Dynamic Stability of Elastic Beams Under Axial Arbitrary Loads

S. Siddique, J. Deng, and E. Mohamedelhassan

1 Introduction

It is a common practice to build structures like tall buildings, wind turbines, transmission towers, oil and gas platforms and bridges using piles as foundations which is actually design as well-known vertical Euler beam. Structures can be affected by various types of dynamic arbitrary loads. These loads can be imposed on the structures by either human activities such as explosions, traffic and machine vibrations or natural phenomena such as seismic activities, wind, waves, water currents and hurricanes. The dynamic instability of simply supported elastic Euler beams has gained tremendous attention by scholars at the very early stage. The renowned Mathieu equation, assuming structural axial loads are harmonic, not arbitrary, is used in the previous studies of literature. Shtokalo [8] authored the stability of linear differential equation with variable coefficients to solve the Mathieu-Hill equation in case of dynamic instability problems of an Euler Beam. The theory and application of the Mathieu equation was reviewed by McLachlan [6]. The most important contribution was conducted by Bolotin [1], who developed the approximate formula of the first three dynamic instability regions for structural dynamic instability. Briseghella et al. [2] studied the dynamic instability regions of the Euler beam by finite-element method. Yang and Fu [12] examined the dynamic instability of the Euler beams with layered composite materials, while Sochacki [9] analysed with additional elastic elements. Besides, Yan et al. [11] studied the effect of cracks on the dynamic instability of Euler beams. It is acknowledged that the loads on the engineering structures are rarely harmonic, but arbitrary which has been overlooked in the previous analyses. Nowadays, little research has been conducted considering arbitrary loading on

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structures. Huang et al. [5] developed an analytical method based on the Mathieu-Hill equation and the dynamic instability regions was calculated by the eigenvalue algorithm for computing the dynamic instability of the Euler beams with an uniform section under unsteady wind loads. Their study showed that the Euler beam become dynamically unstable, if it spotted in the instability regions whether the wind load was large or not. More recently, Deng et al. [3, 4] applied the dynamic bucking to investigate the rockbursts in mining engineering.

This present study attempts to investigate the dynamic instability of a simply supported elastic Euler beam under arbitrary axial loads. This study includes: (1) a new approximate method based on the Mathieu-Hill equation is developed to study the dynamic stability of the Euler beam. In this method, the elastic beam is considered as a continuous system with various simplifying assumptions under the sum of step functions, which is solved using a matrix method involving a set of chain of power of matrices. The governing dynamic equations of motion thus become a matrix or single differential equation being function of time only. (2) Application of the matrix method is discussed based on the real wind forces time-history from a wind tunnel test. (3)The accuracy of the analysis is ensured by comparing the dynamic behaviour of the Euler elastic beam obtained from this analysis with those obtained by other methods available in the literature.

2 Formulation

For a simply supported Euler beam with a uniform section with length, x under the action of the axial arbitrary load, the general reduced form of a second-order differential equation can be denoted as [10]:

$$\frac{d^2x}{dt^2} + q(t) = 0 \tag{1}$$

where $q(t)$ is a periodic function of time period T. When the function $q(t)$ is represented as a sum of step functions in the interval $0 \leq t \leq T$, the matrix method of solution is specially most effective to solve the Mathieu-Hill Equation [7]. $q(t)$ is composed of the sum of n step functions. Each has a time period of T_0, which is the length and heights $h_1, h_2, h_3, ...h_n$, where

$$T = nT_0 \tag{2}$$

For the periodic function $q(t)$, T is the fundamental period which consists of n number of rectangular step functions having different heights. Hence, the Hill Equation in Eq. 1 reduces to

$$\begin{aligned}
&\frac{d^2x}{dt^2} + h_1 x = 0,\, 0 \le t \le T_0 \\
&\frac{d^2x}{dt^2} + h_2 x = 0,\, T_0 \le t \le 2T_0 \\
&\ldots \\
&\frac{d^2x}{dt^2} + h_n x = 0,\, (n-1)T_0 \le t \le nT_0
\end{aligned} \tag{3}$$

