neq 0) \in \mathbb{C}$.

Proof. The non-trivial cases are verified by directly checking

$$\sum_{i+j=1} \mu^i \nu^j \mathcal{M}(z)_{ij}^{kl} |\chi(\tfrac{\mu z}{\nu})\rangle = \mu^k \nu^l |\chi(\tfrac{\mu z}{\nu})\rangle, \tag{18.33}$$

$$\sum_{i+j=1} \mu^i \nu^j \langle\chi(\tfrac{\mu}{\nu z})|\mathcal{M}(z)_{kl}^{ij} = \mu^k \nu^l \langle\chi(\tfrac{\mu}{\nu z})|. \tag{18.34}$$

One can utilize (3.134), (3.135), (3.138) and (3.139) with $q \to -q$. □

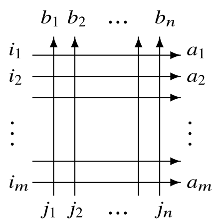
As a corollary of Proposition 18.8, we have the following equality for any $k, l = 0, 1$:

$$\sum_{i,j} \mathcal{M}(z)_{ij}^{kl} |\chi(z)\rangle = |\chi(z)\rangle, \quad \langle\chi(z^{-1})| \sum_{i,j} \mathcal{M}(z)_{kl}^{ij} = \langle\chi(z^{-1})|. \tag{18.35}$$

18.4 Layer Transfer Matrices

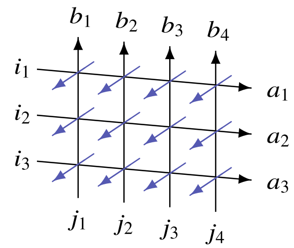
18.4.1 Layer Transfer Matrices with Mixed Boundary Condition

Fix positive integers m, n. Given the arrays $\mathbf{a} = (a_1, \dots, a_m), \mathbf{i} = (i_1, \dots, i_m) \in \{0, 1\}^m$ and $\mathbf{b} = (b_1, \dots, b_n), \mathbf{j} = (j_1, \dots, j_n) \in \{0, 1\}^n$, define a linear operator $T(z)_{\mathbf{i},\mathbf{j}}^{\mathbf{a},\mathbf{b}}$ on $\mathcal{F}_q^{\otimes mn}$ graphically as follows:

$T(z)_{\mathbf{i},\mathbf{j}}^{\mathbf{a},\mathbf{b}} =$ 

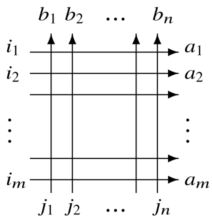
It represents the sums over $\{0, 1\}$ for all the internal edges under the prescribed boundary condition. Each arrow, either horizontal or vertical, carries V. Each vertex represents $\mathcal{L}(z)_{ij}^{ab}$ in (18.25) including the spectral parameter z. Penetrating each vertex from back to front, the Fock space $\mathcal{F}_q$ runs along a blue arrow as in the left diagram in (18.27). When this feature is to be emphasized, we depict $T(z)_{\mathbf{i},\mathbf{j}}^{\mathbf{a},\mathbf{b}}$, say for $(m, n) = (3, 4)$, as

$$T(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} =$$



In our working below, the following object plays the central role:

$$S(z)^{\mathbf{a}}_{\mathbf{j}} = \sum_{\mathbf{i},\mathbf{b}} T(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} = \sum_{\mathbf{i},\mathbf{b}}$$



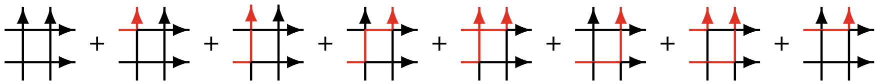
$$\in \mathrm{End}(\mathcal{F}_q^{\otimes mn}). \tag{18.36}$$

The sum $\sum_{\mathbf{i},\mathbf{b}}$ extends over $\mathbf{i} \in \{0, 1\}^m$ and $\mathbf{b} \in \{0, 1\}^n$. The operators $T(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}}$ and $S(z)^{\mathbf{a}}_{\mathbf{j}}$ are the layer transfer matrices of size $m \times n$ with fixed and mixed (NW-free and SE-fixed) boundary conditions, respectively.

Example 18.9 Consider the simplest case $(m, n) = (1, 1)$, where $T(z)^{ab}_{ij} = \mathcal{L}(z)^{ab}_{ij}$. Therefore from (18.24) we have

$$S(z)^0_0 = 1 + z\mathbf{a}^+, \quad S(z)^1_1 = 1 + z^{-1}\mathbf{a}^-, \quad S(z)^0_1 = \mathbf{k}, \quad S(z)^1_0 = -q\mathbf{k}.$$

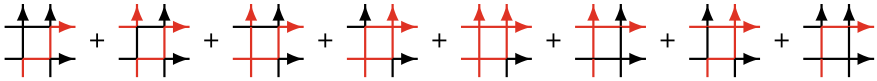
Example 18.10 Consider the case $(m, n) = (2, 2)$. $S(z)^{00}_{00}$ consists of the following 8 terms:



Thus we have

$$\begin{aligned} S(z)^{00}_{00} &= 1\otimes 1\otimes 1\otimes 1 + z\mathbf{a}^+\otimes 1\otimes 1\otimes 1 + z\mathbf{k}\otimes\mathbf{a}^+\otimes 1\otimes 1 + z\mathbf{a}^-\otimes\mathbf{a}^+\otimes\mathbf{a}^+\otimes 1 \\ &+ z^2 1\otimes\mathbf{a}^+\otimes\mathbf{a}^+\otimes 1 - qz1\otimes\mathbf{k}\otimes\mathbf{k}\otimes\mathbf{a}^+ - qz^2\mathbf{a}^+\otimes\mathbf{k}\otimes\mathbf{k}\otimes\mathbf{a}^+ - qz\mathbf{k}\otimes 1\otimes\mathbf{a}^+\otimes 1. \end{aligned} \tag{18.37}$$

