Springer Proceedings in Mathematics & Statistics

Ouayl Chadli
Sourav Das
Ram N. Mohapatra
A. Swaminathan *Editors*

Mathematical Analysis and Applications

MAA 2020, Jamshedpur, India, November 2–4

Springer

Springer Proceedings in Mathematics & Statistics

Volume 381

This book series features volumes composed of selected contributions from workshops and conferences in all areas of current research in mathematics and statistics, including data science, operations research and optimization. In addition to an overall evaluation of the interest, scientific quality, and timeliness of each proposal at the hands of the publisher, individual contributions are all refereed to the high quality standards of leading journals in the field. Thus, this series provides the research community with well-edited, authoritative reports on developments in the most exciting areas of mathematical and statistical research today.

More information about this series at https://link.springer.com/bookseries/10533

Ouayl Chadli · Sourav Das · Ram N. Mohapatra ·
A. Swaminathan
Editors

Mathematical Analysis and Applications

MAA 2020, Jamshedpur, India, November 2–4

Springer

Editors
Ouayl Chadli
Faculty of Economy
Ibn Zohr University
Agadir, Morocco

Sourav Das
Department of Mathematics
National Institute of Technology
Jamshedpur
Jamshedpur, India

Ram N. Mohapatra
Mathematics
University of Central Florida
Orlando, FL, USA

A. Swaminathan
Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee, India

ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-981-16-8176-9 ISBN 978-981-16-8177-6 (eBook)
https://doi.org/10.1007/978-981-16-8177-6

Mathematics Subject Classification: 41-XX, 33-XX, 30-XX, 26-XX, 47-XX

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Committee

Advisory Committee

- Prof. Ram N. Mohapatra, University of Central Florida, USA
- Prof. P. D. Srivastava, Indian Institute of Technology Bhilai, India
- Prof. Henrik L. Pedersen, University of Copenhagen, Denmark
- Prof. G. K. Srinivasan, Indian Institute of Technology Bombay, India
- Prof. A. Swaminathan, Indian Institute of Technology Roorkee, India
- Dr. Armin Straub, University of South Alabama, USA
- Dr. Bappaditya Bhowmik, Indian Institute of Technology Kharagpur, India
- Dr. Khaled Mehrez, Université de Tunis El Manar, Tunisia
- Dr. Ratikanta Behera, University of Central Florida, USA
- Dr. Gavendra Pandey, University of North Carolina, USA
- Dr. A. S. Kelil, Nelson Mandela University, South Africa
- Dr. Kanailal Mahato, Banaras Hindu University, India
- Prof. R. V. Sharma, Dean (R&C), National Institute of Technology Jamshedpur, India
- Prof. Tarni Mandal, Dean (SW), National Institute of Technology Jamshedpur, India
- Prof. Prabha Chand, Dean (FW), National Institute of Technology Jamshedpur, India
- Dr. Sunil Kumar, HOD (Mathematics), National Institute of Technology Jamshedpur, India
- Dr. Sumit Kumar Debnath, National Institute of Technology Jamshedpur, India
- Dr. Sourav Das, National Institute of Technology Jamshedpur, India

Organizing Committee

Chief Patron:

Prof. Karunesh Kumar Shukla, Director, National Institute of Technology Jamshedpur, India

Patron:

Dr. Sunil Kumar, Head, Department of Mathematics, National Institute of Technology Jamshedpur, India

Chairman:

Dr. Sumit Kumar Debnath, National Institute of Technology Jamshedpur, India

Convener & Organizing Secretary:

Dr. Sourav Das, National Institute of Technology Jamshedpur, India

Members:

- Prof. Tarni Mandal, National Institute of Technology Jamshedpur, India
- Dr. Ramayan Singh, National Institute of Technology Jamshedpur, India
- Dr. Sripati Jha, National Institute of Technology Jamshedpur, India
- Dr. Hari Shankar Prasad, National Institute of Technology Jamshedpur, India
- Dr. Mahendra Kumar Gupta, National Institute of Technology Jamshedpur, India
- Dr. Raj Nandkeolyar, National Institute of Technology Jamshedpur, India
- Dr. Rajat Tripathi, National Institute of Technology Jamshedpur, India
- Dr. Ratnesh Kumar Mishra, National Institute of Technology Jamshedpur, India
- Dr. Snehasis Kundu, National Institute of Technology Jamshedpur, India

Schedule of MAA 2020

Time is in IST (UTC+5:30) Format (Indian Time)

Day 01: November 02, 2020	
08:00-09:00	**Inaugural Function** **Chief Guest: Prof. Ram N. Mohapatra** (University of Central Florida, USA)
09:00-09:50	**Prof. Ram N. Mohapatra** (University of Central Florida, USA) **Title of Talk**: Equilibrium Problems Arising in Variational Inequalities (Session Chair: Dr. A. Swaminathan (IIT Roorkee))
10:00-10:50	**Prof. G. K. Srinivasan** (IIT Bombay, India) **Title of Talk**: On the removable singularities theorem for harmonic functions (Session Chair: Dr. A. Swaminathan (IIT Roorkee))
11:00-13:30	Contributed Talks (Parallel Sessions)
18:00-18:50	**Dr. Ratikanta Behera** (University of Central Florida, USA) **Title of Talk**: Wavelets for Evolution Problems with Localized Structures, (Session Chair: Dr. Sunil Kumar, Head, Department of Mathematics, NIT Jamshedpur)
19:00-19:50	**Dr. Armin Straub** (University of South Alabama, USA) **Title of Talk**: Gaussian binomial coefficients with negative arguments (Session Chair: Dr. Sunil Kumar, Head, Department of Mathematics, NIT Jamshedpur)

Day 02: November 03, 2020	
10:00-10:50	**Prof. P. D. Srivastava** (Indian Institute of Technology Bhilai, India) **Title of Talk**: Spectrum and fine spectrum of generalized lower triangular triple band matrices over the sequence space l_p, (Session Chair: Dr. A. Swaminathan (IIT Roorkee))
11:00-11:50	**Dr. A. Swaminathan** (Indian Institute of Technology Roorkee, India) **Title of Talk**: Geometric Properties of Analytic Functions Associated with Nephroid domain, (Session Chair: Prof. P. D. Srivastava (IIT Bhilai))
15:00-17:30	Contributed Talks (Parallel Sessions)
18:30-19:20	**Prof. Henrik L. Pedersen** (University of Copenhagen, Denmark) **Title of Talk**: The generalized Stieltjes transform and special functions, (Session Chair: Dr. A. Swaminathan (IIT Roorkee))

Day 03: November 04, 2020	
11:00-11:50	**Dr. Bappaditya Bhowmik** (Indian Institute of Technology Kharagpur, India) **Title of Talk**: On harmonic univalent mappings with nonzero pole (Session Chair: Dr. Kanailal Mahato, Institute of Science, BHU)
11:50-12:40	**Dr. Kanailal Mahato** (Institute of Science, BHU, India) **Title of Talk**: Composition of wavelet transform on some function spaces (Session Chair: Dr. Sumit Kumar Debnath, NIT Jamshedpur)
12:40-13:30	**Dr. Khaled Mehrez** (Universite Tunis El Manar, Tunisia) **Title of Talk**: Certain generating function involving the incomplete Fox-Wright function and its consequences (Session Chair: Dr. Sumit Kumar Debnath, NIT Jamshedpur)
16:30-17:00	**Valedictory Session** **Chief Guest: Prof. P. D. Srivastava** (Indian Institute of Technology Bhilai, India)

Preface

Welcome to the Proceedings of the International Conference on Mathematical Analysis and Applications (MAA 2020). This international conference was organized by the Department of Mathematics, National Institute of Technology Jamshedpur, India, during November 02–04, 2020. The main objective of this event was to bring together mathematicians and researchers who work in the fields of mathematical analysis and its applications in various aspects of science and engineering and to encourage collaboration and exchange of interdisciplinary ideas among the participants. The main focus of the conference was to demonstrate the versatility, the applicability, and the inherent beauty of mathematical analysis and its applications. The primary aim of this event was to bring together different speakers who could deliver talks in their field of expertise. Another important point that is different from other events is to concentrate on a specific direction in mathematics (mathematical analysis and its applications) instead of diverting the topic to other directions, which added value in comparison to similar events.

The theme of MAA 2020 included the following topics:

Approximation theory, operator theory, fixed point theory, generalized metric spaces, function spaces, differential topology, geometric and univalent function theory, potential theory, value distribution theory, control theory, fractional calculus, orthogonal polynomials, special functions, operation research, theory of inequalities, equilibrium problem, Fourier and wavelet analysis, mathematical physics, graph theory, stochastic orders, and asymptotic analysis.

Internationally reputed speakers and researchers were invited to deliver their talks in this event. Selected list of keynote and invited speakers is given below.

1. Prof. Ram N. Mohapatra, University of Central Florida, USA
2. Prof. P. D. Srivastava, Indian Institute of Technology Bhilai, India
3. Prof. Henrik L. Pedersen, University of Copenhagen, Denmark
4. Prof. G. K. Srinivasan, Indian Institute of Technology Bombay, India
5. Prof. A. Swaminathan, Indian Institute of Technology Roorkee, India
6. Dr. Armin Straub, University of South Alabama, USA
7. Dr. Bappaditya Bhowmik, Indian Institute of Technology Kharagpur, India

8. Dr. Khaled Mehrez, Université de Tunis El Manar, Tunisia
9. Dr. Ratikanta Behera, University of Central Florida, USA
10. Dr. Kanailal Mahato, Institute of Science, BHU, India

Nearly 120 researchers participated from the United States of America, Denmark, Tunisia, Turkey, South Africa, Philippines, Oman, and many other reputed institutions from India. Eighty researchers contributed their research papers for presentation from Institutes of Excellence in India and abroad. The substantial number of abstracts for paper presentation indicates implicitly the success of the conference. A total number of 68 manuscripts were received during call for papers for the proceedings of the conference. Each manuscript was sent to at least three referees, carefully chosen by the Editorial Board of the proceedings of MAA 2020. Based on the comments, suggestions, and recommendations of the referees, our Editorial Board has selected only 22 manuscripts for inclusion in the proceedings.

Mathematical analysis has applications in various fields of science and engineering. In fact, it is the foundation for several applications involving mathematical concepts. It serves as a bridge between pure and applied mathematics. The papers included in this volume explain the recent theory and techniques of mathematical analysis and its applications. Some papers discuss the applications to real-life situations. This proceeding will be beneficial for the researchers and research students working in the field of pure and applied mathematics.

While talking about the success of the conference, it would be incomplete if we forget to thank National Institute of Technology Jamshedpur, India, for its immense support and encouragement to organize this conference. We would like to express our sincere thanks to Prof. Karunesh Kumar Shukla, Director NIT Jamshedpur, India, for his continuous support, and cooperation. We extend our grateful thanks to our keynote speakers and the invited speakers, who in spite of their busy schedules accepted our invitation to share their valuable knowledges with us.

A total number of 76 referees from around the world contributed to the peer review process. We extend our sincere gratitude to the referees for spending their valuable time to review the manuscripts carefully and send their reports within the due date. Our special thanks are due to our participants who have attended this conference. Without their helping hand it would have been impossible to complete this conference. We owe our heartfelt thanks to our organizing committee members, advisory committee members, supporting staffs, students, and faculty members for their support and tireless effort to manage the conference successfully. Once again, we thank everyone who supported directly or indirectly to make this conference a reality.

Our aim will be achieved if the readers find this volume helpful and useful for their further studies and future research. We are grateful to Springer for publishing the proceedings of the conference.

Agadir, Morocco — Ouayl Chadli
Jamshedpur, India — Sourav Das
Orlando, USA — Ram N. Mohapatra
Roorkee, India — A. Swaminathan
February 2021

Contents

A Note on Isolated Removable Singularities of Harmonic Functions

Gopala Krishna Srinivasan

Abstract A proof of the removable singularities theorem for harmonic functions is presented which seems to be different from existing proofs in the literature. This is an important result in analysis with applications to many areas of mathematics. Weyl's lemma which is used in the course of the argument is also proved in a special case to make the note self-contained.

Keywords Harmonic functions · Subharmonic functions · Weyl's lemma

2010 AMS Subject Classification Primary 31-03

1 Introduction

The present note arose as a by-product of an intensive course on Riemann surfaces and complex geometry delivered to a small group of students. The motivation arises from an attempt at constructing harmonic functions on Riemann surfaces with prescribed singularities. These results are needed in the proofs of the uniformization theorem and the Riemann Roch theorem [7]. Although the main result proved in this note is folk-lore, it appears that the result is not as commonly found in books as it ought to be. We provide here an alternate argument which we believe is different from known proofs and is not devoid of interest. Thus the present note is in the main an expository one. We shall make use of Weyl's lemma in the course of the proof. To make this note self-contained we shall give a proof of Weyl's lemma in the special case that we need. We begin by recalling the classical and well-known result in the theory of functions of one complex variable:

Theorem 1 *Assume that f is holomorphic and bounded in the punctured disc $0 < |z - a| < R$. Then f extends as a holomorphic function on the full open disc of radius R centered at a.*

G. K. Srinivasan (✉)
Department of Mathematics, Indian Institute of Technology Bombay, Mumbai, Maharashtra, India
e-mail: gopal@math.iitb.ac.in

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_1

The usual argument proceeds by looking at the Laurant expansion of f at a and showing that the coefficients of the negative powers of $z - a$ all vanish. It would then seem natural to ask the corresponding question for harmonic functions and prove it along similar lines. Such an approach is carried out in the classic work of O. D. Kellogg [4, pp. 269–270] using Fourier expansions in terms of spherical harmonics.

The result can be reformulated but before doing so we introduce some notations. The set of all holomorphic functions on a domain Ω is denoted by $A(\Omega)$ and $A^p(\Omega)$ denotes $A(\Omega) \cap L^p(\Omega)$ where $1 \leq p \leq \infty$. The above theorem may then be expressed as

$$A^\infty(B'_R(0)) = A^\infty(B_R(0)).$$

One can of course try formulating Riemann's removable singularities theorem for the spaces $A^p(\Omega)$ and also look for multi-variable analogues. We shall remain silent on these matters except for citing a couple of good references. The classical cases are dealt with in [3] and the L^p version is available in [5].

In a different direction, the class $A^p(\Omega)$ could be replaced by other function spaces such as the class of harmonic functions on Ω that lie in L^p or the kernel of a more general linear differential operator L. For sure, the nature of the hypothesis involved will then depend on the nature of the operator L as well as the number of space dimensions. The results are often expressed in terms of capacity theory. Here we shall look at the case when L is the Laplace's operator namely, the class of harmonic functions for which a great deal is already known and the main theorem proved here is available in [1, p. 92] as well as in Kellogg's book [4] cited above. The result also holds for subharmonic functions and is available in [2].

2 The Removable Singularities Theorem

We shall prove the theorem in the general $n-$dimensional setting ($n \geq 2$) and begin by recalling that the fundamental solution $E(x)$ of the Laplace's operator in $\mathbb{R}^n$ ($n \geq 2$) is given by

$$E(x) = c_n |x|^{2-n}, \quad n \geq 3 \quad \text{and } E(x) = \log|x| \qquad \text{when } n = 2.$$

Lemma 1 *If u is smooth and harmonic in $B'_R(0)$, and $u = o(E(x))$ in a neighborhood of the origin, then the function*

$$\tilde{I}(r) = \int_{|\omega|=1} u(r\omega) d\omega$$

is constant with respect to r. In particular, denoting by $dS(x)$ the area measure on the sphere of radius r,

$$\int_{|x|=r} u(x)dS(x) = \alpha r^{n-1}.$$

Proof We first obtain a differential equation for $\tilde{I}$. Apply the Gauss divergence theorem to Δu on the shell $r_1 < |x| < r_2$ and we get

$$0 = \int_{r_1<|x|<r_2} \Delta u(x)dx = \int_{|x|=r_2} \frac{\partial u}{\partial \nu} dS(x) - \int_{|x|=r_1} \frac{\partial u}{\partial \nu} dS(x),$$

where ν denotes the unit normal to the boundary of the shell and pointing outside the shell. We get immediately,

$$r_2^{n-1} \int_{|\omega|=1} \omega \cdot \nabla u(r_2\omega)d\omega = r_1^{n-1} \int_{|\omega|=1} \omega \cdot \nabla u(r_1\omega)d\omega.$$

We see that the function $r^{n-1}\tilde{I}'(r)$ is constant whereby differentiation gives the Cauchy Euler equation

$$r^2\tilde{I}''(r) + (n-1)r\tilde{I}'(r) = 0.$$

This immediately gives

$$\tilde{I}(r) = \alpha + \beta r^{2-n}, \quad \text{if } n \geq 3, \quad \tilde{I}(r) = \alpha + \beta \log r, \quad \text{if } n = 2. \tag{1}$$

So far we have not used any hypothesis on u regarding the growth of u as we approach the origin and so (1) holds in general. Now the hypothesis $u = o(E(x))$ gives

$$\tilde{I}(r) = \int_{|\omega|=1} u(r\omega)d\omega = o(r^{2-n}) \ (\text{when } n \geq 3) \quad \text{and} \quad \tilde{I}(r) = o(-\log r) \ (\text{when } n = 2)$$

according as $n \geq 3$ or $n = 2$. This forces $\beta = 0$ and the result follows. □

Comments:
Note that without the hypothesis $u = o(E(x))$, Lemma 1 says that $\tilde{I}(r)$ is linear in $E(r)$. If instead of being harmonic, if u were merely subharmonic namely $\Delta u \geq 0$ then we would get the differential inequality

$$r^2\tilde{I}'' + (n-1)r\tilde{I}' \geq 0.$$

Setting $\log r = s$ in case of $n = 2$ we get that

$$\frac{d^2\tilde{I}}{ds^2} \geq 0.$$

This means $\tilde{I}(r)$ is a convex function of $\log r$ in two dimensions which is the analogue of Hadamard's three circles theorem known in complex analysis. In dimensions three and higher one puts $r^{2-n} = s$ and again $\tilde{I}(r)$ is a convex function of r^{2-n} in dimensions higher than two and this is Hadamard's three sphere's theorem [6, p. 131].

Theorem 2 (Removable singularities theorem) *If u is harmonic in $B'_R(0)$ and $u = o(E(x))$ in a neighborhood of the origin then the origin is a removable singularity of u.*

Proof We apply Cauchy's estimate to the derivative $\dfrac{\partial u}{\partial x_j}$ at a point p such that $|p| = \epsilon$. Taking a sphere of radius[1] ϵ centered at p, we get

$$\left|\frac{\partial u}{\partial x_j}(p)\right| \le \epsilon^{1-n} o(1) \quad n \ge 3; \tag{2}$$

whereas,

$$\left|\frac{\partial u}{\partial x_j}(p)\right| \le \epsilon^{-1} |\log(\epsilon)| o(1) \quad \text{if } n = 2. \tag{3}$$

We now show that $\Delta u = 0$ in the sense of distributions which suffices since Weyl's lemma then says that u is smooth. So let ϕ be a smooth function with compact support in $B_R(0)$.

$$\langle \Delta u, \phi \rangle = \langle u, \Delta \phi \rangle = \int_{B_R(0)} u \Delta \phi dx,$$

since u is locally integrable. Hence, since u is harmonic and smooth on the punctured ball,

$$\begin{aligned} \langle \Delta u, \phi \rangle &= \lim_{\epsilon \to 0} \int_{\epsilon < |x| < R} (u \Delta \phi - \phi \Delta u) dx, \\ &= \lim_{\epsilon \to 0} \int_{|x| = \epsilon} \Big(u \frac{\partial \phi}{\partial \nu} - \phi \frac{\partial u}{\partial \nu} \Big) dS(x), \end{aligned}$$

where ν denotes the unit normal pointing in the direction of the origin. Now, $u dS(x) = \epsilon o(1)$ if $n \ge 3$ and $u dS(x) = \epsilon |\log \epsilon| o(1)$ when $n = 2$, we see that

$$\lim_{\epsilon \to 0} \int_{|x| = \epsilon} u \frac{\partial \phi}{\partial \nu} dS(x) = 0.$$

We need to now examine the limit:

---

[1] Technically one should use a ball of radius ϵ' centered at p where $0 < \epsilon' < \epsilon$. Now use the fact that as $\epsilon \longrightarrow 0$

$$\sup_{B_{\epsilon'}(p)} |u(x)/(2\epsilon)^n| \longrightarrow 0.$$

$$\lim_{\epsilon \to 0} \int_{|x|=\epsilon} \phi \frac{\partial u}{\partial \nu} dS(x),$$

and show that it goes to zero as $\epsilon \to 0$. Observe that along the sphere,

$$\phi dS(x) = \epsilon^{n-1} O(1),$$

whereas by (2) we get, for $n \geq 3$,

$$\phi \frac{\partial u}{\partial \nu} dS(x) = o(1)$$

and the limit is zero as $\epsilon \to 0$. Thus we see directly in case $n \geq 3$ that $\Delta u = 0$ in the sense of distributions on the ball $B_R(0)$ and Weyl's lemma now implies that u is infinitely differentiable there. There remains the case $n = 2$ which is more delicate since we only have $|\log \epsilon| o(1)$ estimate for $\phi \frac{\partial u}{\partial \nu} dS(x)$. We modify the integral as

$$\begin{aligned} \int_{|x|=\epsilon} \phi \frac{\partial u}{\partial \nu} dS(x) &= \int_{|x|=\epsilon} (\phi(x) - \phi(0)) \frac{\partial u}{\partial \nu} dS(x) + \phi(0) \int_{|x|=\epsilon} \frac{\partial u}{\partial \nu} dS(x) \\ &= \epsilon \int_{|x|=\epsilon} O(1) \frac{\partial u}{\partial \nu} dS(x) + \epsilon \phi(0) \int_{|\omega|=1} \nabla u(\epsilon\omega) \cdot \omega d\omega \\ &= \epsilon |\log \epsilon| o(1) + \epsilon \phi(0) \tilde{I}'(r), \end{aligned}$$

where we have used the estimate (3). Since $\tilde{I}'(r) = 0$ by Lemma 1, we are left with $\epsilon |\log \epsilon| o(1)$ which goes to zero with ϵ and the proof is complete. □

Proof of a special case of Weyl's lemma:
For convenience we include a proof of Weyl's lemma in the special case that is relevant to us namely, when u is smooth on $B'_R(0)$ and is a distributional solution of $\Delta u = 0$ on $B_R(0)$ then u is smooth on $B_R(0)$. Well, let ϕ be a smooth function with support in $B_{2R/3}(0)$ and such that ϕ is identically one in a neighborhood of the origin say $B_{R/3}(0)$.

$$\Delta(\phi u) = \phi \Delta u + u \Delta \phi + 2\nabla u \cdot \nabla \phi = u \Delta \phi + 2\nabla u \cdot \nabla \phi = f.$$

Since $\nabla \phi$ and $\Delta \phi$ both vanish in a neighborhood of the origin, f is smooth with compact support and vanishes in a neighborhood of the origin. Further, ϕu also vanishes on the boundary of the ball, so ϕu is the solution of the Poisson's equation with zero boundary data thereby leading to the integral representation:

$$\phi(x) u(x) = \int_{R/3 < |\xi| < 2R/3} G(x, \xi) f(\xi) d\xi,$$

where $G(x, \xi)$ denotes the Green's function for the ball. Hence,

$$u(x) = \int_{R/3<|\xi|<2R/3} G(x, \xi) f(\xi) d\xi, \quad |x| < R/3.$$

Now since $G(x, \xi)$ is infinitely differentiable on the set $x \neq \xi$, we see from the above integral representation that u is infinitely differentiable on $|x| < R/3$ as asserted.

Acknowledgements I would like to thank Anbhu Swaminathan from the Department of Mathematics, IIT Roorkee for encouraging me to write up this piece. This note was a by-product of a series of lectures delivered to a small group of students from IIT Bombay including some from other institutes during the difficult period of covid19. The response and enthusiasm shown by these students has gone a long way in alleviating the agony caused by the pandemic. The author wishes to thank all these participating students. The author wishes to thank the referee whose suggestions have contributed significantly toward enhancing clarity and readability of the paper.

References

1. Egorov, Yu.V., Shubin, M.A.: Foundations of the Classical Theory of Partial Differential Equations. Springer, New York (1998)
2. Gardiner, S.J.: Removable singularities for subharmonic functions. Pac. J. Math. **147**, 71–80 (1991)
3. Gunning, R.C.: Holomorphic functions of several complex variables, vol. 1 (1990)
4. Kellogg, O.D.: Foundations of Potential Theory. Springer (1967)
5. Ohsawa, T.: Analysis of Several Complex Variables. American Mathematical Society, Providence (2002)
6. Protter, M.H., Weinberger, H.F.: Maximum Principles for Partial Differential Equations. Springer, New York (1984)
7. Reyssat, E.: Quelques Aspects Des Surfaces De Riemann. Birkhauser (1989)

Nonlinear Evolution Equations by a Ky Fan Minimax Inequality Approach

Ouayl Chadli, Ram N. Mohapatra, and G. Pany

Abstract Nonlinear evolution equations appear in various fields of sciences including mechanics, physics, engineering, and material sciences. Essential functional methods for the treatment of both the linear as well as nonlinear evolution equations are based on the theory of spectral methods, maximal monotone operators, fixed point theorems, and concept of C_0-semigroups of linear mappings along with the Leray-Schauder degree theory. Recently, the solvability for the evolution equations of nonlinear type has been considered using the Ky Fan minimax inequality. This approach is quite new and different as compared to the traditional approaches. In 1972, Ky Fan (Inequalities. Academic, New York, pp. 103–113, 1972) put forward his pioneering result concerning the existence of solutions for an inequality of minimax type, which is nowadays called as "equilibrium problem" in literature. This kind of model has shown to be a cornerstone result of nonlinear analysis and has gained much interest in the past because it has been used in several contexts such as physics, chemistry, economics, engineering, and so on. This work aims to present a review of recently obtained results on the use of the equilibrium problem theory in the study of nonlinear (implicit) evolution equations. Along with that, we discuss the problem with initial value condition as well as periodic and anti-periodic solutions.

Keywords Nonlinear evolution equations · Equilibrium problems · Initial value problem · Periodic solution · Anti-periodic solutions

Mathematics Subject Classification 34A06 · 34A09 · 47H05 · 90C48 · 90C33 · 49J35

O. Chadli
Department of Economics, Ibn Zohr University, B.P. 8658 Poste Dakhla, Agadir, Morocco
e-mail: o.chadli@uiz.ac.ma

R. N. Mohapatra (✉) · G. Pany
Department of Mathematics, University of Central Florida, Orlando, FL, USA
e-mail: ram.mohapatra@ucf.edu

G. Pany
e-mail: gp11@iitbbs.ac.in

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_2

1 Introduction

Suppose V is a reflexive Banach space which is densely and continuously imbedded in H. Here H is a real Hilbert space, that is identified with its dual. Thus we get $V \subset H \subset V^*$, where V^* denotes the dual of V. Let p, q, and T be constants such that T is positive , $p \geq 1$ and p, q together satisfy $1/p + 1/q = 1$. We consider the following spaces $X = L^p(0, T; V)$, $X^* = L^q(0, T; V^*)$ and $\mathcal{W} = \{u \in X : u' \in X^*\}$ where $u' = \frac{du}{dt}$ is the generalized derivative of u on $]0, T[$, i.e. $\int_0^T u'(t)\phi(t)dt = -\int_0^T u(t)\phi'(t)dt$ for all $\phi \in C_0^\infty([0, T])$. We consider the problem: Find $u \in \mathcal{W}$ such that

$$u'(t) + \mathcal{A}(t)u(t) = f(t), \text{ a.e. } \; t \in (0, T) \tag{1}$$

with

$$u(0) = u_0, \tag{2}$$

where $\mathcal{A}(t)$ is a nonlinear operator from V to V^* and $f : [0, T] \to V^*$ is a measurable functional. When the condition (2) is replaced with $u(0) = u(T)$, then we deal with *periodic solutions* of the problem (1); and when it is replaced with $u(0) = -u(T)$ then we deal with *anti-periodic solutions* of the problem (1).

A particular form of the problem (1) has been widely considered in literature, it consists in the following problem: Find $u \in \mathcal{W}$ such that

$$u'(t) + Au(t) + Gu(t) = f(t), \text{ a.e. } \; t \in (0, T) \tag{3}$$

with

$$u(0) = u_0, \tag{4}$$

where $A : V \to V^*$ is a monotone operator and $G : V \to V^*$ is not. Such issues have been addressed by many authors. The problem (3) was investigated by Browder [2], Pavel [3], Pavel and Vrabie [4] and Pazy [5] for linear operator A, whereas the nonlinear scenario was investigated by Attouch and Damlamian [6], Crandall and Nohel [7], Gutman [8], Hirano [9, 10], and Vrabie [11, 12]. Barbu [13] established the existence of solutions for the Cauchy problem (3)–(4) for $G = 0$, utilizing the theory of monotone operators. Ahmed and Xiang [14], and Liu in [15, 16] studied (3) with zero-initial valued condition (4) using the concept of pseudomonotone operators.

Along with periodic solutions, solvability of anti-periodic solutions to nonlinear evolution equation in the framework of Hilbert spaces has been equally focused and studied by various authors in the last decades. The motivation to study anti-periodic problem mainly comes from physical problems. One may refer to [17–19] to see how the mathematical modeling of a variety of physical processes gives rise to anti-periodic solutions. The first study in this regard was carried out by Okochi [20]. Here the author studied the anti-periodic solutions for evolution equations using the time-independent operator in the framework of Hilbert spaces [21, 22]. Consequently, Haraux [23] established some results on existence as well as uniqueness of solutions to anti-periodic problems using Brouwer's or Schauder's fixed point theorem. Further, Aizicovici and Pavel [24] investigated anti-periodic solutions for second-order evolution equations in the framework of Hilbert and Banach spaces utilizing the theory of monotone and accretive operators, while the nonmonotone cases are considered in [25]. Chen [26] and Chen et al. [27] continued to investigate the anti-periodic solutions with respect to the first-order evolution equations by involving theorems on fixed point in a real Hilbert space H, which is separable. In this study, the evolution equations are connected to a self-adjoint operator A, which is both linear and dense. The domain of A is compactly embedded into H. Recently, Liu [28] investigated the solvability of anti-periodic solutions of evolution equation which is time-independent and nonlinear and involves perturbations of nonmonotone type. utilizing a Browder's surjectivity result on pseudomonotone perturbations of maximal monotone operators in the framework of Banach spaces, that are real and reflexive.

Implicit nonlinear evolution equation consists in the problem below:

$$\frac{d}{dt}(\mathcal{B}u(t)) + \mathcal{A}(t)u(t) = f(t), \text{ a.e. } t \in (0, T), \tag{5}$$

$$\mathcal{B}(u(0)) = \mathcal{B}(u_0), \tag{6}$$

where the operator $\mathcal{B}$ from V to V^* is symmetric,linear, and positive, f is a measurable functional on V^* with domain in $[0, T]$, and $\mathcal{A}(tbounded)$ from V to V^* is a nonlinear time-dependent operator, with t varying over $[0, T]$. Andrews, Kuttler and Schillor [29], Barbu [13], Barbu and Favini [30], Favini and Yagi [31], Liu [32], and Showalter [33] studied implicit evolution equations dealing the Cauchy problem. The operator $\mathcal{A}$ involved in these works was time-invariant and maximal monotone. One may refer to [34, 35] to gain insight on implicit evolution equations dealing with the periodic or anti-periodic problem. Here the resolution technique relies on a convergent approximation procedure and the concept of pseudomonotone perturbations for maximal monotone mappings. We cite also Barbu and Favini [36] and DiBenedetto and Showalter [37], dealing the case where B is nonlinear and monotone. In this context the techniques and the hypotheses vary.

In this paper, we present some recent results on the solvability of nonlinear evolution equations of implicit type by a Ky Fan minimax inequality approach. In 1972, Ky Fan [1] investigated the similar existential results for the solutions of an inequality that has been regarded as a remarkable outcome for nonlinear analysis. We recreate this result in its dual form:

Theorem 1 *Suppose X is a Hausdorff topological vector space with K as a nonempty compact convex subset. Let the following conditions hold for the map Φ defined from $K \times K$ to $\mathbb{R}$:*

(i) $\Phi(u, u) \geq 0$ *for all* $u \in K$*;*
(ii) Φ *is quasi-convex in the second variable for each* $u \in K$*;*
(iii) Φ *is upper semicontinuous in the first variable for each* $v \in K$*;.*

Then, there exists $\bar{u} \in K$ *for which* $\Phi(\bar{u}, v) \geq 0$ *for each* $v \in K$.

Ky Fan named the problem as *minimax inequality*, which is popularly known as *equilibrium problem*, in brief (EP) nowadays in literature. This concept has a significant role in unifying the complex models in the areas of variational inequalities, game theory, mathematical economics, optimization, and fixed point theory to simpler form. The appellation "equilibrium problems" appeared in the pioneering work by Blum and Oettli [38] where they mentioned the unifying aspect of (EP) and provided various basic notions and results. This model has gained much interest in the past as it has been utilized in various contexts as physics, chemistry, economics, engineering, and so on; see [40] for a recent survey.

Suppose K is a nonempty set and Φ is bifunction from $K \times K$ to $\mathbb{R}$. The Ky Fan minimax inequality (or equilibrium problem) is formulated, in a more general way, in the following manner:

$$\bar{u} \in K : \Phi(\bar{u}, v) \geq 0, \text{ for every } v \text{ in } K. \tag{7}$$

A general formulation of equilibrium problem, known as "implicit variational problem," that includes as special cases, fixed point problems, equilibrium problems, Nash equilibria, variational and quasi-variational inequalities was introduced by Mosco [41], Joly and Mosco [42] toward 1975 and later in 1979. This form of EP is expressed as the addition of two bifunctions and was consequently named as *mixed EP*. Further the authors introduced monotonicity in the context bifunctions being motivated by the monotonicity for operator in the sense of Minty. They derived certain results on solvability for this type of EP. These results were established in view of some weaker kind of assumption as compared to the results of Ky Fan [1]. The mixed equilibrium problem takes the following form,

$$\bar{u} \in K : \Phi(\bar{u}, v) + \Psi(\bar{u}, v) \geq 0, \text{ for all } v \in K, \tag{8}$$

where Φ, Ψ are defined from $K \times K$ to $\mathbb{R}$, and K a nonempty set.

In this paper, we establish some existential results for nonlinear (implicit) evolutions equations (1) and (5) by using some recent outcomes on the solution existence of the mixed equilibrium problem (8) where Φ is monotone as well as maximal monotone and the bifunction Ψ is both pseudomonotone, quasimonotone from the topological point of view. Here we consider the problems (1) and (5) with their respective initial value conditions, periodic and anti-periodic conditions. The concept of maximal monotonicity of bifunctions may be considered as an extended version to equilibrium problems of the related one for nonlinear operators. Gwinner [43, 44] first introduced the notion of pseudomonotonicity in the context of bifunctions in the sense of topology and consequently the concept is inspired by a pseudomonotone operator of topological type as per Brézis [45]. In the present work, we adopt the notion of quasimonotonocity for bifunctions in the sense of topology. This gives a further modification to equilibrium bifunctions of the widely known related notions of nonlinear mappings [33, 46, 47]. In these papers quasimonotone mappings defined as per topological view point are considered for certain subclasses of particular category. Further it may be noticed that there have been studies related to pseudomonotonicity and quasimonotonicity from an algebraic point of view. The studies in this regard have been initiated by Karamardian [48] and considered later by many authors for analyzing problems related to equilibrium and variational inequalities. One may refer to [49, 50] and the references therein for more information.

2 Preliminaries and Basic Mathematical Tools

Consider a reflexive Banach space X with X^* as the dual. Let the space X have been renormed in a manner such that X and its dual are locally uniformly convex. We list some standard notations to be used in the sequel as follows,

(i) $\|\cdot\|$: norms of both X and X^*,
(ii) $\text{conv}(E)$: the convex hull of E, for each subset E of X,
(iii) $\text{cl}(E)$: closure of E in X,
(iv) $\mathcal{F}(E)$: family of all finite subsets of E,
(v) 2^E : family of all subsets of E,
(vi) $\langle u^*, u\rangle$: value of u^* at u for $u \in X$ and $u^* \in X^*$,
(vii) $u_n \to u$: sequence $\{u_n\}_{n\in\mathbb{N}}$ in X converges strongly to u in weak topology $\sigma(X, X^*)$ of X,
(viii) $u_n \rightharpoonup u$: sequence $\{u_n\}_{n\in\mathbb{N}}$ in X converges weakly to u in $\sigma(X, X^*)$,
(ix) $\mathcal{D}(T) := \{u \in X : T(u) \neq \emptyset\}$: domain of a multivalued mapping $T : X \to 2^{X^*}$,
(x) $\mathcal{G}(T) := \{(u, u^*) : u \in \mathcal{D}(T) \text{ and } u^* \in T(u)\}$: graph of T.

Definition 1 $T : X \to 2^{X^*}$ is

(a) monotone if, $\langle u^* - v^*, u - v\rangle \geq 0$ for any $u, v \in \mathcal{D}(T)$, and for all $u^* \in T(u)$ and $v^* \in T(v)$;
(b) maximal monotone if, $\langle u^* - v^*, u - v\rangle \geq 0$ for all (v, v^*) in $\mathcal{G}(T)$ implies u in $\mathcal{D}(T)$ and u^* in $T(u)$.

Consider $J : X \to 2^{X^*}$ to be the duality mapping, which is defined as

$$J(x) := \left\{x^* \in X^* : \langle x^*, x\rangle = \|x^*\|^2 \text{ and } \|x^*\| = \|x\|\right\}.$$

It follows from Hahn-Banach theorem that, $J(x)$ is nonempty for any x in X. We may predict that the duality mapping J is continuous, single-valued, monotone and satisfies the (S_+) condition as we have initially assumed that both X and X^* are locally uniformly convex, see e.g. [51, Proposition 32.22].

Definition 2 The mapping $T : X \supset \mathcal{D}(T) \to X^*$ is,

(i) pseudomonotone as per Brézis (in brief B-PMO) if,

$$\liminf \langle T(u_n), u_n - v\rangle_X \geq \langle T(u), u - v\rangle_X, \quad \text{for all } v \text{ in } X;$$

where the sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}(T)$ satisfies $u_n \rightharpoonup u$ in X and $\limsup\langle T(u_n), u_n - u\rangle \leq 0$,

(ii) quasimonotone as per topological point of view (in brief T-QMO) if,

$$\limsup_{n\to\infty}\langle T(u_n), u_n - u\rangle_X \geq 0;$$

where the sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}(T)$ satisfies $u_n \rightharpoonup u$ in X,

(iii) demicontinuous if, for $u_n \to u$ in X, $T(u_n) \rightharpoonup Tu$ in X^*;

(iv) hemicontinuous (or, upper hemicontinuous) if, $t \mapsto \langle T(u + tv), w\rangle_X$ is continuous (or, upper semicontinuous) on [0, 1], for all u, v, w in X,;

(v) bounded if, bounded sets are mapped into bounded sets by T;

(vi) fulfills the (S_+) condition if, for any $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}(T)$ with $u_n \rightharpoonup u \in X$ having the property, $\limsup\langle T(u_n), u_n - u\rangle \leq 0$, we have $u_n \to u$.

Remark 1 Consider $T : X \supset D(T) \to X^*$ to be single-valued

(i) T is T-QMO, if it is monotone.

(ii) T is T-QMO, if it is B-PMO, but in general the converse may not be true. Infact, for the operator $T : X \longrightarrow X^*$ given by

$$T(u) = \begin{cases} 0, & \|u\| < 1, \\ J(u), & \|u\| = 1, \end{cases}$$

we prove that T is monotone. For u, v in X, we have the following choices:

- If $\|u\| < 1$ and $\|v\| < 1$, it is obvious that T is monotone.
- If $\|u\| = 1$, $\|v\| < 1$, we get

$$\begin{aligned}\langle T(u) - T(v), u - v\rangle = \langle J(u), u - v\rangle &= \|u\|^2 - \langle J(u), v\rangle \\ &\geq \|u\|^2 - \|u\|\|v\| = 1 - \|v\| > 0.\end{aligned}$$

- If both the norms become equal to 1, then $\langle T(u) - T(v), u - v\rangle = \langle J(u) - J(v), u - v\rangle \geq 0$.

Thus, T is monotone and as a consequence it is T-QMO. Next, we proceed to show that T is not B-PMO. Take u in $\mathcal{D}(T)$ with norm equal to 1 be fixed and consider $u_n = \frac{n-1}{n}u$, $n \geq 1$. Then $u_n \to u$, therefore $u_n \rightharpoonup u$. But, we have $\langle T(u_n), u_n - u\rangle = 0$. Thus, $\limsup\langle T(u_n), u_n - u\rangle = 0 \leq 0$. Now, suppose v to be 0, then $\liminf\langle T(u_n), u_n - v\rangle = 0$, ahd hence

$$0 = \liminf\langle T(u_n), u_n - v\rangle < 1 = \langle T(u), u - v\rangle.$$

So T is not B-PMO.

Now let us revisit generalized pseudomonotonicity with reference to the domain of a linear maximal monotone operator, see [52, Definition 2.151].

Definition 3 Consider $L : \mathcal{D}(L) \subset X \longrightarrow X^*$ to be maximal monotone mapping, that is densely defined, and linear. A mapping T from X to X^* is called L-generalized pseudomonotone (or L-GPMO) if for a sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}(L)$ satisfying $u_n \rightharpoonup u,\ L(u_n) \rightharpoonup L(u)$, $\limsup\langle T(u_n), u_n - u\rangle_X \leq 0$, we have

$$\liminf\langle T(u_n), u_n - v\rangle_X \geq \langle T(u), u - v\rangle_X \quad \text{for every } \ v \text{ in } X.$$

The L-generalized quasimonotonicity for a mapping T follows similarly.

Next, we recollect some concepts regarding bifunctions, studied earlier [38, 44].

Definition 4 Consider $\emptyset \neq K \subset X$ to be closed as well as convex. The bifunction Θ defined from $K \times K$ to $\mathbb{R}$ is,

(i) monotone if, $\Theta(x, y) + \Theta(y, x) \leq 0$, for all x, y in K;
(ii) pseudomonotone as per of Brézis (in short B-PMB) if, for any sequence $(u_n)_{n\in\mathbb{N}}$ in K satisfying $u_n \rightharpoonup u$ in X and $\liminf\ \Theta(u_n, u) \geq 0$, we obtain $\limsup\ \Theta(u_n, v) \leq \Theta(u, v)$ for each v in K;
(iii) quasimonotone in the topological sense (in short T-QMB) if, for $\{u_n\}_{n\in\mathbb{N}}$ in K satisfying $u_n \rightharpoonup u \in K$, we have

$$\liminf_{n\to\infty} \Theta(u_n, u) \leq 0.$$

(iv) hemicontinuous (respectively upper hemicontinuous) if, $t \mapsto \Theta(tu + (1-t)v, w)$ is continuous (respectively upper semicontinuous) on $[0, 1]$ for every u, v, w in K,
(v) fulfills the (S_+) condition whenever, for $\{u_n\}_{n\in\mathbb{N}}$ in K satisfying $u_n \rightharpoonup u \in K$ and $\liminf \Theta(u_n, u) \geq 0$ we get $u_n \to u$.

Remark 2 (i) If the real-valued bifunction $\Theta(\cdot, v)$ defined on $X \times X$ is upper semicontinuous for the weak topology $\sigma(X, X^*)$, then the bifunction is B-PMB.
(ii) If the operator T from X to X^* is B-PMO (resp., satisfies the (S_+) condition, T-QMO), then the $\Theta : X \times X \longrightarrow \mathbb{R}$ defined by $\Theta(u, v) = \langle T(u), v - u\rangle_X$ is B-PMB (resp., satisfies the (S_+) condition, T-QMB).
(iii) If the bifunctions Φ_1, Φ_2 defined from $K \times K$ to $\mathbb{R}$ are B-PMB for which $\Phi_1(u, u) \leq 0$ and $\Phi_2(u, u) \leq 0$ for all u in K, with $X \supset K$ is closed and convex, then, $\Phi_1 + \Phi_2$ is B-PMB, see [53].
(iv) If Θ is T-QMB, then for $\{u_n\}_{n\in\mathbb{N}}$ in K satisfying $u_n \rightharpoonup u \in X$, we get $\limsup \Theta(u_n, u) \leq 0$. In fact, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ of $\{u_n\}_{n\in\mathbb{N}}$ for which $\limsup \Theta(u_n, u) = \lim \Theta(u_{n_k}, u)$. Since $u_{n_k} \rightharpoonup u$ and Θ is T-QMB, we obtain $\lim \Theta(u_{n_k}, u) = \liminf \Theta(u_{n_k}, u) \leq 0$. Hence, $\limsup \Theta(u_n, u) \leq 0$.

(v) The theory of pseudomonotone operators as per Brézis and its extension to bifunctions combines the monotonicity with the arguments on compactness, for example the prototype of a pseudomonotone mapping as per Brézis is the sum of a monotone hemicontinuous operator and a strongly continuous operator. The novelty of pseudomonotonicity and the (S_+) condition lies in the fact that these conditions are invariant with regard to compact perturbations, see [51, Chapter 27], [54, p. 365].

Now we note the interesting properties as follows.

Proposition 1 *Suppose $K \subset X$ is nonempty and closed and the real-valued bifunction Ψ defined on $K \times K$ satisfies $\Psi(u, u) = 0$ for every u in K. Consider ε greater than 0 and J from X to X^* to be the duality mapping. If Ψ is* B-*PMB, then the bifunction Ψ_ε from $K \times K$ to $\mathbb{R}$ defined as*

$$\Psi_\varepsilon(u, v) := \Psi(u, v) + \varepsilon\langle J(u), v - u\rangle$$

fulfills the (S_+) condition.

Proof Consider $\{u_n\}_{n\in\mathbb{N}}$ in K satisfying $u_n \rightharpoonup u$ in K and $\liminf \Psi_\varepsilon(u_n, u) \geq 0$. Next to show that $u_n \to u$ in K. Since

$$\liminf \Psi_\varepsilon(u_n, u) \geq 0,$$

it follows that

$$\liminf[\Psi(u_n, u) + \varepsilon\langle J(u_n), u - u_n\rangle] \geq 0.$$

Thus,

$$\liminf \Psi(u_n, u) + \limsup \varepsilon\langle J(u_n), u - u_n\rangle \geq 0. \tag{9}$$

Since, J is monotonicity, we obtain,

$$\langle J(u_n), u - u_n\rangle \leq \langle J(u), u - u_n\rangle.$$

Therefore,

$$\limsup\langle J(u_n), u - u_n\rangle \leq \limsup\langle J(u), u - u_n\rangle = 0.$$

Hence, from relation (9), we obtain

$$\liminf \Psi(u_n, u) \geq 0.$$

As Ψ is B-PMB, we have

$$\limsup \Psi(u_n, v) \leq \Psi(u, v), \quad \text{for each } v \text{ in } K.$$

Specifically, putting $v = u$ in the above step, we get

$$\limsup \Psi(u_n, u) \leq 0. \tag{10}$$

Now, since

$$\liminf [\Psi(u_n, u) + \varepsilon \langle J(u_n), u - u_n \rangle] \geq 0,$$

we have

$$\limsup \Psi(u_n, u) + \varepsilon \liminf \langle J(u_n), u - u_n \rangle \geq 0.$$

Considering the relation (10), we obtain

$$\liminf \langle J(u_n), u - u_n \rangle \geq 0.$$

Therefore,

$$\limsup \langle J(u_n), u_n - u \rangle \leq 0.$$

As J fulfills the (S_+) condition, we have $u_n \to u$, which proves the result. □

Proposition 2 *Suppose $\emptyset \neq K \subset X$ is closed and both the bifunctions Ψ, Θ are defined from $K \times K$ to $\mathbb{R}$. If Ψ is $\mathcal{T}$-QMB and Θ fulfills the (S_+) condition, then the same condition is satisfied by the sum of both the bifunctions, that is $\Psi + \Theta$.*

Proof Consider a sequence $\{u_n\}$ in K satisfying $u_n \rightharpoonup u$ in K and $\liminf[\Psi(u_n, u) + \Theta(u_n, u)] \geq 0$. Let $\{u_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence of $\{u_n\}$, then $u_{n_k} \rightharpoonup u$ and $\liminf[\Psi(u_{n_k}, u) + \Theta(u_{n_k}, u)] \geq 0$. Therefore,

$$\liminf \Psi(u_{n_k}, u) + \limsup \Theta(u_{n_k}, u) \geq 0. \tag{11}$$

Since $\liminf \Psi(u_{n_k}, u) \leq 0$, it can be deduced from (11) that $\limsup \Theta(u_{n_k}, u) \geq 0$. Thus, we obtain $\lim \Theta(u_{n_{k_i}}, u) \geq 0$ for a subsequence $\{u_{n_{k_i}}\}_{i\in\mathbb{N}}$ of $\{u_{n_k}\}$. The (S_+) condition of Θ, leads to the fact that $u_{n_{k_i}} \to u$. As a consequence, we can guarantee that each $\{u_{n_k}\}$ admits a subsequence that converges strongly to u. Hence $u_n \to u$, which completes the proof. □

Definition 5 Suppose $L : \mathcal{D}(L) \subset X \longrightarrow X^*$ is a maximal monotone operator which is linear, and densely defined, and $K \subset X$ is closed and convex. A bifunction Θ from $K \times K$ to $\mathbb{R}$ is *L-generalized pseudomonotone* (for short, L-GPMB) if for any sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}(L) \cap K$ satisfying $u_n \rightharpoonup u$, $Lu_n \rightharpoonup Lu$ and $\liminf \Theta(u_n, u) \geq 0$, we get

$$\limsup \Theta(u_n, v) \leq \Theta(u, v), \quad \text{for each } \ v \in K.$$

Following a similar manner, we may define *L-generalized quasimonotone* bifunction (for short, L-GQMB).

In the definition, given below, we recollect the notion of maximal monotonicity as studied in Blum and Oettli [38] for bifunctions.

Definition 6 Suppose $K \subset X$ is nonempty closed and convex and Φ is a real-valued bifunction on $K \times K$, such that $\Phi(u, u) = 0$, for all u in K. Φ is considered as maximal monotone (for short, BO-maximal monotone) if, for each u in K and for each real-valued convex function φ on K with $\varphi(u) = 0$ one has,

$$\Phi(v, u) \leq \varphi(v) \text{ for all } v \text{ in } K \Longrightarrow 0 \leq \Phi(u, v) + \varphi(v) \text{ for all } v \text{ in } K.$$

Remark 3 In an attempt to broaden the concept of maximal monotonicity of operators to bifunctions, the maximal monotonicity concept for bifunctions was introduced by Blum and Oettli [38]. A different notion for monotone bifunction of maximal type has been adopted in [39]. In order to have an idea for the two different perceptions and some other associated properties, one may see [39].

The following propositions present some characteristics of maximal monotone bifunctions.

Proposition 3 *Suppose T is an operator defined from X to X^* and Φ_T is a real-valued bifunction defined from $X \times X$ to $\mathbb{R}$ such that $\Phi_T(u, v) := \langle T(u), v - u\rangle$ for all u, v in X. We obtain the characterizations as follows*

(a) Φ_T is monotone, BO-maximal monotone, when T is hemicontinuous and monotone.
(b) If the functions φ in Definition 6, which are convex get confined to $\varphi(v) = \langle -\xi, v - u\rangle$; $\xi \in X^*$, then

Maximal monotonicity of T implies monotonicity and BO-maximal monotonicity of Φ_T

(c) For Φ_T to be BO-maximal monotone and monotone, the operator T is maximal monotone.

Proof (a) Since T is hemicontinuous as well as monotone, Φ_T is both hemicontinuous and monotone. Suppose u is in X and φ is a real-valued convex function satisfying $\varphi(u) = 0$ and $\Phi_T(v, u) \leq \varphi(v)$ for each v in X. For $t \in (0, 1]$, assign $v_t := tv + (1 - t)u$.

Now,

$$\begin{aligned} 0 = \Phi_T(v_t, v_t) &\leq t\Phi_T(v_t, v) + (1 - t)\Phi_T(v_t, u) \\ &\leq t\Phi_T(v_t, v) + (1 - t)\varphi(v_t) \\ &\leq t\Phi_T(v_t, v) + (1 - t)t\varphi(v). \end{aligned} \tag{12}$$

Thus, $\Phi_T(v_t, v) + (1 - t)\varphi(v) \geq 0$. As t tends to 0, we get $\Phi_T(u, v) + \varphi(v) \geq 0$ for each v in X. The statement (b) can be proved easily. For proving (c), monotonicity of T follows from that of Φ_T. Suppose (u^*, u) in $X^* \times X$ is such that $\langle T(v) - u^*, v - u\rangle \geq 0$. We proceed to prove that $u^* = T(u)$. As $\langle T(v) - u^*, v - u\rangle \geq 0$, we get $\langle T(v), u - v\rangle \leq \langle -u^*, v - u\rangle$. Now put $\varphi(v) = \langle -u^*, v - u\rangle$, then $\varphi(u) = 0$ and $\Phi_T(v, u) \leq \varphi(v)$ for each v in X. As Φ_T is BO-maximal monotone, $\Phi_T(u, v) +$

$\varphi(v) \geq 0$ for each v in X. Thus, $\langle T(u) - u^*, v - u \rangle \geq 0$ for each v in X, that gives $u^* = T(u)$. □

Proposition 4 *Suppose K is both closed and convex and for $\Phi : K \times K \to \mathbb{R}$ $\Phi(u, u) \geq 0$, for each u in K. Let $\Phi(u, \cdot)$ be upper hemicontinuous and convex for all $u \in K$, then Φ is* $\mathcal{BO}$*-maximal monotone.*

Proof Consider $u \in K$ and a convex function $\varphi : K \to \mathbb{R}$ be such that $\varphi(u) = 0$ and $\Phi(v, u) \leq \varphi(v)$ for each $v \in K$. Let us assign $v_t := tv + (1 - t)u \in K$ for t in $(0, 1]$. By proceeding in a similar manner as in (12), we get $\Phi(u, v) + \varphi(v) \geq 0$ for each v in K. □

We give the following theorems which are key tools for the study presented in this paper.

Theorem 2 [55] *Suppose $\emptyset \neq K \subset X$ is convex and closed, where X is a Banach space and Φ, Ψ are two real-valued bifunctions on $K \times K$ that satisfy $\Phi(u, u) = \Psi(u, u) = 0$ for each u in K. Let $J : X \longrightarrow X^*$ be the duality mapping. We assume the following*

(i) *Φ is monotone,* $\mathcal{BO}$*-maximal monotone and weakly lower semicontinuous in the second argument;*
(ii) *both Φ and Ψ are convex in the second argument;*
(iii) *Ψ is $\mathcal{B}$-PMB;*
(iv) *For each $N \in \mathcal{F}(K)$ and v in K, $u \longmapsto \Psi(u, v)$ is upper semicontinuous on* conv(N)*;*
(v) *(Coercivity) $\exists$ a nonempty weakly compact subset W, and for each $\varepsilon > 0$ (small enough), $\exists$ a convex and weakly compact subset B_ε of K such that for all u in $K \setminus W$, one obtains,*

$$\exists v \in B_\varepsilon : \Psi(u, v) + \varepsilon \langle J(u), v - u \rangle < \Phi(v, u).$$

Then, $\bar{u} \in K$ satisfies $\Phi(\bar{u}, v) + \Psi(\bar{u}, v) \geq 0$, for each v in K.

Theorem 3 [55] *Suppose $\emptyset \neq K \subset X$ is convex and closed, where X is a Banach space, $\Phi, \Psi, \Xi : K \times K \to \mathbb{R}$ are bifunctions satisfying $\Phi(u, u) = \Psi(u, u) = \Xi(u, u) = 0$ for all u in K. Consider $J : X \longrightarrow X^*$ to be the duality mapping. Let us assume the following*

(i) *Φ is monotone,* $\mathcal{BO}$*-maximal monotone and weakly lower semicontinuous in the second argument;*
(ii) *Φ, Ψ and Ξ are convex in the second argument;*
(iii) *Ψ is $\mathcal{T}$-QMB;*
(iv) *For every fixed v in K, Ψ is upper semicontinuous in the first argument;*
(v) *Ξ is $\mathcal{B}$-PMB;*
(vi) *For each N in $\mathcal{F}(K)$ and v in K, Ξ is upper semicontinuous in the first argument on* conv(N)*;*

(vii) *(coercivity)* $\exists$ *a nonempty weakly compact subset* W*, and for every* $\varepsilon > 0$ *(small enough),* $\exists$ *a convex and weakly compact subset* B_ε *of* K *such that for every* u *in* $K \setminus W$*,* $\exists$ v *in* B_ε *satisfying*

$$\Psi(u, v) + \Xi(u, v) + \varepsilon\langle J(u), v - u\rangle < \Phi(v, u).$$

Then, $\exists$ $\bar{u}$ *in* K *such that* $\Phi(\bar{u}, v) + \Psi(\bar{u}, v) + \Xi(\bar{u}, v) \geq 0$*, for all* v *in* K.

Remark 4 The coercivity condition (v) in Theorem 2 and the coercivity condition (vii) in Theorem 3 can be removed, when K is compact. If X is a reflexive Banach space with the weak topology $\sigma(X, X^*)$, these coercivity conditions are satisfied with the assumption that $\exists v_0$ in K such that $\Psi_\varepsilon(u, v_0)/\|u - v_0\| \to -\infty$, when $\|u - v_0\| \to +\infty$ uniformly in $\varepsilon > 0$, where the bifunction Ψ_ε takes respectively the following forms: $\Psi_\varepsilon(u, v) := \Psi(u, v) + \varepsilon\langle J(u), v - u\rangle$ for Theorem 2, and $\Psi_\varepsilon(u, v) := \Psi(u, v) + \Xi(u, v) + \varepsilon\langle J(u), v - u\rangle$ for Theorem 3, see [55, Remark 2.5] for details.

3 Existence Results for Nonlinear Evolution Equations

We now present some existential results for the solutions of the nonlinear evolution problem (1) by an approach involving the equilibrium problem theory. We emphasize on the anti-periodic solution existence of the evolution problem (1), where respectively $\mathcal{A}(t)$ is a time-dependent pseudomonotone and quasimonotone operators from the topological point of view. Adopting a similar type approach one may analyze problem (1) with zero-initial condition as well as the periodic problem.

The problem under consideration is as follows

$$u'(t) + \mathcal{A}(t)u(t) = f(t), \quad \text{a.e. } t \in [0, T], \quad u(0) = -u(T). \tag{13}$$

Here the framework of study is the space V, which is a real reflexive Banach space. In the given problem u' represents the generalized derivative of u on $]0, T[$, where $T > 0$ and $\mathcal{A}(t)$ is defined from V to its dual V^*, where as f is defined from $[0, T]$ to V^*. Along with the reflexivity property, another property that is imposed on V is that it is densely and continuously embedded into H, which is a separable Hilbert space.

Consequently, one may be able to observe the evolution triple, formed as follows $V \subset H \subset V^*$, see [51, p.416].

