Testing Quantiles of Two Normal Populations with a Common Mean and Order Restricted Variances

Habiba Khatun and Manas Ranjan Tripathy

Abstract In this study, we consider the problem of testing the hypothesis for the quantile $\theta = \mu + \eta\sigma_1$ (η is known) when independent random samples are available from two normal populations with a common mean μ and ordered restricted variances. Utilizing some of the popular estimators of the common mean under order restricted variances and the generalized p-value approach, we propose several test procedures for the quantiles. All the proposed test procedures are evaluated through their sizes and powers using the Monte Carlo simulation procedure. It has been observed that the proposed tests compete with each other. Finally, two datasets have been considered for illustrating the testing procedures.

Keywords Common mean · Generalized p-value · Generalized test variable · Numerical comparison · Order restricted variances · Power · Size · Testing quantile

1 Introduction

Suppose two normal populations with a common mean μ and unknown different variances σ_1^2 and σ_2^2 are available. Further, it is known in advance that the variances are ordered. Particularly, let $(X_{11}, X_{12}, \ldots, X_{1n_1})$ and $(X_{21}, X_{22}, \ldots, X_{2n_2})$ be independent observations of sizes n_1 and n_2 available from $N(\mu, \sigma_1^2)$ and $N(\mu, \sigma_2^2)$, respectively. It is known that $(\bar{X}_1, \bar{X}_2, S_1^2, S_2^2)$ is the minimal sufficient for $(\mu, \sigma_1^2, \sigma_2^2)$, where $\bar{X}_1 = \sum_{i=1}^{n_1} X_{1i}/n_1 \sim N(\mu, \sigma_1^2/n_1)$, $\bar{X}_2 = \sum_{j=1}^{n_2} X_{2j}/n_2 \sim N(\mu, \sigma_2^2/n_2)$, $S_1^2 = \sum_{i=1}^{n_1}(X_{1i} - \bar{X}_1)^2 \sim \sigma_1^2\chi^2_{n_1-1}$ and $S_2^2 = \sum_{j=1}^{n_2}(X_{2j} - \bar{X}_2)^2 \sim \sigma_2^2\chi^2_{n_2-1}$. All the random variables $\bar{X}_1$, $\bar{X}_2$, S_1^2 and S_2^2 are stochastically independent.

The problem discussed here is to test the hypothesis regarding the quantile $\theta = \mu + \eta\sigma_1$ under the belief that the variances are ordered, that is, $\sigma_1^2 \leq \sigma_2^2$. Here,

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$\eta = \Phi^{-1}(p),\ p \in (0, 1)$, and Φ is the cdf of a standard normal random variable. Specifically, we consider the hypothesis testing

$$H_0 : \theta = \theta_0 \ \text{ against } \ H_1 : \theta \neq \theta_0, \tag{1}$$

where $\theta_0 \in \mathbb{R}$ is a predefined constant.

Note that the quantities like—median, quartiles, deciles, percentiles are obtained from quantiles by considering the different values of η; hence, we consider the testing of the quantile. Several literature pieces are available in the problem of estimating the quantiles of normal and exponential distributions. The first work in this direction was probably considered by Zidek [25], who addressed the problem of estimating quantiles in the case of the normal population and obtained some decision-theoretic results. Further, Rukhin [20] discussed the estimation problem on the quantile of normal populations in decision-theoretic viewpoint and gives some application.

Researchers also considered estimating quantiles in the presence of more than one independent normal populations with an equal mean. For some results in this direction, one can refer to Kumar and Tripathy [12], Tripathy et al. [23] and the references cited therein. Recently, under the order restriction on variances, Nagamani and Tripathy [17] considered estimating quantiles of two Gaussian populations with equal mean. Comparison of quantiles of two or several distributions is also useful, and the same has been considered by Guo and Krishnamoorthy [6] and Li et al. [13]. The problem of interval estimation and testing for quantiles of two normal populations with equal mean and unrestricted variances has been recently investigated by Khatun et al. [10]. The application of quantiles can be seen in the study of life testing, reliability, survival analysis, statistical quality control and related areas. We refer to Saleh [21], Keating and Tripathi [8] and the references cited therein for some application of quantiles.

The current problem has its importance in the sense that the test procedures obtained for the quantiles are based on the estimators of the common mean under order restricted variances. In a particular case, by choosing $\eta = 0$, one can write all the test procedures easily. The problem of estimating common mean of two or more normal populations with unrestricted variances has been considered by Elfessi and Pal [4], Jena et al. [7] and Misra and van der Meulen [15]. The problem of hypothesis testing on the common mean of normal populations without order restriction on variances has drawn several researchers' attention in the past, and probably, Cohen and Sackrowitz [3] was the first to consider this problem. Krishnamoorthy and Lu [11] considered testing the common mean of two Gaussian populations using the generalized variable method. Further, Lin and Lee [14] extended their results to several normal populations. We refer to Chang and Pal [2] and Pal et al. [18] for some review on testing common mean of several Gaussian populations.

The rest of our contributions can be described as follows. In Sect. 2, we discuss some well-known results on estimating common mean μ of two normal populations with and without order restrictions on variances. In Sect. 3, we propose some generalized pivot variable and using these constructed generalized test variable for testing the quantile θ. The generalized test variables have been constructed using some of the well-known estimators of the common mean under order restricted variances. A comprehensive simulation study has been done in Sect. 4 in order to compare the performances of all the proposed tests in terms of size and power for several combinations of sample sizes and parameters. Finally, the article is concluded with some real-life applications.

2 Some Basic Results

In literature, several estimators are available for the common mean of two normal populations when there is no restriction on the variances and also when there is order restriction on variances, that is $\sigma_1^2 \le \sigma_2^2$.