Example 18.11 Consider the case $(m, n) = (2, 2)$. $S(z)^{10}_{10}$ consists of the following 8 terms:



Thus we have

$$S(z)^{10}_{10} = z^{-1}1\otimes\mathbf{a}^-\otimes\mathbf{a}^-\otimes\mathbf{a}^+ + \mathbf{a}^+\otimes\mathbf{a}^-\otimes\mathbf{a}^-\otimes\mathbf{a}^+ + \mathbf{k}\otimes 1\otimes\mathbf{a}^-\otimes\mathbf{a}^+ + \mathbf{a}^-\otimes 1\otimes 1\otimes\mathbf{a}^+ \\ + z1\otimes 1\otimes 1\otimes\mathbf{a}^+ - q1\otimes\mathbf{k}\otimes\mathbf{k}\otimes 1 - q\mathbf{k}\otimes\mathbf{a}^-\otimes 1\otimes\mathbf{a}^+ - qz^{-1}\mathbf{a}^-\otimes\mathbf{k}\otimes\mathbf{k}\otimes 1. \qquad (18.38)$$

18.4.2 Commutativity

Proposition 18.12 *The layer transfer matrices* $S(z)^{\mathbf{a}}_{\mathbf{j}}$ *with the common SE boundary condition* $\mathbf{a}, \mathbf{j}$ *form a commuting family, i.e.*

$$[S(x)^{\mathbf{a}}_{\mathbf{j}}, S(y)^{\mathbf{a}}_{\mathbf{j}}] = 0. \qquad (18.39)$$

Proof. This is a consequence of the tetrahedron equation (18.26) and the trivial eigenvectors of $\mathcal{M}(z)$ in Proposition 18.8. Consider the following two operators on $\mathcal{F}_q^{\otimes mn} \otimes \mathcal{F}_q$:

$$\sum_{\mathbf{b},\mathbf{b}'}\Big(\mathcal{M}(\tfrac{x}{x'})^{a_m,a_m}_{a_m,a_m}\cdots\mathcal{M}(\tfrac{x}{x'})^{a_1,a_1}_{a_1,a_1}\Big)\Big(\mathcal{M}(\tfrac{y}{y'})^{c_n,c'_n}_{b_n,b'_n}\cdots\mathcal{M}(\tfrac{y}{y'})^{c_1,c'_1}_{b_1,b'_1}\Big)T(\tfrac{x}{y})^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}}T(\tfrac{x'}{y'})^{\mathbf{a},\mathbf{b}'}_{\mathbf{i}',\mathbf{j}}, \qquad (18.40)$$

$$\sum_{\mathbf{k},\mathbf{k}'}T(\tfrac{x'}{y'})^{\mathbf{a},\mathbf{c}'}_{\mathbf{k}',\mathbf{j}}T(\tfrac{x}{y})^{\mathbf{a},\mathbf{c}}_{\mathbf{k},\mathbf{j}}\Big(\mathcal{M}(\tfrac{y}{y'})^{j_n,j_n}_{j_n,j_n}\cdots\mathcal{M}(\tfrac{y}{y'})^{j_1,j_1}_{j_1,j_1}\Big)\Big(\mathcal{M}(\tfrac{x}{x'})^{k_m,k'_m}_{i_m,i'_m}\cdots\mathcal{M}(\tfrac{x}{x'})^{k_1,k'_1}_{i_1,i'_1}\Big), \qquad (18.41)$$

where $\mathbf{i} = (i_1, \ldots, i_m)$, etc. The left blocks $(\mathcal{M}(\cdot)^{\bullet\bullet}_{\bullet\bullet}\cdots\mathcal{M}(\cdot)^{\bullet\bullet}_{\bullet\bullet})$ both in (18.40) and (18.41) are actually the identities but it is better to keep them temporarily for the explanation. The operators in (18.40) and (18.41) actually coincide. To see this we depict them as follows.
Here $T(z)^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}}$ acts on $\mathcal{F}_q^{\otimes mn}$ (blue arrows) and $\mathcal{M}(z)^{ab}_{ij}$ acts on the extra single Fock space $\mathcal{F}_q$ (green arrow). In the upper diagram, the front and the back layers correspond to $T(\frac{x}{y})^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}}$ and $T(\frac{x'}{y'})^{\mathbf{a},\mathbf{b}'}_{\mathbf{i}',\mathbf{j}}$ in (18.40), respectively. Similarly, in the lower diagram, the front and the back layers represent $T(\frac{x'}{y'})^{\mathbf{a},\mathbf{c}'}_{\mathbf{k}',\mathbf{j}}$ and $T(\frac{x}{y})^{\mathbf{a},\mathbf{c}}_{\mathbf{k},\mathbf{j}}$ in (18.41), respectively. Starting from the top right corner of the upper diagram, using the tetrahedron equation (Figure 18.2) repeatedly, one can push the green arrow all the way down to the bottom

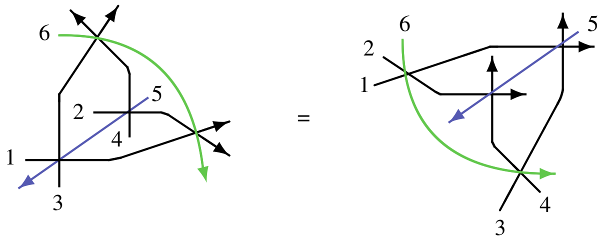


Fig. 18.2 A graphical representation of the tetrahedron equation (18.26). The parameter z_{ij} has been suppressed

left. It transforms the upper diagram into the lower, showing that they are equal as operators on $\mathcal{F}_q^{\otimes mn} \otimes \mathcal{F}_q$.

