Chapter 11
A Model of Consensus and Consensus Building Within the Framework of Committees with Permissible Ranges of Decision Makers

Takehiro Inohara

Abstract A model of consensus and consensus building is proposed within the framework of voting committees with permissible ranges of decision makers. A group decision-making situation is expressed by a voting committee with the unanimous decision rule, and a negotiation process among decision makers in the situation is expressed as a sequence of decision makers' permissible ranges. Consensus is, moreover, defined as a permissible range of decision makers with a stable alternative and consensus building as a sequence of decision makers' permissible ranges from the status quo to consensus. The existence of consensus and relationships between consensus in a committee, the core of the committee, and Nash equilibrium are investigated.

Keywords Group decision and negotiation · Consensus · Committees · Core · Efficiency · Nash equilibrium

11.1 Introduction

A new model of "consensus" and a definition of "consensus building" are proposed in this work within a framework of voting committees (Peleg 1984; Yamazaki et al. 2000). A committee (Peleg 1984) expresses a group decision-making situation. A negotiation process among decision makers (DMs) in the situation is expressed as a sequence of decision makers' permissible ranges, and "consensus" is defined as a permissible range of decision makers with a "stable alternative (Yamazaki et al.

T. Inohara (✉)
Institute for Liberal Arts, Tokyo Institute of Technology, Meguro, Tokyo, Japan

Department of Social and Human Sciences, School of Environment and Society, Tokyo Institute of Technology, Meguro, Tokyo, Japan
e-mail: inohara.t.aa@m.titech.ac.jp

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2000).” Then, “consensus building” is defined as a sequence of decision makers’ permissible ranges from “status quo” to “consensus.”

“Consensus” and “consensus building” are mentioned in the literature as

> Consensus building is a process of seeking unanimous agreement. It involves a good-faith effort to meet the interests of all stakeholders. Consensus has been reached when everyone agrees they can live with whatever is proposed after every effort has been made to meet the interests of all stakeholding parties (page 6 in Susskind (1999)).

and

> A group reaches consensus on a decision when every member can agree to support that decision. Each person may not think it’s the very best decision, but he or she can buy into it and actively support its implementation. No one in the group feels that his or her fundamental interests have been compromised. Consensus is not “almost everybody.” It’s unanimous support for a decision, in the same way that a jury returns a unanimous verdict (page 58 in Straus (2002)).

Since both Susskind (1999) and Straus (2002) deal with the words “agree,” “unanimous,” and “interests,” a model of “consensus” and a definition of “consensus building” should involve these words as keywords. Also, the phrase “live with (Susskind 1999),” which has the meaning “to accept something unpleasant (Oxford Advanced Learner’s Dictionary 2000),” is almost the same meaning as the phrase “buy into (Straus 2002)” accompanied with the sentence “[e]ach person may not think it’s the very best decision (Straus 2002).” A DM, therefore, should be modeled as an agent who may agree to a decision which is not the best for him/her.

With respect to the difference between “consensus” and “agree,” moreover, referring to the sentences “[c]onsensus has been reached when everyone agrees (Susskind 1999),” “A group reaches consensus (Straus 2002),” and “every member can agree (Straus 2002),” the author uses “agree” for describing an individual’s state and “consensus” for expressing a group’s state. More specifically, in this work, a group is said to reach “consensus” on a decision, if and only if every DM in the group “agrees” to the decision.

In the next section, a framework of voting committees with DMs’ permissible ranges is provided on the basis of the framework in Yamazaki et al. (2000). In Sect. 11.3, mathematical definitions of consensus and consensus building are newly proposed, and Sect. 11.4 verifies a relationship between consensus building in a committee and the core of the voting committee. Section 11.5 treats a strategic aspect of consensus building and investigates a relationship between consensus and Nash equilibrium (Nash 1950, 1951). Section 11.6 verifies the existence of consensus in a committee. The last section is devoted to conclusions.

11.2 Preliminaries (Peleg 1984; Yamazaki et al. 2000)

On the basis of the frameworks in Peleg (1984) and Yamazaki et al. (2000), this section gives a framework of voting committees with DMs’ permissible ranges. Mathematical definitions of simple games, properness, unanimity, committees, cores of committees, and efficiency are provided in Sect. 11.2.1. Then, in Sect. 11.2.2, definitions of permissible ranges, stable coalitions, and stable alternatives are provided, and two propositions verified in Yamazaki et al. (2000) are introduced.

11.2.1 Committees and Core (Peleg 1984)

A simple game specifies the set of all DMs in a group decision-making situation and the decision-making rule adopted in the situation.

Definition 1 (Simple Games) A simple game is a pair $(N, \mathbb{W})$ of a set N of all DMs and a set $\mathbb{W}$ of all winning coalitions, where (i) $\emptyset \notin \mathbb{W}$ and $N \in \mathbb{W}$ and (ii) if $S \subseteq T \subseteq N$ and $S \in \mathbb{W}$, then $T \in \mathbb{W}$. □

A winning coalition is assumed to have enough power to make the coalition's opinion be the final decision of the group, if every DM in the coalition agrees to the opinion.

Under the properness of a simple game, no two disjoint winning coalitions can be formed at the same time.

Definition 2 (Properness of Simple Games) A simple game $(N, \mathbb{W})$ is said to be proper if and only if for all $S \subset N$, $S \in \mathbb{W}$ implies $N \backslash S \notin \mathbb{W}$. □

A unanimous decision rule, on which this work concentrates, is expressed by a unanimous simple game.

Definition 3 (Unanimous Simple Games) A simple game $(N, \mathbb{W})$ is said to be unanimous if and only if $\mathbb{W} = \{N\}$. □

Evidently, a unanimous simple game is proper.