Next we proceed to reformulate the problem as a mixed equilibrium problem on a set of suitable type. In order to carry out the reformulation, we put forward the required notations and preliminaries in connection to the nonlinear evolution equations, see e.g. [51, Chapter 30], [10, 28, 52] and the references therein. Let us consider $X = L^p(0, T; V)$, $X^* = L^q(0, T; V^*)$, where $1 < p < +\infty$, and $1/p + 1/q = 1$. We denote the norms of V and H by $\|\cdot\|_V$ and $\|\cdot\|_H$, respectively. Moreover, the notion $\langle x, y\rangle$ refers the duality pairing between x in V^* and y in V, and in the case, where x, y are in H, $\langle x, y\rangle$ is the usual inner product in H, which is a Hilbert space. Now the pairing between $X = L^p(0, T; V)$ and $X^* = L^q(0, T; V^*)$ is given as $\langle\langle\cdot,\cdot\rangle\rangle$. Here J from X^* to X denotes the duality mapping, that is, for v in X^*, $J(v) = \{u \in X : \langle\langle u, v\rangle\rangle = \|u\|_X^2 = \|v\|_{X^*}^2\}$. Next utilizing the Asplund's renorming theorem [56, Theorem 1.105], let us suppose J to be both single-valued demicontinuous and monotone mapping, [13, Theorem 1.2]. We define $\mathcal{W} = \{u \in X : u' \in X^*\}$. Here u' denotes the generalized derivative. Taking $L(u) = u'$ and restricting the generalized derivative to $\mathcal{D}(L) = \{u \in X : u' \in X^* \text{ and } u(0) = -u(T)\} = \{u \in \mathcal{W} : u(0) = -u(T)\}$ we define a linear operator $L : \mathcal{D}(L) \subset X \to X^*$ as $\langle\langle L(u), v\rangle\rangle = \int_0^T \langle u'(t), v(t)\rangle \; dt$ for all v in X. It may be observed that $\mathcal{W}$ is a Banach space with norm $\|u\|_{\mathcal{W}} = \|u\|_X + \|u'\|_{X^*}$ and is real, separable and reflexive. Further the embedding $\mathcal{W} \subset C([0, T]; \mathcal{H})$ is continuous and the subset $\mathcal{D}(L)$ of $\mathcal{W}$ is closed and linear. $\mathcal{D}(L)$ is a reflexive Banach space with the graph norm $\|u\|_L = \|u\|_X + \|u'\|_{X^*}$ (see [52]). In this connection, Liu [28] established that the operator L defined from $\mathcal{D}(L) \subset X$ to X^* is a closed, maximal monotone operator which is densely defined.

Now consider the operator $\widehat{\mathcal{A}}$ associated with $\mathcal{A}$ as

$$\widehat{\mathcal{A}}(u)(t) = \mathcal{A}(t)u(t), \quad t \text{ in } [0, T],$$

It can be connected to the corresponding Nemytskij operator which is generated by the operator-valued function $t \mapsto \mathcal{A}(t)$. Hence, the problem under consideration (13) is as follows

$$u \in \mathcal{D}(L) : \; L(u) + \widehat{\mathcal{A}}(u) = f \quad \text{in } X^*. \tag{14}$$

We now assume the following in the context of the time-dependant operator $\mathcal{A}(t)$ defined from V to V^*.

[H$_1$] $\|\mathcal{A}(t)(u)\|_{V^*} \le k_0[\; \|u\|_V^{p-1} + \alpha_0(t)\;]$ for all u in V and $t \in [0, T]$ with some positive constant k_0 and $\alpha_0 \in L^q(]0, T[)$;

[H$_2$] For $t \in [0, T]$ and v in V, the mapping $u \mapsto \langle\mathcal{A}(t)(u), v - u\rangle$ is upper semicontinuous on conv(D) for every finite subset D in $\mathcal{D}(L)$;

[H$_3$] The function $t \mapsto \langle\mathcal{A}(t)(u), v\rangle$ is measurable on the closed interval $[0, T]$ for all u, vin V;

[H$_4$] $\langle\mathcal{A}(t)(u), u\rangle \ge k_1[\; \|u\|_V^p - \alpha_1(t)\;]$ for all u in V and t in $[0, T]$ with some positive constant k_1 and some function α_1 in $L^1([0, T])$.

We focus here to investigate problem (14) by the equilibrium problem given below:

$$\text{Find } \bar{u} \in \mathcal{D}(L) \text{ satisfying } \Phi(\bar{u}, v) + \Psi(\bar{u}, v) \ge 0, \quad \text{for all } v \in \mathcal{D}(L), \tag{15}$$

where Φ and Ψ are defined for u, v in $\mathcal{D}(L)$ by

$$\Phi(u, v) = \langle\langle L(u), v - u\rangle\rangle \quad \text{and} \quad \Psi(u, v) = \Psi_1(u, v) + \Psi_2(u, v)$$

with

$$\Psi_1(u, v) = \langle\langle \widehat{\mathcal{A}}(u), v - u \rangle\rangle \quad \text{and} \quad \Psi_2(u, v) = \langle\langle f, u - v \rangle\rangle.$$

The bifunction Ψ_1 is given by: $\Psi_1(u, v) = \int_0^T \psi_t(u(t), v(t))dt$ here ψ_t is the bifunction defined for $z, w \in V$ by $\psi_t(z, w) = \langle \mathcal{A}(t)(z), w - z \rangle$. According to assumption [H$_3$], the function $t \mapsto \psi_t(z, w)$ is measurable on $[0, T]$.

From Definition 3, let us recall that for a bifunction $\Theta : \mathcal{D}(L) \times \mathcal{D}(L) \to \mathbb{R}$, here $\mathcal{D}(L)$ is equipped with the graph norm $\|u\|_L = \|u\|_X + \|u'\|_{X^*}$, the concept of pseudomonotonicity in the sense of Brézis (or that Θ is B-PMB) with respect to $\mathcal{D}(L)$ is traduced as the following: If for any $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(L)$ with $u_n \rightharpoonup u$ in X, $Lu_n \rightharpoonup Lu$ in X^* and $\liminf \Theta(u_n, u) \geq 0$, then $\limsup \Theta(u_n, v) \leq \Theta(u, v)$ for every $v \in \mathcal{D}(L)$. The concept of T-QMB along with the condition (S_+) for a bifunction is defined with regard to $\mathcal{D}(L)$ in a comparatively similar manner.

Next, we first establish the existence of solutions to (15), followed by the study on the existence of solutions to (14) from the density of $\mathcal{D}(L)$ in X.

We proceed by deriving some preliminary theoretical results in this regard.

Lemma 1 *Let the assumptions [H$_1$] and [H$_4$] hold. Then for each $z, w \in V$ and $t \in [0, T]$, there exists $\theta \in L^1([0, T])$ independent from z such that $\psi_t(z, w) \leq \theta(t)$.*

Proof By [H$_1$] and [H$_4$], it follows that for each z, w in V and t in $[0, T]$

$$\begin{aligned}
\psi_t(z, w) &\leq \|\mathcal{A}(t)z\|_{V^*}\|w\|_V + k_1(\alpha_1(t) - \|z\|_V^p) \\
&\leq k_0(\|z\|_V^{p-1} + \alpha_0(t))\|w\|_V + k_1(\alpha_1(t) - \|z\|_V^p) \\
&\leq \|z\|_V^{p-1}(k_0\|w\|_V - k_1\|z\|_V) + k_0\alpha_0(t)\|w\|_V + k_1\alpha_1(t).
\end{aligned}$$

- For $k_0\|w\|_V - k_1\|z\|_V \leq 0$, we get $\psi_t(z, w) \leq k_0\alpha_0(t)\|w\|_V + k_1\alpha_1(t)$,
- For $k_0\|w\|_V - k_1\|z\|_V > 0$, we get

  $$\|z\|_V^{p-1} < (k_0/k_1)^{p-1}\|w\|_V^{p-1}$$

  and

  $$\|w\|_V\|z\|_V^{p-1} < (k_0/k_1)^{p-1}\|w\|_V^p.$$

  It follows

  $$\begin{aligned}
  \psi_t(z, w) &\leq k_0(k_0/k_1)^{p-1}\|w\|_V^p + k_0\alpha_0(t)\|w\|_V + k_1\alpha_1(t) - k_1\|z\|_V^{p-1} \\
  &\leq k_0(k_0/k_1)^{p-1}\|w\|_V^p + k_0\alpha_0(t)\|w\|_V + k_1\alpha_1(t),
  \end{aligned}$$

and hence follows the result. □

Lemma 2 *Let the assumption [H$_2$] hold. Then, $u \to \Psi_1(u, v)$ is upper semicontinuous on* conv(D) *for every $D \subset \mathcal{D}(L)$, where D is finite.*

Proof Suppose $D \subset \mathcal{D}(L)$ is finite. Now we may deduce that if for a sequence $\{u_n\}_{n\in\mathbb{N}}$ in conv(D) $u_n \rightharpoonup u \in X$, then $u_n \to u$ in conv(D) as the weak and strong convergence coincide on conv(D). This leads to the fact that, $\lim \int_0^T \|u_n(t) - u(t)\|^p dt = 0$. Thus, there is a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that $u_{n_k}(t) \to u(t)$ for a.e. $t \in [0, T]$. Now by the assumption [H$_2$] we get that for all z in X,

$$\limsup \psi_t(u_{n_k}(t), z(t)) \leq \psi_t(u(t), z(t)).$$

On the other hand, from Lemma 1, $\exists\, \theta \in L^1([0, T])$ satisfying $\psi_t(u(t), z(t)) \leq \theta(t)$. Hence, we get by Fatou's lemma

$$\begin{aligned}\limsup \int_0^T \psi_t(u_{n_k}(t), z(t))dt &\leq \int_0^T \limsup \psi_t(u_{n_k}(t), z(t))dt \\ &\leq \int_0^T \psi_t(u(t), z(t))dt.\end{aligned}$$

The inequality results by using a contradiction argument. □

Lemma 3 *Let the assumptions [H$_1$] and [H$_4$] hold. Suppose the operator $\mathcal{A}(t)$ defined from V to V^* is demicontinuous for all t in the interval* $[0, T]$*, then* $u \to \Psi_1(u, v)$ *is upper semicontinuous.*

Proof Suppose $u_n \to u \in X$, i.e., $\lim \int_0^T \|u_n(t) - u(t)\|^p dt = 0$. So we will have a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that $u_{n_k}(t) \to u(t)$ for almost all t in $[0, T]$. The demicontinuity of $\mathcal{A}(t)$ implies, $\mathcal{A}(t)u_{n_k}(t) \rightharpoonup \mathcal{A}(t)u(t)$ a.e. t in $[0, T]$. Thus, for arbitrary $v \in X$

$$\psi_t(u_{n_k}(t), v(t)) \to \psi_t(u(t), v(t)) \text{ a.e. } t \text{ in } [0, T].$$

As per Lemma 1, applying the dominated convergence theorem, we may obtain $\Psi_1(u_{n_k}, v) \to \Psi_1(u, v)$. We prove the convergence for all the sequence by proceeding through contradiction method. Thus, the bifunction Ψ_1 is continuous with respect to the first argument and hence upper semicontinuous. □

We give the following Hirano's type lemma.

Lemma 4 *Let the assumptions [H$_1$], [H$_3$] and [H$_4$] hold. If $\mathcal{A}(t)$ from V to V^* is* B*-PMO for all t in* $[0, T]$*, then Ψ is* B*-PMB with respect to $\mathcal{D}(L)$.*

Proof Consider u, v in $\mathcal{D}(L)$. Then we can write, $\Psi(u, v) = \Psi_1(u, v) + \Psi_2(u, v)$. Here $u \to \Psi_2(u, v)$ is weakly upper semicontinuous, then it becomes B-PMB. In addition, as the two B-PMB bifunctions also add up to a B-PMB bifunction (see Remark 2 (iii)), we have only to prove that Ψ_1 is B-PMB with reference to $\mathcal{D}(L)$. Now consider, $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(L)$ such that $u_n \rightharpoonup u$ in X, $L(u_n) \rightharpoonup L(u)$ in X^* and $\liminf \Psi_1(u_n, u) \geq 0$. Next, we prove that

$$\limsup \Psi_1(u_n, v) \leq \Psi_1(u, v), \quad \text{for all } v \in \mathcal{D}(L).$$

Now, in view of the evolution triple (see [51]) we may write for any n in $\mathbb{N}$, $u_n(t) = \int_0^t u_n'(s)ds$, where $u_n : [0, T] \to V$ is absolutely continuous. Also for every z in V, which is a subset of H, we obtain

$$\langle u_n(t), z\rangle = \left\langle \int_0^t u_n'(s)ds, z \right\rangle = \int_0^t \left\langle u_n'(s), z\right\rangle ds.$$

As $L(u_n) \rightharpoonup L(u)$ in X^*, that is, $u_n' \rightharpoonup u'$ in X^*, this gives

$$\begin{aligned}\lim\langle u_n(t), z\rangle = \lim \int_0^t \langle u_n'(s), z\rangle ds &= \int_0^t \langle u'(s), z\rangle ds \\ &= \left\langle \int_0^t u'(s)ds, z \right\rangle = \langle u(t), z\rangle,\end{aligned}$$

thus, $u_n(t) \rightharpoonup u(t)$ in V for each t in $[0, T]$.

Now we set $h_n(t) := \psi_t(u_n(t), u(t))$ for $t \in [0, T]$. We proceed to establish that

$$\limsup \int_0^T h_n(t)dt \leq 0. \tag{16}$$

Lemma 1 implies that, there is a non negative function θ in $L^1(]0, T[)$ for which

$$h_n(t) \leq \theta(t), \quad \text{for all } t \text{ in } [0, T]. \tag{17}$$

Next, using Fatou's lemma, we have

$$\limsup \int_0^T h_n(t)dt \leq \int_0^T \limsup h_n(t)dt. \tag{18}$$

Suppose on the contrary that $\exists\, t_0 \in [0, T]$ such that $\limsup h_n(t_0) > 0$. Then, we have $\lim h_{n_k}(t_0) > 0$ for a subsequence h_{n_k}. Using [H_1] and [H_4], we prove that $\{u_{n_k}(t_0)\}$ remains bounded in V. Hence, $\exists\, u_{n_k}$, such that $u_{n_k}(t_0) \rightharpoonup \eta$ for some η in V. Further, utilizing the concept of evolution triple (see [51]) we may have $u_n(t) = \int_0^t u_n'(s)ds$ for any n in $\mathbb{N}$, where u_n defined from $[0, T]$ to V^* is absolutely continuous. For every v in V which further is a subset of H, we have

$$\langle u_n(t), v\rangle = \left\langle \int_0^t u_n'(s)ds, v \right\rangle = \int_0^t \left\langle u_n'(s), v\right\rangle ds.$$

As $L(u_n) \rightharpoonup L(u)$, that is, $u_n' \rightharpoonup u'$ in X^*, we get

$$\lim\langle u_n(t), v\rangle = \lim \int_0^t \left\langle u_n'(s), v\right\rangle ds = \int_0^t \left\langle u'(s), v\right\rangle ds$$
$$= \left\langle \int_0^t u'(s)ds, v \right\rangle = \langle u(t), v\rangle.$$

Thus, $u_n(t) \rightharpoonup u(t)$ for every t in $[0, T]$, it follows that $\eta = u(t_0)$. As $\mathcal{A}(t_0)$ is B-PMO, we get

$$\limsup \psi_{t_0}(u_{n_k}(t_0), v) \leq \psi_{t_0}(u(t_0), v), \quad \text{for every } v \text{ in } V.$$

Specifically, for $v = u_{t_0}$, we get $\limsup h_{n_k}(t_0) \leq 0$, which results in a contradiction. Hence,

$$\limsup h_n(t) \leq 0, \quad \text{for all } t \text{ in } [0, T], \tag{19}$$

and relation (18) implies (16). Thus, from both (18) and the fact that $\liminf \Psi_1(u_n, u) \geq 0$, we have

$$\lim \Psi_1(u_n, u) = \lim \int_0^T \psi_t(u_n(t), u(t))dt = \lim \int_0^T h_n(t)dt = 0.$$

Now setting $z^+(t) = \max\{z(t), 0\}$ and $z^-(t) = z^+(t) - z(t)$, we get

$$\begin{aligned}\limsup \textstyle\int_0^T |h_n(t)|dt &= \limsup \textstyle\int_0^T h_n^+(t) + h_n^-(t)dt\\ &= \limsup \textstyle\int_0^T 2h_n^+(t) - h_n(t)dt\\ &= 2\limsup \textstyle\int_0^T h_n^+(t)dt.\end{aligned}$$

Again (17) implies that $0 \leq h_n^+(t) \leq \theta(t)$ for all t in $[0, T]$ and from (19) we have, $\lim h_n^+(t) = 0$ for all t in $[0, T]$. Thus, we have $\lim \int_0^T h_n^+(t)dt = 0$, as per the dominated convergence theorem and hence, $\limsup \int_0^T |h_n(t)|dt = 0$, that is, $h_n \to 0$ in $L^1(]0, T[)$. Hence, $\exists$ a subsequence $\{h_{n_k}\}$ and $Q \subset [0, T]$ with $meas(Q) = 0$ satisfying $h_{n_k}(t) \to 0$ for every t in $[0, T] \setminus Q$. Let $t_0 \in [0, T] \setminus Q$, then $h_{n_k}(t_0) = \psi_{t_0}(u_{n_k}(t_0), u(t_0)) \to 0$, and as before using [$H_1$] and [$H_4$] we prove that $\{u_{n_k}(t_0)\}$ becomes bounded and $u_{n_k}(t_0) \rightharpoonup u(t_0)$. Since $\mathcal{A}(t_0)$ is B-PMO, it follows that

$$\limsup \psi_{t_0}(u_{n_k}(t_0), z) \leq \psi_{t_0}(u(t_0), z), \quad \text{for all } z \in V.$$

As a consequence, for an arbitrary v in X, we have

$$\Psi_1(u, v) = \int_0^T \psi_t(u(t), v(t))dt \geq \int_0^T \limsup \psi_t(u_{n_k}(t), v(t))dt.$$

From Lemma 1, we may use Fatou's lemma to have

$$\Psi_1(u, v) \geq \limsup \int_0^T \psi_t(u_{n_k}(t), v(t))dt.$$

Hence, $\Psi_1(u, v) \geq \limsup \Psi_1(u_n, v)$ and the result follows. □

Lemma 5 *Assume [H_1], [H_3] and [H_4] hold. If $\mathcal{A}(t) : V \to V^*$ is* T*-QMO for all t in $[0, T]$, then Ψ is* T*-QMB with reference to $\mathcal{D}(L)$.*

Proof We have to show that $\Psi_1(u, v) = \int_0^T \psi_t(u(t), v(t))dt$ is T-QMB with reference to $\mathcal{D}(L)$. Now consider, $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}(L)$ satisfying $u_n \rightharpoonup u \in X$ and $L(u_n) \rightharpoonup L(u) \in X^*$. We have from the proof of Lemma 4 that $u_n(t) \rightharpoonup u(t)$ in V for all t in $[0, T]$. Further, from Lemma 1, we have, $\exists$ a non negative function $\theta \in L^1([0, T])$ for which

$$\psi_t(u_n(t), u(t)) \leq \theta(t), \quad \text{for all } t \in [0, T].$$

Thus, using Fatou's lemma, we get

$$\limsup \int_0^T \psi_t(u_n(t), u(t))dt \leq \int_0^T \limsup \psi_t(u_n(t), u(t))dt. \tag{20}$$

Suppose $\exists\, t_0 \in [0, T]$ for which

$$\limsup \psi_{t_0}(u_n(t_0), u(t_0)) > 0.$$

Thus, for a subsequence $\{u_{n_k}\}$ we obtain

$$\lim \psi_{t_0}(u_{n_k}(t_0), u(t_0)) > 0. \tag{21}$$

We obtain from [H_1] and [H_4] that $\{u_{n_k}(t_0)\}$ is bounded in V. Thus, for a further subsequence $u_{n_k}(t_0) \rightharpoonup u(t_0) \in V$. As $\mathcal{A}(t_0)$ T-QMO, we get

$$\liminf \psi_{t_0}(u_{n_k}(t_0), u(t_0)) \leq 0,$$

which contradicts (21). Thus, $\limsup \psi_t(u_n(t), u(t)) \leq 0$ for all t in $[0, T]$. Therefore, from (20) we get that

$$\limsup \int_0^T \psi_t(u_n(t), u(t))dt \leq 0.$$

Hence, $\liminf \Psi_1(u_n, v) = \liminf \int_0^T \psi_t(u_n(t), u(t))dt \leq 0$ and the proof is complete. □

By applying Theorem 2 and using Lemmas 2, 3 and 4, one has the following result.

Theorem 4 [55] *Suppose the previous assumptions [H_1]-[H_4] hold and that $\mathcal{A}(t) : V \to V^*$ is $\mathbb{B}$-PMO for each $t \in [0, T]$. Then, (15) has at least one solution.*

As a consequence, one has the following result on the solvability of the nonlinear evolution equation (13).

Theorem 5 [55] *Suppose that [H_1]-[H_4] are satisfied and the operator $\mathcal{A}(t)$ from V to V^* is $\mathbb{B}$-PMO for all t in $[0, T]$. Then for f in X^*, $\exists$ a solution u to the problem (13), satisfying $u \in C([0, T]; H) \cap X$ and $u' \in X^*$.*

Proof This may be obtained as an immediate corollary of Theorem 4, as $\mathcal{D}(L)$ is dense in X and $\mathcal{W}$ is continuously embedded in $C([0, T]; H)$ (see [52]). □

For the situation, where $\mathcal{A}(t)$ is $\mathbb{T}$-QMO for each t in $[0, T]$, the following result gives an approximated solution of the nonlinear evolution equation (13). It is obtained by applying [55, Theorem 2.2] and using Lemma 5.

Theorem 6 [55] *Let the assumptions [H_1], [H_3] and [H_4] hold and $\mathcal{A}(t) : V \to V^*$ be $\mathbb{T}$-QMO and demicontinuous for each $t \in [0, T]$. Then for $f \in X^*$ and $\varepsilon > 0$, there exists $u_\varepsilon \in C([0, T]; H)$ such that $u'_\varepsilon \in X^*$ and*

$$u'_\varepsilon(t) + \mathcal{A}(t)(u_\varepsilon)(t) + \varepsilon J(u_\varepsilon(t)) = f(t) \ \textit{for a.e.} \ \ t \in [0, T], \ \ u_\varepsilon(0) = -u_\varepsilon(T).$$

By relaxing the assumptions in the previous theorem, one has the following result.

Theorem 7 [55] *Let the assumptions [H_1], [H_3] and [H_4] hold and $\mathcal{A}(t) : V \to V^*$ be $\mathbb{T}$-QMO and weakly continuous for each $t \in [0, T]$. Then for $f \in X^*$, there exists $u \in C([0, T]; H)$ such that $u' \in X^*$ and*

$$u'(t) + \mathcal{A}(t)u(t) = f(t) \ \textit{for a.e.} \ \ t \in [0, T], \ \ u(0) = -u(T).$$

By applying Theorem 3, one has the following existence result.

Theorem 8 [55] *Let the operator $\mathcal{A}(t)$ from V to V^* be $\mathbb{B}$-PMO for each t in $[0, T]$ and satisfy the conditions [H_1]-[H_4]. Suppose $G(t)$ from V to V^* is $\mathbb{T}$-QMO, weakly continuous for each t in $[0, T]$ and satisfies [H_1], [H_3]. Furthermore, assume that the following condition holds*

$$[H_5] \qquad \langle G(t)u, u\rangle \geq -k_2\|u\|_V^p - \alpha_2(t), \quad \textit{for all } u \in V, \ t \in [0, T]$$

with some $k_2 > 0$ and $\alpha_2 \in L^1(0, T)$. Then, the evolution equation

$$\begin{cases} u'(t) + \mathcal{A}(t)u(t) + G(t)u(t) = f(t), & \textit{for a.e.} \ \ t \in [0, T], \\ u(0) = -u(T), \end{cases}$$

admit a solution $u \in \mathcal{D}(L)$ for any given $f \in X^*$.

Remark 5 Consider the two linear operators $L_i : \mathcal{D}(L_i) \subset X \to X^*,\ i = 1, 2$ defined by

$$L_1(u) = u', \quad \mathcal{D}(L_1) = \{u \in X : u' \in X^* \text{ and } u(0) = 0\},$$
$$L_2(u) = u', \quad \mathcal{D}(L_2) = \{u \in X : u' \in X^* \text{ and } u(0) = u(T)\}.$$

From [51, Proposition 32.10], L_1 and L_2 are maximal monotone operators. It follows, from a characterization of linear maximal monotone operators due to Brézis (see [51, Theorem 32.L]), that $\mathcal{D}(L_i)$ is dense in X and L_i is graph closed, $i = 1, 2$. Therefore, the approach developed in this section may be used for studying the existence of solutions of the nonlinear evolution problem (1) with zero-initial condition as well the periodic problem.

4 Results on Solvability for Nonlinear Implicit Evolution Equations

In this section, we assume V to be a real Hilbert space with V^* topological dual, where $\mathcal{B} : V \to V^*$ is a positive linear operator, which is both bounded and symmetric, $\mathcal{A}(t) : V \to V^*$ is a time-dependent operator of nonlinear type, and $f : [0, T] \to V^*$ is a functional operator. We consider the implicit Cauchy problem as follows

$$\begin{cases} \frac{d}{dt}(\mathcal{B}u(t)) + \mathcal{A}(t)(u(t)) = f(t), \text{ a.e. } t \in (0, T), \\ \mathcal{B}(u(0)) = \mathcal{B}(u_0). \end{cases} \tag{22}$$

We always assume the existence of a real Hilbert space H for which $V \subset H \subset V^*$, the embeddings being continuous and dense. Thus we get an evolution triple $V \subset H \subset V^*$ (see [51, Chapter 13]). The inner product in H and the norm in a Banach space U are denoted by the symbols $(\cdot, \cdot)$ and $\|\cdot\|_U$ respectively. The symbol $\langle \cdot, \cdot \rangle_U$ corresponds to the duality pairing between U and U^*. Let p, q, and T be constants such that $T > 0$, $p \geq 2$ and $1/p + 1/q = 1$. Let $X = L^p(0, T; V)$ and $X^* = L^q(0, T; V^*)$.

Let $\mathcal{T}$ be the canonical isomorphism from V to V^*. Now using the assumptions on $\mathcal{B}$, we may find that $(\varepsilon\mathcal{T} + \mathcal{B}) : V \to V^*$ will be an isomorphism, where $\varepsilon > 0$ is given. The symmetricity of $\mathcal{B}$ leads us to state the inner product on V^* as: $\langle u, v \rangle := \langle u, (\varepsilon\mathcal{T} + \mathcal{B})^{-1} v \rangle_V$ for all $u, v \in V^*$. Now V^* with this inner product is denoted by $W := (V^*, \langle \cdot, \cdot \rangle_W)$ where $\langle u, v \rangle_W := \langle u, v \rangle$. It is obvious that W is a Hilbert space where the norm is given by $\|\cdot\|_W$. It may be observed that the two norms on V^* are equivalent, that is,

$$\|(\varepsilon\mathcal{T} + \mathcal{B})^{-1}\|^{-1/2} \|v\|_W \leq \|v\|_{V^*} \leq \|(\varepsilon\mathcal{T} + \mathcal{B})\|^{1/2} \|v\|_W, \quad \text{for all } v \in V^*.$$

Let $Z = L^p(0, T; W)$. Now W being a Hilbert space, identifying W with its dual, we have $Z^* = L^q(0, T; W)$. For $\varepsilon > 0$, consider the auxiliary equation as follows:

$$\begin{cases} ((\varepsilon\mathcal{T} + \mathcal{B})u(t))' + \mathcal{A}(t)(u(t)) = f(t), & \text{a.e. } t \in (0, T), \\ u(0) = u_0. \end{cases} \tag{23}$$

Define $\mathcal{A}_\varepsilon(t) : W \to W^*$ as

$$\mathcal{A}_\varepsilon(t)(v) = \mathcal{A}(t)((\varepsilon\mathcal{T} + \mathcal{B})^{-1}(v) + u_0), \quad \text{for all } v \in W.$$

By considering $v(t) = (\varepsilon\mathcal{T} + \mathcal{B})(u(t)) - (\varepsilon\mathcal{T} + \mathcal{B})(u_0)$, that is $v(0) = 0$, (23) can be rewritten in the following manner,

$$\begin{cases} v(t)' + \mathcal{A}_\varepsilon(t)(v(t)) = f(t), & \text{a.e. } t \in (0, T), \\ v(0) = 0. \end{cases} \tag{24}$$

Define $L(v) = v'$ and $\mathcal{D}(L) = \{v \in Z : v' \in Z^*, \ v(0) = 0\}$, here v' denotes the generalized derivative of v. Let us define $\widehat{\mathcal{A}}_\varepsilon$ related to $\mathcal{A}_\varepsilon$ as

$$\widehat{\mathcal{A}}_\varepsilon(v)(t) = \mathcal{A}_\varepsilon(t)(v(t)), \quad t \in [0, T],$$

which may be regarded as the related Nemytskij operator generated by the operator-valued function $t \mapsto \mathcal{A}_\varepsilon(t)$. Using the operator L, we may write the auxiliary problem (24) as the following

$$\text{Find } v \in \mathcal{D}(L) \quad \text{such that} \quad L(v) + \widehat{\mathcal{A}}_\varepsilon(v) = f. \tag{25}$$

The approach developed in this section consists first to analyze the solvability of the auxiliary problem (25) using a mixed equilibrium problem (in brief, (MEP)) formulation:

$$\text{Find } \bar{u} \in K \text{ satisfying } \Phi(\bar{u}, v) + \Psi_\varepsilon(\bar{u}, v) \geq 0, \quad \text{for all } v \in K,$$

where K is a closed convex set and $\Phi, \Psi_\varepsilon : K \times K \to \mathbb{R}$ are two bifunctions. In a second stage, based on the results established for the auxiliary problem (25) we present some results on the solvability of (22).

4.1 Solvability for the Auxiliary Evolution Problem

We present some results on the solvability of the auxiliary problem (25) by using an equilibrium problem approach.

We denote the pairing between $Z = L^p(0, T; W)$ and $Z^* = L^q(0, T; W)$ by $\langle\langle \cdot, \cdot \rangle\rangle$. Let $\mathcal{W} = \{v \in Z : v' \in Z^*\}$. The generalized derivative $Lv = v'$ restricted to the subset $\mathcal{D}(L) = \{v \in Z : v' \in Z^* \text{ and } v(0) = 0\} = \{v \in \mathcal{W} : v(0) = 0\}$ denotes a linear mapping $L : \mathcal{D}(L) \subset Z \to Z^*$ given as $\langle\langle Lv, z\rangle\rangle = \int_0^T \langle v'(t), z(t)\rangle_W \, dt$ for all $v, z \in Z$. One may note here that $\mathcal{W}$ is a real reflexive Banach space which is separable with the norm $\|v\|_{\mathcal{W}} = \|v\|_Z + \|v'\|_{Z^*}$, the embedding $\mathcal{W} \subset C([0, T]; W)$ is continuous and $\mathcal{D}(L) \subset \mathcal{W}$ is a linear subspace. $\mathcal{D}(L)$ equipped with the graph norm $\|v\|_L = \|v\|_Z + \|v'\|_{Z^*}$ is a reflexive Banach space. $L : \mathcal{D}(L) \subset Z \to Z^*$ is a maximal monotone operator which is densely defined, and closed. The map $J : Z^* \to Z$ is called as the duality map, that is, for each $v \in Z^*$, $J(v) = \{z \in Z : \langle\langle z, v\rangle\rangle = \|z\|_Z^2 = \|v\|_{Z^*}^2\}$. It may be assumed that J is a single-valued mapping which is monotone and demicontinuous, [13, Theorem 1.2], using the Asplund's renorming theorem [56, Theorem 1.105].

Next, we assume the following.

- [A$_1$] $\mathcal{B} \in L(V, V^*)$, $\langle \mathcal{B}u, u\rangle_V \geq 0$ for every $u \in V$ and $\mathcal{B}$ is symmetric. Here $L(V, V^*)$ is the set of all bounded linear mappings from V to V^*;
- [A$_2$] $\|\mathcal{A}(t)u\|_{V^*} \leq k_0[\, \|u\|_V^{p-1} + \alpha_0(t)\,]$ for all $u \in V$ and $t \in [0, T]$ with some positive constant k_0 and $\alpha_0 \in L^q(]0, T[)$;
- [A$_3$] For $t \in [0, T]$ and $w \in V$, the mapping $u \mapsto \langle \mathcal{A}(t)u, w - u\rangle$ is upper semi-continuous on conv(N) for each finite subset N of V, here conv(N) denotes the convex hull of N;
- [A$_4$] The function $t \mapsto \langle \mathcal{A}(t)u, w\rangle_V$ is measurable on [0,T] for all $u, w \in V$;
- [A$_5$] $\langle \mathcal{A}(t)u, u\rangle_V \geq k_1[\, \|u\|_V^p - \alpha_1(t)\,]$ for all $u \in V$ and $t \in [0, T]$ with some constant $k_1 > 0$ and some function $\alpha_1 \in L^1([0, T])$.

Our purpose in this section is to investigate the solvability of the auxiliary problem (25) using the equilibrium problem formulation, given below:

$$\text{Find } \bar{v} \in \mathcal{D}(L) \text{ such that } \Phi(\bar{v}, v) + \Psi_\varepsilon(\bar{v}, v) \geq 0, \quad \text{for all } v \in \mathcal{D}(L), \tag{26}$$

where Φ and Ψ_ε are defined for $v, z \in \mathcal{D}(L)$ by

$$\Phi(v, z) = \langle\langle L(v), z - v\rangle\rangle \text{ and } \Psi_\varepsilon(v, z) = \Theta_\varepsilon(v, z) + \Xi(v, z),$$

with $\Theta_\varepsilon(v, z) = \langle\langle \widehat{\mathcal{A}}_\varepsilon(v), z - v\rangle\rangle$ and $\Xi(v, z) = \langle\langle f, v - z\rangle\rangle$. The bifunction Θ_ε can be written as the following: $\Theta_\varepsilon(v, z) = \int_0^T \psi_t^\varepsilon(v(t), z(t))dt$ where ψ_t^ε is the bifunction defined for $x, y \in W$ by $\psi_t^\varepsilon(x, y) = \langle \mathcal{A}_\varepsilon(t)(x), y - x\rangle_W$. Note that from [A$_4$], the function $t \mapsto \psi_t^\varepsilon(x, y)$ is measurable on $[0, T]$.

From Definition 3, the concept of pseudomonotonicity in the sense of Brézis for a bifunctions $\Theta : \mathcal{D}(L) \times \mathcal{D}(L) \to \mathbb{R}$, where $\mathcal{D}(L)$ is endowed with the graph norm $\|v\|_L = \|v\|_Z + \|v'\|_{Z^*}$, is traduced as: If for $\{v_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}(L)$ such that $v_n \rightharpoonup v \in Z$, $L(v_n) \rightharpoonup L(v)$ in Z^* and $\liminf \Theta(v_n, v) \geq 0$, we have that $\limsup \Theta(v_n, z) \leq \Psi(v, z)$ for all $z \in \mathcal{D}(L)$. Similarly, one can define the (S_+) condition and the $\mathtt{T}$-QMB notion with respect to $\mathcal{D}(L)$ for a bifunction.

One has the following existence result for the mixed equilibrium problem (26). It is obtained by application of Theorem 2, see [57, Theorem 4.1] for the details of the proof.

Theorem 9 [57] *Let [A_1]-[A_5] be satisfied and $\mathcal{A}(t) : V \to V^*$ is* B*-PMO for all* $t \in [0, T]$. *Then, for each* $\varepsilon > 0$ *the problem (26) has at least one solution.*

One has the following result for the solvability of the auxiliary evolution problem (25).

Theorem 10 [57] *If the conditions [A_1]-[A_5] are satisfied and the operator* $\mathcal{A}(t) : V \to V^*$ *is* B*-PMO for each* $t \in [0, T]$. *Then for* $f \in X^*$ *and* $\varepsilon > 0$, $\exists\ v_\varepsilon$ *of the auxiliary problem (25), such that* $v_\varepsilon \in C([0, T]; W) \cap Z$ *and* $v'_\varepsilon \in Z^*$.

Proof The result follows directly from Theorem 9, as $\mathcal{D}(L)$ is dense in Z and $\mathcal{W}$ is continuously embedded in $C([0, T]; W)$. □

Now, when $\mathcal{A}(t)$ is T-QMO for each $t \in [0, T]$, one has the approximated result on the existence of solutions of the auxiliary problem (25) as follows.

Theorem 11 [57] *Let [A_1], [A_2], [A_4] and [A_5] be satisfied. Furthermore, suppose that* $\mathcal{A}(t) : V \to V^*$ *is* T*-QMO and demicontinuous for every* $t \in [0, T]$. *Then, we have, for* $f \in X^*$, $\varepsilon > 0$, $\lambda > 0$, $\exists\ v \in C([0, T]; W)$ *such as* $v' \in Z^*$ *and*

$$v' + \widehat{\mathcal{A}}_\varepsilon(v) + \lambda J(v) = f \quad \textit{with} \quad v(0) = 0.$$

Proof The proof is obtained by using a Tikhonov regularization procedure of the mixed equilibrium problem (26) and by application of [55, Theorem 2.2]; see [57, Theorem 4.4] for the details of the proof. □

By relaxing the assumptions of Theorem 11, we deduce the existence result as mentioned below for the auxiliary problem (25) when $\mathcal{A}(t)$ is T-QMO for each $t \in [0, T]$.

Theorem 12 [57] *Assume that [A_1], [A_2], [A_4] and [A_5] hold. Moreover, suppose that* $\mathcal{A}(t) : V \to V^*$ *is weakly continuous and* T*-QMO for each* $t \in [0, T]$. *Then for* $f \in X^*$ *and* $\varepsilon > 0$, $\exists\ v \in C([0, T]; W)$ *such as* $v' \in Z^*$ *and*

$$v'(t) + \widehat{\mathcal{A}}_\varepsilon v = f \quad \textit{with} \quad v(0) = 0.$$

The next theorem gives the solvability condition for the auxiliary evolution problem (25) when $\mathcal{A}(t) = \mathcal{M}(t) + \mathcal{N}(t)$ with $\mathcal{M}(t) : V \to V^*$ is B-PMO and $\mathcal{N}(t) : V \to V^*$ is T-QMO. The definitions of the operators $\widehat{\mathcal{M}}_\varepsilon$ and $\widehat{\mathcal{N}}_\varepsilon$ follow the same way as $\widehat{\mathcal{A}}_\varepsilon$.

Theorem 13 [57] *Assume that* $\mathcal{M}(t) : V \to V^*$ *is* B*-PMO for all* $t \in [0, T]$ *and fulfils the conditions [A_2]-[A_5], and that* $\mathcal{N}(t) : V \to V^*$ *is* T*-QMO and satisfies weak continuity for each* $t \in [0, T]$, *satisfies the conditions [A_2], [A_4] and the condition*

$$[A_6] \qquad \langle \mathcal{N}(t)(u), u\rangle \geq -k_2\|u\|_V^p - \alpha_2(t), \quad \text{for all } u \in V,\ t \in [0, T]$$

with some $k_2 > 0$ and $\alpha_2 \in L^1(0, T)$. Then, the auxiliary problem

$$v' + \widehat{\mathcal{M}}_\varepsilon(v) + \widehat{\mathcal{N}}_\varepsilon(v) = f \tag{27}$$

has a solution $v \in \mathcal{D}(L)$ for any $f \in X^*$.

Proof The result is obtained by applying Theorem 3 with
$\Phi(v, z) = \langle\langle L(v), z - v\rangle\rangle$, $\Psi(v, z) = \langle\langle \widehat{\mathcal{N}}_\varepsilon v, z - v\rangle\rangle$ and $\Xi(v, z) = \Xi_1(v, z) + \Xi_2(v, z)$,
where $\Xi_1(v, z) = \langle\langle \widehat{\mathcal{M}}_\varepsilon v, z - v\rangle\rangle$ and $\Xi_2(v, z) = \langle\langle f, v - z\rangle\rangle$. □

4.2 Solvability Criteria for Implicit Nonlinear Evolution Equations

This subsection presents some results on the solvability for the nonlinear implicit evolution equation (22) using the existence results derived for the auxiliary problem (25).

Theorem 14 [57] *Given f in X^* and u_0 in V, assume that [A_1]-[A_5] hold and $\mathcal{A}(t) : V \to V^*$ is $\mathcal{B}$-PMO for each $t \in [0, T]$. Then there exists at least one solution $u \in X$ to (22), such that $\mathcal{B}(u) \in L^p(0, T; V^*)$, $(\mathcal{B}(u))' \in L^q(0, T; V^*)$.*

Proof By Theorem 10, we deduce that there exists a solution v_ε in $\mathcal{D}(L)$ for the auxiliary problem (25) for any $\varepsilon > 0$. This shows that $\exists\, u_\varepsilon \in X$ with $u'_\varepsilon \in X^*$ and

$$\begin{cases} ((\varepsilon\mathcal{T} + \mathcal{B})u_\varepsilon(t))' + \mathcal{A}(t)(u_\varepsilon(t)) = f(t), \text{ a.e. } t \in (0, T), \\ u_\varepsilon(0) = u_0. \end{cases} \tag{28}$$

We write Eq. (28) as

$$\varepsilon\mathcal{T}(u'_\varepsilon(t)) + \mathcal{B}(u'_\varepsilon(t)) + \mathcal{A}(t)(u_\varepsilon(t)) = f(t), \text{ a.e. } t \in (0, T). \tag{29}$$

Multiplying (29) by u_ε we get

$$\begin{aligned} &\tfrac{\varepsilon}{2}\tfrac{d}{dt}\langle \mathcal{T}(u_\varepsilon(t)), u_\varepsilon(t)\rangle_V + \tfrac{1}{2}\tfrac{d}{dt}\langle \mathcal{B}(u_\varepsilon(t)), u_\varepsilon(t)\rangle_V \\ &+\langle \mathcal{A}(t)(u_\varepsilon(t)), u_\varepsilon(t)\rangle_V = \langle f(t), u_\varepsilon(t)\rangle_V \text{ a.e. } t \in (0, T). \end{aligned} \tag{30}$$

Integrating the inequality (30) on $(0, T)$, we obtain from [A_5] and Hölder inequality

$$\frac{\varepsilon}{2}\|u_\varepsilon(T)\|_V^2 - \frac{\varepsilon}{2}\|u_0\|_V^2 + \frac{1}{2}\langle \mathcal{B}(u_\varepsilon(T)), u_\varepsilon(T)\rangle_V - \frac{1}{2}\langle \mathcal{B}(u_0), u_0\rangle_V + k_1\int_0^T \|u_\varepsilon(t)\|_V^p dt$$
$$\leq \left(\int_0^T \|f(t)\|_{V^*}^q dt\right)^{1/q}\left(\int_0^T \|u_\varepsilon(t)\|_V^p dt\right)^{1/p} + k_1\|\alpha_1\|_{L^1(0,T)}.$$

By Young's inequality, we obtain

$$\|u_\varepsilon\|_X^p \leq C, \tag{31}$$

where C is a constant depending on $\|f\|_{X^*}$, $\|u_0\|_V$.

Define $\widehat{\mathcal{A}} : X \to X^*$ related to $\mathcal{A}$ by

$$\widehat{\mathcal{A}}(u)(t) = \mathcal{A}(t)(u(t)), \quad t \in [0, T].$$

By the assumption [A_2] and (31) we have $\{u_\varepsilon\}$ is a bounded sequence in X and $\{\widehat{\mathcal{A}}(u_\varepsilon)\}$ is a bounded sequence in X^*. Thus, for $\{u_\varepsilon\}$, we obtain

$$\begin{aligned} u_\varepsilon &\rightharpoonup u \quad \text{in } X \\ \widehat{\mathcal{A}}(u_\varepsilon) &\rightharpoonup \theta \quad \text{in } X^* \\ \mathcal{B}(u_\varepsilon) &\rightharpoonup \mathcal{B}(u) \quad \text{in } X^* \\ ((\varepsilon\mathcal{T} + \mathcal{B})u_\varepsilon)' &\rightharpoonup (\mathcal{B}(u))' \quad \text{in } X^*. \end{aligned} \tag{32}$$

In view of (32), to complete the proof on existence, we need only to derive that $\theta = \widehat{\mathcal{A}}(u)$. Hence, we proceed by scalar multiplying relation (28) by $u - u_\varepsilon$ and integrating on $(0, T)$, we have

$$\begin{aligned}\langle\langle \widehat{\mathcal{A}}(u_\varepsilon), u - u_\varepsilon\rangle\rangle_X = \langle\langle f, u - u_\varepsilon\rangle\rangle_X + \langle\langle [(\varepsilon\mathcal{T} + \mathcal{B})(u_\varepsilon - u)]', u_\varepsilon - u\rangle\rangle_X + \\ \langle\langle [(\varepsilon\mathcal{T} + \mathcal{B})u]', u_\varepsilon - u\rangle\rangle_X.\end{aligned} \tag{33}$$

Now, let us consider the bifunction $\Psi_1(u, v) := \langle\langle \widehat{\mathcal{A}}(u), v - u\rangle\rangle_X$. As $\mathcal{A}(t)$ is B-PMO, it may be shown by repeating the similar process utilized in the proof of Lemma 4 that Ψ_1 is B-PMO. Using (33) and the fact that $u_\varepsilon \rightharpoonup u$ in X (relation (32)), we obtain

$$\begin{aligned}\liminf \Psi_1(u_\varepsilon, u) &\geq \liminf[\tfrac{\varepsilon}{2}\|u_\varepsilon(T) - u(T)\|_V^2 + \\ &\quad \tfrac{1}{2}\langle \mathcal{B}(u_\varepsilon(T) - u(T)), u_\varepsilon(T) - u(T)\rangle_V] \\ &\geq 0.\end{aligned} \tag{34}$$

Since Ψ_1 is B-PMO, it follows that

$$\limsup \Psi_1(u_\varepsilon, v) \leq \Psi_1(u, v), \quad \text{for all } v \in X. \tag{35}$$

By using relations (34) and (35), we easily get

$$\langle\langle \widehat{\mathcal{A}}(u), v - u\rangle\rangle_X \geq \langle\langle \theta, v - u\rangle\rangle_X, \quad \text{for all } v \in X,$$

Hence, $\theta = \widehat{\mathcal{A}}(u)$. □

In the case where the nonlinear implicit evolution equation (22) is driven by an operator $\mathcal{A}(t)$ which is $\mathtt{T}$-QMO for all $t \in [0, T]$, one has the following existence result.

Theorem 15 [57] *Let $f \in X^*$ and $u_0 \in V$ be given. Let the assumptions [A_1], [A_2], [A_4] and [A_5] hold. Furthermore, suppose that $\mathcal{A}(t) : V \to V^*$ is $\mathtt{T}$-QMO and weakly continuous for each $t \in [0, T]$. Then (22) has at least one solution $u \in X$ satisfying $\mathcal{B}(u) \in L^p(0, T; V^*)$, $(\mathcal{B}u)' \in L^q(0, T; V^*)$.*

Proof By Theorem 12, it is obvious that for any $\varepsilon > 0$ $\exists$ $v_\varepsilon \in \mathcal{D}(L)$ solution of the auxiliary problem (25). It follows that $\exists$ $u_\varepsilon \in X$ with $u'_\varepsilon \in X^*$ and

$$\begin{cases} ((\varepsilon\mathcal{T} + \mathcal{B})u_\varepsilon(t))' + \mathcal{A}(t)(u_\varepsilon(t)) = f(t), \ \text{ a.e. } \ t \in (0, T), \\ u_\varepsilon(0) = u_0. \end{cases} \tag{36}$$

Repeating the similar process as in the above theorem we get $\{u_\varepsilon\}$ and $\{\widehat{\mathcal{A}}(u_\varepsilon)\}$ as bounded in X and X^* respectively. Thus, for $\{u_\varepsilon\}$, it follows

$$\begin{aligned} u_\varepsilon &\rightharpoonup u \quad \text{in } X \\ \widehat{\mathcal{A}}(u_\varepsilon) &\rightharpoonup \theta \quad \text{in } X^* \\ \mathcal{B}(u_\varepsilon) &\rightharpoonup \mathcal{B}(u) \quad \text{in } X^* \\ ((\varepsilon\mathcal{T} + \mathcal{B})u_\varepsilon)' &\rightharpoonup (\mathcal{B}u)' \quad \text{in } X^*. \end{aligned} \tag{37}$$

To complete the proof, it is now to be proved that $\theta = \widehat{\mathcal{A}}(u)$. Now considering [$H_2$] and utilizing the dominated convergence theorem, we have, $\widehat{\mathcal{A}}$ satisfies weak continuity. Thus, $u_\varepsilon \rightharpoonup u$ implies that $\theta = \widehat{\mathcal{A}}(u)$. □

We conclude with the existence result as follows, for the implicit nonlinear evolution equation (22) when $\mathcal{A}(t) = \mathcal{M}(t) + \mathcal{N}(t)$, where both $\mathcal{M}(t)$ and $\mathcal{N}(t)$ defined from V to V^* are $\mathtt{B}$-PMO and $\mathtt{T}$-QMO respectively.

Theorem 16 [57] *Let $f \in X^*$, $u_0 \in V$ be given and the condition [A_1] hold. Assume that $\mathcal{M}(t) : V \to V^*$ is $\mathtt{B}$-PMO for all $t \in [0, T]$ and satisfies the conditions [A_2]-[A_5], and that $\mathcal{N}(t) : V \to V^*$ is $\mathtt{T}$-QMO, weakly continuous for all $t \in [0, T]$ and satisfies the conditions [A_2], [A_4]. Moreover, assume that*

$$[A_6] \qquad \langle \mathcal{N}(t)(u), u \rangle \geq -k_2 \|u\|_V^p - \alpha_2(t), \quad \textit{for all } u \in V, \ t \in [0, T]$$

where $k_2 > 0$ and $\alpha_2 \in L^1(0, T)$. Then there exists at least $u \in X$ such that $\mathcal{B}(u) \in L^p(0, T; V^*)$, $(\mathcal{B}u)' \in L^q(0, T; V^*)$ and

$$\begin{cases} \frac{d}{dt}(\mathcal{B}(u(t))) + \mathcal{M}(t)(u(t)) + \mathcal{N}(t)(u(t)) = f(t), \ \textit{a.e.} \ t \in (0, T), \\ \mathcal{B}(u(0)) = \mathcal{B}(u_0). \end{cases}$$

Proof Using Theorem 13, we obtain that for any $\varepsilon > 0$ $\exists$ $v_\varepsilon \in \mathcal{D}(L)$ satisfying $v'_\varepsilon + \widehat{\mathcal{M}}_\varepsilon(v_\varepsilon) + \widehat{\mathcal{N}}_\varepsilon(v_\varepsilon) = f$. This leads to the fact that there exists u_ε in X with $u'_\varepsilon \in X^*$ and

$$\begin{cases} ((\varepsilon\mathcal{T} + \mathcal{B})u_\varepsilon(t))' + \mathcal{M}(t)(u_\varepsilon(t)) + \mathcal{N}(t)(u_\varepsilon(t)) = f(t), \text{ a.e. } t \in (0, T), \\ u_\varepsilon(0) = u_0. \end{cases} \tag{38}$$

Following the same way as Theorem 14, it can be proved that $\{u_\varepsilon\}$ is bounded in X, and $\{\widehat{\mathcal{M}}(u_\varepsilon)\}$ and $\{\widehat{\mathcal{N}}(u_\varepsilon)\}$ are bounded in X^*. Thus, for a subsequence, $\{u_\varepsilon\}$, we obtain

$$\begin{aligned} u_\varepsilon &\rightharpoonup u \quad \text{in } X \\ \widehat{\mathcal{M}}(u_\varepsilon) &\rightharpoonup \theta \quad \text{in } X^* \\ \widehat{\mathcal{N}}(u_\varepsilon) &\rightharpoonup \tau \quad \text{in } X^* \\ \mathcal{B}(u_\varepsilon) &\rightharpoonup \mathcal{B}(u) \quad \text{in } X^* \\ ((\varepsilon\mathcal{T} + \mathcal{B})(u_\varepsilon))' &\rightharpoonup (\mathcal{B}(u))' \quad \text{in } X^*. \end{aligned} \tag{39}$$

Weak continuity of $\widehat{\mathcal{N}}$ implies that, $\tau = \widehat{\mathcal{N}}(u)$. Next we have to show that $\theta = \widehat{\mathcal{M}}(u)$. For this purpose, we proceed while multiplying relation (38) and integrating on (0,T). Thus we have

$$\begin{aligned} \langle\langle \widehat{\mathcal{M}}(u_\varepsilon), u - u_\varepsilon\rangle\rangle_X + \langle\langle \widehat{\mathcal{N}}(u_\varepsilon), u - u_\varepsilon\rangle\rangle_X = \langle\langle f, u - u_\varepsilon\rangle\rangle_X + \\ \langle\langle [(\varepsilon\mathcal{T} + \mathcal{B})(u_\varepsilon - u)]', u_\varepsilon - u\rangle\rangle_X + \langle\langle [(\varepsilon\mathcal{T} + \mathcal{B})(u)]', u_\varepsilon - u\rangle\rangle_X. \end{aligned} \tag{40}$$

Now consider $\Phi_1(u, v) = \langle\langle \widehat{\mathcal{M}}(u_\varepsilon), u - u_\varepsilon\rangle\rangle_X$ and $\Phi_2(u, v) = \langle\langle \widehat{\mathcal{M}}(u_\varepsilon), u - u_\varepsilon\rangle\rangle_X$ on $X \times X$. Note that both Φ_1 and Φ_2 are B-PMO and T-QMO respectively. Now the relation (40) implies,

$$\begin{aligned} \lim\inf[\Phi_1(u_\varepsilon, u) + \Phi_2(u_\varepsilon, u)] &\geq \lim\inf[\tfrac{\varepsilon}{2}\|u_\varepsilon(T) - u(T)\|_V^2 + \\ &\quad \tfrac{1}{2}\langle \mathcal{B}(u_\varepsilon(T) - u(T)), u_\varepsilon(T) - u(T)\rangle_V] \\ &\geq 0. \end{aligned} \tag{41}$$

Hence,

$$\lim\inf \Phi_1(u_\varepsilon, u) + \lim\sup \Phi_2(u_\varepsilon, u) \geq 0. \tag{42}$$

As Φ_2 is T-QMO, it is deduced from Remark 2(iv) that $\lim\sup \Phi_2(u_\varepsilon, u) \leq 0$. Therefore, from (42), we conclude that $\lim\inf \Phi_1(u_\varepsilon, u) \geq 0$. Since Φ_1 is B-PMO, we obtain that

$$\lim\sup \Phi_1(u_\varepsilon, v) \leq \Phi_1(u, v), \quad \text{for all } v \in X,$$

which leads us to obtain that $\theta = \widehat{\mathcal{M}}(u)$. □

Acknowledgements The second and third authors are grateful to the Mohapatra Family Foundation for its support during this research.

