Convergence and Comparison Theorems for Three-Step Alternating Iteration Method for Rectangular Linear System

Smrutilekha Das, Debadutta Mohanty, and Chinmay Kumar Giri

Abstract The three-step alternating iteration method was introduced by Nandi et al. (J. Appl. Math. Comput. 60:485–515, 2019) for solving a rectangular linear system $Ax = b$ by using proper weak regular splittings of type I. In this article, we expand the theory of such alternating iterations by using proper weak regular splittings of type II.

Keywords Moore–Penrose inverse · Proper splitting · Convergence theorems · Comparison theorems · Three-step alternating iteration method

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1 Introduction

Let us consider a system of linear equations of the form

$$Ax = b, \ \ A \in \mathbb{R}^{m \times n}, \ x \in \mathbb{R}^n, \ b \in \mathbb{R}^m. \tag{1.1}$$

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When the coefficient matrix A is very large and sparse, iterative methods become more efficient. In this direction, by using the notion of proper splitting Berman and Plemmons [4] proposed the following iterative method:

$$x^{k+1} = U^\dagger V x^k + U^\dagger b, \quad k = 0, 1, 2, \dots, \tag{1.2}$$

where $U^\dagger V$ is the iteration matrix and $U^\dagger$ is the Moore–Penrose inverse of U. The same authors also proved that the above iterative method converges to $A^\dagger b$ for any initial guess x^0 if and only if the spectral radius of the iteration matrix $U^\dagger V$ is less than 1 (see Corollary 1, [4] for instance). Therefore, the rate of convergence of the iterative method (1.2) depends on $\rho(U^\dagger V)$ and so, the spectral radius of the iteration matrix plays an important role in the comparison of the rate of convergence of different iterative methods of the same system. Many authors such as Berman and Plemmons [4], Climent et al. [7], Climent and Perea [8], Jena et al. [13], Mishra [14, 18], Mishra and Mishra [17], Baliarsingh and Mishra [1], Giri and Mishra [9–12], Shekhar et al. [21], and others have introduced several convergence and comparison results for different subclasses of a proper splitting.

In particular, Mishra [14] proposed the concept of alternating iteration method for rectangular matrices by extending the work of Benzi and Szyld [2]. Mishra [14] considered two proper splittings of $A \in \mathbb{R}^{m \times n}$, namely $A = M - N = U - V$, and proposed the following iterative method

$$x^{k+1} = U^\dagger V M^\dagger N x^k + U^\dagger (V M^\dagger + I) b, \quad k = 0, 1, 2, \dots$$

to solve (1.1). Recently, Nandi et al. [19] introduced the three-step alternating iteration method by extending the work of Mishra [14]. Now we recall the same. Let $A = M - N = P - Q = U - V$ be proper splittings of $A \in \mathbb{R}^{m \times n}$. The same authors considered

$$x^{k+1/3} = M^\dagger N x^k + M^\dagger b, \tag{1.3}$$
$$x^{k+1/2} = P^\dagger Q x^{k+1/3} + P^\dagger b, \tag{1.4}$$
$$x^{k+1} = U^\dagger V x^{k+1/2} + U^\dagger b, \quad k = 0, 1, 2, \dots. \tag{1.5}$$

By simplifying (1.3), (1.4), and (1.5), one can formulate the following iterative method known as *the three-step alternating iterative method for rectangular matrices*

$$x^{k+1} = U^\dagger V P^\dagger Q M^\dagger N x^k + U^\dagger (V P^\dagger Q M^\dagger + V P^\dagger + I) b, \quad k = 0, 1, 2, \dots, \tag{1.6}$$

where $H = U^\dagger V P^\dagger Q M^\dagger N$ is the iteration matrix of the iterative method (1.6). The same authors [19] then studied the convergence criteria for the above iterative method by assuming the splittings $A = M - N = P - Q = U - V$ are proper weak regular of type I (see Theorem 3.1). The convergence of three-step alternating iterations for a singular linear system using the *Group Inverse* (see [3] for the definition) for type II matrix splitting is studied in [22]. However, convergence of three-step alternating

for a rectangular linear system in the case of type II splittings is not yet considered in the literature.

The main objective of this paper is to study the convergence of three-step alternating iteration method by considering the splittings are proper weak regular of type II. To fulfill this objective, we organize the contents of the paper as follows. In Sect. 2, we introduce notations, definitions, and some preliminary results that are frequently used while proving our main results. In Sect. 3, we prove our main results. Here, we derived convergence and comparison results for the three-step alternating iteration method by considering splittings $A = M - N = P - Q = U - V$ are proper weak regular of type II. The findings are verified through numerical examples in Sect. 4. Finally, we concluded this article in Sect. 5.

2 Preliminaries

Throughout the article, all the matrices are considered as real of order $m \times n$, unless stated otherwise. The symbol $\mathbb{R}^{m \times n}$ denotes the set of all real matrices of order $m \times n$ and by $\mathbb{R}^n$ we mean an n-dimensional Euclidean space. The transpose, the range space and the null space of a matrix A are denoted by A^{T}, $\mathcal{R}(A)$ and $\mathcal{N}(A)$, respectively. Let L and M be complementary subspaces of $\mathbb{R}^n$, and $P_{L,M}$ be a projector onto L along M. Then $P_{L,M}A = A$ if and only if $\mathcal{R}(A) \subseteq L$ and $AP_{L,M} = A$ if and only if $\mathcal{N}(A) \supseteq M$. If $L \perp M$, then we denote $P_{L,M}$ by P_L. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $A \in \mathbb{R}^{n \times n}$. Then the spectral radius of $A \in \mathbb{R}^{n \times n}$ is denoted by $\rho(A)$ and is defined by $\rho(A) = \max\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\}$, whereas $\sigma\{A\}$ denotes the set of all eigenvalues of A.

2.1 Nonnegative Matrices

$A \in \mathbb{R}^{m \times n}$ is called nonnegative (positive) if each entry of A is nonnegative (positive) and is denoted by $A \geq 0$ ($A > 0$). For A, $B \in \mathbb{R}^{m \times n}$, $A \geq B$ means $A - B \geq 0$. The same notation and nomenclature are also used for vectors. The next results deal with the nonnegativity of a matrix and its spectral radius.

Theorem 2.1 (Theorem 2.20, [23])
Let $B \in \mathbb{R}^{n \times n}$ and $B \geq 0$. Then

(i) B has a nonnegative real eigenvalue equal to its spectral radius.
(ii) To $\rho(B)$, there corresponds an eigenvector $x \geq 0$.

Theorem 2.2 (Theorem 2.1.11, [6])
Let $B \in \mathbb{R}^{n \times n}$, $B \geq 0$, $x \geq 0$ ($x \neq 0$), and α is a positive scalar.

(i) If $\alpha x \leq Bx$, then $\alpha \leq \rho(B)$.

(ii) If $Bx \leq \alpha x$, $x > 0$, then $\rho(B) \leq \alpha$.

Theorem 2.3 (Theorem 3.15, [23])
Let $B \in \mathbb{R}^{n \times n}$ and $B \geq 0$. Then $\rho(B) < 1$ if and only if $(I - B)^{-1}$ exists and $(I - B)^{-1} = \sum_{n=0}^{\infty} B^n \geq 0$.

