Shehu-Adomian Decomposition Method for Dispersive KdV-Type Equations

Abey S. Kelil and Appanah R. Appadu

Abstract In this paper, a new method known to be Shehu-Adomian decomposition method is proposed to solve homogeneous and non-homogeneous dispersive KdV-type equations. The Shehu-Adomian decomposition method is a combination of Shehu's transform and Adomian Decomposition method. Some illustrative problems of dispersive KdV-type equations are solved to check the validity of the method. The approximate solutions are given in series form and the proposed method is a reliable and powerful technique to solve numerous physical problems in applications.

Keywords Shehu transform · Adomian decomposition method · Dispersive linear KdV equations

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1 Introduction

The famous Korteweg-de Vries (KdV) equation is a nonlinear dispersive PDE that describes mathematical modeling of traveling wave solution, known to be solitary water waves (also called solitons) in a shallow water domain. This equation is given by the PDE [1]

$$u_t + 6uu_x + u_{xxx} = 0. \quad (1)$$

In 1895, Korteweg and de Vries in [1] derived this equation while studying water waves. Numerical study of KdV equations was pioneered by Zabusky and Kruskal [2] and some recent modifications of the numerical schemes were studied in [3, 4].

There are numerous methods for solving linear/nonlinear partial differential equations. One of these methods is Semi-analytical methods, which can provide

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approximate-analytical solutions for problem considered. Among these methods, we can mention Adomian decomposition method [5–7], Variational iteration method [8–10], and Homotopy perturbation method [11–14]. A literature summary of some semi-analytical methods is given as follows:

(I) Adomian decomposition method (ADM) can be applied to solve linear as well as nonlinear functional equations in [5, 6, 15–17], works by dissecting the equation into linear and nonlinear parts. The method produces series solution whose terms are computed from a recursive relation involving Adomian polynomials. Various modifications of ADM were given in the works of Wazwaz [18].
(II) Homotopy perturbation method (HPM) is used to determine accurate asymptotic solutions of a nonlinear problem. This method is also used effectively to solve PDEs in modeling flows in porous media [19].

Different variants of KdV equation have been investigated in literature [8] (see also [20]). This paper addresses the following problems using some semi-analytic methods [15] and their modifications [21]:

(i) The homogeneous linear KdV equation [18]

$$\begin{cases} u_t + 2u_x + u_{xxx} = 0, \quad (x, t) \in [0, 2\pi] \times [0, 4.0], \\ u(x, 0) = \sin(x). \end{cases} \tag{2}$$

Exact solution for Eq. (2) is given by

$$u(x, t) = \sin(x - t). \tag{3}$$

(ii) The non-homogeneous linear KdV equation with some source term

$$\begin{cases} u_t - u_{xxx} = 2e^{t-x}, \quad (x, t) \in [0, 1.0] \times [0, 2.0], \\ u(x, t) = 1 + e^{t-x}. \end{cases} \tag{4}$$

Exact solution for Eq. (4) is given by

$$u(x, t) = 1 + e^{t-x}. \tag{5}$$

(iii) Homogeneous nonlinear dispersive KdV equation

$$u_t + uu_x + u_{xxx} = 0, \tag{6}$$

with $(x, t) \in [0, 2\pi] \times [0, 0.50]$, and initial condition $u(x, 0) = x$ and the time dependent boundary conditions are

$$u(0, t) = 0, \quad u_x(0, t) = \frac{1}{1 + t}, \ u_{xx}(0, t) = 0. \tag{7}$$

Exact solution is $u(x,t) = \dfrac{x}{1+t}$.

(iv) Inhomogeneous nonlinear dispersive KdV equation [22]

$$u_t - uu_x + u_{xxxxx} = \cos(x) - t\sin(x) + \frac{t^2\sin(2x)}{2}, \tag{8}$$

with $(x,t) \in [0, 2\pi] \times [0, 0.10)$ and initial condition $u(x,0) = 0$, Exact solution is $u(x,t) = t\cos(x)$.

We see that the first term in Eq. (2) refers to time evolution and the third term refers to the dispersion term. Equation (2) is sometimes known as the 'weak dispersion' wave equation. Equation (2) can be represented as the kinematic wave equation, with a dispersive perturbation term of the third order in space. We note that exact solution for the above numerical experiments can be obtained using Ansatz method. (The same also holds for other KdV-type equations considered above).

The objective of this study is to integrate two powerful methods, Shehu transform method and Adomian decomposition method to obtain a better method for solving partial differential equations; in particular on dispersive linear as well as nonlinear KdV-type equations.

2 Adomian Decomposition Method (ADM)

This section recaps some key points of the method ADM to solve linear as well as nonlinear dispersive PDEs.

Let us take the general form of a differential equation as given in [23]:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = G(u, u_x, u_{xx}, \ldots, u_{x^n}) + s(x), \\ u(x,0) = h(x), \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \end{cases} \tag{9}$$

where $u_t = \frac{\partial u}{\partial t}$, $u_{x^i} = \frac{\partial^i u}{\partial x^i}$, $G(\cdot)$ is a polynomial function of its arguments and s is source term.

Following ADM procedures, by splitting the LHS of Eq. (9) into two parts, we have that

$$G[u] = L_G[u] + N_G[u],$$

where $L_G[u]$ is a linear operator with respect to $u, u_x, \ldots, u_{x^n}$ while $N_G[u]$ is nonlinear part of $G[u]$. Then the operator

$$L^{-1}(.) = \int_0^t (.)\, dt,$$

can be introduced to express the solution of Eq. (9) in the form:

$$u = f_0(x) + s(x)\,t + \int_0^t (L_G[u] + N_G[u])\,dt.$$

Let's suppose that

$$u(x;t) = \sum_{n=0}^{\infty} V_n(x;t), \tag{10}$$

and $L_G[u] = \sum_{i\geq 0} L_G[V_i]$ and $N_G[u] = N_G\left[\sum_{i\geq 0} V_i\right] = \sum_{i\geq 0} A_i$, where the newly introduced terms A_i are Adomian polynomials [5, 6, 24]. These polynomials are obtained by using following formulae [10, 24]

$$A_i = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[G\left(\sum_{i=0}^{n}\lambda^i V_i\right)\right]_{\lambda=0}, \tag{11}$$

and some of the first few terms of these polynomials takes the form

$$\begin{aligned}
A_0 &= N(V_0),\\
A_1 &= V_1 N'(V_0),\\
A_2 &= V_2 N'(V_0) + \frac{1}{2}V_1^2 N''(V_0),\\
A_3 &= V_3 N'(V_0) + V_1 V_2 N''(V_0) + \frac{1}{3!}V_1^3 N^{(3)}(V_0),\\
A_4 &= V_4 N'(V_0) + \left(\frac{1}{2}V_2^2 + V_1 V_3\right) N''(V_0) + \frac{1}{2!}V_1^2 V_2 N^{(3)}(V_0) + \frac{1}{4!}V_1^4 N^{(4)}(V_0).
\end{aligned}$$

One can refer to [25, 26] for detailed discussion on Adomain polynomials.

2.1 ADM Applied to Eq. (2)

Let's first rewrite Eq. (2) as

$$\begin{cases} \mathsf{L}_t u + 2u_x + u_{xxx} = 0,\\ u(x,0) = \sin(x), \end{cases} \tag{12}$$

where the differential operator is $\mathsf{L}_t = \frac{\partial}{\partial t}$. By assuming L_t^{-1} exists; that is, $\mathsf{L}_t^{-1}(\cdot) = \int_0^t (\cdot)\,d\tau$, and applying L_t^{-1} on both sides of Eq. (12), we have

$$\mathsf{L}_t^{-1}\mathsf{L}_t u + \mathsf{L}_t^{-1}(2u_x) + \mathsf{L}_t^{-1}(u_{xxx}) = \mathsf{L}_t^{-1}(0),$$

which is equivalently given by

$$u(x,t) = u(x,0) - \left\{\mathsf{L}_t^{-1}(2u_x) + \mathsf{L}_t^{-1}(u_{xxx})\right\}. \tag{13}$$

By employing the decomposition series given in Eq. (10) (cf. [5, 6]), the following recursive approximate values are given as

$$V_0(x) = \sin(x), \tag{14}$$

$$V_1(x;t) = -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_0(x;t)}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_0(x;t)}{\partial x^3}\right)\right\}, \tag{15}$$

$$\vdots$$

$$V_{n+1}(x;t) = -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_n}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_n(x;t)}{\partial x^3}\right)\right\}, \quad n \geq 2. \tag{16}$$

For numerical purpose, $\psi_n(x,t) = \sum_{i=0}^{n} V_i(x,t)$ denotes the n-term approximation to u. The exact solution is $u(x,t) = \lim_{n\to\infty} \psi_n(x,t)$. The number of terms required to obtain an exact solution is considerably small, which will be shown later using the proposed method in this work.