References

1. Fan, K.: A minimax inequality and its application. In: Shisha, O. (ed.) Inequalities, vol. III, pp. 103–113. Academic, New York (1972)
2. Browder, F.E.: Non-linear equations of evolution. Ann. Math. **80**, 485–523 (1964)
3. Pavel, N.H.: Invariant sets for a class of semi linear equations of evolutions. Nonlinear Anal. **1**, 187–196 (1977)
4. Pavel, N.H., Vrabie, I.: Equations d'evolution multivoques dans des espaces de Banach. C. r. hebd. Séanc. Acad. Sci. Paris **287**, 315–317 (1978)
5. Pazy, A.: A class of semi-linear equations of evolution. Israel J. Math. **20**, 23–36 (1978)
6. Attouch, A., Damlamian, A.: On multivalued evolution equations in Hilbert spaces. Israel J. Math. **12**, 373–390 (1972)
7. Crandall, M.G., Nohel, J.A.: An abstract functional differential equation and a related Volterra equation. Israel J. Math. **29**, 313–328 (1979)
8. Gutman, S.: Compact perturbations of m-accretive operators in general Banach spaces. SIAM J. Math. Anal. **13**, 789–800 (1982)
9. Hirano, N.: Local existence theorems for nonlinear differential equations. SIAM J. Math. Anal. **14**, 117–125 (1983)
10. Hirano, N.: Nonlinear evolution equations with nonmonotone perturbations. Nonlinear Anal. **13**, 599–609 (1989)
11. Vrabie, I.: The nonlinear version of Pazy's local existence theorem. Israel J. Math. **32**, 221–235 (1979)
12. Vrabie, I.: An existence result for a class of nonlinear evolution equations in Banach spaces. Nonlinear Anal. **6**, 711–722 (1982)
13. Barbu, V.: Nonlinear Semigroup and Differential Equations in Banach Spaces. Noordhoff, Leyden (1976)
14. Ahmed, N.U., Xiang, X.: Existence of solution for a class of nonlinear equations with non-monotone perturbations. Nonlinear Anal. **22**, 81–89 (1994)
15. Liu, Z.: Nonlinear evolution variational inequalities with nonmonotone perturbations. Nonlinear Anal. **29**, 1231–1236 (1997)
16. Liu, Z.: Existence for implicit differential equations with nonmonotone perturbations. Israel J. Math. **129**, 363–372 (2002)
17. Batchelor, M.T., Baxter, R.J., O'Rourke, M.J., Yung, C.M.: Exact solution and interfacial tension of the six-vertex model with anti-periodic boundary conditions. J. Phys. A **28**, 2759–2770 (1995)
18. Bonilla, L.L., Higuera, F.J.: The onset and end of the Gunn effect in extrinsic semiconductors. SIAM J. Appl. Math. **55**, 1625–1649 (1995)
19. Kulshreshtha, D.S., Liang, J.Q., Muller-Kirsten, H.J.W.: Fluctuation equations about classical field configurations and supersymmetric quantum mechanics. Ann. Phys. **225**, 191–211 (1993)
20. Okochi, H.: On the existence of periodic solutions to nonlinear abstract parabolic equations. J. Math. Soc. Japan **40**, 541–553 (1988)
21. Okochi, H.: On the existence of anti-periodic solutions to a nonlinear evolution equation associated with odd subdifferential operators. J. Funct. Anal. **91**, 246–258 (1990)
22. Okochi, H.: On the existence of anti-periodic solutions to nonlinear parabolic equations in noncylindrical domains. Nonlinear Anal. **14**, 771–783 (1990)
23. Haraux, A.: Anti-periodic solutions of some nonlinear evolution equations. Manuscripta Math. **63**, 479–505 (1989)
24. Aizicovici, S., Pavel, N.H.: Anti-periodic solutions to a class of nonlinear differential equations in Hilbert space. J. Funct. Anal. **99**, 387–408 (1991)
25. Aizicovici, S., McKibben, M., Reich, S.: Anti-periodic solutions to nonmonotone evolution equations with discontinuous nonlinearities. Nonlinear Anal. **43**, 233–251 (2001)
26. Chen, Y.Q.: Anti-periodic solutions for semilinear evolution equations. J. Math. Anal. Appl. **315**, 337–348 (2006)

27. Chen, Y.Q., Nieto, J.J., O'Regan, D.: Anti-periodic solutions for full nonlinear first-order differential equations. Math. Comput. Model. **46**, 1183–1190 (2007)
28. Liu, Z.H.: Anti-periodic solutions to nonlinear evolution equations. J. Funct. Anal. **258**, 2026–2033 (2010)
29. Andrews, K., Kuttler, K., Schillor, M.: Second order evolution equations with dynamic boundary conditions. J. Math. Anal. Appl. **197**, 781–795 (1996)
30. Barbu, V., Favini, A.: Existence for an implicit differential equation. Nonlinear Anal. **32**, 33–40 (1998)
31. Favini, A., Yagi, A.: Multivalued linear operators and degenerate evolution equations. Annali Mat. Pura Appl. **163**, 353–384 (1993)
32. Liu, Z.: Existence for implicit differential equations with nonmonotone perturbations. Israel J. Math. **129**, 363–372 (2002)
33. Showalter, R.: Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations. Mathematical Surveys and Monographs, vol. 49. American Mathematical Society, Providence, RI (1997)
34. Liu, J., Liu, Z.: On the existence of anti-periodic solutions for implicit differential equations. Acta Math. Hung. **132**, 294–305 (2011)
35. Papageorgiou, N.S., Rădulescu, V.D., Repovš, D.D.: Periodic solutions for implicit evolution inclusions. Evol. Equ. Control Theory **8**, 621–631 (2019)
36. Barbu, V., Favini, A.: Existence for implicit differential equations in Banach spaces. Atti Accad. Naz. Lincei Cl. Sci. Fiz. Mat. Natur. Rend. Mat. Appl. **3**, 203–215 (1992)
37. DiBenedetto, E., Showalter, R.: A pseudo-parabolic variational inequality and Stefan problem. Nonlinear Anal. **6**, 279–291 (1982)
38. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Student **63**, 123–145 (1994)
39. Hadjisavvas, N., Khatibzadeh, H.: Maximal monotonicity of bifunctions. Optimization **59**, 147–160 (2010)
40. Bigi, G., Castellani, M., Pappalardo, M., Passacantando, M.: Existence and solution methods for equilibria. European J. Oper. Res. **227**, 1–11 (2013)
41. Mosco, U.: Implicit variational problems and quasi-variational inequalities. In: Gossez, J.P., Lami Dozo, E.J., Mawhin J., et al. (eds.) Nonlinear Operators and the Calculus of Variations, Proceedings of Summer School (Bruxelles 1975). Lecture Notes in Mathematics, vol. 543, pp. 83–156. Springer, Berlin (1976)
42. Joly, J.L., Mosco, U.: A propos de l'existence et de la régularité des solutions de certaines inéquations quasivariationnelles. J. Funct. Anal. **34**, 107–137 (1979)
43. Gwinner, J.: Nichtlineare Variationsungleichungen mit Anwendungen. PhD Thesis, Universität Mannheim, Mannheim, Germany (1978)
44. Gwinner, J.: A note on pseudomonotone functions, regularization, and relaxed coerciveness. Nonlinear Anal. **30**, 4217–4227 (1997)
45. Brézis, H.: Equations et inéquations non linéaires dans les espaces vectoriels en dualité. Ann. Inst. Fourier **18**, 115–175 (1968)
46. Kittilä, A.: On the topological degree for a class of mappings of monotone type and applications to strongly nonlinear elliptic problems. Ann. Acad. Sci. Fenn. Ser. A **91**, 1–47 (1994)
47. Bian, W.: Operator Inclusions and Operator-Differential Inclusions. PhD Thesis, Department of Mathematics, University of Glasgow (1998)
48. Karamardian, S.: Complementarity problems over cones with monotone and pseudomonotone Maps. J. Optim. Theory Appl. **18**, 445–454 (1976)
49. Bianchi, M., Schaible, S.: Generalized monotone bifunctions and equilibrium problems. J. Optim. Theory Appl. **90**, 31–43 (1996)
50. Hadjisavvas, N., Schaible, S.: Quasimonotone variational inequalities in Banach spaces. J. Optim. Theory Appl. **90**, 55–111 (1996)
51. Zeidler, E.: Nonlinear Functional Analysis and Its Applications (II A and II B). Springer, New York, Boston, Berlin (1990)

52. Carl, S., Le, V.K., Motreanu, D.: Nonsmooth Variational Problems and Their Inequalities: Comparison Principles and Applications. Springer Monographs in Mathematics, Springer, New York (2007)
53. Chadli, O., Schaible, S., Yao, J.C.: Regularized equilibrium problems with application to noncoercive hemivariational inequalities. J. Optim. Theory Appl. **121**, 571–596 (2004)
54. Hu, S., Papageorgiou, N.: Handbook of Multivalued Analysis, vol. 1. Kluwer Academic Publishers, Dordrecht (1997)
55. Chadli, O., Ansari, Q.H., Yao, J.C.: Mixed equilibrium problems and anti-periodic solutions for nonlinear evolution equations. J. Optim. Theory Appl. **168**, 410–440 (2016)
56. Barbu, V., Precupanu, Th.: Convexity and Optimization in Banach Spaces. Editura Academiei R.S.R, Bucharest (1975)
57. Chadli, O., Ansari, Q.H., Al-Homidan, S.: Existence of solutions for nonlinear implicit differential equations: an equilibrium problem approach. Numer. Funct. Anal. Optim. **37**, 1385–1419 (2016)

Sufficient Conditions Concerning the Unified Class of Starlike and Convex Functions

Lateef Ahmad Wani and A. Swaminathan

Abstract Let $\mathcal{A}_n$ be the family of analytic functions $f(\xi) = \xi + \sum_{j=n+1}^{\infty} a_j \xi^j$, defined in the open unit disk $\mathbb{D}$. We use differential subordinations to establish sufficient conditions involving third-order differential inequalities for $f \in \mathcal{A}_n$ to be in the unified class of starlike and convex functions

$$\mathcal{S}^*C_n(\alpha, \beta) := \left\{ f \in \mathcal{A}_n : \Re\left(\frac{\xi f'(\xi) + \beta\xi^2 f''(\xi)}{\beta\xi f'(\xi) + (1-\beta) f(\xi)} \right) > \alpha \right\},$$

where $\alpha \in [0, 1)$ and $\beta \in [0, 1]$. As applications, we construct certain members of $\mathcal{S}^*C_n(\alpha, \beta)$ involving triple-integrals and also derive conditions for the Pascu class of functions. Apart from obtaining new results, some of the already known results concerning starlikeness of $f \in \mathcal{A}_n$ are obtained as special cases.

Keywords Starlikeness · Convexity · Differential subordination · Pascu class

1 Introduction

Let $\mathcal{H}$ denotes the set of all analytic functions defined in $\mathbb{D} := \{\xi : |\xi| < 1\}$. Let $\varsigma \in \mathbb{C}$ and $n \in \mathbb{N}$. Define

$$\mathcal{H}_n(\varsigma) := \left\{ f \in \mathcal{H} : f(\xi) = \varsigma + \sum_{j=n}^{\infty} a_j \xi^j,\ a_j \in \mathbb{C} \right\}$$

and

$$\mathcal{A}_n := \left\{ f \in \mathcal{H} : f(\xi) = \xi + \sum_{j=n+1}^{\infty} a_j \xi^j,\ a_j \in \mathbb{C} \right\}.$$

L. A. Wani (✉) · A. Swaminathan
Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee 247667, Uttarakhand, India
e-mail: lateef17304@gmail.com

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_3

Particularly, for $n = 1$ we write $\mathcal{A} := \mathcal{A}_1$. As usual, let $\mathcal{S}_n^*(\alpha)$ and $\mathcal{C}_n(\alpha)$ denote, respectively, the classes of starlike and convex functions of order $\alpha \in [0, 1)$, in $\mathbb{D}$. Analytically,

$$\mathcal{S}_n^*(\alpha) := \left\{ f \in \mathcal{A}_n : \Re\left(\frac{\xi f'(\xi)}{f(\xi)}\right) > \alpha \right\}$$

and

$$\mathcal{C}_n(\alpha) := \left\{ f \in \mathcal{A}_n : \Re\left(1 + \frac{\xi f''(\xi)}{f'(\xi)}\right) > \alpha \right\}.$$

Moreover, $\mathcal{S}^* := \mathcal{S}_1^*(0)$ and $\mathcal{C} := \mathcal{C}_1(0)$ are, respectively, the well-known classes of starlike and convex functions. For more details, one could refer [4].

For $0 \leq \alpha < 1$ and $0 \leq \beta \leq 1$, define the class of functions $\mathcal{S}^*\mathcal{C}_n(\alpha, \beta)$ as

$$\mathcal{S}^*\mathcal{C}_n(\alpha, \beta) := \left\{ f \in \mathcal{A}_n : \Re\left(\frac{\xi f'(\xi) + \beta\xi^2 f''(\xi)}{\beta\xi f'(\xi) + (1-\beta) f(\xi)}\right) > \alpha \right\}.$$

Equivalently, $\mathcal{S}^*\mathcal{C}_n(\alpha, \beta)$ can also be defined as

$$\mathcal{S}^*\mathcal{C}_n(\alpha, \beta) = \left\{ f \in \mathcal{A}_n : \Re\left(\frac{\xi F_\beta'(\xi)}{F_\beta(\xi)}\right) > \alpha \right\},$$

where

$$F_\beta(\xi) = F_\beta[f](\xi) = \beta\xi f'(\xi) + (1-\beta) f(\xi) = \xi + \sum_{j=n+1}^{\infty} \Big(1 + (j-1)\beta\Big) a_j \xi^j.$$

By way of explanation, $\mathcal{S}^*\mathcal{C}_n(\alpha, \beta)$ is the totality of $f \in \mathcal{A}_n$ for which the operator $F_\beta(\xi)$ is starlike of order α. It is clear that $F_\beta(\xi)$ converges in $\mathbb{D}$ as the convex combination of functions analytic in $\mathbb{D}$. For $n = 1$, this class was considered by Altıntaş [1]. Since $\mathcal{S}^*\mathcal{C}_n(\alpha, 0) = \mathcal{S}_n^*(\alpha)$ and $\mathcal{S}^*\mathcal{C}_n(\alpha, 1) = \mathcal{C}_n(\alpha)$, the class $\mathcal{S}^*\mathcal{C}_n(\alpha, \beta)$ is a unification of $\mathcal{S}_n^*(\alpha)$ and $\mathcal{C}_n(\alpha)$. To be explicit, as β varies from 0 to 1, $\mathcal{S}^*\mathcal{C}_n(\alpha, \beta)$ provides a transition from the starlike class $\mathcal{S}_n^*(\alpha)$ to the convex class $\mathcal{C}_n(\alpha)$.

In univalent function theory, one of the important research areas is to establish conditions that sufficiently ensure the starlikeness (or convexity) of an analytic function. These include the conditions in terms of the coefficients a_n $(n \in \mathbb{N})$, for example, see [10, 14, 19], and the conditions in terms of differential inequalities, see, [3, 5, 6, 8, 9, 11, 17, 18]. As, in this paper, we are dealing with the later one, it is imperative to make mention of the historical background and some recent developments in this direction. In 1992, Mocanu [9] considered the problem: For $\xi \in \mathbb{D}$, find

$$\sup\left\{\rho : \left\{f \in \mathcal{A} \text{ s.t. } |f''(\xi)| \leq \rho\right\} \subset \mathcal{S}^*\right\}.$$

The author [9] proved that $\rho = 2/3$ is sufficient to ensure the starlikeness of f. Later, some more authors worked in this direction to improve the result, and finally in 1997, Obradović [11] settled down this problem completely by proving that the result is sharp for $\rho = 1$. Fournier and Mocanu [5] also determined some sufficient conditions for starlikeness, and some of their results were extended by Miller and Mocanu [8] by replacing $\mathcal{A}$ with $\mathcal{A}_n$. Kuroki and Owa [6] and Verma et al. [17] obtained conditions involving differential inequalities that are sufficient to imply the starlikeness of order α. In 2014, Chandrashekar et al. [3] used the results of Kuroki and Owa [6] to establish third-order differential inequalities sufficient for starlikeness of order β. Recently, in 2017, Supramaniam et al. [15] developed second-order and third-order differential inequalities sufficient for the *convexity* of $f \in \mathcal{H}$.

Motivated by the ideas explored by the aforecited papers, here we determine sufficient conditions in terms of differential inequalities which ensure that $f \in \mathcal{A}_n$ is in $\mathcal{S}^*C_n(\alpha, \beta)$. Besides obtaining new conditions concerning the convexity of $f \in \mathcal{A}_n$, some of the already known results for starlikeness are derived as special cases. As applications, we construct functions of the form

$$f(\xi) = \int_0^1 \int_0^1 \int_0^1 \mathcal{J}(s, t, u, \xi) ds dt du,$$

and establish certain conditions on $\mathcal{J}(\xi)$ in order that $f \in \mathcal{S}^*C_n(\alpha, \beta)$. Similar results for a class of starlike functions satisfying a differential inequality have been proved by the authors in [16].

2 Sufficient Conditions

Definition 1 *(Subordination)* Let $f_1, f_2 \in \mathcal{H}$. Then we say that f_1 is subordinate to f_2, written as $f_1 \prec f_2$, if there exists $\omega \in \mathcal{H}$ satisfying $w(0) = 0$ and $|w(\xi)| < 1$, s.t.

$$f_1(\xi) = f_2(\omega(\xi)) \qquad (\xi \in \mathbb{D}).$$

Furthermore, if f_2 is univalent, then

$$f_1 \prec f_2 \iff f_1(0) = f_2(0) \text{ and } f_1(\mathbb{D}) \subset f_2(\mathbb{D}). \tag{1}$$

Definition 2 (Differential Subordination [2, 7]) Let $\Lambda : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ be analytic, and let $u \in \mathcal{H}$ be univalent. If $\wp \in \mathcal{H}$ satisfies

$$\Lambda(\wp(\xi), \xi\wp'(\xi); \xi) \prec u(\xi) \qquad (\xi \in \mathbb{D}), \tag{2}$$

then $\wp$ is called a solution of the first-order differential subordination (2). If $\ell \in \mathcal{H}$ is univalent and $\wp \prec \ell$ for all solutions $\wp$ of (2), then $\ell(\xi)$ is said to be a dominant of (2). A dominant ℓ_1 satisfying $\ell_1 \prec \ell$ for all dominants ℓ of (2) is called the best dominant of (2).

Lemma 1 ([7, p. 71]) *Let u be convex in $\mathbb{D}$ satisfying $u(0) = \varsigma$. Let $\tau \neq 0$ and $\Re(\tau) \geq 0$. If $\wp \in \mathcal{H}_n(\varsigma)$ and*

$$\wp(\xi) + \frac{\xi \wp'(\xi)}{\tau} \prec u(\xi),$$

then

$$\wp(\xi) \prec \ell(\xi) \prec u(\xi),$$

where

$$\ell(\xi) = \frac{\tau}{n\xi^{\tau/n}} \int_0^{\xi} u(\eta)\eta^{\tau/n-1} d\eta.$$

Moreover, $\ell(\xi)$ is convex and is the best dominant.

Lemma 2 ([7, p. 383]) *Let $n \in \mathbb{N}$ and let τ be real with $\tau \in [0, n)$. Let $\ell(\xi) \in \mathcal{H}_n(0)$ with $\ell'(0) \neq 0$ and*

$$\Re\left(1 + \frac{\xi \ell''(\xi)}{\ell'(\xi)}\right) > \frac{\tau}{n}.$$

If $\wp \in \mathcal{H}_n(0)$ satisfies

$$\xi \wp'(\xi) - \tau \wp(\xi) \prec n\xi \ell'(\xi) - \tau \ell(\xi),$$

then $\wp(\xi) \prec \ell(\xi)$ and the result is best possible.

Theorem 1 *Let $\alpha \in [0, 1)$, $\beta \in [0, 1]$, and $\delta \in [0, n)$. If $f \in \mathcal{A}_n$ satisfies*

$$\left|\beta\xi^2 f'''(\xi) + (1 + \beta(1 - \delta))\, \xi f''(\xi) - \delta\left(f'(\xi) - 1\right)\right| < \frac{(1 - \alpha)(n - \delta)(n + 1)}{n + 1 - \alpha}, \tag{3}$$

then $f \in \mathcal{S}^*C_n(\alpha, \beta)$. The result is best possible.

Proof In terms of subordination, the differential inequality (3) can be rewritten as

$$\beta\xi^2 f'''(\xi) + (1 + \beta(1 - \delta))\, \xi f''(\xi) - \delta(f'(\xi) - 1) \prec \frac{(1 - \alpha)(n - \delta)(n + 1)}{n + 1 - \alpha}\xi. \tag{4}$$

On taking

$$\begin{aligned}\wp(\xi) &= \beta\xi f''(\xi) + (1 - \beta(1 + \delta))\, f'(\xi) - (1 - \beta)(1 + \delta)\frac{f(\xi)}{\xi} \\ &= -\delta + (n\beta + 1)(n - \delta)a_{n+1}\xi^n \\ &\quad + ((n + 1)\beta + 1)\,(n + 1 - \delta)a_{n+2}\xi^{n+1} + \cdots \in \mathcal{H}_n(-\delta),\end{aligned}$$

the expression (4) takes the form

$$\wp(\xi) + \xi\wp'(\xi) \prec -\delta + \frac{(1-\alpha)(n-\delta)(n+1)}{n+1-\alpha}\xi =: \ell(\xi). \tag{5}$$

The function $\ell(\xi)$ is convex, as $\Re\left(1 + \xi\ell''(\xi)/\ell'(\xi)\right) = 1 > 0$, and $\ell(0) = -\delta$. Therefore, Lemma 1 is applicable to (5) with $\tau = 1$. Hence, we have

$$\begin{aligned}\wp(\xi) &\prec \frac{1}{n\xi^{\frac{1}{n}}}\int_0^{\xi}\left(-\delta + \frac{(1-\alpha)(n-\delta)(n+1)}{n+1-\alpha}t\right)t^{\frac{1}{n}-1}dt \\ &= -\delta + \frac{(1-\alpha)(n-\delta)}{n+1-\alpha}\xi.\end{aligned}$$

Or, equivalently

$$\beta\xi f''(\xi) + (1-\beta(1+\delta))\, f'(\xi) - (1-\beta)(1+\delta)\frac{f(\xi)}{\xi} \prec -\delta + \frac{(1-\alpha)(n-\delta)}{n+1-\alpha}\xi, \tag{6}$$

so that

$$\left|\beta\xi f''(\xi) + (1-\beta(1+\delta))\, f'(\xi) - (1-\beta)(1+\delta)\frac{f(\xi)}{\xi}\right| < \frac{n(1+\delta-\alpha)}{n+1-\alpha}. \tag{7}$$

Next, if we set

$$\begin{aligned}\wp_0(\xi) = \frac{F_\beta(\xi)}{\xi} - 1 &= \beta f'(\xi) + (1-\beta)\frac{f(\xi)}{\xi} - 1 \\ &= (n+2-\beta)a_{n+1}\xi^n + (n+3-\beta)a_{n+2}\xi^{n+1} + \cdots \in \mathcal{H}_n(0)\end{aligned}$$

and

$$\ell(\xi) = \frac{1-\alpha}{n+1-\alpha}\xi,$$

then from the subordination (6), a computation yields

$$\xi\wp_0'\xi - \delta\wp_0(\xi) \prec \frac{(1-\alpha)(n-\delta)}{n+1-\alpha}\xi = n\xi\ell'(\xi) - \delta\ell(\xi).$$

Since $\ell \in \mathcal{H}_n(0)$, $\ell'(0) \neq 0$, and $\Re\left(1 + \xi\ell''(\xi)/\ell'(\xi)\right) = 1 > \delta/n$, it follows from Lemma 2 that

$$\wp_0(\xi) \prec \ell(\xi), \quad \text{or} \quad \frac{F_\beta(\xi)}{\xi} \prec 1 + \frac{1-\alpha}{n+1-\alpha}\xi.$$

This further gives

$$\left|\frac{F_\beta(\xi)}{\xi}\right| > \frac{n}{n+1-\alpha}. \tag{8}$$

In view of (7) and (8), we have

$$\begin{aligned}
\frac{n\left|\frac{\xi F'_\beta(\xi)}{F_\beta(\xi)} - (1+\delta)\right|}{n+1-\alpha} &< \left|\frac{F_\beta(\xi)}{\xi}\right| \times \left|\frac{\xi F'_\beta(\xi)}{F_\beta(\xi)} - (1+\delta)\right| \\
&= \left|F'_\beta(\xi) - (1+\delta)\frac{F_\beta(\xi)}{\xi}\right| \\
&= \left|\beta\xi f''(\xi) + (1-\beta+\beta\delta)\, f'(\xi) - (1-\beta)(1+\delta)\frac{f(\xi)}{\xi}\right| \\
&< \frac{n(1+\delta-\alpha)}{n+1-\alpha}.
\end{aligned}$$

That is

$$\left|\frac{\xi F'_\beta(\xi)}{F_\beta(\xi)} - (1+\delta)\right| < 1+\delta-\alpha.$$

On using the fact that $|\omega| \le r$ implies $-r \le \Re(\omega) \le r$, the above inequality gives us that $\Re\left(\xi F'_\beta(\xi)/F_\beta(\xi)\right) > \alpha$, and hence $f \in \mathcal{S}^*C_n(\alpha, \beta)$. This completes the proof.

For sharpness of the above result, we supply the following example.

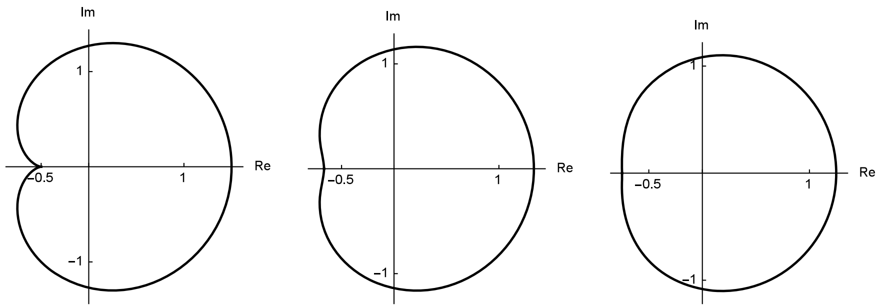
Example 1 For $\alpha \in [0, 1)$ and $\beta \in [0, 1]$, consider the function

$$f_\mu(\xi) = \xi + \frac{(1-\alpha)\mu}{(n+1-\alpha)(n\beta+1)}\xi^{n+1}, \quad |\mu| = 1. \tag{9}$$

Clearly, $f_\mu \in \mathcal{A}_n$ and

$$\begin{aligned}
&\left|\beta\xi^2 f'''_\mu(\xi) + (1+\beta(1-\delta))\,\xi f''_\mu(\xi) - \delta\big(f'_\mu(\xi) - 1\big)\right| \\
&\qquad = \left|\big(\beta n(n-1) + (1+\beta(1-\delta))\,n - \delta\big) \times \frac{(n+1)(1-\alpha)\mu}{(n+1-\alpha)(n\beta+1)}\xi^n\right| \\
&\qquad < \frac{(n+1)(1-\alpha)(n-\delta)}{(n+1-\alpha)}.
\end{aligned}$$

That is f_μ satisfies the condition of Theorem 1, hence $f_\mu \in \mathcal{S}^*C_n(\alpha, \beta)$ for every μ satisfying $|\mu| = 1$. Indeed, for $\xi \in \mathbb{D}$,



(a) $\beta = 0$ (b) $\beta = \frac{1}{2}$ (c) $\beta = 1$

Fig. 1 Boundary curves of $f_1(\xi) = \xi + \xi^2/2(\beta + 1)$, $\xi \in \mathbb{D}$, as β varies from 0 to 1

$$\Re\left(\frac{\xi F'_\beta[f_\mu](\xi)}{F_\beta[f_\mu](\xi)}\right) = \Re\left(\frac{\xi f'_\mu(\xi) + \beta\xi^2 f''_\mu(\xi)}{\beta\xi f'_\mu(\xi) + (1-\beta) f_\mu(\xi)}\right)$$
$$= \Re\left(\frac{\xi + (1+n\beta)\frac{(n+1)(1-\alpha)\mu}{(n+1-\alpha)(n\beta+1)}\xi^{n+1}}{\xi + (\beta(n+1) + (1-\beta))\frac{(1-\alpha)\mu}{(n+1-\alpha)(n\beta+1)}\xi^{n+1}}\right)$$
$$> \frac{1 - \frac{(n+1)(1-\alpha)}{(n+1-\alpha)}}{1 - \frac{1-\alpha}{(n+1-\alpha)}} = \alpha.$$

In Fig. 1, we show the transition of a starlike domain into a convex one as β varies from 0 to 1. We have taken the function $f \in \mathcal{S}^*\mathcal{C}_n(\alpha, \beta)$ as $f_1(\xi) = \xi + \xi^2/2(\beta + 1)$ ($\xi \in \mathbb{D}$), which is obtained by taking $\alpha = 0$ and $n = 1 = \mu$ in (9).

On giving particular values to α, β, δ, and n in Theorem 1, a number of previous, as well as new, results are obtained. Allowing $\beta = 0$ in Theorem 1, we obtain the following starlikeness condition established by Kuroki and Owa [6].

Corollary 1 *Let $\alpha \in [0, 1)$ and $\delta \in [0, n)$. If $f \in \mathcal{A}_n$ satisfies*

$$\left|\xi f''(\xi) - \delta\left(f'(\xi) - 1\right)\right| < \frac{(n+1)(1-\alpha)(n-\delta)}{n+1-\alpha},$$

then f is starlike of order α.

The following result established by Miller and Mocanu [8] is attained for $\beta = \alpha = 0$ in Theorem 1.

Corollary 2 *Let $f \in \mathcal{A}_n$ satisfies*

$$\left|\xi f''(\xi) - \delta(f'(\xi) - 1)\right| < n - \delta, \quad \delta \in [0, n).$$

Then $f \in \mathcal{S}^*$ and the result is best possible for $f(\xi) = \xi + \xi^{n+1}/(n+1)$.

If we set $\beta = 1$ in Theorem 1, we arrive at the following sufficient condition for convexity of order α.

Corollary 3 *Let* $\alpha \in [0, 1)$ *and* $\delta \in [0, n)$. *If* $f \in \mathcal{A}_n$ *satisfies*

$$\left|\xi^2 f'''(\xi) + (2-\delta)\xi f''(\xi) - \delta(f'(\xi) - 1)\right| < \frac{(n+1)(1-\alpha)(n-\delta)}{n+1-\alpha},$$

then f *is convex of order* α. *The result is best possible.*

Further, if we take $n = 3,\ \delta = 2$ and $\alpha = 0$ in Corollary 3, we obtain

Corollary 4 *Let* $f \in \mathcal{A}_3$ *satisfies*

$$\left|\xi^2 f'''(\xi) - 2\big(f'(\xi) - 1\big)\right| < 1.$$

Then f *is convex and the result is best possible for* $f(\xi) = \xi + \xi^4/16$.

Theorem 2 *Let* $\alpha \in [0, 1),\ \beta \in [0, 1]$, *and* $\nu \in [1, n+1)$. *If* $f \in \mathcal{A}_n$ *satisfies*

$$\left|\beta\xi^2 f'''(\xi) + (1+\beta)\xi f''(\xi) - \nu(\nu-1)\left(\frac{F_\beta(\xi)}{\xi} - 1\right)\right| < \frac{(n+1-\nu)(1-\alpha)(n+\nu)}{n+1-\alpha}, \tag{10}$$

then $f \in \mathcal{S}^*C_n(\alpha, \beta)$. *The result is sharp for* $f_\mu(\xi)$ *given by (9).*

Proof Inequality (10) in subordination form can be expressed as

$$\beta\xi^2 f'''(\xi) + (1+\beta)\xi f''(\xi) - \nu(\nu-1)\left(\frac{F_\beta(\xi)}{\xi} - 1\right) \prec \frac{(1-\alpha)(n+\nu)(n+1-\nu)}{n+1-\alpha}\xi,$$

which takes the form

$$\nu p(\xi) + \xi p'(\xi) \prec -\nu(\nu-1) + \frac{(1-\alpha)(n+\nu)(n+1-\nu)}{n+1-\alpha}\xi,$$

for

$$\begin{aligned}\wp(\xi) &= \beta\xi f''(\xi) + (1-\beta\nu)f'(\xi) - (1-\beta)\nu\frac{f(\xi)}{\xi}\\ &= 1-\nu + (n+1-\nu)(n\beta+1)a_{n+1}\xi^n\\ &\qquad +(n+2-\nu)[(n+1)\beta+1]a_{n+2}\xi^{n+1} + \cdots \in \mathcal{H}_n(1-\nu).\end{aligned}$$

A simple verification shows that $\wp$ satisfies the constraints of Lemma 1 and hence, we have

$$\begin{aligned}\wp(\xi) &\prec \frac{1}{n\xi^{\frac{\nu}{n}}}\int_0^{\xi}\left(-\nu(\nu-1)+\frac{(1-\alpha)(n+\nu)(n+1-\nu)}{n+1-\alpha}t\right)t^{\frac{\nu}{n}-1}dt\\ &= -(\nu-1)+\frac{(1-\alpha)(n+1-\nu)}{n+1-\alpha}\xi. \end{aligned} \tag{11}$$

The subordination (11) can be further rewritten as

$$\xi\wp_0'(\xi)-(\nu-1)\wp_0(\xi)\prec\frac{(1-\alpha)(n+1-\nu)}{n+1-\alpha}\xi, \tag{12}$$

where

$$\begin{aligned}\wp_0(\xi) &= \frac{F_\beta(\xi)}{\xi}-1\\ &= (n+2-\beta)a_{n+1}\xi^n+(n+3-\beta)a_{n+2}\xi^{n+1}+\cdots\in\mathcal{H}_n(0).\end{aligned}$$

If we take

$$\ell(\xi)=\frac{(1-\alpha)}{n+1-\alpha}\xi$$

satisfying

$$\ell\in\mathcal{H}_n(0),\quad \ell'(0)\neq 0,\quad \text{and}\quad \Re\left(1+\frac{\xi\ell''(\xi)}{\ell'(\xi)}\right)=1>\frac{\nu-1}{n},$$

then, after simplification, the subordination (12) takes the form

$$\xi\wp_0'(\xi)-(\nu-1)\wp_0(\xi)\prec n\xi\ell'(\xi)-(\nu-1)\ell(\xi).$$

Applying Lemma 2 yields $\wp_0(\xi)\prec\ell(\xi)$ i.e.,

$$\frac{F_\beta(\xi)}{\xi}-1\prec\frac{(1-\alpha)}{n+1-\alpha}\xi.$$

This further implies

$$\left|\frac{F_\beta(\xi)}{\xi}\right|>1-\frac{(1-\alpha)}{n+1-\alpha}=\frac{n}{n+1-\alpha} \tag{13}$$

Also, from (11), we obtain

$$|\wp(\xi)|<\frac{n(\nu-\alpha)}{n+1-\alpha}. \tag{14}$$

Making use of (13) and (14), we have

$$\begin{aligned}\frac{n}{n+1-\alpha}\left|\frac{\xi F_\beta'(\xi)}{F_\beta(\xi)}-\nu\right| &< \left|\frac{F_\beta(\xi)}{\xi}\right|\left|\frac{\xi F_\beta'(\xi)}{F_\beta(\xi)}-\nu\right| \\ &= \left|F_\beta'(\xi)-\nu\frac{F_\beta(\xi)}{\xi}\right| \\ &= |p(\xi)| \\ &< \frac{n(\nu-\alpha)}{n+1-\alpha}.\end{aligned}$$

Or,

$$\left|\frac{\xi F_\beta'(\xi)}{F_\beta(\xi)}-\nu\right| < \nu-\alpha.$$

This further implies $\Re\left(\xi F_\beta'(\xi)/F_\beta(\xi)\right) > \alpha$, and hence $f \in \mathcal{S}^*C_n(\alpha,\beta)$. □

Taking $\beta = 0$ and $\nu(\nu-1) = \vartheta$, $\nu \in [1, n+1)$ in Theorem 2, we obtain the following result established by Verma et al. [17].

Corollary 5 *Let $\vartheta \in [0, n+1)$ and $\alpha \in [0, 1)$. If $f \in \mathcal{A}_n$ satisfies*

$$\left|\xi f''(\xi) - \vartheta\left(\frac{f(\xi)}{\xi}-1\right)\right| < \frac{(1-\alpha)(n^2+n-\vartheta)}{n+1-\alpha},$$

then $f \in \mathcal{S}^*(\alpha)$.

In Corollary 5, if we fix $n = 1$ and $\alpha = 0$, we obtain Theorem 4 of Fournier and Mocanu [5], and if we fix $\vartheta = \alpha = 0$ and $n = 1$, we obtain the following result first introduced by Obradović [11].

Corollary 6 *Let $f \in \mathcal{A}$ satisfies $|\xi f''(\xi)| < 1$ in $\mathbb{D}$. Then $f \in \mathcal{S}^*$ and the result is sharp.*

Now letting $\beta = 1$ in Theorem 2, we have the following result regarding the convexity of order α.

Corollary 7 *Let $\alpha \in [0, 1)$ and $\nu \in [1, n+1)$. If $f \in \mathcal{A}_n$ satisfies*

$$\left|\xi^2 f'''(\xi) + 2\xi f''(\xi) - \nu(\nu-1)\left(f'(\xi)-1\right)\right| < \frac{(n+1-\nu)(1-\alpha)(n+\nu)}{n+1-\alpha},$$

then $f(\xi)$ is convex of order α. The result is sharp for

$$f(\xi) = \xi + \frac{(1-\alpha)\mu}{(n+1-\alpha)(n+1)}\xi^{n+1}, \quad |\mu| = 1.$$

3 Applications

In this section, we use Theorems 1 and 2 to construct functions involving triple integrals, and obtain conditions which are sufficient to ensure that these functions are in the class $\mathcal{S}^*\mathcal{C}_n(\alpha,\beta)$. Consequently, the earlier known results regarding starlikeness are obtained by setting $\beta = 0$.

Theorem 3 *Let $\alpha \in [0, 1)$, $\delta \in [0, n)$, and let $\mathcal{J} \in \mathcal{H}$ satisfies*

$$|\mathcal{J}(\xi)| \leq \frac{(n+1)(1-\alpha)(n-\delta)}{n+1-\alpha}. \tag{15}$$

Then the function

$$f(\xi) = \begin{cases} \xi + \frac{\xi^{n+1}}{\beta} \iiint_0^1 \mathcal{J}(stu\xi)s^n t^{n-1-\delta} u^{n+\frac{1-\beta}{\beta}} dsdtdu, & for\ 0 < \beta \leq 1 \\ \xi + \xi^{n+1} \iint_0^1 \mathcal{J}(st\xi)s^n t^{n-1-\delta} dsdt, & for\ \beta = 0 \end{cases} \tag{16}$$

belongs to the class $\mathcal{S}^*\mathcal{C}_n(\alpha,\beta)$. Furthermore, if equality holds in (15), then the function (16) is

$$f_\mu(\xi) = \xi + \frac{(1-\alpha)\mu}{(n+1-\alpha)(n\beta+1)}\xi^{n+1}, \quad |\mu| = 1,$$

which indeed is in the class $\mathcal{S}^*\mathcal{C}_n(\alpha,\beta)$ (see Example 1).

Proof Let us suppose that $f \in \mathcal{A}_n$ satisfies the third-order differential inequality

$$\beta\xi^2 f'''(\xi) + (1+\beta(1-\delta))\,\xi f''(\xi) - \delta(f'(\xi)-1) = \xi^n \mathcal{J}(\xi). \tag{17}$$

In view of (15), it is clear that

$$\left|\beta\xi^2 f'''(\xi) + (1+\beta(1-\delta))\xi f''(\xi) - \delta(f'(\xi)-1)\right| < \frac{(1-\alpha)(n-\delta)(n+1)}{n+1-\alpha},$$

and hence, from Theorem 1, we conclude that the solution of (17) belongs to $\mathcal{S}^*\mathcal{C}_n(\alpha,\beta)$. Thus, in order to establish the desired result, it is sufficient to verify that the solution of (17) is the function defined in (16). For

$$\phi(\xi) = \beta\xi f''(\xi) + (1-\beta(1+\delta))f'(\xi) - (1-\beta)(1+\delta)\frac{f(\xi)}{\xi},$$

the Eq. (17) takes the form

$$\xi\phi'(\xi) + \phi(\xi) = \xi^n \mathcal{J}(\xi).$$

This, on solving, gives

$$\xi\phi(\xi) = \int_0^\xi \zeta^n \mathcal{J}(\zeta)d\zeta,$$

or

$$\phi(\xi) = \xi^n \int_0^1 \mathcal{J}(s\xi)s^n ds. \tag{18}$$

Taking

$$\psi(\xi) = \beta f'(\xi) + (1-\beta)\frac{f(\xi)}{\xi} - 1,$$

the Eq. (18) can be simplified to

$$\xi\psi'(\xi) - \delta\psi(\xi) = \xi^n \int_0^1 \mathcal{J}(s\xi)s^n ds.$$

The solution ψ of the above differential equation is given by

$$\psi(\xi) = \xi^n \iint_0^1 \mathcal{J}(st\xi)s^n t^{n-1-\delta} dsdt.$$

This gives

$$\beta f'(\xi) + (1-\beta)\frac{f(\xi)}{\xi} = 1 + \xi^n \iint_0^1 \mathcal{J}(st\xi)s^n t^{n-1-\delta} dsdt. \tag{19}$$

Case 1 If $\beta = 0$, then (19) yields

$$f(\xi) = \xi + \xi^{n+1} \iint_0^1 \mathcal{J}(st\xi)s^n t^{n-1-\delta} dsdt.$$

Case 2 If $0 < \beta \leq 1$, then (19) is a first-order differential equation with

$$f(\xi) = \xi + \frac{\xi^{n+1}}{\beta} \iiint_0^1 \mathcal{J}(stu\xi)s^n t^{n-1-\delta} u^{n+\frac{1-\beta}{\beta}} dsdtdu.$$

as its solution. Moreover, if equality holds in (15), then

$$\mathcal{J}(\xi) = \frac{(n+1)(1-\alpha)(n-\delta)}{n+1-\alpha}\mu.$$

for some $\mu \in \mathbb{C}$ with $|\mu| = 1$. Substituting this in (16) and integrating, we obtain the function $f_\mu(\xi)$. □

Remark 1 Setting $\beta = 0$ in Theorem 3, we obtain Theorem 2.6 of Kuroki and Owa [6].

Remark 2 The case, $\beta = \alpha = 0$ in Theorem 3, was considered by Miller and Mocanu [8, Theorem 2.1].

On taking $\beta = 1$ in Theorem 3, we obtain:

Corollary 8 *Let* $0 \leq \alpha < 1$ *and* $0 \leq \delta < n$. *If* $\mathcal{J}(\xi) \in \mathcal{H}$ *satisfies*

$$|\mathcal{J}(\xi)| \leq \frac{(1-\alpha)(n-\delta)(n+1)}{n+1-\alpha},$$

then

$$f(\xi) = \xi + \xi^{n+1} \iiint_0^1 \mathcal{J}(stu\xi) t^{n-1-\delta} (su)^n \, ds dt du$$

is convex of order α.

Further, if we let $\alpha = 0$ and $\delta = n - 1$ in Corollary 8, then the following important result is established.

Corollary 9 *If* $\mathcal{J}(\xi) \in \mathcal{H}$ *satisfies* $|\mathcal{J}(\xi)| \leq 1$, *then the function*

$$f(\xi) = \xi + \xi^{n+1} \iiint_0^1 \mathcal{J}(stu\xi)(su)^n \, ds dt du$$

is convex in $\mathbb{D}$.

The following theorem is an application of Theorem 2.

Theorem 4 *Let* $\nu \in [1, n+1)$ *and* $\alpha \in [0, 1)$. *Also, let* $\mathcal{J} \in \mathcal{H}$ *satisfies*

$$|\mathcal{J}(\xi)| \leq \frac{(n+1-\nu)(1-\alpha)(n+\nu)}{n+1-\alpha}. \tag{20}$$

Then, for $\beta \in (0, 1]$,

$$f(\xi) = \xi + \frac{\xi^{n+1}}{\beta} \iiint_0^1 \mathcal{J}(stu\xi) s^{n+\nu-1} t^{n-\nu} u^{n+\frac{1-\beta}{\beta}} \, ds dt du$$

is a member of the family $\mathcal{S}^*C_n(\alpha, \beta)$. *Furthermore, the function*

$$f(\xi) = \xi + \xi^{n+1} \iint_0^1 \mathcal{J}(st\xi) s^{n+\nu-1} t^{n-\nu} \, ds dt$$

is a member of $\mathcal{S}^*(\alpha)$.

Proof Consider $f \in \mathcal{A}_n$ that satisfies

$$\beta\xi^2 f'''(\xi) + (1+\beta)\xi f''(\xi) - \nu(\nu-1)\left(\frac{F_\beta(\xi)}{\xi} - 1\right) = \xi^n \mathcal{J}(\xi). \tag{21}$$

In the light of (20), it is clear that

$$\left|\beta\xi^2 f'''(\xi) + (1+\beta)\xi f''(\xi) - \nu(\nu-1)\left(\frac{F_\beta(\xi)}{\xi} - 1\right)\right| < \frac{(n+1-\nu)(1-\alpha)(n+\nu)}{n+1-\alpha}.$$

Therefore, it follows from Theorem 2 that the solution of (21) must lie in $\mathcal{S}^*\mathcal{C}_n(\alpha, \beta)$. We now proceed to solve (21). Let us take

$$\varphi(\xi) = \beta\xi f''(\xi) + (1-\beta\nu)f'(\xi) - (1-\beta)\nu\frac{f(\xi)}{\xi},$$

so that (21) becomes

$$\varphi'(\xi) + \frac{\nu}{\xi}\varphi(\xi) = \xi^{n-1}\mathcal{J}(\xi) - \frac{\nu(\nu-1)}{\xi}.$$

This on further simplification gives

$$(\xi^\nu\varphi(\xi))' = \xi^{n+\nu-1}\mathcal{J}(\xi) - \nu(\nu-1)\xi^{\nu-1},$$

or

$$\varphi(\xi) = \xi^n \int_0^1 \mathcal{J}(s\xi)s^{n+\nu-1}ds - (\nu-1).$$

The above equation is equivalent to

$$\beta\xi f''(\xi) + (1-\beta\nu)f'(\xi) - (1-\beta)\nu\frac{f(\xi)}{\xi} + (\nu-1) = \xi^n \int_0^1 \mathcal{J}(s\xi)s^{n+\nu-1}ds,$$

which, after simple calculations, can be rewritten as

$$\xi\left(\frac{F_\beta(\xi)}{\xi} - 1\right)' - (\nu-1)\left(\frac{F_\beta(\xi)}{\xi} - 1\right) = \xi^n \int_0^1 \mathcal{J}(s\xi)s^{n+\nu-1}ds. \tag{22}$$

Solving (22), we obtain

$$\frac{F_\beta(\xi)}{\xi} = 1 + \xi^n \iint_0^1 \mathcal{J}(st\xi)s^{n+\nu-1}t^{n-\nu}ds\,dt. \tag{23}$$

Case I. If $\beta = 0$, then (23) gives

$$f(\xi) = \xi + \xi^{n+1} \iint_0^1 \mathcal{J}(st\xi) s^{n+\nu-1} t^{n-\nu} ds dt,$$

and this function is starlike of order α.
Case II. If $\beta \in (0, 1]$, then the solution of (23) is

$$f(\xi) = \xi + \frac{\xi^{n+1}}{\beta} \iiint_0^1 \mathcal{J}(stu\xi) s^{n+\nu-1} t^{n-\nu} u^{n+\frac{1-\beta}{\beta}} ds dt du.$$

and this is a member of $\mathcal{S}^*\mathcal{C}_n(\alpha, \beta)$. □

Again, Theorem 4 has many consequences, some have been already proved and some are completely new. For example, taking $\beta = 0$ gives us Theorem 4.1 of Verma et al. [17] and $\beta = 1$ gives us results regarding convexity that are not available in the literature.

4 Pascu Class

In this section, we find the differential inequalities sufficient to imply that a function is in the Pascu class $\mathcal{M}(\alpha, \beta)$ of β-convex functions of order α. For details and other related results about this class of functions, we refer to [12, 13].

Definition 3 Let $\alpha \in [0, 1)$, $\beta \in [0, 1]$, and $f \in \mathcal{A}$. Then f is said to belong to the Pascu class $\mathcal{M}(\alpha, \beta)$ if

$$\Re\left(\frac{\beta\xi\left(\xi f'(\xi)\right)' + (1-\beta)\xi f'(\xi)}{\beta\xi f'(\xi) + (1-\beta) f(\xi)}\right) > \alpha.$$

Clearly, $\mathcal{M}(\alpha, 0) = \mathcal{S}^*(\alpha)$ and $\mathcal{M}(\alpha, 1) = \mathcal{C}(\alpha)$. Thus, like $\mathcal{S}^*\mathcal{C}(\alpha, \beta)$, this class also gives a smooth passage between the classes of starlike and convex functions of order α. Observe that $f \in \mathcal{M}(\alpha, \beta)$ if

$$\beta\xi f'(\xi) + (1-\beta) f(\xi) \in \mathcal{S}^*(\alpha), \qquad \beta \in [0, 1].$$

For $\beta \in (0, 1]$, this condition can be further written as

$$f \in \mathcal{M}(\alpha, \beta) \quad \text{if} \quad \beta\xi^{2-\frac{1}{\beta}}\left(\xi^{\frac{1}{\beta}-1} f(\xi)\right)' \in \mathcal{S}^*(\alpha).$$

We now use Corollaries 1 and 5 to prove our results. Note that for $f \in \mathcal{A}_n$, we denote this class by $\mathcal{M}_n(\alpha, \beta)$

Theorem 5 *Let* $\alpha \in [0, 1)$, $\beta \in (0, 1]$ *and* $\delta \in [0, n)$. *Further, for* $f \in \mathcal{A}_n$, *let* $g(\xi) = \xi^{\frac{1}{\beta}-1} f(\xi)$ *satisfies*

$$\left|\xi^{3-\frac{1}{\beta}} g'''(\xi) + \left(4 - \delta - \frac{2}{\beta}\right)\xi^{2-\frac{1}{\beta}} g''(\xi) + \left(2 - \frac{1}{\beta}\right)\left(1 - \delta - \frac{1}{\beta}\right)\xi^{1-\frac{1}{\beta}} g'(\xi) + \delta\right| < \frac{(n+1)(1-\alpha)(n-\delta)}{\beta(n+1-\delta)}.$$

Then f *belongs to the Pascu class* $\mathcal{M}_n(\alpha, \beta)$.

Proof Replacing $f(\xi)$ by

$$\beta\xi^{2-\frac{1}{\beta}}\left(\xi^{\frac{1}{\beta}-1} f(\xi)\right)'$$

in Corollary 1, and doing some calculations yields the desired result. □

Theorem 6 *Let* $\alpha \in [0, 1)$, $\beta \in (0, 1]$ *and* $\mu \in [0, n+1)$. *Further, for* $f \in \mathcal{A}_n$, *let* $g(\xi) = \xi^{\frac{1}{\beta}-1} f(\xi)$ *satisfies*

$$\left|\xi^{3-\frac{1}{\beta}} g'''(\xi) + (4 - \frac{2}{\beta})\xi^{2-\frac{1}{\beta}} g''(\xi) + (2 - \frac{3}{\beta} + \frac{1}{\beta^2})\xi^{1-\frac{1}{\beta}} g'(\xi) - \mu\left(\xi^{1-\frac{1}{\beta}} g(\xi) - 1\right)\right| < \frac{(1-\alpha)\left[n(n+1) - \mu\right]}{\beta(n+1-\alpha)}.$$

Then $f \in \mathcal{M}_n(\alpha, \beta)$.

Proof The result is obtained on replacing $f(\xi)$ by $\beta\xi^{2-\frac{1}{\beta}}\left(\xi^{\frac{1}{\beta}-1} f(\xi)\right)'$ in Corollary 5. □

Acknowledgements This research work of the authors was supported by the Project No. CRG/2019/000200/MS of Science and Engineering Research Board, Department of Science and Technology, New Delhi, India.

References

1. Altıntaş, O.: On a subclass of certain starlike functions with negative coefficients. Math. Japan **36**(3), 489–495 (1991)
2. Bulboacă, T.: Differential Subordinations and Superordinations, Recent Results. House of Scientific Book Publ., Cluj-Napoca (2005)
3. Chandrashekar, R., Ali, R.M., Subramanian, K.G., Swaminathan, A.: Starlikeness of functions defined by third-order differential inequalities and integral operators. Abstr. Appl. Anal. **2014**, Art. ID 723097,6 pp
4. Duren, P.L.: Univalent functions, Grundlehren der Mathematischen Wissenschaften, 259. Springer, New York (1983)
5. Fournier, R., Mocanu, P.: Differential inequalities and starlikeness. Complex Var. Theory Appl. **48**(4), 283–292 (2003)

6. Kuroki, K., Owa, S.: Double integral operators concerning starlike of order β. Int. J. Differ. Equ. **2009**, Art. ID 737129, 13 pp
7. Miller, S.S., Mocanu, P.T.: Differential Subordinations, Monographs and Textbooks in Pure and Applied Mathematics, 225. Marcel Dekker Inc., New York (2000)
8. Miller, S.S., Mocanu, P.T.: Double integral starlike operators. Integral Transforms Spec. Funct. **19**(7–8), 591–597 (2008)
9. Mocanu, P.T.: Two simple sufficient conditions for starlikeness. Mathematica (Cluj) **34(57)**(2), 175–181 (1992)
10. Mustafa, N.: Some subclasses of analytic functions of complex order. Turkish J. Math. **42**(5), 2423–2435 (2018)
11. Obradović, M.: Simple sufficient conditions for univalence. Mat. Vesnik **49**(3–4), 241–244 (1997)
12. Pascu, N.N., Podaru, V.: On the radius of alpha-starlikeness for starlike functions of order beta. In: Complex Analysis—Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), pp. 336–349, Lecture Notes in Mathematics, vol. 1013. Springer, Berlin (1981)
13. Raghavendar, K., Swaminathan, A.: Integral transforms of functions to be in certain class defined by the combination of starlike and convex functions. Comput. Math. Appl. **63**(8), 1296–1304 (2012)
14. Ruscheweyh, S.: Coefficient conditions for starlike functions. Glasgow Math. J. **29**(1), 141–142 (1987)
15. Supramaniam, S., Chandrashekar, R., Lee, S.K., Subramanian, K.G.: Convexity of functions defined by differential inequalities and integral operators. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **111**(1), 147–157 (2017)
16. Swaminathan, A., Wani, L.A.: Sufficient conditions and radii problems for a starlike class involving a differential inequality. Bull. Korean Math. Soc. **57**(6), 1409–1426 (2020)
17. Verma, S., Gupta, S., Singh, S.: A differential inequality and starlikeness of a double integral. Rocky Mountain J. Math. **44**(5), 1653–1659 (2014)
18. Wani, L.A., Anbhu, S.: Inclusion properties of hypergeometric type functions and related integral transforms. Stud. Univ. Babeş-Bolyai Math. **65**(2), 211–227 (2020)
19. Wani, L.A., Swaminathan, A.: Starlike and convex functions associated with a nephroid domain. Bull. Malays. Math. Sci. Soc. **44**(1), 79–104 (2021)

One Dimensional Parametrized Test Functions Space of Entire Functions

Sheila M. Menchavez and Irene Mae Y. Antabo

Abstract Inspired by the construction of Kondratiev test functions in infinite dimensional analysis, this paper constructs a nuclear space of entire test functions of minimal type, endowed with the projective limit topology.

Keywords White noise theory · Topological spaces of test functions · Special classes of entire functions and growth estimates

2010 Mathematics Subject Classification: 60H40 · 46F05 · 30D15

1 Introduction

In the 1970s, the theory of generalized functionals of infinitely many variables with a dual pairing between spaces of test and generalized functions generated by Gaussian measures was introduced independently by Yu. G. Kondratiev in [1] and series of papers [2–7]; and by T. Hida in [9–11]. On the other hand, at the same time in [12, 13], Yu. M. Berezansky and coauthors have developed a more general theory of generalized functionals of infinitely many variables with the pairing generated by non-Gaussian measures. The underlying principle for this kind of analysis is the construction of suitable Gelfand triples of test and generalized functions (see e.g., [1, 7, 11, 14]). As stated in [15], the fundamental approach is to embed polynomials into a countably Hilbert space, depending on the specific choice of these Hilbert spaces one thus obtains the spaces of Hida or the Kondratiev test functions. The latter extend the polynomials to a topological space of entire functions [7]. The explicit form of Kondratiev spaces of test functions in infinite dimensional analysis is given in [16], as a Hilbert space of formal power series. From the norm used to construct the Kondratiev spaces, the parametrized Kondratiev spaces of test functions can be defined

S. M. Menchavez (✉) · I. M. Y. Antabo
MSU-Iligan Institute of Technology, 9200 Iligan City, Philippines
e-mail: sheila.menchavez@g.msuiit.edu.ph

I. M. Y. Antabo
e-mail: irenemae.antabo@g.msuiit.edu.ph

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_4

analogously. Specifically, when the norm is parametrized with $(n!)^{1+\beta}$, $\beta \in [0, 1]$ instead of $(n!)^2$ in the definition of the Kondratiev spaces, or by analogy with [17], more general spaces of test functions can be obtained. But the consequence of generalizing deteriorates the properties of the mentioned spaces and of the corresponding dual spaces in comparison with the case of the non parametrized spaces. Studies on this area may involve a characterization of the spaces considered in terms of analytic and growth properties of the corresponding S-transforms, which is established in [8]. Kondratiev test and generalized functions of one complex variable were studied in [18]. The one dimensional Kondratiev space of test functions is studied in [15] with norm given by the sum

$$\sum_{n=0}^{\infty} |a_n|^2 e^{pn} (n!)^2 < \infty, \; p \in \mathbb{N}_0. \tag{1}$$

This paper constructs the one dimensional parametrized Kondratiev spaces of test functions. More precisely, we use $(n!)^{1+\beta}$ and $\beta \in [0, 1]$, instead of $(n!)^2$ in the definition of the norm in (1) to define a countable system of nondecreasing Hilbertian norms and construct a countable family of Hilbert spaces $\{\mathcal{H}_p\}_{p\geq 0}$ of entire functions such that their intersection $\mathcal{E}^\beta = \bigcap_{p\geq 0} \mathcal{H}_p$ is a space of entire functions of minimal type, endowed with the projective limit topology. To this end, in Sect. 2, we shall collect necessary concepts regarding properties of entire functions. Please see [19–21] on the notion of countably Hilbert spaces. Section 3 is dedicated to the construction of parametrized Kondratiev test functions space.

2 Preliminaries

In this section, we recall some facts and notations on order of growth and type of the entire functions that are essential in this paper (see e.g., [22, 23]).

Theorem 1 [22] *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of order*

$$\rho := \inf\left\{K : \max_{|z|=r} |f(z)| \overset{as}{<} \exp(r^K)\right\},$$

where $\overset{as}{<}$ means "for sufficiently large argument." Then

$$\rho = \limsup_{n\to\infty} \frac{n \ln n}{\ln(1/|a_n|)}. \tag{2}$$

Moreover, if $f(z)$ is an entire function of order of growth ρ and type

$$\tau := \inf\left\{A : \max_{|z|=r} |f(z)| \overset{as}{<} \exp(Ar^{\rho})\right\}$$

then

$$\tau = \frac{1}{\rho e} \limsup_{n\to\infty}\left(n\sqrt[n]{|a_n|^{\rho}}\right). \tag{3}$$

Lemma 1 [23] *If the asymptotic inequality*

$$\max_{|z|=r} |f(z)| \overset{as}{<} \exp(Ar^{\rho})$$

is fulfilled, then

$$|a_n| \overset{as}{<} \left(\frac{eA\rho}{n}\right)^{\frac{n}{\rho}}. \tag{4}$$

Furthermore, if the asymptotic inequality (4) is fulfilled, then

$$\max_{|z|=r} |f(z)| \overset{as}{<} \exp((A+\epsilon)r^{\rho}), \quad \forall\epsilon > 0.$$

3 Main Results and Proofs

In this section, we first consider an inner product space of entire functions that contains polynomials and the space of polynomials forms its dense subset. Moreover, its completion with respect to increasing sequence of norms produces a chain of Hilbert spaces such that their intersection is a nuclear space of entire test functions of minimal type.

Definition 1 For $0 \le \beta \le 1$ and $p \in \mathbb{N}_0$. We define a linear space of power series

$$\mathcal{F} = \left\{f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 e^{pn} (n!)^{1+\beta} < \infty\right\}. \tag{5}$$

Lemma 2 *The space $\mathcal{F}$ given in (5) is a complete inner product space of entire functions with inner product defined by*

$$(\cdot,\cdot) : \mathcal{F}\times\mathcal{F} \to \mathbb{C} \;\; \textit{defined by}\; (f,g)_{p,\beta} = \sum_{n=0}^{\infty} a_n \bar{b}_n e^{pn} (n!)^{1+\beta} \tag{6}$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n,\ a_n \in \mathbb{C}$ *and* $g(z) = \sum_{m=0}^{\infty} b_m z^m,\ b_m \in \mathbb{C}$.