When there is no order restriction on the variances, some well-known estimators for the common mean μ are proposed by Graybill and Deal [5], Khatri and Shah [9], Moore and Krishnamoorthy [16], Tripathy and Kumar [22] and the grand sample mean, which are given as follows,

$$\mu_{\text{GD}} = \frac{(n_1-1)n_1S_2^2\bar{X}_1 + (n_2-1)n_2S_1^2\bar{X}_2}{(n_1-1)n_1S_2^2 + (n_2-1)n_2S_1^2}$$

$$\mu_{\text{KS}} = \frac{(n_1-3)n_1S_2^2\bar{X}_1 + (n_2-3)n_2S_1^2\bar{X}_2}{(n_1-3)n_1S_2^2 + (n_2-3)n_2S_1^2}$$

$$\mu_{\text{MK}} = \frac{\sqrt{(n_1-1)n_1S_2^2}\bar{X}_1 + \sqrt{(n_2-1)n_2S_1^2}\bar{X}_2}{\sqrt{(n_1-1)n_1S_2^2} + \sqrt{(n_2-1)n_2S_1^2}}$$

$$\mu_{\text{TK}} = \frac{\sqrt{n_1S_2^2}b_{n_2}\bar{X}_1 + \sqrt{n_2S_1^2}b_{n_1}\bar{X}_2}{\sqrt{n_1S_2^2}b_{n_2} + \sqrt{n_2S_1^2}b_{n_1}}$$

$$\mu_{\text{GM}} = \frac{n_1\bar{X}_1 + n_2\bar{X}_2}{n_1+n_2}$$

where $b_{n_1} = \frac{\Gamma((n_1-1)/2)}{\sqrt{2}\Gamma(n_1/2)}$ and $b_{n_2} = \frac{\Gamma((n_2-1)/2)}{\sqrt{2}\Gamma(n_2/2)}$.

Further, Elfessi and Pal [4] proposed an estimator for the common mean when the variances are ordered as $\sigma_1^2 \le \sigma_2^2$ or equivalently $\sigma_1 \le \sigma_2$ which is

$$\hat{\mu}_{\text{GD}} = \begin{cases} (1-C)\bar{X}_1 + C\bar{X}_2, & \text{if } \frac{S_1^2}{n_1-1} \le \frac{S_2^2}{n_2-1} \\ C^*\bar{X}_1 + (1-C^*)\bar{X}_2, & \text{if } \frac{S_1^2}{n_1-1} > \frac{S_2^2}{n_2-1}, \end{cases} \tag{2}$$

where

$$C = \frac{n_2(n_2-1)S_1^2}{n_1(n_1-1)S_2^2 + n_2(n_2-1)S_1^2}$$

and

$$C^* = \begin{cases} C, & \text{if } n_1 = n_2 \\ \frac{n_1}{n_1+n_2}, & \text{if } n_1 \neq n_2. \end{cases}$$

When $\sigma_1^2 \leq \sigma_2^2$, the estimator $\hat{\mu}_{\text{GD}}$ dominates μ_{GD} stochastically and universally for two normal populations. The results have been extended for several Gaussian populations by Misra and van der Meulen [15].

Jena et al. [7] considered the same problem and proposed some alternative estimators of the common mean μ which improve upon the estimators μ_{KS}, μ_{MK} and μ_{TK} when $\sigma_1^2 \leq \sigma_2^2$. These improved estimators are given by

$$\hat{\mu}_{\text{KS}} = \begin{cases} (1-C_1)\bar{X}_1 + C_1\bar{X}_2, & \text{if } \frac{S_1^2}{S_2^2} \leq \frac{n_1-3}{n_2-3} \\ C_1^*\bar{X}_1 + (1-C_1^*)\bar{X}_2, & \text{if } \frac{S_1^2}{S_2^2} > \frac{n_1-3}{n_2-3}, \end{cases} \tag{3}$$

$$\hat{\mu}_{\text{MK}} = \begin{cases} (1-C_2)\bar{X}_1 + C_2\bar{X}_2, & \text{if } \frac{\sqrt{(n_2-1)S_1^2}}{\sqrt{(n_1-1)S_2^2}} \leq \frac{\sqrt{n_2}}{\sqrt{n_1}} \\ C_2^*\bar{X}_1 + (1-C_2^*)\bar{X}_2, & \text{if } \frac{\sqrt{(n_2-1)S_1^2}}{\sqrt{(n_1-1)S_2^2}} > \frac{\sqrt{n_2}}{\sqrt{n_1}}, \end{cases} \tag{4}$$

$$\hat{\mu}_{\text{TK}} = \begin{cases} (1-C_3)\bar{X}_1 + C_3\bar{X}_2, & \text{if } \frac{S_1}{S_2} \leq \frac{\sqrt{n_2}b_{n_2}}{\sqrt{n_1}b_{n_1}} \\ C_3^*\bar{X}_1 + (1-C_3^*)\bar{X}_2, & \text{if } \frac{S_1}{S_2} > \frac{\sqrt{n_2}b_{n_2}}{\sqrt{n_1}b_{n_1}}, \end{cases} \tag{5}$$

where $C_1 = \frac{n_2(n_2-3)S_1^2}{n_1(n_1-3)S_2^2+n_2(n_2-3)S_1^2}$, $C_2 = \frac{\sqrt{n_2(n_2-1)}S_1}{\sqrt{n_1(n_1-1)}S_2+\sqrt{n_2(n_2-1)}S_1}$, $C_3 = \frac{\sqrt{n_2}b_{n_1}S_1}{\sqrt{n_1}b_{n_2}S_2+\sqrt{n_2}b_{n_1}S_1}$ and

$$C_i^* = \begin{cases} C_i, & \text{if } n_1 = n_2 \\ \frac{n_1}{n_1+n_2}, & \text{if } n_1 \neq n_2. \end{cases}$$

for $i = 1, 2, 3$.