By using the recursive relations in Eqs. (15)–(16) and the linearity property of the operator L_t, we have the first few terms of $V_n(x,t)$:

$$\begin{cases} V_1(x;t) &= -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_0(x;t)}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_0(x;t)}{\partial x^3}\right)\right\} = -t\cos(x), \\ V_2(x;t) &= -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_1(x;t)}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_1(x;t)}{\partial x^3}\right)\right\} = -\frac{t^2}{2!}\sin(x), \\ V_3(x;t) &= -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_2(x;t)}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_2(x;t)}{\partial x^3}\right)\right\} = \frac{t^3}{3!}\cos(x), \\ V_4(x;t) &= -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_3(x;t)}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_3(x;t)}{\partial x^3}\right)\right\} = \frac{t^4}{4!}\sin(x), \\ V_5(x;t) &= -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_4(x;t)}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_4(x;t)}{\partial x^3}\right)\right\} = -\frac{t^5}{5!}\cos(x), \\ V_6(x;t) &= -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_5(x;t)}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_5(x;t)}{\partial x^3}\right)\right\} = -\frac{t^6}{6!}\sin(x), \\ V_7(x;t) &= -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_6(x;t)}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_6(x;t)}{\partial x^3}\right)\right\} = \frac{t^7}{7!}\cos(x) \end{cases} \tag{17}$$

and higher order V_j values are obtained from iteration formula Eq. (16). The ADM solution up to seventh order terms is

$$\psi_7(x,t) = \sum_{j=0}^{7} V_j(x,t) = \left(-t\cos(x) + \frac{t^3}{3!}\cos(x) - \frac{t^5}{5!}\cos(x) + \frac{t^7}{7!}\cos(x)\right) + \left(\sin(x) - \frac{t^2}{2!}\sin(x) + \frac{t^4}{4!}\sin(x) - \frac{t^6}{6!}\sin(x)\right). \quad (18)$$

By using Taylor's expansion and Eq. (18), we have $V_{2n}(x;t) = \frac{(-1)^n t^{2n}}{(2n)!}\sin(x),\ n \in \mathbb{N}_0$, and applying the principle of Mathematical Induction gives

$$\begin{aligned} V_{2n+1}(x;t) &= -\left\{\mathsf{L}_t^{-1}(2V_{2n,x}) + \mathsf{L}_t^{-1}(V_{2n,xxx})\right\} \\ &= \frac{(-1)^{n+1}}{(2n)!}\cos(x)\int_0^t \tau^{2n}\,d\tau = -\frac{(-1)^n t^{2n+1}}{(2n+1)!}\cos(x),\ n \in \mathbb{N}_0. \end{aligned}$$

Thus, from the convergence of ADM in [27], we have that

$$\begin{aligned} u(x;t) &= \sum_{n=0}^{\infty} V_{2n}(x;t) + \sum_{n=0}^{\infty} V_{2n+1}(x;t) \\ &= \sin(x)\left(\sum_{n\geq 0}\frac{(-1)^n t^{2n}}{(2n)!}\right) - \cos(x)\left(\sum_{n\geq 0}\frac{(-1)^n t^{2n+1}}{(2n+1)!}\right) = \sin(x-t). \end{aligned}$$

We note that same approximate-analytical solution for Eq. (2) via LADM have been obtained in [28] and the result coincides with the results of ADM. See Fig. 1 for the graphical illustration and Table 1 for the numerical results of experiment 1.

2.2 ADM Applied to Eq. (4)

We now rewrite Eq. (4) as

$$\mathsf{L}_t u - u_{xxx} = 2e^{t-x}, \quad (19)$$

with $\mathsf{L}_t = \frac{\partial}{\partial t}$, the linear differential operator, which is assumed to be invertible; i.e., $\mathsf{L}_t^{-1}(\cdot) = \int_0^t(\cdot)\,d\tau$. By applying L_t^{-1} on both sides of Eq. (19),

$$\mathsf{L}_t^{-1}\mathsf{L}_t u = 2\mathsf{L}_t^{-1}(e^{t-x}) - \mathsf{L}_t^{-1}(u_{xxx}),$$

which is equivalently

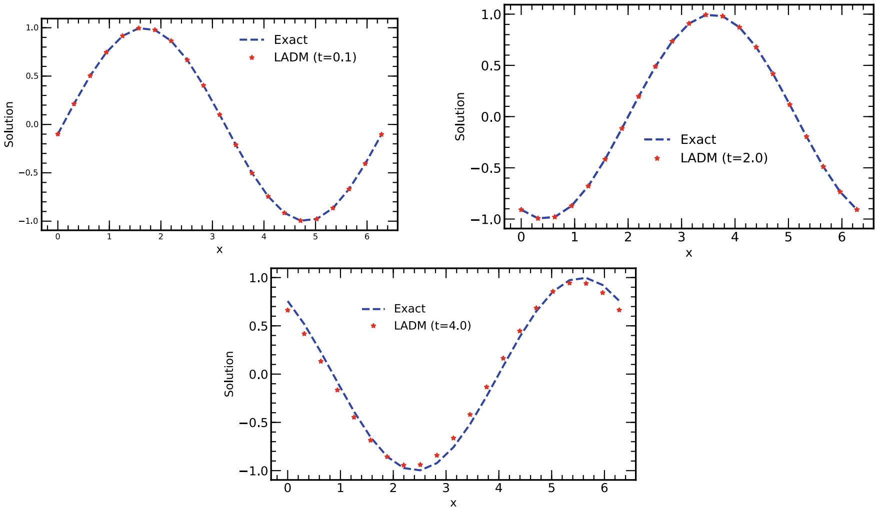

Fig. 1 Plots for Exact solution and ADM (LADM) using ten terms versus x at times 0.1, 2.0 and 4.0

$$u(x,t) = u(x,0) + 2\mathsf{L}_t^{-1}(e^{t-x}) - \mathsf{L}_t^{-1}(u_{xxx}). \tag{20}$$

By employing the decomposition series given in Eq. (10) together with Eq. (20), we get

$$\begin{cases} V_0(x;t) &= u(x,0) + 2\mathsf{L}_t^{-1}(e^{t-x}) = 1 + e^{-x} + 2e^{-x}\mathsf{L}_t^{-1}(e^t) = 1 + 2e^{t-x} - e^{-x}, \\ V_1(x;t) &= -\mathsf{L}_t^{-1}(V_{0,xxx}) = -2e^{t-x} + te^{-x} + 2e^{-x}, \\ V_2(x;t) &= -\mathsf{L}_t^{-1}(V_{1,xxx}) = 2e^{t-x} + e^{-x}\frac{t^2}{2!} - 2te^{-x} - 2e^{-x}, \\ V_3(x;t) &= -\mathsf{L}_t^{-1}(V_{2,xxx}) = -2e^{t-x} + 2te^{-x} + 2e^{-x} + t^2e^{-x} + \frac{t^3}{3!}e^{-x}, \end{cases} \tag{21}$$

and so on.

We see the self-cancelling 'noise' terms in Eq.(21) gives the exact solution

$$u(x,t) = 1 + e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) = 1 + e^{t-x}. \tag{22}$$

Remark 1 An approximate series solution terms given in Eq. (21) for the inhomogeneous KdV-type equation obey self-cancelling behavior; which are also known in the literature as 'noise terms' [29, 30]. A necessary condition for the appearance of noise terms for inhomogeneous problems is that the zeroth component V_0 must possess the exact solution u among other terms [24]. One can refer to [29] for more on noise terms.