Proof First, we show that $(f, g)_{p,\beta} = \sum_{l=0}^{\infty} a_l \bar{b_l} e^{pl}(l!)^{1+\beta} < \infty$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{m=0}^{\infty} b_m z^m$ are in $\mathcal{F}$. Then corresponding to $0 < \epsilon < 1$, there exist natural numbers N_1, N_2 such that $\left| \sum_{n=N_1+1}^{\infty} |a_n|^2 e^{pn}(n!)^{1+\beta} \right| < \frac{\epsilon}{2}$ for all $n \geq N_1$ and $\left| \sum_{m=N_2+1}^{\infty} |a_m|^2 e^{pn}(m!)^{1+\beta} \right| < \frac{\epsilon}{2}$ for all $n \geq N_2$. Now, choose $N = \max\{N_1, N_2\}$ such that for all $l > N$, we get

$$\left| \sum_{l=N+1}^{\infty} a_l \bar{b_l} e^{pl}(l!)^{1+\beta} \right| \leq \left| \sum_{l=N+1}^{\infty} |a_l|^2 e^{pl}(l!)^{1+\beta} \right| \left| \sum_{l=N+1}^{\infty} |b_l|^2 e^{pl}(l!)^{1+\beta} \right| < \epsilon.$$

Also, $(f, f)_{p,\beta}$ is positive-definite, since $|a_n|^2 e^{pn}(n!)^{1+\beta} > 0$ for all $n \in \mathbb{N}_0$. Moreover, by definition of f, we have $|a_n|^2 \leq \frac{\epsilon}{(n!)^{1+\beta} e^{pn}}$, for all $n \geq N_1$. By the Stirling formula, $\lim_{n\to\infty} |a_n|^{1/n} = 0$. Thus it can be verified that $\mathcal{F}$ is an inner product space of entire functions. To this end, we show that $\mathcal{F}$ is complete. For $0 \leq \beta \leq 1$ and $p \in \mathbb{N}_0$, define

$$\|x - y\|_{p,\beta} = \left(\sum_{j=0}^{\infty} |x_j - y_j|^2 e^{pj}(j!)^{1+\beta} \right)^{\frac{1}{2}},$$

where $x = (x_j)_{j=0}^{\infty}$ and $y = (y_j)_{j=0}^{\infty} \in \mathcal{F}$. Let $(f^{(n)})_{n=0}^{\infty}$ be a Cauchy sequence in $\mathcal{F}$, where $f^{(n)} = \{f_1^{(n)}, f_2^{(n)}, \dots\}$. Then there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$\|f^{(m)} - f^{(n)}\|_{p,\beta} < \epsilon.$$

Thus for any $j \in \mathbb{N}$, we get

$$\left| f_j^{(m)} - f_j^{(n)} \right|^2 e^{pj}(j!)^{1+\beta} \leq \sum_{j=0}^{\infty} \left| f_j^{(m)} - f_j^{(n)} \right|^2 e^{pj}(j!)^{1+\beta} < \epsilon^2,$$

which implies that all $m, n \geq N$, $\left| f_j^{(m)} - f_j^{(n)} \right| < \epsilon$. That is for any $j \in \mathbb{N}$, the sequence $\left(f_j^{(n)} \right)_{n=0}^{\infty}$ is Cauchy. Since $\mathbb{C}$ is complete, for all $j \in \mathbb{N}$, there exists $f_j \in \mathbb{C}$ such that

$$\lim_{n\to\infty} f_j^{(n)} = f_j.$$

Next, define $f = \{f_1, f_2, \dots\}$ and show that $f \in \mathcal{F}$ and $f^{(n)}$ converges to f. Fix $k \in \mathbb{N}$, so that for all $m, n \geq N$, we have

$$\sum_{j=0}^{k} \left| f_j^{(m)} - f_j^{(n)} \right|^2 e^{pj} (j!)^{1+\beta} \leq \sum_{j=0}^{\infty} \left| f_j^{(m)} - f_j^{(n)} \right|^2 e^{pj} (j!)^{1+\beta} < \epsilon^2. \quad (7)$$

Letting $n \to \infty$ in Eq. (7), we get

$$\sum_{j=0}^{k} \left| f_j^{(m)} - f_j \right|^2 e^{pj} (j!)^{1+\beta} < \epsilon^2 \quad (8)$$

for all $m \geq N$. Now, as $k \to \infty$ we obtain all $m \geq N$,

$$\sum_{j=0}^{\infty} \left| f_j^{(m)} - f_j \right|^2 e^{pj} (j!)^{1+\beta} < \epsilon^2. \quad (9)$$

Thus we have shown that $f^{(n)} - f \in \mathcal{F}$. Moreover, using Minkowski's inequality,

$$\begin{aligned}\left(\sum_{j=0}^{\infty} |f_j|^2 e^{pj} (j!)^{1+\beta} \right)^{\frac{1}{2}} &= \left(\sum_{j=0}^{\infty} \left| f_j^{(m)} - f_j + f_j^{(m)} \right|^2 e^{pj} (j!)^{1+\beta} \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{j=0}^{\infty} \left| f_j^{(m)} - f_j \right|^2 e^{pj} (j!)^{1+\beta} \right)^{\frac{1}{2}} + \left(\sum_{j=0}^{\infty} \left| f_j^{(m)} \right|^2 e^{pj} (j!)^{1+\beta} \right)^{\frac{1}{2}}\end{aligned}$$

Hence, $f \in \mathcal{F}$. Again letting $k \to \infty$ in Eq. (8), we obtain

$$\| f^{(m)} - f \|_{p,\beta} = \sum_{j=0}^{\infty} \left| f_j^{(m)} - f_j \right|^2 e^{pj} (j!)^{1+\beta} < \epsilon^2.$$

Hence, $f^{(m)}$ converges to f. □

The inner product given in Eq. (6) defines a Hilbertian norm on $\mathcal{F}$. In what follows, we introduce a countable system of nondecreasing Hilbertian norms $\ldots \leq \|\cdot\|_{p,\beta} \leq \|\cdot\|_{p+1,\beta} \leq \|\cdot\|_{p+2,\beta} \ldots$ with parameter $p = 0, 1, 2, \ldots$ and $0 \leq \beta \leq 1$ corresponding to countable family of Hilbert spaces.

Definition 2 For $0 \leq \beta \leq 1$ and $p = 0, 1, 2, \ldots$, we define Hilbert spaces

$$\mathcal{H}_p^\beta := \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \middle| \|f\|_{p,\beta}^2 := \sum_{n=0}^{\infty} |a_n|^2 e^{pn} (n!)^{1+\beta} < \infty \right\}.$$

Since $\|\cdot\|_{p,\beta} \leq \|\cdot\|_{p+1,\beta}$ for all $p \in \mathbb{N}_0$, we have $\mathcal{H}_{p+1}^\beta \subset \mathcal{H}_p^\beta$ for all $p \in \mathbb{N}_0$.

Lemma 3 *The monomials*

$$e_n^{(p)}(z) = e^{\frac{-pn}{2}}(n!)^{\frac{1+\beta}{-2}} z^n \quad with \quad n = 0, 1, 2, \ldots \tag{10}$$

are an orthonormal basis in $\mathcal{H}_p^\beta$*. Moreover, the set* $\left\{e_n^{(p)}(z) \ : \ n \in \mathbb{N}_0\right\}$ *is total in* $\mathcal{H}_p^\beta$*.*

Proof Let $f \in \mathcal{H}_p^\beta$, $p = 0, 1, 2, \ldots$ and $0 \le \beta \le 1$. If $n = m$, we have

$$\left(e_n^{(p)}, e_m^{(p)}\right) = \left\|e_n^{(p)}\right\|_{p,\beta}^2 = 1.$$

Suppose $n \neq m$ then with

$$a_i = \begin{cases} 0, \ \text{if } i \neq n \\ e^{\frac{-pn}{2}}(n!)^{-\frac{1+\beta}{2}}, \ \text{if } i = n \end{cases}$$

and

$$b_j = \begin{cases} 0, \ \text{if} j \neq m \\ e^{\frac{-pm}{2}}(m!)^{\frac{1+\beta}{2}}, \ \ \text{if} j = m \end{cases}.$$

we have

$$\left(e_n^{(p)}, e_m^{(p)}\right) = \left(\sum_{i=0}^{\infty} a_i z^i, \sum_{j=0}^{\infty} b_j z^j\right) = \sum_{k=0}^{\infty} a_k \bar{b_k} e^{pk}(k!)^{1+\beta}.$$

Since $n \neq m$, $a_k \bar{b_k} = 0$, for all $k \ge 0$. Hence $\left(e_n^{(p)}, e_m^{(p)}\right) = 0$ if $n \neq m$. Moreover, the completeness of $\mathcal{H}_p^\beta$ is sufficient to conclude that $\left\{e_n^{(p)}(z) \ : \ n \in \mathbb{N}_0\right\}$ is total in $\mathcal{H}_p^\beta$, that is, $\overline{\text{span}\,\{e_n^{(p)}(z) : \ n \in \mathbb{N}_0\}} = \mathcal{H}_p^\beta$. □

The above result implies that for $0 \le \beta \le 1$, $\mathcal{H}_p^\beta$ is separable Hilbert space for any $p = 0, 1, 2, \ldots$.

Lemma 4 *The Hilbert spaces form a chain of dense continuous embedded spaces*

$$\ldots \subset \mathcal{H}_{p+1}^\beta \subset \mathcal{H}_p^\beta \subset \mathcal{H}_{p-1}^\beta \ldots .$$

Proof Clearly, we can define the identity map from $\mathcal{H}_{p+1}^\beta$ to $\mathcal{H}_p^\beta$ as the inclusion map. Next, let $N_p(0, \epsilon)$ be a neighborhood of zero in $\mathcal{H}_p^\beta$ defined by

$$N_p(0, \epsilon) = \{g \in \mathcal{H}_p^\beta : \|g\|_{p,\beta} < \epsilon\}$$

and let $i_{p,p-1} : \mathcal{H}_p^\beta \to \mathcal{H}_{p-1}^\beta$ be the inclusion map. Then

$$i_{p,p-1}^{-1}N_{p-1}(0,\epsilon) = \{g \in \mathcal{H}_p^\beta : i_{p,p-1}(g) = g \in N_{p-1}(0,\epsilon)\}$$
$$= \{g \in \mathcal{H}_p^\beta : \|g\|_{p-1,\beta} < \epsilon\}$$

and

$$N_p(0,\epsilon) \subset i_{p,p-1}^{-1}N_{p-1}^\beta(0,\epsilon) = \{g \in \mathcal{H}_p^\beta : \|g\|_{p-1,\beta} < \epsilon\}.$$

Hence, $i_{p,p-1}$ is continuous. Finally, let $f \in \mathcal{H}_{p-1}^\beta$, and let $\epsilon > 0$. Since $\overline{\text{span}\,\{e_n^{(p-1)}(z) : n \in \mathbb{N}_0\}} = \mathcal{H}_{p-1}^\beta$, there exists $g \in \text{span}\,\{e_n^{(p-1)}(z) : n \in \mathbb{N}_0\}$ for which $\|f - g\|_{p-1,\beta} < \epsilon$. Note that for any k, $e_k^{(p-1)}(z) = e^{\frac{-(p-1)k}{2}}(k!)^{\frac{1+\beta}{2}} z^k = e^{\frac{k}{2}} e_k^{(p)}(z)$ so that for some n and constants $a_0, a_1, a_2, \ldots, a_n$, $g = \sum_{k=0}^{n} a_k e^{\frac{k}{2}} e_k^{(p)}(z)$. Moreover, $\|g\|_{p,\beta}^2 = \sum_{k=0}^{n} |a_k|^2 e^k < \infty$. Thus $g \in \mathcal{H}_p^\beta$, and hence $\mathcal{H}_p^\beta$ is dense in $\mathcal{H}_{p-1}^\beta$.

Lemma 5 *For $p > q$, the corresponding norms of the Hilbert spaces $\mathcal{H}_p^\beta$ and $\mathcal{H}_q^\beta$ are compatible.*

Proof Clearly, from Lemma 4, the embedding $\mathcal{H}_p^\beta \subset \mathcal{H}_q^\beta$ for $p > q$ is injective and since the inclusion map from $\mathcal{H}_p^\beta$ onto $\mathcal{H}_q^\beta$ is continuous, the corresponding norms are compatible.

Proposition 1 *For $p = 2, 3, \ldots$ and $0 \le \beta \le 1$. The functions f in $\mathcal{H}_p^\beta$ are of at most $\frac{2}{1+\beta}$ order of growth and type $\tau \le \frac{(1+\beta)}{2} e^{\frac{-p}{1+\beta}}$.*

Proof Let $f \in \mathcal{H}_p^\beta$. Then take $0 < \epsilon < 1$. Thus, by the Stirling formula we obtain

$$|a_n|^2 \le \frac{\epsilon}{\left(\sqrt{2\pi}\, n^{n+1/2} e^{-n}\right)^{1+\beta} e^{pn}} \tag{11}$$

for sufficiently large n. Now using (11) to get

$$2\ln|a_n| \le \ln\epsilon - (1+\beta)\ln\sqrt{2\pi} - (1+\beta)(n+\tfrac{1}{2})\ln n - (p-(1+\beta))n$$
$$\le -(1+\beta)n\ln n \;\text{ for some } 0 < \epsilon < 1.$$

It follows from (2) that f has order of growth at most $\frac{2}{1+\beta}$. Now, for $p > 1$, we have the bound

$$|a_n|^{\rho/n} \le \frac{\epsilon^{1/(1+\beta)n}}{(\sqrt{2\pi}\, n^{n+1/2} e^{-n})^{1/n} e^{p/(1+\beta)}}.$$

Thus using (3)

$$\tau \leq \frac{(1+\beta)}{2e} \lim_{n\to\infty} \sup n \frac{\epsilon^{1/(1+\beta)n}}{(\sqrt{2\pi})^{1/n}\, n e^{(1/2n)\ln n} e^{-1} e^{p/(1+\beta)}}$$
$$= \frac{(1+\beta)}{2} e^{\frac{-p}{(1+\beta)}}.$$

This means that f is of order at most $\frac{2}{1+\beta}$ and type no more than $\frac{(1+\beta)}{2}e^{\frac{-p}{1+\beta}}$ for that corresponding order. □

Definition 3 Define the space $\mathcal{E}^\beta$

$$\mathcal{E}^\beta := \operatorname{proj}\lim_{p\to\infty} \mathcal{H}_p^\beta = \bigcap_{p\geq 0} \mathcal{H}_p^\beta,$$

called the projective limit of the spaces $\mathcal{H}_p^\beta$.

In the projective limit topology a neighborhood basis for the linear space $\mathcal{E}^\beta$ is given by

$$U_{p,\epsilon} = \{f \in \mathcal{E}^\beta : \|f\|_{p,\beta} < \epsilon\}, \quad p \in \mathbb{N}, \quad \epsilon > 0. \tag{12}$$

Lemma 6 *The set $U_{p,\epsilon}$ given in (12) is convex, balanced and absorbing local base of $\mathcal{E}^\beta$.*

Proof Let $h \in tU_{p,\epsilon} + (1-t)U_{p,\epsilon}$ and $0 \leq t \leq 1$, then there exists $f, g \in U_{p,\epsilon}$ such that $h = tf + (1-t)g$. Thus, for all $t \in [0, 1]$,

$$\|h\|_{p,\beta} = \|tf + (1-t)g\|_{p,\beta} \leq \|tf\|_{p,\beta} + \|(1-t)g\|_{p,\beta} < \epsilon.$$

Next, let λ be a scalar such that $|\lambda| \leq 1$ and $f \in U_{p,\epsilon}$. Then

$$\|\lambda f\|_{p,\beta} = |\lambda|\ \|f\|_{p,\beta} \leq \|f\|_{p,\beta} < \epsilon.$$

Lastly, let $f \in \mathcal{E}^\beta$ and $\|f\|_{p,\beta} < s\epsilon$ for some $s > 0$. Then

$$\left\|s^{-1} f\right\|_{p,\beta} = s^{-1}\ \|f\|_{p,\beta} < \epsilon.$$

This means that $s^{-1}f \in U_{p,\epsilon}$. Thus $f \in sU_{p,\epsilon}$. Hence, the assertion follows. □

Moreover, the space $\mathcal{E}^\beta$ is a locally convex topological linear space. For more detail on such topological linear spaces see [19–21], and for a related construction see Chap. 3 and Appendix A5 of [11].

Theorem 2 *The space $\mathcal{E}^\beta$ is countably Hilbert and Fréchet space.*

Proof From the compatibility of the above norms, $\mathcal{E}^\beta$ is countably Hilbert and complete. Now, the topology in $\mathcal{E}^\beta$ generated by the metric defined by

$$d(f, g) = \sum_{p=0}^{\infty} \frac{1}{2^p} \frac{\|f - g\|_{p,\beta}}{1 + \|f - g\|_{p,\beta}},$$

which is identical with the original topology generated by the neighborhood basis given in Eq. (12). Thus, $\mathcal{E}^\beta$ is a metrizable space. Since, $\mathcal{E}^\beta$ is locally convex, it follows that it is a Fréchet space. □

Theorem 3 *The linear space $\mathcal{E}^\beta$ is a nuclear space.*

Proof It suffices to show that for all $p \in \mathbb{N}_0$ the embedding

$$i_{r,p} : \mathcal{H}_r^\beta \to \mathcal{H}_p^\beta$$

is of Hilbert-Schmidt type for some $r \in \mathbb{N}_0$. Let $p \in \mathbb{N}_0$ then for some $q \in \mathbb{N}$, define the embedding by the inclusion map

$$i_{p+q,p} : \mathcal{H}_{p+q}^\beta \to \mathcal{H}_p^\beta.$$

Now,

$$\begin{aligned} i_{p+q,p} e_n^{(p+q)}(z) &= \sum_m \big(i_{p+q,p} e_n^{(p+q)}(z), e_m^{(p)}(z)\big)_p e_m^{(p)}(z) \\ &= e^{-qn/2} e_n^{(p)}(z). \end{aligned}$$

Thus

$$\|i_{p+q,p}\|_{p,\beta}^2 = \sum_n e^{-qn} < \infty.$$

□

Theorem 4 *The space $\mathcal{E}^\beta = \operatorname{proj}\lim_{p\to\infty} \mathcal{H}_p^\beta = \bigcap_{p\geq 0} \mathcal{H}_p^\beta$ is the space $\mathcal{E}_{\min}^{\frac{2}{1+\beta}}$ of entire functions of order at most $\rho = \frac{2}{1+\beta}$ and minimal type.*

Proof Let $f(z) \in \bigcap_{p\geq 0} \mathcal{H}_p^\beta$. Then $f(z) \in \mathcal{H}_p^\beta$ for all p. By Proposition 1, for each p, there exists $r_0(p) > 0$ such that the asymptotic inequality holds

$$\max_{|z|=r} |f(z)| < \exp\left(\frac{1+\beta}{2} e^{\frac{-p}{1+\beta}} |z|^{\frac{2}{1+\beta}}\right)$$

for all $|z| > r_0(p)$. Given $\epsilon > 0$, choose p such that $\frac{1+\beta}{2} e^{\frac{-p}{1+\beta}} < \epsilon$. Then the following asymptotic inequality holds,

$$\max_{|z|=r} |f(z)| < \exp\left(\epsilon |z|^{\frac{2}{1+\beta}}\right)$$

for all $|z| > r_0(p)$. Conversely, consider now $f(z) = \sum_{n=0}^{\infty} a_n z^n$, an entire function of order growth at most $\rho = \frac{2}{1+\beta}$ and minimal type. Let $p \geq 0$. The first part of Lemma 1 implies that for any $A > 0$

$$|a_n| \overset{as}{<} \left(\frac{2eA}{n(1+\beta)}\right)^{\frac{n(1+\beta)}{2}}.$$

Hence, using Stirling's formula

$$e^{np}(n!)^{1+\beta}|a_n|^2 \overset{as}{<} e^{np}(n^n e^{-n}\sqrt{2\pi n})^{1+\beta}\left(\frac{2eA}{n(1+\beta)}\right)^{(1+\beta)n}$$

and

$$\|f\|_p^2 = \sum_n e^{np}(n!)^{1+\beta}|a_n|^2 < \sum_n (2\pi n)^{\frac{1+\beta}{2}} (e^p(2A)^{1+\beta})^n.$$

Choose A for which $e^p(2A)^{1+\beta} < 1$. Hence $f \in \mathcal{H}_p^\beta$. □

The reader may find it interesting to explore the dual spaces of the family of Hilbert spaces $\{\mathcal{H}_p^\beta\}_{p\geq 0}$, which can be easily deduced from the above construction. Also, the inductive limit of the dual spaces is straightforward to obtain.

References

1. Berezansky, Y., Kondratiev, Y.: Spectral Methods in Infinite Dimensional Analysis. Naukova Dumka, Kiev (1988) English translation. Kluwer Academic Publishers, Dordrecht (1995)
2. Kondratiev, Y.: Wick powers of Gaussian generalizes random processes. In: Methods of Functional Analysis in Problems of Mathematical Physics, pp. 129–158. Institute of Mathematics, NANU, Kiev (1978)
3. Kondratiev, Y.: Nuclear spaces of entire functions in problems of infinite dimensional analysis. Soviet Math. Dokl. **254**(1), 1325–1329 (1980)
4. Kondratiev, Y.: Spaces of infinite dimensional entire functions connected with Fock space riggings. In: Spectral Analysis of Differential Operators, pp. 25–53. Institute of Mathematics, Kiev (1980)
5. Kondratiev, Y., Samoilenko, Y.: Spaces of test and generalized functions of infinite number of variables. Rep. Math. Phys. **14**(3), 325–350 (1978)
6. Kondratiev, Y., Streit, L.: Spaces of white noise distributions: constructions, descriptions, applications I. Rep. Math. Phys. **33**, 341–366 (1993)
7. Kondratiev, Yu.G., Streit, L., Westerkamp, W., Yan, J.: Generalized functions in infinite dimensional analysis. Hiroshima Math. J. **28**, 213–260 (1998)
8. Kondratiev, Y., Leukert, P., Potthoft, J., Streit, L., Westerkamp, W.: Generalized functionals in Gaussian spaces: the characterization theorem revisited. J. Funct. Anal. **141**, 301–318 (1996)
9. Hida, T.: Analysis of Brownian Functionals, Carleton Math. Lecture Notes, no. **13**, Carleton University, Ottawa (1975)
10. Hida, T.: Browian Motion, Applied Mathematics, vol. 11. Springer, New York (1980)

11. Hida, T., Kuo, H., Potthoft, J., Streit, L.: White Noise. An infinite Dimensional Calculus. Kluwer Academic Publishers, Dordrecht (1993)
12. Berezanskii, Y.: Selfadjoint Operators in Spaces of Functions of Many Variables. American Mathematical Society, Providence, RI (1986)
13. Berezansky, Y., Samoilenko, Y.: Nuclear spaces of functions of infinitely many variables. Ukrain. Mat. Zh. **25**(6), 719–732 (1973)
14. Minlos, R.: Rigged Hilbert Space, Encyclopedia of Mathematics. EMS Press (2001)
15. Jamil, F., Kondratiev, Y., Menchavez, S., Streit, L.: Automorphisms generated by umbral calculus on a nuclear space of entire test functions. Methods Funct. Anal. Topology **24**(4), 339–348 (2018)
16. Kachanovsky, N.: On Kondratiev spaces of test functions in the non-Gaussian infinite-dimensional analysis. Methods Funct. Anal. Topology **19**(4), 301–309 (2013)
17. Kubo, I., Kuo, H., Sengupta, A.: White noise analysis on a new space of Hida distributions. Infin. Dimens. Anal. Quantum Probab. Relat. Top. **2**, 315–335 (1999)
18. Ferreira, J., Gouveia, D., Reis, M., Silva J.l.: Funcões teste e funcões generalizadas em dimensão 1. Descricão e caracterizacão. Boletim da SPM **52**, 59–75 (2005)
19. Gelfand, I., Shilov, G.: Generalized Functions II. Academic Press, N.Y. (1968)
20. Gelfand, I., Vilenkin, Y.: Generalized Functions IV. Academic Press, N.Y. (1964)
21. Obata, N.: White Noise Calculus and Fock Space, LNM, vol. 1577. Springer, Berlin, Heidelberg, and New York (1994)
22. Boas, R.: Entire Functions. Academic Press Inc., N.Y. (1954)
23. Levin, B.: Lectures on Entire Functions. Mathematical Monographs, vol. 150. American Mathematical Society (1996)

Extremal Mild Solutions of Hilfer Fractional Impulsive Systems

Divya Raghavan and N. Sukavanam

Abstract The well-established monotone iterative technique that is used to study the existence and uniqueness of fractional impulsive system is extended to Hilfer fractional order in this paper. The results are derived by using the method of upper and lower solution and Gronwall inequality. Also, conditions on non-compactness of measure are used effectively to prove the main result.

Keywords Upper and lower solutions · Hilfer fractional derivative · Non-compactness measure

1 Literature Motivation

Over the years, the urge of finding the extremals of a function evolved in many problems, especially in, geometry, history and mechanics. Du and Lakshmikantham [5] investigated the initial value problem given as,

$$x' = g(t, x); \quad x(0) = x_0$$

in the Banach space E with norm $\|\cdot\|$, where $x_0, x, g \in E$ and developed a monotone iterative technique to find the existence of the extremal solutions. Ladde et al. [12] rendered a substantial theory of monotone method using upper and lower solutions for nonlinear equations in their monograph. The basic discussion in this monograph focused on the first-and second-order partial differential equation. The authors of this monograph constructed two monotone sequences on the basis of quasi-monotone property. Further they claim that the converging limits $\bar{x}$ and $\underline{x}$ of the two sequences are same for the parabolic system and left an open problem regarding elliptic systems.

D. Raghavan (✉) · N. Sukavanam
Department of Mathematics, I.I.T Roorkee, Roorkee, India
e-mail: divyar@ma.iitr.ac.in

N. Sukavanam
e-mail: n.sukavanam@ma.iitr.ac.in

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_5

Nieto and Cabada [19] studied in detail the existence of extremal solution of periodic boundary value second-order system with boundary conditions,

$$-x'' = g(t, x);\;\; x(0) = x(2\pi);\;\; x'(0) = x'(2\pi).$$

Here the authors utilized monotone iterative procedures without the usual required boundary conditions $\alpha'(0) \geq \alpha'(2\pi)$, $\beta'(0) \leq \beta'(2\pi)$, where α and β are lower and upper solutions. For both initial value problem and boundary value problem, the impulsive system finds itself a significant role in the past as well as in the present. Almost in all physical problems, the movement of the state of the system is discontinuous. Hence, widening the idea of monotone iterative technique to impulsive system was unavoidable. Liz and Nieto [15] extended this approach to impulsive periodic boundary value second-order system. The authors had put forward a maximum principle exclusively for impulsive functions.

Due to the advantage over integer order in many practical problems, fractional calculus is more appreciated in the past few decades. Basic theory related to fractional calculus and its applications are available in the literature in numerous research articles, books, and monographs. For interested readers, [21] by Podlubny, [11] by Kilbas et al. and [22] by Stamova and Stamov for impulsive systems can be referred. Due to the fact that the systems with fractional order is inevitable, the study of existence and uniqueness of extremals of differential system using monotone iterative method for fractional system is vital. Lakshmikantham and Vatsala [13] enhanced the monotone iterative theory to fractional order initial value problem given by

$$D^{\mu}(x - x(0)) = f(t, x);\;\; x(0) = x_0,$$

where $0 < \mu < 1$, $f \in C[\mathbb{R}_0, \mathbb{R}]$. Here, $\mathbb{R}_0 = (t, x) : 0 \leq t \leq \alpha$ and $|x - x_0| \leq \beta$. Here α and β are lower and upper solutions. Subsequently, many researchers focused their interest in finding the extremals using upper and lower solution method along with monotone iterative technique for both initial value and boundary value problem of fractional order differential equations. McRae [16] discussed the existence result using fractional monotone iterative method exclusively for differential system with Riemann-Liouville fractional order. Several authors including Denton and Vatsala [4] on finite fractional system, Liu et al. [14] on fractional integral systems with advanced arguments and Wang [27] on boundary value fractional system with deviating arguments studied the existence and uniqueness using this method. Recently Agarwal et al. [1] analyzed various cases of upper and lower solutions with initial time differences and discussed the different algorithms for distinct cases, some cases using Mittag-Leffler functions and some cases using mathematical software. The work of Mu cannot be excluded. His solitary work [17] and the work along with Li [18] on monotone iterative technique for impulsive fractional system given by

$$\begin{cases} D^{\alpha}x(t) + Ax(t) = g(t, x(t)),\;\; t \in J,\;\; t \neq t_k, \\ \Delta x|_{t=t_k} = J_k(x(t_k)),\;\; k = 1, 2, \ldots n, \\ x(0) = x_0, \end{cases}$$

using non-compactness measure and generalized Gronwall inequality is noteworthy. Here D^{α} is the Caputo fractional derivative of order $0 < \alpha < 1$, $-A$ an infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ and g and J_k are continuous functions. Zhang and Liang [29] employed monotone iterative technique in the presence of coupled lower and upper L-quasi solution and Sadovskii's fixed point theorem.

In this regard, Gou and Li [8] investigated the existence of extremal solution with the aid of lower and upper solution method for Hilfer fractional differential system. Driven by the fact that the monotone technique has not reached the impulsive Hilfer fractional differential system, this paper is projected to bridge the void. Therefore, in this paper an impulsive system with Hilfer fractional derivative is considered as follows:

$$\begin{cases} {}_{t_0}D_t^{\mu,\nu}x(t) + Ax(t) = g(t, x(t)), & t \in J, \ t \neq t_k \\ \Delta I_{t_k}^{(1-\lambda)}x(t_k) = \phi_k(x(t_k)), & k = 1, 2, \dots l \\ I_{t_{0+}}^{(1-\lambda)}[x(t)]_{t=0} = x_0. \end{cases} \tag{1}$$

Here, $D_0^{\mu,\nu}$ denotes the Hilfer fractional derivative of order $0 < \mu < 1$, type $0 \leq \nu \leq 1$ and $\lambda = \mu + \nu - \mu\nu$. $-A$ is the infinitesimal generator of an analytic-semigroup of uniformly bounded linear operators $Q(t)(t \geq 0)$ on a Banach space E, and for $M \geq 1$, $\sup_{t\in[0,\infty)} |Q(t)| \leq M$. If the impulse effect occurs at $t = t_k$, for $(k = 1, 2, \dots, l)$, then $\phi_k : E \to E$ is the mapping of the solution before the impulse effect, $x(t_k^-)$, to after the impulse effect, $x(t_k^+)$. It determines the size of the jump at time t_k. In other words, the impulsive moments meet the relation $\Delta I_{t_k}^{1-\lambda}x(t_k) = I_{t_k^+}^{1-\lambda}x(t_k^+) - I_{t_k^-}^{1-\lambda}x(t_k^-)$, where $I_{t_k^+}^{1-\lambda}x(t_k^+)$ and $I_{t_k^-}^{1-\lambda}x(t_k^-)$ denotes the right and the left limit of $I_{t_k}^{1-\lambda}x(t)$ at $t = t_k$ with $0 = t_0 < t_1 \dots < t_l < t_{l+1} = T$. In the given impulsive system, let $J = [0, T]$ and $J' = J \backslash \{t_1, t_2, t_3, \dots, t_l\}$, for $T > 0$ and g is a continuous nonlinear operator such that $g : J \times E \to E$.

The rest of the paper is framed as follows: Sect. 2 gives a revisit to definitions on fractional calculus and certain necessary basic theorems. Section 3 includes the proof of the main theorem and few other results related to the existence of extremal solutions and Sect. 4 gives the conclusion of the paper.

2 Essential Notions

This section covers the basic results, definitions, and theorems that are essential throughout this paper.

Definition 1 [5] In an ordered Banach space E, let N be a proper subset of E. Then N is said to be a cone if for $\eta \geq 0$, $\eta N \subset N$, $N + N \subset N$, $N \cap (-N) = \{0\}$ and $N = \overline{N}$ where $\overline{N}$ denotes the closure of N.

Definition 2 [5] A cone N is said to be normal if there exists a real number $D > 0$ such that for $0 \leq y \leq z$ implies $\| \cdot \| \leq D\|z\|$. Here D is independent of y and z.

For detailed definition and explanation regarding positive cone of an ordered Banach space the reader may refer to [5]. Let the space of all continuous functions from J to E be denoted by $C(J, E)$, where E is an ordered Banach space with partial order $\leq$, norm $\|\cdot\|$ and whose positive cone $N = \{x \in E : x(t) \geq \theta\}$ is normal with normal constant D, where θ is the zero element of E. $C(J, E)$ is also an ordered Banach space with norm stated as $\|x\|_C = \max \|x(t)\|$. Apparently, $PC(J, E)$ is an ordered Banach Space along with the norm $\|x\|_{PC} = \sup_{t\in J} \|x(t)\|$. Also, $PC_{1-\lambda}$ is an ordered Banach space with partial order $\leq$, defined as $PC_{1-\lambda}(J, E) = \{x \in PC(J, E) : (t - t_k)^{1-\lambda} x(t) \in PC(J, E)\}$ with norm $\|x\|_{PC} = \sup_{t\in J} \|(t - t_k)^{1-\lambda} x(t)\|$ whose positive cone $N_{PC_{1-\lambda}} = \{x \in PC_{1-\lambda}(J, E) : x \geq \theta\}$ is normal with the same normal constant D. From the defined system (1), $x(t)$ is continuous in each J_k, where $J_k = (t_k, t_{k+1}]$, for $k = 1, 2, \dots, l$ with $t_0 = 0$ and $t_{l+1} = T$.

The fractional integral of order μ and for an integrable function g is given as [21],

$$I_t^{\mu} g(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t - s)^{\mu-1} g(s) ds, \quad 0 < \mu < 1.$$

Here $\Gamma(\cdot)$ is the gamma function. Also, the fractional derivative of the two classical derivatives, Caputo and Riemann-Liouville of order μ, respectively is given by [21],

$${}^{C}D_{0+}^{\mu} g(t) = \frac{1}{\Gamma(1-\mu)} \int_0^t \frac{g'(s)}{(t-s)^{\mu}} ds, \;\; t > 0, \;\; 0 < \mu < 1,$$

and

$${}^{L}D_{0+}^{\mu} g(t) = \frac{1}{\Gamma(1-\mu)} \left(\frac{d}{dt}\right) \int_0^t \frac{g(s)}{(t-s)^{\mu}} ds, \;\; t > 0, \;\; 0 < \mu < 1.$$

The Hilfer fractional derivative of order $0 < \mu < 1$ and type $0 \leq \nu \leq 1$ of function $g(t)$ is defined by Hilfer [10],

$$D_{0+}^{\mu,\nu} g(t) = I_{0+}^{\nu(1-\mu)} D I_{0+}^{(1-\nu)(1-\mu)} g(t)$$

where $D := \frac{d}{dt}$. The existence of solution for fractional system with Hilfer fractional derivative which was established by Furati et al. in [6] and Gu and Trujillo in [7] unlatched the flow of research on differential system with Hilfer fractional derivative. Riemann-Liouville and Caputo can be regarded as a special case of Hilfer fractional derivative, respectively as

$$D_{0+}^{\mu,\nu} = \begin{cases} D I_{0+}^{1-\mu} = {}^{L}D_{0+}^{\mu}, & \nu = 0 \\ I_{0+}^{1-\mu} D = {}^{C}D_{0+}^{\mu}, & \nu = 1. \end{cases}$$

The parameter λ satisfies $\lambda = \mu + \nu - \mu\nu, \;\; 0 < \lambda \leq 1$.

For $y, z \in PC_{1-\lambda}(J, E)$, the interval $[y, z] = \{x \in PC_{1-\lambda}(J, E) | y \leq x \leq z\}$ is ordered in $PC_{1-\lambda}(J, E)$ for $y \leq z$ and $[y(t), z(t)] = \{w \in E \; | y(t) \leq w \leq z(t), \;\; t \in J\}$. Fix $C^{\mu,\nu}(J, E) = \{x \in C(J, E) | D^{\mu,\nu} x$ exists and $D^{\mu,\nu} x \in C(J, E)\}$. The graph norm or the A-norm of the Banach space denoted by E_A, $dom(A)$ is defined as $\|\cdot\|_A = \|\cdot\|_E + \|A(\cdot)\|_E$. If any $x \in PC_{1-\lambda}(J, E) \cap C^{\mu,\nu}(J', E) \cap C(J', E_A)$ satisfies all the equalities of (1), then such an abstract function is said to be the solution of (1).

Definition 3 [8] If $z_0 \in PC_{1-\lambda}(J, E) \cap C^{\mu,\nu}(J', E) \cap C(J', E_A)$ satisfies all the inequalities of

$$\begin{cases} {}_{t_0}D_t^{\mu,\nu} z_0(t) + Az_0(t) \geq g(t, z_0(t)), \;\; t \in J, \;\; t \neq t_k \\ \Delta I_{t_k}^{(1-\lambda)} z_0(t)|_{t=t_k} \geq \phi_k(x(t_k)), \;\; k = 1, 2, \ldots l \\ I_{t_{0+}}^{(1-\lambda)}[z_0(t)]_{t=0} \geq x_0 \end{cases}$$

then z_0 is called the upper solution of the problem (1).

Remark 1 [8] If all the inequalities of Definition 3 are satisfied by y_0 in the reverse order, then it is a lower solution of the problem (1).

Definition 4 [18]

An operator family $Q(t) : E \to E$ for $t \geq 0$ is supposedly positive if, for any $u \geq N$ and $t \geq 0$, the inequality $Q(t)u \geq \theta$ holds.

It can be referred [9] that the Kuratowski measure of non-compactness measure denoted by $\alpha(\cdot)$ is defined on a bounded set. For any $t \in J$ and $B \subset C(J, E)$, define $B(t) = \{x(t) : x \in B\}$. If B is bounded in $C(J, E)$, then B is bounded in E. Also, $\alpha(B(t)) \leq (B)$.

The following two lemmas are imperative for the proof of the main theorem in the next section.

Lemma 1 [9] *Let $B_p = \{x_p\} \subset C(J, E)$, $(p = 1, 2, \ldots)$ be a bounded and countable set. Then, $\alpha(B_p(t))$ is Lebesgue integral on J. And*

$$\alpha\left(\left\{\int_J x_p(t)dt|_{p=1,2,\ldots,}\right\}\right) \leq 2\int_J \alpha(B_p(t))dt.$$

The subsequent lemma is with reference to the generalized Gronwall inequality for fractional differential equation.

Lemma 2 [28] *Suppose $b \geq 0$, $\beta > 0$ and $a(t)$ is a nonnegative function locally integrable on $0 \leq t < T$ (some $T \leq +\infty$), and suppose $x(t)$ is nonnegative and locally integrable on $0 \leq t < T$ with*

$$x(t) \leq a(t) + b\int_0^t (t-s)^{\beta-1} x(s)ds$$

on this interval; then

$$x(t) \leq a(t) + \int_0^t \Big[\sum_{n=1}^{\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} a(s)\Big] ds, \;\; 0 \leq t < T.$$

Definition 5 [3] A function $x \in PC_{1-\lambda}(J, E)$ is called the mild solution of system (1), if for $t \in J$ it satisfies the following integral equation

$$x(t) = S_{\mu,\nu}(t)x_0 + \sum_{i=1}^{k} S_{\mu,\nu}(t-t_i)\phi_i(x(t_i)) + \int_0^t (t-s)^{\mu-1} P_\mu(t-s)g(s, x(s))ds \tag{2}$$

where,

$$S_{\mu,\nu}(t) = I_{0+}^{\nu(1-\mu)} P_\mu(t), \ \ P_\mu(t) = \int_0^\infty \mu\theta\xi_\mu(\theta)Q(t^\mu\theta)d\theta,$$

$$\varpi_\mu(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\mu-1}\frac{\Gamma(n\mu+1)}{n!}\sin(n\pi\mu), \ \ \theta \in (0, \infty)$$

and $\xi_\mu(\theta) = \frac{1}{\mu}\theta^{-1-\frac{1}{\mu}}\varpi_\mu(\theta^{-\frac{1}{\mu}})$ is a probability density function defined on $(0, \infty)$, that is

$$\xi_\mu(\theta) \geq 0 \text{ and } \int_0^\infty \xi_\mu(\theta)d\theta = 1.$$

Remark 2 1. From [6], when $\nu = 0$, the solution reduces to the solution of classical Riemann-Liouville fractional derivative, that is, $S_{\mu,0}(t) = P_\mu(t)$.
2. Similarly when $\nu = 1$, the solution reduces to the solution of classical Caputo fractional derivative, that is $S_{\mu,1(t)} = S_\mu(t)$.

Lemma 3 [3] *If the analytic semigroup $Q(t)(t \geq 0)$ is bounded uniformly, then the operator, $P_\mu(t)$ and $S_{\mu,\nu}(t)$ satisfies the following bounded and continuity conditions. $S_{\mu,\nu}(t)$ and $P_\mu(t)$ are linear bounded operators and for any $x \in E$*

$$\|S_{\mu,\nu}(t)x\|_E \leq \frac{Mt^{\lambda-1}}{\Gamma(\lambda)}\|x\|_E \text{ and } \|P_\mu(t)x\|_E \leq \frac{M}{\Gamma(\mu)}\|x\|_E.$$

3 Main Results

To prove the main theorem of this paper, an equivalent system given below is discussed. The perturbed equivalent system is valid as the constant $C \geq 0$.

$$\begin{cases} {}_{t_0}D_t^{\mu,\nu}x(t) + (A + CI)x(t) = g(t, x(t)) + Cx(t), \ \ t \in J, \ \ t \neq t_k \\ \Delta I_{t_k}^{(1-\lambda)}x(t_k) = \phi_k(x(t_k)), \ \ k = 1, 2, \ldots l \\ I_{t_{0+}}^{(1-\lambda)}[x(t)]_{t=0} = x_0. \end{cases} \tag{3}$$

Remark 3 1. With reference to [20], for any $C \geq 0$, $-(A + CI)$ generates an analytic semigroup $R(t) = e^{-Ct}Q(t)$ and for $t \geq 0$, $R(t)$ is positive and $\sup_{t\in[0,\infty)} \|R(t)\| \leq M^*$ for $M^* \geq 1$.

2. Let $S^*_{\mu,\nu}(t)$ and $P^*_\mu(t)$ for $t \geq 0$ be two families of operators defined by

$$S^*_{\mu,\nu}(t) = I_{0+}^{\nu(1-\mu)} P^*_\mu(t), \;\; P^*_\mu(t) = \int_0^\infty \mu\theta\xi_\mu(\theta)R(t^\mu\theta)d\theta.$$

3. The above two operators are positive for $(t \geq 0)$ and for any $x \in E$,

$$\|S^*_{\mu,\nu}(t)\| \leq \frac{M^* t^{\lambda-1}}{\Gamma(\lambda)} \text{ and } \|P^*_\mu(t)\| \leq \frac{M^*}{\Gamma(\mu)}.$$

Definition 6 A function $x \in PC_{1-\lambda}(J, E)$ is said to be a mild solution of the problem (3) if for any $x \in PC_{1-\lambda}(J, E)$, the integral equation

$$x(t) = S^*_{\mu,\nu}(t)x_0$$

$$+\sum_{i=1}^{k} S^*_{\mu,\nu}(t-t_i)\phi_i(x(t_i)) + \int_0^t (t-s)^{\mu-1} P^*_\mu(t-s)\Big[g(s, x(s)) + Cx(s)\Big]ds.$$

The following theorem guarantees the existence of the extremal mild solution of the impulsive system (1).

Theorem 1 *Let E be an ordered Banach space with the positive cone N. Assume that $Q(t) \geq 0$ and the impulsive system (1) has both lower and upper solution, given by y_0 and z_0 respectively, where $y_0, z_0 \in PC_{1-\lambda}$ and $y_0 \leq z_0$. By adopting the monotone iterative procedure and presuming the following conditions, the impulsive system (1) has the extremal solution between y_0 and z_0.*

- $[A(1)]$*:- For $x \in [y_0(t), z_0(t)]$ the function $g(t, x) + Cx$ is increasing in x, precisely, there exists a constant $C \geq 0$ such that*

$$g(t, x_2) - g(t, x_1) \geq -C(x_2 - x_1)$$

  and $y_0(t) \leq x_1 \leq x_2 \leq z_0(t)$ for any $t \in J$.
- $[A(2)]$*: For $x \in [y_0(t), z_0(t)]$, the impulsive function is increasing. It implies*

$$\phi_k(x_1) \leq \phi_k(x_2), \;\; k = 1, 2, \ldots, l.$$

- $[A(3)]$*: The sequence $\{x_p\} \subset [y_0(t), z_0(t)]$, for $t \in J$ is either decreasing or increasing monotonic sequence, in particular, there exists a constant $L \geq 0$ such that*

$$\alpha\Big(\{g(t, x_p)\}\Big) \leq L\alpha\Big(\{x_p\}\Big), \;\; p = 1, 2, \ldots, .$$

Proof As $C > 0$, the problem (1) can be presented in the form of problem (3). So, it is sufficient to prove the existence of a unique solution of the problem (3). For a fixed x_0, define the operator $\mathcal{G} : [y_0, z_0] \to PC_{1-\lambda}(J, E)$ by

$$(\mathcal{G}x)(t) = \begin{cases} S^*_{\mu,\nu}(t)x_0 + \int_0^t (t-s)^{\mu-1} P^*_\mu(t-s)\Big[g(s,x(s)) + Cx(s)\Big]ds, \ \ t \in [0,t_1] \\ S^*_{\mu,\nu}(t)x_0 + \sum_{i=1}^{k} S^*_{\mu,\nu}(t-t_i)\phi_i(x(t_i)) \\ +\int_0^t (t-s)^{\mu-1} P^*_\mu(t-s)\Big[g(s,x(s)) + Cx(s)\Big]ds, \\ t \in (t_k, t_{k+1}], \ \ k = 1, 2, \ldots l. \end{cases} \tag{4}$$

The map $\mathcal{G}(x)(t)$ is continuous since g is continuous. By the Definition 5, the fixed points of the operator $\mathcal{G}$ are equivalent to the mild solution of the system given in (2). It means,

$$\mathcal{G}x(t) = x(t). \tag{5}$$

Now it is to be proved that the operator $\mathcal{G}$ is an increasing monotonic operator. The following steps lead to the completion of the proof.
Step1:- To show $\mathcal{G}(x_1) \leq \mathcal{G}(x_2)$:- The condition $A(1)$, can be presented in the following ways, which can be directly used in the proof. That is $\forall t \in J^{'}$,

$$\begin{aligned} y_0(t) \leq x_1(t) \leq x_2(t) \leq z_0(t). \\ g(t, x_1(t)) + Cx_1(t) \leq g(t, x_2(t)) + Cx_2(t). \end{aligned} \tag{6}$$

Considering the case for $t \in J_0^{'}$, for $J_0^{'} = [0, t_1]$:- As the operators $S^*_{\mu,\nu}(t)$ and $P^*_\mu(t)$ are positive operators, when the mild solutions are compared, using (6), the following inequality is obtained.

$$\int_0^t (t-s)^{\mu-1} P^*_\mu(t-s)\Big[g(s,x_1(s)) + Cx_1(s)\Big]ds \leq$$

$$\int_0^t (t-s)^{\mu-1} P^*_\mu(t-s)\Big[g(s,x_2(s)) + Cx_2(s)\Big]ds.$$

In which case, for $\forall t \in J_k^{'}$, with $J_k^{'} = (t_k, t_{k+1}], k = 1, 2, \ldots l$, applying the condition $A(2)$ yields

$$\begin{aligned} S^*_{\mu,\nu}(t)x_1(0) &+ \sum_{i=1}^{k} S^*_{\mu,\nu}(t-t_i)\phi_i(x_1(t_i)) \\ &+ \int_0^t (t-s)^{\mu-1} P^*_\mu(t-s)\Big[g(s,x_1(s)) + Cx_1(s)\Big]ds \leq \\ S^*_{\mu,\nu}(t)x_2(0) &+ \sum_{i=1}^{k} S^*_{\mu,\nu}(t-t_i)\phi_i(x_2(t_i)) \\ &+ \int_0^t (t-s)^{\mu-1} P^*_\mu(t-s)\Big[g(s,x_2(s)) + Cx_2(s)\Big]ds. \end{aligned}$$

Eventually, $\mathcal{G}(x_1) \le \mathcal{G}(x_2)$.
Step2:- To show $y_0 \le \mathcal{G}(y_0)$; $\mathcal{G}(z_0) \le z_0$:-

For the case for which $t \in J_0^{'}$:-

Let $D^{\mu,\nu}z_0(t) + Az_0(t) + Cz_0(t) = \xi(t)$. By the Definition 3 of the upper solution, the mild solution of the system (1) can be written as

$$\begin{aligned} z_0(t) &= S^*_{\mu,\nu}(t)z_0(0) + \int_0^t (t-s)^{\mu-1}P^*_\mu(t-s)\xi(s)ds \\ &\ge S^*_{\mu,\nu}(t)x_0 + \int_0^t (t-s)^{\mu-1}P^*_\mu(t-s)\Big[g(s,z_0(s)) + Cz_0(s)\Big]ds \end{aligned}$$

From (4), it can be observed that $z_0(t) \ge \mathcal{G}(z_0)$.
For $t \in J_1^{'}$:-

$$\begin{aligned} z_0(t) &= S^*_{\mu,\nu}(t)z_0(0) + S^*_{\mu,\nu}(t-t_1)\phi_1(z_0(t_1)) + \int_0^t (t-s)^{\mu-1}P^*_\mu(t-s)\xi(s)ds \\ &\ge S^*_{\mu,\nu}(t)x_0 + S^*_{\mu,\nu}(t-t_1)\phi_1(z_0(t_1)) \\ &+ \int_0^t (t-s)^{\mu-1}P^*_\mu(t-s)\Big[g(s,z_0(s)) + Cz_0(s)\Big]ds. \end{aligned}$$

Hence $z_0(t) \ge \mathcal{G}(z_0)$. Progressing in the same way, for every $J_k^{'}$, yields, in general $z_0(t) \ge \mathcal{G}(z_0)$. In the same manner, it can be proved that $y_0(t) \le \mathcal{G}(y_0)$. Altogether, it can be deduced that

$$y_0(t) \le \mathcal{G}(y_0) \le \mathcal{G}(x) \le \mathcal{G}(z_0) \le z_0(t).$$

Whereby the conclusion may be drawn that $\mathcal{G} : [y_0, z_0] \to PC_{1-\lambda}(J, E)$ is an increasing monotonic operator. Through the iterative pattern, two sequence $\{y_p\}$ and $\{z_p\}$ can be defined as,

$$y_p = \mathcal{G}(y_{p-1}); \;\; z_p = \mathcal{G}z_{p-1}; \;\; p = 1, 2, \dots. \tag{7}$$

Eventually, due to the monotonicity property of $\mathcal{G}$, an increasing sequence is derived as,

$$y_0 \le y_1 \le y_2 \le \dots \le y_p \le \dots \le z_p \le \dots \le z_2 \le z_1 \le z_0. \tag{8}$$

Step3:- Convergence of sequences $\{y_p\}$ and $\{z_p\}$ in $J^{'}$:-

Let $B_p = \{y_p | p \in \mathbb{N}\}$ and $B_{p-1} = \{y_{p-1} | p \in \mathbb{N}\}$. The pattern (7) gives the relation $B_p = \mathcal{G}(B_{p-1})$ and as B_{p-1} can be written as $B_{p-1} = B_p \cup \{y_0\}$ for $t \in J^{'}$, it follows that $\alpha(B_{p-1}(t)) = \alpha(B_p(t))$. Let $\psi(t) := \alpha(B_p(t))$. By proving that $\psi(t) \equiv 0$ on every interval $J_k^{'}$, it means that $\alpha(B(t_k)) \equiv 0$ for $k = 1, 2, \dots, l$, and hence $\{y_p\}$

is precompact in E for every $t \in J$. Ultimately, by the definition of precompact,$\{y_p\}$ has a converging subsequence in E. Thus it is necessary to prove that $\psi(t) \equiv 0$.

For $t \in J_0^{'}$ for $J_0^{'} = (0, t_1]$:-

$\psi(t) = \alpha(B_p(t)) = \alpha(\mathcal{G}B_{p-1}(t))$

$$= \alpha\left(\left\{\int_0^t (t-s)^{\mu-1} P_\mu^*(t-s)\Big[g(s, y_{p-1}(s)) + Cy_{p-1}(s)\Big]ds\right\} : p = 1, 2, \ldots\right)$$

By using Lemma 1 gives

$$\psi(t) \le 2\int_0^t \alpha\left(\left\{(t-s)^{\mu-1} P_\mu^*(t-s)\Big[g(s, y_{p-1}(s)) + Cy_{p-1}(s)\Big]ds\right\} : p = 1, 2, \ldots\right)$$

Applying the presumed conditions along with Lemma 3 results in

$$\begin{aligned}\psi(t) &\le \frac{2M^*}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\left[(L+C)\alpha(B_{p-1}(s))\right]ds\\ &= \frac{2M^*}{\Gamma(\mu)}(L+C)\int_0^t (t-s)^{\mu-1}\psi(s)ds.\end{aligned}$$

By Lemma 2, $\psi(t) \equiv 0$ on $J_0^{'}$. Since this holds true for all $t \in J_0^{'}$, in particular it holds for $t = t_1$. Hence $\alpha(B_p(t_1)) = \alpha(B_{p-1}(t_1)) = \psi(t_1) = 0$. Therefore $B_P(t_1)$ and $B_{p-1}(t_1)$ are precompact and subsequently $\phi(B_{p-1}(t_1)) = 0$. Now for $t \in J_1^{'}$, for $J_1^{'} = (t_1, t_2]$:-

$$\begin{aligned}\psi(t) &= \alpha(B_p(t)) = \alpha(\mathcal{G}B_{p-1}(t))\\ &= \alpha\Bigg(\Big\{S_{\mu,\nu}^*(t)y_{p-1}(t_1) + S_{\mu,\nu}^*(t)\phi_1(y_{p-1}(t_1))\\ &\quad + \int_0^t (t-s)^{\mu-1} P_\mu^*(t-s)\Big[g(s, y_{p-1}(s)) + Cy_{p-1}(s)\Big]ds\Big\} : p = 1, 2, \ldots\Bigg)\\ &\le \frac{M^* b^{1-\lambda}}{\Gamma(\lambda)}[\alpha B_{p-1}(t_1)] + \frac{2M^*}{\Gamma(\mu)}(L+C)\int_0^t (t-s)^{\mu-1}\psi(s)ds\\ \psi(t) &\le \frac{2M^*}{\Gamma(\mu)}(L+C)\int_0^t (t-s)^{\mu-1}\psi(s)ds.\end{aligned}$$

By Lemma 2 $\psi(t) \equiv 0$ on $J_1^{'}$. Proceeding the same process interval by interval, it can be proved that $\psi(t) \equiv 0$ on every interval $J_k^{'}, k = 1, 2, \ldots l$. Thus $\{y_p\}$ is precompact and eventually for $p = 1, 2, \ldots,$ $\{y_p\}$ has a converging subsequence and from (8), it can be observed that $\{y_p\}$ is itself is a converging sequence and hence there exists $\underline{x}(t) \in E$ such that $\{y_p\} \to \underline{x}(t)$ as $p \to \infty$, for every $t \in J$. By the Definition (4) the operator $\mathcal{G}$ and the fact that $y_p = \mathcal{G}y_{p-1}$, it can be written as

$$y_p(t) = \begin{cases} S^*_{\mu,\nu}(t)x_0 + \int_0^t (t-s)^{\mu-1} P^*_\mu(t-s)\Big[g(s, y_{p-1}(s)) + Cy_{p-1}(s)\Big]ds, \\ \quad \text{for } t \in [0, t_1] \\ S^*_{\mu,\nu}(t)x_0 + \sum_{i=1}^{k} S^*_{\mu,\nu}(t-t_i)\phi_i(y_{p-1}(t_i)) \\ \quad + \int_0^t (t-s)^{\mu-1} P^*_\mu(t-s)\Big[g(s, y_{p-1}(s)) + Cy_{p-1}(s)\Big]ds, \\ \quad \text{for } t \in (t_k, t_{k+1}], \ \ k = 1, 2, \ldots l. \end{cases}$$

Using Lebesgue dominated convergence theorem, as $p \to \infty$

$$\underline{x}(t) = \begin{cases} S^*_{\mu,\nu}(t)x_0 + \int_0^t (t-s)^{\mu-1} P^*_\mu(t-s)\Big[g(s, \underline{x}(s)) + C\underline{x}(s)\Big]ds, \ \ t \in [0, t_1] \\ S^*_{\mu,\nu}(t)x_0 + \sum_{i=1}^{k} S^*_{\mu,\nu}(t-t_i)\phi_i(\underline{x}(t_i)) \\ \quad + \int_0^t (t-s)^{\mu-1} P^*_\mu(t-s)\Big[g(s, \underline{x}(s)) + C\underline{x}(s)\Big]ds, \\ \quad \text{for } t \in (t_k, t_{k+1}], \ \ k = 1, 2, \ldots l. \end{cases}$$

It can be observed that $\underline{x} \in PC_{1-\lambda}$ and $\underline{x} = \mathcal{G}\underline{x}$. In a similar manner, it can be proved that $\exists\, \overline{x} \in PC_{1-\lambda}$ such that $\overline{x} = \mathcal{G}\overline{x}$. With the monotonicity property of $\mathcal{G}$, it can be concluded that $y_0 \leq \underline{x} \leq \overline{x} \leq z_0$. This proves that there exists minimal and maximal solutions $\underline{x}$ and $\overline{x}$ respectively in $[y_0, z_0]$ for the given impulsive system (1).

Remark 4 The above proved theorem holds for the case when the positive cone N which is normal is replaced with positive cone which is regular. For detailed proof [18, Corollary 3.3] may be referred.

Corollary 1 *In an ordered Banach space E, let N be the positive cone with normal constant D. Supposing that the operator $Q(t)$ is positive for $t \in J$. If the conditions $A(1)$ and $A(2)$ are satisfied combined with the following condition, then the condition $A(3)$ is automatically true.*

1. $[A(4)]$*:- There exists a constant C^* such that*

$$g(t, x_2) - g(t, x_1) \leq C^*(x_2 - x_1)$$

and $y_0(t) \leq x_1 \leq x_2 \leq z_0(t)$ for any $t \in J$.

Proof Let $\{x_p\}$ and $\{x_q\}$ be two increasing sequences such that $\{x_p\}, \{x_q\} \subset [y_0(t), z_0(t)]$, for $t \in J$ and $p \leq q$. By the condition $A(1)$ and $A(4)$,

$$\theta \leq g(t, x_q) - g(t, x_p) + C(x_q - x_p) \leq (C^* + C)(x_q - x_p).$$

Using the normality constant of the positive cone N, it reduces to,

$$\|g(t, x_q) - g(t, x_p)\| \le (DC^* + DC + C)\|x_q - x_p\|.$$

Let $L = (DC^* + DC + C)$. By the definition of measure of non-compactness the above equation reduces to, $\alpha\left(\{g(t, x_p)\}\right) \le L\alpha(\{x_p\})$. Thus the condition $A(3)$ is reduced. □

Now it is necessary to prove the uniqueness of the mild solution that lies in $[y_0, z_0]$.

Theorem 2 *An impulsive fractional system (1) is said to have an unique mild solution that lies between* $[y_0, z_0]$*, where* $y_0 \in PC_{1-\lambda}$ *and* $z_0 \in PC_{1-\lambda}$ *are the lower and upper solution with* $y_0 \le z_0$*, if the conditions* $A(1)$*,* $A(2)$ *and the Corollary 1 holds.*

Proof If $\overline{x}$ and $\underline{x}$ are the maximal and the minimal solution of the impulsive system (1), then to prove the uniqueness, it has to be proved that $\overline{x} = \underline{x}$. Like in the previous proof, the theorem is proved interval by interval. Let $t \in J_0^{'}$. Using (5) for both the solutions results in,

$$\begin{aligned}\theta \le\ & \overline{x}(t) - \underline{x}(t) = \mathcal{G}\overline{x}(t) - \mathcal{G}\underline{x}(t)\\ =& \int_0^t (t-s)^{\mu-1} P_\mu^*(t-s)\big[(g(s, \overline{x}(t)) - g(s, \underline{x}(t))) + C(\overline{x}(t) - \underline{x}(t))\big]ds\\ \le& \int_0^t (t-s)^{\mu-1} P_\mu^*(t-s)(C^* + C)(\overline{x}(t) - \underline{x}(t))ds.\end{aligned}$$

Using the normality of the positive cone N,

$$\|\overline{x}(t) - \underline{x}(t)\| \le \frac{DM^*}{\Gamma(\mu)}(C^* + C)\int_0^t (t-s)^{\mu-1}\|\overline{x}(t) - \underline{x}(t)\|ds.$$

By Gronwall inequality, $\|\overline{x}(t) - \underline{x}(t)\| = 0$. Which implies $\overline{x}(t) \equiv \underline{x}(t)$. Calculating in the similar way results in $\overline{x}(t) \equiv \underline{x}(t)$ for $t \in J_k^{'}$, for $k = 1, 2, \ldots, l$, that is for every interval $J_k^{'}$. The uniqueness is thus proved. □

4 Conclusion

The study of existence of upper and lower solution of impulsive fractional system with Hilfer fractional derivative is not yet done so far. The objective of this paper is to study the existence of the mild solutions for an impulsive Hilfer fractional evolution equation where the operator generates positive analytic semigroup. This paper is expected to give rise to many open problems. A few problems listed below for further scope in this direction.