A group decision-making situation is represented by a committee.

Definition 4 (Committees) A committee C is a 4-tuple $(N, \mathbb{W}, A, (\succsim_i)_{i \in N})$ of a set N of all DMs, a set $\mathbb{W}$ of all winning coalitions, a set A of all alternatives, and a list $(\succsim_i)_{i \in N}$ of preferences $\succsim_i$ on A of DM i for each $i \in N$, where $(N, \mathbb{W})$ is a simple game, $2 \leq |N| < \infty$, and $2 \leq |A| < \infty$. For any $i \in N$, the preference $\succsim_i$ on A of DM i is an element of the set $L(A)$ of all linear orderings on A. □

A relation $\succsim$ on A is said to be a linear ordering on A if and only if $\succsim$ is complete, transitive, and anti-symmetric; that is, (i) for x and y in A, $x \succsim y$ or $y \succsim x$ (complete), (ii) for x, y, and z in A, if $x \succsim y$ and $y \succsim z$, then $x \succsim z$ (transitive), and (iii) for x and y in A, if $x \succsim y$ and $y \succsim x$, then $x = y$ (anti-symmetric). Therefore, $L(A)$ is the set of all complete, transitive, and anti-symmetric relations on A.

When we see a linear ordering $\succsim$ on A as a DM's preference, for x and y in A, $x \succsim y$ means that x is equally or more preferred to y. $x \succ y$ is defined as $x \succsim y$ and $\neg(y \succsim x)$, where $\neg$ denotes "not." If $x \neq y$, then $x \succsim y$ implies $x \succ y$, because $\succsim$ is a linear ordering (in particular, an anti-symmetric relation). For $\succsim \in L(A)$, moreover, $max \succsim$ denotes the most preferred alternative in A in terms of $\succsim$, that is, $max \succsim = a$ if and only if for all $x \in A$, $a \succsim x$. $max \succsim$ is uniquely determined, because $\succsim$ is a linear ordering. Furthermore, for $\succsim \in L(A)$, $\succsim = [x, y, z]$ denotes that x is more preferred to y and y is more preferred to z (and hence x is more preferred to z by the transitivity of r), that is, $x \succ y$ and $y \succ z$ (and hence $x \succ z$), with respect to $\succsim$.

Definition 5 (Cores of Committees) Consider a committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i \in N})$, and the relation Dom on A, which is defined as, for all a and b in A, $a\text{Dom}b$ if and only if there exists $S \in \mathbb{W}$ such that $a \succsim_i b$ for all $i \in S$. For all a and b in A, moreover, $a\text{D}\!\!\not{o}\text{m}b$ means "not $a\text{Dom}b$." The core of C, denoted by $Core(C)$, is defined as the set $\{a \in A | \forall b \in A \backslash \{a\}, b\text{D}\!\!\not{o}\text{m}a\}$. □

As shown in Appendix, in the case that the simple game $(N, \mathbb{W})$ of the committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i \in N})$ is unanimous, the following three propositions are mutually equivalent: (i) $x \in Core(C)$, (ii) x is Pareto efficient, and (iii) x is strongly Pareto efficient, where Pareto efficiency and strong Pareto efficiency are defined as follows:

Definition 6 (Pareto Efficiency (p. 7 in Osborne and Rubinstein (1994))) Consider a committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i \in N})$. x is said to be Pareto efficient if and only if no $b \in A$ satisfies that $b \succ_i x$ for all $i \in N$. More, x is said to be strongly Pareto efficient if and only if no $b \in A$ satisfies that $b \succsim_i x$ for all $i \in N$ and $b \succ_i x$ for some $i \in N$.

11.2.2 Committees with Permissible Ranges (Yamazaki et al. 2000)

Permissible ranges of DMs allow us to treat the flexibility of the DMs and make it possible to model agents who may agree to a decision which is not the best for them.

Definition 7 (Committees with Permissible Ranges) A committee C with permissible range P, denoted by $C(P)$, is a pair of a committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i \in N})$ and a list $P = (P_i)_{i \in N}$ of permissible ranges P_i of DM i for each $i \in N$, where $\succsim_i \in P_i \subseteq L(A)$ for each $i \in N$. It is assumed that for all $i \in N$ and all x and y in A, if $x \succsim_i y$ and there exists $\succsim \in P_i$ such that $max \succsim = y$, then there exists $\succsim' \in P_i$ such that $max \succsim' = x$. The set of all permissible ranges P_i of DM i is denoted by $\mathbb{P}_i$. □

For $i \in N$ and $a \in A$, DM i is said to have a as one of his/her permissible alternatives if and only if there exists $\succsim \in P_i$ such that $max \succsim = a$. The set of all DM i's permissible alternatives is denoted by $maxP_i$, that is, $maxP_i = \{a \in A | \exists \succsim \in P_i, max \succsim = a\}$. The assumption in Definition 7 can be expressed as follows: if $x \succsim_i y$ and $y \in maxP_i$, then $x \in maxP_i$, which can be regarded as a kind of monotonicity. This assumption reflects the idea that each DM considers his/her interests even when he/she agrees to an alternative which is not the best for him/her.