2.2 The Moore–Penrose Inverse of a Matrix

For $A \in \mathbb{R}^{m \times n}$, the unique matrix $X \in \mathbb{R}^{n \times m}$ satisfying the following four equations known as Penrose equations:

$$AXA = A, \quad XAX = X, \quad (AX)^{\mathrm{T}} = AX \text{ and } (XA)^{\mathrm{T}} = XA$$

is called the Moore–Penrose inverse of A. It always exists and is denoted by $A^{\dagger}$ (see [20]). Next, we collect some well-known properties of the Moore–Penrose inverse of $A \in \mathbb{R}^{m \times n}$ which will be used frequently in this article, namely: $\mathcal{R}(A^{\dagger}) = \mathcal{R}(A^{\mathrm{T}})$; $\mathcal{N}(A^{\dagger}) = \mathcal{N}(A^{\mathrm{T}})$; $AA^{\dagger} = P_{\mathcal{R}(A^{\dagger})}$; $A^{\dagger}A = P_{\mathcal{R}(A^{\mathrm{T}})}$. In particular, if $x \in \mathcal{R}(A)$, then $x = A^{\dagger}Ax$ (for more details, see [3]). $A \in \mathbb{R}^{m \times n}$ is called semimonotone, if $A^{\dagger} \geq 0$.

2.3 Proper Splittings

A splitting $A = U - V$ of $A \in \mathbb{R}^{m \times n}$ is called a proper splitting if $\mathcal{R}(U) = \mathcal{R}(A)$ and $\mathcal{N}(U) = \mathcal{N}(A)$. A few properties of a proper splitting are summarized below.

Theorem 2.4 (Theorem 1, [4])
Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{m \times n}$. Then

(a) $A = U(I - U^{\dagger}V)$,
(b) $(I - U^{\dagger}V)$ is nonsingular,
(c) $A^{\dagger} = (I - U^{\dagger}V)^{-1}U^{\dagger}$.

Theorem 2.5 (Theorem 1, [7])
Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{m \times n}$. Then

(a) $A = (I - VU^{\dagger})U$,
(b) $(I - VU^{\dagger})$ is nonsingular,
(c) $A^{\dagger} = U^{\dagger}(I - VU^{\dagger})^{-1}$.

Theorem 2.6 (Theorem 1, [15])
Let $A = U - V$ be a proper splitting of $A \in \mathbb{R}^{m \times n}$. Then

(a) $AA^{\dagger} = UU^{\dagger}$ and $A^{\dagger}A = U^{\dagger}U$,

(b) $U^\dagger V A^\dagger = A^\dagger V U^\dagger$,
(c) $U^\dagger V A^\dagger V = A^\dagger V U^\dagger V$,
(d) $V U^\dagger V A^\dagger = V A^\dagger V U^\dagger$.

We refer [5, 16, 17, 21] for methods of construction of a proper splitting for a given $A \in \mathbb{R}^{m \times n}$. Different subclasses of a proper splitting are recalled next.

Definition 2.7 A proper splitting $A = U - V$ of $A \in \mathbb{R}^{m \times n}$ is called a

(i) proper regular splitting [13], if $U^\dagger \geq 0$ and $V \geq 0$.
(ii) proper weak regular splitting of type I [13], if $U^\dagger \geq 0$ and $U^\dagger V \geq 0$.
(iii) proper weak regular splitting of type II [12], if $U^\dagger \geq 0$ and $V U^\dagger \geq 0$.

For the above class of proper splitting, we have the following convergence result.

Theorem 2.8 (Theorem 2.4, [17])
Let $A = U - V$ be any of the above class of splittings of $A \in \mathbb{R}^{m \times n}$. Then $A^\dagger \geq 0$ if and only if $\rho(V U^\dagger) = \rho(U^\dagger V) < 1$.

It is well known that matrix splittings having a smaller radius of iteration matrix gives a faster rate of convergence for (1.2). Therefore, we have the following comparison result for proper weak regular splittings of different types.

Theorem 2.9 (Theorem 3.3, [9])
Let $A = U_1 - V_1 = U_2 - V_2$ be two proper weak regular splitting of different types a semimonotone matrix $A \in \mathbb{R}^{m \times n}$. If $U_1^\dagger \geq U_2^\dagger$, then $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$.

3 Classical Three-Step Alternating Iteration Method

This section deals with the convergence of the three-step alternating iteration method, when the splittings $A = M - N = P - Q = U - V$ are proper weak regular of type II. Before proving our main results, we recall the following results.

Theorem 3.1 (Theorem 15, [19])
Let $A = M - N = P - Q = U - V$ be three proper weak regular splittings of type I of a semimonotone matrix $A \in \mathbb{R}^{m \times n}$. Then $\rho(H) = \rho(U^\dagger V P^\dagger Q M^\dagger N) < 1$.

Theorem 3.2 (Theorem 16, [19])
Let $A = M - N = P - Q = U - V$ be three proper weak regular splittings of type I of a semimonotone matrix $A \in \mathbb{R}^{m \times n}$. Then the unique splitting $A = B - C$ induced by H with $B = M(M + U - A + V P^\dagger N)^\dagger U$ is a proper weak regular splitting of type I if $\mathcal{R}(M + U - A + V P^\dagger N) = \mathcal{R}(A)$ and $\mathcal{N}(M + U - A + V P^\dagger N) = \mathcal{N}(A)$.

A few properties of the matrix B mentioned in the above theorem are obtained next.

Lemma 3.3 *Let $A = M - N = P - Q = U - V$ be three proper splittings of $A \in \mathbb{R}^{m \times n}$. If $B = M(M + U - A + VP^{\dagger}N)^{\dagger}U$, $\mathcal{R}(M + U - A + VP^{\dagger}N) = \mathcal{R}(A)$ and $\mathcal{N}(M + U - A + VP^{\dagger}N) = \mathcal{N}(A)$, then $B^{\dagger} = U^{\dagger}(M + U - A + VP^{\dagger}N)M^{\dagger}$, $\mathcal{R}(B) = \mathcal{R}(A)$ and $\mathcal{N}(B) = \mathcal{N}(A)$.*

Proof Let $X = U^{\dagger}(M + U - A + VP^{\dagger}N)M^{\dagger}$. Since $\mathcal{R}(M + U - A + VP^{\dagger}N) = \mathcal{R}(A)$, $\mathcal{N}(M + U - A + VP^{\dagger}N) = \mathcal{N}(A)$ and $A = M - N = P - Q = U - V$ are proper splittings, we have $(M + U - A + VP^{\dagger}N)(M + U - A + VP^{\dagger}N)^{\dagger} = AA^{\dagger} = MM^{\dagger} = PP^{\dagger} = UU^{\dagger}$. Hence,

$$
\begin{aligned}
XB &= U^{\dagger}(M + U - A + VP^{\dagger}N)M^{\dagger}M(M + U - A + VP^{\dagger}N)^{\dagger}U \\
&= U^{\dagger}(M + U - A + VP^{\dagger}N)(M + U - A + VP^{\dagger}N)^{\dagger}U \\
&= U^{\dagger}U \\
&= (U^{\dagger}U)^{\mathrm{T}} \\
&= (XB)^{\mathrm{T}},
\end{aligned}
$$

and

$$
\begin{aligned}
BX &= M(M + U - A + VP^{\dagger}N)^{\dagger}UU^{\dagger}(M + U - A + VP^{\dagger}N)M^{\dagger} \\
&= M(M + U - A + VP^{\dagger}N)^{\dagger}(M + U - A + VP^{\dagger}N)M^{\dagger} \\
&= MM^{\dagger} \\
&= (MM^{\dagger})^{\mathrm{T}} \\
&= (BX)^{\mathrm{T}}.
\end{aligned}
$$

Therefore, $XBX = U^{\dagger}UU^{\dagger}(M + U - A + VP^{\dagger}N)M^{\dagger} = U^{\dagger}(M + U - A + VP^{\dagger}N)M^{\dagger} = X$ and $BXB = MM^{\dagger}M(M + U - A + VP^{\dagger}N)^{\dagger}U = M(M + U - A + VP^{\dagger}N)^{\dagger}U = B$. Hence, $X = B^{\dagger}$.