They proved that these estimators dominate the unrestricted estimators μ_{KS}, μ_{MK} and μ_{TK} in terms of stochastic domination and Pitman nearness criteria. Applying Brewster and Zidek [1] technique, they have also improved the estimators μ_{GD}, μ_{KS}, μ_{MK} and μ_{TK} under order restricted variances. These improved estimators are, respectively, given by

$$\mu_{\text{GD}}^a = \begin{cases} \mu_{\text{GD}}, & \text{if } \frac{S_1^2}{n_1-1} \leq \frac{S_2^2}{n_2-1} \\ \mu_{\text{GM}}, & \text{if } \frac{S_1^2}{n_1-1} > \frac{S_2^2}{n_2-1}, \end{cases} \tag{6}$$

$$\mu_{\text{KS}}^a = \begin{cases} \mu_{\text{KS}}, & \text{if } \frac{S_1^2}{n_1-3} \leq \frac{S_2^2}{n_2-3} \\ \mu_{\text{GM}}, & \text{if } \frac{S_1^2}{n_1-3} > \frac{S_2^2}{n_2-3}, \end{cases} \tag{7}$$

$$\mu^a_{\text{MK}} = \begin{cases} \mu_{\text{MK}}, \text{ if } \sqrt{\frac{S_1^2}{n_1-1}} \leq \sqrt{\frac{S_2^2}{n_2-1}} \\ \mu_{\text{GM}}, \text{ if } \sqrt{\frac{S_1^2}{n_1-1}} > \sqrt{\frac{S_2^2}{n_2-1}}, \end{cases} \tag{8}$$

$$\mu^a_{\text{TK}} = \begin{cases} \mu_{\text{TK}}, \text{ if } \frac{S_1}{S_2} \leq \frac{\sqrt{n_2}b_{n_2}}{\sqrt{n_1}b_{n_1}} \\ \mu_{\text{GM}}, \text{ if } \frac{S_1}{S_2} > \frac{\sqrt{n_2}b_{n_2}}{\sqrt{n_1}b_{n_1}}. \end{cases} \tag{9}$$

Jena et al. [7] noted that for unequal sample sizes ($n_1 \neq n_2$), $\mu^a_{\text{GD}} = \hat{\mu}_{\text{GD}}$, $\mu^a_{\text{KS}} = \hat{\mu}_{\text{KS}}$, $\mu^a_{\text{MK}} = \hat{\mu}_{\text{MK}}$ and $\mu^a_{\text{TK}} = \hat{\mu}_{\text{TK}}$.

In the next section, we propose some generalized test variables using these improved estimators under order restriction on the variances, to test the quantile $\theta = \mu + \eta\sigma_1$.

3 Generalized Test Variable with P-Value

In this section, we will apply the generalized variable method proposed by Tsui and Weerahandi [24] to test the hypothesis (1) regarding the quantile θ. We note that Krishnamoorthy and Lu [11] and Lin and Lee [14] successfully applied this method to test the common mean of two and more than two normal populations. In order to obtain the test statistics using this approach, we first state the following definitions.

Let X be any random variable, and the distribution of X only depends on (δ, β), where we want to test the parameter δ, which is the parameter of interest and β is the nuisance parameter. Further, suppose one is interested in testing the hypothesis

$$H_0^* : \delta \leq \delta_0 \text{ against } H_1^* : \delta > \delta_0, \tag{10}$$

where δ_0 is a known constant.

Definition 1 A random variable $P = P(X; x, \delta, \beta)$ will be called a generalized pivot variable of δ if it satisfies the following conditions

(a) The distribution of $P(X; x, \delta, \beta)$ is free of all unknown parameters for a fixed $X = x$.
(b) The value of P at $X = x$, is δ, that is, $P(X; x, \delta, \beta) = \delta$, the parameter of interest.

Definition 2 A variable $T = T(X; x, \delta, \beta)$ will be called a generalized test variable for testing the hypothesis (10), if it satisfies the conditions (a)–(c).

(a) The distribution of $T = T(X; x, \delta, \beta)$ is free from the nuisance parameter β for a given x.
(b) The value of $T = T(X; x, \delta, \beta)$ is free of any unknown parameters when $X = x$ fixed.

(c) For fixed x and β, the distribution of $T = T(X; x, \delta, \beta)$ is either stochastically increasing or stochastically decreasing as a function of δ.

where x is the observed value of X.

Definition 3 Let $t = T(x; x, \delta, \beta)$, the value of T for fixed $X = x$. In the case of stochastically increasing T with respect to δ, the generalized p-value for testing the hypothesis (10) is given by

$$\sup_{H_0^*} \Pr\{T(X; x, \delta, \beta) \geq t\} = \Pr\{T(X; x, \delta_0, \beta) \geq t\}, \tag{11}$$

and if $T(X; x, \delta, \beta)$ is stochastically decreasing in δ, the generalized p-value for testing the hypothesis (10) is given by

$$\sup_{H_0^*} \Pr\{T(X; x, \delta, \beta) \leq t\} = \Pr\{T(X; x, \delta_0, \beta) \leq t\}. \tag{12}$$

3.1 Generalized Test Variable for the Quantile

In this subsection, we propose generalized test variables using the estimators of common mean μ to test the hypothesis (1). Let the observed value of $(\bar{X}_1, \bar{X}_2, S_1^2, S_2^2)$ is $(\bar{x}_1, \bar{x}_2, s_1^2, s_2^2)$. Further, suppose $Z_1 = (\bar{X}_1 - \mu)/(\sigma_1/\sqrt(n_1))$ and $Z_2 = (\bar{X}_2 - \mu)/(\sigma_2/\sqrt(n_2))$ where Z_1 and Z_2 follows $N(0, 1)$. Now, denote $U_1^2 = S_1^2/\sigma_1^2 \sim \chi^2_{n_1-1}$ and $U_2^2 = S_2^2/\sigma_2^2 \sim \chi^2_{n_2-1}$ which are independent of Z_1 and Z_2.