- In practical problems the impulse experienced by the system is not necessarily always instantaneous. Hence it is much necessary to study the extremal solutions of such non-instantaneous system also. Sousa et al. [26] and [2] can be referred for system with non-instantaneous impulses with Hilfer fractional derivative.

- The existence can be further studied when the operator A generates a C_0-semigroup.
- The results can be further extended and studied for ψ-Hilfer operator. For detailed work on ψ-Hilfer, the authors may refer to [23–25].

Acknowledgements The work of the first author is supported by the Women Scientist Scheme A (WOS-A) of the Department of Science and Technology, India, through Project No. SR/WOS-A/PM-18/2016.

References

1. Agarwal, R., Golev, A., Hristova, S., O'Regan, D., Stefanova, K.: Iterative techniques with computer realization for the initial value problem for Caputo fractional differential equations. J. Appl. Math. Comput. **58**(1–2), 433–467 (2018)
2. Ahmed et al., H.M.: Approximate controllability of noninstantaneous impulsive Hilfer fractional integrodifferential equations with fractional Brownian motion. Bound. Value Probl. **2020**, Paper No. 120, 25 pp
3. Debbouche, A., Antonov, V.: Approximate controllability of semilinear Hilfer fractional differential inclusions with impulsive control inclusion conditions in Banach spaces. Chaos Solitons Fractals **102**(3), 140–148 (2017)
4. Denton, Z., Vatsala, A.S.: Monotone iterative technique for finite systems of nonlinear Riemann-Liouville fractional differential equations. Opuscula Math. **31**(3), 327–339 (2011)
5. Du, S.W., Lakshmikantham, V.: Monotone iterative technique for differential equations in a Banach space. J. Math. Anal. Appl. **87**(2), 454–459 (1982)
6. Furati, K.M., Kassim, M.D., Tatar, N.: Existence and uniqueness for a problem involving Hilfer fractional derivative. Comput. Math. Appl. **64**(6), 1616–1626 (2012)
7. Gu, H., Trujillo, J.J.: Existence of mild solution for evolution equation with Hilfer fractional derivative. Appl. Math. Comput. **257**, 344–354 (2015)
8. Gou, H., Li, Y.: Upper and lower solution method for Hilfer fractional evolution equations with nonlocal conditions. Bound. Value Probl. **2019**, Paper No. 187, 25 pp
9. Heinz, H.-P.: On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions. Nonlinear Anal. **7**(12), 1351–1371 (1983)
10. Hilfer, R. (ed.): Applications of Fractional Calculus in Physics. World Scientific, Singapore (2000)
11. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies, vol. 204. Elsevier Science B.V, Amsterdam (2006)
12. Ladde, G.S., Lakshmikantham, V., Vatsala, A.S.: Monotone iterative techniques for nonlinear differential equations. Monographs, Advanced Texts and Surveys in Pure and Applied Mathematics, vol. 27, Pitman (Advanced Publishing Program), Boston, MA (1985)
13. Lakshmikantham, V., Vatsala, A.S.: General uniqueness and monotone iterative technique for fractional differential equations. Appl. Math. Lett. **21**(8), 828–834 (2008)
14. Liu, Z., Sun, J., Szántó, I.: Monotone iterative technique for Riemann-Liouville fractional integro-differential equations with advanced arguments. Results Math. **63**(3–4), 1277–1287 (2013)
15. Liz, E., Nieto, J.J.: The method of upper and lower solutions for a periodic boundary value problem of second order integro-differential equations. In: International Conference on Differential Equations (Lisboa, 1995), pp. 426–430. World Scientific Publishing, River Edge, NJ
16. McRae, F.A.: Monotone iterative technique and existence results for fractional differential equations. Nonlinear Anal. **71**(12), 6093–6096 (2009)

17. Mu, J.: Monotone iterative technique for fractional evolution equations in Banach spaces. J. Appl. Math. **2011**, Art. ID 767186, 13 pp
18. Mu, J., Li, Y.: Monotone iterative technique for impulsive fractional evolution equations. J. Inequal. Appl. **2011**, 125, 12 pp (2011)
19. Nieto, J.J., Cabada, A.: A generalized upper and lower solutions method for nonlinear second order ordinary differential equations. J. Appl. Math. Stoch. Anal. **5**(2), 157–165 (1992)
20. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, vol. 44. Springer, New York (1983)
21. Podlubny, I.: Fractional Differential Equations, Mathematics in Science and Engineering, vol. 198. Academic Press Inc., San Diego, CA (1999)
22. Stamova, I.M., Stamov, G.Tr.: Functional and Impulsive Differential Equations of Fractional Order. CRC Press, Boca Raton, FL (2017)
23. Sousa, J.V.C., Benchohra, M., N'Guérékata, G.M.: Attractivity for differential equations of fractional order and ψ-Hilfer type. Fract. Calc. Appl. Anal. **23**(4), 1188–1207 (2020)
24. Sousa, J.V.C., Capelas de Oliveira, E.: On the ψ-Hilfer fractional derivative. Commun. Nonlinear Sci. Numer. Simul. **60**, 72–91 (2018)
25. Sousa, J.V.C., de Oliveira, E.C.: A Gronwall inequality and the Cauchy-type problem by means of ψ-Hilfer operator. Differ. Equ. Appl. **11**(1), 87–106 (2019)
26. Sousa, J.V.C., Jarad, F., Abdeljawad, T.: Existence of mild solutions to Hilfer fractional evolution equations in Banach space. Ann. Funct. Anal. **12** (2021). https://doi.org/10.1007/s43034-020-00095-5
27. Wang, G.: Monotone iterative technique for boundary value problems of a nonlinear fractional differential equation with deviating arguments. J. Comput. Appl. Math. **236**(9), 2425–2430 (2012)
28. Ye, H., Gao, J., Ding, Y.: A generalized Gronwall inequality and its application to a fractional differential equation. J. Math. Anal. Appl. **328**(2), 1075–1081 (2007)
29. Zhang, L., Liang, Y.: Monotone iterative technique for impulsive fractional evolution equations with noncompact semigroup. Adv. Differ. Equ. **2015**, 324, 15 pp (2015)

On Integral Solutions for a Class of Mixed Volterra-Fredholm Integro Differential Equations with Caputo Fractional Derivatives

Bandita Roy and Swaroop Nandan Bora

Abstract This work studies the existence of integral solution for a class of neutral integro-differential equation of mixed type involving Caputo fractional derivative under the assumption that the associated operator A is not dense. Utilizing semigroup theory, fractional calculus, Darbo-Sadovskii's fixed point theorem and measure of noncompactness, we have established some sufficient conditions which ensure the existence of integral solutions of our problem.

Keywords Volterra Fredholm integrodifferential equation · Hille Yosida condition · Integral solution · Fixed point theorem · Measure of noncompactness

1 Introduction

Fractional differential equation has garnered a lot of attention due to its growing number of applications in various areas of applied sciences and engineering [1, 2]. It is mainly because of the fact that the differential equations involving fractional derivatives are more realistic for describing many physical phenomena than the classical derivatives. Fractional differential equation provides

a powerful tool for modeling numerous real life dynamic processes as it can describe their behavior more accurately. One can find its applications in signal and image processing, atmospheric diffusion of pollution, transmission of ultrasound waves, cellular diffusion processes, feedback amplifiers, the effect of speculation on the profitability of stocks in financial markets, and many more. For more details on this topic, we refer the reader to [3, 4] and the references therein.

B. Roy (✉) · S. N. Bora
Department of Mathematics, Indian Institute of Technology Guwahati,
Guwahati 781039, India
e-mail: bandita.roy@iitg.ac.in

S. N. Bora
e-mail: swaroop@iitg.ac.in

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_6

There are various ways of interpolating the definition of integer order to non-integer order. Among them, the most widely known are Riemann-Liouville and Caputo derivatives. There are several works in literature, involving the mild solutions of various differential equations with fractional derivatives. However, many authors defined the mild solution of fractional evolution equation by generalizing the mild solution definition of integer order evolution equations [5, 6], which is not appropriate [7]. A suitable concept of mild solution for fractional evolution equations of order $\alpha \in (0, 1)$, is given by Zhou et al. [8, 9], wherein they used Laplace transform and probability density function $M_\alpha(\varphi)$ (defined only for $\alpha \in (0, 1)$). Subsequently, many authors have used this approach in their study of fractional evolution equations of order $\alpha \in (0, 1)$. For mild solutions of order $\alpha \in (1, 2)$, we refer to [10–12].

The fractional evolution equation

$$\begin{aligned} {}^C D^{\alpha}_{0^+} y(\upsilon) &= Ay(\upsilon) + f(\upsilon, y(\upsilon)), \quad \upsilon \in [0, b],\ \alpha \in (0, 1), \\ y(0) &= y_0, \end{aligned}$$

has been extensively studied, for the case when A is densely defined. The study of initial value problems with nondense domain was initiated by Da Prato and Sinestrari [13]. They introduced the concept of integral solutions of the abstract Cauchy problem

$$\begin{aligned} y'(\upsilon) &= Ay(\upsilon) + f(\upsilon), \quad \upsilon \in [0, b], \\ y(0) &= y_0. \end{aligned}$$

For more details, the readers are referred to [14–17]. The following fractional semilinear equation was considered by Gu et al. in [18]:

$$\begin{aligned} {}^C D^{\alpha}_{0^+} y(\upsilon) &= Ay(\upsilon) + f(\upsilon, y(\upsilon)), \quad \upsilon \in (0, b],\ \alpha \in (0, 1), \\ y(0) &= y_0. \end{aligned}$$

They studied the existence of integral solutions by using measure of noncompactness. Motivated by the works carried out in [18] and [19], we consider the following neutral fractional integro-differential equation of mixed type:

$$\left.\begin{aligned} {}^C D^{\alpha}_{0^+}[y(\upsilon) - u(\upsilon, y(\upsilon))] &= A[y(\upsilon) - u(\upsilon, y(\upsilon))] + f(\upsilon, y(\upsilon), (Hy)(\upsilon), (Gy)(\upsilon)), \\ &\quad \upsilon \in [0, b], \\ y(0) &= y_0, \end{aligned}\right\} \tag{1}$$

where

$$(Hy)(\upsilon) = \int_0^{\upsilon} h(\upsilon, \varepsilon, y(\varepsilon))d\varepsilon \ \text{ and } \ (Gy)(\upsilon) = \int_0^{b} g(\upsilon, \varepsilon, y(\varepsilon))d\varepsilon,$$

with $\alpha \in (0, 1)$, $J = [0, b]$, $A\colon D(A) \subseteq Y \to Y$ a closed linear operator on Y, which is not necessarily densely defined. The state $y(.)$ takes values in a Banach space Y with norm $\|.\|_Y$, $u\colon J \times Y \to Y$ is a function satisfying some assumptions which will be specified later. $h\colon \Delta \times Y \to Y$, $g\colon J \times J \times Y \to Y$ and $f\colon J \times Y \times Y \times Y \to Y$ are given abstract functions and here $\Delta = \{(\upsilon, \varepsilon) \in J \times J | \varepsilon \leq \upsilon\}$.

The rest of this paper is organized as follows: In Sect. 2, we recall some definitions, theorems, and lemmas, which are used throughout our work. The existence theorems for the integral solution of our problem (1) are stated in Sect. 3.

2 Preliminaries

We use the following notations:

$C(J, Y)$ denote the collection of all continuous functions from J to Y which is a Banach space with respect to the norm $\|y\|_C = \sup_{\upsilon \in J} \|y(\upsilon)\|_Y$. $B(Y)$ denotes the Banach space of all bounded linear operators on Y and Ω_Y, the set of all bounded subsets of Y.

Definition 1 [3] The Caputo derivative of order α is defined as

$$^{C}D^{\alpha}_{0^+} f(\upsilon) = \frac{1}{\Gamma(n-\alpha)} \int_0^{\upsilon} (\upsilon - \varepsilon)^{n-\alpha-1} f^{(n)}(\varepsilon) d\varepsilon, \quad \upsilon > 0,$$

where n is the least integer $\geq \alpha$.

If f is an abstract function then the above integral is taken in Bochner's sense.

Definition 2 [20] The Kuratowski measure of noncompactness β is a non-negative real-valued function defined on Ω_Y by

$$\beta(D) = \inf\{\epsilon > 0 | D \subseteq \cup_{i=1}^{m} D_i \text{ and } \operatorname{diam}(D_i) \leq \epsilon \text{ for i} = 1, \ldots, m\},$$

where $D \in \Omega_Y$ and $diam(D_i) = \sup\{\|y_1 - y_2\| : y_1, y_2 \in D_i\}$.

Theorem 1 *[20] β satisfies the following properties:*

1. *$\beta(D) = 0$ iff $\bar{D}$ is compact,*
2. *$D_1 \subset D_2 \implies \beta(D_1) \leq \beta(D_2)$,*
3. *$\beta(D_1 + D_2) \leq \beta(D_1) + \beta(D_1)$.*

Theorem 2 *[21] Let $D \in \Omega_Y$. Then $\exists$ a countable set $D_0 \subset D$ such that*

$$\beta(D) \leq 2\beta(D_0).$$

Theorem 3 *[22] Let $D \subset C(J, Y)$ be equicontinuous and bounded, then $\beta(D(\upsilon)) \in C(J, [0, \infty))$ and*

$$\beta(D) = \max_{\upsilon \in J} \beta(D(\upsilon)).$$

Theorem 4 *[23] Let $\{y_n\}_{n=1}^{\infty}$ be Bochner integrable functions from J into Y with*

$$\|y_n(\upsilon)\| \leq j(\upsilon) \text{ for almost all } \upsilon \in J \text{ and } \forall\, n \in \mathbf{N},$$

where $j \in L(J, [0, \infty))$. Then $\phi(\upsilon) = \beta\Big(\{y_n\}_{n=1}^{\infty}\Big) \in L(J, [0, \infty))$ and satisfies

$$\beta\left(\left\{ \int_0^{\upsilon} y_n(\varepsilon)d\varepsilon \middle| n \in \mathbf{N} \right\}\right) \leq 2\int_0^{\upsilon} \phi(\varepsilon)d\varepsilon.$$

Theorem 5 *[24] Let S be a closed, convex, and bounded subset of a Banach space $\mathbb{W}$. If $Q: S \to S$ is a condensing map, which means that $\beta(Q(S)) \leq \beta(S)$. Then Q has a fixed point in S.*

Theorem 6 (Darbo-Sadovskii's fixed point theorem) *[25] If S is a closed, convex, and bounded subset of a Banach space $\mathbb{W}$ and $Q: S \to S$ is continuous mapping and a β-contraction, then Q has atleast one fixed point in S.*

Assume that $A: D(A) \subset Y \to Y$ satisfies the Hille-Yosida condition, i.e., $\exists\, \overline{P} \geq 0$ and a constant $w \in \mathbf{R}$ such that $(w, +\infty) \subseteq \rho(A)$ and

$$\sup\left\{(\lambda - w)^n \|R(\lambda : A)^n\|_{B(Y)} \middle| n \in \mathbf{N}, \lambda > w\right\} \leq \overline{P},$$

where $\rho(A) = \{\lambda : \lambda I - A \text{ is invertible}\}$ is the resolvent set of A, and $R(\lambda : A) = \{(\lambda I - A)^{-1} : \lambda \in \rho(A)\}$ denotes the resolvent of A.

Let A_0 be the part of A in $\overline{D(A)}$ defined by

$$D(A_0) = \left\{y \in D(A) \middle| Ay \in \overline{D(A)}\right\},$$
$$A_0 y = Ay.$$

Then A_0 generates a C_0-semigroup $\{P(\upsilon)\}_{\upsilon \geq 0}$ on $\overline{D(A)}$. Assume that $\exists\, P > 1$ such that $\sup_{\upsilon \in [0,\infty)} \|P(\upsilon)\|_{B(Y)} < P$.

Taking cue from [18], we present the following definition and results: Assuming f to be continuous and $y_0 \in \overline{D(A)}$, the integral solution of our problem (1) is defined as follows:

Definition 3 A function $y: J \to Y$ is said to be an integral solution of (1) if

$$y \in C(J, Y), \quad I_{0+}^{\alpha}[y(\upsilon) - u(\upsilon, y(\upsilon))] \in D(A) \text{ for } \upsilon \in [0, b],$$

and

$$y(\upsilon) = y_0 - u(0, y(0)) + u(\upsilon, y(\upsilon)) + A\frac{1}{\Gamma(\alpha)}\int_0^{\upsilon} (\upsilon - \varepsilon)^{\alpha-1}[y(\varepsilon) - u(\varepsilon, y(\varepsilon))]d\varepsilon$$
$$+ \frac{1}{\Gamma(\alpha)}\int_0^{\upsilon} (\upsilon - \varepsilon)^{\alpha-1} f(\varepsilon, y(\varepsilon), (Hy)(\varepsilon), (Gy)(\varepsilon))d\varepsilon, \quad \upsilon \in [0, b].$$

Note. If y is an integral solution of our problem (1), it can be shown that $y(\upsilon) - u(\upsilon, y(\upsilon)) \in \overline{D(A)}$ for $\upsilon \in J$.

Now, consider the auxiliary problem

$$\left.\begin{array}{c} {}^{C}D_{0^+}^{\alpha}[y(\upsilon) - u(\upsilon, y(\upsilon))] = A_0[y(\upsilon) - u(\upsilon, y(\upsilon))] + f(\upsilon, y(\upsilon), (Hy)(\upsilon), (Gy)(\upsilon)), \\ \upsilon \in [0, b], \\ y(0) = y_0. \end{array}\right\} \tag{2}$$

The integral solution of (2) can be written as

$$\begin{aligned} y(\upsilon) = {} & y_0 - u(0, y(0)) + u(\upsilon, y(\upsilon)) + A_0 I_{0^+}^{\alpha}[y(\upsilon) - u(\upsilon, y(\upsilon))] \\ & + I_{0^+}^{\alpha} f(\upsilon, y(\upsilon), (Hy)(\upsilon), (Gy)(\upsilon)). \end{aligned} \tag{3}$$

Theorem 7 *The integral solution of (3) can be written as*

$$\begin{aligned} y(\upsilon) = {} & u(\upsilon, y(\upsilon)) + S_{\alpha}(\upsilon)[y_0 - u(0, y(0))] + \int_0^{\upsilon} K_{\alpha}(\upsilon - \varepsilon) \\ & \times f(\varepsilon, y(\varepsilon), (Hy)(\varepsilon), (Gy)(\varepsilon))d\varepsilon, \end{aligned}$$

where

$$S_{\alpha}(\upsilon) = I_{0^+}^{1-\alpha} K_{\alpha}(\upsilon), \quad K_{\alpha}(\upsilon) = \upsilon^{\alpha-1} P_{\alpha}(\upsilon), \quad P_{\alpha}(\upsilon) = \int_0^{\infty} \alpha\varphi M_{\alpha}(\varphi) P(\upsilon^{\alpha}\varphi)d\varphi.$$

Here

$$\begin{aligned} M_{\alpha}(\varphi) &= \frac{1}{\alpha}\varphi^{-1-\frac{1}{\alpha}}\psi_{\alpha}(\varphi^{-\frac{1}{\alpha}}), \\ \psi_{\alpha}(\varphi) &= \frac{1}{\pi}\sum_{i=0}^{\infty}(-1)^{i-1}\varphi^{(-\alpha i-1)}\frac{\Gamma(\alpha i + 1)}{i!}\sin(i\pi\alpha), \ \varphi \in (0, +\infty). \end{aligned}$$

The probability density function $M_{\alpha}(\varphi)$ *defined on* $(0, +\infty)$ *satisfies*

$$M_{\alpha}(\varphi) \geq 0, \ \int_0^{\infty} M_{\alpha}(\varphi)d\varphi = 1, \ \int_0^{\infty} \varphi M_{\alpha}(\varphi)d\varphi = \frac{1}{\Gamma(1 + \alpha)}.$$

Let $B_{\lambda} = \lambda(\lambda I - A)^{-1}$. Then, since $B_{\lambda}y \to y$ as $\lambda \to +\infty$ for $y \in \overline{D(A)}$, therefore (4) can be written as

$$\begin{aligned} y(\upsilon) = {} & u(\upsilon, y(\upsilon)) + S_{\alpha}(\upsilon)[y_0 - u(0, y(0))] \\ & + \lim_{\lambda\to\infty}\int_0^{\upsilon} K_{\alpha}(\upsilon - \varepsilon)B_{\lambda} f(\varepsilon, y(\varepsilon), (Hy)(\varepsilon), (Gy)(\varepsilon))d\varepsilon, \quad \upsilon \in [0, b]. \end{aligned}$$

Theorem 8 *[18] For any fixed $t > 0$, $\{K_\alpha(\upsilon)\}_{\upsilon>0}$ and $\{S_\alpha(\upsilon)\}_{\upsilon>0}$ are linear operators, and for any $y \in \overline{D(A)}$,*

$$\|K_\alpha(\upsilon)y\|_Y \leq \frac{P\upsilon^{\alpha-1}}{\Gamma(\alpha)}\|y\|_Y \ \ \textit{and} \ \ \|S_\alpha(\upsilon)y\|_Y \leq P\|y\|_Y.$$

Theorem 9 *[18] $\{K_\alpha(\upsilon)\}_{\upsilon>0}$ and $\{S_\alpha(\upsilon)\}_{\upsilon>0}$ are strongly continuous, i.e., for any $y \in \overline{D(A)}$ and $0 < \upsilon_1 < \upsilon_2 \leq b$,*

$$\|K_\alpha(\upsilon_2)y - K_\alpha(\upsilon_1)y\|_Y \longrightarrow 0 \ \ \textit{and} \ \ \|S_\alpha(\upsilon_2)y - S_\alpha(\upsilon_1)y\|_Y \longrightarrow 0,$$

as $\upsilon_2 \to \upsilon_1$.

Theorem 10 *[25] For any fixed $\upsilon > 0$, $P_\alpha(\upsilon)$ is a linear and bounded operator, and*

$$\|P_\alpha(\upsilon)y\|_Y \leq \frac{P}{\Gamma(\alpha)}\|y\|_Y \ \ \textit{for any } y \in \overline{D(A)}.$$

Theorem 11 *[17] Assume that $\{P(\upsilon)\}_{\upsilon>0}$ is compact. Then $\{P(\upsilon)\}_{\upsilon>0}$ is continuous in the uniform operator topology.*

3 Existence Results

First, we introduce the following assumptions:

(H1) $\{P(\upsilon)\}_{\upsilon>0}$ is compact.

(H2)(i) for each $\upsilon \in [0, b]$, $f(\upsilon, ., ., .) \colon Y \times Y \times Y \to Y$ is continuous; and for each $(y_1, y_2, y_3) \in Y \times Y \times Y$, $f(., y_1, y_2, y_3) \colon J \to Y$ is strongly measurable.

(ii) $\exists$ a function $l \in L^\infty(J, [0, \infty))$ such that

$$\|f(\upsilon, y_1, y_2, y_3)\|_Y \leq l(\upsilon), \text{ for all } y_1, y_2, y_3 \in Y \text{ and for } \upsilon \in [0, b].$$

(iii) $\exists$ a constant $\alpha_1 \in (0, \alpha)$ and a function $l \in L^{\frac{1}{\alpha_1}}(J, [0, \infty))$ such that

$$\|f(\upsilon, y_1, y_2, y_3)\|_Y \leq l(\upsilon)(\|y_1\|_Y + \|y_2\|_Y + \|y_3\|_Y),$$

for all $y_1, y_2, y_3 \in Y$ and for $\upsilon \in [0, b]$.

(iv) there exist $l_1, l_2, l_3 \in C(J, [0, \infty))$ such that

$$\beta(f(\upsilon, D_1, D_2, D_3)) \leq l_1(\upsilon)\beta(D_1) + l_2(\upsilon)\beta(D_2) + l_3(\upsilon)\beta(D_3)$$

for any $D_1, D_2, D_3 \in \Omega_Y$ and $\upsilon \in J$. Let $l_i^* = \sup_{\upsilon \in J} |l_i(\upsilon)|$, $i = 1, 2, 3$.

(H3)(i) for each $(\upsilon, \varepsilon) \in \Delta$, $h(\upsilon, \varepsilon, .) \colon Y \to Y$ is continuous; and for each $y \in Y$, $h(., ., y) \colon \Delta \to Y$ is strongly measurable.

(ii) $\exists$ a function $m_h(\upsilon, \varepsilon) \in C(\Delta, [0, \infty))$ such that

$$\|h(\upsilon, \varepsilon, y)\|_Y \le m_h(\upsilon, \varepsilon)\|y\|_Y, \text{ for } (\upsilon, \varepsilon) \in \Delta, \ y \in Y$$

and $H^* = \sup_{\upsilon \in J} \int_0^{\upsilon} m_h(\upsilon, \varepsilon) d\varepsilon < \infty$.

(iii) for any $D \in \Omega_Y$, and $(\upsilon, \varepsilon) \in \Delta$, $\exists$ a function $m\colon \Delta \to [0, \infty)$ such that

$$\beta(h(\upsilon, \varepsilon, D)) \le m(\upsilon, \varepsilon)\beta(D) \tag{4}$$

with $m^* = \sup_{\upsilon \in J} \int_0^{\upsilon} m(\upsilon, \varepsilon) d\varepsilon < \infty$.

(H4)(i) for each $(\upsilon, \varepsilon) \in J \times J$, $g(\upsilon, \varepsilon, .)\colon Y \to Y$ is continuous; and for each $y \in Y$, $g(., ., y)\colon J \times J \to Y$ is strongly measurable.

(ii) $\exists$ a function $m_g(\upsilon, \varepsilon) \in C(J \times J, [0, \infty))$ such that

$$\|g(\upsilon, \varepsilon, y)\|_Y \le m_g(\upsilon, \varepsilon)\|y\|_Y, \text{ for } (\upsilon, \varepsilon) \in J \times J, \ y \in Y$$

and $G^* = \sup_{\upsilon \in J} \int_0^{b} m_g(\upsilon, \varepsilon) d\varepsilon < \infty$.

(iii) for any $D \in \Omega_Y$, and $(\upsilon, \varepsilon) \in J \times J$, $\exists$ a function $n\colon J \times J \to [0, \infty)$ such that

$$\beta(g(\upsilon, \varepsilon, D)) \le n(\upsilon, \varepsilon)\beta(D) \tag{5}$$

with $n^* = \sup_{\upsilon \in J} \int_0^{b} n(\upsilon, \varepsilon) d\varepsilon < \infty$.

(H5) for the function $u : J \times Y \to Y$, $\exists$ a constant $L_1 > 0$ such that

$$\|u(\upsilon_1, y_1) - u(\upsilon_2, y_2)\|_Y \le L_1(|\upsilon_1 - \upsilon_2| + \|y_1 - y_2\|_Y), \tag{6}$$

for all $\upsilon_1, \upsilon_2 \in [0, b]$ and all $y_1, y_2 \in Y$. Further, let $P_0 = \sup_{\upsilon \in J} \|u(\upsilon, 0)\|_Y$.

(H6) for each $\{P(\upsilon)\}_{\upsilon > 0}$ is equicontinuous.

Our first result is based on Darbo-Sadovskii's fixed point theorem.

Theorem 12 *Suppose that (H1), (H2)(i), (H2)(ii), (H3)(i), (H3)(ii), (H4)(i), (H4)(ii) and (H5) are satisfied. Then (1) has an integral solution in* $C(J, \overline{D(A)})$ *provided*

$$L_1 < 1.$$

Proof Let $B_r = \{y \in C(J, \overline{D(A)}) | \|y\|_C \le r\}$ where $r = \frac{\xi_3}{1 - L_1}$, $\xi_3 = P_0 + P\|y_0\|_Y + P\|u(0, y(0))\|_Y + \frac{P\overline{P}}{\Gamma(\alpha+1)} b^{\alpha} \|l\|_{L^{\infty}}$. Define $Q\colon C(J, \overline{D(A)}) \to C(J, \overline{D(A)})$ by

$$(Qy)(\upsilon) = (Q_1 y)(\upsilon) + (Q_2 y)(\upsilon),$$

where

$$(Q_1 y)(\upsilon) = S_\alpha(\upsilon)[y_0 - u(0, y(0))] + \lim_{\lambda\to\infty} \int_0^{\upsilon} K_\alpha(\upsilon - \varepsilon) \times B_\lambda f(\varepsilon, y(\varepsilon), (Hy)(\varepsilon), (Gy)(\varepsilon)) d\varepsilon,$$
$$(Q_2 y)(\upsilon) = u(\upsilon, y(\upsilon)).$$

***Step 1*:** To show $Q\colon B_r \to B_r$.

It follows from the fact that

$$\|(Qy)(\upsilon)\|_Y \le L_1 \|y\|_C + P_0 + P\|y_0\|_Y + P\|u(0, y(0))\|_Y + \frac{P\overline{P}}{\Gamma(\alpha+1)} b^\alpha \|l\|_{L^\infty} \le r.$$

***Step 2*:** To show Q_1 is completely continuous.

Q_1 is equicontinuous on B_r: Let $y \in B_r$ and $0 \le \upsilon_1 < \upsilon_2 \le b$, then

$$\|(Q_1 y)(\upsilon_2) - (Q_1 y)(\upsilon_1)\|_Y \le I_1 + I_2,$$

where

$$I_1 = \|S_\alpha(\upsilon_2)[y_0 - u(0, y(0))] - S_\alpha(\upsilon_1)[y_0 - u(0, y(0))]\|_Y,$$
$$I_2 = \| \lim_{\lambda\to\infty} \int_0^{\upsilon_2} K_\alpha(\upsilon_2 - \varepsilon) B_\lambda f(\varepsilon, y(\varepsilon), (Hy)(\varepsilon), (Gy)(\varepsilon)) d\varepsilon - \lim_{\lambda\to\infty} \int_0^{\upsilon_1} K_\alpha(\upsilon_1 - \varepsilon) B_\lambda f(\varepsilon, y(\varepsilon), (Hy)(\varepsilon), (Gy)(\varepsilon)) d\varepsilon \|_Y.$$

For I_1, by Lemma 9 we have $I_1 \to 0$ as $\upsilon_2 \to \upsilon_1$. For $\upsilon_1 = 0$, $0 < \upsilon_2 \le b$, we get $I_2 \le \frac{P\overline{P}}{\Gamma(\alpha)} \upsilon_2^\alpha \|l\|_{L^\infty} \longrightarrow 0$,as $\upsilon_2 \to 0$. And for $0 < \upsilon_1 < \upsilon_2 \le b$, we have $I_2 \le I_1^* + I_2^* + I_3^*$, where

$$I_1^* = \frac{P\overline{P}}{\Gamma(\alpha)} \int_{\upsilon_1}^{\upsilon_2} (\upsilon_2 - \varepsilon)^{\alpha-1} l(\varepsilon) d\varepsilon,$$
$$I_2^* = \frac{P\overline{P}}{\Gamma(\alpha)} \int_0^{\upsilon_1} [(\upsilon_1 - \varepsilon)^{\alpha-1} - (\upsilon_2 - \varepsilon)^{\alpha-1}] l(\varepsilon) d\varepsilon,$$
$$I_3^* = \overline{P} \int_0^{\upsilon_1} (\upsilon_1 - \varepsilon)^{\alpha-1} \|P_\alpha(\upsilon_2 - \varepsilon) - P_\alpha(\upsilon_1 - \varepsilon)\|_{B(Y)} l(\varepsilon) d\varepsilon.$$

Also,

$$I_1^* \le \frac{P\overline{P}}{\Gamma(\alpha)} \|l\|_{L^\infty} (\upsilon_2 - \upsilon_1)^\alpha \longrightarrow 0, \quad I_2^* \le \frac{P\overline{P}}{\Gamma(\alpha)} \|l\|_{L^\infty} (\upsilon_2 - \upsilon_1)^\alpha \longrightarrow 0 \text{ as } \upsilon_2 \to \upsilon_1.$$

Next,

$$I_3^* = \overline{P} \int_0^{\upsilon_1} (\upsilon_1 - \varepsilon)^{\alpha-1} \|P_\alpha(\upsilon_2 - \varepsilon) - P_\alpha(\upsilon_1 - \varepsilon)\|_{B(Y)} l(\varepsilon) d\varepsilon.$$

For $\epsilon > 0$ small enough,

$$I_3^* \leq \overline{P}\int_0^{\upsilon_1}(\upsilon_1-\varepsilon)^{\alpha-1}l(\varepsilon)d\varepsilon \sup_{\varepsilon\in[0,\upsilon_1-\epsilon]}\|P_\alpha(\upsilon_2-\varepsilon)-P_\alpha(\upsilon_1-\varepsilon)\|_{B(Y)} + \frac{2P\overline{P}}{\Gamma(\alpha+1)}\|l\|_{L^\infty}\epsilon^\alpha = I_{31}^* + I_{32}^*.$$

From (H1), it follows that $I_{31}^* \to 0$ as $\upsilon_2 \to \upsilon_1$ and also $I_{32}^* \to 0$ as $\epsilon \to 0$. Thus, $\|(Q_1y)(\upsilon_2)-(Q_1y)(\upsilon_1)\|_Y \to 0$ as $\upsilon_2 \to \upsilon_1$, independent of $y \in B_r$, which implies that $\{Q_1y | y \in B_r\}$ is equicontinuous.

Q_1 is continuous on B_r: Let $(y_n) \subset B_r$ such that $y_n \to y$ in B_r.

Using (H2)(i), (H3), (H4) and Dominated convergence theorem, it follows that

$$f(\varepsilon, y_n(\varepsilon), (Hy_n)(\varepsilon), (Gy_n)(\varepsilon)) \longrightarrow f(\varepsilon, y(\varepsilon), (Hy)(\varepsilon), (Gy)(\varepsilon)), \text{ as } n \to \infty.$$

Now for each $\upsilon \in J$, using (H2)(ii), we have

$$(\upsilon-\varepsilon)^{\alpha-1}\|f(\varepsilon, y_n(\varepsilon), (Hy_n)(\varepsilon), (Gy_n)(\varepsilon)) - f(\varepsilon, y(\varepsilon), (Hy)(\varepsilon), (Gy)(\varepsilon))\|_Y \leq 2(\upsilon-\varepsilon)^{\alpha-1}l(\varepsilon) \in L^1(J, [0,\infty)), \text{ for } \varepsilon \in [0,\upsilon],\ \upsilon \in J.$$

Therefore, by Lebesgue's dominated convergence theorem, we obtain

$$\int_0^{\upsilon}(\upsilon-\varepsilon)^{\alpha-1}\|f(\varepsilon, y_n(\varepsilon), (Hy_n)(\varepsilon), (Gy_n)(\varepsilon)) - f(\varepsilon, y(\varepsilon), (Hy)(\varepsilon), (Gy)(\varepsilon))\|_Y d\varepsilon \longrightarrow 0 \text{ as } n \to \infty.$$

Therefore, Q_1 is continuous.

For any $\upsilon \in J$, $\{Q_1y(\upsilon) | y \in B_r\}$ is relatively compact in Y: For $\upsilon = 0$, it is obvious. So, fix $\upsilon \in (0, b]$ then for $\epsilon \in (0,\upsilon)$ and $\forall\, \delta > 0$, define

$$(Q_1^{\epsilon,\delta}y)(\upsilon) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^{\upsilon-\epsilon}\int_\delta^\infty \varphi M_\alpha(\varphi)(\upsilon-\varepsilon)^{-\alpha}\varepsilon^{\alpha-1}P(\varepsilon^\alpha\varphi) \times [y_0 - u(0, y(0))]d\varphi d\varepsilon + \alpha P(\epsilon^\alpha\delta)\lim_{\lambda\to\infty}\int_0^{\upsilon-\epsilon}\int_\delta^\infty \varphi(\upsilon-\varepsilon)^{\alpha-1} \times M_\alpha(\varphi)P((\upsilon-\varepsilon)^\alpha\varphi - \epsilon^\alpha\delta)B_\lambda f(\varepsilon, y(\varepsilon), (Hy)(\varepsilon), (Gy)(\varepsilon))d\varphi d\varepsilon.$$

From the compactness of $T(\epsilon^\alpha\delta)$, $(\epsilon^\alpha\delta > 0)$, we obtain that $\{(Q_1^{\epsilon,\delta}y)(\upsilon) | y \in B_r\}$ is relatively compact in Y $\forall\, \epsilon \in (0,\upsilon)$ and $\forall\, \delta > 0$. Moreover, for any $y \in B_r$, we have

$$\|(Q_1 y)(\upsilon) - (Q_1^{\epsilon,\delta} y)(\upsilon)\|_Y$$
$$\leq \frac{P\alpha}{\Gamma(1-\alpha)} B(\alpha, 1-\alpha)[\|y_0\|_Y + \|u(0, y(0))\|_Y] \int_0^\delta \varphi M_\alpha(\varphi) d\varphi + \frac{P}{\Gamma(1-\alpha)\Gamma(\alpha)}$$
$$\times [\|y_0\|_Y + \|u(0, y(0))\|_Y] \int_{\upsilon-\epsilon}^{\upsilon} (\upsilon-\varepsilon)^{-\alpha} \varepsilon^{\alpha-1} d\varepsilon + \alpha P\overline{P} \int_0^{\upsilon} (\upsilon-\varepsilon)^{\alpha-1} l(\varepsilon) d\varepsilon$$
$$\times \int_0^\delta \varphi M_\alpha(\varphi) d\varphi + \alpha P\overline{P} \int_{\upsilon-\epsilon}^{\upsilon} (\upsilon-\varepsilon)^{\alpha-1} l(\varepsilon) d\varepsilon \int_0^\infty \varphi M_\alpha(\varphi) d\varphi.$$

Therefore,
$$\|(Q_1 y)(\upsilon) - (Q_1^{\epsilon,\delta} y)(\upsilon)\|_Y \leq J_1 + J_2 + J_3 + J_4.$$

Using the inequality $\int_0^\infty \varphi M_\alpha(\varphi) d\varphi = \frac{1}{\Gamma(1+\alpha)}$, we conclude that J_1, J_3 and J_4 tend to 0 as $\epsilon, \delta \to 0$. Also, applying the absolute continuity of Lebesgue integral, $J_2 \to 0$ as $\epsilon, \delta \to 0$. Therefore, $\{(Q_1 y)(\upsilon) | y \in B_r\}, \upsilon > 0$ is relatively compact. Consequently, $\{Q_1 y | y \in B_r\}$ is a relatively compact set in Y.

***Step 3*:** To show Q is continuous on B_r.

Proceeding similarly as in ***Step 2***, it can be shown that Q is continuous on B_r.

***Step 4*:** To show Q_2 is a contraction on B_r.

For any $y_1, y_2 \in B_r$, we have $\|(Q_2 y_1)(\upsilon) - (Q_2 y_2)(\upsilon)\|_Y \leq L_1 \|y_1(\upsilon) - y_2(\upsilon)\|_Y$. Thus,
$$\|Q_2 y_1 - Q_2 y_2\|_C \leq L_1 \|y_1 - y_2\|_C,$$

which implies that $\beta(Q_2 B_r) \leq L_1 \beta(B_r)$. Also, $Q_1 B_r$ is relatively compact in Y which gives $\beta(Q_1 B_r) = 0$. Therefore, $\beta(Q B_r) \leq \beta(Q_1 B_r) + \beta(Q_2 B_r) \leq L_1 \beta(B_r)$. As $L_1 < 1$, Q is an β-contraction on B_r. Hence, from Theorem 6, it follows that Q has atleast one fixed point on B_r. □

Our next result for problem (1) is for the case where the associated C_0-semigroup is not compact.

Theorem 13 *Assume that (H2)(i),(iii),(iv), (H3)(i),(ii),(iii), (H4)(i),(ii),(iii), (H5) and (H6) hold. Then (1) has an integral solution provided*

$$\xi_4 = L_1 + \frac{P\overline{P}}{\Gamma(\alpha)}(1 + H^* + G^*)\|l\|_{L^{\frac{1}{\alpha_1}}} \frac{b^{\alpha-\alpha_1}}{(\frac{\alpha-\alpha_1}{1-\alpha_1})^{1-\alpha_1}} < 1$$

and

$$2L_1(1+P) + \frac{4P\overline{P}}{\Gamma(\alpha+1)} b^\alpha (l_1^* + 2l_2^* m^* + 2l_3^* n^*) < 1.$$

Proof Choose $r = \frac{\phi}{1-\xi_4}$, where $\phi = P_0 + P\|y_0\|_Y + P\|u(0, y(0))\|_Y$ and let $B_r = \{y \in C(J, \overline{D(A)}) \mid \|y\|_C \leq r\}$. Define $Q : C(J, \overline{D(A)}) \to C(J, \overline{D(A)})$ by

$$(Qy)(\upsilon) = u(\upsilon, y(\upsilon)) + S_\alpha(\upsilon)[y_0 - u(0, y(0))] + \lim_{\lambda\to\infty}\int_0^\upsilon K_\alpha(\upsilon-\varepsilon) \times B_\lambda f(\varepsilon, y(\varepsilon), (Hy)(\varepsilon), (Gy)(\varepsilon))d\varepsilon, \quad \upsilon \in J.$$

Then proceeding similarly as in Theorem 12, it can be shown that $Q\colon B_r \to B_r$ is continuous as well as equicontinuous. Now, it remains to show that $Q\colon B_r \to B_r$ is a condensing operator.

For all $D \subset B_r$, $Q(D)$ is bounded and equicontinuous. Hence, by Lemma 2, there exists a countable set $D_0 = \{y_n\}_{n=1}^\infty \subset D$ such that

$$\beta(Q(D)) \le 2\beta(Q(D_0)). \tag{7}$$

Since $Q(D_0) \subset Q(B_r)$ is equicontinuous, so by using Lemma 3, we get

$$\beta(Q(D_0)) = \max_{\upsilon\in J}\beta(Q(D_0(\upsilon))). \tag{8}$$

Now, let

$$(Qy)(\upsilon) = (Q_1 y)(\upsilon) + (Q_2 y)(\upsilon),$$

where

$$(Q_1 y)(\upsilon) = u(\upsilon, y(\upsilon)) + S_\alpha(\upsilon)[y_0 - u(0, y(0))],$$
$$(Q_2 y)(\upsilon) = \lim_{\lambda\to\infty}\int_0^\upsilon K_\alpha(\upsilon-\varepsilon)B_\lambda f(\varepsilon, y(\varepsilon), (Hy)(\varepsilon), (Gy)(\varepsilon))d\varepsilon.$$

For $y_1, y_2 \in D_0$, we have $\|Q_1 y_1 - Q_1 y_2\|_C \le L_1(1+P)\|y_1 - y_2\|_C$. Therefore, it follows that $\beta(Q_1(D_0)) \le L_1(1+P)\beta(D_0)$. Next, for $\upsilon \in J$, we get

$$\begin{aligned}
\beta(\{Q_2 y_n(\upsilon)\}_{n=1}^\infty) &= \beta\Big(\Big\{\lim_{\lambda\to\infty}\int_0^\upsilon K_\alpha(\upsilon-\varepsilon)B_\lambda f(\varepsilon, y(\varepsilon), (Hy)(\varepsilon), (Gy)(\varepsilon))d\varepsilon\Big\}_{n=1}^\infty\Big)\\
&\le \frac{2P\overline{P}}{\Gamma(\alpha)}\int_0^\upsilon (\upsilon-\varepsilon)^{\alpha-1}\beta\Big(\{f(\varepsilon, y(\varepsilon), (Hy)(\varepsilon), (Gy)(\varepsilon))\}_{n=1}^\infty\Big)d\varepsilon\\
&\le \frac{2P\overline{P}}{\Gamma(\alpha+1)}\beta(D)b^\alpha(l_1^* + l_2^* 2m^* + l_3^* 2n^*).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\beta(Q(D_0)(\upsilon)) &\le \beta(Q_1(D_0)(\upsilon)) + \beta(Q_2(D_0)(\upsilon))\\
&\le \Big[L_1(1+P) + \frac{2P\overline{P}}{\Gamma(\alpha+1)}b^\alpha(l_1^* + l_2^* 2m^* + l_3^* 2n^*)\Big]\beta(D).
\end{aligned}$$

From Eqs. (7) and (8), we have

$$\beta(QD) \leq 2\Big[L_1(1+P) + \frac{2P\overline{P}}{\Gamma(\alpha+1)} b^{\alpha}(l_1^* + l_2^* 2m^* + l_3^* 2n^*)\Big]\beta(D) < \beta(D).$$

Thus $Q\colon B_r \to B_r$ is a condensing operator and therefore from Lemma 5, we conclude that Q has a fixed point on B_r. □

4 Examples

Consider the following fractional partial differential system

$$\begin{aligned}
{}^{C}D_{0^+}^{\alpha}[y(\upsilon,x) - u(\upsilon, y(\upsilon,x))] &= \frac{\partial^2}{\partial x^2}[y(\upsilon,x) - u(\upsilon, y(\upsilon,x))] \\
&\quad + f\Big(\upsilon, y(\upsilon,x), \int_0^{\upsilon} h(\upsilon,\varepsilon,y(\varepsilon,x))d\varepsilon, \int_0^{b} g(\upsilon,\varepsilon,y(\varepsilon,x))d\varepsilon\Big), \\
&\qquad \upsilon \in [0,b], \quad x \in \Omega = [0,\pi], \\
y(\upsilon,0) = 0 = y(\upsilon,\pi), &\quad \upsilon \in [0,b], \\
y(0,x) = y_0(x), &\quad x \in \Omega,
\end{aligned}$$

where $b > 0$ is finite and $y_0 \in C(\Omega, \mathbf{R})$ with $y_0(0) = 0 = y_0(\pi)$.

Next, let $Y = C(\Omega, \mathbf{R})$ and consider $A\colon D(A) \subset Y \to Y$ defined by

$$Aw = \frac{\partial^2 w}{\partial x^2}$$

with its domain of definition,

$$D(A) = \Big\{w \in Y\colon \frac{\partial^2 w}{\partial x^2} \in Y \text{ and } w = 0 \text{ on } \partial\Omega\Big\}.$$

Then,

$$\overline{D(A)} = \{w \in Y\colon w = 0 \text{ on } \partial\Omega\} \neq Y.$$

Also, from [13], it is known that A satisfies Hille-Yosida condition with $(0,\infty) \subset \rho(A)$, $\|R(\lambda : A)\| \leq \lambda^{-1}$ and $\overline{P} = 1$ and generates a compact C_0-semigroup $\{P(\upsilon)\}_{\upsilon>0}$ on $\overline{D(A)}$ with $P = 1$.

Set,

$$\begin{aligned}
&y(\upsilon)(x) = y(\upsilon,x), \\
&f(\upsilon, y(\upsilon), (Hy)(\upsilon), (Gy)(\upsilon))(x) \\
&= f\Big(\upsilon, y(\upsilon,x), \int_0^{\upsilon} h(\upsilon,\varepsilon,y(\varepsilon,x))d\varepsilon, \int_0^{b} g(\upsilon,\varepsilon,y(\varepsilon,x))d\varepsilon\Big)
\end{aligned}$$

for $\upsilon \in [0, b]$, $x \in \Omega$ then (1) is the abstract formulation of the above considered problem.

Also, take

$$h(\upsilon, \varepsilon, y(\varepsilon, x)) = \upsilon \sin y(\varepsilon, x) \text{ and } g(\upsilon, \varepsilon, y(\varepsilon, x)) = \varepsilon \sin y(\varepsilon, x),$$

then h and g satisfies (H3)(i), (ii) and (H4)(i), (ii) respectively with $H^* = b^2$ and $G^* = \frac{b^2}{2}$.

Consider

$$\begin{aligned} &f\big(\upsilon, y(\upsilon, x), \int_0^\upsilon h(\upsilon, \varepsilon, y(\varepsilon, x))d\varepsilon, \int_0^b g(\upsilon, \varepsilon, y(\varepsilon, x))d\varepsilon\big) \\ &= \exp(-\upsilon)\cos\left(\frac{|y(\upsilon, x)|}{1 + |y(\upsilon, x)|} + \int_0^\upsilon h(\upsilon, \varepsilon, y(\varepsilon, x))d\varepsilon + \int_0^b g(\upsilon, \varepsilon, y(\varepsilon, x))d\varepsilon\right). \end{aligned}$$

Here, choose $l(\upsilon) = \exp(-\upsilon)$ and assuming u to be a suitable function satisfying (H5), Theorem 12 implies the existence of integral solution of our problem.

5 Conclusion

This paper is concerned with the existence of integral solution of a class of neutral fractional integro-differential equation of mixed type when the operator A is not dense. Here, we have used fixed point theorems, fractional calculus, and measure of noncompactness, to obtain some sufficient conditions which ensure the existence of integral solutions of our problem, when the associated C_0-semigroup is generated by the part of A in $\overline{D(A)}$ is compact or non-compact.

Acknowledgements The first author expresses her gratitude to Indian Institute of Technology Guwahati for providing her fellowship to pursue her PhD.

References

1. Baleanu, D., Chen, W., Chen, Y.Q., Sun, H.G., Zhang, Y.: A new collection of real world applications of fractional calculus in science and engineering. Commun. Nonlinear Sci. Numer. Simul. **64**, 213–231 (2018)
2. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
3. Diethelm, K.: The Analysis of Fractional Differential Equations. Springer, New York (2010)
4. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, New York (2006)
5. Al-Omari, A., Jaradat, O.K., Momani, S.: Existence of the mild solution for fractional semilinear initial value problems. Nonlinear Anal. **69**, 3153–3159 (2008)
6. Mophou, G.: Existence and uniqueness of mild solutions to impulsive fractional differential equations. Nonlinear Anal. **72**, 1604–1615 (2010)

7. Jia, J., Li, K.: Existence and uniqueness of mild solutions for abstract delay fractional differential equations. Comput. Math. Appl. **62**, 1398–1404 (2011)
8. Jiao, F., Zhou, Y.: Nonlocal Cauchy problem for fractional evolution equations. Nonlinear Anal. Real World Appl. **11**, 4465–4475 (2010)
9. Shen, X.H., Zhang, L., Zhou, Y.: Existence of mild solutions for fractional evolution equations. J. Int. Equ. Appl. **25**, 557–586 (2013)
10. Shu, X.B., Wang, Q.: The existence and uniqueness of mild solutions for fractional differential equations with nonlocal conditions of order $1<\alpha<2$. Comput. Math. Appl. **64**, 2100–2110 (2012)
11. Jia, J., Li, K., Peng, J.: Cauchy problems for fractional evolution equations with Riemann-Liouville fractional derivatives. J. Funct. Anal. **263**, 476–510 (2012)
12. Feng, Z., Li, Y.N., Sun, H.R.: Fractional abstract Cauchy problem with order $\alpha \in (1, 2)$. Dyn. Partial Differ. Equ. **13**, 155–177 (2016)
13. Prato, G.D., Sinestrari, E.: Differential operators with nondense domain. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14**, 285–344 (1987)
14. Ezzinbi, K., Liu, J.H.: Nondensely defined evolution equations with nonlocal conditions. Math. Comput. Model. **36**, 1027–1038 (2002)
15. Fan, Z.: Existence of nondensely defined evolution equations with nonlocal conditions. Nonlinear Anal. **70**, 3829–3836 (2009)
16. Guérékata, G.M.N., Mophou, G.M.: On integral solutions of some nonlocal fractional differential equations with nondense domain. Nonlinear Anal. **71**, 4668–4675 (2009)
17. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York (1983)
18. Ahmad, B., Alsaedi, A., Gu, H., Zhou, Y.: Integral solutions of fractional evolution equations with nondense domain. Electron. J. Differ. Equ. **2017**, 1–15 (2017)
19. Fu, X.: On solutions of neutral nonlocal evolution equations with nondense domain. J. Math. Anal. Appl. **299**, 392–410 (2004)
20. Benchohra, M., Seba, D.: Impulsive fractional differential equations in Banach spaces. Electron. J. Qual. Theory Differ. Equ. **Special Edition I**, 1–14 (2009)
21. Chen, P., Li, Y.: Monotone iterative technique for a class of semilinear evolution equations with nonlocal conditions. Results Math. **63**, 731–744 (2013)
22. Banas, J., Goebel, K.: Measure of Noncompactness in Banach spaces. In: Lecture Notes in Pure and Applied Mathematics. Marcel Dekker, Inc., New York (1980)
23. Liu, Y.L., Lv, J.Y.: Existence results for Riemann-Liouville fractional neutral evolution equations. Adv. Differ. Equ. **2014**, 1–16 (2014)
24. Deimling, K.: Nonlinear Functional Analysis. Springer, New York (1985)
25. Zhou, Y.: Basic Theory of Fractional Differential Equations. World Scientific, Singapore (2014)

Trajectory Controllability of Integro-Differential Systems of Fractional Order $\gamma \in (1, 2]$ in a Banach Space with Deviated Argument

Parveen Kumar, Ankit Kumar, Ramesh K. Vats, and Avadhesh Kumar

Abstract In this paper, the fractional integro-differential control system of order $\gamma \in (1, 2]$ in a Banach space with deviated argument is considered. In order to study the trajectory controllability for the proposed control problem, the theory of fractional calculus, Gronwall's inequality, and fractional order cosine family are used. Finally, we provide an example to illustrate our main results.

Keywords Integro-differential equation · Trajectory controllability · Gronwall's inequality · Fractional order cosine family · Deviated argument

2010 AMS Subject Classification: 34A08 · 34K30 · 93B05

1 Introduction

Controllability is a crucial concept in mathematical control theory. Particularly, it has importance in the classical theory of dynamical control systems. There are various types of controllability such as exact controllability, approximate controllability, null controllability, complete controllability and trajectory controllability/T-controllability. Many authors (see [1–6]) investigated the trajectory controllability which is a stronger notion of controllability. On the other hand, fractional calculus has gained great interest from researchers and many authors worked on it (for more, see [7–11]).

P. Kumar · A. Kumar (✉) · R. K. Vats
Department of Mathematics and Scientific Computing, National Institute of Technology Hamirpur (H.P.), Hamirpur 177 005, India
e-mail: ankitramkumar620@gmail.com

R. K. Vats
e-mail: rkvats@nith.ac.in

A. Kumar
Department of Mathematics and Computer Science, Sri Sathya Sai Institute of Higher Learning (A.P.), Anantapur 515134, India

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_7

M. Muslim et al. [5] studied exact and trajectory controllability with deviated argument of second-order nonlinear systems in a Banach space. Exact controllability of fractional integro-differential systems with deviated argument was investigated in [12]. Motivated by the above observation, we consider fractional integro-differential equation in a Banach Space $\mathcal{V}$ with deviated argument as follows:

$$\begin{cases} {}^{c}\mathcal{D}_t^{\gamma}\vartheta(t) = \mathscr{A}\vartheta(t) + \mathscr{W}(t) + \mathscr{L}(t, \vartheta(t), \vartheta[\mathscr{H}(\vartheta(t), t)]) \\ \qquad\qquad + \mathscr{G}(t, \vartheta(t), \int_0^t \mathscr{Y}(t, \varpi, \vartheta(\varpi))d\varpi), \quad t \in (0, T], \\ \vartheta(0) = y_0, \quad \vartheta'(0) = y_1, \end{cases} \tag{1}$$

where ${}^{c}\mathcal{D}_t^{\gamma}$, $\gamma \in (1, 2]$ is the Caputo fractional derivative, $\vartheta : \mathcal{I}(= [0, T]) \to \mathcal{V}$ is the state function, $\mathscr{A}$ is the infinitesimal generator of a strongly continuous γ-order cosine family $(\mathscr{C}_\gamma(t))_{t\geq 0}$ on $\mathcal{V}$ and the control function $\mathscr{W}(\cdot) \in L^2(\mathcal{I}, \mathcal{W})$, $\mathcal{W}$ is a Hilbert space of control functions known as control space. Continuous functions $\mathscr{L}$, $\mathscr{H}$, $\mathscr{G}$ and $\mathscr{Y}$ are to be specified later.

2 Preliminaries

In this section, we introduce a few notations, definitions and assumptions needed to establish the results. Let $\mathcal{V}$ be a Banach space with norm $||.||$, and $L(\mathcal{V})$ denote the space of all bounded linear operators form $\mathcal{V}$ into $\mathcal{V}$. $L^p([0, T], \mathcal{V})$, $1 \leq p < \infty$ be the space of $\mathcal{V}$-valued functions $\tilde{\mathscr{L}} : [0, T] \to \mathcal{V}$ in the Bochner sense endowed with the norm

$$||\tilde{\mathscr{L}}||_{L^p} = \Big(\int_0^T ||\tilde{\mathscr{L}}(t)||^p dt\Big)^{\frac{1}{p}}.$$

The spaces $C([0, T], \mathcal{V})$, and $C^1([0, T], \mathcal{V})$ are the space of continuous and continuously differentiable functions, respectively, endowed with the norms

$$||\tilde{\mathscr{L}}||_C = \sup_{t\in\mathcal{I}} ||\tilde{\mathscr{L}}(t)||, \quad ||\tilde{\mathscr{L}}||_{C^1} = \sup_{t\in\mathcal{I}} \sum_{k=0}^{1} ||\tilde{\mathscr{L}}^k(t)||.$$

Now, we define the set $C_L(\mathcal{I}, \mathcal{V}) = \{\vartheta \in C([0, T], \mathcal{V})\}$ endowed with supremum norm

$$||\vartheta(t) - \vartheta(\varpi)|| \leq \mathcal{L}|t - \varpi|, \forall\, t,\ \varpi \in \mathcal{I}\,, \mathcal{L} > 0.$$

Definition 2.1 The Riemann-Liouville fractional integral operator of order $\gamma > 0$ is defined by

$$I_t^{\gamma}\vartheta(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t - s)^{\gamma-1}\vartheta(\varpi)d\varpi,$$

where $\vartheta(t) \in L^1([0,T],\mathcal{V})$ and $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 The Riemann-Liouville fractional derivative of order $\gamma \in (1,2]$ is defined as

$$\mathcal{D}_t^{\gamma}\vartheta(t) = \frac{d^2}{dt^2} I_t^{2-\gamma}\vartheta(t),$$

where $\vartheta(t),\ \mathcal{D}_t^{\gamma}\vartheta(t) \in L^1([0,T],\mathcal{V})$.

Definition 2.3 For $\gamma \in (1,2]$, the Caputo fractional derivative is defined by

$${}^{c}\mathcal{D}_t^{\gamma}\vartheta(t) = I_t^{2-\gamma}\frac{d^2}{dt^2}\vartheta(t),$$

where $\vartheta(t) \in L^1([0,T],\mathcal{V}) \cap C^1([0,T],\mathcal{V})$.