Next we will show that $\mathcal{R}(B) = \mathcal{R}(A)$ and $\mathcal{N}(B) = \mathcal{N}(A)$. To do this, we will first prove that $\mathcal{N}(U) = \mathcal{N}(B)$. Clearly, $\mathcal{N}(U) \subseteq \mathcal{N}(B)$. Let $x \in \mathcal{N}(B)$, i.e., $Bx = 0$. By pre-multiplying $M^{\dagger}$ to $Bx = 0$ and using the fact that $M^{\dagger}M = P_{\mathcal{R}((M+U-A+VP^{\dagger}N)^{\mathrm{T}})} = P_{\mathcal{R}(A^{\mathrm{T}})}$, we get $M^{\dagger}M(M + U - A + VP^{\dagger}N)^{\dagger}Ux = (M + U - A + VP^{\dagger}N)^{\dagger}Ux = 0$. Again, pre-multiplying $(M + U - A + VP^{\dagger}N)$ and using the fact $(M + U - A + VP^{\dagger}N)(M + U - A + VP^{\dagger}N)^{\dagger} = P_{\mathcal{R}(M+U-A+VP^{\dagger}N)} = P_{\mathcal{R}(A)} = P_{\mathcal{R}(U)}$, we get $(M + U - A + VP^{\dagger}N)(M + U - A + VP^{\dagger}N)^{\dagger}Ux = Ux = 0$, i.e., $x \in \mathcal{N}(U)$. Hence, $\mathcal{N}(B) \subseteq \mathcal{N}(U)$. Next, we will show that $\mathcal{R}(B) = \mathcal{R}(A)$ which is equivalent to prove $\mathcal{N}(B^{\mathrm{T}}) = \mathcal{N}(A^{\mathrm{T}})$. But $B = M(M + U - A + VP^{\dagger}N)^{\dagger}U$ implies $\mathcal{N}(M^{\mathrm{T}}) \subseteq \mathcal{N}(B^{\mathrm{T}})$. So, we need to show the other way, i.e., $\mathcal{N}(B^{\mathrm{T}}) \subseteq \mathcal{N}(M^{\mathrm{T}})$. Let $x \in \mathcal{N}(B^{\mathrm{T}})$, then $(M(M + U - A + VP^{\dagger}N)^{\dagger}U)^{\mathrm{T}}x = 0$. By pre-multiplying $(U^{\dagger})^{\mathrm{T}}$, we get $(UU^{\dagger})^{\mathrm{T}}((M + U - A + VP^{\dagger}N)^{\dagger})^{\mathrm{T}}M^{\mathrm{T}}x = ((M + U - A + VP^{\dagger}N)^{\dagger}UU^{\dagger})^{\mathrm{T}}M^{\mathrm{T}}x = 0$ and using the fact $UU^{\dagger} = P_{\mathcal{R}(A)} = P_{\mathcal{R}(M+U-A+VP^{\dagger}N)} = (M + U - A + VP^{\dagger}N)(M + U - A + VP^{\dagger}N)^{\dagger}$, we get $((M + U - A + VP^{\dagger}N)^{\dagger})^{\mathrm{T}}M^{\mathrm{T}}x = 0$. Again, pre-multiplying $(M + U - A + VP^{\dagger}N)^{\mathrm{T}}$ and using the fact $(M + U - A + VP^{\dagger}N)^{\dagger}(M +$

$U - A + VP^\dagger N) = P_{\mathcal{R}((M+U-A+VP^\dagger N)^\mathrm{T})} = P_{\mathcal{R}(A^\mathrm{T})} = M^\dagger M$, we obtain $(M + U - A + VP^\dagger N)^\mathrm{T}((M + U - A + VP^\dagger N)^\dagger)^\mathrm{T} M^\mathrm{T} x = (M^\dagger M)^\mathrm{T} M^\mathrm{T} x = M^\mathrm{T} x = 0$. Hence, $\mathcal{N}(B^\mathrm{T}) \subseteq \mathcal{N}(M^\mathrm{T}) = \mathcal{N}(A^\mathrm{T})$. $\qquad\square$

We have the following two expressions for $B^\dagger$,

$$
\begin{aligned}
B^\dagger &= U^\dagger (M + U - A + VP^\dagger N) M^\dagger \\
&= U^\dagger MM^\dagger + U^\dagger UM^\dagger - U^\dagger AM^\dagger + U^\dagger VP^\dagger NM^\dagger \\
&= U^\dagger + U^\dagger VM^\dagger + U^\dagger VP^\dagger NM^\dagger
\end{aligned}
$$

and

$$
\begin{aligned}
B^\dagger &= U^\dagger (M + U - A + VP^\dagger N) M^\dagger \\
&= U^\dagger MM^\dagger + U^\dagger UM^\dagger - U^\dagger AM^\dagger + U^\dagger VP^\dagger NM^\dagger \\
&= M^\dagger + U^\dagger NM^\dagger + U^\dagger VP^\dagger NM^\dagger.
\end{aligned}
$$

The example given below shows that the alternating scheme is convergent even though $A = M - N = P - Q = U - V$ are not proper weak regular splittings of type I.

Example 3.4 Let $A = \begin{bmatrix} 5 & -3 & 5 \\ -3 & 5 & -3 \end{bmatrix}$. Then, $A^\dagger = \begin{bmatrix} 0.1563 & 0.0938 \\ 0.1875 & 0.3125 \\ 0.1563 & 0.0938 \end{bmatrix} \geq 0$. Consider

$A = \begin{bmatrix} 15 & -9 & 15 \\ -3 & 10 & -3 \end{bmatrix} - \begin{bmatrix} 10 & -6 & 10 \\ 0 & 5 & 0 \end{bmatrix} = M - N$. Then $\mathcal{R}(M) = \mathcal{R}(A)$, $\mathcal{N}(M) = \mathcal{N}(A)$,

$M^\dagger = \begin{bmatrix} 0.0407 & 0.0366 \\ 0.0244 & 0.1220 \\ 0.0407 & 0.0366 \end{bmatrix} \geq 0$, $M^\dagger N = \begin{bmatrix} 0.4065 & -0.0610 & 0.4065 \\ 0.2439 & 0.4634 & 0.2439 \\ 0.4065 & -0.0610 & 0.4065 \end{bmatrix} \ngeq 0$

and $NM^\dagger = \begin{bmatrix} 0.6667 & 0 \\ 0.1220 & 0.6098 \end{bmatrix} \geq 0$. Hence, $A = M - N$ is a proper weak regular splitting of type II but not I.

Further consider $A = \begin{bmatrix} 20 & -12 & 20 \\ -3 & 10 & -3 \end{bmatrix} - \begin{bmatrix} 15 & -9 & 15 \\ 0 & 5 & 0 \end{bmatrix} = P - Q$. Then $\mathcal{R}(P) = \mathcal{R}(A)$, $\mathcal{N}(P) = \mathcal{N}(A)$, $P^\dagger = \begin{bmatrix} 0.0305 & 0.0366 \\ 0.0183 & 0.1220 \\ 0.0305 & 0.0366 \end{bmatrix} \geq 0$, $P^\dagger Q = \begin{bmatrix} 0.4573 & -0.0915 & 0.4573 \\ 0.2744 & 0.4451 & 0.2744 \\ 0.4573 & -0.0915 & 0.4573 \end{bmatrix} \ngeq 0$ and $QP^\dagger = \begin{bmatrix} 0.7500 & 0 \\ 0.0915 & 0.6098 \end{bmatrix} \geq 0$. Hence, $A = P - Q$ is a proper weak regular splitting of type II but not I.