Assume $x(t)$ is the solution of Eq. 1 in the fundamental interval $0 \le t \le T_0$ and $v(t)$ is the first derivative and these can be expressed as:

$$x(t) = A_1 u_1(t) + A_2 u_2(t) \tag{4}$$

$$v(t) = A_1 \dot{u}_1(t) + A_2 \dot{u}_2(t) \tag{5}$$

where A_1 and A_2 are arbitrary constants and u_1 and u_2 are linearly independent functions. Let the following notations be introduced:

$$\omega_n = \sqrt{h_n},\, \theta_n = T_0\sqrt{h_n} = T_0\omega_n \tag{6}$$

The general solution of Eq. 1 in the interval of $0 \le t \le T_0$ can be written in a matrix form as

$$\begin{bmatrix} x_t \\ v_t \end{bmatrix} = \begin{bmatrix} \cos\theta_1 & \sin\theta_1/\omega_1 \\ -\omega_1\sin\theta_1 & \cos\theta_1 \end{bmatrix} . \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} \tag{7}$$

where x_0 and v_0 are the initial values at $t = 0$. Since the values of x and v at $t = T_0$ are the initial values of x and v in the next interval, $T_0 \le t \le 2T_0$. The solution will take the form as

$$\begin{bmatrix} x_T \\ v_T \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & \sin\theta_2/\omega_2 \\ -\omega_2\sin\theta_2 & \cos\theta_2 \end{bmatrix} . \begin{bmatrix} \cos\theta_1 & \sin\theta_1/\omega_1 \\ -\omega_1\sin\theta_1 & \cos\theta_1 \end{bmatrix} . \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} \tag{8}$$

where $\theta_2 = \omega_2 T = T\sqrt{h_2}$. If the indicated multiplication is performed at the end of n complete periods of the rectangular tipple of Fig. 1, the solution can be written in the form as follows:

$$\begin{bmatrix} x_{(nT)} \\ v_{(nT)} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} . \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} = [\, M \,]^n . \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} \tag{9}$$

where M is the monodromy matrix and the nature of the solution depends on the form taken by the powers of the matrix $[M]$. If $(A + D) \ne \pm 2$, the matrix $[M]$ has

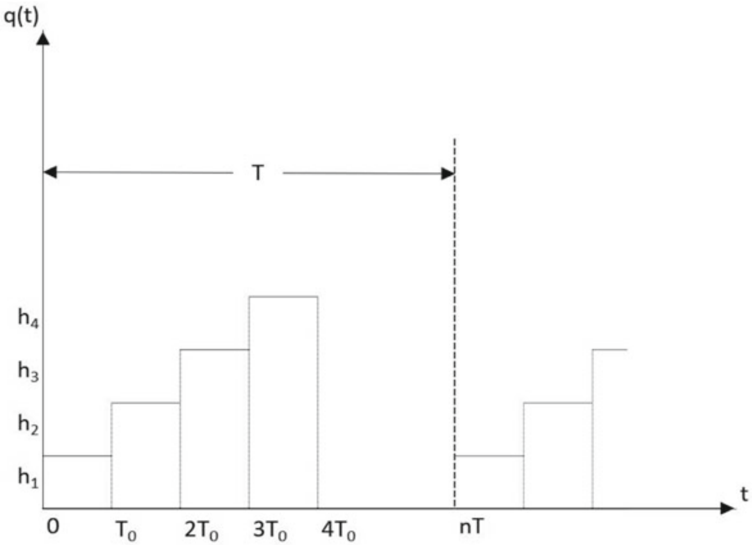

Fig. 1 The periodic input function, $q(t)$

two distinct latent roots. The stability of solution can be determined by satisfying the following conditions:

$$\frac{|A+D|}{2} > 1 \quad unstable \quad \text{and} \quad \frac{|A+D|}{2} < 1 \quad stable \tag{10}$$

The matrix $[M]$ can be acquired by considering the solution for two different initial conditions and also matrix $[M]$ can be obtained after one period T by the multiplication of the chain of matrices in the following form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = [M]_n.[M]_{(n-1)}......[M]_3.[M]_2.[M]_1 = [M] \tag{11}$$

$$[M]_n = \begin{bmatrix} P_n & Q_n \\ R_n & P_n \end{bmatrix} = \begin{bmatrix} \cos\theta_n & \sin\theta_n/\omega_n \\ -\omega_n \sin\theta_n & \cos\theta_n \end{bmatrix} \tag{12}$$

This matrix method gives the solution of the equations of the type in Eq. 1 subject to prescribed initial conditions (initial-value problems), when the solutions are not restricted to be periodic ones. The current method reduces the solution of Eq. 1 subject to given initial-value conditions to that of computing powers of matrices and is applicable to the solution of dynamic stability of beams in practical engineering problems.