Table 1 Absolute/relative errors between ADM (LADM) and exact solution

t	x	Exact	Numerical	Absolute error	Relative error
$t = 0.1$	0.314	0.212370	0.212370	0.000000	0.000000
	0.942	0.745977	0.745977	1.110223×10^{-16}	1.488281×10^{-16}
	1.570	0.994924	0.994924	1.110223×10^{-16}	1.115887×10^{-16}
	2.826	0.403732	0.403732	1.110223×10^{-16}	2.749900×10^{-16}
	3.454	−0.210814	−0.210814	8.326673×10^{-17}	3.949777×10^{-16}
	4.082	−0.744915	−0.744915	0.000000	0.000000
	4.710	−0.994763	−0.994763	0.000000	0.000000
	5.966	−0.405189	−0.405189	2.775558×10^{-16}	6.850036×10^{-16}
	6.280	−0.103002	−0.103002	3.608225×10^{-16}	3.503053×10^{-15}
$t = 2.0$	0.314	−0.993371	−0.993422	5.015442×10^{-5}	5.048909×10^{-5}
	0.942	−0.871376	−0.871412	3.618402×10^{-5}	4.152515×10^{-5}
	1.570	−0.416871	−0.416879	8.406101×10^{-6}	2.016476×10^{-5}
	2.826	0.735226	0.735271	4.494898×10^{-5}	6.113628×10^{-5}
	3.454	0.993187	0.993237	5.016629×10^{-5}	5.051041×10^{-5}
	4.082	0.872156	0.872193	3.624056×10^{-5}	4.155283×10^{-5}
	4.710	0.418318	0.418326	8.485738×10^{-6}	2.028538×10^{-5}
	5.966	−0.734146	−0.734190	4.491153×10^{-5}	6.117524×10^{-5}
	6.280	−0.907967	−0.908017	4.998895×10^{-5}	5.505590×10^{-5}
$t = 4.0$	0.314	0.517911	0.417558	1.003528×10^{-1}	1.937646×10^{-1}
	0.942	−0.083495	−0.165409	8.191365×10^{-2}	9.810567×10^{-1}
	1.570	−0.653041	−0.685258	3.221691×10^{-2}	4.933369×10^{-2}
	2.826	−0.922304	−0.841901	8.040265×10^{-2}	8.717588×10^{-2}
	3.454	−0.519273	−0.418922	1.003508×10^{-1}	1.932525×10^{-1}
	4.082	0.081908	0.163914	8.200590×10^{-2}	0.1001194×10^{1}
	4.710	0.651834	0.684202	3.236825×10^{-2}	4.965722×10^{-2}
	5.966	0.922918	0.842611	8.030689×10^{-2}	8.701410×10^{-2}
	6.280	0.758881	0.663909	9.497124×10^{-2}	1.251465×10^{-1}

3 A New Laplace-Type Transform: Shehu's Transform Method for Solving PDEs

A new Laplace-type integral transform, known to be Shehu's transform, is introduced in [21] to solve both ODEs and PDEs. This method is efficient in the sense that it has great mathematical simplicity and ease of formulations as it is also generalization of many of the well-known integral transforms. Some of the advantages of this method are its simple application to a class of ordinary or partial differential equations; for instance, for some of the dispersive KdV-type equations.

Generally speaking, Shehu's transform can be perceived as a corner stone to the Sumudu transform, the natural transform, the Elzaki transform, and the Laplace transform [21].

Definition 1 The Shehu transform of the function $v(t)$ of exponential order is defined over the set of functions,

$A = \left\{ v(t) : \exists\ N,\ \eta_1,\ \eta_2 > 0,\ |v(t)| < N \exp\left(\frac{|t|}{\eta_i}\right),\ \text{if } t \in (-1)^i \times [0, \infty) \right\}$,
by the following integral

$$\mathbb{S}\left[v(t)\right] = V(s, \rho) = \int_0^\infty \exp\left(\frac{-st}{\rho}\right) v(t) dt$$
$$= \lim_{\alpha \to \infty} \int_0^\alpha \exp\left(\frac{-st}{\rho}\right) v(t) dt; \quad s > 0,\ \rho > 0. \tag{23}$$

Equation (23) converges when the limit value of the above integral is finite and diverges if this is not the case.

Let's denote the inverse Shehu transform, for $t \geq 0$, by

$$\mathbb{S}^{-1}\left[V(s, \rho)\right] = v(t). \tag{24}$$

Equation (24) is equivalently expressed as

$$v(t) = \mathbb{S}^{-1}\left[V(s, \rho)\right] = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{1}{\rho} \exp\left(\frac{st}{\rho}\right) V(s, \rho)\, ds, \tag{25}$$

where $\alpha \in \mathbb{R}$, s and u are Shehu variables [21] and the integral in Eq. (25) is taken along $s = \alpha$ in the complex plane $s = x + iy$.

Theorem 1 ([21]) *If $v(t)$ is piecewise continuous on $t \in [0, \beta]$ and of exponential order α for $t > \beta$, then Shehu's transform exists.*

Theorem 2 ([21]) *Let $v^{(n)}(t)$ denotes the nth derivative of the function $v(t) \in A$ with respect to t. The Shehu transform of $v^{(n)}(t)$ is given by*

$$\mathbb{S}\left[v^{(n)}(t)\right] = \frac{s^n}{\rho^n} \cdot V(s, \rho) - \sum_{k=0}^{n-1} \left(\frac{s}{\rho}\right)^{n-(k+1)} v^{(k)}(0). \tag{26}$$

Fpr $n = 1,\ 2,$ and 3 in Eq. (26), we have the following derivatives with respect to t:

$$\mathbb{S}\left[v'(t)\right] = \frac{s}{\rho} \cdot V(s, \rho) - v(0),$$

$$\mathbb{S}\left[v''(t)\right] = \frac{s^2}{\rho^2} \cdot V(s, \rho) - \frac{s}{\rho} v(0) - v'(0),$$

$$\mathbb{S}\left[v'''(t)\right] = \frac{s^3}{\rho^3} V(s, \rho) - \frac{s^2}{\rho^2} v(0) - \frac{s}{\rho} v'(0) - v''(0).$$

By employing Leibniz's rule, some properties are noted as follows:

$$\mathbb{S}\left[\frac{\partial v(x,t)}{\partial x}\right] = \int_0^\infty \exp\left(\frac{-st}{\rho}\right)\frac{\partial v(x,t)}{\partial x}\,dt = \frac{\partial}{\partial x}\int_0^\infty \exp\left(\frac{-st}{\rho}\right)v(x,t)\,dt$$
$$= \frac{\partial}{\partial x}\left[V(x,s,\rho)\right] \Rightarrow \mathbb{S}\left[\frac{\partial v(x,t)}{\partial x}\right] = \frac{d}{dx}\left[V(x,s,\rho)\right],$$

$$\mathbb{S}\left[\frac{\partial^2 v(x,t)}{\partial x^2}\right] = \int_0^\infty \exp\left(\frac{-st}{\rho}\right)\frac{\partial^2 v(x,t)}{\partial x^2}\,dt = \frac{\partial^2}{\partial x^2}\int_0^\infty \exp\left(\frac{-st}{\rho}\right)v(x,t)\,dt$$
$$= \frac{\partial^2}{\partial x^2}\left[V(x,s,\rho)\right] \Rightarrow \mathbb{S}\left[\frac{\partial^2 v(x,t)}{\partial x^2}\right] = \frac{d^2}{dx^2}\left[V(x,s,\rho)\right].$$

Some important properties of this transform are given as follows:

(i) Linearity property of Shehu transform:

$$\mathbb{S}\left[\alpha v(t) + \beta w(t)\right] = \alpha\mathbb{S}\left[v(t)\right] + \beta\mathbb{S}\left[w(t)\right].$$

(ii) Scaling property of Shehu transform:

$$\mathbb{S}\left[v(\beta t)\right] = \frac{\rho}{\beta}\cdot V\left(\frac{s}{\beta},\rho\right).$$

Proposition 1 ([21]) *Suppose* $\frac{\partial v(x,t)}{\partial t}$ *and* $\frac{\partial^2 v(x,t)}{\partial x^2}$ *exist, then*

$$\mathbb{S}\left[\frac{\partial v(x,t)}{\partial t}\right] = \frac{s}{\rho}\cdot V(x,s,\rho) - v(x,0),$$
$$\mathbb{S}\left[\frac{\partial^2 v(x,t)}{\partial x^2}\right] = \frac{s^2}{\rho^2}\cdot V(s,\rho) - \frac{s}{\rho}\cdot v(0) - \frac{\partial v(x,0)}{\partial t}.$$

Our next section introduces SADM, which is a combination of ADM and Shehu's transform, and some illustrative examples are also provided.

Table 2 Some essential properties of Shehu's transform for SADM

Function form $f(\tilde{X}, t)$	Transformed form $F_k(\tilde{X})$
1	$\frac{\rho}{s}$
$\frac{t^n}{n!}$	$\left(\frac{\rho}{s}\right)^{n+1}$
e^{at}	$\frac{\rho}{s-a\rho}$
te^{at}	$\frac{\rho^2}{(s-a\rho)^2}$
$\frac{t^n e^{at}}{n!}$	$\frac{\rho^{n+1}}{(s-a\rho)^{n+1}}$
$\sin(at)$	$\frac{a\rho^2}{s^2+a^2}$
$\cos(at)$	$\frac{\rho s}{s^2+a^2\rho^2}$
$e^{bt}\cos(at)$	$\frac{\rho(s-a\rho)}{(s-b\rho)^2+a^2\rho^2}$
$\frac{e^{at}}{b-a}$	$\frac{\rho^2}{(s-a\rho)(s-b\rho)}$
$\frac{be^{bt}-ae^{at}}{b-a}$	$\frac{\rho s}{(s-a\rho)(s-b\rho)}$