Let S_a be the set $\{i \in S | \exists \succsim \in P_i, max \succsim = a\}$, that is, S_a denotes the set of all DMs who are members of coalition S and have a as one of their permissible alternatives. Moreover, $\mathbb{W}_{C(P)} = \{S \in W | \exists a \in A, S_a \in \mathbb{W}\}$, that is, $\mathbb{W}_{C(P)}$ denotes the set of all winning coalitions S such that S_a forms a winning coalition for some $a \in A$. In other words, $\mathbb{W}_{C(P)}$ is the set of all winning coalitions which have possibility of cooperation to make their permissible alternatives be chosen as the final decision of the group. An alternative must be permissible for all members in a winning coalition in $\mathbb{W}_{C(P)}$, in order to be the final decision of the group, and such alternatives constitute the set $A_{C(P)}$, that is, $A_{C(P)} = \{a \in A | \exists S \in \mathbb{W}, S_a \in \mathbb{W}\}$, or equivalently, $A_{C(P)} = \{a \in A | N_a \in \mathbb{W}\}$.

There may exist a winning coalition $S \in \mathbb{W}_{C(P)}$ such that all DMs in S have an alternative $a \in A$ as their common permissible alternative, and for each DM $i \in S$, the alternative a is the best for him/her among the alternatives in $A_{C(P)}$. Such a coalition is quite stable in the group decision situation, because each DM in S has no incentives to deviate from the coalition, and there is no need for the DMs in S to invite other DMs into S in order to obtain bigger power. This type of winning coalitions is said to be stable, in this work.

Definition 8 (Stable Coalitions) Consider a committee C with permissible range P, that is, $C(P)$, where $C = (N, \mathbb{W}, A, (\succsim_i)_{i \in N})$, and $\mathbb{W}_{C(P)}$. A winning coalition $S \in \mathbb{W}_{C(P)}$ is said to be stable if and only if there exists $a \in A$ such that (i) $S_a = S$, and (ii) for all $i \in S$ and all $b \in A \backslash \{a\}$, if $b \succsim_i a$, then $b \notin A_{C(P)}$. The set of all stable coalitions in $C(P)$ is denoted by $\overline{\mathbb{W}_{C(P)}}$. □

An alternative that has possibility to be selected as the final choice by some stable coalitions is called a stable alternative, and the set of all stable alternatives, that is, the set

$$\{a \in A \,| \exists S \in \mathbb{W}_{C(P)}, S_a = S \wedge (\forall i \in S, \forall b \in A \backslash \{a\}, b \succsim_i a \rightarrow b \notin A_{C(P)})\},$$

is denoted by $\overline{A_{C(P)}}$.

The next proposition validates that the number of stable alternatives in a committee with a proper simple game is at most one.

Proposition 1 (Coincidence of Final Choice (Yamazaki et al. 2000)) *Consider a committee C with permissible range P, that is, $C(P)$, where $C = (N, \mathbb{W}, A, (\succsim_i)_{i \in N})$, and assume that the simple game $(N, \mathbb{W})$ is proper. Then, the number of elements in $\overline{A_{C(P)}}$ is one, at most.* □

Consider a committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i \in N})$ and an alternative $x \in A$. For $i \in N$, P_i^x denotes the set $\{\succsim \in L(A) \mid (max \succsim) \succsim_i x\}$, which expresses that the DM i's permissible alternatives are those that he/she equally or more prefers to x in terms of DM i's preference $\succsim_i$. P^x denotes the list $(P_i^x)_{i \in N}$ of P_i^x for each $i \in N$, and $C(P^x)$ is a committee with permissible range P^x. In this case, in particular, all DMs have x as one of their permissible alternatives.

The next proposition gives a characterization of the stable alternatives in a committee C with permissible range P^x with respect to the core $Core(C)$ of C.

Proposition 2 (Yamazaki et al. 2000) *Consider a committee* $C = (N, \mathbb{W}, A, (\succsim_i)_{i \in N})$*, and assume that the simple game* $(N, \mathbb{W})$ *is proper. For an alternative* $x \in A$*, it is satisfied that* $\overline{A_{C(P^x)}} = \{x\}$ *if and only if* $x \in Core(C)$*.* □

11.3 Consensus and Consensus Building

This section proposes mathematical definitions of consensus and consensus building.

A negotiation process in a group decision situation is expressed by a sequence of DMs' permissible ranges in a committee. It is assumed in this work that the negotiation process starts from the state, called the status quo, in which each DM agrees only to his/her best alternative. In the process, however, each DM may change his/her permissible range and may come to agree to an alternative which is not the best for him/her.

Definition 9 (Negotiation Processes in Committees) Consider a committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i \in N})$. A negotiation process in C is a sequence $(P^t)_{t \in T}$ of DMs' permissible ranges $P^t = (P_i^t)_{i \in N}$ at time t for each $t \in T$, where $T = \{0, 1, 2, \ldots\}$. $P^0 = (P_i^0)_{i \in N} = (\{\succsim_i\})_{i \in N}$ is called status quo. □

Consensus is defined as a state with a stable alternative in a negotiation process, and a sequence of DMs' permissible ranges from the status quo to the consensus is called consensus building.

Definition 10 (Consensus and Consensus Building) Consider a committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i \in N})$. A negotiation process $(P^t)_{t \in T}$ in C is said to reach consensus at $t^* \in T$ on $x \in A$ if and only if either (i) $t^* = 0$ and $\overline{A_{C(P^0)}} = \{x\}$ or (ii) $t^* > 0$, $\overline{A_{C(P^t)}} = \emptyset$ for all t such that $0 \leq t < t^*$, and $\overline{A_{C(P^{t^*})}} = \{x\}$. In either cases, the sequence $(P^0, P^1, \ldots, P^{t^*})$ is called the consensus building on x in C, and x is said to be consensus through the sequence $(P^0, P^1, \ldots, P^{t^*})$ in C. □

The next example shows that the consensus may change depending on the consensus building process.