Consider the differential equation of fractional order as follows:

$${}^{c}\mathcal{D}_t^{\gamma}\vartheta(t) = \mathscr{A}\vartheta(t), \quad \vartheta(0) = \eta,\ \vartheta'(0) = 0, \tag{2}$$

where $\gamma \in (1,2],\ \mathscr{A} : D(\mathscr{A}) \subset \mathcal{V} \to \mathcal{V}$ is a linear operator in Banach space $\mathcal{V}$.

Definition 2.4 ([10]) A family $(\mathscr{C}_\gamma(t))_{t\geq 0} \subset L(\mathcal{V}),\ \gamma \in (1,2]$ is called a strongly continuous cosine family of fractional order for (2) and $\mathscr{A}$ is the infinitesimal generator of $\mathscr{C}_\gamma(t)$, if they hold the following:

(i) $\mathscr{C}_\gamma(t)$ is strongly continuous for $t \geq 0$ and $\mathscr{C}_\gamma(0) = I$, where I is identity operator;

(ii) $\mathscr{C}_\gamma(t)D(\mathscr{A}) \subset D(\mathscr{A})$ and $\mathscr{A}\mathscr{C}_\gamma(t)\upsilon = \mathscr{C}_\gamma(t)\mathscr{A}\upsilon$ for all $\upsilon \in D(\mathscr{A}),\ t \geq 0$;

(iii) $\mathscr{C}_\gamma(t)\upsilon$ is the solution of $\vartheta(t) = \eta + \frac{1}{\Gamma(\gamma)}\int_0^t (t-\varpi)^{\gamma-1}\mathscr{A}\vartheta(\varpi)d\varpi \quad \forall \quad \upsilon \in D(\mathscr{A})$.

Definition 2.5 The fractional sine family $\mathscr{S}_\gamma : [0,\infty) \to L(\mathcal{V})$ associated with $\mathscr{C}_\gamma$ is defined as

$$\mathscr{S}_\gamma(t) = \int_0^t \mathscr{C}_\gamma(\varpi)d\varpi,\ t \geq 0. \tag{3}$$

Definition 2.6 The fractional Riemann Liouville family $\mathscr{P}_\gamma : [0,\infty) \to L(\mathcal{V})$ of order γ associated with $\mathscr{C}_\gamma$ is defined as

$$\mathscr{P}_\gamma(t) = I^{\gamma-1}\mathscr{C}_\gamma(t). \tag{4}$$

Definition 2.7 The cosine family $\mathscr{C}_\gamma(t)$ of order γ is called exponentially bounded if $\exists\ \hat{\mathcal{N}}, \delta \geq 0$ such that

$$||\mathscr{C}_\gamma(t)|| \leq \hat{\mathcal{N}} e^{\delta t}, \ t \geq 0. \tag{5}$$

Infinitesimal generator $\mathscr{A}$ belongs to $C^\gamma(X; \hat{\mathcal{N}}, \delta)$, if the problem (2) has an solution operator $\mathscr{C}_\gamma(t)$ satisfying (4).

Definition 2.8 ([5]) A function $\vartheta(\cdot) \in C_L([0, T], \mathcal{V})$ is the mild solution of fractional integro-differential control system (1) if $\vartheta(t)$ is the solution of the following equation:

$$\begin{aligned}\vartheta(t) \ &= \mathscr{C}_\gamma(t) y_0 + \mathscr{S}_\gamma(t) y_1 + \textstyle\int_0^t \mathscr{P}_\gamma(t-\varpi) \big[\mathscr{L}(\varpi, \vartheta(\varpi), \vartheta[\mathscr{H}(\upsilon(\varpi), \varpi)]) d\varphi \\ &\quad + \mathscr{G}\big(\varpi, \vartheta(\varpi), \textstyle\int_0^\varpi \mathscr{Y}(\varpi, \zeta, \vartheta(\zeta)) d\zeta\big) + \mathscr{W}(\varpi)\big] d\varpi. \end{aligned} \tag{6}$$

Let $\mathcal{T}$ be the set of all feasible trajectories $\Theta(\cdot)$ defined on the interval $\mathcal{I}$, where $\mathcal{I} = [0, T]$ such that $\Theta(0) = y_0$, $\Theta'(0) = y_1$ and $\Theta(T) = \vartheta_T \ \ \forall\, t \in \mathcal{I}$ for the system (1) and ${}^c\mathcal{D}_t^\gamma \Theta(t)$ exists almost everywhere on $\mathcal{I}$.

Definition 2.9 The fractional integro-differential control system (1) is said to be trajectory-controllable (T-controllable) on $\mathcal{I}$, if for $\Theta \in \mathcal{T}$ such that (6) satisfies $\vartheta(t) = \Theta(t)$ almost everywhere.

Definition 2.10 The system (1) is said to be exactly controllable on $\mathcal{I}$, if for every $\vartheta_0, y_0, \vartheta_T \in \mathcal{V}$ there exists a control $\mathscr{W}(\cdot) \in L^2(\mathcal{I}, \mathcal{W})$ such that the mild solution of (6) satisfies $\vartheta(T) = \vartheta_T$.

Definition 2.11 The system (1) is said to be totally controllable on $\mathcal{I}$, if for all subintervals $[t_k, t_{k+1}]$ of $\mathcal{I}$, the system (1) is exactly controllable.

Remark: Trajectory controllability $\Rightarrow$ Total controllability $\Rightarrow$ Exact controllability.

In order to establish the main result for system (1), we required the following hypotheses:

(H1) $\mathscr{A}$ is the infinitesimal generator of a strongly continuous γ-order cosine family $||\mathscr{C}_\gamma(t)$ on $\mathcal{V}$ and there exists a constant $\bar{\mathcal{N}}_\mathscr{C} \geq 1$, such that $||\mathscr{C}_\gamma(t)|| \leq \bar{\mathcal{N}}_\mathscr{C}, \ \ \bar{\mathcal{N}}_\mathscr{C} \geq 1$.

(H2) $\mathscr{L}, \mathscr{G} \ : \mathcal{I} \times \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ are a continuous function and $\exists$, $\mathcal{K}^*, \mathcal{K}, \mathcal{M}^*{}_\mathscr{Y}, \mathcal{M}_\mathscr{G}$ and $\mathcal{N}_\mathscr{G}$ positive constants, such that

(i)

$$||\mathscr{L}(t, \vartheta_1, \mu_1) - \mathscr{L}(t, \vartheta_2, \mu_2)|| \leq \mathcal{K}^*(||\vartheta_1 - \vartheta_2|| + ||\mu_1 - \mu_2||), \ \text{forall} \ \vartheta_1, \vartheta_2, \mu_1, \mu_2 \in \mathcal{V},$$

and $max_{t \in \mathcal{I}} ||\mathscr{L}(t, 0, \vartheta(0)|| = \mathcal{K}$.

(ii)

$$||\mathscr{G}(t, \vartheta_1, \mu_1) - \mathscr{G}(t, \vartheta_2, \mu_2)|| \leq \mathcal{M}_\mathscr{G} ||\vartheta_1 - \vartheta_2|| + \mathcal{N}_\mathscr{G} ||\mu_1 - \mu_2||).$$

(iii)

$$\int_0^t ||\mathscr{Y}(t, \varpi, \vartheta(\varpi)) - \mathscr{Y}(t, \varpi, \mu(\varpi))|| \leq \mathcal{M}^*{}_\mathscr{Y} ||\vartheta(t) - \mu(t)||.$$

(H3) $\mathscr{H} : \mathcal{V} \times \mathcal{I} \to \mathcal{I}$ is a uniformly continuous and $\exists$, $\mathcal{C}_{\mathscr{H}} = \mathcal{C}_{\mathscr{H}}(t)$, $\mathcal{C}_{\mathscr{H}} > 0$, such that

$$|\mathscr{H}(\vartheta_1, \varpi) - \mathscr{H}(\vartheta_2, \varpi)| \leq \mathcal{C}_{\mathscr{H}} ||\vartheta_1 - \vartheta_2||, \forall \vartheta_1, \vartheta_2 \in \mathcal{V} \text{ whenever } 0 \leq \varpi \leq t$$

and hold $\mathscr{H}(\cdot, 0) = 0$ for each $t > 0$.

3 Trajectory Controllability

Theorem 3.1 *Assume that all (H1)–(H3) are satisfied, then the integro-differential system (1) of order $\gamma \in (1, 2]$ is trajectory-controllable on $\mathcal{I}$.*

Proof Let $\Theta(t)$ be any given trajectory in $\mathcal{T}$ and the feedback control function $\mathscr{W}(t)$ as follows:

$$\mathscr{W}(t) = {}^{c}\mathcal{D}_t^{\gamma}\Theta(t) - \mathscr{A}\Theta(t) - \mathscr{L}(t, \Theta(t), \Theta[\mathscr{H}(\Theta(t), t)]) - \mathscr{G}\left(t, \Theta(t), \int_0^t \mathscr{Y}(t, \varpi, \Theta(\varpi))d\varpi\right). \tag{7}$$

From Eq. (7) in Eq. (1) and we get

$$\begin{aligned} {}^{c}\mathcal{D}_t^{\gamma}\vartheta(t) &= \mathscr{A}\vartheta(t) + \mathscr{L}(t, \vartheta(t), \vartheta[\mathscr{H}(\vartheta(t), t)]) + \mathscr{G}\left(t, \vartheta(t), \int_0^t \mathscr{Y}(t, \varpi, \vartheta(\varpi))d\varpi\right) \\ &+ {}^{c}\mathcal{D}_t^{\gamma}\Theta(t) - \mathscr{A}\Theta(t) - \mathscr{L}(t, \Theta(t), \Theta[\mathscr{H}(\Theta(t), t)]) - \mathscr{G}\left(t, \Theta(t), \int_0^t \mathscr{Y}(t, \varpi, \Theta(\varpi))d\varpi\right). \end{aligned}$$

Hence, we have

$$\begin{aligned} {}^{c}\mathcal{D}_t^{\gamma}[\vartheta(t) - \Theta(t)] &= \mathscr{A}[\vartheta(t) - \Theta(t)] + \mathscr{L}(t, \vartheta(t), \vartheta[\mathscr{H}(\vartheta(t), t)]) - \mathscr{L}(t, \Theta(t), \Theta[\mathscr{H}(\Theta(t), t)]) \\ &+ \mathscr{G}\left(t, \vartheta(t), \int_0^t \mathscr{Y}(t, \varpi, \vartheta(\varpi))d\varpi\right) - \mathscr{G}\left(t, \Theta(t), \int_0^t \mathscr{Y}(t, \varpi, \Theta(\varpi))d\varpi\right). \end{aligned}$$

Choose $\varkappa(t) = \vartheta(t) - \Theta(t)$ considering the following IVP:

$$\begin{cases} {}^{c}\mathcal{D}_t^{\gamma}\varkappa(t) = \mathscr{A}\varkappa(t) + \mathscr{L}(t, \vartheta(t), \vartheta[\mathscr{H}(\vartheta(t), t)]) - \mathscr{L}(t, \Theta(t), \Theta[\mathscr{H}(\Theta(t), t)]) \\ \qquad + \mathscr{G}\left(t, \vartheta(t), \int_0^t \mathscr{Y}(t, \varpi, \vartheta(\varpi))d\varpi\right) - \mathscr{G}\left(t, \Theta(t), \int_0^t \mathscr{Y}(t, \varpi, \Theta(\varpi))d\varpi\right) \\ \varkappa(0) = 0, \quad \varkappa'(0) = 0. \end{cases} \tag{8}$$

The mild solution for system (8) is given by

$$\begin{aligned} \varkappa(t) &= \int_0^t \mathscr{P}_{\gamma}(t - \varpi)\Big[\mathscr{L}(t, \vartheta(t), \vartheta[\mathscr{H}(\vartheta(t), t)]) - \mathscr{L}(t, \Theta(t), \Theta[\mathscr{H}(\Theta(t), t)]) \\ &+ \mathscr{G}\left(t, \vartheta(t), \int_0^{\varpi} \mathscr{Y}(\varpi, \zeta, \vartheta(\zeta))d\zeta\right) - \mathscr{G}\left(t, \Theta(t), \int_0^{\varpi} \mathscr{Y}(\varpi, \zeta, \Theta(\zeta))d\zeta\right)\Big]ds. \end{aligned}$$

Hence, we have

$$
\begin{aligned}
\|\varkappa(t)\| = & \Big\| \int_0^t \mathscr{P}_\gamma(t-\varpi)\Big[\mathscr{L}(t,\vartheta(t),\vartheta[\mathscr{H}(\vartheta(t),t)]) - \mathscr{L}(t,\Theta(t),\Theta[\mathscr{H}(\Theta(t),t)]) \\
& + \mathscr{G}\left(t,\vartheta(t),\int_0^\varpi \mathscr{Y}(\varpi,\zeta,\vartheta(\zeta))d\zeta\right) - \mathscr{G}\left(t,\Theta(t),\int_0^\varpi \mathscr{Y}(\varpi,\zeta,\Theta(\zeta))d\zeta\right)\Big]ds\Big\| \\
\leq & \int_0^t \|\mathscr{P}_\gamma(t-\varpi)\|\|\mathscr{L}(t,\vartheta(t),\vartheta[\mathscr{H}(\vartheta(t),t)]) - \mathscr{L}(t,\Theta(t),\Theta[\mathscr{H}(\Theta(t),t)])\| \\
& + \|\mathscr{G}\left(t,\vartheta(t),\int_0^\varpi \mathscr{Y}(\varpi,\zeta,\vartheta(\zeta))d\zeta\right) - \mathscr{G}\left(t,\Theta(t),\int_0^\varpi \mathscr{Y}(\varpi,\zeta,\Theta(\zeta))d\zeta\right)\|\Big]ds \\
\leq & \frac{\bar{\mathcal{N}}_{\mathscr{C}} T^{\gamma-1}}{\Gamma(\gamma)} \int_0^t \mathcal{K}^*(2+\mathcal{L}.\mathcal{C}_{\mathscr{H}})\|\vartheta(t)-\Theta(t)\|ds \\
& + \frac{\bar{\mathcal{N}}_{\mathscr{C}} T^{\gamma-1}}{\Gamma(\gamma)} \int_0^t (\mathcal{M}_{\mathscr{G}} + \mathcal{N}_{\mathscr{G}}\mathcal{M}^*{}_{\mathscr{Y}})\|\vartheta(t)-\Theta(t)\|ds, \\
\leq & \frac{\bar{\mathcal{N}}_{\mathscr{C}} T^{\gamma-1}}{\Gamma(\gamma)} \int_0^t [\mathcal{K}^*(2+\mathcal{L}.\mathcal{C}_{\mathscr{H}}) + (\mathcal{M}_{\mathscr{G}} + \mathcal{N}_{\mathscr{G}}\mathcal{M}^*{}_{\mathscr{Y}})]\|\varkappa(t)\|ds \\
\leq & \frac{\bar{\mathcal{N}}_{\mathscr{C}} T^{\gamma-1}}{\Gamma(\gamma)} \hat{\Delta} \int_0^t \|\varkappa(t)\|ds
\end{aligned} \tag{9}
$$

where $\hat{\Delta} = \mathcal{K}^*(2+\mathcal{L}.\mathcal{C}_{\mathscr{H}}) + (\mathcal{M}_{\mathscr{G}} + \mathcal{N}_{\mathscr{G}}\mathcal{M}^*{}_{\mathscr{Y}})$. By using Gronwall's inequality in Eq. (9), we obtain $\varkappa(t) = 0$. Hence, $\vartheta(t) = \Theta(t)$ almost everywhere. Thus, for $\gamma \in (1,2]$ fractional integro-differential system (1) is trajectory-controllable. □

4 Application

Let $\mathcal{V} = L^2(0,\pi)$. Consider the following system of fractional order:

$$
\begin{cases}
{}^c\mathcal{D}_t^\gamma \aleph(t,\varsigma) = \partial_\varsigma(t,\varsigma) + \mathscr{U}(t,\varsigma) + \mathscr{Q}(\varsigma,\aleph(t,\varsigma)), + \mathscr{R}(t,\varsigma,\aleph(t,\varsigma)), \\
\aleph(t,0) = \aleph(t,\pi) = 0, \ t \in [0,T], \ 0 < T < \infty, \\
\aleph(0,\varsigma) = y_0, \ \varsigma \in (0,\pi), \\
\partial_t\aleph(0,\varsigma) = \varsigma_0, \varsigma \in (0,\pi),
\end{cases} \tag{10}
$$

where

$$
\gamma \in (1,2], \quad \mathscr{R}(t,\varsigma,\aleph(t,\varsigma)) = \int_0^\varsigma \mathscr{K}(\varsigma,\varpi)\aleph(\varpi, \mathscr{J}(t)(c_1|\aleph(t,\varpi)| + c_2|\aleph(t,\varpi)|))d\varpi.
$$

We assume that $c_1, c_2 \geq 0$, $(c_1,c_2) \neq (0,0)$, $\mathscr{J} : \mathcal{I} \to \mathcal{I}$ is locally Hölder continuous in t with $\mathscr{J}(0) = 0$ and $\mathscr{K} : [0,\pi] \times [0,\pi] \to \mathbb{R}$. The operator $\mathscr{U}$ is defined as follows:

$$
\mathscr{A}\vartheta = \vartheta'' \quad \text{with} \quad \vartheta \in D(\mathscr{A}) = \{\vartheta \in H_0^1(0,\pi) \cap H^2(0,\pi) : \ \vartheta'' \in \mathcal{V}\}. \tag{11}
$$

Then, $\mathscr{A}$ is represented as in the following series:

$$\mathscr{A}\vartheta = \sum_{n=1}^{\infty} -n^2(\vartheta, \vartheta_n)\vartheta_n, \quad \vartheta \in D(\mathscr{A}),$$

where the orthonormal set of eigenfunctions of $\mathscr{A}$ is $\vartheta_n(\varpi) = \sqrt{2/\pi}\sin n\varpi$, $n = 1, 2, 3 \ldots$. Moreover, the operator $\mathscr{A}$ is the infinitesimal generator of $\mathscr{C}(t)_{t\in\mathbb{R}}$ on $\mathcal{V}$ (see [7, 13] for more about cosine family). It is given by

$$\mathscr{C}(t)\vartheta = \sum_{n=1}^{\infty} \cos nt(\vartheta, \vartheta_n)\vartheta_n, \quad \vartheta \in \mathcal{V},$$

and the sine family $\mathscr{S}(t)_{t\in\mathbb{R}}$ associated with $\mathscr{C}(t)_{t\in\mathbb{R}}$ on $\mathcal{V}$ is as follows:

$$\mathscr{S}(t)\vartheta = \sum_{n=1}^{\infty} \frac{1}{n}\sin nt(\vartheta, \vartheta_n)\vartheta_n, \quad \vartheta \in \mathcal{V}.$$

For $\gamma = 2$, Eq. (10) can be converted into the following in $\mathcal{V} = L^2(0, \pi)$:

$$\begin{cases} \vartheta''t) = \mathscr{A}\vartheta(t) + \mathscr{W}(t) + \mathscr{L}(t, \vartheta(t), \vartheta[\mathscr{H}(\vartheta(t), t)]) \\ \qquad\qquad +\mathscr{G}(t, \vartheta(t), \int_0^t \mathscr{Y}(t, \varpi, \vartheta(\varpi)d\varpi), \quad t > 0, \\ \vartheta(0) = y_0, \quad \vartheta'(0) = y_1, \end{cases}$$

where $\vartheta(t) = \aleph(t, .)$, that is, $\vartheta(t)(\varsigma) = \aleph(t, \varsigma)$, $u(t)(\varsigma) = \mathscr{U}(t, \varsigma)$, $\varsigma \in (0, \pi)$ and $\mathscr{A}$ is the same as in Eq. (11). The function $\mathscr{L} : \mathcal{I} \times \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ is given by

$$\mathscr{L}(t, \varrho, \xi)(\varsigma) = \mathscr{Q}(\varsigma, \xi) + \mathscr{R}(t, \varsigma, \varrho),$$

where $\mathscr{Q} : [0, \pi] \times \mathcal{V} \to H_0^1(0, \pi)$ is given by

$$\mathscr{Q}(\varsigma, \xi) = \int_0^{\varsigma} \mathscr{K}(\varsigma, \vartheta)\xi(\vartheta)d\vartheta,$$

and

$$\|\mathscr{R}(t, \varsigma, \varrho)\| \le \mathscr{V}(\varsigma, t)(1 + \|\varrho\|_{H^2(0,\pi)})$$

with $\mathscr{V}(., t) \in \mathcal{V}$ being continuous with respect to the second argument (see [14]). Thus, Theorem (3.1) can be applied in differential system (10).

For $\gamma \in (1, 2)$, $\mathscr{A}$ is the infinitesimal generator of $\mathscr{C}(t)_{t\in\mathbb{R}}$, form the subordinate principle (Theorem 3.1, [10]), it follows that $\mathscr{A}$ infinitesimal generator of $\mathscr{C}_\gamma(t)$ such that $\mathscr{C}_\gamma(0) = I$, and

$$\mathscr{C}_\gamma(t) = \int_0^{\infty} \sigma_{t,\gamma/2}(\varpi)C(\varpi)d\varpi, \quad t > 0,$$

where $\sigma_{t,\gamma/2}(\varpi) = t^{-\gamma/2}\phi_{\gamma/2}(\varpi t^{-\gamma/2})$, and

$$\phi_\beta(\varsigma) = \sum_{n=0}^{\infty} \frac{(-\varsigma)^n}{n!\Gamma(-\beta n + 1 - \beta)}, \quad 0 < \beta < 1.$$

Then Eq. (10) can be reformulated into the following equation of order $\gamma \in (1, 2]$ in $\mathcal{V} = L^2(0, \pi)$:

$$\begin{cases} {}^c\mathcal{D}_t^\gamma \vartheta(t) = \mathscr{A}\vartheta(t) + \mathscr{W}(t) + \mathscr{L}(t, \vartheta(t), \vartheta[\mathscr{H}(\vartheta(t), t)]) \\ \qquad\qquad + \mathscr{G}(t, \vartheta(t), \int_0^t \mathscr{Y}(t, \varpi, \vartheta(\varpi))d\varpi), \quad t > 0, \\ \vartheta(0) = y_0, \quad \vartheta'(0) = y_1. \end{cases}$$

Acknowledgements The third author would like to acknowledge CSIR-HRDG, Government of India, for supporting this research work under Grant 25(0268)/17/EMR-II.

References

1. Kumar, S., Sukavanam, N.: Approximate controllability of fractional order semilinear systems with bounded delay. J. Differ. Equ. **252**(11), 6163–6174 (2012)
2. Muslim, M., George, R.K.: Trajectory controllability of the nonlinear systems governed by fractional differential equations. Differ. Equ. Dyn. Syst. (2016). https://doi.org/10.1007/s12591-016-0292-z
3. Chalishajar, D.N., George, R.K., Nandakumaran, A.K., Acharya, F.S.: Trajectory controllability of nonlinear integro-differential system. J. Frankl. Inst. **347**, 1065–1075 (2010)
4. Kumar, A., Vats, R.K., Kumar, A.: Approximate controllability of second-order non-autonomous system with finite delay. J. Dyn. Control Syst. **26**, 611–627 (2020)
5. Li, K., Peng, J., Gao, J.: Controllability of nonlocal fractional differential systems of order $\gamma \in (1, 2]$, in Banach spaces. Rep. Math. Phys. Appl. **71**(1), 33–43 (2013)
6. Muslim, M., Kumar, A., Agarwal, R.P.: Exact and trajectory controllability of second order nonlinear systems with deviated argument. Funct. Differ. Equ. **23**, 27–41 (2016)
7. Travis, C.C., Webb, G.F.: Cosine families and abstract nonlinear second order differential equations. Acta Math. Acad. Sci. Hung. **32**, 76–96 (1978)
8. Podlubny, I.: Fractional Differential Equations. Academic Press, New York (1999)
9. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
10. Bazhlekova, E.: Fractional Evolution Equations in Banach Spaces. University Press Facilities, Eindhoven University of Technology (2001)
11. Hernández, E., O'Regan, D., Balachandran, K.: On recent developments in the theory of abstract differential equations with fractional derivatives. Nonlinear Anal. **73**, 3462–3471 (2010)
12. Muslim, M., Kumar, A., Agarwal, R.P.: Exact controllability of fractional integro-differential systems of order $\alpha \in$ (1,2] with deviated argument, Analele Universit at, ii Oradea Fasc. Matematica **XXIV**, Issue no. 1, 171 (2017)
13. Travis, C.C., Webb, G.F.: Compactness, regularity and uniform continuity properties of strongly continuous cosine family. Houstan J. Math. **3**, 555–567 (1977)
14. Gal C.G.: Nonlinear abstract differential equations with deviated argument. J. Math. Anal. Appl. 177–189 (2007)

Shehu-Adomian Decomposition Method for Dispersive KdV-Type Equations

Abey S. Kelil and Appanah R. Appadu

Abstract In this paper, a new method known to be Shehu-Adomian decomposition method is proposed to solve homogeneous and non-homogeneous dispersive KdV-type equations. The Shehu-Adomian decomposition method is a combination of Shehu's transform and Adomian Decomposition method. Some illustrative problems of dispersive KdV-type equations are solved to check the validity of the method. The approximate solutions are given in series form and the proposed method is a reliable and powerful technique to solve numerous physical problems in applications.

Keywords Shehu transform · Adomian decomposition method · Dispersive linear KdV equations

2010 Mathematics Subject Classification: 35A25, 35A22, 34A45

1 Introduction

The famous Korteweg-de Vries (KdV) equation is a nonlinear dispersive PDE that describes mathematical modeling of traveling wave solution, known to be solitary water waves (also called solitons) in a shallow water domain. This equation is given by the PDE [1]

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

In 1895, Korteweg and de Vries in [1] derived this equation while studying water waves. Numerical study of KdV equations was pioneered by Zabusky and Kruskal [2] and some recent modifications of the numerical schemes were studied in [3, 4].

There are numerous methods for solving linear/nonlinear partial differential equations. One of these methods is Semi-analytical methods, which can provide

A. S. Kelil (✉) · A. R. Appadu
Nelson Mandela University, Gqeberha, South Africa
e-mail: abeysh2001@gmail.com

A. R. Appadu
e-mail: rao.appadu@mandela.ac.za

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_8

approximate-analytical solutions for problem considered. Among these methods, we can mention Adomian decomposition method [5–7], Variational iteration method [8–10], and Homotopy perturbation method [11–14]. A literature summary of some semi-analytical methods is given as follows:

(I) Adomian decomposition method (ADM) can be applied to solve linear as well as nonlinear functional equations in [5, 6, 15–17], works by dissecting the equation into linear and nonlinear parts. The method produces series solution whose terms are computed from a recursive relation involving Adomian polynomials. Various modifications of ADM were given in the works of Wazwaz [18].

(II) Homotopy perturbation method (HPM) is used to determine accurate asymptotic solutions of a nonlinear problem. This method is also used effectively to solve PDEs in modeling flows in porous media [19].

Different variants of KdV equation have been investigated in literature [8] (see also [20]). This paper addresses the following problems using some semi-analytic methods [15] and their modifications [21]:

(i) The homogeneous linear KdV equation [18]

$$\begin{cases} u_t + 2u_x + u_{xxx} = 0, \quad (x,t) \in [0, 2\pi] \times [0, 4.0], \\ u(x,0) = \sin(x). \end{cases} \tag{2}$$

Exact solution for Eq. (2) is given by

$$u(x,t) = \sin(x - t). \tag{3}$$

(ii) The non-homogeneous linear KdV equation with some source term

$$\begin{cases} u_t - u_{xxx} = 2e^{t-x}, \quad (x,t) \in [0, 1.0] \times [0, 2.0], \\ u(x,t) = 1 + e^{t-x}. \end{cases} \tag{4}$$

Exact solution for Eq. (4) is given by

$$u(x,t) = 1 + e^{t-x}. \tag{5}$$

(iii) Homogeneous nonlinear dispersive KdV equation

$$u_t + uu_x + u_{xxx} = 0, \tag{6}$$

with $(x,t) \in [0, 2\pi] \times [0, 0.50]$, and initial condition $u(x,0) = x$ and the time dependent boundary conditions are

$$u(0,t) = 0, \quad u_x(0,t) = \frac{1}{1+t}, \; u_{xx}(0,t) = 0. \tag{7}$$

Exact solution is $u(x,t) = \dfrac{x}{1+t}$.

(iv) Inhomogeneous nonlinear dispersive KdV equation [22]

$$u_t - uu_x + u_{xxxxx} = \cos(x) - t\sin(x) + \frac{t^2\sin(2x)}{2}, \tag{8}$$

with $(x,t) \in [0, 2\pi] \times [0, 0.10)$ and initial condition $u(x,0) = 0$, Exact solution is $u(x,t) = t\cos(x)$.

We see that the first term in Eq. (2) refers to time evolution and the third term refers to the dispersion term. Equation (2) is sometimes known as the 'weak dispersion' wave equation. Equation (2) can be represented as the kinematic wave equation, with a dispersive perturbation term of the third order in space. We note that exact solution for the above numerical experiments can be obtained using Ansatz method. (The same also holds for other KdV-type equations considered above).

The objective of this study is to integrate two powerful methods, Shehu transform method and Adomian decomposition method to obtain a better method for solving partial differential equations; in particular on dispersive linear as well as nonlinear KdV-type equations.

2 Adomian Decomposition Method (ADM)

This section recaps some key points of the method ADM to solve linear as well as nonlinear dispersive PDEs.

Let us take the general form of a differential equation as given in [23]:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = G(u, u_x, u_{xx}, \ldots, u_{x^n}) + s(x), \\ u(x,0) = h(x), \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \end{cases} \tag{9}$$

where $u_t = \frac{\partial u}{\partial t}$, $u_{x^i} = \frac{\partial^i u}{\partial x^i}$, $G(\cdot)$ is a polynomial function of its arguments and s is source term.

Following ADM procedures, by splitting the LHS of Eq. (9) into two parts, we have that

$$G[u] = L_G[u] + N_G[u],$$

where $L_G[u]$ is a linear operator with respect to $u, u_x, \ldots, u_{x^n}$ while $N_G[u]$ is nonlinear part of $G[u]$. Then the operator

$$L^{-1}(.) = \int_0^t (.)\, dt,$$

can be introduced to express the solution of Eq. (9) in the form:

$$u = f_0(x) + s(x)\,t + \int_0^t (L_G[u] + N_G[u])\,dt.$$

Let's suppose that

$$u(x;t) = \sum_{n=0}^{\infty} V_n(x;t), \tag{10}$$

and $L_G[u] = \sum_{i\geq 0} L_G[V_i]$ and $N_G[u] = N_G\left[\sum_{i\geq 0} V_i\right] = \sum_{i\geq 0} A_i$, where the newly introduced terms A_i are Adomian polynomials [5, 6, 24]. These polynomials are obtained by using following formulae [10, 24]

$$A_i = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[G\left(\sum_{i=0}^{n}\lambda^i V_i\right)\right]_{\lambda=0}, \tag{11}$$

and some of the first few terms of these polynomials takes the form

$$\begin{aligned}
A_0 &= N(V_0),\\
A_1 &= V_1 N'(V_0),\\
A_2 &= V_2 N'(V_0) + \frac{1}{2}V_1^2 N''(V_0),\\
A_3 &= V_3 N'(V_0) + V_1 V_2 N''(V_0) + \frac{1}{3!}V_1^3 N^{(3)}(V_0),\\
A_4 &= V_4 N'(V_0) + \left(\frac{1}{2}V_2^2 + V_1 V_3\right) N''(V_0) + \frac{1}{2!}V_1^2 V_2 N^{(3)}(V_0) + \frac{1}{4!}V_1^4 N^{(4)}(V_0).
\end{aligned}$$

One can refer to [25, 26] for detailed discussion on Adomain polynomials.

2.1 ADM Applied to Eq. (2)

Let's first rewrite Eq. (2) as

$$\begin{cases} \mathsf{L}_t u + 2u_x + u_{xxx} = 0,\\ u(x,0) = \sin(x), \end{cases} \tag{12}$$

where the differential operator is $\mathsf{L}_t = \dfrac{\partial}{\partial t}$. By assuming L_t^{-1} exists; that is, $\mathsf{L}_t^{-1}(\cdot) = \int_0^t (\cdot)\,d\tau$, and applying L_t^{-1} on both sides of Eq. (12), we have

$$\mathsf{L}_t^{-1}\mathsf{L}_t u + \mathsf{L}_t^{-1}(2u_x) + \mathsf{L}_t^{-1}(u_{xxx}) = \mathsf{L}_t^{-1}(0),$$

which is equivalently given by

$$u(x,t) = u(x,0) - \left\{\mathsf{L}_t^{-1}(2u_x) + \mathsf{L}_t^{-1}(u_{xxx})\right\}. \tag{13}$$

By employing the decomposition series given in Eq. (10) (cf. [5, 6]), the following recursive approximate values are given as

$$V_0(x) = \sin(x), \tag{14}$$

$$V_1(x;t) = -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_0(x;t)}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_0(x;t)}{\partial x^3}\right)\right\}, \tag{15}$$

$$\vdots$$

$$V_{n+1}(x;t) = -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_n}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_n(x;t)}{\partial x^3}\right)\right\}, \quad n \geq 2. \tag{16}$$

For numerical purpose, $\psi_n(x,t) = \sum_{i=0}^{n} V_i(x,t)$ denotes the n-term approximation to u. The exact solution is $u(x,t) = \lim_{n\to\infty} \psi_n(x,t)$. The number of terms required to obtain an exact solution is considerably small, which will be shown later using the proposed method in this work.

By using the recursive relations in Eqs. (15)–(16) and the linearity property of the operator L_t, we have the first few terms of $V_n(x,t)$:

$$\begin{cases} V_1(x;t) &= -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_0(x;t)}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_0(x;t)}{\partial x^3}\right)\right\} = -t\cos(x), \\ V_2(x;t) &= -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_1(x;t)}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_1(x;t)}{\partial x^3}\right)\right\} = -\frac{t^2}{2!}\sin(x), \\ V_3(x;t) &= -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_2(x;t)}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_2(x;t)}{\partial x^3}\right)\right\} = \frac{t^3}{3!}\cos(x), \\ V_4(x;t) &= -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_3(x;t)}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_3(x;t)}{\partial x^3}\right)\right\} = \frac{t^4}{4!}\sin(x), \\ V_5(x;t) &= -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_4(x;t)}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_4(x;t)}{\partial x^3}\right)\right\} = -\frac{t^5}{5!}\cos(x), \\ V_6(x;t) &= -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_5(x;t)}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_5(x;t)}{\partial x^3}\right)\right\} = -\frac{t^6}{6!}\sin(x), \\ V_7(x;t) &= -\left\{\mathsf{L}_t^{-1}\left(2\frac{\partial V_6(x;t)}{\partial x}\right) + \mathsf{L}_t^{-1}\left(\frac{\partial^3 V_6(x;t)}{\partial x^3}\right)\right\} = \frac{t^7}{7!}\cos(x) \end{cases} \tag{17}$$

and higher order V_j values are obtained from iteration formula Eq. (16). The ADM solution up to seventh order terms is

$$\psi_7(x,t) = \sum_{j=0}^{7} V_j(x,t) = \left(-t\cos(x) + \frac{t^3}{3!}\cos(x) - \frac{t^5}{5!}\cos(x) + \frac{t^7}{7!}\cos(x)\right) + \left(\sin(x) - \frac{t^2}{2!}\sin(x) + \frac{t^4}{4!}\sin(x) - \frac{t^6}{6!}\sin(x)\right). \quad (18)$$

By using Taylor's expansion and Eq. (18), we have $V_{2n}(x;t) = \dfrac{(-1)^n t^{2n}}{(2n)!}\sin(x),\ n \in \mathbb{N}_0$, and applying the principle of Mathematical Induction gives

$$\begin{aligned} V_{2n+1}(x;t) &= -\left\{\mathsf{L}_t^{-1}(2V_{2n,x}) + \mathsf{L}_t^{-1}(V_{2n,xxx})\right\} \\ &= \frac{(-1)^{n+1}}{(2n)!}\cos(x)\int_0^t \tau^{2n}\,d\tau = -\frac{(-1)^n t^{2n+1}}{(2n+1)!}\cos(x),\ n \in \mathbb{N}_0. \end{aligned}$$

Thus, from the convergence of ADM in [27], we have that

$$\begin{aligned} u(x;t) &= \sum_{n=0}^{\infty} V_{2n}(x;t) + \sum_{n=0}^{\infty} V_{2n+1}(x;t) \\ &= \sin(x)\left(\sum_{n\geq 0}\frac{(-1)^n t^{2n}}{(2n)!}\right) - \cos(x)\left(\sum_{n\geq 0}\frac{(-1)^n t^{2n+1}}{(2n+1)!}\right) = \sin(x-t). \end{aligned}$$

We note that same approximate-analytical solution for Eq. (2) via LADM have been obtained in [28] and the result coincides with the results of ADM. See Fig. 1 for the graphical illustration and Table 1 for the numerical results of experiment 1.

2.2 ADM Applied to Eq. (4)

We now rewrite Eq. (4) as

$$\mathsf{L}_t u - u_{xxx} = 2e^{t-x}, \quad (19)$$

with $\mathsf{L}_t = \dfrac{\partial}{\partial t}$, the linear differential operator, which is assumed to be invertible; i.e., $\mathsf{L}_t^{-1}(\cdot) = \int_0^t(\cdot)\,d\tau$. By applying L_t^{-1} on both sides of Eq. (19),

$$\mathsf{L}_t^{-1}\mathsf{L}_t u = 2\mathsf{L}_t^{-1}(e^{t-x}) - \mathsf{L}_t^{-1}(u_{xxx}),$$

which is equivalently

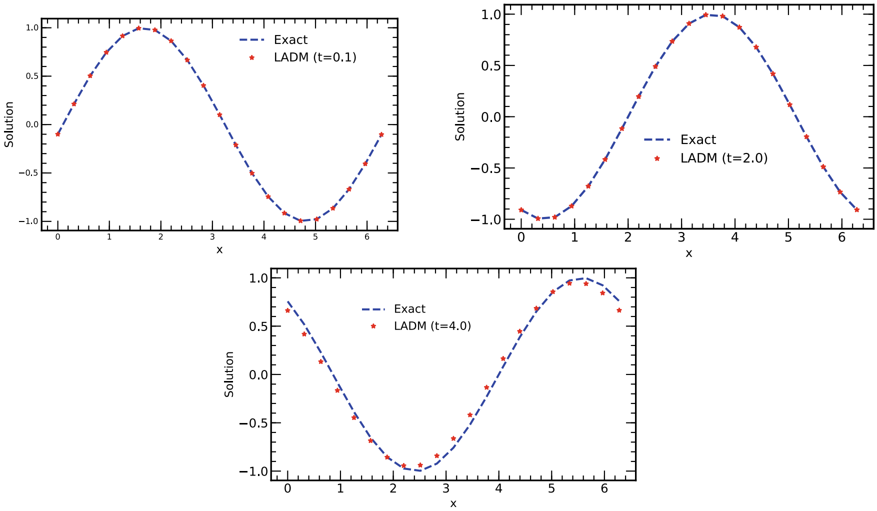


Fig. 1 Plots for Exact solution and ADM (LADM) using ten terms versus x at times 0.1, 2.0 and 4.0

$$u(x,t) = u(x,0) + 2\mathsf{L}_t^{-1}(e^{t-x}) - \mathsf{L}_t^{-1}(u_{xxx}). \tag{20}$$

By employing the decomposition series given in Eq. (10) together with Eq. (20), we get

$$\begin{cases} V_0(x;t) &= u(x,0) + 2\mathsf{L}_t^{-1}(e^{t-x}) = 1 + e^{-x} + 2e^{-x}\mathsf{L}_t^{-1}(e^t) = 1 + 2e^{t-x} - e^{-x}, \\ V_1(x;t) &= -\mathsf{L}_t^{-1}(V_{0,xxx}) = -2e^{t-x} + te^{-x} + 2e^{-x}, \\ V_2(x;t) &= -\mathsf{L}_t^{-1}(V_{1,xxx}) = 2e^{t-x} + e^{-x}\frac{t^2}{2!} - 2te^{-x} - 2e^{-x}, \\ V_3(x;t) &= -\mathsf{L}_t^{-1}(V_{2,xxx}) = -2e^{t-x} + 2te^{-x} + 2e^{-x} + t^2e^{-x} + \frac{t^3}{3!}e^{-x}, \end{cases} \tag{21}$$

and so on.

We see the self-cancelling ‘noise’ terms in Eq.(21) gives the exact solution

$$u(x,t) = 1 + e^{-x}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right) = 1 + e^{t-x}. \tag{22}$$

Remark 1 An approximate series solution terms given in Eq. (21) for the inhomogeneous KdV-type equation obey self-cancelling behavior; which are also known in the literature as ‘noise terms’ [29, 30]. A necessary condition for the appearance of noise terms for inhomogeneous problems is that the zeroth component V_0 must possess the exact solution u among other terms [24]. One can refer to [29] for more on noise terms.

Table 1 Absolute/relative errors between ADM (LADM) and exact solution

t	x	Exact	Numerical	Absolute error	Relative error
$t = 0.1$	0.314	0.212370	0.212370	0.000000	0.000000
	0.942	0.745977	0.745977	1.110223×10^{-16}	1.488281×10^{-16}
	1.570	0.994924	0.994924	1.110223×10^{-16}	1.115887×10^{-16}
	2.826	0.403732	0.403732	1.110223×10^{-16}	2.749900×10^{-16}
	3.454	−0.210814	−0.210814	8.326673×10^{-17}	3.949777×10^{-16}
	4.082	−0.744915	−0.744915	0.000000	0.000000
	4.710	−0.994763	−0.994763	0.000000	0.000000
	5.966	−0.405189	−0.405189	2.775558×10^{-16}	6.850036×10^{-16}
	6.280	−0.103002	−0.103002	3.608225×10^{-16}	3.503053×10^{-15}
$t = 2.0$	0.314	−0.993371	−0.993422	5.015442×10^{-5}	5.048909×10^{-5}
	0.942	−0.871376	−0.871412	3.618402×10^{-5}	4.152515×10^{-5}
	1.570	−0.416871	−0.416879	8.406101×10^{-6}	2.016476×10^{-5}
	2.826	0.735226	0.735271	4.494898×10^{-5}	6.113628×10^{-5}
	3.454	0.993187	0.993237	5.016629×10^{-5}	5.051041×10^{-5}
	4.082	0.872156	0.872193	3.624056×10^{-5}	4.155283×10^{-5}
	4.710	0.418318	0.418326	8.485738×10^{-6}	2.028538×10^{-5}
	5.966	−0.734146	−0.734190	4.491153×10^{-5}	6.117524×10^{-5}
	6.280	−0.907967	−0.908017	4.998895×10^{-5}	5.505590×10^{-5}
$t = 4.0$	0.314	0.517911	0.417558	1.003528×10^{-1}	1.937646×10^{-1}
	0.942	−0.083495	−0.165409	8.191365×10^{-2}	9.810567×10^{-1}
	1.570	−0.653041	−0.685258	3.221691×10^{-2}	4.933369×10^{-2}
	2.826	−0.922304	−0.841901	8.040265×10^{-2}	8.717588×10^{-2}
	3.454	−0.519273	−0.418922	1.003508×10^{-1}	1.932525×10^{-1}
	4.082	0.081908	0.163914	8.200590×10^{-2}	0.1001194×10^{1}
	4.710	0.651834	0.684202	3.236825×10^{-2}	4.965722×10^{-2}
	5.966	0.922918	0.842611	8.030689×10^{-2}	8.701410×10^{-2}
	6.280	0.758881	0.663909	9.497124×10^{-2}	1.251465×10^{-1}

3 A New Laplace-Type Transform: Shehu's Transform Method for Solving PDEs

A new Laplace-type integral transform, known to be Shehu's transform, is introduced in [21] to solve both ODEs and PDEs. This method is efficient in the sense that it has great mathematical simplicity and ease of formulations as it is also generalization of many of the well-known integral transforms. Some of the advantages of this method are its simple application to a class of ordinary or partial differential equations; for instance, for some of the dispersive KdV-type equations.

Generally speaking, Shehu's transform can be perceived as a corner stone to the Sumudu transform, the natural transform, the Elzaki transform, and the Laplace transform [21].

Definition 1 The Shehu transform of the function $v(t)$ of exponential order is defined over the set of functions,

$A = \left\{ v(t) : \exists\ N,\ \eta_1,\ \eta_2 > 0,\ \ |v(t)| < N \exp\left(\frac{|t|}{\eta_i}\right),\ \text{ if } t \in (-1)^i \times [0, \infty) \right\}$,
by the following integral

$$\mathbb{S}\,[v(t)] = V(s, \rho) = \int_0^{\infty} \exp\left(\frac{-st}{\rho}\right) v(t)dt$$
$$= \lim_{\alpha \to \infty} \int_0^{\alpha} \exp\left(\frac{-st}{\rho}\right) v(t)dt; \quad s > 0,\ \ \rho > 0. \tag{23}$$

Equation (23) converges when the limit value of the above integral is finite and diverges if this is not the case.

Let's denote the inverse Shehu transform, for $t \geq 0$, by

$$\mathbb{S}^{-1}\,[V(s, \rho)] = v(t). \tag{24}$$

Equation (24) is equivalently expressed as

$$v(t) = \mathbb{S}^{-1}\,[V(s, \rho)] = \frac{1}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{1}{\rho} \exp\left(\frac{st}{\rho}\right) V(s, \rho)\, ds, \tag{25}$$

where $\alpha \in \mathbb{R}$, s and u are Shehu variables [21] and the integral in Eq. (25) is taken along $s = \alpha$ in the complex plane $s = x + iy$.

Theorem 1 ([21]) *If $v(t)$ is piecewise continuous on $t \in [0, \beta]$ and of exponential order α for $t > \beta$, then Shehu's transform exists.*

Theorem 2 ([21]) *Let $v^{(n)}(t)$ denotes the nth derivative of the function $v(t) \in A$ with respect to t. The Shehu transform of $v^{(n)}(t)$ is given by*

$$\mathbb{S}\left[v^{(n)}(t)\right] = \frac{s^n}{\rho^n} \cdot V(s, \rho) - \sum_{k=0}^{n-1} \left(\frac{s}{\rho}\right)^{n-(k+1)} v^{(k)}(0). \tag{26}$$

Fpr $n = 1,\ 2,$ and 3 in Eq. (26), we have the following derivatives with respect to t:

$$\mathbb{S}\left[v'(t)\right] = \frac{s}{\rho} \cdot V(s, \rho) - v(0),$$

$$\mathbb{S}\left[v''(t)\right] = \frac{s^2}{\rho^2} \cdot V(s, \rho) - \frac{s}{\rho} v(0) - v'(0),$$

$$\mathbb{S}\left[v'''(t)\right] = \frac{s^3}{\rho^3} V(s, \rho) - \frac{s^2}{\rho^2} v(0) - \frac{s}{\rho} v'(0) - v''(0).$$

By employing Leibniz's rule, some properties are noted as follows:

$$\mathbb{S}\left[\frac{\partial v(x,t)}{\partial x}\right] = \int_0^\infty \exp\left(\frac{-st}{\rho}\right)\frac{\partial v(x,t)}{\partial x}\,dt = \frac{\partial}{\partial x}\int_0^\infty \exp\left(\frac{-st}{\rho}\right) v(x,t)\,dt$$
$$= \frac{\partial}{\partial x}\left[V(x,s,\rho)\right] \Rightarrow \mathbb{S}\left[\frac{\partial v(x,t)}{\partial x}\right] = \frac{d}{dx}\left[V(x,s,\rho)\right],$$

$$\mathbb{S}\left[\frac{\partial^2 v(x,t)}{\partial x^2}\right] = \int_0^\infty \exp\left(\frac{-st}{\rho}\right)\frac{\partial^2 v(x,t)}{\partial x^2}\,dt = \frac{\partial^2}{\partial x^2}\int_0^\infty \exp\left(\frac{-st}{\rho}\right) v(x,t)\,dt$$
$$= \frac{\partial^2}{\partial x^2}\left[V(x,s,\rho)\right] \Rightarrow \mathbb{S}\left[\frac{\partial^2 v(x,t)}{\partial x^2}\right] = \frac{d^2}{dx^2}\left[V(x,s,\rho)\right].$$

Some important properties of this transform are given as follows:

(i) Linearity property of Shehu transform:

$$\mathbb{S}\left[\alpha v(t) + \beta w(t)\right] = \alpha\mathbb{S}\left[v(t)\right] + \beta\mathbb{S}\left[w(t)\right].$$

(ii) Scaling property of Shehu transform:

$$\mathbb{S}\left[v(\beta t)\right] = \frac{\rho}{\beta}\cdot V\left(\frac{s}{\beta},\rho\right).$$

Proposition 1 ([21]) *Suppose* $\frac{\partial v(x,t)}{\partial t}$ *and* $\frac{\partial^2 v(x,t)}{\partial x^2}$ *exist, then*

$$\mathbb{S}\left[\frac{\partial v(x,t)}{\partial t}\right] = \frac{s}{\rho}\cdot V(x,s,\rho) - v(x,0),$$
$$\mathbb{S}\left[\frac{\partial^2 v(x,t)}{\partial x^2}\right] = \frac{s^2}{\rho^2}\cdot V(s,\rho) - \frac{s}{\rho}\cdot v(0) - \frac{\partial v(x,0)}{\partial t}.$$

Our next section introduces SADM, which is a combination of ADM and Shehu's transform, and some illustrative examples are also provided.

Table 2 Some essential properties of Shehu's transform for SADM

Function form $f(\tilde{X}, t)$	Transformed form $F_k(\tilde{X})$
1	$\frac{\rho}{s}$
$\frac{t^n}{n!}$	$\left(\frac{\rho}{s}\right)^{n+1}$
e^{at}	$\frac{\rho}{s-a\rho}$
te^{at}	$\frac{\rho^2}{(s-a\rho)^2}$
$\frac{t^n e^{at}}{n!}$	$\frac{\rho^{n+1}}{(s-a\rho)^{n+1}}$
$\sin(at)$	$\frac{a\rho^2}{s^2+a^2}$
$\cos(at)$	$\frac{\rho s}{s^2+a^2\rho^2}$
$e^{bt}\cos(at)$	$\frac{\rho(s-a\rho)}{(s-b\rho)^2+a^2\rho^2}$
$\frac{e^{at}}{b-a}$	$\frac{\rho^2}{(s-a\rho)(s-b\rho)}$
$\frac{be^{bt}-ae^{at}}{b-a}$	$\frac{\rho s}{(s-a\rho)(s-b\rho)}$

3.1 *Outline of the Method: SADM*

To illustrate the basic concepts of SADM, let's us consider the following equation

$$\begin{cases} \mathsf{L}_t u(x,t) + Mu(x,t) + Nu(x,t) = g(x,t), \\ u(x,0) = h(x), \end{cases} \tag{27}$$

where N is a nonlinear operator, $\mathsf{L}_t = \frac{\partial}{\partial t}$ is the linear operator, M is a linear operator w.r.t x and g is the source term, which doesn't rely on u. By first applying Laplace transform on both sides of Eq. (27), we get

$$\mathbb{S}\Big\{\mathsf{L}_t u(x,t)\Big\} = \mathbb{S}\Big\{g(x,t) - Mu(x,t) - Nu(x,t)\Big\} \tag{28}$$

and by rewriting Eq. (28) equivalently as

$$\frac{s}{\rho}\cdot\mathbb{S}\Big\{u(x,t)\Big\} - u(x,0) = \mathbb{S}\Big\{g(x,t) - Mu(x,t) - Nu(x,t)\Big\}. \tag{29}$$

In the homogeneous case, $g(x,t) = 0$, and therefore we have that

$$u(x,s) = \frac{\rho}{s} \cdot h(x) - \frac{\rho}{s} \cdot \mathbb{S}\Big\{Mu(x,t) + Nu(x,t)\Big\}.$$

Employing inverse Shehu's transform to Eq. (29) gives

$$u(x,t) = h(x) - \mathbb{S}^{-1}\Big[\frac{\rho}{s} \cdot \mathbb{S}\Big\{Mu(x,t) + Nu(x,t)\Big\}\Big]. \tag{30}$$

Let us consider SADM decomposition series by

$$u(x,t) = \sum_{n=0}^{\infty} V_n(x,t), \tag{31}$$

and the nonlinear term by

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n(V_0, V_1, \ldots, V_n), \tag{32}$$

where the sequence $\{A_n\}_{n=0}^{\infty}$ are the well-known Adomian polynomials (see [5, 6, 30]). Using Eqs. (31) and (32) into Eq. (30), we obtain

$$\sum_{n=0}^{\infty} V_n(x,t) = h(x) - \mathbb{S}^{-1}\Big[\frac{\rho}{s} \cdot \mathbb{S}\{M\sum_{n=0}^{\infty} V_n(x,t) + \sum_{n=0}^{\infty} A_n(V_0, V_1, \ldots, V_n)\}\Big]. \tag{33}$$

The following recursive formulae follows from Eq. (33) as follows.

$$\begin{cases} V_0(x,t) = h(x), \\ V_{n+1}(x,t) = -\mathbb{S}^{-1}\Big[\frac{\rho}{s} \cdot \mathbb{S}\Big\{MV_n(x,t) + A_n(V_0, V_1, \ldots, V_n)\Big\}\Big], \quad n = 0, 1, 2, \ldots. \end{cases} \tag{34}$$

Using Eq. (34), an approximate solution of Eq. (27) takes the form

$$u(x,t) \approx \sum_{r=0}^{n} V_r(x,t), \quad \text{where} \quad \lim_{n\to\infty} \sum_{r=0}^{n} V_r(x,t) = u(x,t). \tag{35}$$

The following Shehu's transformation results are given in [21].

4 Some Applications: SADM

In this section, SADM is applied to dispersive linear and nonlinear KdV-type equations to show the reliability of the method.

4.1 Implementation of SADM for Eq. (2)

The linearized homogeneous equation in [18] takes the form

$$\begin{cases} u_t + 2u_x + u_{xxx} = 0, \quad (x,t) \in [0, 2\pi] \times [0, 2.75], \\ u(x,0) = \sin(x). \end{cases} \tag{36}$$

By applying Shehu's transform $\mathbb{S}$ in given Eqs. (23)–(36), we have

$$\mathbb{S}\{u_t\} = \frac{s}{\rho} \cdot \mathbb{S}\Big\{u(x,t)\Big\} - u(x,0) = -2\mathbb{S}\{u_x\} - \mathbb{S}\{u_{xxx}\}, \quad t > 0. \tag{37}$$

By employing inverse Shehu's transform to Eq. (37), we obtain

$$u(x,t) = u(x,0) - \mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \left[\mathbb{S}\{2u_x\} - \mathbb{S}\{u_{xxx}\}\right]\right]. \tag{38}$$

By using SADM's series given in Eq. (31) into Eq. (38), the following recursive values are given as follows.

$$\begin{cases} V_0(x,t) = \sin(x), \\ V_1(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \left[\mathbb{S}\{-2V_{0,x}\} - \mathbb{S}\{V_{0,xxx}\}\right]\right] \\ V_2(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \left[\mathbb{S}\{-2V_{1,x}\} - \mathbb{S}\{V_{1,xxx}\}\right]\right] \\ \vdots \\ V_n(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \left[\mathbb{S}\{-2V_{n-1,x}\} - \mathbb{S}\{V_{n-1,xxx}\}\right]\right] \end{cases}. \tag{39}$$

By using Eq. (39) and some of properties of Shehu's transform given in Table 2, we have that

$$\begin{cases} V_0(x,t) = \sin(x), \\ V_1(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\{-2V_{0,x}\} - \mathbb{S}\{V_{0,xxx}\}\right]\right] = -t\cos(x), \\ V_2(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\{-2V_{1,x}\} - \mathbb{S}\{V_{1,xxx}\}\right]\right] = -\frac{t^2}{2!}\sin(x), \\ V_3(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\{-2V_{2,x}\} - \mathbb{S}\{V_{2,xxx}\}\right]\right] = -\frac{t^3}{3!}\cos(x), \\ V_4(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\{-2V_{3,x}\} - \mathbb{S}\{V_{3,xxx}\}\right]\right] = \frac{t^4}{4!}\sin(x), \\ V_5(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\{-2V_{4,x}\} - \mathbb{S}\{V_{4,xxx}\}\right]\right] = -\frac{t^5}{5!}\cos(x), \\ V_6(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\{-2V_{5,x}\} - \mathbb{S}\{V_{5,xxx}\}\right]\right] = -\frac{t^6}{6!}\sin(x), \\ V_7(x;t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\{-2V_{6,x}\} - \mathbb{S}\{V_{6,xxx}\}\right]\right] = -\frac{t^7}{7!}\cos(x). \end{cases} \tag{40}$$

The rest of the components can be obtained from Eq. (40) in a similar way. The 7-term approximate SADM solution is

$$\begin{aligned} \Psi_7(x,t) = \sum_{i=0}^{7} V_i(x,t) &= \left(\sin(x) - \frac{t^2}{2!}\sin(x) + \frac{t^4}{4!}\sin(x) - \frac{t^6}{6!}\sin(x)\right) \\ &+ \left(-t\cos(x) + \frac{t^3}{3!}\cos(x) - \frac{t^5}{5!}\cos(x) + \frac{t^7}{7!}\cos(x)\right). \end{aligned} \tag{41}$$

In view of Eq. (41) and using Taylor's expansion, we have

$$V_{2n}(x;t) = \frac{(-1)^n t^{2n}}{(2n)!}\sin(x), \text{ for } n \in \mathbb{N}_0,$$

and thus

$$\begin{aligned} V_{2n+1}(x;t) &= -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\{-2V_{2n,x}\} - \mathbb{S}\{V_{2n,xxx}\}\right]\right] \\ &= -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\left[\mathbb{S}\left(2\frac{(-1)^n t^{2n}}{(2n)!}\cos(x) - \frac{(-1)^n t^{2n}}{(2n)!}\cos(x)\right)\right]\right] \\ &= \cos(x)(-1)^{n+1}\frac{t^{2n+1}}{(2n+1)!}. \end{aligned}$$

4.2 Implementation of SADM for Eq. (6)

Applying Shehu transform on both sides of Eq. (6), we get

$$\mathbb{S}(u(x,t)) = x - \left[\frac{\rho}{s} \cdot \mathbb{S}\Big(uu_x + u_{xxx}\Big)\right]. \tag{42}$$

Taking inverse Shehu transform on both sides of Eq. (42), we obtain

$$u(x,t) = x - \mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \mathbb{S}\Big(uu_x + u_{xxx}\Big)\right]. \tag{43}$$

By applying the aforesaid decomposition method, we have

$$\sum_{n=0}^{\infty} u_n(x,t) = x - \mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \mathbb{S}\left\{\sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n) + \sum_{n=0}^{\infty} (u_n)_{xxx}\right\}\right]. \tag{44}$$

Comparing both sides of Eq. (44) gives

$$\begin{cases} u_0(x,t) = x, \\ u_1(x,t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \mathbb{S}\left\{A_0(u_0) + (u_0)_{xxx}\right\}\right], \\ u_2(x,t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \mathbb{S}\left\{A_1(u_0,u_1) + (u_1)_{xxx}\right\}\right], \\ u_3(x,t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s} \cdot \mathbb{S}\left\{A_2(u_0,u_1,u_2) + (u_2)_{xxx}\right\}\right], \\ \qquad \vdots \end{cases} \tag{45}$$

The first few components of Adomain polynomials $A_n(u_0, u_1, \ldots, u_n)$ (cf. [25, 26]) are given by

$$\begin{cases} A_0(u_0) = u_0 u_{0,x} = x, \\ A_1(u_0,u_1) = u_0 u_{1,x} + u_1 u_{0,x} = -xt, \\ A_2(u_0,u_1,u_2) = u_0 u_{2,x} + u_2 u_{0,x} + u_1 u_{1,x} = xt^2, \\ A_3(u_0,u_1,u_2,u_3) = u_3 u_{0,x} + u_1 u_{2,x} + u_2 u_{1,x} + u_0 u_{3,x} = -4xt^3, \\ \qquad \ddots \end{cases} \tag{46}$$

Using the iteration formulae (45) and Adomian polynomials in (46), we obtain

$$u_0(x,t) = x, \quad u_1(x,t) = -xt, \quad u_2(x,t) = xt^2, \quad u_3(x,t) = -xt^3, \quad u_4(x,t) = xt^4. \tag{47}$$

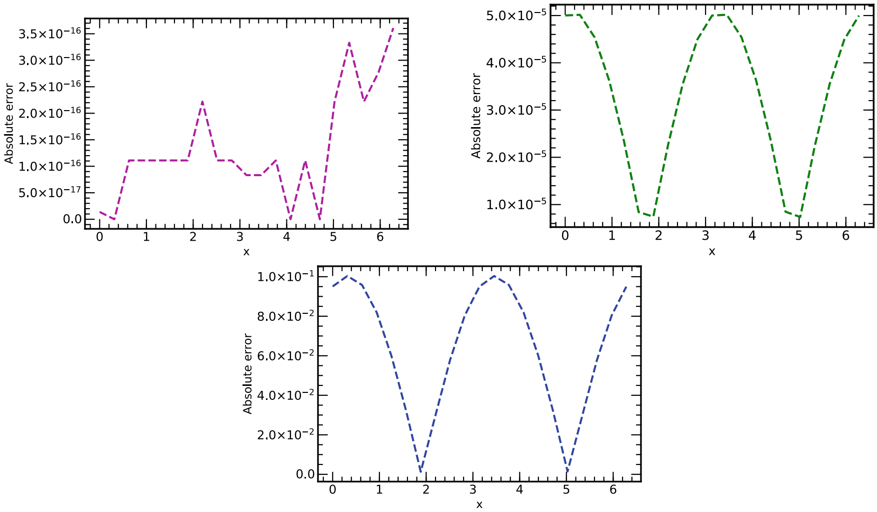


Fig. 2 Error plots versus x at times $t = 0.1, 2.0, 4.0$ (LADM)

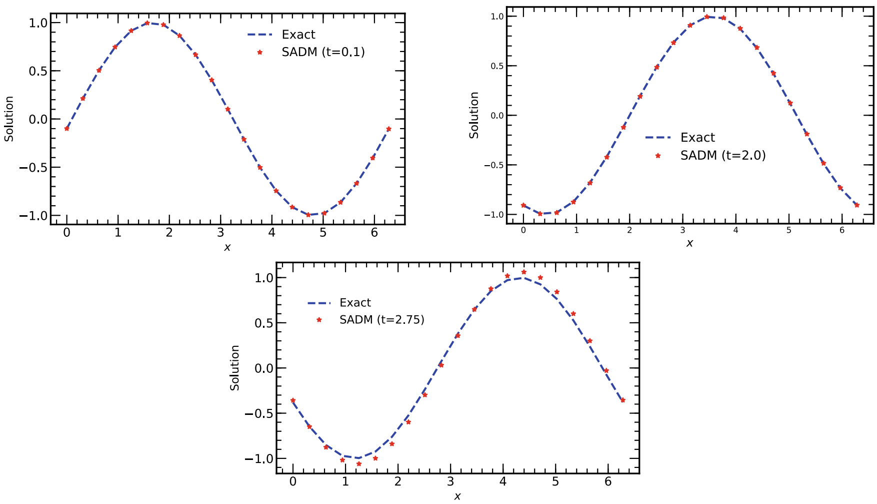


Fig. 3 Plot for Exact solution and SADM at $0 \leq x \leq 2\pi$ and times $t = 0.1, 2.0, 2.75$, respectively,

Thus, an approximate-analytical solution for $u(x, t)$ is given by

$$u_{\text{STADM}}(x, t) = x - xt + xt^2 - xt^3 + xt^4 + \cdots, \tag{48}$$

which gives the exact solution $u(x, t) = \dfrac{x}{1+t}$ with $|-t| < 1$ (Table 3).