3.1 *Outline of the Method: SADM*

To illustrate the basic concepts of SADM, let's us consider the following equation

$$\begin{cases} \mathsf{L}_t u(x,t) + Mu(x,t) + Nu(x,t) = g(x,t), \\ u(x,0) = h(x), \end{cases} \tag{27}$$

where N is a nonlinear operator, $\mathsf{L}_t = \frac{\partial}{\partial t}$ is the linear operator, M is a linear operator w.r.t x and g is the source term, which doesn't rely on u. By first applying Laplace transform on both sides of Eq. (27), we get

$$\mathbb{S}\Big\{\mathsf{L}_t u(x,t)\Big\} = \mathbb{S}\Big\{g(x,t) - Mu(x,t) - Nu(x,t)\Big\} \tag{28}$$

and by rewriting Eq. (28) equivalently as

$$\frac{s}{\rho}\cdot\mathbb{S}\Big\{u(x,t)\Big\} - u(x,0) = \mathbb{S}\Big\{g(x,t) - Mu(x,t) - Nu(x,t)\Big\}. \tag{29}$$

In the homogeneous case, $g(x,t) = 0$, and therefore we have that

$$u(x,s)=\frac{\rho}{s}\cdot h(x)-\frac{\rho}{s}\cdot\mathbb{S}\Big\{Mu(x,t)+Nu(x,t)\Big\}.$$

Employing inverse Shehu's transform to Eq. (29) gives

$$u(x,t)=h(x)-\mathbb{S}^{-1}\Bigg[\frac{\rho}{s}\cdot\mathbb{S}\Big\{Mu(x,t)+Nu(x,t)\Big\}\Bigg]. \tag{30}$$

Let us consider SADM decomposition series by

$$u(x,t)=\sum_{n=0}^{\infty}V_n(x,t), \tag{31}$$

and the nonlinear term by

$$Nu(x,t)=\sum_{n=0}^{\infty}A_n(V_0,V_1,\ldots,V_n), \tag{32}$$

where the sequence $\{A_n\}_{n=0}^{\infty}$ are the well-known Adomian polynomials (see [5, 6, 30]). Using Eqs. (31) and (32) into Eq. (30), we obtain

$$\sum_{n=0}^{\infty}V_n(x,t)=h(x)-\mathbb{S}^{-1}\Big[\frac{\rho}{s}\cdot\mathbb{S}\{M\sum_{n=0}^{\infty}V_n(x,t)+\sum_{n=0}^{\infty}A_n(V_0,V_1,\ldots,V_n)\}\Big]. \tag{33}$$

The following recursive formulae follows from Eq. (33) as follows.

$$\begin{cases} V_0(x,t)=h(x), \\ V_{n+1}(x,t)=-\mathbb{S}^{-1}\Big[\frac{\rho}{s}\cdot\mathbb{S}\Big\{MV_n(x,t)+A_n(V_0,V_1,\ldots,V_n)\Big\}\Big], \quad n=0,1,2,\ldots. \end{cases} \tag{34}$$

Using Eq. (34), an approximate solution of Eq. (27) takes the form

$$u(x,t)\approx\sum_{r=0}^{n}V_r(x,t), \quad \text{where} \quad \lim_{n\to\infty}\sum_{r=0}^{n}V_r(x,t)=u(x,t). \tag{35}$$

The following Shehu's transformation results are given in [21].

4 Some Applications: SADM

In this section, SADM is applied to dispersive linear and nonlinear KdV-type equations to show the reliability of the method.

4.1 Implementation of SADM for Eq. (2)

The linearized homogeneous equation in [18] takes the form

$$\begin{cases} u_t + 2u_x + u_{xxx} = 0, \quad (x,t) \in [0, 2\pi] \times [0, 2.75], \\ u(x,0) = \sin(x). \end{cases} \tag{36}$$

By applying Shehu's transform $\mathbb{S}$ in given Eqs. (23)–(36), we have

$$\mathbb{S}\{u_t\} = \frac{s}{\rho} \cdot \mathbb{S}\Big\{u(x,t)\Big\} - u(x,0) = -2\mathbb{S}\{u_x\} - \mathbb{S}\{u_{xxx}\}, \quad t > 0. \tag{37}$$

By employing inverse Shehu's transform to Eq. (37), we obtain

$$u(x,t) = u(x,0) - \mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \left[\mathbb{S}\{2u_x\} - \mathbb{S}\{u_{xxx}\}\right]\right]. \tag{38}$$

By using SADM's series given in Eq. (31) into Eq. (38), the following recursive values are given as follows.

$$\begin{cases} V_0(x,t) = \sin(x), \\ V_1(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \left[\mathbb{S}\{-2V_{0,x}\} - \mathbb{S}\{V_{0,xxx}\}\right]\right] \\ V_2(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \left[\mathbb{S}\{-2V_{1,x}\} - \mathbb{S}\{V_{1,xxx}\}\right]\right] \\ \vdots \\ V_n(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \left[\mathbb{S}\{-2V_{n-1,x}\} - \mathbb{S}\{V_{n-1,xxx}\}\right]\right] \end{cases}. \tag{39}$$

By using Eq. (39) and some of properties of Shehu's transform given in Table 2, we have that

$$\begin{cases} V_0(x,t) = \sin(x), \\ V_1(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\{-2V_{0,x}\} - \mathbb{S}\{V_{0,xxx}\}\right]\right] = -t\cos(x), \\ V_2(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\{-2V_{1,x}\} - \mathbb{S}\{V_{1,xxx}\}\right]\right] = -\frac{t^2}{2!}\sin(x), \\ V_3(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\{-2V_{2,x}\} - \mathbb{S}\{V_{2,xxx}\}\right]\right] = -\frac{t^3}{3!}\cos(x), \\ V_4(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\{-2V_{3,x}\} - \mathbb{S}\{V_{3,xxx}\}\right]\right] = \frac{t^4}{4!}\sin(x), \\ V_5(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\{-2V_{4,x}\} - \mathbb{S}\{V_{4,xxx}\}\right]\right] = -\frac{t^5}{5!}\cos(x), \\ V_6(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\{-2V_{5,x}\} - \mathbb{S}\{V_{5,xxx}\}\right]\right] = -\frac{t^6}{6!}\sin(x), \\ V_7(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\{-2V_{6,x}\} - \mathbb{S}\{V_{6,xxx}\}\right]\right] = -\frac{t^7}{7!}\cos(x). \end{cases} \tag{40}$$

The rest of the components can be obtained from Eq. (40) in a similar way. The 7-term approximate SADM solution is

$$\Psi_7(x,t) = \sum_{i=0}^{7} V_i(x,t) = \left(\sin(x) - \frac{t^2}{2!}\sin(x) + \frac{t^4}{4!}\sin(x) - \frac{t^6}{6!}\sin(x)\right) + \left(-t\cos(x) + \frac{t^3}{3!}\cos(x) - \frac{t^5}{5!}\cos(x) + \frac{t^7}{7!}\cos(x)\right). \tag{41}$$

In view of Eq. (41) and using Taylor's expansion, we have

$$V_{2n}(x;t) = \frac{(-1)^n t^{2n}}{(2n)!}\sin(x), \text{ for } n \in \mathbb{N}_0,$$

and thus

$$\begin{aligned} V_{2n+1}(x;t) &= -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\{-2V_{2n,x}\} - \mathbb{S}\{V_{2n,xxx}\}\right]\right] \\ &= -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\left(2\frac{(-1)^n t^{2n}}{(2n)!}\cos(x) - \frac{(-1)^n t^{2n}}{(2n)!}\cos(x)\right)\right]\right] \\ &= \cos(x)(-1)^{n+1}\frac{t^{2n+1}}{(2n+1)!}. \end{aligned}$$

4.2 Implementation of SADM for Eq. (6)

Applying Shehu transform on both sides of Eq. (6), we get

$$\mathbb{S}(u(x,t)) = x - \left[\frac{\rho}{s} \cdot \mathbb{S}\Big(uu_x + u_{xxx}\Big)\right]. \tag{42}$$

Taking inverse Shehu transform on both sides of Eq. (42), we obtain

$$u(x,t) = x - \mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \mathbb{S}\Big(uu_x + u_{xxx}\Big)\right]. \tag{43}$$

By applying the aforesaid decomposition method, we have

$$\sum_{n=0}^{\infty} u_n(x,t) = x - \mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \mathbb{S}\left\{\sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n) + \sum_{n=0}^{\infty} (u_n)_{xxx}\right\}\right]. \tag{44}$$

Comparing both sides of Eq. (44) gives

$$\begin{cases} u_0(x,t) = x, \\ u_1(x,t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \mathbb{S}\left\{A_0(u_0) + (u_0)_{xxx}\right\}\right], \\ u_2(x,t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \mathbb{S}\left\{A_1(u_0, u_1) + (u_1)_{xxx}\right\}\right], \\ u_3(x,t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \mathbb{S}\left\{A_2(u_0, u_1, u_2) + (u_2)_{xxx}\right\}\right], \\ \qquad \vdots \end{cases} \tag{45}$$