3 Verifications

In order to verify the precision of the matrix method of the present study, comparison of the first instability region graph of a real wind force time-history was conducted with the graph of [5]. Figure 2 compares the first instability regions obtained by the matrix method, [5] eigenvalue method, the first order approximation of the Mathieu equation which is calculated by the following formula [10] respectively:

$$\frac{\nu}{2\omega} \approx 1 \pm \frac{1}{2}\mu \tag{13}$$

It has been seen that the results from the three methods are almost similar at smaller excitation parameters (μ), however, bit different when the excitation amplitude is large. According to [1] the difference of the results of the matrix method and Huang et al.'s eigenvalue method with the approximation formula is because the influence of the ith harmonic wave on the width of the kth instability region is the order of $(\mu_\kappa + \mu_\iota)^2$, while under the wind load the value of $(\mu_\kappa + \mu_\iota)^2$ can be fairly large. Hence, it has been suggested that during the wind-induced instability analysis, the interaction between various types of harmonic waves should be considered carefully.

To verify the precision of the matrix method of this study, two points at A $(0.65, 0.80)$ and B $(1.31, 0.80)$ are chosen on the parametric plane, which are the same as [5]. The reason of selecting these points is, as the graph of the present study is a bit different from the graph of the eigenvalue method at the larger excitation parameters, selecting points of that zone is more appropriate to verify the precision of the matrix method. Point A is located inside the stable region of the matrix method, while it is in unstable region of the first-order approximations of the Mathieu equation. Point B is in the unstable region for all these three methods. According to the Modal response graph 2 [5] (see Fig. 3), it has been clearly seen that modal response

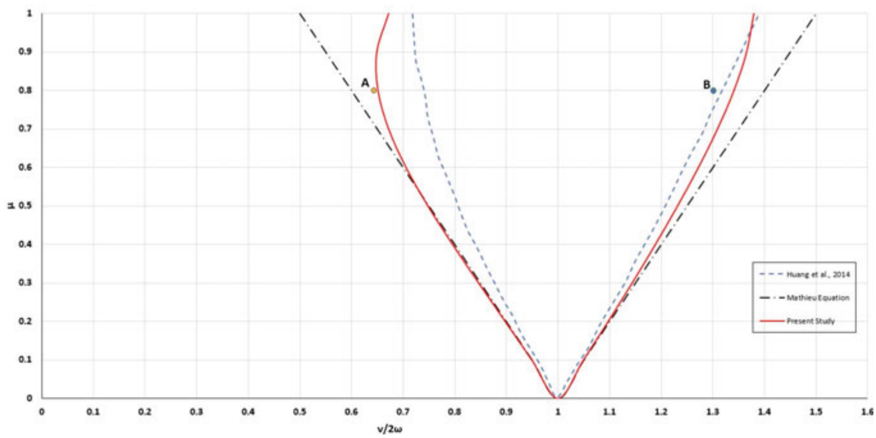

Fig. 2 Comparison of the first instability regions

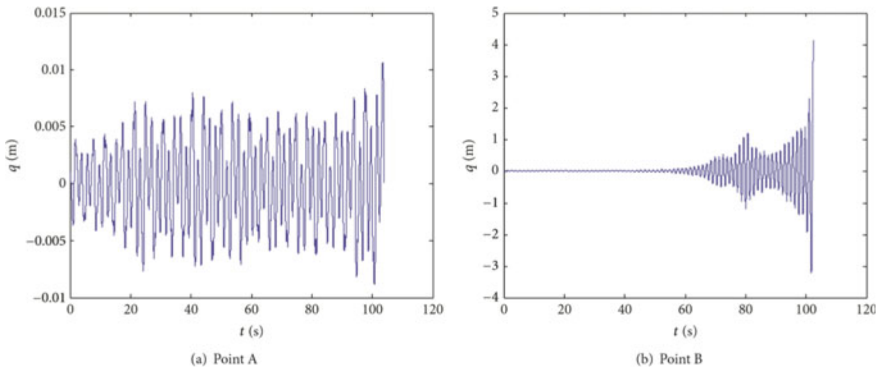

Fig. 3 Modal response of the parametric points A and B [5]

of point A is confined in a range of about 0.01 m from the beginning ($t = 0$ s) to 100 s, on the other hand, in case of point B, modal response suddenly increases to about 5 m in the same direction at $t = 100$s. In other words, it can be said that point A behaves dynamically stable, while point B is dynamically unstable. This characteristics of these two points is perfectly captured by the proposed matrix method. So, it is proved that the matrix method of this study provides more accurate results to find the dynamic instability region of a simply supported Euler beam with uniform sections under any arbitrary dynamic axial load (Fig. 2).

4 Case Study and Discussions

In order to describe the application of the matrix method, a case study has been conducted with the data of real wind force. The model tests of a cylindrical reticulated shell, with the span of 103 m, the longitudinal length of 140 m, and the height of 40 m, were carried in a TJ-2 wind tunnel of Tongji University, China, in order to obtain the time-history of the wind pressure on the shell (Fig. 4).