When sample sizes are equal ($n_1 = n_2$) using the alternative estimators of common mean μ given in equations (2) to (5), we propose the generalized pivot variable for the quantile $\theta = \mu + \eta\sigma_1$ as

$$P_{\mathrm{GD}} = \begin{cases} \bar{\mu}_{\mathrm{GD}} - \frac{(\sqrt{(n_1-1)n_1}s_1s_2^2t_1)+(\sqrt{(n_2-1)n_2}s_1^2s_2t_2)}{((n_2-1)n_2s_1^2)+((n_1-1)n_1s_2^2)} + \eta\frac{s_1}{U_1}, & \text{if } \frac{s_1^2}{n_1-1} \leq \frac{s_2^2}{n_2-1} \\ \bar{\mu}_{\mathrm{GD}} - \frac{((n_1-1)n_1s_2^3t_2/\sqrt{(n_2-1)n_2})+((n_2-1)n_2s_1^3t_1/\sqrt{(n_1-1)n_1})}{((n_2-1)n_2s_1^2)+((n_1-1)n_1s_2^2)} + \eta\frac{s_1}{U_1}, & \text{if } \frac{s_1^2}{n_1-1} > \frac{s_2^2}{n_2-1}, \end{cases}$$

$$P_{\mathrm{KS}} = \begin{cases} \bar{\mu}_{\mathrm{KS}} - \frac{(\sqrt{n_1/(n_1-1)}(n_1-3)s_1s_2^2t_1)+(\sqrt{n_2/(n_2-1)}(n_2-3)s_1^2s_2t_2)}{((n_2-3)n_2s_1^2)+((n_1-3)n_1s_2^2)} + \eta\frac{s_1}{U_1}, & \text{if } \frac{s_1^2}{n_1-3} \leq \frac{s_2^2}{n_2-3} \\ \bar{\mu}_{\mathrm{KS}} - \frac{((n_1-3)n_1s_2^3t_2/\sqrt{(n_2-1)n_2})+((n_2-3)n_2s_1^3t_1/\sqrt{(n_1-1)n_1})}{((n_2-3)n_2s_1^2)+((n_1-3)n_1s_2^2)} + \eta\frac{s_1}{U_1}, & \text{if } \frac{s_1^2}{n_1-3} > \frac{s_2^2}{n_2-3}, \end{cases}$$

$$P_{\mathrm{MK}} = \begin{cases} \bar{\mu}_{\mathrm{MK}} - \frac{(t_1+t_2)s_1s_2}{(\sqrt{(n_2-1)n_2}s_1)+(\sqrt{(n_1-1)n_1}s_2)} + \eta\frac{s_1}{U_1}, & \text{if } \frac{\sqrt{(n_2-1)s_1^2}}{\sqrt{(n_1-1)s_2^2}} \leq \frac{\sqrt{n_2}}{\sqrt{n_1}} \\ \bar{\mu}_{\mathrm{MK}} - \frac{(\sqrt{(n_2/n_1)((n_2-1)/(n_1-1))}s_1^2t_1)+(\sqrt{(n_1/n_2)((n_1-1)/(n_2-1))}s_2^2t_2)}{(\sqrt{(n_2-1)n_2}s_1)+(\sqrt{(n_1-1)n_1}s_2)} + \eta\frac{s_1}{U_1}, & \text{if } \frac{\sqrt{(n_2-1)s_1^2}}{\sqrt{(n_1-1)s_2^2}} > \frac{\sqrt{n_2}}{\sqrt{n_1}}, \end{cases}$$

$$P_{\mathrm{TK}} = \begin{cases} \bar{\mu}_{\mathrm{TK}} - \frac{(b_{n_2}s_1s_2t_1/\sqrt{n_1-1})+(b_{n_1}s_1s_2t_2/\sqrt{n_2-1})}{(\sqrt{n_2}s_1b_1)+(\sqrt{n_1}s_2b_2)} + \eta\frac{s_1}{U_1}, & \text{if } \frac{s_1}{s_2} \leq \frac{\sqrt{n_2}b_{n_2}}{\sqrt{n_1}b_{n_1}} \\ \bar{\mu}_{\mathrm{TK}} - \frac{(\sqrt{n_2/(n_1-1)}b_{n_1}s_1t_1)+(\sqrt{n_1/(n_2-1)}b_{n_2}s_2t_2)}{(\sqrt{n_2}s_1b_1)+(\sqrt{n_1}s_2b_2)} + \eta\frac{s_1}{U_1}, & \text{if } \frac{s_1}{s_2} > \frac{\sqrt{n_2}b_{n_2}}{\sqrt{n_1}b_{n_1}}, \end{cases}$$