Again consider $A = \begin{bmatrix} 25 & -15 & 25 \\ -3 & 10 & -3 \end{bmatrix} - \begin{bmatrix} 20 & -12 & 20 \\ 0 & 5 & 0 \end{bmatrix} = U - V$. Then $\mathcal{R}(U) = \mathcal{R}(A)$, $\mathcal{N}(U) = \mathcal{N}(A)$, $U^\dagger = \begin{bmatrix} 0.0244 & 0.0366 \\ 0.0146 & 0.1220 \\ 0.0244 & 0.0366 \end{bmatrix} \geq 0$,

$$U^\dagger V = \begin{bmatrix} 0.4878 & -0.1098 & 0.4878 \\ 0.2927 & 0.4341 & 0.2927 \\ 0.4878 & -0.1098 & 0.4878 \end{bmatrix} \not\geq 0 \text{ and } V U^\dagger = \begin{bmatrix} 0.8000 & 0 \\ 0.0732 & 0.6098 \end{bmatrix} \geq 0. \text{ Hence,}$$

$A = U - V$ is a proper weak regular splitting of type II but not I. But $\rho(H) = \rho(U^\dagger V P^\dagger Q M^\dagger N) = 0.4 < 1$, and A is semimonotone.

In the above example, we can see that $H = U^\dagger V P^\dagger Q M^\dagger N$

$$= \begin{bmatrix} 0.3046 & -0.1147 & 0.3046 \\ 0.3486 & 0.0176 & 0.3486 \\ 0.3046 & -0.1147 & 0.3046 \end{bmatrix} \not\geq 0 \text{ but } S = V U^\dagger Q P^\dagger N M^\dagger =$$

$$\begin{bmatrix} 0.4 & 0 \\ 0.1191 & 0.2267 \end{bmatrix} \geq 0.$$

Motivated by the above example, we will now introduce a similar result as Theorem 3.1 by considering the given splittings are proper weak regular of type II. With this objective, we first derived the following properties of H and S which are useful to prove further results.

Theorem 3.5 *Let* $A = M - N = P - Q = U - V$ *be three proper splittings of* $A \in \mathbb{R}^{m \times n}$ *and* $S = V U^\dagger Q P^\dagger N M^\dagger$. *Then,*

  (i) $AA^\dagger S = S = SAA^\dagger$ *and* $A^\dagger A H = H = HA^\dagger A$, *where* H *is the iteration matrix of the iterative method* (1.6).
  (ii) $S = AHA^\dagger$, $H = A^\dagger SA$ *and* $\rho(S) = \rho(H)$.
  (iii) $I - S$ *and* $I - H$ *are invertible if* $\mathcal{R}(M + U - A + V P^\dagger N) = \mathcal{R}(A)$ *and* $\mathcal{N}(M + U - A + V P^\dagger N) = \mathcal{N}(A)$.

  Proof (i) $AA^\dagger S = AA^\dagger V U^\dagger Q P^\dagger N M^\dagger = S$ using the fact that $\mathcal{R}(S) \subseteq \mathcal{R}(V) \subseteq \mathcal{R}(A)$. Again $SAA^\dagger = V U^\dagger Q P^\dagger N M^\dagger AA^\dagger = V U^\dagger Q P^\dagger N M^\dagger M M^\dagger = S$. Similar argument yields the other equality.
  (ii) By Theorem 2.6, we have $U^\dagger V A^\dagger = A^\dagger V U^\dagger$, $P^\dagger Q A^\dagger = A^\dagger Q P^\dagger$ and $M^\dagger N A^\dagger = A^\dagger N M^\dagger$. So

$$\begin{aligned} S &= AA^\dagger S \\ &= AA^\dagger V U^\dagger Q P^\dagger N M^\dagger \\ &= AU^\dagger V A^\dagger Q P^\dagger N M^\dagger \\ &= AU^\dagger V P^\dagger Q A^\dagger N M^\dagger \\ &= AU^\dagger V P^\dagger Q M^\dagger N A^\dagger \\ &= AHA^\dagger. \end{aligned}$$

We then have $A^\dagger S = A^\dagger AHA^\dagger = HA^\dagger$. Again, post-multiplying A, we get $H = A^\dagger SA$.

Consider $Sx = \lambda x$, where λ is an eigenvalue of S, and x is its corresponding eigenvector. Then $x \in \mathcal{R}(S) \subseteq \mathcal{R}(A)$. Now $\lambda x = Sx = AHA^\dagger x$. Pre-multiplying $A^\dagger$, we get $\lambda y = Hy$, where $y = A^\dagger x$. Therefore, λ is an eigenvalue of H if, $y \neq 0$. Suppose that $y = A^\dagger x = 0$. Then $x \in \mathcal{N}(A^\dagger) = \mathcal{N}(A^T)$. So, we have $x \in \mathcal{R}(A) \cap \mathcal{N}(A^T) =$

$\{0\}$ a contradiction. Hence, $y \neq 0$ and so $\sigma\{S\} \subseteq \sigma\{H\}$. For the other way, consider $Hy = \mu y$, where μ is an eigenvalue of H, and y is its corresponding eigenvector. Then $y \in \mathcal{R}(H) \subseteq \mathcal{R}(A^T)$. Also $\mu y = A^\dagger S A y$. Pre-multiplying A, we get $\mu z = S z$, where $z = Ay$. Suppose that $z = 0$. Then $y \in \mathcal{N}(A)$. So $y \in \mathcal{R}(A^T) \cap \mathcal{N}(A)$ which yields $y = 0$, a contradiction. Hence, $\sigma\{H\} \subseteq \sigma\{S\}$. Therefore, $\rho(S) = \rho(H)$.

(iii) We will prove this by the method of contradiction. Suppose that $I - S$ is not invertible. Then 1 is an eigenvalue of S. Therefore, $x = VU^\dagger QP^\dagger NM^\dagger x \in \mathcal{R}(VU^\dagger QP^\dagger NM^\dagger) \subseteq \mathcal{R}(V) \subseteq \mathcal{R}(U) = \mathcal{R}(P) = \mathcal{R}(M) = \mathcal{R}(A)$. So, $x = UU^\dagger x = PP^\dagger x = MM^\dagger x$ and hence