The first few components of Adomain polynomials $A_n(u_0, u_1, \ldots, u_n)$ (cf. [25, 26]) are given by

$$\begin{cases} A_0(u_0) = u_0 u_{0,x} = x, \\ A_1(u_0, u_1) = u_0 u_{1,x} + u_1 u_{0,x} = -xt, \\ A_2(u_0, u_1, u_2) = u_0 u_{2,x} + u_2 u_{0,x} + u_1 u_{1,x} = xt^2, \\ A_3(u_0, u_1, u_2, u_3) = u_3 u_{0,x} + u_1 u_{2,x} + u_2 u_{1,x} + u_0 u_{3,x} = -4xt^3, \\ \qquad \vdots \end{cases} \tag{46}$$

Using the iteration formulae (45) and Adomian polynomials in (46), we obtain

$$u_0(x,t) = x, \quad u_1(x,t) = -xt, \quad u_2(x,t) = xt^2, \quad u_3(x,t) = -xt^3, \quad u_4(x,t) = xt^4. \tag{47}$$

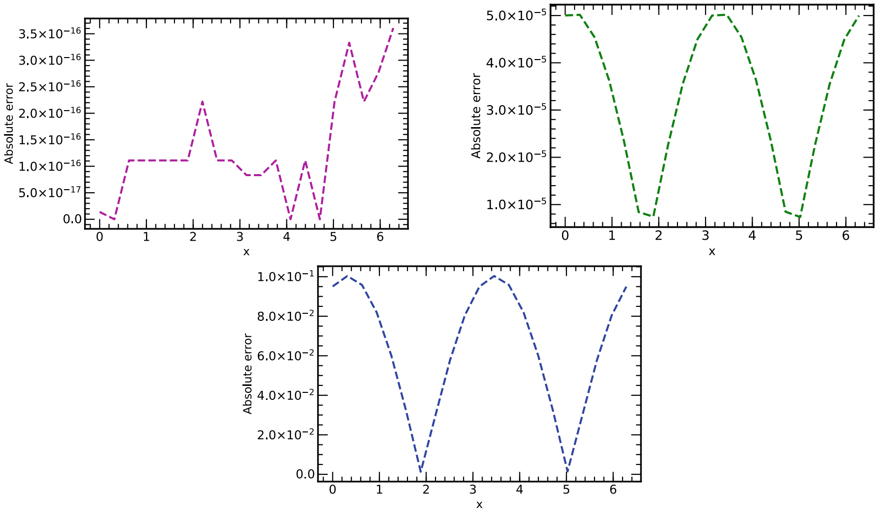

Fig. 2 Error plots versus x at times $t = 0.1, 2.0, 4.0$ (LADM)

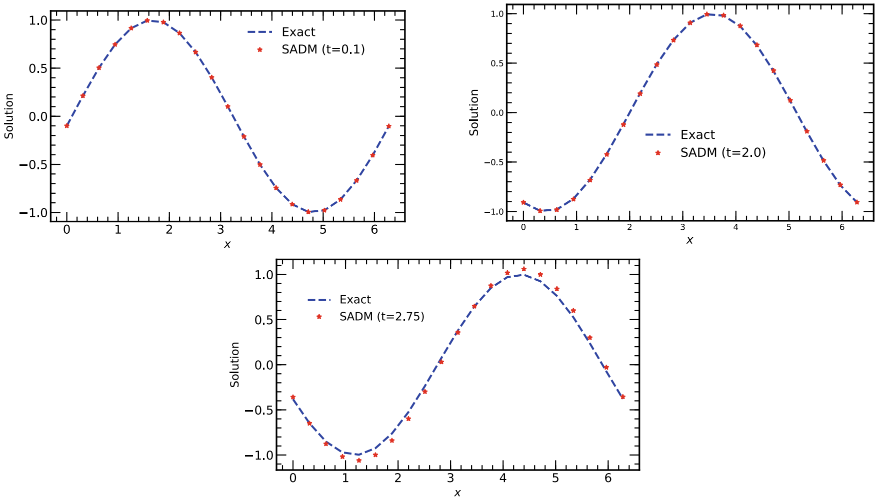

Fig. 3 Plot for Exact solution and SADM at $0 \leq x \leq 2\pi$ and times $t = 0.1, 2.0, 2.75$, respectively,

Thus, an approximate-analytical solution for $u(x, t)$ is given by

$$u_{\text{STADM}}(x, t) = x - xt + xt^2 - xt^3 + xt^4 + \cdots, \tag{48}$$

which gives the exact solution $u(x, t) = \dfrac{x}{1 + t}$ with $|-t| < 1$ (Table 3).

Table 3 Absolute/relative errors at some values of x and at times 0.1, 2.0, 2.75 using 7-terms of SADM

t	values of x	Exact	Numerical	Absolute error	Relative error
	0.000	−0.099833	−0.099833	2.747802×10^{-15}	2.752387×10^{-14}
	0.314	0.212370	0.212370	7.399636×10^{-14}	3.484308×10^{-13}
	0.942	0.745977	0.745977	1.988409×10^{-13}	2.665512×10^{-13}
	1.570	0.994924	0.994924	2.481348×10^{-13}	2.494007×10^{-13}
	2.198	0.864217	0.864217	2.022826×10^{-13}	2.340645×10^{-13}
t = 0.10	2.826	0.403732	0.403732	7.971401×10^{-14}	1.974428×10^{-13}
	3.454	−0.210814	−0.210814	7.352452×10^{-14}	3.487653×10^{-13}
	4.082	−0.744915	−0.744915	1.987299×10^{-13}	2.667820×10^{-13}
	4.710	−0.994763	−0.994763	2.480238×10^{-13}	2.493296×10^{-13}
	5.338	−0.865018	−0.865018	2.023937×10^{-13}	2.339764×10^{-13}
	5.966	−0.405189	−0.405189	7.965850×10^{-14}	1.965960×10^{-13}
	6.280	−0.103002	−0.103002	3.191891×10^{-15}	3.098854×10^{-14}
	0.000	−0.909297	−0.907937	1.360919×10^{-3}	1.496671×10^{-3}
	0.314	−0.993371	−0.993953	5.820994×10^{-4}	5.859837×10^{-4}
	0.942	−0.871376	−0.875489	4.112929×10^{-3}	4.720040×10^{-3}
	1.570	−0.416871	−0.422945	6.074300×10^{-3}	1.457118×10^{-2}
	2.198	0.196709	0.190991	5.717768×10^{-3}	2.906717×10^{-2}
t = 2.0	2.826	0.735226	0.732047	3.179384×10^{-3}	4.324363×10^{-3}
	3.454	0.993187	0.993759	5.722263×10^{-4}	5.761516×10^{-4}
	4.082	0.872156	0.876262	4.105480×10^{-3}	4.707276×10^{-3}
	4.710	0.418318	0.424390	6.072117×10^{-3}	1.451556×10^{-2}
	5.338	−0.195147	−0.189425	5.721685×10^{-3}	2.931987×10^{-2}
	5.966	−0.734146	−0.730958	3.187906×10^{-3}	4.342335×10^{-3}
	6.280	−0.907967	−0.906587	1.380264×10^{-3}	1.520169×10^{-3}
	0.000	−0.381661	−0.358498	2.316300×10^{-2}	6.068998×10^{-2}
	0.314	−0.648485	−0.649521	1.035837×10^{-3}	1.597318×10^{-3}
	0.942	−0.971999	−1.018772	4.677316×10^{-2}	4.812059×10^{-2}
	1.570	−0.924606	−0.999268	7.466224×10^{-2}	8.075033×10^{-2}
	2.198	−0.524391	−0.598452	7.406083×10^{-2}	1.412320×10^{-1}
t = 2.75	2.826	0.075927	0.030728	4.519843×10^{-2}	5.952891×10^{-1}
	3.454	0.647272	0.648183	9.113162×10^{-4}	1.407934×10^{-3}
	4.082	0.971623	1.018297	4.667331×10^{-2}	4.803642×10^{-2}
	4.710	0.925212	0.999837	7.462516×10^{-2}	8.065740×10^{-2}
	5.338	0.525747	0.599847	7.410067×10^{-2}	1.409437×10^{-1}
	5.966	−0.074339	−0.029039	4.529999×10^{-2}	6.093728×10^{-1}
	6.280	−0.378715	−0.355314	2.340076×10^{-2}	6.178992×10^{-2}

