A Numerical Study of Free Vibration Behaviour of Shear Deformable Functionally Graded Beam

Maitreya Limkar, Nimitya Phalke, and Kanishk Sharma

Abstract A numerical study to analyse the vibration characteristics of the shear deformable graded beam is presented in this paper. The material properties of the beam are assumed to be varied in thickness and/or axial direction in accordance with the power law. The governing differential equations for free vibration analysis of FGM beam are derived using Hamilton's Principle. The finite element formulation is then employed to obtain the numerical solution of derived differential equations. A convergence study is conducted to fix the number of elements for discretization of finite element model of FGM beam. The accuracy of model is verified by comparing the present results with that available in the literature. Parametric studies are conducted to investigate the effect of material properties, boundary conditions and geometrical parameters on the free vibration behaviour of FGM beam. Vibration characteristics of the FGM beam are presented in the form of natural frequencies and corresponding mode shapes. It is found that the vibration response of FGM beam is significantly affected by the material gradation profile.

Keywords Vibration · Functionally graded materials · Finite element method · Shear deformable · Mori–Tanaka scheme and power law

1 Introduction

In recent years, a new class of composites namely functionally graded materials (FGMs) has gained great attention in many modern engineering applications such as military, aerospace, automotive, biomedical, marine and civil engineering. FGM is advanced class of composites which combines favourable properties of both ceramic and metal by providing smooth and gradual spatial variation of its constituents. Due to its superior properties over composites such as lower transverse shear stresses, high

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resistance to temperature shocks and no interface problems through the layer interfaces, the researchers have extensively examined the static, vibration and buckling responses of these structures.

The literature devoted to predict the structural response of FGMs can be characterized into study of FGM beams, plates and shells. It is also worth to notice that as compared to FGM plate and shell, a smaller number of studies are available on FEM study of continuous FGM beam. The literature on FGM beam can be segregated based on the used beam theories. It is well-known that the classical beam theory, known as Euler–Bernoulli theory, ignores the effects of shear deformation, is oldest beam theory. Studies such as [1–3] investigated the response of FGM beams using classical beam theory. In addition to this researchers [4–11] also used first-order beam theory or Timoshenko theory, which takes shear deformation into account in determining the flexural behaviour. For instance, Aydogdu and Taskin [9] examined the effect of material inhomogeneity on free vibration response of FGM beam using Timoshenko beam theory. The Young's Modulus and density were varied along the thickness of beam while Poisson's ratio was kept constant. Another study by Pradhan and Chakraverty [4] also investigated the effects of constituent volume fractions, slenderness ratios and the beam theories on the natural frequencies on FGM beam. Ziane, Meftah and Belhadj [12] analysed thin and thick functionally graded material box beams under free vibration. Chen, Kitipornchai and Yang [11] investigated the non-linear free vibration behaviour of shear deformable sandwich porous beam by employing Ritz method and von Kármán type non-linear strain–displacement relationships. The effects of porosity coefficient, slenderness ratio was observed in order to improve its vibration behaviour. Sharma [13] developed a generalized beam theory to study the linear-static behaviour of an Aluminium-Zirconia functionally graded beam under thermomechanical loading conditions. Celebi et al. [14] used complementary functions method to convert the problem into initial-value problem for free vibration analysis of FGM beams. Furthermore, functionally graded beams were analysed using shear deformation theories of different orders [5, 11, 15]. Li et al. [16] focused on vibration analysis of a variable thickness beam made of functionally graded materials, which are submerged in water. Babaei et al. [17] examined the effects of large amplitude free vibrations on FGM shallow arches on non-linear elastic foundations.