$$
\begin{aligned}
x &= VU^\dagger QP^\dagger NM^\dagger x \\
&= (U - A)U^\dagger(P - A)P^\dagger(M - A)M^\dagger x \\
&= (UU^\dagger - AU^\dagger)(PP^\dagger - AP^\dagger)(MM^\dagger - AM^\dagger)x \\
&= (AA^\dagger - AU^\dagger)(AA^\dagger - AP^\dagger)(AA^\dagger - AM^\dagger)x \\
&= (AA^\dagger - AU^\dagger)(AA^\dagger - AP^\dagger)(AA^\dagger x - AM^\dagger x) \\
&= (AA^\dagger - AU^\dagger)(AA^\dagger - AP^\dagger)(x - AM^\dagger x) \\
&= (AA^\dagger - AU^\dagger)(AA^\dagger x - AA^\dagger AM^\dagger x - AP^\dagger x + AP^\dagger AM^\dagger x) \\
&= (AA^\dagger - AU^\dagger)(x - AM^\dagger x - AP^\dagger x + AP^\dagger AM^\dagger x) \\
&= (x - AM^\dagger x - AP^\dagger x + AP^\dagger AM^\dagger x - AU^\dagger x + AU^\dagger AM^\dagger x \\
&\quad + AU^\dagger AP^\dagger x - AU^\dagger AP^\dagger AM^\dagger x) \\
&= x - A(M^\dagger + P^\dagger - P^\dagger AM^\dagger + U^\dagger - U^\dagger AM^\dagger - U^\dagger AP^\dagger \\
&\quad + U^\dagger AP^\dagger AM^\dagger)x \\
&= x - A(M^\dagger MM^\dagger + P^\dagger PP^\dagger + U^\dagger - P^\dagger AM^\dagger - U^\dagger AM^\dagger - U^\dagger AP^\dagger \\
&\quad + U^\dagger AP^\dagger AM^\dagger)x \\
&= x - A(U^\dagger UM^\dagger + U^\dagger UP^\dagger + U^\dagger - P^\dagger AM^\dagger - U^\dagger AM^\dagger - U^\dagger AP^\dagger \\
&\quad + U^\dagger AP^\dagger AM^\dagger)x \\
&= x - A(U^\dagger UM^\dagger + U^\dagger UP^\dagger PP^\dagger + U^\dagger UU^\dagger - P^\dagger PP^\dagger AM^\dagger - U^\dagger AM^\dagger \\
&\quad - U^\dagger AP^\dagger PP^\dagger + U^\dagger AP^\dagger AM^\dagger)x \\
&= x - A(U^\dagger UM^\dagger + U^\dagger UP^\dagger MM^\dagger + U^\dagger MM^\dagger - U^\dagger UP^\dagger AM^\dagger - U^\dagger AM^\dagger \\
&\quad - U^\dagger AP^\dagger M^\dagger + U^\dagger AP^\dagger AM^\dagger)x \\
&= x - A(U^\dagger(U + M - A + UP^\dagger M - UP^\dagger A - AP^\dagger M + AP^\dagger A)M^\dagger)x \\
&= x - A(U^\dagger(U + M - A + VP^\dagger N)M^\dagger)x \\
&= x - AB^\dagger x.
\end{aligned}
$$

Then $AB^\dagger x = 0$. Thus, $B^\dagger x \in \mathcal{N}(A) = \mathcal{N}(B)$ and so $BB^\dagger x = 0$. But $x \in \mathcal{R}(A) = \mathcal{R}(B)$. (We have used the facts that $\mathcal{N}(B) = \mathcal{N}(A)$ and $\mathcal{R}(B) = \mathcal{R}(A)$ which follows from Lemma 3.3.) Therefore, $x = BB^\dagger x = 0$, a contradiction. Thus, $I - S$ is invertible.

The other proof is explained next. Let $(I - H)x = (I - U^\dagger VP^\dagger QM^\dagger N)x = 0$. So, $x \in \mathcal{R}(U^\dagger) = \mathcal{R}(U^T) = \mathcal{R}(A^T)$. Substituting $V = U - A$, $Q = P - A$ and $N = M - A$ in $(I - H)x = 0$ and then simplifying, we get $B^\dagger Ax = 0$. Pre-multiplying B and using the fact $\mathcal{R}(B) = \mathcal{R}(A)$, we get $x \in \mathcal{N}(A)$. Hence, $x = 0$ yielding a contradiction. Thus, $I - H$ is invertible. $\square$

Theorem 3.6 *Let $A = M - N = P - Q = U - V$ be three proper weak regular splittings of type II of a semimonotone matrix $A \in \mathbb{R}^{m \times n}$ and $S = VU^\dagger QP^\dagger NM^\dagger$. Then $\rho(H) = \rho(U^\dagger VP^\dagger QM^\dagger N) < 1$.*

Proof We have

$$
\begin{aligned}
0 \le S &= VU^\dagger QP^\dagger NM^\dagger \\
&= (U - A)U^\dagger(P - A)P^\dagger(M - A)M^\dagger \\
&= (UU^\dagger - AU^\dagger)(PP^\dagger - AP^\dagger)(MM^\dagger - AM^\dagger) \\
&= (AA^\dagger - AU^\dagger)(AA^\dagger - AP^\dagger)(AA^\dagger - AM^\dagger) \\
&= AA^\dagger - AM^\dagger - AP^\dagger - AU^\dagger + AP^\dagger AM^\dagger + AU^\dagger AM^\dagger \\
&\quad + AU^\dagger AP^\dagger - AU^\dagger AP^\dagger AM^\dagger
\end{aligned}
$$

since $A = M - N = P - Q = U - V$ are proper weak regular splittings of type II. Now

$$
\begin{aligned}
A^\dagger(I - S) &= A^\dagger - A^\dagger S \\
&= A^\dagger AA^\dagger - A^\dagger S \\
&= A^\dagger(AA^\dagger - S) \\
&= A^\dagger(AA^\dagger - AA^\dagger + AM^\dagger + AP^\dagger + AU^\dagger - AP^\dagger AM^\dagger - AU^\dagger AM^\dagger \\
&\quad - AU^\dagger AP^\dagger + AU^\dagger AP^\dagger AM^\dagger) \\
&= A^\dagger AM^\dagger + A^\dagger AP^\dagger + A^\dagger AU^\dagger - A^\dagger AP^\dagger AM^\dagger - A^\dagger AU^\dagger AM^\dagger \\
&\quad - A^\dagger AU^\dagger AP^\dagger + A^\dagger AU^\dagger AP^\dagger AM^\dagger \\
&= M^\dagger + P^\dagger + U^\dagger - P^\dagger AM^\dagger - U^\dagger AM^\dagger - U^\dagger AP^\dagger + U^\dagger AP^\dagger AM^\dagger \\
&= M^\dagger + P^\dagger(M - A)M^\dagger + U^\dagger(P - A)P^\dagger + U^\dagger(AP^\dagger A - A)M^\dagger \\
&= M^\dagger + P^\dagger NM^\dagger + U^\dagger QP^\dagger + U^\dagger(AP^\dagger A - PP^\dagger A)M^\dagger \\
&= M^\dagger + P^\dagger NM^\dagger + U^\dagger QP^\dagger - U^\dagger(P - A)P^\dagger AM^\dagger \\
&= M^\dagger + P^\dagger NM^\dagger + U^\dagger QP^\dagger - U^\dagger QP^\dagger(M - N)M^\dagger \\
&= M^\dagger + P^\dagger NM^\dagger + U^\dagger QP^\dagger NM^\dagger \ge 0.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
0 \le A^\dagger(I - S)(I + S + S^2 + S^3 + \cdots + S^m) \\
= A^\dagger(I - S^{m+1}) \\
\le A^\dagger
\end{aligned}
$$

for each $m \in \mathbb{N}$. So, the partial sums of the series $\sum_{m=0}^{\infty} S^m$ is uniformly bounded. Hence, $\rho(S) < 1$. Thus $\rho(H) = \rho(S) < 1$ by Theorem 3.5 (ii). $\qquad\square$

The next result showing that the matrix B in the splitting $A = B - C$ induced by H can also be expressed as the product of A and $(I - H)^{-1}$.

Lemma 3.7 *Let $A = M - N = P - Q = U - V$ be three proper splittings of $A \in \mathbb{R}^{m \times n}$ such that $\rho(H) < 1$, $\mathcal{R}(M + P - A + VP^\dagger N) = \mathcal{R}(A)$ and $\mathcal{N}(M + P - A + VP^\dagger N) = \mathcal{N}(A)$. Then the unique splitting $A = B - C$ induced by H is a proper splitting such that $H = B^\dagger C$, where $B = A(I - H)^{-1}$.*

Proof Let $B = A(I - H)^{-1}$ and $C = B - A$. Then

$$
\begin{aligned}
B^\dagger C &= B^\dagger (B - A) \\
&= B^\dagger B - B^\dagger A \\
&= B^\dagger B - (I - H)A^\dagger A \\
&= A^\dagger A - A^\dagger A + H A^\dagger A \\
&= H
\end{aligned}
$$

using the fact that $A = B - C$ is a proper splitting which is shown next.