Table 3 Absolute/relative errors at some values of x and at times 0.1, 2.0, 2.75 using 7-terms of SADM

t	values of x	Exact	Numerical	Absolute error	Relative error
t = 0.10	0.000	−0.099833	−0.099833	2.747802×10^{-15}	2.752387×10^{-14}
	0.314	0.212370	0.212370	7.399636×10^{-14}	3.484308×10^{-13}
	0.942	0.745977	0.745977	1.988409×10^{-13}	2.665512×10^{-13}
	1.570	0.994924	0.994924	2.481348×10^{-13}	2.494007×10^{-13}
	2.198	0.864217	0.864217	2.022826×10^{-13}	2.340645×10^{-13}
	2.826	0.403732	0.403732	7.971401×10^{-14}	1.974428×10^{-13}
	3.454	−0.210814	−0.210814	7.352452×10^{-14}	3.487653×10^{-13}
	4.082	−0.744915	−0.744915	1.987299×10^{-13}	2.667820×10^{-13}
	4.710	−0.994763	−0.994763	2.480238×10^{-13}	2.493296×10^{-13}
	5.338	−0.865018	−0.865018	2.023937×10^{-13}	2.339764×10^{-13}
	5.966	−0.405189	−0.405189	7.965850×10^{-14}	1.965960×10^{-13}
	6.280	−0.103002	−0.103002	3.191891×10^{-15}	3.098854×10^{-14}
t = 2.0	0.000	−0.909297	−0.907937	1.360919×10^{-3}	1.496671×10^{-3}
	0.314	−0.993371	−0.993953	5.820994×10^{-4}	5.859837×10^{-4}
	0.942	−0.871376	−0.875489	4.112929×10^{-3}	4.720040×10^{-3}
	1.570	−0.416871	−0.422945	6.074300×10^{-3}	1.457118×10^{-2}
	2.198	0.196709	0.190991	5.717768×10^{-3}	2.906717×10^{-2}
	2.826	0.735226	0.732047	3.179384×10^{-3}	4.324363×10^{-3}
	3.454	0.993187	0.993759	5.722263×10^{-4}	5.761516×10^{-4}
	4.082	0.872156	0.876262	4.105480×10^{-3}	4.707276×10^{-3}
	4.710	0.418318	0.424390	6.072117×10^{-3}	1.451556×10^{-2}
	5.338	−0.195147	−0.189425	5.721685×10^{-3}	2.931987×10^{-2}
	5.966	−0.734146	−0.730958	3.187906×10^{-3}	4.342335×10^{-3}
	6.280	−0.907967	−0.906587	1.380264×10^{-3}	1.520169×10^{-3}
t = 2.75	0.000	−0.381661	−0.358498	2.316300×10^{-2}	6.068998×10^{-2}
	0.314	−0.648485	−0.649521	1.035837×10^{-3}	1.597318×10^{-3}
	0.942	−0.971999	−1.018772	4.677316×10^{-2}	4.812059×10^{-2}
	1.570	−0.924606	−0.999268	7.466224×10^{-2}	8.075033×10^{-2}
	2.198	−0.524391	−0.598452	7.406083×10^{-2}	1.412320×10^{-1}
	2.826	0.075927	0.030728	4.519843×10^{-2}	5.952891×10^{-1}
	3.454	0.647272	0.648183	9.113162×10^{-4}	1.407934×10^{-3}
	4.082	0.971623	1.018297	4.667331×10^{-2}	4.803642×10^{-2}
	4.710	0.925212	0.999837	7.462516×10^{-2}	8.065740×10^{-2}
	5.338	0.525747	0.599847	7.410067×10^{-2}	1.409437×10^{-1}
	5.966	−0.074339	−0.029039	4.529999×10^{-2}	6.093728×10^{-1}
	6.280	−0.378715	−0.355314	2.340076×10^{-2}	6.178992×10^{-2}

Table 4 Absolute/relative errors at some values of x and at times 0.1, 2.0, 2.75 using 7-terms of SADM

t	Values of x	Exact	Numerical	Absolute error	Relative error
	0.000	0.000000	0.000000	0.000000	–
	0.628	0.615686	0.615686	1.970196×10^{-9}	3.200000×10^{-9}
	1.256	1.231373	1.231373	3.940392×10^{-9}	3.200000×10^{-9}
	1.884	1.847059	1.847059	5.910588×10^{-9}	3.200000×10^{-9}
	2.512	2.462745	2.462745	7.880784×10^{-9}	3.200000×10^{-9}
t = 0.02	3.140	3.078431	3.078431	9.850980×10^{-9}	3.200000×10^{-9}
	3.768	3.694118	3.694118	1.182118×10^{-8}	3.200000×10^{-9}
	4.396	4.309804	4.309804	1.379137×10^{-8}	3.200000×10^{-9}
	5.024	4.925490	4.925490	1.576157×10^{-8}	3.200000×10^{-9}
	5.652	5.541176	5.541176	1.773177×10^{-8}	3.200000×10^{-9}
	6.280	6.156863	6.156863	1.970196×10^{-8}	3.200000×10^{-9}
	0.000	0.000000	0.000000	0.000000	–
	0.628	0.592453	0.592453	4.606913×10^{-7}	7.776000×10^{-7}
	1.256	1.184906	1.184907	9.213826×10^{-7}	7.776000×10^{-7}
	1.884	1.777358	1.777360	1.382074×10^{-6}	7.776000×10^{-7}
	2.512	2.369811	2.369813	1.842765×10^{-6}	7.776000×10^{-7}
t = 0.06	3.140	2.962264	2.962266	2.303457×10^{-6}	7.776000×10^{-7}
	3.768	3.554717	3.554720	2.764148×10^{-6}	7.776000×10^{-7}
	4.396	4.147170	4.147173	3.224839×10^{-6}	7.776000×10^{-7}
	5.024	4.739623	4.739626	3.685531×10^{-6}	7.776000×10^{-7}
	5.652	5.332075	5.332080	4.146222×10^{-6}	7.776000×10^{-7}
	6.280	5.924528	5.924533	4.606913×10^{-6}	7.776000×10^{-7}
	0.000	0.000000	0.000000	0.000000	–
	0.628	0.570909	0.570915	5.709091×10^{-6}	1.000000×10^{-5}
	1.256	1.141818	1.141830	1.141818×10^{-5}	1.000000×10^{-5}
	1.884	1.712727	1.712744	1.712727×10^{-5}	1.000000×10^{-5}
	2.512	2.283636	2.283659	2.283636×10^{-5}	1.000000×10^{-5}
t = 0.10	3.140	2.854545	2.854574	2.854545×10^{-5}	1.000000×10^{-5}
	3.768	3.425455	3.425489	3.425455×10^{-5}	1.000000×10^{-5}
	4.396	3.996364	3.996404	3.996364×10^{-5}	1.000000×10^{-5}
	5.024	4.567273	4.567318	4.567273×10^{-5}	1.000000×10^{-5}
	5.652	5.138182	5.138233	5.138182×10^{-5}	1.000000×10^{-5}
	6.280	5.709091	5.709148	5.709091×10^{-5}	1.000000×10^{-5}
	0.000	0.000000	0.000000	0.000000	–
	0.628	0.418667	0.431750	1.308333×10^{-2}	3.125000×10^{-2}
	1.256	0.837333	0.863500	2.616667×10^{-2}	3.125000×10^{-2}
	1.884	1.256000	1.295250	3.925000×10^{-2}	3.125000×10^{-2}
	2.512	1.674667	1.727000	5.233333×10^{-2}	3.125000×10^{-2}
t = 0.50	3.140	2.093333	2.158750	6.541667×10^{-2}	3.125000×10^{-2}
	3.768	2.512000	2.590500	7.850000×10^{-2}	3.125000×10^{-2}
	4.396	2.930667	3.022250	9.158333×10^{-2}	3.125000×10^{-2}
	5.024	3.349333	3.454000	1.046667×10^{-1}	3.125000×10^{-2}
	5.652	3.768000	3.885750	1.177500×10^{-1}	3.125000×10^{-2}
	6.280	4.186667	4.317500	1.308333×10^{-1}	3.125000×10^{-2}

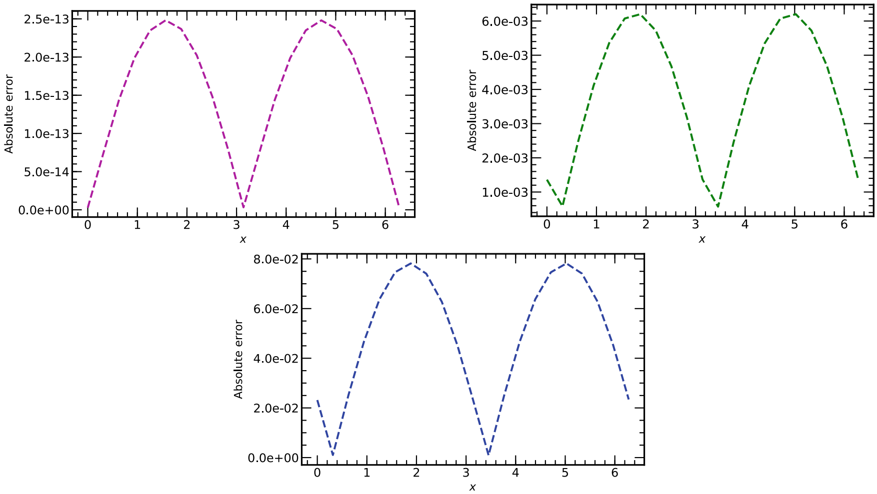


Fig. 4 Error plots versus x (SADM) at times $t = 0.1, 2.0, 2.75$ respectively

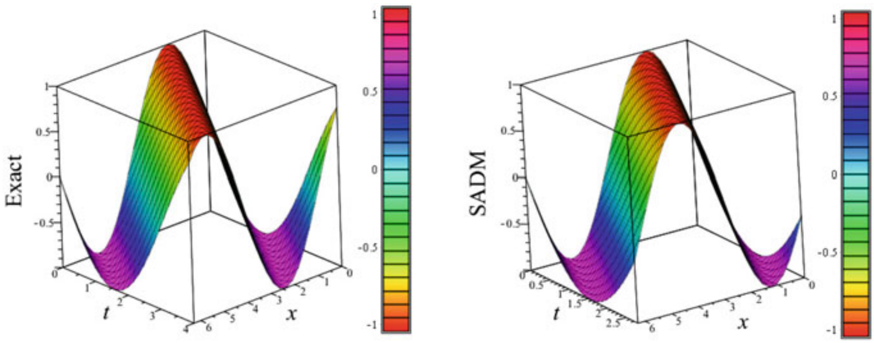


Fig. 5 Three-dimensional representation for Exact solution and SADM at $0 \leq x \leq 2\pi$ and $0 \leq t \leq 4.0$

Plots of exact and numerical solution vs x are displayed in Fig. 6. We obtain plots of absolute error vs x at four different values of time in Fig. 2. We also compare the absolute and relative errors at some values of x at four different times in Table 4. We note that same approximate-analytical solution have been obtained using using SADM for the considered numerical experiments in this paper as shown in Figs. 3, 4, 5 and 7.

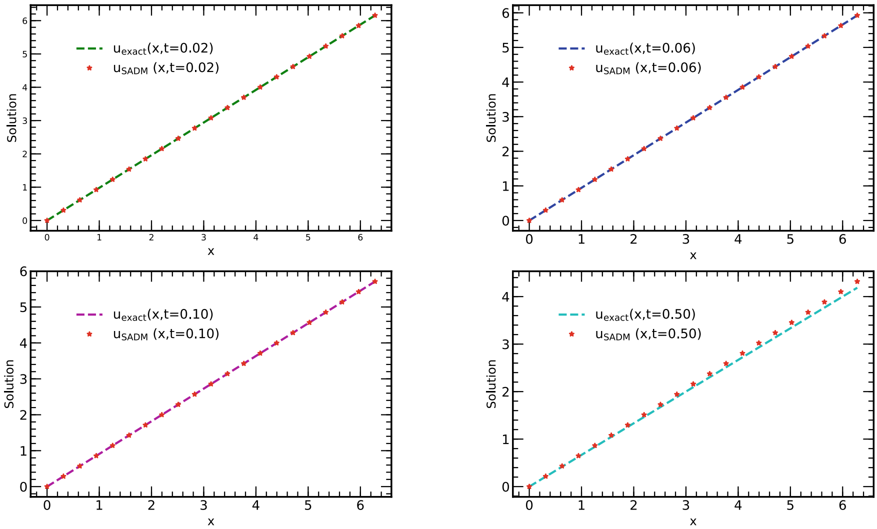


Fig. 6 Plots of exact solution and approximate solution using SADM (4-terms) versus x at times 0.02, 0.06, 0.10, and 0.50 (The space interval used for these plots is $\frac{\pi}{10} \approx 0.314$)

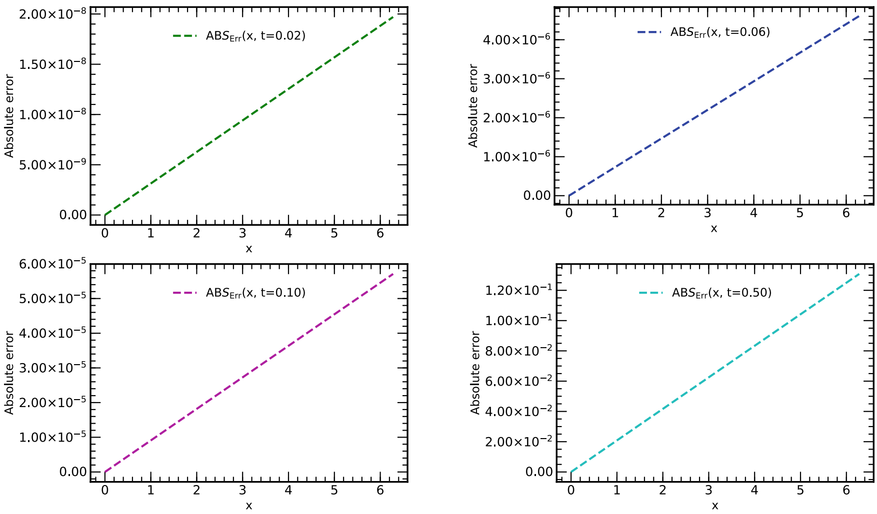


Fig. 7 Plots of absolute errors versus x at different values of time ($t = 0.02, 0.06, 0.10, 0.50$) using SADM (4-terms)

4.3 Implementation of SADM for Eq. (8)

By consider the inhomogeneous equation in Eq. (8), we apply Shehu transform on both sides of Eq. (8) to get

$$\mathbb{S}(u(x,t)) = \frac{\rho}{s}.u(x,0) + \frac{\rho}{s}.\left\{\mathbb{S}\left[\cos(x) + 2t\sin(x) + \frac{t^2\sin(x)}{2}\right] - \mathbb{S}\Big[-uu_x + u_{xxxxx}\Big]\right\}. \quad (49)$$

Taking inverse Shehu transform on both sides of Eq. (49), we obtain

$$u(x,t) = u(x,0) - \mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\mathbb{S}\left[\cos(x) + 2t\sin(x) + \frac{t^2\sin(x)}{2}\right] - \mathbb{S}\Big[-uu_x + u_{xxxxx}\Big]\right]. \quad (50)$$

By applying the aforesaid decomposition method, we have

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,0) - \mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\ \mathbb{S}\left[\cos(x) + 2t\sin(x) + \frac{t^2\sin(x)}{2}\right]\right.$$
$$\left. - \mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\ \mathbb{S}\left\{\sum_{n=0}^{\infty} A_n(u_0,u_1,\ldots,u_n) + \sum_{n=0}^{\infty}(u_n)_{xxxxx}\right\}\right]\right]. \quad (51)$$

On comparing both sides of Eq. (51), we obtain

$$u_0(x,t) = u(x,0) + \mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\mathbb{S}\left[\cos(x) + 2t\sin(x) + \frac{t^2}{2}\sin(x)\right]\right] \quad (52)$$

$$u_1(x,t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\mathbb{S}\left[(u_0)_{xxxxx} - A_0(u_0)\right]\right], \quad (53)$$

$$u_2(x,t) = -\mathbb{S}^{-1}\left[\frac{\rho}{s}\cdot\mathbb{S}\left[(u_1)_{xxxxx} - A_1(u_0,u_1)\right]\right]. \quad (54)$$

$$\vdots$$

The first few components of Adomain polynomials $A_n(u)$ are obtained using formulae (cf. [25, 26])

$$A_0(u_0) = u_0u_{0,x}$$
$$= -t^2\cos(x)\sin(x) + \left(\cos^2(x) - \sin^2(x)\right)t^3 + \left(\frac{1}{6}\cos^2(x) + \cos(x)\sin(x) - \frac{1}{6}\sin^2(x)\right)t^4$$
$$+ \frac{1}{3}t^5\sin(x)\cos(x) + \frac{1}{36}t^6\sin(x)\cos(x), \quad (55)$$

$$
\begin{aligned}
A_1(u_0, u_1) &= u_0 u_{1,x} + u_1 u_{0,x} \\
&= \left(-\frac{\sin(x)}{1512} + \frac{\cos^2(x)\sin(x)}{504}\right) t^{10} + \left(-\frac{5\ \sin(x)}{378} + \frac{5\cos^2(x)\sin(x)}{126}\right) t^9 \\
&+ \left(\frac{\cos^3(x)}{126} - \frac{\cos(x)}{252} - \frac{1}{18}\sin(x) + \frac{1}{6}\cos^2(x)\sin(x) - \frac{\cos(x)\sin^2(x)}{252}\right) t^8 \\
&+ \left(-\frac{\cos^2(x)}{144} + \frac{7\cos^3(x)}{36} - \frac{7\ \cos(x)}{72} + \frac{\sin^2(x)}{144} - \frac{2}{9}\cos(x)\sin^2(x)\right) t^7 \\
&+ \left(\frac{\cos(2x)}{72} + \frac{1}{18}\sin(x) + \frac{1}{2}\cos^3(x) - \frac{1}{4}\cos(x) - \cos(x)\sin^2(x) - \frac{1}{6}\cos^2(x)\sin(x)\right) t^6 \\
&+ \left(-\frac{1}{3}\sin^2(x) + \frac{7\sin(x)}{12} + \frac{1}{3}\cos^2(x) + \frac{1}{4}\cos(x)\sin(x) - \frac{5}{2}\cos^2(x)\sin(x)\right) t^5 \\
&+ \left(-\frac{2}{3}\cos^3(x) + \frac{1}{3}\cos(x) + \frac{1}{3}\cos(x)\sin(x) + \frac{1}{3}\cos(x)\sin^2(x)\right) t^4 + \frac{\cos(2x)}{2} t^3.
\end{aligned} \tag{56}
$$

The polynomials $A_2(u_0, u_1, u_2)$ and $A_3(u_0, u_1, u_2, u_3)$ are obtained by

$$
\begin{aligned}
A_2(u_0, u_1, u_2) &= u_0 u_{2,x} + u_2 u_{0,x} + u_1 u_{1,x}, \\
A_3(u_0, u_1, u_2, u_3) &= u_3 u_{0,x} + u_1 u_{2,x} + u_2 u_{1,x} + u_0 u_{3,x},
\end{aligned}
$$

and the higher order ones are obtained by

$$
A_n(u_0, u_1, u_2, \ldots, u_n) = \sum_{j=0}^{n-1} u_j \frac{\partial u_{n-j}}{\partial x}. \tag{57}
$$

Employing Eqs. (56), (55) together with Eq. (52) yields

$$
u_0(x, t) = t\cos(x) + t^2\sin(x) + \frac{t^3}{3!}\sin(x), \tag{58a}
$$

$$
\begin{aligned}
u_1(x, t) &= \frac{1}{2}t^2\sin(x) + \frac{1}{6}\left(2\ \cos(x) - \sin(2x)\right) t^3 \\
&+ \frac{1}{4}\left(\cos(2x) - \frac{1}{6}\cos(x)\right) t^4 + \frac{1}{36}\sin(2x)\, t^6 + \frac{\sin(2x)\, t^7}{504},
\end{aligned} \tag{58b}
$$

$$\begin{aligned} u_2(x,t) = &\left(\frac{\cos^2(x)\sin(x)}{5544} - \frac{\sin(x)}{16632}\right)t^{11} + \left(-\frac{\sin(x)}{756} + \frac{\cos^2(x)\sin(x)}{252}\right)t^{10} \\ &+ \left(-\frac{\sin(x)}{162} + \frac{\cos^2(x)\sin(x)}{54} - \frac{\cos(x)\ \sin^2(x)}{2268} + \frac{\cos^3(x)}{1134} - \frac{\cos(x)}{2268}\right)t^9 \\ &+ \left(\frac{7\cos^3(x)}{288} - \frac{7\cos(x)}{576} + \frac{(\sin(x))^2}{1152} + \frac{1}{126} - \frac{1}{36}\cos(x)\ \sin^2(x) - \frac{15\ (\cos(x))^2}{896}\right)t^8 \\ &+ \left(-\frac{127\cos^2(x)}{504} + \frac{1}{14}\cos^3(x) - \frac{1}{28}\cos(x) - \frac{(\sin(x))^2}{504} + \frac{8}{63} - \frac{1}{7}\cos(x)\sin^2(x)\right. \\ &\qquad \left. - \frac{1}{42}\cos^2(x)\sin(x) + \frac{\sin(x)}{126}\right)t^7 \\ &+ \left(\frac{1}{18}\cos^2(x) - \frac{1}{18}\ \sin^2(x) + \frac{7\sin(x)}{72} + \frac{1}{24}\cos(x)\sin(x) - \frac{5\cos^2(x)\sin(x)}{12}\right)t^6 \\ &+ \left(-\frac{\sin(x)}{120} + \frac{49\ \cos(x)\sin(x)}{15} + \frac{1}{15}\cos(x)\sin^2(x) - \frac{2}{15}\cos^3(x) + \frac{1}{15}\cos(x)\right)t^5 \\ &+ \left(\frac{67\cos^2(x)}{24} - \frac{1}{8}\sin^2(x) + \frac{1}{12}\sin(x) - \frac{4}{3}\right)t^4 - \frac{1}{6}t^3\cos(x). \end{aligned}$$

Thus, the sum of first three iterates to build an approximate-analytical solution for $u(x,t)$ of Eq. (8) is given by

$$u_{\text{SADM}}(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t). \tag{59}$$

Remark 2 Fig. 8 shows exact and SADM solution whereas Fig. 9 demonstrates Absolute error at different times. From numerical experiments above, we see that SADM is a promising semi-analytical method for solving PDEs. Comparison of SADM with other traditional semi-analytic methods such HPM, VIM, RDTM will be prominent continuation of this work, as this is not studied yet.

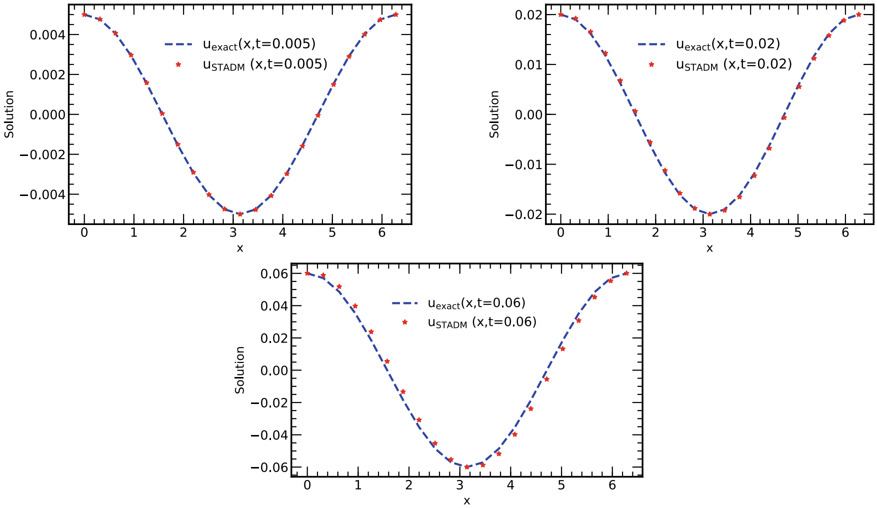


Fig. 8 Plots of Exact solution and approximate solution using 3-terms of SADM versus x at times 0.005, 0.02, and 0.06. (The space step size used for these plots is $\frac{\pi}{10} \approx 0.314$)

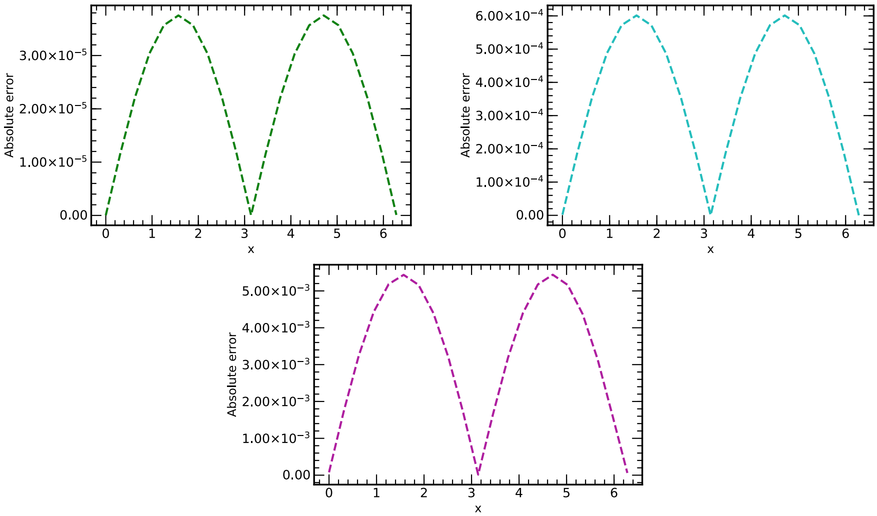


Fig. 9 Plots of absolute errors versus x at times $t = 0.005, 0.02, 0.06$ using SADM

Table 5 Absolute and relative errors at some values of x obtained at times $t = 0.005, 0.02, 0.06$ for Numerical Experiment 2

t	Values of x	Exact solution	Numerical solution	Absolute error	Relative error
t = 0.005	0.000	0.005000	0.005000	2.286458×10^{-8}	4.572917×10^{-6}
	0.628	0.004046	0.004068	2.206271×10^{-5}	5.452940×10^{-3}
	1.256	0.001548	0.001584	3.568366×10^{-5}	2.304976×10^{-2}
	1.884	−0.001541	−0.001505	3.568926×10^{-5}	2.316672×10^{-2}
	2.512	−0.004041	−0.004019	2.207738×10^{-5}	5.462890×10^{-3}
	3.140	−0.005000	−0.005000	4.100777×10^{-8}	8.201564×10^{-6}
	3.768	−0.004051	−0.004073	2.201170×10^{-5}	5.434056×10^{-3}
	4.396	−0.001556	−0.001591	3.566489×10^{-5}	2.292553×10^{-2}
	5.024	0.001533	0.001497	3.570741×10^{-5}	2.329308×10^{-2}
	5.652	0.004037	0.004015	2.212304×10^{-5}	5.480554×10^{-3}
	6.280	0.005000	0.005000	9.665089×10^{-8}	1.933028×10^{-5}
t = 0.02	0.000	0.020000	0.020002	1.853337×10^{-6}	9.266683×10^{-5}
	0.628	0.016184	0.016539	3.547192×10^{-4}	2.191778×10^{-2}
	1.256	0.006192	0.006765	5.722393×10^{-4}	9.240910×10^{-2}
	1.884	−0.006162	−0.005590	5.717151×10^{-4}	9.277835×10^{-2}
	2.512	−0.016165	−0.015812	3.533471×10^{-4}	2.185830×10^{-2}
	3.140	−0.020000	−0.020000	1.577400×10^{-7}	7.887008×10^{-6}
	3.768	−0.016203	−0.016556	3.532682×10^{-4}	2.180294×10^{-2}
	4.396	−0.006223	−0.006795	5.718787×10^{-4}	9.190149×10^{-2}
	5.024	0.006132	0.005560	5.719482×10^{-4}	9.327499×10^{-2}
	5.652	0.016147	0.015793	3.534486×10^{-4}	2.189001×10^{-2}
	6.280	0.020000	0.020000	6.214620×10^{-8}	3.107326×10^{-6}
t = 0.06	0.000	0.060000	0.060078	7.812199×10^{-5}	1.302033×10^{-3}
	0.628	0.048552	0.051803	3.250500×10^{-3}	6.694850×10^{-2}
	1.256	0.018577	0.023761	5.183495×10^{-3}	2.790220×10^{-1}
	1.884	−0.018486	−0.013322	5.164223×10^{-3}	2.793513×10^{-1}
	2.512	−0.048496	−0.045296	3.200081×10^{-3}	6.598642×10^{-2}
	3.140	−0.060000	−0.059984	1.586109×10^{-5}	2.643518×10^{-4}
	3.768	−0.048608	−0.051797	3.188704×10^{-3}	6.559995×10^{-2}
	4.396	−0.018668	−0.023844	5.175574×10^{-3}	2.772400×10^{-1}
	5.024	0.018396	0.013234	5.161817×10^{-3}	2.806014×10^{-1}
	5.652	0.048440	0.045287	3.152560×10^{-3}	6.508212×10^{-2}
	6.280	0.060000	0.060060	6.080264×10^{-5}	1.013382×10^{-3}

5 Conclusions

In this paper, we have obtained an approximate-analytical solution to homogeneous as well as non-homogeneous dispersive KdV equations with some initial approximation using modified Adomian decomposition method using Shehu's transform. For the homogeneous KdV equation in Eq. (2), results obtained by methods, standard ADM, LADM, and SADM, are equivalent and therefore give the same results. The LADM and ADM are also powerful methods for solving both linear as well as nonlinear PDEs as these methods do not need any form of transformation, perturbation, or linearization. However, rigorous computation of Adomian polynomials is one of the requirement, which can sometimes result in intensive computations for nonlinear problems.

As our main contribution, we have applied a reliable method, SADM, which combines Shehu's transform with Adomian Decomposition Method to both linear as well as nonlinear homogeneous and non-homogeneous dispersive KdV-type equation and the numerical results using SADM are given in Tables 3 and 5. The obtained numerical results in this paper confirm that SADM is an effective method, as it allows us to know the exact solution after computing first few terms only. Therefore, this method an be considered as an alternative method to solve numerous linear and nonlinear problems efficiently.

Acknowledgements A. S. Kelil kindly acknowledges NMU council postdoctoral fellowship programme (2020–2021). He is also grateful to the top-up funding support of the DST-NRF Centre of Excellence in Mathematical and Statistical Sciences(CoE-MaSS) towards the research. Opinions expressed and conclusions arrived at are those of the author and are not necessarily to be attributed to the CoE-MaSS.

References

1. Korteweg, D.J., De Vries, G.: On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. Philos. Mag. **39**, 422–443 (1895)
2. Zabusky, N.J., Kruskal, M.D.: Interaction of "solitons" in a collisionless plasma and the recurrence of initial states. Phys. Rev. Lett. **15**, 240 (1965)
3. Appadu, A.R., Chapwanya, M., Jejeniwa, O.A.: Some optimised schemes for 1D Korteweg-de-Vries equation. Prog. Comput. Fluid Dyn. **17**, 250–266 (2017)
4. Wang, H., Wang, Y., Hu, Y.: An explicit scheme for the Korteweg-de-Vries equation. Chin. Phys. Lett. **25**, 2335–2338 (2008)
5. Adomian, G.: Solving Frontier Problems of Physics: The Decomposition Method. Kluwer Academic Publishers (1994)
6. Adomian, G.: A review of decomposition method and some recent results for nonlinear equation. Math. Comput. Model. **13**, 17–43 (1992)
7. Wazwaz, A.M.: The combined Laplace transform-Adomian decomposition method for handling nonlinear Volterra integro-differential equations. Appl. Math. Comput. **216**, 1304–1309 (2010)
8. Abassy, T.A., El-Tawil, M.A., Saleh, H.K.: The solution of KdV and mKdV equations using Adomian Padé approximation. Int. J. Nonlinear Sci. Numer. Simul. **5**, 327–339 (2004)
9. He, J.H.: Variational iteration method-a kind of non-linear analytical technique: some examples. Int. J. Non-Linear Mech. **34**, 699–708 (1999)
10. Wazwaz, A.M.: A study on linear and nonlinear Schrödinger equations by the variational iteration method. Chaos Solitons Fractals **37**, 1136–1142 (2008)
11. He, J.H.: Homotopy perturbation technique. Comput. Methods Appl. Mech. Eng. **178**, 257–262 (1999)
12. He, J.H.: Homotopy perturbation method: a new nonlinear analytical technique. Appl. Math. Comput. **135**, 73–79 (2003)
13. He, J.H.: The homotopy perturbation method for nonlinear oscillators with discontinuities. Appl. Math. Comput. **151**, 287–292 (2004)

14. He, J.H.: Application of homotopy perturbation method to nonlinear wave equations. Chaos Solitons Fractals **26**, 695–700 (2005)
15. Kaya, D.: A review of the semi-analytic/numerical methods for higher order nonlinear partial equations. Contemp. Anal. Appl. Math. Off. J. Grad. Sch. Sci. Eng. **133**
16. Abbaoui, K., Cherruault, Y.: Convergence of Adomian's method applied to differential equations. Comput. Math. Appl. **28**, Elsevier, 103–109 (1994)
17. Abbaoui, K., Cherruault, Y.: New ideas for proving convergence of decomposition methods. Comput. Math. Appl. **29**, Elsevier, 103–108 (1995)
18. Wazwaz, A.M.: An analytic study on the third-order dispersive partial differential equations. Appl. Math. Comput. **142**, 511–520 (2003)
19. Dehghan, Z.M., Shakeri, F.: Use of He's Homotopy perturbation method for solving a partial differential equation arising in modeling of flow in porous media. J. Porous Media **11**, 765–778 (2008)
20. Helal, M.A., Mona Samir, M.: A comparative study between two different methods for solving the general Korteweg de Vries equation. Chaos Solitons Fractals **33**, 725–739 (2007)
21. Maitama, S., Zhao, W.: New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations. arXiv:1904.11370 (2019)
22. Goswami, A., Singh, J., Kumar, D.: Numerical simulation of fifth order KdV equation occurring in magneto-acoustic waves. Ain Shams Eng. J. **9**, 2265–2273 (2018)
23. Zachmanoglou, E.C., Thoe, D.W.: Introduction to Partial Differential Equations with Applications. Courier Corporation (1986)
24. Adomian, G., Rach, R.: Noise terms in decomposition solution series. Comp. Math. Appl. **24**, 61–64 (1992)
25. Duan, J.S.: Convenient analytic recurrence algorithms for the Adomian polynomials. Appl. Math. Comput. **217**, 6337–6348 (2011)
26. Duan, J.S.: New recurrence algorithms for the nonclassic Adomian polynomials. Appl. Math. Comput. **62**, 2961–2977 (2011)
27. Wazwaz, A.M., El-Sayed, S.M.: A new modification of the Adomian decomposition method for linear and nonlinear operators. Appl. Math. Comput. **122**, 393–405 (2001)
28. Appadu, A.R., Kelil, A.S.: On semi-analytical solutions for linearized dispersive KdV equation. Mathematics **8**, 1769 (2020)
29. Wazwaz, A.M.: Partial Differential Equations: Methods and Applications. Balkema Publishers, Lisse (2002)
30. Wazwaz, A.M.: Necessary conditions for the appearance of noise terms in decomposition solution series. J. Math. Anal. Appl. **5**, 265–274 (1997)

On Certain Properties of Perturbed Freud-Type Weight: A Revisit

Abey S. Kelil, Appanah R. Appadu, and Sama Arjika

Abstract In this paper, monic polynomials orthogonal with deformation of the Freud-type weight function are considered. These polynomials fulfill linear differential equations with some polynomial coefficients in their holonomic form. The aim of this work is to explore certain characterizing properties of perturbed Freud-type polynomials such as nonlinear recursion relations, finite moments, differential-recurrence, and differential relations satisfied by the recurrence coefficients as well as the corresponding semiclassical orthogonal polynomials. We note that the obtained differential equation fulfilled by the considered semiclassical polynomials are used to study an electrostatic interpretation for the distribution of zeros based on the original ideas of Stieltjes.

Keywords Orthogonal polynomial · Freud-type · Three-term recurrence · Differential-recurrence equation · Electrostatic zeros

2010 Mathematics Subject Classification: 33C45

1 Introduction

Suppose we have a family of polynomials $\{\psi_m(x)\}_{m=1}^{\infty}$ which are monic of degree m and that are orthogonal with respect to the positive weight $w(x)$ on the interval $[c, d]$, i.e.,

A. S. Kelil (✉) · A. R. Appadu
Nelson Mandela University, Gqeberha, South Africa
e-mail: abeysh2001@gmail.com

A. R. Appadu
e-mail: rao.appadu@mandela.ac.za

S. Arjika
University of Agadez, P.O. Box 199, 8000 Agadez, Niger

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_9

$$\langle \psi_m, \psi_k \rangle_w = \int_c^d \psi_m(x)\psi_k(x)w(x)\mathrm{d}x = \Gamma_m \delta_{m,k}, \quad m, k = 0, 1, 2, \ldots,$$

where $\Gamma_m > 0$ denotes the normalization constant [7, 29]. This value can be obtained from the square of the weighted L^2-norm of $\psi_m(x)$ over $[c, d]$. Monic polynomial representation takes the form

$$\psi_n(x) = x^n + p(n)x^{n-1} + \cdots$$

It is known that $\det\left(x_j^{i-1}\right)_{i,j=1}^N = \prod_{1\leq i<j\leq N}(x_i - x_j) = \det\left(\psi_{i-1}(x_j)\right)_{i,j=1}^N$. The polynomials $\psi_n(x)$ can be generated by the Gram-Schmidt orthogonalization process [7, 18].

As it is known in [7, 18, 29], classical orthogonal polynomials obey Pearson's differential equation

$$\frac{d\left(\lambda(x)w(x)\right)}{\mathrm{d}x} = \tau(x)w(x), \tag{1}$$

where the polynomials $\lambda(x)$ and $\tau(x)$ are of degrees two and one, respectively. Whereas polynomials for which the weight fulfills Eq. (1) with $\deg(\lambda) \geq 2$ or $\deg(\tau) \neq 1$ are said to be Semiclassical orthogonal polynomials [17].

For deformed orthogonality weight, if the moments exist and the corresponding monic orthogonal polynomials $\psi_n(z)$ for $n = 0, 1, 2, \ldots$ obey linear recursive relation

$$\begin{cases} z\psi_n(z) = \psi_{n+1}(z) + \gamma_n\psi_{n-1}(z) + \alpha_n\psi_n(z), \\ \psi_0(z) = 1, \quad \gamma_0\psi_{-1}(z) = 0. \end{cases}$$

The following relations in [3] are valid for a semiclassical weight w with $w(a) = w(b) = 0$.

Lemma 1 ([3]) *Suppose that $v(x) = -\ln w(x)$ has a derivative in some Lipschitz order with a positive exponent [27]. The differential-difference coefficients obey the following formulas:*

$$\psi_n'(z) = \gamma_n \mathcal{A}_n(z)\psi_{n-1}(z) - \mathcal{B}_n(z)\psi_n(z), \tag{2}$$

$$\psi_{n-1}'(z) = -\mathcal{A}_{n-1}(z)\psi_n(z) + \left[\mathcal{B}_n(z) + v'(z)\right]\psi_{n-1}(z), \tag{3}$$

where

$$\mathcal{A}_n(z) := \frac{1}{\Gamma_n}\int_a^b \frac{v'(z) - v'(\tau)}{z - \tau}\psi_n^2(y)w(\tau)\mathrm{d}\tau, \tag{4}$$

$$\mathcal{B}_n(z) := \frac{1}{\Gamma_{n-1}}\int_a^b \frac{v'(z) - v'(\tau)}{z - \tau}\psi_n(\tau)\psi_{n-1}(\tau)w(\tau)\mathrm{d}\tau. \tag{5}$$

Lemma 2 ([3]) *The coefficients* $\mathcal{A}_n(z)$ *and* $\mathcal{B}_n(z)$ *defined by Eqs. (4) and (5) obeys*

$$\begin{cases} \mathcal{B}_{n+1}(z) + \mathcal{B}_n(z) = -v'(z) + (z - \alpha_n)\mathcal{A}_n(z), & (M_1) \\ 1 + (z - \alpha_n)[\mathcal{B}_{n+1}(z) - \mathcal{B}_n(z)] = -\gamma_n \mathcal{A}_{n-1}(z) + \gamma_{n+1}\mathcal{A}_{n+1}(z). & (M_2) \end{cases}$$

We also mention another supplementary condition, that involves $\sum_{j=0}^{n-1} \mathcal{A}_j(z)$ and we will denote it by (M_2') as this relation helps to obtain recurrence coefficients α_n and γ_n, as

$$v'(z)\mathcal{B}_n(z) + \sum_{j=0}^{n-1} \mathcal{A}_j(z) + \mathcal{B}_n^2(z) = \gamma_n \mathcal{A}_n(z)\mathcal{A}_{n-1}(z). \qquad (M_2')$$

Eq. (M_2') can be perceived as an equation for $\sum_{j=0}^{n-1} \mathcal{A}_j(z)$. See, for instance, [2, 4]. The differential equation fulfilled by $\psi_n(z)$ is generated by eliminating $\psi_{n-1}(z)$ from ladder operators, and it is given as

$$\psi_n''(z) - \left(v'(z) + \frac{\mathcal{A}_n'(z)}{\mathcal{A}_n(z)}\right)\psi_n'(z) + \left(\mathcal{B}_n'(z) - \mathcal{B}_n(z)\frac{\mathcal{A}_n'(z)}{\mathcal{A}_n(z)} + \sum_{j=0}^{n-1} \mathcal{A}_j(z)\right)\psi_n(z) = 0, \tag{7}$$

where $\sum_{j=0}^{n-1} \mathcal{A}_j(z)$ is obtained from (M_2').

Lemma 3 *Suppose we have a symmetric semiclassical weight* $W_\sigma(x;t) = \exp(tx^2)\, w_0(x)$*, with* $t \in \mathbb{R}$ *such that the moments of for* w_0 *is finite. The recursive coefficient* $\gamma_n(t)$ *obeys the Volterra, or the Langmuir lattice, equation [31]*

$$\frac{d\gamma_n(t)}{dt} = \gamma_n(t)\left(\gamma_{n+1}(t) - \gamma_{n-1}(t)\right). \tag{8}$$

Proof See, for example, [31, Theorem 2.4]. □

In this paper, we consider studying semiclassical perturbed Freud-type measure

$$\begin{cases} d\mu_\sigma(x) = W_\sigma(x;t)\,\mathrm{d}x = |x|^{2\sigma+1}\exp\left(-[cx^6 + t(x^4 - x^2)]\right)\,\mathrm{d}x, \\ \sigma > 0,\ c > 0,\ t \in \mathbb{R}, \end{cases} \tag{9}$$

involving parameters t, σ, which will be used to represent the polynomials and in the L^2 norm. For simplicity, we may not sometimes display the parameters in the polynomials.

The motives for the choice of the perturbed orthogonality measure in (9) is as follows: First, from some of the classical orthogonal polynomials, a new class of semiclassical (non-classical) orthogonal polynomials can be obtained by means of slight modifications on their orthogonality measure [25, 26]. Such measure deformation usually results in some difficulties, most of which have not been handled yet as noted in [25, 26]. Motivated by the works of P. Nevai et al. [26], a slight modification of a new orthogonality measure on non-compact support presents a new class of orthogonal polynomials if certain characterizing properties associated with the considered polynomials are successfully obtained. Secondly, the choice of modified Freud-type measure is reasonable in the sense that this orthogonality measure emanates from quadratic transformation and Chihara's symmetrization of the modified Airy-type measure (cf. [7] for symmetrization process). This also leads to an investigation of certain fresh properties such as nonlinear differential-recurrence and differential equations satisfied by the recurrence coefficients as well as the perturbed polynomials themselves. The results obtained also motivate considerable applications; for instance, in modeling nonlinear phenomena, Soliton Theory and Random matrix theory [4] and in the crystal structure in solid-state physics, to mention a few.

2 Semiclassical Perturbed Freud-Type Polynomials

Semiclassical perturbed Freud polynomials $\{\mathcal{S}_n(x;t)\}_{n=0}^{\infty}$ on $\mathbb{R}$ are real polynomials with their orthogonality weight given by

$$\begin{cases} d\mu_\sigma(x) = W_\sigma(x;t)\,\mathrm{d}x = |x|^{2\sigma+1}\exp\left(-[cx^6 + t(x^4 - x^2)]\right)\,\mathrm{d}x, \\ \sigma > 0,\ c > 0,\ t \in \mathbb{R}, \end{cases}$$

and the orthogonality condition is given by

$$\langle \mathcal{S}_n, \mathcal{S}_m\rangle_{W_\sigma} = \int_{-\infty}^{\infty} \mathcal{S}_n(x;t)\,\mathcal{S}_m(x;t)\,W_\sigma(x;t)\,\mathrm{d}x = \hat{\Gamma}_n\,\delta_{mn}, \tag{10}$$

where δ_{mn} denotes the Kronecker delta function. It follows from Eq. (10) that the recursion relation takes the form

$$\begin{cases} \mathcal{S}_{n+1}(x;t) = -\gamma_n(t;\sigma)\,\mathcal{S}_{n-1}(x;t) + x\mathcal{S}_n(x;t), \quad n \in \mathbb{N}, \\ \mathcal{S}_0 := 1 \quad \text{and} \quad \gamma_0\mathcal{S}_{-1} := 0. \end{cases} \tag{11}$$

If we multiply Eq. (11) with $\mathcal{S}_{n-1}(x;t)W_\sigma(x;t)$ and then integrate with respect to x and using orthogonality given in Eq. (10), we obtain

$$\gamma_n(t;\sigma) = \frac{1}{\hat{\Gamma}_{n-1}(t)}\langle x\mathcal{S}_n, \mathcal{S}_{n-1}\rangle_{W_\sigma} = \frac{\hat{\Gamma}_n(t)}{\hat{\Gamma}_{n-1}} > 0. \tag{12}$$

Observe that $\mathcal{S}_n(x;t)$ comprises the terms $x^{n-r},\ r \le n$ and is symmetric so that

$$\begin{cases} \mathcal{S}_n(-x;t) = (-1)^n\mathcal{S}_n(x;t), \\ \mathcal{S}_n(0;t)\,\mathcal{S}_{n-1}(0;t) = 0, \end{cases}$$

as the weight $W_\sigma(x;t)$ is even on $\mathbb{R}$. Using monic representation of considered polynomials $\mathcal{S}_n(x;t)$, associated with $W_\sigma(x;t)$, we have that

$$\mathcal{S}_n(x;t) = x^n + \chi(n;t)\, x^{n-2} + \ldots + \mathcal{S}_n(0;t), \tag{13}$$

which can be expressed equivalently as [7],

$$\begin{cases} \mathcal{S}_{2i}(x;t) = x^{2i} + \chi(2i;t)\, x^{2i-2} + \cdots + \mathcal{S}_{2i}(0), \\ \mathcal{S}_{2i+1}(x;t) = x^{2i+1} + \chi(2i+1;t)\, x^{2i-1} + \cdots + \mathrm{s}.x = x\left(x^{2i} + \chi(2i+1;t)\, x^{2i-2} + \cdots + \mathrm{s}\right), \end{cases}$$

where $s \in \mathbb{R}$. By substituting Eq. (13) into Eq. (11), we obtain

$$\begin{cases} \gamma_n(t) = \chi(n;t) - \chi(n+1;t), \\ \chi(0) := 0. \end{cases} \tag{14}$$

Imposing a telescoping iteration of terms of Eq. (14) gives

$$\sum_{k=0}^{n-1} \gamma_k(t,\sigma) = -\chi(n;t).$$

3 Certain Properties of the Considered Semiclassical Polynomials

In this section, we explore certain characterizing properties for perturbed semiclassical Freud-type polynomials.

3.1 Finite Moments

For certain semiclassical weights, it is known in [8, 9, 21] that the moments make a link between the weight function and the theory of integrable equations, in particular, Painlevé-type equations [31].

Theorem 1 *Suppose $x, t \in \mathbb{R}$ and $c, \sigma > 0$. The first moment $\eta_0(t;\sigma)$ associated with the weight (10) is finite.*

Proof For the weight given in Eq. (9), the moment $\eta_0(t;\sigma)$ takes the form

$$\eta_0(t;\sigma) = \int_{-\infty}^{\infty} W_\sigma(x;t)\,\mathrm{d}x = 2\int_0^{\infty} W_\sigma(x;t)\,\mathrm{d}x. \tag{15}$$

For $\sigma > 0$ and $c > 0$, the function $W_\sigma(x;t) = x^{2\sigma+1}\exp\left(-[cx^6 + t(x^4 - x^2)]\right)$ is continuous on $[0,\infty)$, and hence is integrable on $[0,\mathcal{K}]$ for any $\mathcal{K} > 0$. In order to show $\int_{\mathcal{K}}^{\infty} W_\sigma(x;t)\,\mathrm{d}x$ is finite, we first note that $\lim_{x\to\infty} x^2 W_\sigma(x;t) = 0$; that is, there exists an $N > 0$ such that $x^2 W_\sigma(x;t) < 1$ whenever $x > N$ by definition. As $\int_N^\infty \frac{dx}{x^2} < \infty$, it follows, for $N > 0$, that $\int_N^\infty W_\sigma(x;t)\,\mathrm{d}x < \infty$, particularly when $N = \mathcal{K}$. Hence, $\int_0^\infty W_\sigma(x;t)\,\mathrm{d}x < \infty$. □

The following result presents some conditions for differentiation and integration order for functions of two variables [20].

Lemma 4 [20, Theorem 16.11] *Let $J = (a,b) \subset \mathbb{R}$ be an open interval and $g : \mathbb{R}\times J \to \mathbb{R}$. Assume that*

(i) $g(x,t)$ has a derivative on $\mathbb{R}$ with respect to t for almost all $x \in \mathbb{R}$,
(ii) for every fixed $t \in J$, $\int_{-\infty}^{\infty} g(x,t)\,\mathrm{d}x < \infty$,
(iii) $\exists$ an integrable function $h : \mathbb{R} \to \mathbb{R}$ such that $\forall t \in J$, $\left|\frac{\partial g(x,t)}{\partial t}\right| \le h(x)$, which is true for almost all $x \in \mathbb{R}$.

It then follows that

$$\frac{d}{dt}\int_{-\infty}^{\infty} g(x,t)\,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\partial g(x,t)}{\partial t}\,\mathrm{d}x.$$

The following result shows how moments of high order behave for the weight function in Eq. (9).

Theorem 2 *For $n \in \mathbb{N}_0$, the moments associated with the perturbed Freud weight given in (9) obey the following formulations*

$$\begin{cases} \eta_{2n}(t;\sigma) &= \dfrac{d^n}{dt^n}\int_{-\infty}^{\infty} |x|^{2\sigma+1}\exp\left(-[cx^6 + t(x^4 - x^2)]\right)\,\mathrm{d}x \\ &= \sum_{k=0}^{n}(-1)^{n+k}\binom{n}{k}\eta_{4n-2k}(t;\sigma) = \dfrac{d^n}{dt^n}\eta_0(t;\sigma), \\ \eta_{2n+1}(t;\sigma) &= 0. \end{cases} \tag{16}$$

Proof Taking into account the weight in Eq. (9) is even on $\mathbb{R}$, let's take Freud-type weight defined on the positive x-axis; that is,

$$W_\sigma(x;t) := x^{2\sigma+1}\exp\left(-[cx^6 + t(x^4 - x^2)]\right), \quad x \in (0,\infty), \quad \sigma > 0, \quad t \in J \subset \mathbb{R}.$$

One can see that W_σ is a rapidly decreasing function [20].

Using Theorem 1, we can easily see that

$$\frac{\partial W_\sigma(x;t)}{\partial t} = (x^4 - x^2)\,x^{2\sigma+1}\exp\left(-[cx^6 + t(x^4 - x^2)]\right), \tag{17}$$

is continuous on $\mathbb{R}^+$. For $t \le 0$ and $x \in (1, \infty)$, we have $\exp\left(t(x^4 - x^2)\right) \le 1$, since $ty^2 \le 0$ for $y \in \mathbb{R}$. Thus,

$$\left|\frac{\partial W_\sigma(x;t)}{\partial t}\right| = \left|x^{2\sigma+1}(x^4 - x^2)\exp\left(-[cx^6 + t(x^4 - x^2)]\right)\right| \le x^{2\sigma+k}\exp\left(-cx^6\right) := G(x), \tag{18}$$

for some bounding $k \in \mathbb{R}^+$ and $\sigma > 0$, with

$$\int_0^\infty G(x)\,\mathrm{d}x = \int_0^\infty x^{2\sigma+k}\exp\left(-cx^6\right)\,\mathrm{d}x = \frac{1}{6}\left(\frac{1}{c}\right)^{\frac{\sigma+4}{k}}\Gamma\left(\frac{2\sigma+8}{6}\right) < \infty,$$

where $\Gamma(z)$ denotes the Gamma function.