The test data was given by [5] for the present study and details information about the test procedure was provided in [13]. The number of the wind load data is $N = 6000$ and sampling frequency is $F_s = 9.58$Hz. Hence the load duration is $t_{\max} = \frac{N}{F_s} = 626.30$s. To conduct the present study, the calculation of first 10 data is shown. The fundamental interval $T = 1$ is divided into ten equal segments of length $T_0 = \frac{1}{10}$ in order to represent the function $q(t)$ by step functions. A graphical representation of the $q(t)$ and the step function are shown in Fig. 5.

This matrix method is used to find the instability boundaries for the Mathieu-Hill equation in Eq. 1 with input function in the form of a sum of step functions. The solution is based on a procedure involving a set of chain of different power matrices. It can be clearly seen that the steps of Fig. 5 are not followed any specific patterns. Hence, the general integration method is used to calculated the area. And then, the

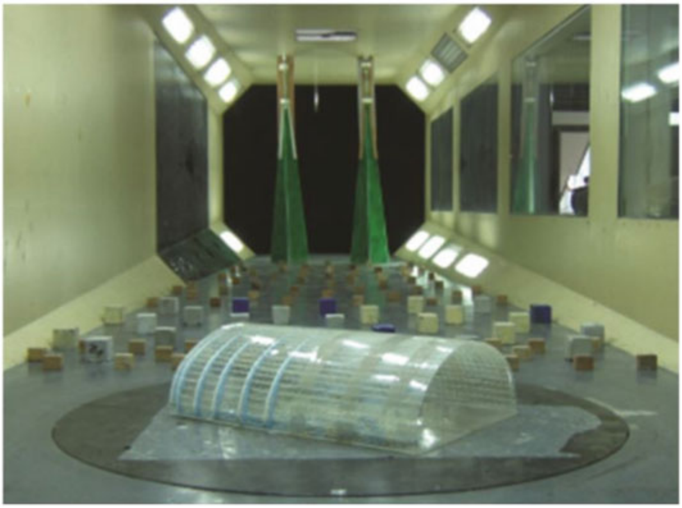

Fig. 4 Rigid model in the wind tunnel tests [5, 13]

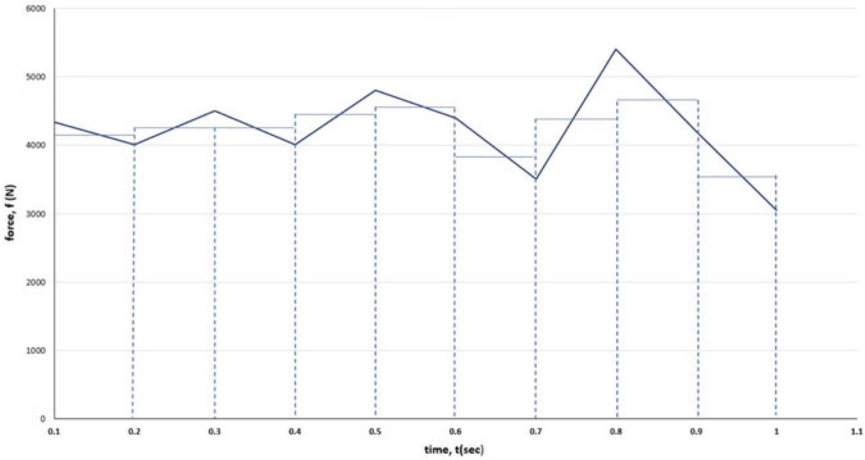

Fig. 5 The periodic input function $q(t)$ with a period of $T = 1.0$

actually height of each segment of the fundamental interval $0 \leq t \leq 1.0$ with a periodic function of fundamental period $T = 1$ is calculated. The values of h_k, ω_k and the matrix elements of $[M]_k$ are calculated using Eqs. 6 and 12. The calculated data of the parameters are represented in Table 1.

The values of x_k and v_k of the solution at the end of the kth subinterval considering the initial conditions $x_0 = 1$ and its first derivative $v_0 = \frac{1}{2}$ are presented in Table 2. The matrix $[M]$ is obtained by multiplication of the chain of matrices in the form

Table 1 The values of area A_k, height h_k, $w_k = \sqrt{h_k}$ and the matrix elements of the representative step functions in the interval of $0 \le t \le 1.0$