Let $Z = (I - H)A^\dagger$. Then $BZ = AA^\dagger$ which is symmetric and $ZBZ = (I - H)A^\dagger A A^\dagger = (I - H)A^\dagger = Z$. Again

$$
\begin{aligned}
ZB &= (I - H)A^\dagger A(I - H)^{-1} \\
&= (A^\dagger A - H A^\dagger A)(I - H)^{-1} \\
&= (A^\dagger A - A^\dagger A H)(I - H)^{-1} \\
&= A^\dagger A
\end{aligned}
$$

which is symmetric and $BZB = AA^\dagger A(I - H)^{-1} = A(I - H)^{-1} = B$. So, we have

$$
\begin{aligned}
B^\dagger &= (I - H)A^\dagger \\
&= A^\dagger - U^\dagger V P^\dagger Q M^\dagger N A^\dagger \\
&= A^\dagger - (U^\dagger(U - A)P^\dagger(P - A)M^\dagger(M - A)A^\dagger) \\
&= A^\dagger - (P^\dagger - U^\dagger A P^\dagger)(P M^\dagger - A M^\dagger)(M A^\dagger - A A^\dagger) \\
&= A^\dagger - (M^\dagger - P^\dagger A M^\dagger - U^\dagger A M^\dagger + U^\dagger A P^\dagger A M^\dagger)(M A^\dagger - A A^\dagger) \\
&= M^\dagger + P^\dagger - P^\dagger A M^\dagger + U^\dagger - U^\dagger A M^\dagger - U^\dagger A P^\dagger + U^\dagger A P^\dagger A M^\dagger \\
&= M^\dagger + U^\dagger - U^\dagger A M^\dagger + P^\dagger - P^\dagger(P - Q)M^\dagger - U^\dagger(U - V)P^\dagger \\
&\quad + U^\dagger(U - V)P^\dagger(M - N)M^\dagger \\
&= M^\dagger + U^\dagger - U^\dagger A M^\dagger + U^\dagger V P^\dagger N M^\dagger.
\end{aligned}
$$

Hence, $A = B - C$ is a proper splitting, by Lemma 3.3.

For uniqueness, suppose that there exists another splitting $A = \bar{B} - \bar{C}$ such that $H = \bar{B}^\dagger \bar{C}$. Then $\bar{B}H = \bar{B}\bar{B}^\dagger \bar{C} = \bar{C} = \bar{B} - A$. So $\bar{B}(I - H) = A$. Hence $\bar{B} = A(I - H)^{-1} = B$. $\square$

Theorem 3.8 *Let $A = M - N = P - Q = U - V$ be three proper weak regular splittings of type II of a semimonotone matrix $A \in \mathbb{R}^{m \times n}$ and $S = VU^\dagger QP^\dagger NM^\dagger$. Then, H and S induce the same proper splitting $A = B - C$ if $\mathcal{R}(M + U - A + VP^\dagger N) = \mathcal{R}(A)$ and $\mathcal{N}(M + U - A + VP^\dagger N) = \mathcal{N}(A)$. Furthermore, the unique proper splitting $A = X - Y$ induced by matrix S is also a proper weak regular splitting of type II.*

Proof By Lemma 3.7, we have $B = A(I - H)^{-1}$. Let us consider $X = (I - S)^{-1}A$ and $Y = X - A$. We will show that the matrices H and S induce the same proper splitting $A = B - C$. Since $H = HA^\dagger A$ and $S = AHA^\dagger$, so $S^k = AH^k A^\dagger$, for any nonnegative integer k. By Theorem 3.6, we have $\rho(S) < 1$. Also, $S \geq 0$. Therefore, Theorem 2.3 yields

$$X = (I - S)^{-1}A$$
$$= \sum_{k=o}^{\infty} S^k A$$
$$= \sum_{k=o}^{\infty} AH^k A^\dagger A$$
$$= \sum_{k=o}^{\infty} AH^k$$
$$= A(I - H)^{-1}$$
$$= B.$$

Then $\mathcal{R}(X) = \mathcal{R}(B) = \mathcal{R}(A)$ and $\mathcal{N}(X) = \mathcal{N}(B) = \mathcal{N}(A)$ which in turn yields $A = X - Y$ is a proper splitting.

To prove $A = X - Y$ is a proper weak regular splitting of type II, consider $Z = A^\dagger(I - S)$. Then $ZX = A^\dagger(I - S)(I - S)^{-1}A = A^\dagger A$. Hence, ZX is symmetric and $ZXZ = A^\dagger AA^\dagger(I - S) = A^\dagger(I - S) = Z$. Using the property $AA^\dagger S = S = SAA^\dagger$, we obtain

$$XZ = (I - S)^{-1}AA^\dagger(I - S)$$
$$= (I - S)^{-1}(AA^\dagger - AA^\dagger S)$$
$$= (I - S)^{-1}(AA^\dagger - SAA^\dagger)$$
$$= AA^\dagger.$$

So, XZ is symmetric and

$$XZX = AA^\dagger(I - S)^{-1}A$$

$$= AA^\dagger \sum_{k=0}^{\infty} S^k A$$

$$= \sum_{k=0}^{\infty} S^k A$$

$$= (I - S)^{-1}A$$

$$= X.$$

Hence, $X^\dagger = A^\dagger(I - S) \geq 0$ (see the proof of Theorem 3.6 for $A^\dagger(I - S) \geq 0$). Therefore,

$$YX^\dagger = (X - A)X^\dagger$$

$$= XX^\dagger - AX^\dagger$$

$$= AA^\dagger - AA^\dagger(I - S)$$

$$= S \geq 0.$$

Thus $A = X - Y$ is a proper weak regular splitting of type II induced by S. Let $A = X_1 - Y_1$ be another splitting induced by S such that $S = Y_1 X_1^\dagger$. Then $SX_1 = Y_1 X_1^\dagger X_1 = Y_1 = X_1 - A$. So $A = X_1 - SX_1 = (I - S)X_1$ which implies $X_1 = (I - S)^{-1}A = X$. Hence, $A = X - Y$ is the unique proper weak regular splitting of type II induced by S. □

The next results confirm that the proposed alternating iterative scheme converges faster than (1.2) under suitable assumptions.

Theorem 3.9 *Let $A = M - N = P - Q = U - V$ be three proper regular splittings of a semimonotone matrix $A \in \mathbb{R}^{m \times n}$ with $\mathcal{R}(M + U - A + VP^\dagger N) = \mathcal{R}(A)$ and $\mathcal{N}(M + U - A + VP^\dagger N) = \mathcal{N}(A)$. Then $\rho(H) \leq \min\{\rho(M^\dagger N), \ \rho(P^\dagger Q), \ \rho(U^\dagger V)\} < 1$.*

Proof By Theorem 3.8, $A = B - C$ is a proper weak regular splitting of type II induced by H, and from (1.6), we have

$$B^\dagger = U^\dagger(VP^\dagger QM^\dagger + VP^\dagger + I) = U^\dagger VP^\dagger QM^\dagger + U^\dagger VP^\dagger + U^\dagger \geq U^\dagger.$$

and

$$B^\dagger = U^\dagger VP^\dagger QM^\dagger + U^\dagger VP^\dagger + U^\dagger$$

$$= U^\dagger VP^\dagger QM^\dagger + U^\dagger UP^\dagger - U^\dagger AP^\dagger + U^\dagger MM^\dagger$$

$$= U^\dagger VP^\dagger QM^\dagger + P^\dagger + U^\dagger(M - A)M^\dagger$$

$$= U^\dagger VP^\dagger QM^\dagger + P^\dagger + U^\dagger NM^\dagger \geq P^\dagger.$$