where $\bar{\mu}_{\mathrm{GD}}$, $\bar{\mu}_{\mathrm{KS}}$, $\bar{\mu}_{\mathrm{MK}}$ and $\bar{\mu}_{\mathrm{TK}}$ are the observed values of μ_{GD}, μ_{KS}, μ_{MK} and μ_{TK}, respectively. Observe that all the four statistics P_{GD}, P_{KS}, P_{MK} and P_{TK} satisfy the conditions given in Definition 1. Now, we construct the generalized test variables for the quantile θ using these four alternative estimators of the common mean μ under the condition $\sigma_1^2 \le \sigma_2^2$ as $T_{\text{GD}} = P_{\text{GD}} - \theta$, $T_{\text{KS}} = P_{\text{KS}} - \theta$, $T_{\text{MK}} = P_{\text{MK}} - \theta$ and $T_{\text{TK}} = P_{\text{TK}} - \theta$. These test variables satisfy the conditions (a) and (b) in Definition 2 and also stochastically decreasing in θ. Utilizing the Definition 3, we compute the p-values of all the generalized test variables as

$$2\min(\Pr\{P_{\text{GD}} \ge \theta_0\}, \Pr\{P_{\text{GD}} \le \theta_0\}) \tag{13}$$
$$2\min(\Pr\{P_{\text{KS}} \ge \theta_0\}, \Pr\{P_{\text{KS}} \le \theta_0\}) \tag{14}$$
$$2\min(\Pr\{P_{\text{MK}} \ge \theta_0\}, \Pr\{P_{\text{MK}} \le \theta_0\}) \tag{15}$$
$$2\min(\Pr\{P_{\text{TK}} \ge \theta_0\}, \Pr\{P_{\text{TK}} \le \theta_0\}). \tag{16}$$

The null hypothesis H_0 will be rejected if the p-values are less than α, the level of significance.

Next, we construct the generalized pivot variables and generalized test variables for the quantile θ utilizing the improved estimators of the common mean under order restriction $\sigma_1^2 \le \sigma_2^2$ given in (6)–(9). The generalized pivot variables are given by

$$P_{\text{GD}}^a = \begin{cases} \bar{\mu}_{\text{GD}}^a - \frac{(\sqrt{(n_1-1)n_1}s_1 s_2^2 t_1)+(\sqrt{(n_2-1)n_2}s_1^2 s_2 t_2)}{((n_2-1)n_2 s_1^2)+((n_1-1)n_1 s_2^2)} + \eta\frac{s_1}{U_1}, & \text{if } \frac{s_1^2}{n_1-1} \le \frac{s_2^2}{n_2-1} \\ \bar{\mu}_{\text{GD}}^a - \frac{(\sqrt{(n_1-1)n_1}s_1 t_1)+(\sqrt{(n_2-1)n_2}s_2 t_2)}{n_1+n_2} + \eta\frac{s_1}{U_1}, & \text{if } \frac{s_1^2}{n_1-1} > \frac{s_2^2}{n_2-1}, \end{cases}$$

$$P_{\text{KS}}^a = \begin{cases} \bar{\mu}_{\text{KS}}^a - \frac{(\sqrt{n_1/(n_1-1)}(n_1-3)s_1 s_2^2 t_1)+(\sqrt{n_2/(n_2-1)}(n_2-3)s_1^2 s_2 t_2)}{((n_2-3)n_2 s_1^2)+((n_1-3)n_1 s_2^2)} + \eta\frac{s_1}{U_1}, & \text{if } \frac{s_1^2}{n_1-3} \le \frac{s_2^2}{n_2-3} \\ \bar{\mu}_{\text{KS}}^a - \frac{(\sqrt{(n_1-1)n_1}s_1 t_1)+(\sqrt{(n_2-1)n_2}s_2 t_2)}{n_1+n_2} + \eta\frac{s_1}{U_1}, & \text{if } \frac{s_1^2}{n_1-3} > \frac{s_2^2}{n_2-3}, \end{cases}$$

$$P_{\text{MK}}^a = \begin{cases} \bar{\mu}_{\text{MK}}^a - \frac{(t_1+t_2)s_1 s_2}{(\sqrt{(n_2-1)n_2}s_1)+(\sqrt{(n_1-1)n_1}s_2)} + \eta\frac{s_1}{U_1}, & \text{if } \sqrt{\frac{s_1^2}{n_1-1}} \le \sqrt{\frac{s_2^2}{n_2-1}} \\ \bar{\mu}_{\text{MK}}^a - \frac{(\sqrt{(n_1-1)n_1}s_1 t_1)+(\sqrt{(n_2-1)n_2}s_2 t_2)}{n_1+n_2} + \eta\frac{s_1}{U_1}, & \text{if } \sqrt{\frac{s_1^2}{n_1-1}} > \sqrt{\frac{s_2^2}{n_2-1}}, \end{cases}$$

$$P_{\text{TK}}^a = \begin{cases} \bar{\mu}_{\text{TK}}^a - \frac{(b_{n_2} s_1 s_2 t_1/\sqrt{n_1-1})+(b_{n_1} s_1 s_2 t_2/\sqrt{n_2-1})}{(\sqrt{n_2}s_1 b_1)+(\sqrt{n_1}s_2 b_2)} + \eta\frac{s_1}{U_1}, & \text{if } \frac{s_1}{s_2} \le \frac{\sqrt{n_2}b_{n_2}}{\sqrt{n_1}b_{n_1}} \\ \bar{\mu}_{\text{TK}}^a - \frac{(\sqrt{(n_1-1)n_1}s_1 t_1)+(\sqrt{(n_2-1)n_2}s_2 t_2)}{n_1+n_2} + \eta\frac{s_1}{U_1}, & \text{if } \frac{s_1}{s_2} > \frac{\sqrt{n_2}b_{n_2}}{\sqrt{n_1}b_{n_1}}. \end{cases}$$