Now we rephrase the equality of (18.40) and (18.41) as

$$\sum_{\mathbf{b},\mathbf{b}'} \Big(\mathcal{M}(\tfrac{y}{y'})^{c_n,c_n'}_{b_n,b_n'} \cdots \mathcal{M}(\tfrac{y}{y'})^{c_1,c_1'}_{b_1,b_1'}\Big) T(\tfrac{x}{y})^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} T(\tfrac{x'}{y'})^{\mathbf{a},\mathbf{b}'}_{\mathbf{i}',\mathbf{j}}$$
$$= \sum_{\mathbf{k},\mathbf{k}'} T(\tfrac{x'}{y'})^{\mathbf{a},\mathbf{c}'}_{\mathbf{k}',\mathbf{j}} T(\tfrac{x}{y})^{\mathbf{a},\mathbf{c}}_{\mathbf{k},\mathbf{j}} \Big(\mathcal{M}(\tfrac{x}{x'})^{k_m,k_m'}_{i_m,i_m'} \cdots \mathcal{M}(\tfrac{x}{x'})^{k_1,k_1'}_{i_1,i_1'}\Big) \tag{18.42}$$

removing the identity parts. Evaluate (18.42) between $\langle\chi(\frac{y'}{y})| \in \mathcal{F}_q^*$ and $|\chi(\frac{x}{x'})\rangle \in \mathcal{F}_q$, where these vectors are on the green arrows on which only the block of $\mathcal{M}(z)$'s act. Taking a further sum over $\mathbf{i}, \mathbf{i}', \mathbf{c}, \mathbf{c}'$ on both sides eliminates $\mathcal{M}(z)$'s by means of (18.35), leading to

$$\langle\chi(\tfrac{y}{y'})|\chi(\tfrac{x}{x'})\rangle \sum_{\mathbf{i},\mathbf{i}',\mathbf{b},\mathbf{b}'} T(\tfrac{x}{y})^{\mathbf{a},\mathbf{b}}_{\mathbf{i},\mathbf{j}} T(\tfrac{x'}{y'})^{\mathbf{a},\mathbf{b}'}_{\mathbf{i}',\mathbf{j}} = \langle\chi(\tfrac{y}{y'})|\chi(\tfrac{x}{x'})\rangle \sum_{\mathbf{k},\mathbf{k}',\mathbf{c},\mathbf{c}'} T(\tfrac{x'}{y'})^{\mathbf{a},\mathbf{c}'}_{\mathbf{k}',\mathbf{j}} T(\tfrac{x}{y})^{\mathbf{a},\mathbf{c}}_{\mathbf{k},\mathbf{j}}. \tag{18.43}$$

Since $\langle\chi(\frac{y'}{y})|\chi(\frac{x}{x'})\rangle = \sum_{m\ge 0} \frac{(q^2;q^2)_m}{(-q;-q)_m^2}(\frac{xy'}{x'y})^m \neq 0$ by (3.16), it can be removed. From the definition of $S(z)^{\mathbf{a}}_{\mathbf{j}}$ in (18.36), the resulting equality is stated as $S(\frac{x}{y})^{\mathbf{a}}_{\mathbf{j}} S(\frac{x'}{y'})^{\mathbf{a}}_{\mathbf{j}} = S(\frac{x'}{y'})^{\mathbf{a}}_{\mathbf{j}} S(\frac{x}{y})^{\mathbf{a}}_{\mathbf{j}}$. □

One can check the commutativity (18.39) for those $S(z)^{\mathbf{a}}_{\mathbf{j}}$ in Examples 18.9, 18.10 and 18.11. The latter two are already quite non-trivial.

18.4.3 Bilinear Identities of Layer Transfer Matrices

In the proof of Proposition 18.12, we have only used the trivial eigenvectors of $\mathcal{M}(z)$ given in Proposition 18.8. A similar argument utilizing the non-trivial eigenvectors (the rightmost ones including μ and ν) leads to a family of bilinear identities of $S(z)^{\mathbf{a}}_{\mathbf{j}}$

mixing different boundary conditions $\mathbf{a}, \mathbf{j}$. They include the commutativity (18.39) as the simplest case. To describe the general case we prepare some notation.

Recall that m and n are any positive integers representing the size of the layer as in (18.36). For a subset $I \subseteq \{1, \ldots, m\}$ with the complement $\overline{I} = \{1, \ldots, m\} \setminus I$ and the sequences $\boldsymbol{\alpha} \in \{0, 1\}^{\#I}$, $\boldsymbol{\beta} \in \{0, 1\}^{\#\overline{I}}$, let $\boldsymbol{\alpha}_I \boldsymbol{\beta}_{\overline{I}} \in \{0, 1\}^m$ be the sequence in which the subsequence with indices from I is $\boldsymbol{\alpha}$ and the rest is $\boldsymbol{\beta}$.[6] For instance, for $m = 5$ and $I = \{1, 3, 4\}$, we set[7]

$$\boldsymbol{\alpha}_I \boldsymbol{\beta}_{\overline{I}} = \boldsymbol{\alpha}_{\{1,3,4\}} \boldsymbol{\beta}_{\{2,5\}} = (\alpha_1, \beta_1, \alpha_2, \alpha_3, \beta_2). \tag{18.44}$$

Likewise for $J \sqcup \overline{J} = \{1, \ldots, n\}$ and $\boldsymbol{\gamma} \in \{0, 1\}^{\#J}$, $\boldsymbol{\delta} \in \{0, 1\}^{\#\overline{J}}$, the array $\boldsymbol{\gamma}_J \boldsymbol{\delta}_{\overline{J}} \in \{0, 1\}^n$ denotes a similar sequence. For any sequence $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \{0, 1\}^k$, we set $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_k$ and $\overline{\boldsymbol{\alpha}} = (1 - \alpha_1, \ldots, 1 - \alpha_k)$.