Present study highlights the effects of various material properties, boundary conditions and geometrical parameters on free vibration behaviour of FGM beam. The primary objective of the present study is to demonstrate an efficient and accurate solution method. Material properties, like Young's Modulus and density, vary in thickness direction according to Mori–Tanaka scheme and power law. Poisson's ratio is kept constant. Hamilton's principle is used to derive the governing differential equations. The numerical solution of the derived differential equations is obtained by employing finite element formulation. Convergence study is conducted and accuracy of model is verified by comparing the results with that in literature. Also, vibration characteristics of the FGM beam are displayed in the form of mode shapes of natural frequencies.

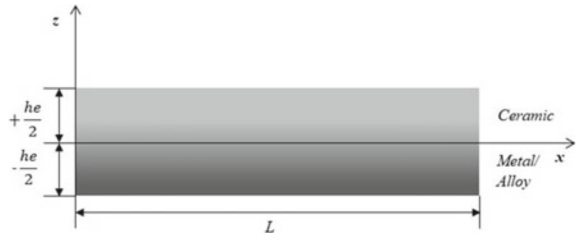

Fig. 1 Material gradation of FGM beam

FEM Formulation

Material Gradation. A functionally graded beam with a uniform material distribution on a Cartesian coordinate system is shown in Fig. 1. The beam has a length L, width b and thickness he. Material properties of the beam are Young's modulus E, Poisson's ratio, shear modulus G and mass density ρ. It is assumed that the effective material properties $P(z)$, satisfying all the material properties, vary continuously in the thickness direction (z) according to the following power law distribution [1, 2]:

$$P(z) = (P_c - - P_m)V_c + P_m \tag{1}$$

where P_c and P_m are, respectively, the material properties at the top and bottom surfaces of the FG beam, V_c is the volume fraction of the top constituent ceramic of the beam defined as:

$$V_c = \left(\frac{z}{h_e} + \frac{1}{2}\right)^n \text{ for } n \geq 0 \tag{2}$$

Following governing equations for shear deformable are obtained using Hamilton's principle

$$-I_0\ddot{v} + A_0 v'' + I_1\ddot{\phi} - A_1\phi'' = 0 \tag{3}$$

$$-I_0 + A_3 w'' - A_3\phi' = 0 \tag{4}$$

$$I_1\ddot{v} - A_1 \upsilon'' + A_3 w' - I_2\ddot{\phi} + A_2\phi'' - A_3\phi = 0 \tag{5}$$

wherein I_i ($i = 0, 1, 2$) and A_j ($j = 0, 1, 2, 3$) are defined as:

$$I_i = \int z^i \rho(z)dA,\ A_i = \int z^i E(z)dA (i = 0.1.2)\ A_3 \int G(z)dA \tag{6}$$

2 Convergence Study

A convergence study has been conducted to fix the number of elements in FEA model of FGM beam, and the results of convergence study are presented in Tables 1, 2. For instance, Table 1 shows the variation of the first five natural frequency parameters $\left(\lambda = \frac{\omega L^2}{h_e}\sqrt{\frac{\text{rhom}}{E_m}}\right)$ with the corresponding number of elements for FGM beam with simply supported edges (S–S). Similarly, the effect of number of elements on the calculated frequency parameters for functionally graded beam with both edges free (i.e. F–F) is shown in Table 2. It is to be noted that the convergence study is conducted with 20 slenderness ratios (i.e. $L/h_e = 20$) whereas the value of power law exponent is kept unity.

The frequency parameter (λ) is expressed by normalizing the obtained eigenfrequencies using the following expression:

$$\lambda = \frac{\omega L^2}{h_e}\sqrt{\frac{\rho_m}{E_m}}$$

where

ω = Natural frequency of beam
L = Length of beam
h_e = Height of beam
ρ_m = Mass Density of metal

Table 1 Convergence of first five frequency parameters of S–S beam (slenderness ratio = 20; power law exponent = 1)

ndiv	λ_1	λ_2	λ_3	λ_4	λ_5
2	1.235	29.661	65.846	231.12	233.03
4	1.156	5.105	10.878	26.915	41.708
8	1.121	4.508	10.255	18.548	26.894
16	1.115	4.414	9.770	17.003	25.918
24	1.113	4.393	9.664	16.678	25.147