where $\bar{\mu}_{\text{GD}}^a$, $\bar{\mu}_{\text{KS}}^a$, $\bar{\mu}_{\text{MK}}^a$ and $\bar{\mu}_{\text{TK}}^a$ are the observed values of μ_{GD}^a, μ_{KS}^a, μ_{MK}^a and μ_{TK}^a, respectively. In a similar manner, as discussed before, we construct the generalized test variables for the quantile θ to test hypothesis (1), which are given by $T_{\text{GD}}^a = P_{\text{GD}}^a - \theta$, $T_{\text{KS}}^a = P_{\text{KS}}^a - \theta$, $T_{\text{MK}}^a = P_{\text{MK}}^a - \theta$ and $T_{\text{TK}}^a = P_{\text{TK}}^a - \theta$. All these test variables are stochastically decreasing in θ and satisfy the first two conditions in Definition 2. The p-values for these tests are computed as

$$2\min(\Pr\{P_{\text{GD}}^a \ge \theta_0\}, \Pr\{P_{\text{GD}}^a \le \theta_0\}) \tag{17}$$
$$2\min(\Pr\{P_{\text{KS}}^a \ge \theta_0\}, \Pr\{P_{\text{KS}}^a \le \theta_0\}) \tag{18}$$
$$2\min(\Pr\{P_{\text{MK}}^a \ge \theta_0\}, \Pr\{P_{\text{MK}}^a \le \theta_0\}) \tag{19}$$
$$2\min(\Pr\{P_{\text{TK}}^a \ge \theta_0\}, \Pr\{P_{\text{TK}}^a \le \theta_0\}). \tag{20}$$

If the p-values are less than significance level α, one should reject the null hypothesis H_0 given in (1), otherwise accept it.

For unequal sample sizes $(n_1 \neq n_2)$, $T_{\text{GD}} = T^a_{\text{GD}}$, $T_{\text{KS}} = T^a_{\text{KS}}$, $T_{\text{MK}} = T^a_{\text{MK}}$ and $T_{\text{TK}} = T^a_{\text{TK}}$ as their corresponding estimators are equal.

Remark 1 Using the above pivot variables, one can also construct the generalized confidence intervals. The $(1-\psi)100\%$ generalized confidence interval is $(P(\psi/2), P(1-\psi/2))$.

4 Simulation Study

In Sect. 3, we have proposed several generalized test variables for the quantile θ under the condition that $\sigma_1^2 \leq \sigma_2^2$. In this section, we will compare the performances of all those proposed test procedures for various combinations of sample sizes in terms of size and power.

To compute the size and power, we have generated 10,000 random samples from each normal population with a common mean and ordered variances, using the Monte Carlo simulation method. In order to compute the generalized test statistics, the inner loop is repeated 5000 times. Though several choices of parameters and sample sizes have been considered in the simulation study, in Tables 3 and 4, we have presented the size and power for a few specific choices of parameters and sample sizes. Throughout the simulation study, we have taken $\eta = 1.96$. For computing the size and power, we have taken $\theta_0 = 1.96$. Observe that all the tests are location invariant; hence, its size only depends on $\rho = \sigma_1/\sigma_2 > 0$ through σ_2. The values of ρ have been varied from 0 to 1 by fixing $\sigma_1 = 1$, so that the condition $\sigma_1^2 \leq \sigma_2^2$ is satisfied.

In Table 3, we present the size of all the eight test procedures for the specific sample sizes. In Table 3, the first column presents the choice of ρ. Corresponding to one value of ρ, there are eight values from the second column onward that present sizes of tests in the given order of sample sizes. In each cell, the size values will be read vertically downward. It is to be noted that for equal sample sizes, that is, when $n_1 = n_2$ as the estimators $\hat{\mu}_{\text{GD}} = \hat{\mu}_{\text{KS}}$, $\hat{\mu}_{\text{MK}} = \hat{\mu}_{\text{TK}}$, $\hat{\mu}^a_{\text{GD}} = \hat{\mu}^a_{\text{KS}}$ and $\hat{\mu}^a_{\text{MK}} = \hat{\mu}^a_{\text{TK}}$, so their corresponding tests, as well as sizes and powers, are also equal. For the unequal sample sizes that is when $n_1 \neq n_2$, $\mu^a_{\text{GD}} = \hat{\mu}_{\text{GD}}$, $\mu^a_{\text{KS}} = \hat{\mu}_{\text{KS}}$, $\mu^a_{\text{MK}} = \hat{\mu}_{\text{MK}}$ and $\mu^a_{\text{TK}} = \hat{\mu}_{\text{TK}}$, so similarly their corresponding tests, sizes and powers are also equal. The maximum bound for the simulation error has been seen up to 0.003 to attain a high level of accuracy. The following observations have been made from our simulation study and also from Tables 3 and 4, which we write in the forms of a remark.

Remark 2 1. It has been observed that all the tests attain the nominal level within 20% of the specified level of significance $\alpha = 0.05$. All the tests are qualified for further power comparison.

2. The powers of all the tests have been computed by varying θ from θ_0, through $\mu = 0.4, 0.6, 0.8, 1, 2$ and fixing $\sigma_1 = \sigma_2 = 1$.

Table 1 p-values for all the proposed tests

Method	T_{GD}	T_{KS}	T_{MK}	T_{TK}	T^a_{GD}	T^a_{KS}	T^a_{MK}	T^a_{TK}
p-value	0.0036	0.0036	0.0034	0.0034	0.0038	0.0038	0.0037	0.0037

Table 2 p-values for all the proposed tests

Method	T_{GD}	T_{KS}	T_{MK}	T_{TK}	T^a_{GD}	T^a_{KS}	T^a_{MK}	T^a_{TK}
p-value	0.7048	0.6972	0.7432	0.7492	0.7048	0.6972	0.7432	0.7492

3. It is further observed that as the difference between θ and θ_0 increases, the powers of all the tests increase up to 1. Moreover, when the sample size increases, the powers of all the tests increase.
4. In the case of equal sample sizes, the test based on the improved estimator μ^a_{GD} and μ^a_{KS} has the best performance in terms of power, whereas all other tests are competing with each other. However, in the case of unequal sample sizes, none of the tests dominates others; that is, all the tests compete well with each other.