Theorem 18.13 *For any subsets $I \subseteq \{1, \ldots, m\}$, $J \subseteq \{1, \ldots, n\}$ and sequences $\boldsymbol{\alpha} \in \{0, 1\}^{\#I}$ and $\boldsymbol{\gamma} \in \{0, 1\}^{\#J}$, the bilinear relation*[8]

$$\sum_{\boldsymbol{\beta}, \boldsymbol{\delta}} y^{|\boldsymbol{\beta}| + |\boldsymbol{\delta}|} x^{|\overline{\boldsymbol{\beta}}| + |\overline{\boldsymbol{\delta}}|} \, S(y)^{\boldsymbol{\alpha}_I \boldsymbol{\beta}_{\overline{I}}}_{\boldsymbol{\gamma}_J \boldsymbol{\delta}_{\overline{J}}} \, S(x)^{\boldsymbol{\alpha}_I \overline{\boldsymbol{\beta}}_{\overline{I}}}_{\boldsymbol{\gamma}_J \overline{\boldsymbol{\delta}}_{\overline{J}}} = (x \leftrightarrow y) \tag{18.45}$$

holds, where the sum runs over $\boldsymbol{\beta} \in \{0, 1\}^{\#\overline{I}}$ and $\boldsymbol{\delta} \in \{0, 1\}^{\#\overline{J}}$.

The commutativity (Proposition 18.12) is the simplest case of Theorem 18.13 corresponding to $I = \{1, \ldots, m\}$, $J = \{1, \ldots, n\}$, where the sum reduces to a single term. As another example, when $(m, n) = (4, 3)$, $I = \{1, 3\}$, $J = \{2, 3\}$, $\boldsymbol{\alpha} = (0, 1)$, $\boldsymbol{\gamma} = (1, 0)$, the relation (18.45) reads as

$$\begin{aligned} & x^3 S(y)^{0010}_{010} S(x)^{0111}_{110} + yx^2 S(y)^{0011}_{010} S(x)^{0110}_{110} + yx^2 S(y)^{0110}_{010} S(x)^{0011}_{110} \\ & + y^2 x S(y)^{0111}_{010} S(x)^{0010}_{110} + yx^2 S(y)^{0010}_{110} S(x)^{0111}_{010} + y^2 x S(y)^{0011}_{110} S(x)^{0110}_{010} \\ & + y^2 x S(y)^{0110}_{110} S(x)^{0011}_{010} + y^3 S(y)^{0111}_{110} S(x)^{0010}_{010} = (x \leftrightarrow y). \end{aligned} \tag{18.46}$$

We will present a proof of Theorem 18.13 only for the special case considered in Corollary 18.14 below since the general case is easily inferred from it. It corresponds to the choice $I = \{2, 3, \ldots, m\}$, $\boldsymbol{\alpha} = \mathbf{a}$, $J = \{2, 3, \ldots, n\}$, $\boldsymbol{\gamma} = \mathbf{j}$, which will suffice for the proof of Theorem 18.5.

Corollary 18.14 *For any sequences $\mathbf{a} \in \{0, 1\}^{m-1}$ and $\mathbf{j} \in \{0, 1\}^{n-1}$, we have*

$$\begin{aligned} & x^2 S(y)^{0\mathbf{a}}_{0\mathbf{j}} S(x)^{1\mathbf{a}}_{1\mathbf{j}} + yx S(y)^{0\mathbf{a}}_{1\mathbf{j}} S(x)^{1\mathbf{a}}_{0\mathbf{j}} \\ & + yx S(y)^{1\mathbf{a}}_{0\mathbf{j}} S(x)^{0\mathbf{a}}_{1\mathbf{j}} + y^2 S(y)^{1\mathbf{a}}_{1\mathbf{j}} S(x)^{0\mathbf{a}}_{0\mathbf{j}} = (x \leftrightarrow y). \end{aligned} \tag{18.47}$$

---

[6] $\#I$ denotes the cardinality of the set I.

[7] Note that it is *not* $(\alpha_1, \alpha_3, \alpha_4, \beta_2, \beta_5)$.

[8] $(x \leftrightarrow y)$ is shorthand for $\mathrm{LHS}_{x \leftrightarrow y}$.

Proof. The proof is a slight and natural modification of the one for Proposition 18.12. Consider the following equality of operators on $\mathcal{F}_q^{\otimes mn} \otimes \mathcal{F}_q$:

$$
\begin{aligned}
&\sum_{\substack{\mathbf{b},\mathbf{b}'\\ a_1''+a_1'''=1}} \mathcal{M}(\tfrac{x}{x'})^{a_1,a_1'}_{a_1'',a_1'''}\Big(\mathcal{M}(\tfrac{y}{y'})^{c_n,c_n'}_{b_n,b_n'}\cdots\mathcal{M}(\tfrac{y}{y'})^{c_1,c_1'}_{b_1,b_1'}\Big)T(\tfrac{x}{y})^{\mathbf{a}'',\mathbf{b}}_{\mathbf{i},\mathbf{j}}T(\tfrac{x'}{y'})^{\mathbf{a}''',\mathbf{b}'}_{\mathbf{i}',\mathbf{j}'}\\
&=\sum_{\substack{\mathbf{k},\mathbf{k}'\\ j_1''+j_1'''=1}} T(\tfrac{x'}{y'})^{\mathbf{a}',\mathbf{c}'}_{\mathbf{k}',\mathbf{j}'''}T(\tfrac{x}{y})^{\mathbf{a},\mathbf{c}}_{\mathbf{k},\mathbf{j}''}\mathcal{M}(\tfrac{y}{y'})^{j_1'',j_1'''}_{j_1,j_1'}\Big(\mathcal{M}(\tfrac{x}{x'})^{k_m,k_m'}_{i_m,i_m'}\cdots\mathcal{M}(\tfrac{x}{x'})^{k_1,k_1'}_{i_1,i_1'}\Big),
\end{aligned}
\tag{18.48}
$$

where $\mathbf{a}, \mathbf{a}', \mathbf{a}'', \mathbf{a}'''$ (resp. $\mathbf{j}, \mathbf{j}', \mathbf{j}'', \mathbf{j}'''$)[9] differ from each other only in the first components a_1, a_1', a_1'', a_1''' (resp. j_1, j_1', j_1'', j_1'''). We take $a_1 + a_1' = 1$ and $j_1 + j_1' = 1$ and exhibit the constraints $a_1'' + a_1''' = 1$, $j_1'' + j_1''' = 1$ coming from $\mathcal{M}(z)^{ab}_{ij} = 0$ unless $a + b = i + j$. Unlike the previous (18.40) = (18.41), the identity operators $\mathcal{M}(z)^{i,i}_{i,i} = 1$ have been omitted already. The diagram for (18.48) is Fig. 18.3 except that the (a_1, a_1) on the end of the top horizontal arrows are replaced by (a_1, a_1') and (j_1, j_1) at the bottom of the leftmost vertical arrows are changed into (j_1, j_1').

Substitution of $\mu = xy'$, $\nu = x'y$ into (18.33) and (18.34) lead to

$$
\sum_{i+j=1} \mu^i \nu^j \mathcal{M}(\tfrac{y}{y'})^{kl}_{ij} |\chi(\tfrac{x}{x'})\rangle = \mu^k \nu^l |\chi(\tfrac{x}{x'})\rangle, \tag{18.49}
$$

$$
\sum_{i+j=1} \mu^i \nu^j \langle\chi(\tfrac{y'}{y})| \,\mathcal{M}(\tfrac{x}{x'})^{ij}_{kl} = \mu^k \nu^l \langle\chi(\tfrac{y'}{y})|. \tag{18.50}
$$

Multiply (18.48) by $\mu^{a_1+j_1}\nu^{a_1'+j_1'}$ and take the sum over $\mathbf{i}, \mathbf{i}', \mathbf{c}, \mathbf{c}'$ and a_1, a_1', j_1, j_1' with the constraints $a_1 + a_1' = 1$, $j_1 + j_1' = 1$. Sandwich the resulting operator identity by $\langle\chi(\frac{y'}{y})|(\cdots)|\chi(\frac{x}{x'})\rangle$. Thanks to the identities (18.35), (18.49) and (18.50), all $\mathcal{M}(z)$'s disappear. After canceling $\langle\chi(\frac{y'}{y})|\chi(\frac{x}{x'})\rangle \neq 0$ from both sides we are left with

$$
\begin{aligned}
&\sum_{\mathbf{i},\mathbf{i}',\mathbf{b},\mathbf{b}';a_1''+a_1'''=1,j_1+j_1'=1} \mu^{a_1''+j_1}\nu^{a_1'''+j_1'} T(\tfrac{x}{y})^{\mathbf{a}'',\mathbf{b}}_{\mathbf{i},\mathbf{j}} T(\tfrac{x'}{y'})^{\mathbf{a}''',\mathbf{b}'}_{\mathbf{i}',\mathbf{j}'}\\
&= \sum_{\mathbf{k},\mathbf{k}',\mathbf{c},\mathbf{c}';a_1+a_1'=1,j_1''+j_1'''=1} \mu^{a_1+j_1''}\nu^{a_1'+j_1'''} T(\tfrac{x'}{y'})^{\mathbf{a}',\mathbf{c}'}_{\mathbf{k}',\mathbf{j}'''} T(\tfrac{x}{y})^{\mathbf{a},\mathbf{c}}_{\mathbf{k},\mathbf{j}''}.
\end{aligned}
\tag{18.51}
$$

[9] The arrays $\mathbf{a}$ and $\mathbf{j}$ here have a slightly different meaning from those in (18.47) since the final form we will reach is (18.52).

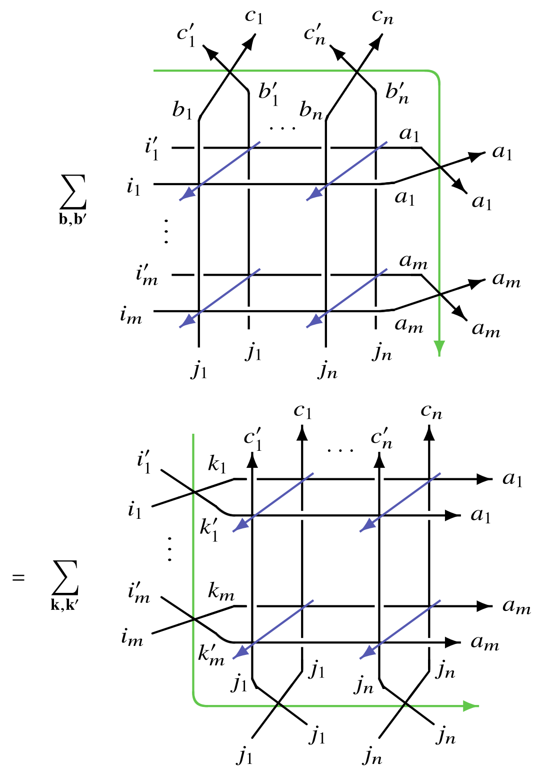


Fig. 18.3 Diagrams representing (18.40) and (18.41)