Step	A_k	h_k	w_k	P_k	Q_k	R_k
1	435.5504	4172.57	64.60	0.89625	0.006866	−28.65
2	445.5586	4268.65	65.33	0.85946	0.007824	−33.40
3	445.5631	4268.50	65.33	0.85944	0.007825	−33.40
4	464.3215	4448.20	66.70	0.77839	0.009413	−41.87
5	476.5704	4565.54	67.57	0.71796	0.010302	−47.03
6	407.2275	3901.24	62.46	0.97213	0.003753	−14.64
7	490.8977	4702.80	68.58	0.64087	0.011194	−52.64
8	479.1060	4589.83	67.75	0.70479	0.010471	−48.06
9	335.4349	3213.47	56.69	0.93380	−0.006312	20.28
10	350.8659	3361.30	57.98	0.97336	−0.003955	13.29

$$[M] = [M]_{10}.[M]_9.[M]_8.........[M]_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} -0.323312 & -0.014125 \\ 58.96415 & -0.516855 \end{bmatrix} \tag{14}$$

From Eq. 14, we can write

$$\frac{|A+D|}{2} = 0.839 < 1.0 \quad stable \tag{15}$$

It can be clearly seen that the solution is stable in this case study, which is based on the stability criterion of Eq. 10. This calculation is only the first 10 data of the real wind force. There are $N = 6000$ data available from the mentioned test. Similar calculation can be conducted to find the matrix $[M]_{N=6000}$.

5 Conclusions

A new approach, i.e., the matrix method, for analysing the dynamic stability of a simply supported elastic Euler beam under arbitrary axial loads is established based on a class of second-order linear differential equations with periodic coefficients of Mathieu-Hill equation with a sum of step functions. This system is solved numerically using the monodromy matrix method based on a procedure involving a set of chain of different power matrices. The procedure is adequate for the study of a large class of dynamic stability of beams in practical engineering problems, such as piles under earthquake, and mine pillars excited by underground blasting. The accuracy of the analysis is ensured by comparing the dynamic behaviour of an elastic beam obtained from the matrix method with other methods available in the literature. The present method accurately obtained similar graphs like other methods. Application examples

Table 2 The values of x_k and v_k of the solution at the end of the kth subinterval considering $x_0 = 1$ and $v_0 = \frac{1}{2}$

step	$[M]_k = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$	$\begin{bmatrix} x \\ v \end{bmatrix}$
1	$\begin{bmatrix} 0.89625 & 0.00687 \\ -28.6513 & 0.89625 \end{bmatrix}$	$\begin{bmatrix} 1.0000 \\ 0.5000 \end{bmatrix}$
2	$\begin{bmatrix} 0.85946 & 0.00782 \\ -33.3986 & 0.85946 \end{bmatrix}$	$\begin{bmatrix} 0.8996 \\ -28.2032 \end{bmatrix}$
3	$\begin{bmatrix} 0.85944 & 0.00782 \\ -33.4007 & 0.85944 \end{bmatrix}$	$\begin{bmatrix} 0.5526 \\ -54.2877 \end{bmatrix}$
4	$\begin{bmatrix} 0.77839 & 0.00941 \\ -44.8696 & 0.77840 \end{bmatrix}$	$\begin{bmatrix} 0.5009 \\ -65.1130 \end{bmatrix}$
5	$\begin{bmatrix} 0.71796 & 0.01030 \\ -47.0335 & 0.71796 \end{bmatrix}$	$\begin{bmatrix} -0.5739 \\ -52.7809 \end{bmatrix}$
6	$\begin{bmatrix} 0.97213 & 0.00375 \\ -14.6428 & 0.97213 \end{bmatrix}$	$\begin{bmatrix} -0.9557 \\ -10.9024 \end{bmatrix}$
7	$\begin{bmatrix} 0.64087 & 0.01119 \\ -52.6429 & 0.64087 \end{bmatrix}$	$\begin{bmatrix} -0.9701 \\ 3.3966 \end{bmatrix}$
8	$\begin{bmatrix} 0.70479 & 0.01047 \\ -48.0615 & 0.70479 \end{bmatrix}$	$\begin{bmatrix} -0.5834 \\ 53.2434 \end{bmatrix}$
9	$\begin{bmatrix} 0.93380 & -0.00631 \\ 20.2830 & 0.93380 \end{bmatrix}$	$\begin{bmatrix} 0.1462 \\ 65.5769 \end{bmatrix}$
10	$\begin{bmatrix} 0.97336 & -0.00396 \\ 13.2930 & 0.97336 \end{bmatrix}$	$\begin{bmatrix} -0.2774 \\ 58.8024 \end{bmatrix}$

are provided, and limitations of this approach are also discussed. The study provides an excellent analytical knowledge to enhance the understanding of dynamic stability of elastic beams under axial arbitrary loads, which can be used to develop software and modify the relevant design codes.

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