Example 1 Consider a committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i \in N})$ such that $N = \{1, 2, 3\}$; $\mathbb{W} = \{\{1, 2, 3\}\}$; $A = a, b, c$; $\succsim_1 = [a, b, c]$; $\succsim_2 = [b, c, a]$; $\succsim_3 = [c, b, a]$. Note that the simple game $(N, \mathbb{W})$ is unanimous. Consider, moreover, the following permission ranges of DMs:

$$P_{11} = \{[a,b,c]\};\ P_{12} = \{[a,b,c],[b,a,c]\};\ P_{13} = \{[a,b,c],[b,c,a],[c,a,b]\};$$
$$P_{21} = \{[b,c,a]\};\ P_{22} = \{[b,c,a],[c,b,a]\};\ P_{23} = \{[b,c,a],[c,a,b],[a,b,c]\};$$
$$P_{31} = \{[c,b,a]\};\ P_{32} = \{[c,b,a],[b,c,a]\};\ P_{33} = \{[c,b,a],[b,a,c],[a,c,b]\}.$$

Then, (i) $P^0 = (P_{11}, P_{21}, P_{31})$, $P^1 = (P_{12}, P_{21}, P_{31})$, $P^2 = (P_{12}, P_{21}, P_{32})$ is a consensus building on $b \in A$ in C; in fact, $\overline{A_{C(P^0)}} = \overline{A_{C(P^1)}} = \emptyset$ and $\overline{A_{C(P^2)}} = \{b\}$; (ii) $P^0 = (P_{11}, P_{21}, P_{31})$, $P^1 = (P_{12}, P_{21}, P_{31})$, $P^2 = (P_{13}, P_{21}, P_{31})$, $P^3 = (P_{13}, P_{22}, P_{31})$ is a consensus building on $c \in A$ in C; and (iii) $P^0 = (P_{11}, P_{21}, P_{31})$, $P^1 = (P_{11}, P_{21}, P_{32})$, $P^2 = (P_{11}, P_{21}, P_{33})$, $P^3 = (P_{11}, P_{22}, P_{33})$, $P^4 = (P_{11}, P_{23}, P_{33})$ is a consensus building on $a \in A$ in C.

11.4 Consensus and Core

The following proposition gives a relationship between consensus building in a committee and the core of the committee.

Proposition 3 *Consider a committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i\in N})$. Assume that the simple game $(N, \mathbb{W})$ is unanimous. Then, for an alternative $x \in A$, there exists a negotiation process $(P^t)_{t\in T}$ in C which reaches consensus at $t^* \in T$ on $x \in A$ for some $t^* \in T$ if and only if $x \in Core(C)$.* □

Proof Consider a committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i\in N})$, where the simple game $(N, \mathbb{W})$ is unanimous, and an alternative $x \in A$.

First, assume that there exists a negotiation process $(P^t)_{t\in T}$ in C which reaches consensus at $t^* \in T$ on $x \in A$ for some $t^* \in T$. Then, we immediately have from the definition of consensus (Definition 10) that $\overline{A_{C(P^{t^*})}} = \{x\}$. Since the simple game $(N, \mathbb{W})$ is unanimous, that is, $\mathbb{W} = \{N\}$, it is implied that $N_x = N$, that is,

$$\text{for all } i \in N, x \in maxP_i, \tag{11.1}$$

and

$$\text{for all } i \in N \text{ and all } b \in A\backslash\{x\}, \text{ if } b \succsim_i x, \text{ then } b \notin A_{C(P)} \tag{11.2}$$

(see Definition 8).

If $x \notin Core(C)$, then the unanimity of the simple game $(N, \mathbb{W})$ implies that

$$\text{there exists } b \in A\backslash\{x\} \text{ such that for all } i \in N, b \succsim_i x. \tag{11.3}$$

The alternative b in (11.3) satisfies that for all $i \in N$, $b \in maxP_i$, which implies $b \in A_{C(P)}$, by (11.1) and the assumption on DMs' permissible ranges (see Definition 7). Equation (11.3) and $b \in A_{C(P)}$ imply the existence of $b \in A\backslash\{x\}$ such that for all $i \in N$, $b \succsim_i x$, and $b \in A_{C(P)}$, which contradicts (11.2).

Thus, if there exists a negotiation process $(P^t)_{t\in T}$ in C which reaches consensus at $t^* \in T$ on $x \in A$ for some $t^* \in T$, then we have that $x \in Core(C)$.

Second, assume that $x \in Core(C)$. If $\overline{A_{C(P^0)}} = \{x\}$, where $P^0 = (P_i^0)_{i\in N} = (\{\succsim_i\})_{i\in N}$, then the negotiation process $(P^t)_{t\in T}$ reaches consensus on $x \in A$ at $t^* = 0$ (see Definition 10). That is, the sequence (P^0) is the consensus building on x.

If $\overline{A_{C(P^0)}} \neq \{x\}$, then consider the sequence (P^0, P^1), where $P^0 = (P_i^0)_{i\in N} = (\{\succsim_i\})_{i\in N}$ and $P^1 = (P_i^1)_{i\in N} = (P_i^x)_{i\in N}$. Then, we have, by Proposition 2 and the assumption that $x \in Core(C)$, that $\overline{A_{C(P^1)}} = \overline{A_{C(P^x)}} = \{x\}$, which implies that the negotiation process $(P^t)_{t\in T}$ reaches consensus on $x \in A$ at $t^* = 1$.