After dividing by $(yy')^2$, this is identified with

$$\sum_{a_1''+a_1'''=1,\, j_1+j_1'=1} (\tfrac{x}{y})^{a_1''+j_1} (\tfrac{x'}{y'})^{a_1'''+j_1'} S(\tfrac{x}{y})^{\mathbf{a}''}_{\mathbf{j}} S(\tfrac{x'}{y'})^{\mathbf{a}'''}_{\mathbf{j}'}$$

$$= \sum_{a_1+a_1'=1,\, j_1''+j_1'''=1} (\tfrac{x}{y})^{a_1+j_1''} (\tfrac{x'}{y'})^{a_1'+j_1'''} S(\tfrac{x'}{y'})^{\mathbf{a}'}_{\mathbf{j}'''} S(\tfrac{x}{y})^{\mathbf{a}}_{\mathbf{j}''} \tag{18.52}$$

in terms of $S(z)^{\mathbf{a}}_{\mathbf{j}}$ in (18.36), which completes the proof. □

Remark 18.15 The bilinear relation (18.45) can further be generalized by introducing inhomogeneity of the parameters. In (18.36) we consider horizontal arrows as carrying $x_1, \ldots, x_m$ from the top to the bottom and vertical ones do $y_1, \ldots, y_n$ from the left to the right. Set $\mathbf{x} = (x_1, \ldots, x_m)$ and $\mathbf{y} = (y_1, \ldots, y_n)$. Define $S(\mathbf{x}; \mathbf{y})^{\mathbf{a}}_{\mathbf{j}}$ by putting $\mathcal{L}(x_i/y_j)$ on the intersection of the ith horizontal and the jth vertical arrows. As in Theorem 18.13, let I, J be subsets of $\{1, \ldots, m\}, \{1, \ldots, n\}$ and take $\boldsymbol{\alpha} \in \{0, 1\}^{\#I}$, $\boldsymbol{\gamma} \in \{0, 1\}^{\#J}$. Suppose that $(\mathbf{x}; \mathbf{y})$ and $(\mathbf{x}'; \mathbf{y}')$ satisfy

$$x_1/x_1' = \cdots = x_m/x_m' = u, \quad y_1/y_1' = \cdots = y_n/y_n' = v. \tag{18.53}$$

Then the following relation is valid:

$$\sum_{\beta,\delta} (\tfrac{u}{v})^{|\beta|+|\delta|}\, S(\mathbf{x};\mathbf{y})^{\alpha_I \beta_{\bar{I}}}_{\gamma_J \delta_{\bar{J}}}\, S(\mathbf{x}';\mathbf{y}')^{\alpha_I \overline{\beta}_{\bar{I}}}_{\gamma_J \overline{\delta}_{\bar{J}}}$$
$$= \sum_{\beta,\delta} (\tfrac{u}{v})^{|\overline{\beta}|+|\overline{\delta}|}\, S(\mathbf{x}';\mathbf{y}')^{\alpha_I \beta_{\bar{I}}}_{\gamma_J \delta_{\bar{J}}}\, S(\mathbf{x};\mathbf{y})^{\alpha_I \overline{\beta}_{\bar{I}}}_{\gamma_J \overline{\delta}_{\bar{J}}}, \quad (18.54)$$

where the sums are over $\beta \in \{0, 1\}^{\#\bar{I}}$ and $\delta \in \{0, 1\}^{\#\bar{J}}$ as in (18.45). The derivation is similar and outlined in [90, Remark 5.4].

18.5 Proof of Theorem 18.5

We are ready to prove Theorem 18.5 by using the special case $m = n$ and $q = 0$ of the preceding results. Note that the layer transfer matrix $S(z)^{\mathbf{a}}_{\mathbf{j}}$ (18.36) remains well defined at $q = 0$. In fact, comparison of (18.25) and (18.19) shows that $q = 0$ is achieved just by excluding the rightmost vertex in the former and replacing $\mathbf{a}^{\pm}, \mathbf{k}$ with $\mathbf{b}^{\pm}, \mathbf{t}$, respectively. See (18.22). For distinction we prepare the notation of it as

$$\mathbb{S}(z)^{\mathbf{a}}_{\mathbf{j}} = \lim_{q\to 0} S(z)^{\mathbf{a}}_{\mathbf{j}}|_{m=n}. \quad (18.55)$$

It is still a non-trivial operator on $\mathcal{F}^{\otimes n^2}$ on which $\mathbf{b}^{\pm}, \mathbf{t}$ in each component act as (18.20).