Table 4 Absolute/relative errors at some values of x and at times 0.1, 2.0, 2.75 using 7-terms of SADM

t	Values of x	Exact	Numerical	Absolute error	Relative error
	0.000	0.000000	0.000000	0.000000	–
	0.628	0.615686	0.615686	1.970196×10^{-9}	3.200000×10^{-9}
	1.256	1.231373	1.231373	3.940392×10^{-9}	3.200000×10^{-9}
	1.884	1.847059	1.847059	5.910588×10^{-9}	3.200000×10^{-9}
	2.512	2.462745	2.462745	7.880784×10^{-9}	3.200000×10^{-9}
t = 0.02	3.140	3.078431	3.078431	9.850980×10^{-9}	3.200000×10^{-9}
	3.768	3.694118	3.694118	1.182118×10^{-8}	3.200000×10^{-9}
	4.396	4.309804	4.309804	1.379137×10^{-8}	3.200000×10^{-9}
	5.024	4.925490	4.925490	1.576157×10^{-8}	3.200000×10^{-9}
	5.652	5.541176	5.541176	1.773177×10^{-8}	3.200000×10^{-9}
	6.280	6.156863	6.156863	1.970196×10^{-8}	3.200000×10^{-9}
	0.000	0.000000	0.000000	0.000000	–
	0.628	0.592453	0.592453	4.606913×10^{-7}	7.776000×10^{-7}
	1.256	1.184906	1.184907	9.213826×10^{-7}	7.776000×10^{-7}
	1.884	1.777358	1.777360	1.382074×10^{-6}	7.776000×10^{-7}
	2.512	2.369811	2.369813	1.842765×10^{-6}	7.776000×10^{-7}
t = 0.06	3.140	2.962264	2.962266	2.303457×10^{-6}	7.776000×10^{-7}
	3.768	3.554717	3.554720	2.764148×10^{-6}	7.776000×10^{-7}
	4.396	4.147170	4.147173	3.224839×10^{-6}	7.776000×10^{-7}
	5.024	4.739623	4.739626	3.685531×10^{-6}	7.776000×10^{-7}
	5.652	5.332075	5.332080	4.146222×10^{-6}	7.776000×10^{-7}
	6.280	5.924528	5.924533	4.606913×10^{-6}	7.776000×10^{-7}
	0.000	0.000000	0.000000	0.000000	–
	0.628	0.570909	0.570915	5.709091×10^{-6}	1.000000×10^{-5}
	1.256	1.141818	1.141830	1.141818×10^{-5}	1.000000×10^{-5}
	1.884	1.712727	1.712744	1.712727×10^{-5}	1.000000×10^{-5}
	2.512	2.283636	2.283659	2.283636×10^{-5}	1.000000×10^{-5}
t = 0.10	3.140	2.854545	2.854574	2.854545×10^{-5}	1.000000×10^{-5}
	3.768	3.425455	3.425489	3.425455×10^{-5}	1.000000×10^{-5}
	4.396	3.996364	3.996404	3.996364×10^{-5}	1.000000×10^{-5}
	5.024	4.567273	4.567318	4.567273×10^{-5}	1.000000×10^{-5}
	5.652	5.138182	5.138233	5.138182×10^{-5}	1.000000×10^{-5}
	6.280	5.709091	5.709148	5.709091×10^{-5}	1.000000×10^{-5}
	0.000	0.000000	0.000000	0.000000	–
	0.628	0.418667	0.431750	1.308333×10^{-2}	3.125000×10^{-2}
	1.256	0.837333	0.863500	2.616667×10^{-2}	3.125000×10^{-2}
	1.884	1.256000	1.295250	3.925000×10^{-2}	3.125000×10^{-2}
	2.512	1.674667	1.727000	5.233333×10^{-2}	3.125000×10^{-2}
t = 0.50	3.140	2.093333	2.158750	6.541667×10^{-2}	3.125000×10^{-2}
	3.768	2.512000	2.590500	7.850000×10^{-2}	3.125000×10^{-2}
	4.396	2.930667	3.022250	9.158333×10^{-2}	3.125000×10^{-2}
	5.024	3.349333	3.454000	1.046667×10^{-1}	3.125000×10^{-2}
	5.652	3.768000	3.885750	1.177500×10^{-1}	3.125000×10^{-2}
	6.280	4.186667	4.317500	1.308333×10^{-1}	3.125000×10^{-2}

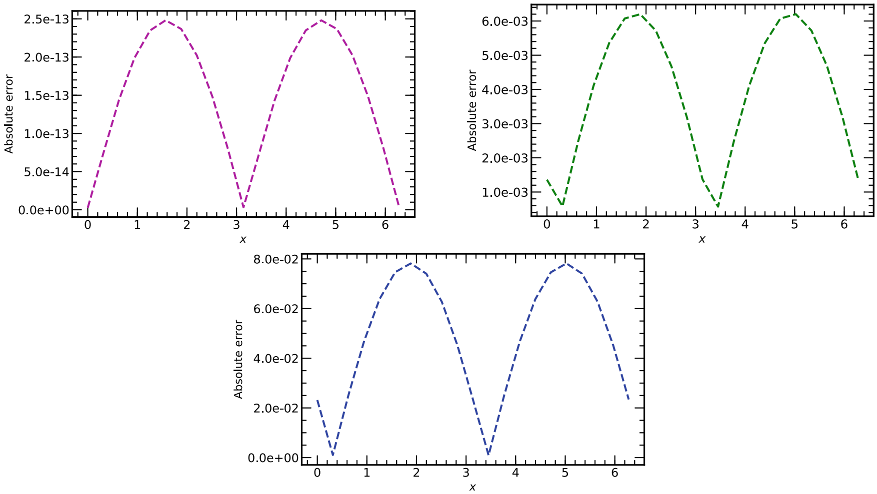

Fig. 4 Error plots versus x (SADM) at times $t = 0.1, 2.0, 2.75$ respectively

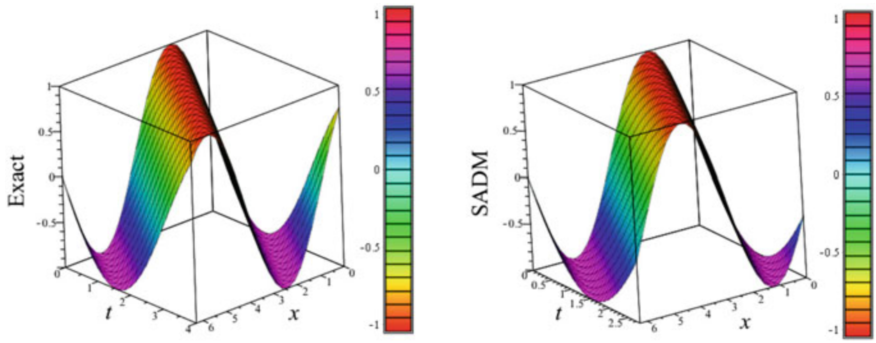

Fig. 5 Three-dimensional representation for Exact solution and SADM at $0 \leq x \leq 2\pi$ and $0 \leq t \leq 4.0$

Plots of exact and numerical solution vs x are displayed in Fig. 6. We obtain plots of absolute error vs x at four different values of time in Fig. 2. We also compare the absolute and relative errors at some values of x at four different times in Table 4. We note that same approximate-analytical solution have been obtained using using SADM for the considered numerical experiments in this paper as shown in Figs. 3, 4, 5 and 7.

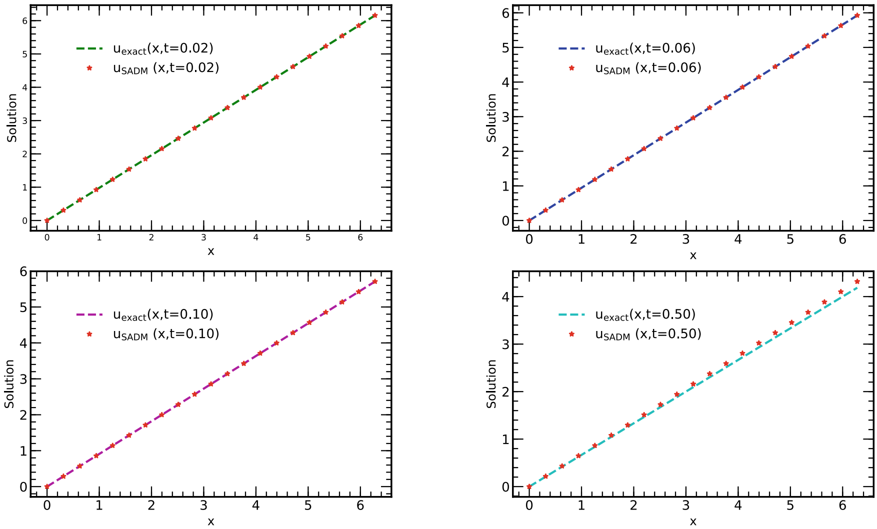

Fig. 6 Plots of exact solution and approximate solution using SADM (4-terms) versus x at times 0.02, 0.06, 0.10, and 0.50 (The space interval used for these plots is $\frac{\pi}{10} \approx 0.314$)