5 Concluding Remarks with Real-life Examples

In this article, we have derived several generalized test procedures to test the hypothesis regarding the quantile θ of the first normal population among two normal populations with a common mean and order restricted variances. Utilizing some estimators proposed by Jena et al. [7], the generalized variable as well as test statistics have been constructed to test a hypothesis regarding the quantile. All the proposed test procedures have been compared in terms of their sizes and powers numerically using the Monte Carlo simulation method.

From our simulation study, we have concluded that all the proposed tests attain the nominal level within 20% of the level of significance. It is also concluded that the test based on the estimators μ^a_{GD} and μ^a_{KS} (i.e., T^a_{GD} and T^a_{KS}) have the best performance in terms of power for equal sample sizes, whereas for unequal sample sizes, all the tests compete with each other. We hope that the current research work will enlighten the inference on the quantiles, which have many real-world applications. Below, we discuss two examples, which will illustrate the methods of tests proposed in this paper.

Example 1 We consider the two datasets of equal sample size 10 as given in Jena et al. [7] which have a common mean and ordered variances $\sigma_1^2 \leq \sigma_2^2$. The sufficient statistics for this data is $\bar{x}_1 = 25.36$, $\bar{x}_2 = 25.04$, $s_1^2 = 57.69$ and $s_2^2 = 36.46$. It has been noted that $s_1^2 > s_2^2$ for this data. Suppose, one is interested to test hypothesis $H_0 : \theta = 40$ against the alternative $H_1 : \theta \neq 40$ at the level of significance $\alpha = 0.05$. Table 1 presents the the p-values of all the proposed tests to test this hypothesis.

Table 3 Sizes of the proposed tests for the sample sizes (5, 5), (12, 12), (25, 25), (40, 40), (5, 10), (10, 5), (12, 20), (20, 12) and $\alpha = 0.05$

ρ	T_{GD}	T_{KS}	T_{MK}	T_{TK}	T^a_{GD}	T^a_{KS}	T^a_{MK}	T^a_{TK}
0.2	0.0452	0.0452	0.0472	0.0472	0.0452	0.0452	0.0472	0.0472
	0.0500	0.0500	0.0516	0.0516	0.0500	0.0500	0.0516	0.0516
	0.0512	0.0512	0.0524	0.0524	0.0512	0.0512	0.0524	0.0524
	0.0560	0.0560	0.0500	0.0500	0.0560	0.0560	0.0500	0.0500
	0.0368	0.0412	0.0428	0.0420	0.0368	0.0412	0.0428	0.0420
	0.0352	0.0356	0.0400	0.0400	0.0352	0.0356	0.0400	0.0400
	0.0420	0.0412	0.0452	0.0444	0.0420	0.0412	0.0452	0.0444
	0.0480	0.0476	0.0472	0.0472	0.0480	0.0476	0.0472	0.0472
0.4	0.0384	0.0384	0.0428	0.0428	0.0384	0.0384	0.0428	0.0428
	0.0448	0.0448	0.0456	0.0456	0.0448	0.0448	0.0456	0.0456
	0.0448	0.0448	0.0472	0.0472	0.0448	0.0448	0.0472	0.0472
	0.0516	0.0516	0.0544	0.0544	0.0516	0.0516	0.0544	0.0544
	0.0400	0.0412	0.0404	0.0404	0.0400	0.0412	0.0404	0.0404
	0.0456	0.0464	0.0468	0.0468	0.0456	0.0464	0.0468	0.0468
	0.0496	0.0496	0.0452	0.0444	0.0496	0.0496	0.0452	0.0444
	0.0528	0.0528	0.0560	0.0560	0.0528	0.0528	0.0560	0.0560
0.6	0.0420	0.0420	0.0424	0.0424	0.0436	0.0436	0.0436	0.0436
	0.0496	0.0496	0.0484	0.0484	0.0504	0.0504	0.0484	0.0484
	0.0448	0.0448	0.0432	0.0432	0.0448	0.0448	0.0432	0.0432
	0.0468	0.0468	0.0472	0.0472	0.0468	0.0468	0.0472	0.0472
	0.0440	0.0444	0.0420	0.0416	0.0440	0.0444	0.0420	0.0416
	0.0468	0.0484	0.0480	0.0480	0.0468	0.0484	0.0480	0.0480
	0.0400	0.0408	0.0396	0.0396	0.0400	0.0408	0.0396	0.0396
	0.0524	0.0520	0.0532	0.0512	0.0524	0.0520	0.0532	0.0512
0.8	0.0456	0.0456	0.0464	0.0464	0.0500	0.0500	0.0492	0.0492
	0.0404	0.0404	0.0408	0.0408	0.0456	0.0456	0.0448	0.0448
	0.0444	0.0444	0.0432	0.0432	0.0440	0.0440	0.0432	0.0432
	0.0496	0.0496	0.0492	0.0492	0.0500	0.0500	0.0492	0.0492
	0.0412	0.0408	0.0412	0.0408	0.0412	0.0408	0.0412	0.0408
	0.0488	0.0508	0.0516	0.0504	0.0488	0.0508	0.0516	0.0504
	0.0432	0.0448	0.0412	0.0404	0.0432	0.0448	0.0412	0.0404
	0.0516	0.0536	0.0488	0.0484	0.0516	0.0536	0.0488	0.0484
1.0	0.0372	0.0372	0.0372	0.0372	0.0424	0.0424	0.0400	0.0400
	0.0380	0.0380	0.0385	0.0385	0.0432	0.0432	0.0464	0.0464
	0.0488	0.0488	0.0480	0.0480	0.0496	0.0496	0.0484	0.0484
	0.0468	0.0468	0.0488	0.0488	0.0484	0.0484	0.0480	0.0480
	0.0452	0.0436	0.0436	0.0428	0.0452	0.0436	0.0436	0.0428
	0.0552	0.0552	0.0560	0.0560	0.0552	0.0552	0.0560	0.0560
	0.0376	0.0380	0.0348	0.0348	0.0376	0.0380	0.0348	0.0348
	0.0504	0.0500	0.0492	0.0492	0.0504	0.0500	0.0492	0.0492