Table 2 Convergence of first five frequency parameters of F–F beam (slenderness ratio = 20; power law exponent = 1)

ndiv	λ_1	λ_2	λ_3	λ_4	λ_5
2	11.767	–	–	–	–
4	11.601	32.213	63.909	115.48	378.35
8	6.498	14.540	19.214	24.324	35.440
16	6.456	14.380	19.194	23.832	34.158
24	6.435	14.308	19.184	23.645	33.724

E_m = Young's Modulus of metal.

By this study, it is found that by increasing the number of discretized elements of either beam, the difference between the consecutive frequencies for any mode decreases considerably. This shows that the results obtained would be of higher accuracy when the number of discretized elements for FG beam is increased. Thus, it can be observed that the difference between frequency parameters becomes minimum as we consider higher values of ndiv. Conclusively, the number of discretized elements (ndiv) is set as '8' for this complete study.

3 Validation Study

To perform the present study, the accuracy of FEM formulation must be validated with the results reported in the relevant literature.

In this section, first five frequency parameters for the free vibration of FGM beam subjected to different sets of boundary conditions are compared. To validate the analysis, results for simply supported (S–S) as well as Fixed (C–C) beam are compared with numerical convergence studies of frequency parameters with the literature published. The results for S–S FGM beam are compared with that reported by Aydogdu and Taskin [9] and presented in Table 3. Table 4 shows the comparison of natural frequencies for C–C FGM beam with Pradhan and Chakraverty [4].

Table 3 Comparison of natural frequency parameters for S–S Al/Al_2O_3FGM beam with Aydogdu and Taskin [9]

Slenderness ratio	Reference of study	$n = 0$	$n = 0.1$	$n = 1$	$n = 2$	$n = \infty$
$L/h_e = 5$	Present study	6.622	6.438	5.095	4.135	2.849
	Aydogdu and Taskin [9]	6.847	6.499	4.821	4.251	2.938
$L/h_e = 20$	Present study	6.638	6.454	5.107	4.135	2.849
	Aydogdu and Taskin [9]	6.951	6.599	4.907	4.334	2.983

Table 4 Comparison of natural frequency parameters for F–F Al/Al_2O_3FGM beam with Aydogdu and Pradhan and Chakraverty [4]

Slenderness ratio	Reference of study	$n = 0$	$n = 0.1$	$n = 1$	$n = 2$	$n = \infty$
$L/h_e = 5$	Present study	15.048	14.231	13.128	11.579	9.375
	Pradhan and Chakraverty[4]	15.460	14.001	12.450	10.909	9.629
$L/h_e = 20$	Present study	15.048	14.231	13.128	11.579	9.375
	Pradhan and Chakraverty[4]	15.754	14.268	12.689	11.161	9.864

Material and geometrical parameters for FGM beam are taken from references [9, 10] and mentioned below for ready reference.

$$\begin{aligned} &\text{Al} : E_m = 70 \text{ GPa}, \\ &\text{rhom} = 2702 \text{ kg/m}^3 \\ &\text{Al}_2\text{O}_3 : E_c = 380 \text{ GPa} \\ &\text{rhoc} = 3800 \text{ kg/m}^3 \end{aligned}$$

The Poisson's ratio for both materials is kept constant as 0.3. From Tables 3 and 4, a good agreement between the results of the frequency parameters for different values of the power law exponent can be observed.