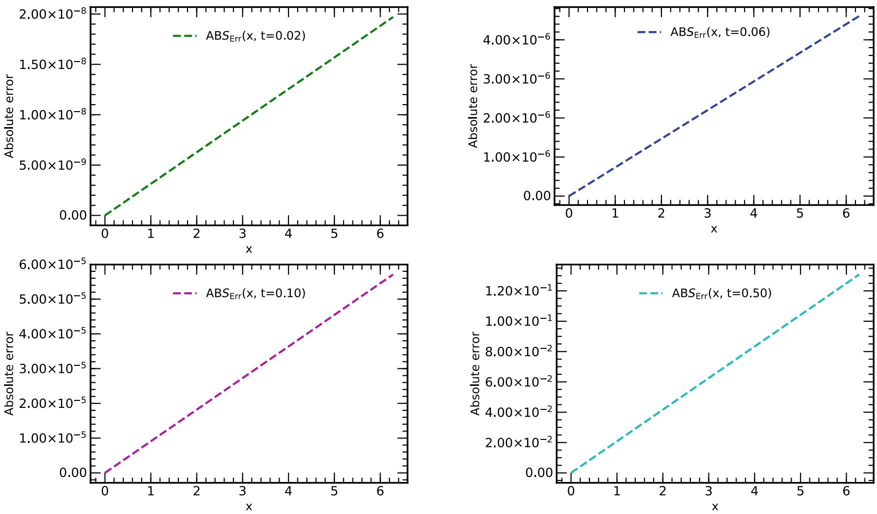

Fig. 7 Plots of absolute errors versus x at different values of time ($t = 0.02, 0.06, 0.10, 0.50$) using SADM (4-terms)

4.3 Implementation of SADM for Eq. (8)

By consider the inhomogeneous equation in Eq. (8), we apply Shehu transform on both sides of Eq. (8) to get

$$\mathbb{S}(u(x,t)) = \frac{\rho}{s}.u(x,0) + \frac{\rho}{s}.\left\{\mathbb{S}\left[\cos(x) + 2t\sin(x) + \frac{t^2\sin(x)}{2}\right] - \mathbb{S}\Big[-uu_x + u_{xxxxx}\Big]\right\}. \quad (49)$$

Taking inverse Shehu transform on both sides of Eq. (49), we obtain

$$u(x,t) = u(x,0) - \mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\mathbb{S}\left[\cos(x) + 2t\sin(x) + \frac{t^2\sin(x)}{2}\right] - \mathbb{S}\Big[-uu_x + u_{xxxxx}\Big]\right]. \quad (50)$$

By applying the aforesaid decomposition method, we have

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) - \mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\ \mathbb{S}\left[\cos(x) + 2t\sin(x) + \frac{t^2\sin(x)}{2}\right]\right.$$
$$\left. - \mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\ \mathbb{S}\left\{\sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n) + \sum_{n=0}^{\infty}(u_n)_{xxxxx}\right\}\right]\right]. \quad (51)$$

On comparing both sides of Eq. (51), we obtain

$$u_0(x,t) = u(x,0) + \mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\mathbb{S}\left[\cos(x) + 2t\sin(x) + \frac{t^2}{2}\sin(x)\right]\right] \quad (52)$$

$$u_1(x,t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\mathbb{S}\left[(u_0)_{xxxxx} - A_0(u_0)\right]\right], \quad (53)$$

$$u_2(x,t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\mathbb{S}\left[(u_1)_{xxxxx} - A_1(u_0,u_1)\right]\right]. \quad (54)$$

$$\vdots$$

The first few components of Adomain polynomials $A_n(u)$ are obtained using formulae (cf. [25, 26])

$$\begin{aligned} A_0(u_0) &= u_0 u_{0,x} \\ &= -t^2\cos(x)\sin(x) + \left(\cos^2(x) - \sin^2(x)\right)t^3 + \left(\frac{1}{6}\cos^2(x) + \cos(x)\sin(x) - \frac{1}{6}\sin^2(x)\right)t^4 \\ &\quad + \frac{1}{3}t^5\sin(x)\cos(x) + \frac{1}{36}t^6\sin(x)\cos(x), \end{aligned} \quad (55)$$

$$\begin{aligned}
A_1(u_0, u_1) &= u_0 u_{1,x} + u_1 u_{0,x} \\
&= \left(-\frac{\sin(x)}{1512} + \frac{\cos^2(x)\sin(x)}{504}\right)t^{10} + \left(-\frac{5\ \sin(x)}{378} + \frac{5\cos^2(x)\sin(x)}{126}\right)t^9 \\
&+ \left(\frac{\cos^3(x)}{126} - \frac{\cos(x)}{252} - \frac{1}{18}\sin(x) + \frac{1}{6}\cos^2(x)\sin(x) - \frac{\cos(x)\sin^2(x)}{252}\right)t^8 \\
&+ \left(-\frac{\cos^2(x)}{144} + \frac{7\cos^3(x)}{36} - \frac{7\ \cos(x)}{72} + \frac{\sin^2(x)}{144} - \frac{2}{9}\cos(x)\sin^2(x)\right)t^7 \\
&+ \left(\frac{\cos(2x)}{72} + \frac{1}{18}\sin(x) + \frac{1}{2}\cos^3(x) - \frac{1}{4}\cos(x) - \cos(x)\sin^2(x) - \frac{1}{6}\cos^2(x)\sin(x)\right)t^6 \\
&+ \left(-\frac{1}{3}\sin^2(x) + \frac{7\sin(x)}{12} + \frac{1}{3}\cos^2(x) + \frac{1}{4}\cos(x)\sin(x) - \frac{5}{2}\cos^2(x)\sin(x)\right)t^5 \\
&+ \left(-\frac{2}{3}\cos^3(x) + \frac{1}{3}\cos(x) + \frac{1}{3}\cos(x)\sin(x) + \frac{1}{3}\cos(x)\sin^2(x)\right)t^4 + \frac{\cos(2x)}{2}t^3.
\end{aligned} \tag{56}$$

The polynomials $A_2(u_0, u_1, u_2)$ and $A_3(u_0, u_1, u_2, u_3)$ are obtained by

$$\begin{aligned}
A_2(u_0, u_1, u_2) &= u_0 u_{2,x} + u_2 u_{0,x} + u_1 u_{1,x}, \\
A_3(u_0, u_1, u_2, u_3) &= u_3 u_{0,x} + u_1 u_{2,x} + u_2 u_{1,x} + u_0 u_{3,x},
\end{aligned}$$

and the higher order ones are obtained by

$$A_n(u_0, u_1, u_2, \ldots, u_n) = \sum_{j=0}^{n-1} u_j \frac{\partial u_{n-j}}{\partial x}. \tag{57}$$

Employing Eqs. (56), (55) together with Eq. (52) yields

$$u_0(x, t) = t\cos(x) + t^2\sin(x) + \frac{t^3}{3!}\sin(x), \tag{58a}$$

$$\begin{aligned}
u_1(x, t) &= \frac{1}{2}t^2\sin(x) + \frac{1}{6}\left(2\ \cos(x) - \sin(2x)\right)t^3 \\
&+ \frac{1}{4}\left(\cos(2x) - \frac{1}{6}\cos(x)\right)t^4 + \frac{1}{36}\sin(2x)\,t^6 + \frac{\sin(2x)\,t^7}{504},
\end{aligned} \tag{58b}$$

$$
\begin{aligned}
u_2(x,t) = &\left(\frac{\cos^2(x)\sin(x)}{5544} - \frac{\sin(x)}{16632}\right)t^{11} + \left(-\frac{\sin(x)}{756} + \frac{\cos^2(x)\sin(x)}{252}\right)t^{10} \\
&+ \left(-\frac{\sin\,(x)}{162} + \frac{\cos^2(x)\sin(x)}{54} - \frac{\cos(x)\ \sin^2(x)}{2268} + \frac{\cos^3(x)}{1134} - \frac{\cos(x)}{2268}\right)t^9 \\
&+ \left(\frac{7\cos^3(x)}{288} - \frac{7\cos(x)}{576} + \frac{(\sin(x))^2}{1152} + \frac{1}{126} - \frac{1}{36}\cos(x)\ \sin^2(x) - \frac{15\ (\cos(x))^2}{896}\right)t^8 \\
&+ \left(-\frac{127\cos^2(x)}{504} + \frac{1}{14}\cos^3(x) - \frac{1}{28}\cos(x) - \frac{(\sin(x))^2}{504} + \frac{8}{63} - \frac{1}{7}\cos(x)\sin^2(x)\right. \\
&\qquad \left. - \frac{1}{42}\cos^2(x)\sin(x) + \frac{\sin(x)}{126}\right)t^7 \\
&+ \left(\frac{1}{18}\cos^2(x) - \frac{1}{18}\ \sin^2(x) + \frac{7\sin(x)}{72} + \frac{1}{24}\cos(x)\sin(x) - \frac{5\cos^2(x)\sin(x)}{12}\right)t^6 \\
&+ \left(-\frac{\sin(x)}{120} + \frac{49\ \cos(x)\sin(x)}{15} + \frac{1}{15}\cos(x)\sin^2(x) - \frac{2}{15}\cos^3(x) + \frac{1}{15}\cos(x)\right)t^5 \\
&+ \left(\frac{67\cos^2(x)}{24} - \frac{1}{8}\sin^2(x) + \frac{1}{12}\sin(x) - \frac{4}{3}\right)t^4 - \frac{1}{6}t^3\cos(x).
\end{aligned}
$$

Thus, the sum of first three iterates to build an approximate-analytical solution for $u(x,t)$ of Eq. (8) is given by

$$u_{\text{SADM}}(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t). \tag{59}$$

Remark 2 Fig. 8 shows exact and SADM solution whereas Fig. 9 demonstrates Absolute error at different times. From numerical experiments above, we see that SADM is a promising semi-analytical method for solving PDEs. Comparison of SADM with other traditional semi-analytic methods such HPM, VIM, RDTM will be prominent continuation of this work, as this is not studied yet.