Also,

$$
\begin{aligned}
B^\dagger &= U^\dagger (M + U - A + V P^\dagger N) M^\dagger \\
&= U^\dagger M M^\dagger + U^\dagger U M^\dagger - U^\dagger (M - N) M^\dagger + U^\dagger V P^\dagger N M^\dagger \\
&= U^\dagger + M^\dagger - U^\dagger M M^\dagger + U^\dagger N M^\dagger + U^\dagger V P^\dagger N M^\dagger \\
&= U^\dagger + M^\dagger - U^\dagger + U^\dagger N M^\dagger + U^\dagger V P^\dagger N M^\dagger \\
&= M^\dagger + U^\dagger N M^\dagger + U^\dagger V P^\dagger N M^\dagger \geq M^\dagger.
\end{aligned}
$$

Hence, by applying Theorem 2.9 to the pair of the splittings $A = B - C$ and $A = P - Q$, $A = B - C$ and $A = U - V$, and $A = B - C$ and $A = B - C$, we have $\rho(H) \leq \rho(P^\dagger Q) < 1$, $\rho(H) \leq \rho(M^\dagger N) < 1$ and $\rho(H) \leq \rho(U^\dagger V) < 1$, respectively. Therefore, $\rho(H) \leq \min\{\rho(P^\dagger Q), \ \rho(M^\dagger N), \ \rho(U^\dagger V)\} < 1$. □

Theorem 3.10 *Let $A = P - Q = M - N = U - V$ be three proper weak regular splittings of type II of a semimonotone matrix $A \in \mathbb{R}^{n \times n}$ with $\mathcal{R}(M + U - A + V P^\dagger N) = \mathcal{R}(A)$ and $\mathcal{N}(M + U - A + V P^\dagger N) = \mathcal{N}(A)$. Let $A = B - C$ be the proper weak regular splitting of type II induced by H (or S). If $P B^\dagger \geq I$, $M B^\dagger \geq I$ and $U B^\dagger \geq I$, then $\rho(H) \leq \min\{\rho(P^\dagger Q), \ \rho(M^\dagger N), \ \rho(U^\dagger V)\} < 1$.*

Proof Consider the pair of splittings $A = B - C$ and $A = P - Q$. By Theorem 2.5, we have

$$
B^\dagger (I - C B^\dagger)^{-1} = P^\dagger (I - Q P^\dagger)^{-1}. \tag{3.1}
$$

Pre-multiplying (3.1) by P, we obtain

$$
P B^\dagger (I - C B^\dagger)^{-1} = P P^\dagger (I - Q P^\dagger)^{-1} = (I - Q P^\dagger)^{-1}. \tag{3.2}
$$

As $C B^\dagger \geq 0$, there exists an eigenvector $x \geq 0$ such that $C B^\dagger x = \rho(C B^\dagger) x$ by Theorem 2.1 . Post-multiplying (3.2) by x, we get $P B^\dagger (I - C B^\dagger)^{-1} x = (I - Q P^\dagger)^{-1} x$, i.e., $\frac{P B^\dagger x}{1 - \rho(B^\dagger C)} = (I - Q P^\dagger)^{-1} x$. Using the fact $P B^\dagger \geq I$, we have

$$
\frac{x}{1 - \rho(C B^\dagger)} \leq \frac{P B^\dagger x}{1 - \rho(C B^\dagger)} = (I - Q P^\dagger)^{-1} x.
$$

Thus, $\rho(B^\dagger C) \leq \rho(P^\dagger Q)$ by Theorem 2.2 (i). Similarly, for the pair of splittings $A = B - C$ and $A = M - N$, and $A = B - C$ and $A = U - V$, we have $\rho(H) \leq \rho(M^\dagger N) < 1$ and $\rho(H) \leq \rho(U^\dagger V) < 1$, respectively. Hence, $\rho(H) \leq \min\{\rho(P^\dagger Q), \ \rho(M^\dagger N), \ \rho(U^\dagger V)\} < 1$. □

Theorem 3.11 *Let $A = M - N = P - Q = U - V$ be three proper regular splittings of a semimonotone matrix $A \in \mathbb{R}^{m \times n}$ with $\mathcal{R}(M + U - A + V P^\dagger N) = \mathcal{R}(A)$ and $\mathcal{N}(M + U - A + V P^\dagger N) = \mathcal{N}(A)$. Then $\rho(H) \leq \min\{\rho(P^\dagger Q M^\dagger N), \ \rho(U^\dagger V P^\dagger Q), \ \rho(U^\dagger V M^\dagger N)\} < 1$.*

Proof Suppose $A = B - C$ be the splitting induced by the matrix H. Let $A = B_1 - C_1$, $A = B_2 - C_2$, and $A = B_3 - C_3$ be the splitting induced by the matrices $M^\dagger N P^\dagger Q$, $M^\dagger N U^\dagger V$, and $P^\dagger Q U^\dagger V$, respectively. Then

$$
\begin{aligned}
B^\dagger &= M^\dagger + U^\dagger - U^\dagger A M^\dagger + U^\dagger V P^\dagger N M^\dagger \\
&\geq M^\dagger + U^\dagger - U^\dagger A M^\dagger \\
&= U^\dagger (M + U - A) M^\dagger = B_2^\dagger
\end{aligned}
$$

and by Theorems 3.8 and 2.9, we have $\rho(B^\dagger C) \leq \rho(B_2^\dagger C_2)$, i.e., $\rho(H) \leq \rho(U^\dagger V M^\dagger N)$. Again,

$$
\begin{aligned}
B^\dagger &= M^\dagger + P^\dagger + U^\dagger - P^\dagger A M^\dagger - U^\dagger A P^\dagger + U^\dagger A P^\dagger A M^\dagger \\
&= M^\dagger + P^\dagger - P^\dagger A M^\dagger + U^\dagger - U^\dagger A M^\dagger - U^\dagger A P^\dagger + U^\dagger A P^\dagger A M^\dagger \\
&= M^\dagger + P^\dagger - P^\dagger A M^\dagger + U^\dagger - U^\dagger (M - N) M^\dagger - U^\dagger (P - Q) P^\dagger \\
&\quad + U^\dagger (P - Q) P^\dagger (M - N) M^\dagger \\
&= M^\dagger + P^\dagger - P^\dagger A M^\dagger + U^\dagger N M^\dagger - U^\dagger + U^\dagger Q P^\dagger \\
&\quad + U^\dagger (P P^\dagger - Q P^\dagger)(M M^\dagger - N M^\dagger) \\
&= M^\dagger + P^\dagger - P^\dagger A M^\dagger + U^\dagger Q P^\dagger N M^\dagger \\
&\geq M^\dagger + P^\dagger - P^\dagger A M^\dagger \\
&= P^\dagger (M + P - A) M^\dagger = B_1^\dagger
\end{aligned}
$$

and by Theorems 3.8 and 2.9, we have $\rho(B^\dagger C) \leq \rho(B_1^\dagger C_1)$, i.e., $\rho(H) \leq \rho(P^\dagger Q M^\dagger N)$.

Similarly, we can obtain

$$
\begin{aligned}
B^\dagger &= U^\dagger + P^\dagger - U^\dagger A P^\dagger + U^\dagger V P^\dagger Q M^\dagger \\
&\geq U^\dagger + P^\dagger - U^\dagger A P^\dagger \\
&= B_3^\dagger
\end{aligned}
$$

and $\rho(H) \leq \rho(U^\dagger V P^\dagger Q)$. Hence $\rho(H) \leq \min\{\rho(P^\dagger Q M^\dagger N), \rho(U^\dagger V P^\dagger Q), \rho(U^\dagger V M^\dagger N)\} < 1$. $\qquad\square$

The following example demonstrates the above results.