4 Present Study

In the present study, the first five frequency responses of a functionally graded (FG) beam for three different scenarios are investigated and presented. Functionally graded material of the beam is basically composed of Silicon Carbide at the top of the beam (i.e. z = +h/2) and Titanium Aluminide ($Ti_{48}Al_2Cr_2Nb$) at the bottom of the beam (i.e. z = −h/2) with the following properties varying or as per the power law through the thickness of the beam.

$$\begin{aligned} \text{Silicon Carbide} : E_c &= 410 \text{ GPa} \\ \rho_c &= 3100 \text{ kg/m}^3 \\ \text{Titanium Aluminide} : E_m &= 160 \text{ GPa} \\ \rho_m &= 3600 \text{ kg/m}^3 \end{aligned}$$

For the first study, the behaviour of frequency parameters against various values of power law exponent is observed as shown in Table 5. The value of L/h_e is set as 20 and boundary condition taken as simply supported. It can be clearly seen that the natural frequency of the FG beam for irrespective of mode shapes decreases with the

Table 5 Effect of material inhomogeneity on the first five natural frequencies parameters for a TiAl/SiC S–S FG beam with slenderness ratio = 20

Mode	$n = 0$	$n = 0.1$	$n = 0.2$	$n = 1$	$n = 5$	$n = 10$	$n = \infty$
λ_1	5.115	4.967	4.829	4.013	2.927	2.851	2.849
λ_2	20.467	19.874	19.323	15.059	11.711	11.409	11.399
λ_3	46.099	44.762	43.521	36.171	26.378	25.697	25.674
λ_4	82.171	79.790	77.577	64.476	47.020	45.805	45.765
λ_5	113.540	110.249	107.191	89.089	64.970	63.290	63.236

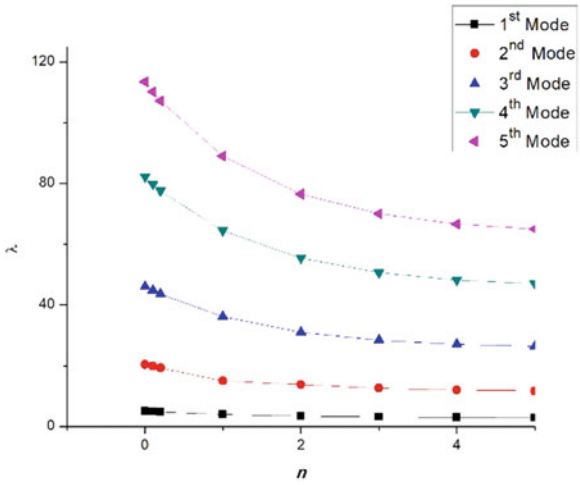

Fig. 2 Variation of frequency parameters for different power law exponents

increase in the value of *n*. This behaviour has been portrayed by a 1-D plot graph as shown in Fig. 2.

For the second study, the behaviour of frequency parameters for four different boundary conditions is examined as shown in Table 6. Similar to the previous condition, the value of L/h_e is set as 20 and the value of power law exponent *n* is 1. Out of all four boundary conditions, the value of normalized frequency of C–C beam for any mode shapes is the largest.

For the third study, the behaviour of frequency parameters for different values of slenderness ratio is examined as shown in Table 7. By keeping power law exponent as $n = 1$ and boundary condition as simply supported beam, the eigen frequencies (f) and respective dimensionless frequency parameters (λ) are evaluated for variable slenderness ratios. Values of eigenfrequencies vary considerably for this part of study.

Table 6 Effect of boundary conditions on the first five natural frequency parameters of TiAl/SiC FG beam (slenderness ratio = 20; power law exponent = 1)

Frequency parameters	Boundary conditions			
	S–S	C–C	C-F	C-S
λ_1	4.013	9.099	1.429	6.270
λ_2	15.059	25.097	8.961	20.328
λ_3	36.171	49.287	25.107	42.471
λ_4	64.476	81.789	44.331	72.864
λ_5	89.089	89.089	49.280	89.089

Table 7 Effect of slenderness ratio on the first five natural frequency/frequency parameters of TiAl/SiC S–S FG beam WITH power law exponent = 1