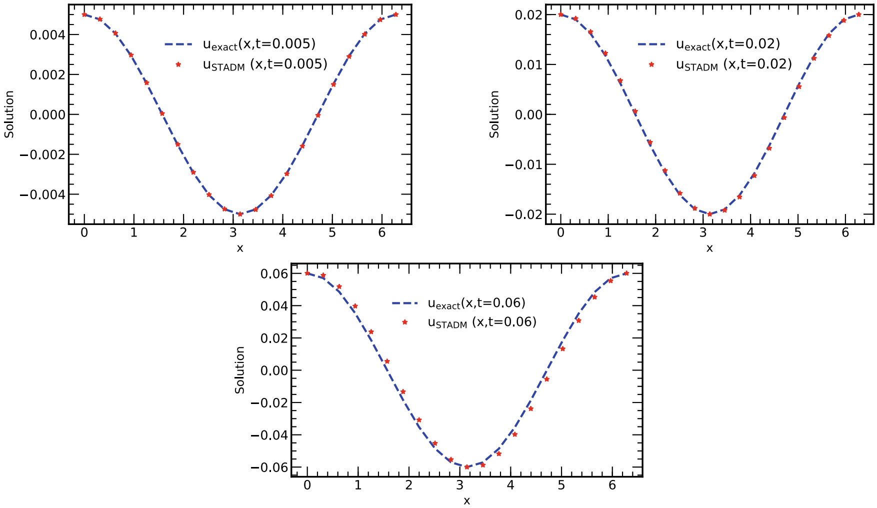

Fig. 8 Plots of Exact solution and approximate solution using 3-terms of SADM versus x at times 0.005, 0.02, and 0.06. (The space step size used for these plots is $\frac{\pi}{10} \approx 0.314$)

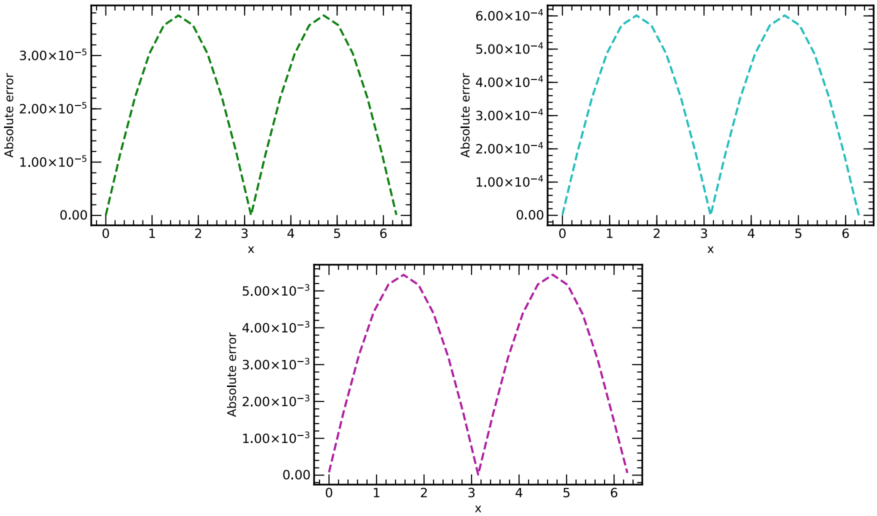

Fig. 9 Plots of absolute errors versus x at times $t = 0.005, 0.02, 0.06$ using SADM

Table 5 Absolute and relative errors at some values of x obtained at times $t = 0.005, 0.02, 0.06$ for Numerical Experiment 2

t	Values of x	Exact solution	Numerical solution	Absolute error	Relative error
t = 0.005	0.000	0.005000	0.005000	2.286458×10^{-8}	4.572917×10^{-6}
	0.628	0.004046	0.004068	2.206271×10^{-5}	5.452940×10^{-3}
	1.256	0.001548	0.001584	3.568366×10^{-5}	2.304976×10^{-2}
	1.884	−0.001541	−0.001505	3.568926×10^{-5}	2.316672×10^{-2}
	2.512	−0.004041	−0.004019	2.207738×10^{-5}	5.462890×10^{-3}
	3.140	−0.005000	−0.005000	4.100777×10^{-8}	8.201564×10^{-6}
	3.768	−0.004051	−0.004073	2.201170×10^{-5}	5.434056×10^{-3}
	4.396	−0.001556	−0.001591	3.566489×10^{-5}	2.292553×10^{-2}
	5.024	0.001533	0.001497	3.570741×10^{-5}	2.329308×10^{-2}
	5.652	0.004037	0.004015	2.212304×10^{-5}	5.480554×10^{-3}
	6.280	0.005000	0.005000	9.665089×10^{-8}	1.933028×10^{-5}
t = 0.02	0.000	0.020000	0.020002	1.853337×10^{-6}	9.266683×10^{-5}
	0.628	0.016184	0.016539	3.547192×10^{-4}	2.191778×10^{-2}
	1.256	0.006192	0.006765	5.722393×10^{-4}	9.240910×10^{-2}
	1.884	−0.006162	−0.005590	5.717151×10^{-4}	9.277835×10^{-2}
	2.512	−0.016165	−0.015812	3.533471×10^{-4}	2.185830×10^{-2}
	3.140	−0.020000	−0.020000	1.577400×10^{-7}	7.887008×10^{-6}
	3.768	−0.016203	−0.016556	3.532682×10^{-4}	2.180294×10^{-2}
	4.396	−0.006223	−0.006795	5.718787×10^{-4}	9.190149×10^{-2}
	5.024	0.006132	0.005560	5.719482×10^{-4}	9.327499×10^{-2}
	5.652	0.016147	0.015793	3.534486×10^{-4}	2.189001×10^{-2}
	6.280	0.020000	0.020000	6.214620×10^{-8}	3.107326×10^{-6}
t = 0.06	0.000	0.060000	0.060078	7.812199×10^{-5}	1.302033×10^{-3}
	0.628	0.048552	0.051803	3.250500×10^{-3}	6.694850×10^{-2}
	1.256	0.018577	0.023761	5.183495×10^{-3}	2.790220×10^{-1}
	1.884	−0.018486	−0.013322	5.164223×10^{-3}	2.793513×10^{-1}
	2.512	−0.048496	−0.045296	3.200081×10^{-3}	6.598642×10^{-2}
	3.140	−0.060000	−0.059984	1.586109×10^{-5}	2.643518×10^{-4}
	3.768	−0.048608	−0.051797	3.188704×10^{-3}	6.559995×10^{-2}
	4.396	−0.018668	−0.023844	5.175574×10^{-3}	2.772400×10^{-1}
	5.024	0.018396	0.013234	5.161817×10^{-3}	2.806014×10^{-1}
	5.652	0.048440	0.045287	3.152560×10^{-3}	6.508212×10^{-2}
	6.280	0.060000	0.060060	6.080264×10^{-5}	1.013382×10^{-3}

5 Conclusions

In this paper, we have obtained an approximate-analytical solution to homogeneous as well as non-homogeneous dispersive KdV equations with some initial approximation using modified Adomian decomposition method using Shehu's transform. For the homogeneous KdV equation in Eq. (2), results obtained by methods, standard ADM, LADM, and SADM, are equivalent and therefore give the same results. The LADM and ADM are also powerful methods for solving both linear as well as nonlinear PDEs as these methods do not need any form of transformation, perturbation, or linearization. However, rigorous computation of Adomian polynomials is one of the requirement, which can sometimes result in intensive computations for nonlinear problems.

As our main contribution, we have applied a reliable method, SADM, which combines Shehu's transform with Adomian Decomposition Method to both linear as well as nonlinear homogeneous and non-homogeneous dispersive KdV-type equation and the numerical results using SADM are given in Tables 3 and 5. The obtained numerical results in this paper confirm that SADM is an effective method, as it allows us to know the exact solution after computing first few terms only. Therefore, this method an be considered as an alternative method to solve numerous linear and nonlinear problems efficiently.

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