Thus, if $x \in Core(C)$, then there exists a negotiation process $(P^t)_{t\in T}$ in C which reaches consensus at $t^* \in T$ on $x \in A$ for some $t^* \in T$. ■

This proposition implies that for an alternative in a committee, being a consensus through some sequences is equivalent to be an element of the core of the committee.

11.5 Consensus and Nash Equilibrium

Consider a committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i\in N})$. Then, we can define a game $G_C = (N, (\mathbb{P}_i)_{i\in N}, (\succsim_i')_{i\in N})$ in normal form by defining $\succsim_i'$ for each $i \in N$ based on C as follows: for $P = (P_i)_{i\in N}$ and $P' = (P_i')_{i\in N}$ in $\mathbb{P} = \prod_{i\in N} \mathbb{P}_i$, $P \succsim_i' P'$, if and only if either

- $\overline{A_{C(P)}} = \{a\}$, $\overline{A_{C(P')}} = \{b\}$, and $a \succsim_i b$,
- $\overline{A_{C(P)}} = \overline{A_{C(P')}} = \emptyset$,
- $\overline{A_{C(P)}} = \{a\}$, $\overline{A_{C(P')}} = \emptyset$, and $a \in maxP_i$, or
- $\overline{A_{C(P)}} = \emptyset$, $\overline{A_{C(P')}} = \{a\}$, and $a \notin maxP_i'$.

Among these four conditions, fourth one cannot hold for a committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i\in N})$ such that the simple game $(N, \mathbb{W})$ is unanimous, that is, $\mathbb{W} = \{N\}$, because we always have $a \in maxP_i'$ if $\overline{A_{C(P')}} = \{a\}$. Moreover, if the simple game $(N, \mathbb{W})$ in the committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i\in N})$ is unanimous, then we have the next lemma, which is used in the proof of Proposition 4.

Lemma 1 *Consider $P = (P_i)_{i\in N} = (P_i, P_{-i})$ and $P' = (P_i')_{i\in N} = (P_i', P_{-i}')$ in $\prod_{i\in N} \mathbb{P}_i$, and assume that $P_{-i} = P_{-i}'$. If $maxP_i' \subseteq maxP_i$, then $A_{C(P')} \subseteq A_{C(P)}$.* □

Proof For $x \in A$, consider the sets N_x and N_x', which are defined as $\{i \in N | x \in maxP_i\}$ and $\{i \in N | x \in maxP_i'\}$, respectively. Since it is assumed that $P_{-i} = P_{-i}'$, we have $N_x' \subseteq N_x$ from $maxP_i' \subseteq maxP_i$. Therefore, it is satisfied that if $N_x' = N$, then $N_x = N$, which implies by the unanimity of the simple game $(N, \mathbb{W})$ that $A_{C(P')} = \{a \in A | N'_a = N\} \subseteq \{a \in A | N_a = N\} = A_{C(P)}$. ■

The next proposition shows that if a sequence $(P^0, P^1, \ldots, P^{t^*})$ is consensus building on some alternative $x \in A$ in a committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i \in N})$, then $P^{t^*} = (P_i^{t^*})_{i \in N} \in \prod_{i \in N} \mathbb{P}_i$ is a Nash equilibrium in the game $G_C = (N, (\mathbb{P}_i)_{i \in N}, (\succsim'_i)_{i \in N})$, where $P = (P_i)_{i \in N} \in \prod_{i \in N} \mathbb{P}_i$ is said to be a Nash equilibrium (Nash 1950, 1951) in G, if and only if $(P_i, P_{-i}) \succsim'_i (P'_i, P_{-i})$ for all $i \in N$ and all $P'_i \in \mathbb{P}_i$, where $P_{-i} = (P_1, P_2, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n) \in \prod_{j \neq i} \mathbb{P}_j$.

Proposition 4 *Consider a committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i \in N})$ and the game $G_C = (N, (\mathbb{P}_i)_{i \in N}, (\succsim'_i)_{i \in N})$. Assume that the simple game $(N, \mathbb{W})$ is unanimous. Then, for $P = (P_i)_{i \in N} \in \prod_{i \in N} \mathbb{P}_i$, if $\overline{A_{C(P)}} = \{x\}$ for some $x \in A$ in C, then P is Nash equilibrium in G_C.* □

Proof It suffices to verify $P \succsim'_i P'$ for $i \in N$ and $P' = (P'_i)_{i \in N} = (P'_i, P'_{-i}) \in \prod_{i \in N} \mathbb{P}_i$ such that $P_{-i} = P'_{-i}$. Consider the following three cases: (a) $maxP'_i \subseteq maxP_i$ and $x \in maxP'_i$, (b) $maxP'_i \subseteq maxP_i$ and $x \notin maxP'_i$, and (c) $maxP'_i \supseteq maxP_i$.

(a) Cases $maxP'_i \subseteq maxP_i$ and $x \in maxP'_i$:
First, $x \in maxP'_i$ and $P_{-i} = P'_{-i}$ implies that

$$\text{if } N_x = N \text{ then } N'_x = N, \tag{11.4}$$

where for $x \in A$, N_x and N'_x are defined as the sets $\{j \in N | x \in maxP_j\}$ and $\{j \in N | x \in maxP'_i\}$, respectively.
From $maxP'_i \subseteq maxP_i$ and Lemma 1, we have $A_{C(P')} \subseteq A_{C(P)}$, which implies that

$$\text{if } b \notin A_{C(P)} \text{ then } b \notin A_{C(P')}. \tag{11.5}$$

Then, from the unanimity of the simple game $(N, \mathbb{W})$,

$$\begin{aligned}
\overline{A_{C(P)}} = \{x\} &\Rightarrow N_x = N \text{ and } \forall i \in N, \forall b \in A \backslash \{x\}, (b \succsim_i x \rightarrow b \notin A_{C(P)}) \\
&\Rightarrow N'_x = N \text{ and } \forall i \in N, \forall b \in A \backslash \{x\}, (b \succsim_i x \rightarrow b \notin A_{C(P')}) \\
&\quad \text{(by (11.4) and (11.5))} \\
&\Rightarrow \overline{A_{C(P')}} = \{x\} \quad \text{(by Proposition 1)}
\end{aligned}$$

Thus, in this case, $P \succsim'_i P'$ holds by the definition of $\succsim'_i$.