Example 3.12 Consider the matrices in Example 3.4. Then $S = V U^\dagger Q P^\dagger N M^\dagger$
$= \begin{bmatrix} 0.4000 & 0 \\ 0.1191 & 0.2267 \end{bmatrix}$ and the splitting induced by S is $A = X - Y$
$= \begin{bmatrix} 8.3333 & -5 & 8.3333 \\ -2.5960 & 5.6957 & -2.5960 \end{bmatrix} - \begin{bmatrix} 3.3333 & -2 & 3.3333 \\ 0.4040 & 0.6957 & 0.4040 \end{bmatrix}$. Here, $\mathcal{R}(A) = \mathcal{R}(X)$,

$\mathcal{N}(A) = \mathcal{N}(X)$, $X^{\dagger} = \begin{bmatrix} 0.0826 & 0.0725 \\ 0.0753 & 0.2417 \\ 0.0826 & 0.0725 \end{bmatrix} \geq 0$ and $YX^{\dagger} = \begin{bmatrix} 0.4000 & 0 \\ 0.1191 & 0.2267 \end{bmatrix} \geq 0$.

Hence, the induced splitting $A = X - Y$ is a proper weak regular splitting of type II.

Also, $PX^{\dagger} = \begin{bmatrix} 2.4000 & 0 \\ 0.2573 & 1.9816 \end{bmatrix} \geq I$, $MX^{\dagger} = \begin{bmatrix} 1.8000 & 0 \\ 0.2573 & 1.9816 \end{bmatrix} \geq I$, $UX^{\dagger} = \begin{bmatrix} 3 & 0 \\ 0.2573 & 1.9816 \end{bmatrix} \geq I$, respectively. Further $\rho(H) = 0.4 < \min\{\rho(M^{\dagger}N), \rho(P^{\dagger}Q), \rho(U^{\dagger}V)\} = \min\{0.6666, 0.75, 0.8\} < 1$ and $\rho(H) = 0.4 < \min\{\rho(P^{\dagger}QM^{\dagger}N), \rho(U^{\dagger}VP^{\dagger}Q), \rho(U^{\dagger}VM^{\dagger}N)\} = \min\{0.5, 0.6, 0.5333\} < 1$.

4 Numerical Computation

In this section, we demonstrate a few numerical examples to validate the proposed theory. The estimation of error bounds, time in seconds, the number of iterations (IT), and the spectral radius of the corresponding iteration matrix are evaluated using MATLAB. We use MATLAB R2015a for the numerical computations in a Windows operating system with configurations: Intel(R) Xenon(R) E-2224 with 16 GB RAM. To terminate the iterative process, we use the stopping criteria $||x_k - x_{k-1}|| < \epsilon$, where $\epsilon = 10^{-7}$.

   To show that the three-step iterative scheme converges faster than the two-step and single-step iterative schemes, we consider two different inconsistent linear system and we consider three proper splittings of the coefficient matrices.

Example 4.1 Let us consider a linear system of the form (1.1) with $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 8 \\ 11 & 11 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

and $b = [1, 1, 1, 1]^{T}$. We have

$A^{\dagger} = \begin{bmatrix} 0.0041 & 0 & 0.0451 & 0 \\ 0.0041 & 0 & 0.0451 & 0 \\ 0 & 0.1176 & 0 & 0.0294 \end{bmatrix} \geq 0$.   Setting $M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 12 \\ 11 & 11 & 0 \\ 0 & 0 & 3 \end{bmatrix}$,

$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 28 \\ 11 & 11 & 0 \\ 0 & 0 & 7 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 16 \\ 11 & 11 & 0 \\ 0 & 0 & 16 \end{bmatrix}$, we get three proper regular splittings

$A = M - N = P - Q = U - V$ of A. The comparison of rate of convergence of single-step, two-step, and three-step is given in Table 1.

Table 1 Comparison table for Example 4.1

Alternate iterations	Splittings	IT	$||A^\dagger b - x_n||_2$	Time	ρ
Three-step	$A = M - N = P - Q = U - V$	7	4.9835e−08	0.000261	0.1190
Two-step	$A = M - N = P - Q$	10	8.6099e−08	0.000369	0.2381
Two-step	$A = M - N = U - V$	8	8.7555e−08	0.000349	0.1667
Two-step	$A = P - Q = U - V$	14	8.0775e−08	0.000512	0.3571
Single-step	$A = M - N$	13	9.2239e−08	0.000469	0.3333
Single-step	$A = P - Q$	43	7.6557e−08	0.001516	0.7143
Single-step	$A = U - V$	21	7.0123e−08	0.000753	0.5

Example 4.2 Let us consider another linear system of the form (1.1) with $A = \begin{bmatrix} 1.41 & 0 & 1.41 \\ 0 & 1.41 & 0 \\ 1.41 & 0 & 1.41 \\ 2.82 & 0 & 2.82 \end{bmatrix}$ and $b = [2, 4, 1, 1]^{\mathrm{T}}$. We have

$A^\dagger = \begin{bmatrix} 0.0591 & 0 & 0.0591 & 0.1182 \\ 0.0000 & 0.7092 & 0 & 0 \\ 0.0591 & 0 & 0.0591 & 0.1182 \end{bmatrix} \geq 0.$ Setting $M = \begin{bmatrix} 2.115 & 0 & 2.115 \\ 0 & 2.115 & 0 \\ 2.115 & 0 & 2.115 \\ 4.23 & 0 & 4.23 \end{bmatrix}$,

$P = \begin{bmatrix} 4.512 & 0 & 4.512 \\ 0 & 4.5120 & 0 \\ 4.512 & 0 & 4.512 \\ 9.024 & 0 & 9.024 \end{bmatrix}$ and $U = \begin{bmatrix} 3.525 & 0 & 3.525 \\ 0 & 3.525 & 0 \\ 3.525 & 0 & 3.525 \\ 7.05 & 0 & 7.05 \end{bmatrix}$, we get three proper

regular splittings $A = M - N = P - Q = U - V$ of A. The comparison of rate of convergence of single-step, two-step, and three-step is given in Table 2.

From Tables 1 and 2, we observe that the spectral radius of the iteration matrix of three-step alternating iteration schemes is significantly less which ultimately leads to faster converges of these schemes. One may note that for a small size linear system 4×3, the iteration number (IT) and time in seconds is quite less in case of three-step. So, for large inconsistent linear system, one can expect a significantly less iterations and time consumption which eventually leads to a faster convergence and thus computationally more feasible.

Table 2 Comparison table for Example 4.2

Alternate iterations	Splittings	IT	$\lVert A^\dagger b - x_n \rVert_2$	Time	ρ
Three-step	$A = M - N = P - Q = U - V$	9	5.0376e−08	0.000332	0.1375
Two-step	$A = M - N = P - Q$	12	6.0162e−08	0.000434	0.2292
Two-step	$A = M - N = U - V$	11	5.8726e−08	0.000408	0.2
Two-step	$A = P - Q = U - V$	14	5.8342e−08	0.000730	0.4125
Single-step	$A = M - N$	16	8.2157e−08	0.001231	0.6
Single-step	$A = P - Q$	46	9.3766e−08	0.001643	0.6875
Single-step	$A = U - V$	34	8.2157e−08	0.001231	0.6

5 Conclusion

In this paper, we have settled the problem of finding the least-squares solution of minimum norm of a given inconsistent linear system using iterative method. In this direction, we have derived some suitable sufficient conditions for the convergence of the three-step alternating iterative scheme in case of proper weak regular splittings of type II (Theorem 3.6) and then we have shown that the three-step alternating iteration scheme converges faster than the usual iteration scheme and the two-step alternating iteration scheme (Theorems 3.9, 3.10 and 3.11). Finally, we have validated our theoretical results by performing numerical computations in Sect. 4. Our results further expand the existing theory and give a faster numerical solution.

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