Proposition 18.16 *The matrix product operator $X_i(z)$ (18.18) is contained in the layer transfer matrices at $q = 0$ as follows:*

$$\mathbb{S}(z)^{00\cdots 0}_{00\cdots 0} = \sum_{i=0}^{n} X_i(z) \otimes \underbrace{\overbrace{\mathbf{b}^+ \otimes \cdots \otimes \mathbf{b}^+}^{i} \otimes \overbrace{1 \otimes \cdots \otimes 1}^{n-i}}_{\text{diagonal}} \otimes 1 \otimes \cdots \otimes 1, \quad (18.56)$$

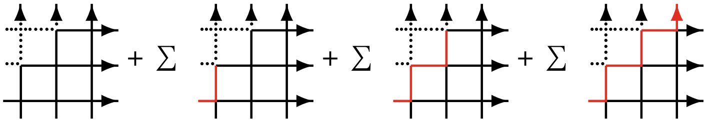
$$\mathbb{S}(z)^{10\cdots 0}_{10\cdots 0} = z^{-1} \sum_{i=0}^{n} X_i(z) \otimes \underbrace{\overbrace{1 \otimes \cdots \otimes 1}^{i} \otimes \overbrace{\mathbf{b}^- \otimes \cdots \otimes \mathbf{b}^-}^{n-i}}_{\text{diagonal}} \otimes \overbrace{\mathbf{b}^+ \otimes \cdots \otimes \mathbf{b}^+}^{n-1} \otimes 1 \otimes \cdots \otimes 1. \quad (18.57)$$

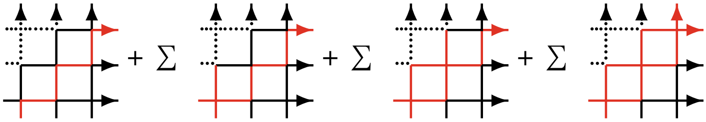
Here "diagonal" signifies the part of the tensor components corresponding to the vertices on the NE–SW diagonal in (18.36)$|_{m=n}$.[10]

Proof. We regard the triangular region in (18.18) as embedded into the $n \times n$ square lattice in (18.36)$|_{m=n}$. Since the rightmost vertex of $\mathcal{L}(z)$ in (18.25) is absent at $q = 0$,

[10] For the ordering of the components, see the explanation in Example 18.4.

the red lines tend to be confined to the upper left region. Also, once an edge on the diagonal boundary in (18.18) becomes black, then the subsequent ones continue to be black in its further NE region. These properties imply the claimed expansion formulas. See the following example from $n = 3$, where the dotted ones are to be summed over 0 and 1.[11] The four diagrams correspond to $i = 0, \dots, 3$ terms in (18.56) and (18.57) from the left to the right. The general case is similar.

$\mathbb{S}(z)^{000}_{000} = \sum$ 

$\mathbb{S}(z)^{100}_{100} = \sum$ 

From (18.25), notice that the weight of z for $\mathbb{S}(z)^{\mathbf{a}}_{\mathbf{j}}$ is calculated by #(1 on the top edges) − #(1 on the bottom edges), whereas the one for $X_i(z)$ is just #(1 on the top edges). This explains the extra overall factor z^0 and z^{-1} in (18.56) and (18.57). □

Example 18.17 Consider the case $n = 2$. Setting $q = 0$ in Example 18.10, we have

$$\begin{aligned}\mathbb{S}(z)^{00}_{00} &= (1 + z\mathbf{b}^+)\otimes 1\otimes 1\otimes 1 + z\mathbf{t}\otimes\mathbf{b}^+\otimes 1\otimes 1 + (z\mathbf{b}^- + z^2 1)\otimes\mathbf{b}^+\otimes\mathbf{b}^+\otimes 1\\ &= X_0(z)\otimes 1\otimes 1\otimes 1 + X_1(z)\otimes\mathbf{b}^+\otimes 1\otimes 1 + X_2(z)\otimes\mathbf{b}^+\otimes\mathbf{b}^+\otimes 1\end{aligned}$$

by Example 18.3 in agreement with (18.56). Similarly, Example 18.11 leads to

$$\begin{aligned}z\mathbb{S}(z)^{10}_{10} &= (1 + z\mathbf{b}^+)\otimes\mathbf{b}^-\otimes\mathbf{b}^-\otimes\mathbf{b}^+ + z\mathbf{t}\otimes 1\otimes\mathbf{b}^-\otimes\mathbf{b}^+ + (z\mathbf{b}^- + z^2 1)\otimes 1\otimes 1\otimes\mathbf{b}^+\\ &= X_0(z)\otimes\mathbf{b}^-\otimes\mathbf{b}^-\otimes\mathbf{b}^+ + X_1(z)\otimes 1\otimes\mathbf{b}^-\otimes\mathbf{b}^+ + X_2(z)\otimes 1\otimes 1\otimes\mathbf{b}^+\end{aligned}$$

in agreement with (18.57).

Proof of Theorem 18.5. Substituting (18.56) into the commutativity (18.39) and collecting the coefficient of

$$\overbrace{(\mathbf{b}^+)^2\otimes\cdots\otimes(\mathbf{b}^+)^2}^{j}\otimes\overbrace{\mathbf{b}^+\otimes\cdots\otimes\mathbf{b}^+}^{i-j}\otimes 1\otimes\cdots\otimes 1 \qquad (0 \le j \le i \le n), \tag{18.58}$$

we get (18.14).

[11] Some of them are actually fixed to 0 or 1, but they are left dotted for the sake of exposition.