(b) Cases $maxP'_i \subseteq maxP_i$ and $x \notin maxP'_i$:
Since $x \notin maxP'_i$ implies $i \notin N'_x$, we have $x \notin \overline{A_{C(P')}}$.
Assume that $\overline{A_{C(P')}} = \{y\}$ for some $y \in A$ such that $y \neq x$. Then, we have to have $N'_y = N$, where N'_y is defined as the set $\{j \in N | y \in maxP'_j\}$. $N'_y = N$ implies $y \in A_{C(P')}$, and $y \in A_{C(P)}$ follows by $maxP'_i \subseteq maxP_i$ and Lemma 1.

$N'_y = N$ implies $y \in maxP'_i$, too. Then, we have $y \succsim_i x$ under the completeness of $\succsim_i$. In fact, if we do not have $y \succsim_i x$, then we need to have $x \succsim_i y$ by the completeness of $\succsim_i$. By the assumption on DMs' permissible ranges (see Definition 7), $x \succsim_i y$ and $y \in maxP'_i$ imply $x \in maxP'_i$, which contradicts the assumption that $x \notin maxP'_i$.

We see that $y \in A$ satisfies that $y \succsim_i x$ and $y \in A_{C(P)}$, which contradict $\overline{A_{C(P)}} = \{x\}$ and $y \neq x$.

Therefore, $\overline{A_{C(P')}} = \{y\}$ for some $y \in A$ such that $y \neq x$ cannot be satisfied, and then, we have $\overline{A_{C(P')}} = \emptyset$.

Thus, in this case, $P \succsim'_i P'$ holds by the definition of $\succsim'_i$.

(c) Cases $maxP'_i \supseteq maxP_i$:

It suffices to show that $\overline{A_{C(P')}} = \{y\}$ for some $y \in A$ such that $y \neq x$ cannot be satisfied, because this implies from Proposition 1 that either $\overline{A_{C(P')}} = \emptyset$ or $\overline{A_{C(P')}} = \{x\}$, and thus, we have $P \succsim'_i P'$.

Assume that $\overline{A_{C(P')}} = \{y\}$ for some $y \in A$ such that $y \neq x$. If it is satisfied that $x \succsim_i y$ and $x \in A_{C(P')}$, then it contradicts $\overline{A_{C(P')}} = \{y\}$ by the definition of $\overline{A_{C(P')}}$.

If we do not have $x \succsim_i y$, then we need to have $y \succsim_i x$ by the completeness of $\succsim_i$. The assumption $\overline{A_{C(P)}} = \{x\}$ means that $x \in A_{C(P)}$, which implies $x \in maxP_i$. By the assumption on DMs' permissible ranges (see Definition 7), $y \succsim_i x$ and $x \in maxP_i$ imply $y \in maxP_i$.

$\overline{A_{C(P')}} = \{y\}$ implies $y \in A_{C(P')}$, which means $N'_y = N$, where N'_y is defined as the set $\{j \in N | y \in maxP'_j\}$. Since $P_{-i} = P'_{-i}$, we have that $y \in A_{C(P)}$ from $y \in maxP_i$.

$y \succsim_i x$ and $y \in A_{C(P)}$ contradict $x \in \overline{A_{C(P)}}$, and thus, we have $x \succsim_i y$.

From the assumption of $maxP'_i \supseteq maxP_i$ and Lemma 1, we have $A_{C(P')} \supseteq A_{C(P)}$, which implies $x \in A_{C(P')}$, because $x \in A_{C(P)}$ follows $\overline{A_{C(P)}} = \{x\}$.

Consequently, we have both $x \succsim_i y$ and $x \in A_{C(P')}$.

■

By this proposition, we see that a stable alternative, and consequently, consensus (see Definition 10), in a committee is actually stable when we see the committee as a strategic decision situation.

11.6 Existence of Consensus

This section deals with the existence of consensus.

Consider a committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i \in N})$. Then, we can define $max(\succsim_i)_{i \in N}$ as a set $\{x \in A \mid \exists i \in N, x = max \succsim_i\}$ of alternatives. Then, we have the next proposition.

Proposition 5 *Consider a committee* $C = (N, \mathbb{W}, A, (\succsim_i)_{i\in N})$*, and assume that the simple game* $(N, \mathbb{W})$ *of the committee* $C = (N, \mathbb{W}, A, (\succsim_i)_{i\in N})$ *is unanimous. Then, we have that* $\emptyset \neq max(\succsim_i)_{i\in N} \subseteq Core(C)$. □

Proof From the argument in the proof of Proposition 6 in Appendix, it is satisfied, in the setting of this work, that $Core(C) = \{x \in A | \forall b \in A\backslash\{x\}, \exists i \in N, x \succ_i b\}$.

If $x \in max(\succsim_i)_{i\in N}$, then for some $i \in N$, $x = \max \succsim_i$, that is, $\exists i \in N, \forall b \in A\backslash\{x\}, x \succ_i b$, which logically implies that $\forall b \in A\backslash\{x\}, \exists i \in N, x \succ_i b$. Thus, we have $max(\succsim_i)_{i\in N} \subseteq Core(C)$.

