Chapter 46
Multiparton Webs Beyond Three Loops

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Abstract In QCD, the soft function exponentiate in terms of diagrams known as webs. We have defined Cwebs or correlator webs which are useful in the calculation of soft function exponentiation at higher perturbative orders. We review the results of the four-loop Cweb mixing matrices. We also provide a direct construction of a few of the mixing matrices without applying the complicated steps of the replica trick.

46.1 Introduction

In non-abelian gauge theory the studies of infrared singularities have a rich history and have produced remarkable insights in all order results. These singulartites get canceled in a well-defined (infrared safe) physical observable but they leave their signatures in the form of large logarithms of the kinematic variables. In the IR limit, the scattering amplitude factorizes into a universal soft function, a collinear jet function, and an infrared finite hard function. Our object of interest, the soft function for a n parton scattering process is defined as

$$\mathcal{S}_n\Big(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\Big) \equiv \langle 0| \prod_{k=1}^{n} \Phi_{\beta_k}(\infty, 0)\, |0\rangle \,, \tag{46.1}$$

where $\Phi_{\beta_i}(\infty, 0)$ are semi-infinite Wilson lines along β_i (velocity of the i-th parton), $\alpha_s = g_s^2/4\pi$ and $\epsilon = (4-d)/2$. As a consequence of factorization, the soft function

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obeys renormalization group equation, solving which leads to the exponentiation in terms of soft anomalous dimension $\mathbf{\Gamma}_{\rm n}$. The soft function in terms of the soft anomalous dimension is given by

$$\mathcal{S}_n\Big(\beta_i\cdot\beta_j,\alpha_s(\mu^2),\epsilon\Big) = \mathcal{P}\exp\left[-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\mathbf{\Gamma}_n\Big(\beta_i\cdot\beta_j,\alpha_s(\lambda^2),\epsilon\Big)\right]. \quad (46.2)$$

The soft anomalous dimension was computed recently at three loops in [1, 2] and the current frontier is to calculate the soft anomalous dimension at four loops.

The soft function $\mathcal{S}_n$ follows a diagramatic exponentiation such that

$$\mathcal{S}_n = \exp\big[\mathcal{W}_n\big], \quad (46.3)$$

where $\mathcal{W}_n$ are collectively known as webs. Thus, one can directly compute the soft anomalous dimension matrix $\mathbf{\Gamma}_{\rm n}$ using webs. The diagrammatic exponentiation was first observed in QED, where $\mathcal{W}_n$ contains only connected photon sub-diagrams. In QCD for the general case of n Wilson lines, a *web* is defined as a set of diagrams which are related to one another by the permutation of the gluons on each Wilson line. If $\mathcal{K}(D)$ and $C(D)$ denote the kinematics and color of a diagram D in a web, then the exponent of the soft function is given by

$$\mathcal{S}_n = \exp\left[\sum_{D,D'}\mathcal{K}(D)R(D,D')C(D')\right], \quad (46.4)$$

where R is called the web mixing matrix and

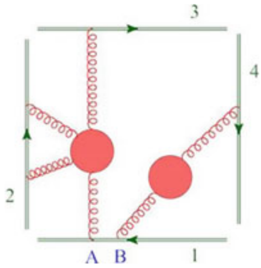
$$\widetilde{C}(D) = \sum_{D'} R(D,D')\,C(D'), \quad (46.5)$$

is called the exponentiated colour factor for a diagram D. The general properties of the web mixing matrices were studied in [3–6] and are given by

1. The web mixing matrices are idempotent, i.e., $R^2 = R$.
2. The row-sum of the matrices are zero.
3. The elements of web mixing matrices obey the column sum rule $\sum_D s(D)\,R(D,D') = 0$, where $s(D)$ denotes the number of ways that the gluons can be sequentially shrunk to the hard interaction vertex.

46.2 Cwebs at Four Loops

We define a *correlator web*, or a *Cweb* as a set diagrams, built out of connected gluon correlators attached to Wilson lines, and closed under shuflles of the gluon attachments to each Wilson line. As compared to webs, Cwebs have their own perturbative



Diagrams	Sequences	S-factors
C_1	$\{\{BA\}\}$	1
C_2	$\{\{AB\}\}$	1

Fig. 46.1 Diagrams for $W_4^{(1,0,1)}(2, 2, 1, 1)$

expansions and thus useful in the enumeration of webs at higher orders. A Cweb connecting n Wilson lines with c_m number of m-point gluon correlators and with k_l number of attachments on l-th Wilson line is denoted by $W_n^{(c_2,\ldots,c_p)}(k_1, \ldots, k_n)$. As described in [8], one can generate all the Cwebs at $\mathcal{O}(g^{2n})$ from the Cwebs at $\mathcal{O}(g^{2n-2})$ by performing the following moves:

1. Add a two-gluon correlator connecting any two Wilson lines.
2. Connect an existing m-point correlator to any Wilson line, turning it into an $(m + 1)$-point correlator.
3. Connect an existing m-point correlator to an existing n-point correlator, resulting in an $(n + m)$-point correlator.

Using the above steps, we have generated all the four-loop Cwebs [7, 8]. We have developed an in-house Mathematica code which computes the mixing matrices of all the Cwebs at four loops following the steps of the replica trick algorithm [3].

We show an example of a mixing matrix of a four-loop Cweb $W_4^{(1,0,1)}(2, 2, 1, 1)$ which connects 4 Wilson lines and has one 2-point gluon correlator and a 4-point gluon correlator.

The mixing matrix for this Cweb is given by

$$R = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{46.6}$$

This mixing matrix follows all the properties of a general mixing matrix. Using (46.5), one can easily calculate the exponentiated color factors. The mixing matrices for all the four-loop Cwebs connecting 4 and 5 Wilson lines are presented in [8] and for 2 and 3 Wilson lines in [7]. We have checked the correctness of our results by checking the known properties of the mixing matrices: idempotence, zero row-sum rule, and the conjectured column sum rule.

46.3 Direct Construction of Mixing Matrices

In this section, we will describe the construction of the web mixing matrices without applying the replica trick algorithm. All the elements of the possible two-dimensional mixing matrices arising at all perturbative orders are fixed by using the row-sum, column-sum, and the idempotence property. A detail calculation is presented in [7].

The next step is to calculate the three-dimensional mixing matrices using the known properties. The column weight vector of a Cweb with three diagrams is $s = \{1, 0, 1\}$. The diagram which has s = 0, cannot be generated from diagrams which have s = 1, by the action of the replica ordering operator. Taking this into consideration, the three-dimensional mixing matrix takes the form

$$R = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} . \tag{46.7}$$

This is the only three-dimensional mixing matrix that can appear in any perturbative order. Proceeding further, we find that the mixing matrices for any prime dimension p are unique at all perturbative orders and are given by [7]

$$R = \begin{pmatrix} \frac{1}{2} & 0 \; 0 \ldots 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \; 0 \ldots 0 & -\frac{1}{2} \\ & \ldots & \\ -\frac{1}{2} & 0 \; 0 \ldots 1 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \; 0 \ldots 0 & \frac{1}{2} \end{pmatrix} . \tag{46.8}$$

We believe that the exponentiation of soft function in terms of Cwebs will make the enumeration of Cwebs at higher orders much simpler as compared to webs. The exponentiated color factors presented in [7, 8] complete the full list of color factors, which will be instrumental in the calculation of the soft anomalous dimension at $\mathcal{O}(g^8)$ in the future.

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