Slenderness Ratio	Frequency parameters									
	λ_1	f_1 (Hz)	λ_2	f_2 (Hz)	λ_3	f_3 (Hz)	λ_4	f_4 (Hz)	λ_5	f_5 (Hz)
5	4.0139	818.379	16.059	3274.314	22.272	4540.933	36.171	7374.771	45.404	9257.124
10	4.0140	409.189	16.0599	1637.157	36.1718	3687.385	44.544	4540.933	64.476	6572.769
20	4.0140	204.594	16.0599	818.578	36.1718	1843.692	64.476	3286.384	89.089	4540.933
40	4.0140	102.297	16.0599	409.289	36.1718	921.846	64.476	1643.192	101.278	2581.103
50	4.0139	81.837	16.0599	327.431	36.1718	737.477	64.476	1314.553	101.278	2064.882

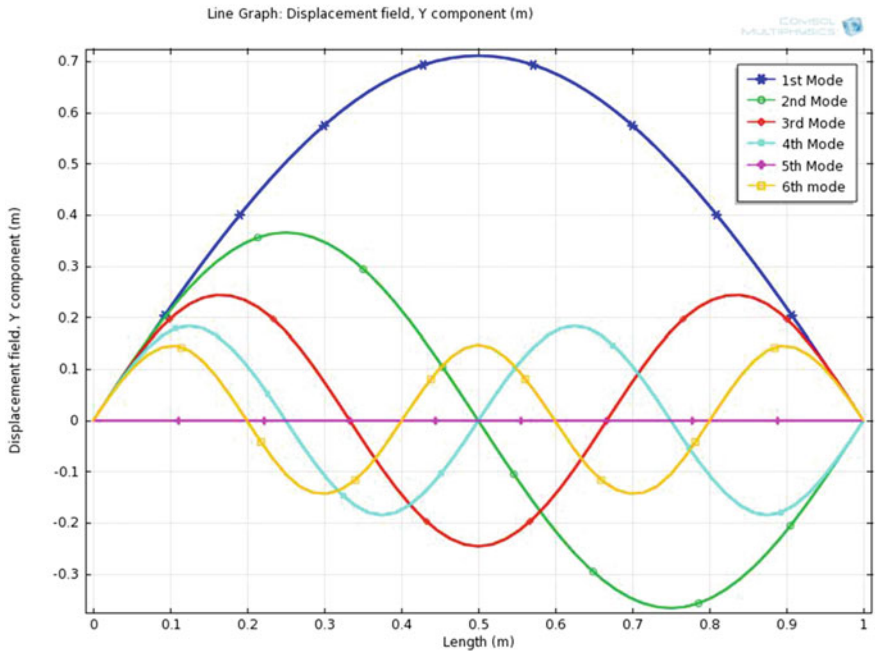

Fig. 3 Variation of displacements for first five-mode shapes

5 Conclusion

A finite element formulation for free vibration analysis of FGM beam is carried out in the present study. The accuracy of model is verified by comparing the present results with that available in the literature and various numerical studies are conducted to investigate the effect of material properties, boundary conditions and geometrical parameters on the free vibration behaviour of FGM beam. Based on the present study, following important conclusions can be drawn:

- The behaviour observed in the first study showed the variation in the frequency parameter λ with respect to the power law exponent n for our parametric considerations. It is observed that with the increase in n, a subsequent decrease in λ is depicted. We can thus imply that when the volumetric fraction starts leaning more towards metal, then the corresponding natural frequency for the FG beam also increases.
- It is found that out of four boundary conditions the highest natural frequency was attained by the C–C beam whereas the lowest natural frequency was attained by the C–F beam. This indicates that for lowest probability of resonance condition, an FG beam with highest possible natural frequency for any mode should be used, i.e. the C–C FG beam.

- It is observed that the values of eigenfrequency for respective mode shapes were decreasing with increase in L/h_e ratio. This proves that a thin or slender beam has lower natural frequency compared to that of a thick or rigid beam and more susceptible towards failure.

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