We have $max(\succsim_i)_{i\in N}$ is non-empty, because $max \succsim_i$ is uniquely determined for each $i \in N$ from the assumption that $\succsim_i$ is a linear ordering for each $i \in N$. ■

Proposition 5 together with Proposition 3 implies that in a committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i\in N})$ with a unanimous simple game $(N, \mathbb{W})$, there always exists a negotiation process $(P^t)_{t\in T}$ in C which reaches consensus at t^* on x for some $t^* \in T$ and some $x \in A$.

The next example shows that $max(\succsim_i)_{i\in N} = Core(C)$ is not always true.

Example 2 Consider a committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i\in N})$ such that $N = \{1, 2, 3\}; \mathbb{W} = \{\{1, 2, 3\}\}; A = \{a, b, c, d\}; \succsim_1 = [b, a, c, d], \succsim_2 = [c, a, d, b], \succsim_3 = [d, a, b, c]$. Then, we see that $max(\succsim_i)_{i\in N} = \{b, c, d\}$ and $Core(C) = \{a, b, c, d\}$. In fact, $a \in A$ is not dominated by any one of the other alternatives. □

11.7 Conclusions

This work proposed a new model of consensus and a definition of consensus building on the basis of the frameworks in Peleg (1984) and Yamazaki et al. (2000) (Sect. 11.3) and verified some relationships between consensus and core (Proposition 3), between consensus and Nash equilibrium (Proposition 4), and the existence of consensus (Proposition 5). More, Proposition 6 in Appendix indirectly showed a relationship between consensus and efficiency.

Through these propositions, we obtained the following insights on consensus and consensus building:

- For an alternative in committee, being a consensus through some sequences is equivalent to be an element of the core of the committee (Proposition 3).
- Consensus is stable in the sense that it constitutes a Nash equilibrium in the game in normal form, which describes the strategic aspect of the committee (Proposition 4).
- There always exists a negotiation process which reaches consensus (Propositions 3 and 5).
- Consensus is efficient (Propositions 3 and 6).

This work treated stability of consensus as a state in a group decision situation in Sect. 11.5. Instead, we should investigate stability of consensus building as a negotiation process in future research opportunities. This requires modelling a group decision situation as a game in extensive form game (Eichberger 1993; Osborne and Rubinstein 1994) or a graph model within the framework of the Graph Model for Conflict Resolution (Fang et al. 1993; Yasui and Inohara 2007). In order to generalize the existence result in Proposition 5, we need to think of Nakamura's theorem (Nakamura 1979) on the relationship between the non-emptiness of cores of committees and the cardinality of the set of all alternatives. Moreover, strategic information exchange should be involved in the model, and the models by Gibbard (1973) and Satterthwaite (1975) and that by Inohara (2002) may be useful.

Appendix: Core and Efficiency

For $i \in N$, a relation $\succsim_i$ on A is said to be complete, if and only if for x and y in A, $x \succsim_i y$ or $y \succsim_i x$. Also, $\succsim_i$ is said to be anti-symmetric, if and only if for x and y in A, if $x \succsim_i y$ and $y \succsim_i x$ then $x = y$. Note that for x and y in A, $x \succ_i y$ is defined as to satisfiy that $x \succsim_i y$ and $\neg(y \succsim_i x)$, where $\neg$ denotes "not," and that if $x \neq y$ and $\succsim_i$ is anti-symmetric, then $x \succsim_i y$ implies $x \succ_i y$.

Lemma 2 *Assume that $\succsim_i$ is complete. Then, for all b and x in A, we have (i) $\neg(b \succsim_i x)$ if and only if $x \succ_i b$, and (ii) $\neg(b \succ_i x)$ if and only if $x \succsim_i b$.* □

Proof First, assume that $\neg(b \succsim_i x)$. By the completeness of $\succsim_i$, $\neg(b \succsim_i x)$ implies that $x \succsim_i b$. Then, $x \succsim_i b$ together with $\neg(b \succsim_i x)$ means $x \succ_i b$. Second, assume that $x \succ_i b$. By the definition of $\succ_i$, we have that $x \succsim_i b$ and $\neg(b \succsim_i x)$. This implies, in particular, that $\neg(b \succsim_i x)$.

The contraposition of the proposition "$\neg(b \succsim_i x)$ if and only if $x \succ_i b$" is the proposition "$\neg(x \succ_i b)$ if and only if $b \succsim_i x$." Replacing b and x with each other, we have "$\neg(b \succ_i x)$ if and only if $x \succsim_i b$." ■

Lemma 3 *Assume that $\succsim_i$ is complete and anti-symmetric. Then, for b and x in A such that $b \neq x$, we have that $\neg(b \succ_i x)$ if and only if $x \succ_i b$.* □

Proof If $\neg(b \succ_i x)$, then, by the definition of $\succ_i$, we have $\neg(b \succsim_i x \wedge \neg(x \succsim_i b))$, which implies $\neg(b \succsim_i x) \vee x \succsim_i b$. In the case of $\neg(b \succsim_i x)$, by Lemma 2, we have $x \succ_i b$. If $x \succsim_i b$, then we have $x \succ_i b$ by the assumptions of $x \neq b$ and anti-symmetry of $\succsim_i$. Thus, in both cases, we have $x \succ_i b$.