Table 4 Powers of the proposed tests for the sample sizes (5, 5), (12, 12), (25, 25), (40, 40), (5, 10), (12, 20), (10, 5), (20, 12) and $\alpha = 0.05$

θ	T_{GD}	T_{KS}	T_{MK}	T_{TK}	T^a_{GD}	T^a_{KS}	T^a_{MK}	T^a_{TK}
2.36	0.1072	0.1072	0.1156	0.1156	0.1300	0.1300	0.1292	0.1292
	0.1488	0.1488	0.1552	0.1552	0.1612	0.1612	0.1604	0.1604
	0.3044	0.3044	0.3044	0.3044	0.3048	0.3048	0.3036	0.3036
	0.4432	0.4432	0.4424	0.4424	0.4464	0.4464	0.4444	0.4444
	0.1056	0.1048	0.1064	0.1064	0.1056	0.1048	0.1064	0.1064
	0.1820	0.1828	0.1824	0.1828	0.1820	0.1828	0.1824	0.1828
	0.1176	0.1200	0.1116	0.1128	0.1176	0.1200	0.1116	0.1128
	0.2364	0.2376	0.2332	0.2340	0.2364	0.2376	0.2332	0.2340
2.56	0.2224	0.2224	0.2316	0.2316	0.2428	0.2428	0.2432	0.2432
	0.3308	0.3308	0.3380	0.3380	0.3508	0.3508	0.3480	0.3480
	0.6068	0.6068	0.6076	0.6076	0.6100	0.6100	0.6072	0.6072
	0.7820	0.7820	0.7808	0.7808	0.7832	0.7832	0.7812	0.7812
	0.1876	0.1864	0.1852	0.1832	0.1876	0.1864	0.1852	0.1832
	0.3512	0.3484	0.3520	0.3504	0.3512	0.3484	0.3520	0.3504
	0.2388	0.2428	0.2328	0.2348	0.2388	0.2428	0.2328	0.2348
	0.4776	0.4800	0.4696	0.4720	0.4776	0.4800	0.4696	0.4720
2.76	0.4136	0.4136	0.4200	0.4200	0.4336	0.4336	0.4296	0.4296
	0.5444	0.5444	0.5496	0.5496	0.5608	0.5608	0.5552	0.5552
	0.8564	0.8564	0.8552	0.8552	0.8564	0.8564	0.8552	0.8552
	0.9612	0.9612	0.9612	0.9612	0.9612	0.9612	0.9612	0.9612
	0.2904	0.2856	0.2884	0.2872	0.2904	0.2856	0.2884	0.2872
	0.5812	0.5816	0.5792	0.5784	0.5812	0.5816	0.5792	0.5784
	0.3992	0.4040	0.3732	0.3816	0.3992	0.4040	0.3732	0.3816
	0.7368	0.7388	0.7272	0.7292	0.7368	0.7388	0.7272	0.7292
2.96	0.6140	0.6140	0.6224	0.6224	0.6312	0.6312	0.6304	0.6304
	0.7380	0.7380	0.7376	0.7376	0.7440	0.7440	0.7412	0.7412
	0.9688	0.9688	0.9684	0.9684	0.9688	0.9688	0.9684	0.9684
	0.9984	0.9984	0.9984	0.9984	0.9984	0.9984	0.9984	0.9984
	0.4432	0.4388	0.4376	0.4328	0.4432	0.4388	0.4376	0.4328
	0.8012	0.8020	0.7996	0.7992	0.8012	0.8020	0.7996	0.7992
	0.6212	0.6204	0.6020	0.6056	0.6212	0.6204	0.6020	0.6056
	0.9084	0.9084	0.9100	0.9108	0.9084	0.9084	0.9100	0.9108
3.96	0.9988	0.9988	1.0000	1.0000	0.9988	0.9988	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.9760	0.9816	0.9784	0.9788	0.9760	0.9816	0.9784	0.9788
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
	0.9988	0.9988	0.9980	0.9980	0.9988	0.9988	0.9980	0.9980
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Since all the p-values are less than 0.05, the null hypothesis H_0 is rejected with significance level 0.05.

Example 2 Rohatgi and Saleh [19] (p. 515) took up an example for a two-sample problem on the mean life of bulbs (in hours). The mean life of the first sample of nine light bulbs is ($\bar{x}_1$) 1309 h and standard deviation 420 h. The second sample of 16 bulbs from other population has a mean ($\bar{x}_2$) 1205 h and a standard deviation 390 h. Equality of means has been tested by two-sample t-test. Further, we apply the F-test and observed that the variances are ordered, that is, $\sigma_1^2 \leq \sigma_2^2$. Therefore, these datasets are considered for our model. It is our interest to test hypothesis $H_0 : \theta = 2040$ against the alternative $H_1 : \theta \neq 2040$ at significance level 0.05. and present the p-values in Table 2.

We can conclude from the above p-values that hypothesis H_0 cannot be rejected using all the test procedures at level 0.05 (Tables 3 and 4).

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