Next we show (18.15). Set $\mathbf{a} = (0, \dots, 0)$, $\mathbf{j} = (0, \dots, 0)$ in Corollary 18.14 and use the obvious property $S(z)^{10\cdots 0}_{00\cdots 0} = 0$ to derive

$$\begin{aligned} &x^2 \mathbb{S}(y)^{00\cdots 0}_{00\cdots 0} \mathbb{S}(x)^{10\cdots 0}_{10\cdots 0} + y^2 \mathbb{S}(y)^{10\cdots 0}_{10\cdots 0} \mathbb{S}(x)^{00\cdots 0}_{00\cdots 0} \\ &- y^2 \mathbb{S}(x)^{00\cdots 0}_{00\cdots 0} \mathbb{S}(y)^{10\cdots 0}_{10\cdots 0} - x^2 \mathbb{S}(x)^{10\cdots 0}_{10\cdots 0} \mathbb{S}(y)^{00\cdots 0}_{00\cdots 0} = 0. \end{aligned} \tag{18.59}$$

Write the diagonal parts in (18.56) and (18.57) as Y_i and W_i, i.e. we set

$$\mathbb{S}(z)^{00\cdots 0}_{00\cdots 0} = \sum_{i=0}^{n} X_i(z) \otimes Y_i \otimes 1 \otimes \cdots \otimes 1, \quad Y_i = (\mathbf{b}^+)^{\otimes i} \otimes 1^{\otimes n-i}, \tag{18.60}$$

$$z\mathbb{S}(z)^{10\cdots 0}_{10\cdots 0} = \sum_{i=0}^{n} X_i(z) \otimes W_i \otimes (\mathbf{b}^+)^{\otimes n-1} \otimes 1 \otimes \cdots \otimes 1, \quad W_i = 1^{\otimes i} \otimes (\mathbf{b}^-)^{\otimes n-i}. \tag{18.61}$$

Substitution of them into (18.59) generates the terms all having the common off-diagonal tail $(\mathbf{b}^+)^{\otimes n-1} \otimes 1 \otimes \cdots \otimes 1$. It therefore reduces to the identity without the tail. Explicitly it is given by

$$\begin{aligned} \sum_{0\le i,j\le n} \Big(& x X_i(y) X_j(x) \otimes Y_i W_j + y X_i(y) X_j(x) \otimes W_i Y_j \\ & - y X_i(x) X_j(y) \otimes Y_i W_j - x X_i(x) X_j(y) \otimes W_i Y_j \Big) = 0, \end{aligned} \tag{18.62}$$

where Y_k, W_k correspond to the diagonal part in Proposition 18.16. Now let us pick the coefficients of the terms whose diagonal part is

$$\overbrace{\mathbf{b}^+ \otimes \cdots \otimes \mathbf{b}^+}^{j} \otimes \overbrace{\mathbf{t} \otimes \cdots \otimes \mathbf{t}}^{i-j} \otimes \overbrace{\mathbf{b}^- \otimes \cdots \otimes \mathbf{b}^-}^{n-i} \qquad (0 \le j < i \le n). \tag{18.63}$$

In view of (18.21), such a term does not arise from $W_i Y_j$ but only comes from the expansion of

$$\begin{aligned} Y_i W_j &= (\overbrace{\mathbf{b}^+ \otimes \cdots \otimes \mathbf{b}^+}^{i} \otimes \overbrace{1 \otimes \cdots \otimes 1}^{n-i})(\overbrace{1 \otimes \cdots \otimes 1}^{j} \otimes \overbrace{\mathbf{b}^- \otimes \cdots \otimes \mathbf{b}^-}^{n-j}) \\ &= \overbrace{\mathbf{b}^+ \otimes \cdots \otimes \mathbf{b}^+}^{j} \otimes \overbrace{(1-\mathbf{t}) \otimes \cdots \otimes (1-\mathbf{t})}^{i-j} \otimes \overbrace{\mathbf{b}^- \otimes \cdots \otimes \mathbf{b}^-}^{n-i} \end{aligned} \tag{18.64}$$

with a fixed coefficient $(-1)^{i-j}$. Thus (18.62) gives $x X_i(y) X_j(x) = y X_i(y) X_j(x)$ for $0 \le j < i \le n$, which is (18.15). This completes the proof of Theorem 18.5. □

18.6 Bibliographical Notes and Comments

Matrix product construction of the stationary probability was initiated in [39] for the single species TASEP subject to non-trivial boundary reservoirs. For a general introduction to the subject, see for example [23, 32, 140] and the references therein. As mentioned in the main text, the stationary probabilities become non-trivial even under the periodic boundary condition for the n-TASEP with $n \geq 2$. The first systematic result about it was obtained in [47], where the combinatorial construction, called the Ferrari–Martin (FM) algorithm, was put forward. Many works followed it, seeking an operator formulation and/or generalization to multispecies *partially* asymmetric simple exclusion processes, e.g. [45, 124].

This chapter, which is mainly based on [90], presents a unique approach from the 3D integrability. It identifies the tetrahedron equation of type $MMLL = LLMM$ (Sects. 2.6 and 3.5.3) as the ultimate structure validating the matrix product formula based on $X_i(z)$ in (18.18). As noted in Remark 18.6, the quantum group theoretical origin of the FM algorithm is a composition of the quantum R matrices (11.26) at $q = 0$ [89].

There is another class of stochastic models known as the totally asymmetric zero range process (TAZRP). See for instance [46] for a general background. Among them, there is a special example, n-TAZRP, which admits results quite parallel to this chapter [91, 92]. A contrasting feature of the n-TAZRP is that it allows occupancy of more than one particle at a site with some combinatorial constraint on their hopping rule. The n-TASEP and the n-TAZRP are sister models. The quantum R matrices relevant to the FM-like algorithms are those associated with the antisymmetric tensor representations (11.26) and the symmetric tensor representations (13.10), respectively. The solutions to the tetrahedron equation relevant to the layer transfer matrices (matrix product operators) are those consisting of the 3D L (Theorem 3.25) and the 3D R (Theorem 3.26), respectively. The n-TAZRP [91, 92] is a special limit of the integrable Markov process associated with the stochastic R matrix [87] quoted in Sect. 13.7. The latter contains numerous models which have been studied extensively. A bird's eye view of their degeneration scheme is given in [81, Figs. 1 and 2]. A survey from the 3D viewpoint is available in [100].

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