If $x \succ_i b$, then we have, in particular, $x \succsim_i b$, by the definition of $\succ_i$. Then, by Lemma 2, $\neg(b \succ_i x)$. ■

Proposition 6 *Consider a committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i \in N})$, and assume that the simple game $(N, \mathbb{W})$ of the committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i \in N})$ is unanimous. Assume, moreover, that $\succsim_i$ is complete and anti-symmetric for all $i \in N$. Then, for $x \in A$, the following three propositions are mutually equivalent:*

1. *$x \in Core(C)$,*
2. *x is Pareto efficient, and*
3. *x is strongly Pareto efficient.*

Proof By Definition 5, $Core(C)$ is defined as the set $\{x \in A | \forall b \in A \backslash \{x\}, \neg(\exists S \in \mathbb{W}, \forall i \in S, b \succsim_i x)\}$. In the case that the simple game $(N, \mathbb{W})$ of the committee $C = (N, \mathbb{W}, A, (\succsim_i)_{i \in N})$ is unanimous, that is, $\mathbb{W} = \{N\}$,

$$\begin{aligned} Core(C) &= \{x \in A | \forall b \in A \backslash \{x\}, \neg(\forall i \in N, b \succsim_i x)\} \\ &= \{x \in A | \forall b \in A \backslash \{x\}, \exists i \in N, \neg(b \succsim_i x)\}. \end{aligned}$$

We have, moreover, by Lemma 2 and completeness of $\succsim_i$,

$$Core(C) = \{x \in A | \forall b \in A \backslash \{x\}, \exists i \in N, x \succ_i b\}. \tag{11.6}$$

By Definition 6, we have that x is Pareto efficient, if and only if

$$\begin{aligned} \neg(\exists b \in A, \forall i \in N, b \succ_i x) &\iff \forall b \in A, \exists i \in N, \neg(b \succ_i x) \\ &\iff \forall b \in A \backslash \{x\}, \exists i \in N, \neg(b \succ_i x), \end{aligned}$$

since we always have $\neg(x \succ_i x)$ for all $x \in A$ and $i \in N$ from the completeness of $\succsim_i$. We have, moreover, by Lemma 3, and the completeness and anti-symmetry of $\succsim_i$, the above statements are all equivalent to

$$\forall b \in A \backslash \{x\}, \exists i \in N, x \succ_i b. \tag{11.7}$$

Similarly, from Definition 6, x is strongly Pareto efficient, if and only if

$$\begin{aligned} &\neg(\exists b \in A, (\forall i \in N, b \succsim_i x) \wedge (\exists i \in N, b \succ_i x)) \\ &\iff \forall b \in A, \neg((\forall i \in N, b \succsim_i x) \wedge (\exists i \in N, b \succ_i x)) \\ &\iff \forall b \in A, (\exists i \in N, \neg(b \succsim_i x)) \vee (\forall i \in N, \neg(b \succ_i x)) \\ &\iff \forall b \in A \backslash \{x\}, (\exists i \in N, \neg(b \succsim_i x)) \vee (\forall i \in N, \neg(b \succ_i x)) \end{aligned}$$

Lemma 2 and the completeness of $\succsim_i$ imply that the above statements are all equivalent to

$$\forall b \in A \backslash \{x\}, (\exists i \in N, x \succ_i b) \vee (\forall i \in N, \neg(b \succ_i x)),$$

and moreover, Lemma 3, and the completeness and anti-symmetry of $\succsim_i$ imply that the previous statement is equivalent to

$$\forall b \in A \backslash \{x\}, (\exists i \in N, x \succ_i b) \vee (\forall i \in N, x \succ_i b),$$

which is equivalent to

$$\forall b \in A \backslash \{x\}, \exists i \in N, x \succ_i b. \tag{11.8}$$

Therefore, from Eqs. (11.6), (11.7), and (11.8), we have the result. ■

References

Eichberger, J. (1993). *Game theory for economists*. Academic Press, Inc.

Fang, L., Hipel, K. W., & Kilgour, D. M. (1993). *Interactive decision making: The graph model for conflict resolution*. Wiley.

Gibbard, A. (1973). Manipulation of voting schemes: A general result. *Econometrica*, *41*, 587–601.

Inohara, T. (2002). Generalizations of the concepts of core of simple games and their characterization in terms of permission of voters. *Applied Mathematics and Computation*, *132*(1), 47–62.

Nakamura, K. (1979). The vetoers in a simple game with ordinal preferences. *International Journal of Game Theory*, *8*, 55–61.

Nash, J. F. (1950). Equilibrium points in n-person games. *Proceedings of National Academy of Sciences of the USA*, *36*, 48–49.

Nash, J. F. (1951). Non-cooperative games. *Annals of Mathematics*, *54*(2), 286–295.

Osborne, M. J. & Rubinstein, A. (1994). *A course in game theory*. The MIT Press.

Oxford Advanced Learner's Dictionary. (2000). Oxford University Press.

Peleg, B. (1984). *Game theoretic analysis of voting in committees*. Cambridge University Press.

Satterthwaite M. A. (1975). Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, *10*, 187–217.

Straus, D. (2002). *How to make collaboration work: Powerful ways to build consensus, solve problems, and make decisions*. Berrett-Koehler Pub.

Susskind, L. E. (1999). *The consensus building handbook: A comprehensive guide to reaching agreement*. California: Sage Publications, Inc.

Yamazaki, A., Inohara, T., & Nakano, B. (2000). New interpretation of the core of simple games in terms of voters' permission. *Applied Mathematics and Computation*, *108*(2–3), 115–127.

Yasui, S. & Inohara, T. (2007). A graph model of unanimous decision systems. In *2007 IEEE International Conference on Systems, Man and Cybernetics (Montreal, Canada, October 7–10, 2007)* (pp. 1752–1757).