It then follows from Eq. (17) that

$$\left|\frac{\partial W_\sigma(x;t)}{\partial t}\right| = \left|x^{2\sigma+3}\exp\left(-[cx^6 + t(x^4 - x^2)]\right)\right| \le x^{2\sigma+3}\exp\left(-cx^6 + Ax^2\right) := K(x),$$

for $t \in [0, A]$, $A \in \mathbb{R}^+$ and $K(x)$ is integrable for $x \in \mathbb{R}^+$. We see that all the conditions of Lemma 4 are fulfilled so that Eq. (16) can be proved using the principles of mathematical induction. For $n = 1$, we have

$$\begin{aligned}\frac{d}{dt}\eta_0(t,\sigma) &= \frac{d}{dt}\int_{-\infty}^\infty |x|^{2\sigma+1}\exp\left(-[cx^6 + t(x^4 - x^2)]\right)\,\mathrm{d}x\\ &= (-1)\int_{-\infty}^\infty (x^4 - x^2)W_\sigma(x;t)\,\mathrm{d}x = (-1)\left(\eta_4(t,\sigma) - \eta_2(t,\sigma)\right).\end{aligned}$$

We suppose, for inductive assumption, that

$$\frac{d^n}{dt^n}\eta_0(t,\sigma) = \sum_{k=0}^n (-1)^{n+k}\binom{n}{k}\eta_{4n-2k}(t;\sigma) := \eta_{2n}(t,\sigma).$$

We need to show that

$$\eta_{2n+2}(t,\sigma) = \frac{d^{n+1}}{dt^{n+1}}\eta_0(t,\sigma).$$

We note that $x^{2n+2\sigma+1}\exp\left(-[cx^6 + t(x^4 - x^2)]\right),\ x \in \mathbb{R}^+,\ t \in J$, also obeys the conditions of Lemma 4. Then, by applying binomial expansion, we have

$$\begin{aligned}\frac{d^{n+1}}{dt^{n+1}}\eta_0(t,\sigma) &= \frac{d}{dt}\left(\frac{d^n}{dt^n}\eta_0(t,\sigma)\right)\\ &= \frac{d}{dt}\int_{\mathbb{R}}(-1)^n\left(x^4 - x^2\right)^n W_\sigma(x;t)\,\mathrm{d}x = \int_{\mathbb{R}}(-1)^n(-1)\left(x^4 - x^2\right)^{n+1}W_\sigma(x;t)\,\mathrm{d}x\end{aligned}$$

$$= \sum_{k=0}^{n} (-1)^{n+1} \binom{n+1}{k} \int_{-\infty}^{\infty} \left(x^4\right)^{n+1-k} \left(-x^2\right)^k \; W_\sigma(x;t)\,\mathrm{d}x$$
$$= \sum_{k=0}^{n} (-1)^{n+k+1} \binom{n+1}{k} \eta_{4n+4-2k}(t;\sigma) = \eta_{2n+2}(t,\sigma) \equiv \eta_0(t; n+\sigma+1).$$

Besides, moments of odd order vanish; i.e.,

$$\eta_{2n+1}(t;\sigma) = \int_{-\infty}^{\infty} x^{2n+1}\, W_\sigma(x;t)\mathrm{d}x = 0, \quad n \in \mathbb{N}, \tag{12}$$

as the expression in the above integral is an odd function. □

3.2 Concise Formulation

The following result gives a concise formulation for perturbed Freud-type polynomials $\mathcal{S}_n(x;t)$. For a similar result, [19, Lemma 3.2].

Lemma 5 *Suppose we have the perturbed Freud-type weight given in (9). Concise formulation of the corresponding polynomials, in terms of recurrence coefficient $\gamma_j(t;\sigma)$, is given by*

$$\begin{cases} \mathcal{S}_q(x;t) = \sum_{k=0}^{\lfloor \frac{q}{2} \rfloor} \Psi_k(q)\, x^{q-2k}, \\ \Psi_0(q) = 1, \quad \textit{for} \quad k \in \{1,2,\ldots,\lfloor \frac{q}{2} \rfloor\}, \quad q \in \mathbb{N}, \end{cases} \tag{13a}$$

where

$$\Psi_k(q) = (-1)^k \sum_{j_1=1}^{q+1-2k} \gamma_{j_1}(t;\sigma) \sum_{j_2=j_1+2}^{q+3-2k} \gamma_{j_2}(t;\sigma) \sum_{j_3=j_2+2}^{q+5-2k} \gamma_{j_3}(t;\sigma) \cdots \sum_{j_k=j_{k-1}+2}^{q-1} \gamma_{j_k}(t;\sigma). \tag{13b}$$

Proof Since the perturbed Freud-type polynomials $\mathcal{S}_q(x;t)$ are symmetric and monic of degree q, and for a fixed $t \in \mathbb{R}$, we have $\mathcal{S}_q(-x) = (-1)^q \mathcal{S}_q(x)$, so that

$$\mathcal{S}_{2q}(x;t) = \sum_{j=0}^{q} g_{2q-2j}\, x^{2q-2j}; \qquad \mathcal{S}_{2q+1}(x;t) = \sum_{j=0}^{q} g_{2q-2j+1}\, x^{2q-2j+1}, \tag{14}$$

where $g_{q-2k} = \Psi_k(q)$ with $\Psi_0(q) = 1$ and $\Psi_k(q) = 0$ for $k > \lfloor \frac{q}{2} \rfloor$. If we substitute Eq. (13a) into Eq. (11) and if we compare the coefficients of x, we obtain

$$\begin{cases} \Psi_k(q+1) - \Psi_k(q) = -\gamma_q(t;\sigma)\Psi_{k-1}(q-1), \\ \Psi_0(q) = 1. \end{cases} \tag{15}$$

Equation (13b) can be proved by employing induction on k. For $k = 1$, we see that

$$\Psi_1(q) - \Psi_1(q-1) = -\gamma_{q-1}, \tag{16}$$

By employing a telescoping sum of terms in Eq. (16), we obtain

$$\Psi_1(q) = -\sum_{j_1=0}^{q-1} \gamma_{j_1}(t;\sigma), \ \forall q \geq 1.$$

Let's assume that, for every $q \in \mathbb{N}$, Eq. (13b) holds true for values up to $k-1$, i.e.,

$$\Psi_{k-1}(n) = (-1)^{k-1} \sum_{j_1=1}^{q+3-2k} \gamma_{j_1}(t;\sigma) \sum_{j_2=j_1+2}^{q+5-2k} \gamma_{j_2}(t;\sigma) \sum_{j_3=j_2+2}^{q+7-2k} \gamma_{j_3}(t;\sigma) \cdots \sum_{j_{k-1}=j_{k-2}+2}^{q-1} \gamma_{j_{k-1}}(t;\sigma). \tag{17}$$

Equation (15) can be repeatedly used to obtain

$$\begin{aligned}
\Psi_k(q) &= \Psi_k(q-1) - \gamma_{q-1}\Psi_{k-1}(q-2),\\
&= \Psi_k(q-2) - \gamma_{q-2}\Psi_{k-1}(q-3) - \gamma_{q-1}\Psi_{k-1}(q-2),\\
&= \Psi_k(q-3) - \gamma_{q-3}\Psi_{k-1}(q-4) - \gamma_{q-2}\Psi_{k-1}(q-3) - \gamma_{q-1}\Psi_{k-1}(q-2),\\
&\ \ \vdots\\
&= -\gamma_{2k-1}\Psi_{k-1}(2k-2) - \gamma_{2k}\Psi_{k-1}(2k-1) - \cdots - \gamma_{q-2}\Psi_{k-1}(q-3) - \gamma_{q-1}\Psi_{k-1}(q-2).
\end{aligned} \tag{18}$$

Substituting Eq. (17) into Eq. (18) yields Eq. (13b) and hence the required result. □

Lemma 5 is alternately given as follows.

Proposition 1 *The following formulation also holds for monic perturbed Freud-type polynomials $\mathcal{S}_q(x;t)$:*

$$\mathcal{S}_q(x;t) = x^q + \sum_{r=1}^{\lfloor \frac{q}{2} \rfloor} (-1)^r \left(\sum_{k \in W(q,r)} \gamma_{k_1}\gamma_{k_2} \cdots \gamma_{k_{r-1}}\gamma_{k_r} \right) x^{q-2r},$$

where $W(q,r) = \{k \in \mathbb{N}^r \mid k_{j+1} \geq k_j + 2 \ \textit{for} \ 1 \leq j \leq r-1, \ \ 1 \leq k_1, k_r < q\}$, *and* $\lfloor \frac{q}{2} \rfloor = \begin{cases} \frac{q}{2}, & q \textit{ is even}, \\ \frac{q-1}{2}, & q \textit{ is odd}. \end{cases}$

3.3 Normalization Constant

The normalization constant $\hat{\Gamma}_m$ in Eq. (10) for the weight in Eq. (9) takes the form

$$\hat{\Gamma}_m = \langle \mathcal{S}_m, \mathcal{S}_m \rangle_{W_\sigma} = \|\mathcal{S}_m\|^2_{W_\sigma} = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \Psi_k(m)\ \eta_{2m-2k}(t;\sigma), \tag{19}$$

where $\Psi_k(m)$ is given in Eq. (13b). Equation (19) is equivalently given by

$$\hat{\Gamma}_m(t) = \int_{-\infty}^{\infty} \mathcal{S}_m^2(x,t)\ W_\sigma(x;t)\,\mathrm{d}x.$$

By using variable transformation $x^2 = \xi$, we have different normalization parties as follows:

$$\begin{aligned}\hat{\Gamma}_{2m}(t) &= \int_{-\infty}^{\infty} \mathcal{S}_{2m}^2(x,t)\ W_\sigma(x;t)\mathrm{d}x \\ &= 2\int_0^{\infty} \mathcal{S}_{2m}^2(\sqrt{\xi},t)\ |\xi|^{\sigma+\frac{1}{2}} \exp\left(-[c\xi^3 + t(\xi^2-\xi)]\right) \frac{1}{2\sqrt{\xi}}\mathrm{d}\xi \\ &= \int_0^{\infty} \widetilde{P}_m^2(\xi,t)s^{-\frac{1}{2}}\ |\xi|^{\sigma+\frac{1}{2}} \exp\left(-[c\xi^3 + t(\xi^2-\xi)]\right)\ \mathrm{d}\xi =: \widetilde{h}_m(t),\end{aligned}$$

and

$$\begin{aligned}\hat{\Gamma}_{2m+1}(t) &= \int_{-\infty}^{\infty} \mathcal{S}_{2m+1}^2(x,t)\ W_\sigma(x;t)\ \mathrm{d}x \\ &= 2\int_0^{\infty} \mathcal{S}_{2m+1}^2(\sqrt{\xi},t)\ |\xi|^{\sigma+\frac{1}{2}} \exp\left(-[cs^3 + t(s^2-s)]\right) \frac{1}{2\sqrt{\xi}}\ \mathrm{d}\xi \\ &= \int_0^{\infty} \widehat{P}_n^2(\xi,t)\ \xi^{\frac{1}{2}}\ |\xi|^{\sigma+\frac{1}{2}} \exp\left(-[c\xi^3 + t(\xi^2-\xi)]\right)\ \mathrm{d}\xi =: \widehat{h}_m(t),\end{aligned}$$

We now see that

$$\begin{aligned}\mathcal{S}_{2m}(\sqrt{\xi},t) &= (\sqrt{\xi})^{2m} + \chi(2m,t)(\sqrt{\xi})^{2m-2} + \cdots + \mathcal{S}_{2m}(0,t) \\ &= \xi^n + \widetilde{p}(m,t)\xi^{m-1} + \cdots + \widetilde{P}_m(0,t) := \widetilde{P}_m(\xi,t),\end{aligned}$$

and

$$\begin{aligned}\mathcal{S}_{2m+1}(\sqrt{\xi},t) &= (\sqrt{\xi})^{2m+1} + \chi(2m,t)(\sqrt{\xi})^{2m-1} + \cdots + k\cdot\sqrt{\xi}, \quad k \in \mathbb{R}, \\ &= \sqrt{\xi}\left(\xi^m + \widehat{p}(m,t)\xi^{m-1} + \cdots + k\right) := \sqrt{\xi}\,\widehat{P}_m(\xi,t).\end{aligned}$$

The above polynomials $\widetilde{P}_m(\xi,t)$ and $\widehat{P}_m(\xi,t)$ are recognized as monic semiclassical Airy-type polynomials with corresponding orthogonality weights

$$w_1(x;t) = \xi^{-\frac{1}{2}}|\xi|^{\sigma+\frac{1}{2}} \exp\left(-[c\xi^3 + t(\xi^2-\xi)]\right), \tag{20a}$$

$$w_2(x;t) = \xi^{\frac{1}{2}}|\xi|^{\sigma+\frac{1}{2}} \exp\left(-[c\xi^3 + t(\xi^2 - \xi)]\right), \tag{20b}$$

both defined over $(0, \infty)$, respectively. (See [7] for symmetrization process and quadratic transformation).

The corresponding Hankel determinants for the weights in Eq. (20) can be given by

$$\widetilde{D}_m(t) := \det\left(\int_0^\infty \xi^{i+j-\frac{1}{2}}\, \xi^{\sigma+\frac{1}{2}} \exp\left(-[c\xi^3 + t(\xi^2 - \xi)]\right)\, d\xi\right)_{i,j=0}^{n-1} = \prod_{l=0}^{m-1} \widetilde{h}_l(\xi),$$

$$\widehat{D}_m(t) := \det\left(\int_0^\infty \xi^{i+j+\frac{1}{2}}\, \xi^{\sigma+\frac{1}{2}} \exp\left(-[c\xi^3 + t(\xi^2 - \xi)]\right)\, d\xi\right)_{i,j=0}^{n-1} = \prod_{l=0}^{m-1} \widehat{h}_l(\xi)$$

respectively. Hence,

$$\Delta_n(t) = \prod_{j=0}^{n-1} \Gamma_j(t) = \begin{cases} \widetilde{D}_{k+1}\widehat{D}_k & n = 2k+1, \\ \widetilde{D}_k\widehat{D}_k & n = 2k. \end{cases}$$

It is good to mention here that investigation of asymptotics of the Hankel determinants when n is large has been an interesting subject for many years; for instance, for Gaussian weight is studied in Chen et al. in [23]. See also the monograph by Szegö [29] as we will not address this as it goes beyond the scope of the paper.

3.4 Nonlinear Recursion Relation

In this section, we explore certain nonlinear recurrence relations associated with the semiclassical weight given in (9).

Theorem 3 *For the semiclassical weight in (10), the recurrence coefficient $\gamma_n(t;\sigma)$ fulfills the following difference relations*

$$6c\left[\gamma_n\left(\Xi_{n-1} + \Xi_n + \Xi_{n+1}\right) + \gamma_{n-1}\gamma_n\gamma_{n+1}\right] + 4t\,\Xi_n - 2t\gamma_n = n + (2\sigma+1)\Omega_n, \tag{21}$$

with initial conditions given by

$$\begin{cases} \gamma_1(t;\sigma) = \dfrac{\|x^2\|_t^2}{\|1\|_t^2} = \dfrac{\eta_2(t;\sigma)}{\eta_0(t;\sigma)} = \dfrac{\int_{-\infty}^{\infty} x^2 W_\sigma(x;t)dx}{\int_{-\infty}^{\infty} W_\sigma(x;t)\, dx}, \\ \gamma_0 = 0, \end{cases} \tag{22}$$

where Ξ_n and Ω_n are, respectively, given by

$$\Xi_n = \gamma_n(t;\sigma)\left[\gamma_{n-1}(t;\sigma) + \gamma_n(t;\sigma) + \gamma_{n+1}(t;\sigma)\right], \tag{23}$$

and

$$\Omega_n = \frac{1-(-1)^n}{2} = \begin{cases} 1, & \textit{for} \quad n \textit{ is odd} \\ 0, & \textit{for} \quad n \textit{ is even}. \end{cases} \tag{24}$$

Proof (i) Applying similar procedure due to Freud as given in [30, Section 2] (see also [26]), let's consider the following integral

$$\mathbb{J}_n = \frac{1}{\hat{\Gamma}_n}\int_{-\infty}^{\infty}\left[\mathcal{S}_n(x;t)\ \mathcal{S}_{n-1}(x;t)\right]'\ W_\sigma(x;t)\ \mathrm{d}x, \tag{25}$$

where $\hat{\Gamma}_n$ is given in (19). Equation (25) is equivalently given by

$$\begin{aligned}\mathbb{J}_n &= \frac{1}{\hat{\Gamma}_n}\left[\langle \mathcal{S}_n', \mathcal{S}_{n-1}\rangle_{W_\sigma} + \langle \mathcal{S}_n, \mathsf{S}_{n-1}'\rangle_{W_\sigma}\right] \\ &= \frac{1}{\hat{\Gamma}_n}\int_{-\infty}^{\infty}\left(nx^{n-1} + V_{n-2}\right)\ \mathcal{S}_{n-1}(x;t)\ W_\sigma(x;t)\ \mathrm{d}x = \frac{\hat{\Gamma}_{n-1}}{\hat{\Gamma}_n}n,\end{aligned} \tag{26}$$

where $V_{n-2} \in \mathbb{P}_{n-2}$. We also see that by evaluating Eq. (25) using technique of integration, we arrive at

$$\begin{aligned}\mathbb{I}_n\hat{\Gamma}_n &= \left[\mathcal{S}_n(x;t)\ \mathcal{S}_{n-1}(x;t)W_\sigma(x;t)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\mathcal{S}_n(x;t)\ \mathcal{S}_{n-1}(x;t)\ W_\sigma'(x;t)\ \mathrm{d}x \\ &= -(2\sigma+1)\int_{-\infty}^{\infty}\frac{\mathcal{S}_n(x;t)\ \mathcal{S}_{n-1}(x;t)}{x}\ W_\sigma(x;t)\ \mathrm{d}x + 6c\int_{-\infty}^{\infty}x^5\mathcal{S}_n(x;t)\ \mathcal{S}_{n-1}(x;t)\ W_\sigma(x;t)\ \mathrm{d}x \\ &\quad + 4t\int_{-\infty}^{\infty}x^3\mathcal{S}_n(x;t)\ \mathcal{S}_{n-1}(x;t)\ \ W_\sigma(x;t)\ \mathrm{d}x - 2t\int_{-\infty}^{\infty}x\mathcal{S}_n(x;t)\ \mathcal{S}_{n-1}(x;t)\ \ W_\sigma(x;t)\ \mathrm{d}x,\end{aligned} \tag{27}$$

in consideration of the fact that $\left[\mathcal{S}_n(x;t)\ \mathcal{S}_{n-1}(x;t)W_\sigma(x;t)\right]_{-\infty}^{\infty} = 0$ as the weight (10) vanishes at the boundary terms when $x \to \pm\infty$ due to symmetry property of the weight W_σ; hence it follows that

$$\int_{-\infty}^{\infty}\mathcal{S}_n(x;t)\ \frac{1}{x}\ \mathcal{S}_{n-1}(x;t)\ W_\sigma(x;t)\ \mathrm{d}x = 0, \tag{28a}$$

for n is even and, when n is odd, we have that

$$\int_{-\infty}^{\infty}\mathcal{S}_{n-1}(x;t)\ \frac{\mathcal{S}_n(x;t)}{x}\ \ W_\sigma(x;t)\ \mathrm{d}x = \hat{\Gamma}_{n-1}, \tag{28b}$$

as $\dfrac{\mathcal{S}_n(x;t)}{x}$ is a polynomial of degree $n-1$. Thus, we have

$$\int_{-\infty}^{\infty}\frac{\mathcal{S}_{n-1}(x;t)\mathcal{S}_n(x;t)}{x}\ W_\sigma(x;t)\ \mathrm{d}x = \Omega_n\hat{\Gamma}_{n-1}, \tag{28c}$$

where Ω_n is given in (24). Let us employ the following iterated recurrence relation from Eq. (11) to obtain

$$\begin{aligned} x^5\mathcal{S}_n(x;t) = \mathcal{S}_{n+5}(x;t) &+ (\gamma_n+\gamma_{n+1}+\gamma_{n+2}+\gamma_{n+3}+\gamma_{n+4})\,\mathcal{S}_{n+3}(x;t) \\ &+ \left[\gamma_n\left(\Xi_{n-1}+\Xi_n+\Xi_{n+1}\right)+\gamma_{n-1}\gamma_n\gamma_{n+1}\right]\mathcal{S}_{n+1}(x;t) \\ &+ \left[\gamma_n\gamma_{n-2}\Xi_{n-1}+\gamma_{n-2}\gamma_{n-1}\gamma_n\gamma_{n+1}+\gamma_n\gamma_{n-1}\gamma_{n-2}\gamma_{n-3}\right]\mathcal{S}_{n-3}(x;t) \\ &+ (\gamma_n\gamma_{n-1}\gamma_{n-2}\gamma_{n-3}\gamma_{n-4})\,\mathcal{S}_{n-5}(x;t), \end{aligned} \tag{29a}$$

$$\begin{aligned} x^4\mathcal{S}_n(x;t) = \mathcal{S}_{n+4}(x;t) &+ (\gamma_n+\gamma_{n+1}+\gamma_{n+2}+\gamma_{n+3})\mathcal{S}_{n+2}(x;t) \\ &+ \left[\gamma_n(\gamma_{n-1}+\gamma_n+\gamma_{n+1})+\gamma_{n+1}(\gamma_n+\gamma_{n+1}+\gamma_{n+2})\right]\mathcal{S}_n(x;t) \\ &+ \gamma_n\gamma_{n-1}(\gamma_{n-2}+\gamma_{n-1}+\gamma_n+\gamma_{n+1})\mathcal{S}_{n-2}(x;t)+(\gamma_n\gamma_{n-1}\gamma_{n-2}\gamma_{n-3})\mathcal{S}_{n-4}(x;t), \end{aligned} \tag{29b}$$

$$\begin{aligned} x^3\mathcal{S}_n(x;t) = (\gamma_n+\gamma_{n+1}+\gamma_{n+2})\,\mathcal{S}_{n+1}(x;t) &+ \mathcal{S}_{n+3}(x;t) \\ &+ \gamma_n\gamma_{n-1}\,\gamma_{n-2}\mathcal{S}_{n-3}(x;t)+\gamma_n\left(\gamma_{n-1}+\gamma_n+\gamma_{n+1}\right)\mathcal{S}_{n-1}(x;t), \end{aligned} \tag{29c}$$

$$x^2\mathcal{S}_n(x;t) = (\gamma_n+\gamma_{n+1})\,\mathcal{S}_n(x;t)+\gamma_n\gamma_{n-1}\mathcal{S}_{n-2}(x;t)+\mathcal{S}_{n+2}(x;t). \tag{29d}$$

By using the identities (29) and Eq. (1) for the weight (9) together with Eqs. (28) into (27), we obtain

$$\begin{aligned} n\hat{\Gamma}_{n-1} = \mathbb{I}_n\hat{\Gamma}_n = 6c&\left[(\gamma_n+\gamma_{n-1})\,\Xi_n+(\gamma_n\Xi_{n+1}+\gamma_n\gamma_{n-1}\gamma_{n-2})\right]\hat{\Gamma}_{n-1} \\ &-2t\gamma_n\hat{\Gamma}_{n-1}-(2\sigma+1)\Omega_n\hat{\Gamma}_{n-1}+4t\left[\gamma_n\left(\gamma_{n-1}+\gamma_n+\gamma_{n+1}\right)\right]\hat{\Gamma}_{n-1}, \end{aligned} \tag{30}$$

which simplifies, using the fact that $\hat{\Gamma}_{n-1}\neq 0$, to

$$\begin{aligned} n+(2\sigma+1)\Omega_n = 6c&\left[(\gamma_n+\gamma_{n-1})\,\Xi_n+(\gamma_n\Xi_{n+1}+\gamma_n\gamma_{n-1}\gamma_{n-2})\right] \\ &+4t\left[\gamma_n\left(\gamma_{n-1}+\gamma_n+\gamma_{n+1}\right)\right]-2t\gamma_n, \end{aligned} \tag{31}$$

where Ω_n is given in (24). Note that Eqs. (30) and (26) yield Eq. (21).

□

Remark 1 Quite similar nonlinear discrete equations like Eq. (31) can be obtained in [13, Eq. (23), p. 5] and we also refer to [1, 9, 31].

The following result gives the differential-recurrence relation for the weight (9).

Theorem 4 *For the semiclassical weight in (10), the coefficients $\gamma_n(t;\sigma)$ obey Toda-type formulation*

$$\frac{d\gamma_n}{dt} = \gamma_n \big[(\gamma_{n+1} - \Xi_{n+1}) - (\gamma_{n-1} - \Xi_{n-1}) \big], \tag{32}$$

where Ξ_n is given in Eq. (23).

Proof In order to prove this result, we first differentiate the normalization constant $\hat{\Gamma}_n(t)$ with respect to t as

$$\begin{aligned} \frac{d\hat{\Gamma}_n}{dt} &= 2\langle \frac{d\mathcal{S}_n}{dt}, \mathcal{S}_n \rangle_{W_\sigma} + \langle (x^2 - x^4)\mathsf{S}_n, \mathsf{S}_n \rangle_{W_\sigma}, \\ &= 2\int_{-\infty}^{\infty} \frac{d\mathcal{S}_n(x;t)}{dt}\, \mathcal{S}_n(x;t)\, W_\sigma(x;t)\, \mathrm{d}x + \int_{-\infty}^{\infty} x^2\, \mathsf{S}_n^2(x;t)\, W_\sigma(x;t)\, \mathrm{d}x \\ & - \int_{-\infty}^{\infty} x^4\, \mathsf{S}_n^2(x;t)\, W_\sigma(x;t)\, \mathrm{d}x. \end{aligned} \tag{33}$$

We see from Eq. (33) that the first integral vanishes by orthogonality as $\dfrac{d\mathcal{S}_n}{dt} \in \mathcal{P}_{n-1}$. Using the recursive relation in Eq. (11) and orthogonality fact, we now have

$$\frac{d}{dt}\hat{\Gamma}_n = (\gamma_n + \gamma_{n+1})\,\hat{\Gamma}_n - (\Xi_n + \Xi_{n+1})\,\hat{\Gamma}_n = \big[(\gamma_n - \Xi_n) + (\gamma_{n+1} - \Xi_{n+1})\big]\,\hat{\Gamma}_n, \tag{34}$$

Besides, if we differentiate Eq. (12) with respect to t, we obtain

$$\frac{d}{dt}\gamma_n = \frac{d}{dt}\left(\frac{\hat{\Gamma}_n}{\hat{\Gamma}_{n-1}}\right) = \gamma_n\left[\frac{d}{dt}\ln\hat{\Gamma}_n - \frac{d}{dt}\ln\hat{\Gamma}_{n-1}\right] = \gamma_n\big[(\gamma_{n+1} - \gamma_{n-1}) - [\Xi_{n+1} - \Xi_{n-1}]\big], \tag{35}$$

and substituting Eq. (34) into (35) leads to the required result. □

The following result presents nonlinear differential-recurrence relation of high order associated with the weight (10); we quote ideas of the proof from [22].

Theorem 5 *The coefficients $\gamma_n(t;\sigma)$ for the weight in Eq. (9) fulfills the following nonlinear differential-recurrence equation*

$$
\begin{cases}
\frac{d^2\gamma_n}{dt} = \frac{1}{6c}\left[n+(2\sigma+1)\Omega_n-\vartheta(t)\right]+\left(-\gamma_{n-1}-\gamma_{n+1}\right)\gamma_n^4 \\
\quad +\left(-\gamma_{n-2}\gamma_{n-1}-\gamma_{n-1}^2-6\gamma_{n-1}\gamma_{n+1}-\gamma_{n+1}^2-\gamma_{n+1}\gamma_{n+2}+2\gamma_{n-1}+2\gamma_{n+1}\right)\gamma_n^3 \\
\quad +\Big(\gamma_{n-3}\gamma_{n-2}\gamma_{n-1}+\gamma_{n-2}^2\gamma_{n-1}+2\gamma_{n-2}\gamma_{n-1}^2-4\gamma_{n-2}\gamma_{n-1}\gamma_{n+1}+\gamma_{n-1}^3-5\gamma_{n-1}^2\gamma_{n+1}-4\gamma_{n-1}\gamma_{n+1}\gamma_{n+2} \\
\qquad -5\gamma_{n-1}\gamma_{n+1}^2+\gamma_{n+1}^3+2\gamma_{n+1}^2\gamma_{n+2}+\gamma_{n+1}\gamma_{n+2}^2+\gamma_{n+1}\gamma_{n+2}\gamma_{n+3}+8\gamma_{n-1}\gamma_{n+1}-\gamma_{n-1}-\gamma_{n+1}\Big)\gamma_n^2 \\
\quad +\Big(\gamma_{n-4}\gamma_{n-3}\gamma_{n-2}\gamma_{n-1}+\gamma_{n-3}^2\gamma_{n-2}\gamma_{n-1}+2\gamma_{n-3}\gamma_{n-2}^2\gamma_{n-1}+2\gamma_{n-3}\gamma_{n-2}\gamma_{n-1}^2+\gamma_{n-2}^3\gamma_{n-1} \\
\qquad +3\gamma_{n-2}^2\gamma_{n-1}^2+3\gamma_{n-2}\gamma_{n-1}^3-2\gamma_{n-2}\gamma_{n-1}\gamma_{n+1}^2-2\gamma_{n-2}\gamma_{n-1}\gamma_{n+1}\gamma_{n+2}+\gamma_{n-1}^4-2\gamma_{n-1}^2\gamma_{n+1}^2 \\
\qquad -2\gamma_{n-1}^2\gamma_{n+1}\gamma_{n+2}+\gamma_{n+1}^4+3\gamma_{n+1}^3\gamma_{n+2}+3\gamma_{n+1}^2\gamma_{n+2}^2+2\gamma_{n+1}^2\gamma_{n+2}\gamma_{n+3}+\gamma_{n+1}\gamma_{n+2}^3 \\
\qquad +2\gamma_{n+1}\gamma_{n+2}^2\gamma_{n+3}+\gamma_{n+1}\gamma_{n+2}\gamma_{n+3}^2-2\gamma_{n-2}^2\gamma_{n-1}+\gamma_{n+1}\gamma_{n+2}\gamma_{n+3}\gamma_{n+4}-2\gamma_{n-3}\gamma_{n-2}\gamma_{n-1} \\
\qquad -4\gamma_{n-2}\gamma_{n-1}^2+2\gamma_{n-2}\gamma_{n-1}\gamma_{n+1}-2\gamma_{n-1}^3+2\gamma_{n-1}^2\gamma_{n+1}+2\gamma_{n-1}\gamma_{n+1}^2+2\gamma_{n-1}\gamma_{n+1}\gamma_{n+2}-2\gamma_{n+1}^3 \\
\qquad -4\gamma_{n+1}^2\gamma_{n+2}-\gamma_n^2-2\gamma_{n+1}\gamma_{n+2}^2-2\gamma_{n+1}\gamma_{n+2}\gamma_{n+3}-2\gamma_{n-1}\gamma_{n+1}-2\gamma_n\gamma_{n-1}-2\gamma_n\gamma_{n+1}-\gamma_{n+1}\gamma_{n-1}\Big)\gamma_n, \\
\vartheta(t) = 4t\,\Xi_n - 2\gamma_n t = 2t\gamma_n\left[2(\gamma_{n-1}+\gamma_n+\gamma_{n+1})-1\right].
\end{cases}
$$

where Ω_n and Ξ_n are given in Eqs. (24) and (23) respectively.

Proof For the proof, we refer similar ideas in [22]. □

3.5 Differential-Recurrence Relation

Chen and Feigin [6] obtained ladder operators for a semiclassical weight $\widetilde{w}(x)|x-t|^{\Theta}$, where $x, \Theta, t \in \mathbb{R}$ and $\widetilde{w}(x)$ is classical weight function. In Filipuk et al. [12], it is shown that the recurrence coefficients for the quartic Freud weight $|x|^{2\alpha+1}\mathrm{e}^{-x^4+tx^2}$, $x, t \in \mathbb{R}$, $\alpha > -1$ are related to the solutions of the Painlevé IV and the first discrete Painlevé equation. Clarkson et al. [9] provided a systematic study on Freud weights and some generalized work for [6].

Lemma 6 ([22]) *The monic orthogonal polynomials $P_n(x; t)$ with respect to the semiclassical Freud-type weight (9)*

$$w_\alpha(x) = |x|^\alpha w_0(x),$$

where

$$w_0(x) := \mathrm{e}^{-v_0(x)} \quad \text{with} \quad v_0(x) := cx^6 + t(x^4 - x^2).$$

on $\mathbb{R}$ satisfy the differential-difference–recurrence relation

$$P_n'(x) = \gamma_n(t)\mathcal{A}_n(x)P_{n-1}(x) - \mathcal{B}_n(x)P_n(x),$$

where

$$\mathcal{A}_n(x) := \frac{1}{\Gamma_n}\int_{-\infty}^{\infty}\frac{v_0'(x)-v_0'(\tau)}{x-\tau}\,P_n^2(\tau)\,w(\tau)\,d\tau, \tag{36a}$$

$$\mathcal{B}_n(x) := \frac{1}{\Gamma_{n-1}}\int_{-\infty}^{\infty}\frac{v_0'(x)-v_0'(\tau)}{x-\tau}\,P_n(\tau)\,P_{n-1}(\tau)\,w(\tau)\,d\tau + \frac{\alpha\left[1-(-1)^n\right]}{2x}. \tag{36b}$$

Proof For the proof, we refer to [22]. See also similar works in [5]. □

Lemma 7 *$\mathcal{A}_n(z)$ and $\mathcal{B}_n(z)$ defined by Lemma 6 satisfy the relation:*

$$\mathcal{A}_n(z) = \frac{v_0'(z)}{z} + \frac{\mathcal{B}_n(z) + \mathcal{B}_{n+1}(z)}{z} - \frac{\alpha}{z^2}. \tag{37}$$

Proof Be the definition of $\mathcal{A}_n(z)$, we rewrite it as

$$\begin{aligned}
\mathcal{A}_n(z) =& \frac{1}{z\Gamma_n}\left\{\int_{-\infty}^{\infty} \frac{v_0'(z) - v_0'(\tau)}{z-\tau} y P_n^2(\tau)w(\tau)\mathrm{d}\tau + \int_{-\infty}^{\infty} \left[v_0'(z) - v_0'(\tau)\right] P_n^2(\tau)w(\tau)\mathrm{d}\tau\right\} \\
=& \frac{1}{z\Gamma_n}\left\{\int_{-\infty}^{\infty} \frac{v_0'(z) - v_0'(\tau)}{z-\tau} \left[P_{n+1}(\tau) + \gamma_n P_{n-1}(\tau)\right] P_n(\tau)w(\tau)\mathrm{d}\tau + v_0'(z)\Gamma_n \right. \\
& \left. - \int_{-\infty}^{\infty} P_n^2(\tau)\left[\frac{\alpha}{\tau}w(\tau) - w'(\tau)\right]\mathrm{d}\tau\right\} \\
=& \frac{1}{z}\left\{\mathcal{B}_{n+1}(z) - \frac{\alpha}{2z}\left[1-(-1)^{n+1}\right] + \mathcal{B}_n(z) - \frac{\alpha}{2z}\left[1-(-1)^n\right]\right\} + \frac{v_0'(z)}{z}, \\
=& \frac{\mathcal{B}_n(z) + \mathcal{B}_{n+1}(z)}{z} - \frac{\alpha}{z^2} + \frac{v_0'(z)}{z},
\end{aligned}$$

which completes the proof. □

Lemma 8 [18, Chapter 3] *The functions $\mathcal{A}_n(z)$, $\mathcal{B}_n(z)$, and $\sum_{k=0}^{n-1}\mathcal{A}_k(z)$ satisfy the identity*

$$\mathcal{B}_n^2(z) + v'(z)\mathcal{B}_n(z) + \sum_{k=0}^{n-1}\mathcal{A}_k(z) = \gamma_n\mathcal{A}_n(z)\mathcal{A}_{n-1}(z). \tag{38}$$

We, next, apply the ladder coefficients to the case of perturbed Freud weight as follows.

3.5.1 Ladder Operator Relations for the Weight (9)

For the perturbed Freud-type weight (9),

$$v(x) = -\ln W_\sigma(x;t) = -(2\sigma+1)\ln|x| + cx^6 + t(x^4 - x^2), \quad x \in \mathbb{R}, \tag{39}$$

we have

$$v'(x) = -\frac{(2\sigma+1)}{x} + 6cx^5 + t(4x^3 - 2x),$$

and hence

$$\frac{v'(x) - v'(\tau)}{x-\tau} = \frac{2\sigma+1}{x\tau} + 6c\{x^4 + x^3\tau + x^2\tau^2 + x\tau^3 + \tau^4\} + 4t(x^2 + x\tau + \tau^2) - 2t.$$

Theorem 6 *The monic orthogonal polynomials $\mathcal{S}_n(x;t)$ with respect to the weight in (9) defined on $\mathbb{R}$ obey the relation*

$$\mathsf{S}_n'(x;t) = \gamma_n(t)\mathcal{A}_n(x;t)\mathcal{S}_{n-1}(x;t) - \mathcal{B}_n(x;t)\mathcal{S}_n(x;t)$$

where

$$\mathcal{A}_n(x;t) = 6cx^4 + 6c(\gamma_n + \gamma_{n+1})x^2 + 6c\,(\Xi_{n+1} + \Xi_n) + 4tx^2 + 4t(\gamma_n + \gamma_{n+1}) - 2t, \tag{40a}$$

$$\mathcal{B}_n(x;t) = \left(\frac{2\sigma+1}{x}\right)\Omega_n + 6c\gamma_n x^3 + 6c\,\Xi_n x + 4tx\gamma_n, \tag{40b}$$

where the expressions Ξ_n and Ω_n are given in (23) and (24), respectively.

Proof From (36a), we obtain

$$\begin{aligned}
\mathcal{A}_n(x;t) &= \frac{1}{\hat{\Gamma}_n}\int_{\mathbb{R}} \mathsf{S}_n^2(\tau)\left(\frac{v'(x)-v'(\tau)}{x-\tau}\right)W_\sigma(\tau;t)\mathrm{d}\tau\\
&= \frac{1}{\hat{\Gamma}_n}\int_{\mathbb{R}} \mathsf{S}_n^2(\tau)\Big(\frac{2\sigma+1}{x\tau} + 6c\{x^4 + x^3\tau + x^2\tau^2 + x\tau^3 + \tau^4\} + 4t(x^2 + x\tau + \tau^2) - 2t\Big)W_\sigma(\tau;t)\mathrm{d}\tau\\
&= 6cx^4 + 6c(\gamma_n + \gamma_{n+1})x^2 + 6c\,(\Xi_{n+1} + \Xi_n) + 4tx^2 + 4t(\gamma_n + \gamma_{n+1}) - 2t,
\end{aligned} \tag{41}$$

and the integral in (41) vanishes due to symmetry of W_σ.

Besides, by using Eq. (36b), orthogonality and Eq. (11), we have that

$$\begin{aligned}
\mathcal{B}_n(x;t) = \frac{1}{\hat{\Gamma}_{n-1}}\int_{\mathbb{R}} \mathcal{S}_n(\tau)\mathcal{S}_{n-1}(\tau)\Big(&\frac{2\sigma+1}{xy} + 6c\{x^4 + x^3y + x^2y^2 + xy^3 + y^4\}\\
&+ 4t(x^2 + xy + y^2) - 2t\Big)W_\sigma(y;t)\,dy\\
= 6c\gamma_n x^3 + 6c\,\Xi_n x + 4tx\gamma_n &+ \left(\frac{2\sigma+1}{x}\right)\Omega_n,
\end{aligned} \tag{42}$$

where Ξ_n and Ω_n are given respectively in (23) and (24). □

Remark 2 It is good to mention that there is a similar result in [10] for differential-recurrence relation for sextic Freud-type weight; whereas our considered weight in Eq. (9) can be perceived as generalized measure deformation using $d\mu(x;t) = e^{t(x^4-x^2)}d\mu(x;0)$, For a similar procedure, one can see [16] where the authors used classical measure deformation via $d\mu(x;t) = e^{tx^2}d\mu(x;0)$ for Laguerre-type weight.

3.6 Shohat's Quasi-Orthogonality Method

Shohat [28] studied a strategy using quasi-orthogonality, to find differential-difference relation for a general semiclassical weight function. Bonan, Freud, Mhaskar, and

Nevai are renowned experts who used this method in their work [26]. The idea of quasi-orthogonality is well articulated in [11, 24, 28]). Our goal in this section is to apply this method to the case of perturbed Freud-type weight in (9) [9, Section 4.5]. Following the ideas in [26], we notice that monic perturbed Freud-type polynomials obey quasi-orthogonality of order $m = 7$ and therefore

$$x\,\frac{d\mathcal{S}_n(x;\tau)}{\mathrm{d}x} = \sum_{k=n-6}^{n} \mathrm{u}_{n,k}\,\mathcal{S}_k(x;\tau), \tag{43}$$

where the expression $\mathrm{u}_{n,k}$ is obtained by

$$\mathrm{u}_{n,k} = \frac{1}{\Gamma_k}\int_{-\infty}^{\infty} x\,\frac{d\mathcal{S}_n}{\mathrm{d}x}(x;\tau)\,\mathcal{S}_k(x;t)\,W_\sigma(x;\tau)\,\mathrm{d}x, \tag{44}$$

with $n-6 \leq k \leq n$ and $\Gamma_k \neq 0$. By employing integration techniques, for $n-6 \leq j \leq n-1$, we have

$$\begin{aligned}
\Gamma_k\,\mathrm{u}_{n,k} &= \Big[x\,\mathcal{S}_k(x;t)\,\mathcal{S}_n(x;t)\,W_\sigma(x;t)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{d}{\mathrm{d}x}\left(x\mathcal{S}_k(x;t)W_\sigma(x;t)\right)\,\mathcal{S}_n(x;t)\,\mathrm{d}x\\
&= -\int_{-\infty}^{\infty}\left[\mathcal{S}_n(x;t)\,\mathcal{S}_k(x;t) + x\,\mathcal{S}_n(x;t)\,\frac{\mathcal{S}_k}{x}(x;t)\right]\,W_\sigma(x;t)\,\mathrm{d}x\\
&\quad -\int_{-\infty}^{\infty} x\mathcal{S}_n(x;t)\,\mathcal{S}_j(x;t)\,\frac{dW_\sigma(x,t)}{\mathrm{d}x}(x;t)\,\mathrm{d}x, \qquad (45)\\
&= -\int_{-\infty}^{\infty}\mathcal{S}_n(x;t)\,\mathcal{S}_j(x;t)\,\left(-6cx^6 - 4tx^4 + 2tx^2 + 2\sigma + 1\right)W_\sigma(x;t)\,\mathrm{d}x\\
&= \int_{-\infty}^{\infty}\left(6cx^6 + 4tx^4 - 2tx^2 - (2\sigma+1)\right)\mathcal{S}_n(x;t)\,\mathcal{S}_j(x;t)\,W_\sigma(x;t)\,\mathrm{d}x, \qquad (46)
\end{aligned}$$

since

$$x\,\frac{dW_\sigma(x,t)}{\mathrm{d}x} = \big[-6cx^6 - 4tx^4 + 2tx^2 + 2\sigma + 1\big]W_\sigma(x;t).$$

The following relations follow from iterating the recurrence given in Eq. (11):

$$\begin{aligned}
x^6\mathcal{S}_n(x;t) &= \mathcal{S}_{n+6}(x;t) + \left(\gamma_n + \gamma_{n+1} + \gamma_{n+2} + \gamma_{n+3} + \gamma_{n+4} + \gamma_{n+5}\right)\mathcal{S}_{n+4}(x;t)\\
&+ \Big[\gamma_{n+3}\left(\gamma_n + \gamma_{n+1} + \gamma_{n+2} + \gamma_{n+3} + \gamma_{n+4}\right) + \gamma_{n+2}\left(\gamma_n + \gamma_{n+1} + \gamma_{n+2} + \gamma_{n+3}\right) + \Xi_n + \Xi_{n+1}\Big]\mathcal{S}_{n+2}(x;t)\\
&+ \Big[\gamma_{n+1}\gamma_{n+2}\left[\gamma_n + \gamma_{n+1} + \gamma_{n+2} + \gamma_{n+3}\right] + \left[(\gamma_n + \gamma_{n+1})(\Xi_n + \Xi_{n+1})\right]\\
&\qquad + \gamma_n\gamma_{n-1}\left(\gamma_{n-2} + \gamma_{n-1} + \gamma_n + \gamma_{n+1}\right)\Big]\mathcal{S}_n(x;t)\\
&+ \gamma_n\gamma_{n-1}\Big[\Xi_{n-1} + \Xi_n + \Xi_{n+1} + \gamma_{n-1}\gamma_{n+1} + \gamma_{n-2}\left(\gamma_{n-3} + \gamma_{n-2} + \gamma_{n-1} + \gamma_n + \gamma_{n+1}\right)\Big]\mathcal{S}_{n-2}(x;t)\\
&+ \gamma_n\gamma_{n-1}\gamma_{n-2}\gamma_{n-3}\Big[\gamma_{n-4} + \gamma_{n-3} + \gamma_{n-2} + \gamma_{n-1} + \gamma_n + \gamma_{n+1}\Big]\mathcal{S}_{n-4}(x;t)\\
&+ \left(\gamma_n\gamma_{n-1}\gamma_{n-2}\gamma_{n-3}\gamma_{n-4}\gamma_{n-5}\right)\mathcal{S}_{n-6}(x;t), \qquad (47a)
\end{aligned}$$

$$\begin{aligned} x^4\mathcal{S}_n(x;t) &= \mathcal{S}_{n+4}(x;t) + (\gamma_n+\gamma_{n+1}+\gamma_{n+2}+\gamma_{n+3})\,\mathcal{S}_{n+2}(x;t) \\ &\quad + \big[\gamma_n(\gamma_{n-1}+\gamma_n+\gamma_{n+1}) + \gamma_{n+1}(\gamma_n+\gamma_{n+1}+\gamma_{n+2})\big]\mathcal{S}_n(x;t) \\ &\quad + \gamma_n\gamma_{n-1}(\gamma_{n-2}+\gamma_{n-1}+\gamma_n+\gamma_{n+1})\mathcal{S}_{n-2}(x;t) + (\gamma_n\gamma_{n-1}\gamma_{n-2}\gamma_{n-3})\mathcal{S}_{n-4}(x;t), \end{aligned} \tag{47b}$$

$$x^2\mathcal{S}_n(x;t) = \mathcal{S}_{n+2}(x;t) + (\gamma_n+\gamma_{n+1})\,\mathcal{S}_n(x;t) + \gamma_n\gamma_{n-1}\mathcal{S}_{n-2}(x;t), \tag{47c}$$

By substituting Eq. (47) into Eq. (46), we obtain the coefficients $\{\mathfrak{f}_{n,j}\}_{j=n-4}^{n-1}$ in Eq. (43) as:

$$\mathfrak{u}_{n,n-6} = 6c\left(\prod_{j=0}^{5}\gamma_{n-j}\right) = 6c\Big[\gamma_n\gamma_{n-1}\gamma_{n-2}\gamma_{n-3}\gamma_{n-4}\gamma_{n-5}\Big], \qquad \mathfrak{u}_{n,n-5}=0, \tag{48a}$$

$$\mathfrak{u}_{n,n-4} = 6c\left(\prod_{j=0}^{3}\gamma_{n-j}\right)\Big[\gamma_{n-4}+\gamma_{n-3}+\gamma_{n-2}+\gamma_{n-1}+\gamma_n+\gamma_{n+1}\Big], \qquad \mathfrak{u}_{n,n-3}=0, \tag{48b}$$

$$\begin{aligned} \mathfrak{u}_{n,n-2} = \gamma_n\gamma_{n-1}\Big[& 6c\big\{\Xi_{n-2}+\Xi_{n-1}+\Xi_n+\Xi_{n+1}+\gamma_{n-1}\gamma_{n-2}+\gamma_{n+1}\,(\gamma_{n-2}+\gamma_{n-1})\big\} \\ & + 4t\,(\gamma_{n-2}+\gamma_{n-1}+\gamma_n+\gamma_{n+1}) - 2t\Big], \end{aligned} \tag{48c}$$

$$\mathfrak{u}_{n,n-1} = 0. \tag{48d}$$

For the case when $k=n$, we use integration technique in Eq. (44) to obtain

$$\begin{aligned} \Gamma_n\mathfrak{f}_{n,n} &= \int_{-\infty}^{\infty} x\,\frac{d\mathcal{S}_n(x;t)}{dx}\mathcal{S}_n(x;t)W_\sigma(x;t)\,\mathrm{d}x = -\tfrac{1}{2}\int_{-\infty}^{\infty}\mathcal{S}_n^2(x;t)\left[W_\sigma(x;t)+x\,\frac{dW_\sigma(x;t)}{dx}\right]\mathrm{d}x \\ &= -\tfrac{1}{2}\Gamma_n + \int_{-\infty}^{\infty}\mathcal{S}_n^2(x;t)\big(3cx^6-2tx^4+tx^2-\sigma-\tfrac{1}{2}\big)W_\sigma(x;t)\,\mathrm{d}x \\ &= 3c\int_{-\infty}^{\infty}x^6\mathcal{S}_n^2(x;t)W_\sigma(x;t)\,\mathrm{d}x - 2t\int_{-\infty}^{\infty}x^4\mathcal{S}_n^2(x;t)W_\sigma(x;t)\,\mathrm{d}x \\ &\quad + t\int_{-\infty}^{\infty}x^2\mathcal{S}_n^2(x;t)W_\sigma(x;t)\,\mathrm{d}x - (\sigma+1)\Gamma_n. \end{aligned} \tag{49}$$

By using the recursive relation given in Eq. (11) for Eq. (49), we have that

$$x^2\mathcal{S}_n^2 = (\mathcal{S}_{n+1}+\gamma_n\mathcal{S}_{n-1})^2 = \mathcal{S}_{n+1}^2 + 2\gamma_n\mathcal{S}_{n+1}\mathcal{S}_{n-1} + \gamma_n^2\mathcal{S}_{n-1}^2, \tag{50a}$$

$$\begin{aligned} x^4\mathcal{S}_n^2 &= x^2(\mathcal{S}_{n+1}^2 + 2\gamma_n\mathcal{S}_{n+1}\mathcal{S}_{n-1} + \gamma_n^2\mathcal{S}_{n-1}^2) = x^2\mathcal{S}_{n+1}^2 + 2\gamma_n(x\mathcal{S}_{n+1})(x\mathcal{S}_{n-1}) + \gamma_n^2x^2\mathcal{S}_{n-1}^2 \\ &= \big(\mathcal{S}_{n+2}+\gamma_{n+1}\mathcal{S}_n\big)^2 + 2\gamma_n\big(\mathcal{S}_{n+2}+\gamma_{n+1}\mathcal{S}_n\big)\big(\mathcal{S}_n+\gamma_{n-1}\mathcal{S}_{n-2}\big) + \gamma_n^2\big(\mathcal{S}_n+\gamma_{n-1}\mathcal{S}_{n-2}\big)^2 \\ &= \mathcal{S}_{n+2}^2 + 2(\gamma_{n+1}+\gamma_n)\mathcal{S}_{n+2}\mathcal{S}_n + (\gamma_{n+1}+\gamma_n)^2\mathcal{S}_n^2 + 2\gamma_n\gamma_{n-1}\mathcal{S}_{n+2}\mathcal{S}_{n-2} \\ &\quad + 2\gamma_n\gamma_{n-1}(\gamma_n+\gamma_{n+1})\mathcal{S}_n\mathcal{S}_{n-2} + \gamma_n^2\gamma_{n-1}^2\mathcal{S}_{n-2}^2, \end{aligned} \tag{50b}$$

and so by orthogonality, we have that

$$\int_{-\infty}^{\infty} x^6 \mathcal{S}_n^2(x;t)\, W_\sigma(x;t)\,\mathrm{d}x = (\Gamma_{n+3} + \gamma_{n+2}^2 \Gamma_{n+1}) + 2(\gamma_{n+1}+\gamma_n)\gamma_{n+1}\gamma_{n+2}\Gamma_n + (\gamma_{n+1}+\gamma_n)^2(\Gamma_{n+1} + \gamma_n^2\Gamma_{n-1})$$

$$= (\gamma_n\gamma_{n+1}\gamma_{n+2})\Gamma_n + \Xi_{n+2}\gamma_{n+1}\Gamma_n + \gamma_{n+1}(\Xi_n + \Xi_{n+1})\Gamma_n + \gamma_n(\Xi_n + \Xi_{n+1})\Gamma_n + \gamma_{n-1}\gamma_n\gamma_{n+1}\Gamma_n + \gamma_n\Xi_{n-1}\Gamma_n, \tag{51a}$$

$$\int_{-\infty}^{\infty} x^2\, \mathcal{S}_n^2(x;t)\, W_\sigma(x;t)\ \mathrm{d}x = \Gamma_{n+1} + \gamma_n^2\, \Gamma_{n-1} = (\gamma_{n+1}+\gamma_n)\,\Gamma_n, \tag{51b}$$

$$\begin{aligned}\int_{-\infty}^{\infty} x^4 \mathcal{S}_n^2(x;t)\, W_\sigma(x;t)\ \mathrm{d}x &= \Gamma_{n+2} + (\gamma_{n+1}+\gamma_n)^2\Gamma_n + \gamma_n^2\gamma_{n-1}^2\ \Gamma_{n-2}\\ &= \big[(\gamma_{n+1}+\gamma_n+\gamma_{n-1})\gamma_n + (\gamma_{n+2}+\gamma_{n+1}+\gamma_n)\gamma_{n+1}\big]\Gamma_n\\ &= (\Xi_n + \Xi_{n+1})\,\Gamma_n,\end{aligned} \tag{51c}$$

using $\Gamma_{n+1} = \gamma_{n+1}\Gamma_n$, the difference equation Eq. (21) and Ξ_n is given by Eq. (23). By rearranging Eq. (21) and taking $n \to n-1$ in Eq. (21), we have

$$2t\,\Xi_n - t\gamma_n = \frac{n + (2\sigma+1)\Omega_n}{2} - 3c\left[\gamma_n\,(\Xi_{n-1} + \Xi_n + \Xi_{n+1}) + \gamma_{n-1}\gamma_n\gamma_{n+1}\right], \tag{52a}$$

$$2t\,\Xi_{n+1} - t\gamma_{n+1} = \frac{n + 1 + (2\sigma+1)\Omega_{n+1}}{2} - 3c\left[\gamma_{n+1}\,(\Xi_n + \Xi_{n+1} + \Xi_{n+2}) + \gamma_n\gamma_{n+1}\gamma_{n+2}\right]. \tag{52b}$$

By combining Eqs. (52a) and (52b), we obtain

$$\begin{aligned}&-2t\int_{-\infty}^{\infty} x^4 \mathcal{S}_n^2(x;t)\, W_\sigma(x;t)\,\mathrm{d}x + t\int_{-\infty}^{\infty} x^2 \mathcal{S}_n^2(x;t)\, W_\sigma(x;t)\,\mathrm{d}x\\ &\quad= -(t\gamma_n - 2t\,\Xi_n) - (t\gamma_{n+1} - 2t\,\Xi_{n+1})\\ &\quad= -3c\Big[\gamma_n\,(\Xi_{n-1}+\Xi_n+\Xi_{n+1}) + \gamma_{n-1}\gamma_n\gamma_{n+1} + \gamma_{n+1}\,(\Xi_n+\Xi_{n+1}+\Xi_{n+2}) + \gamma_n\gamma_{n+1}\gamma_{n+2}\Big]\\ &\qquad+ \frac{2n+1+(2\sigma+1)\,(\Omega_n+\Omega_{n+1})}{2}\\ &\quad= -3c\Big[\gamma_n\,(\Xi_{n-1}+\Xi_n+\Xi_{n+1}) + \gamma_{n-1}\gamma_n\gamma_{n+1} + \gamma_{n+1}\,(\Xi_n+\Xi_{n+1}+\Xi_{n+2}) + \gamma_n\gamma_{n+1}\gamma_{n+2}\Big]\\ &\qquad+ n + (\sigma+1),\end{aligned} \tag{53}$$

Hence from Eq. (51a) and Eq. (53), Eq. (49) becomes

$$\begin{aligned}
\mathfrak{u}_{n,n} &= \frac{1}{\Gamma_n}\left\{3c\int_{-\infty}^{\infty} x^6 \mathcal{S}_n^2(x;t)\, W_\sigma(x;t)\,\mathrm{d}x - (\sigma+1)\Gamma_n - 2t\int_{-\infty}^{\infty} x^4 \mathcal{S}_n^2(x;t)\, W_\sigma(x;t)\,\mathrm{d}x \right.\\
&\qquad \left. + t\int_{-\infty}^{\infty} x^2 \mathcal{S}_n^2(x;t)\, W_\sigma(x;t)\,\mathrm{d}x\right\}\\
&= 3c\Big[(\gamma_n\gamma_{n+1}\gamma_{n+2}) + \Xi_{n+2}\gamma_{n+1} + \gamma_{n+1}(\Xi_n + \Xi_{n+1})\Gamma_n + \gamma_n(\Xi_n + \Xi_{n+1}) + \gamma_{n-1}\gamma_n\gamma_{n+1}\Gamma_n + \gamma_n\Xi_{n-1}\Big]\\
&- (\sigma+1) - 3c\Big[\gamma_n\left(\Xi_{n-1} + \Xi_n + \Xi_{n+1}\right) + \gamma_{n-1}\gamma_n\gamma_{n+1} + \gamma_{n+1}\left(\Xi_n + \Xi_{n+1} + \Xi_{n+2}\right) + \gamma_n\gamma_{n+1}\gamma_{n+2}\Big]\\
&\quad + n + (\sigma+1)\\
&= n.
\end{aligned} \tag{54}$$

Combining Eq. (48) with Eq. (43) gives

$$x\frac{d\mathcal{S}_n}{\mathrm{d}x} = \mathfrak{u}_{n,n-6}\,\mathcal{S}_{n-6}(x;t) + \mathfrak{u}_{n,n-4}\,\mathcal{S}_{n-4}(x;t) + \mathfrak{u}_{n,n-2}\,\mathcal{S}_{n-2}(x;t) + \mathfrak{u}_{n,n}\,\mathcal{S}_n(x;t). \tag{55a}$$

Rewriting $\mathcal{S}_{n-4}$ and $\mathcal{S}_{n-2}$ into Eq. (55a) in terms of $\mathcal{S}_n$ and $\mathcal{S}_{n-1}$ using Eq. (11), we obtain

$$\mathcal{S}_{n-2}(x;t) = \frac{x\mathcal{S}_{n-1}(x;t) - \mathcal{S}_n(x;t)}{\gamma_{n-1}}, \tag{55b}$$

$$\mathcal{S}_{n-3}(x;t) = \frac{x\mathcal{S}_{n-2}(x;t) - \mathcal{S}_{n-1}(x;t)}{\gamma_{n-2}} = \frac{x^2-\gamma_{n-1}}{\gamma_{n-1}\gamma_{n-2}}\,\mathcal{S}_{n-1}(x;t) - \frac{x}{\gamma_{n-1}\gamma_{n-2}}\,\mathcal{S}_n(x;t), \tag{55c}$$

$$\mathcal{S}_{n-4}(x;t) = \frac{x\mathcal{S}_{n-3}(x;t) - \mathcal{S}_{n-2}(x;t)}{\gamma_{n-3}} = \frac{x^3-(\gamma_{n-1}+\gamma_{n-2})x}{\gamma_{n-1}\gamma_{n-2}\gamma_{n-3}}\,\mathcal{S}_{n-1}(x;t) - \frac{x^2-\gamma_{n-2}}{\gamma_{n-1}\gamma_{n-2}\gamma_{n-3}}\,\mathcal{S}_n(x;t), \tag{55d}$$

$$\begin{aligned}
\mathcal{S}_{n-6}(x;t) =& \left\{\frac{x^5-(\gamma_{n-1}+\gamma_{n-2}+\gamma_{n-3}+\gamma_{n-4})x + (\gamma_{n-1}\gamma_{n-3}+\gamma_{n-1}\gamma_{n-4}+\gamma_{n-2}\gamma_{n-4})}{\gamma_{n-1}\gamma_{n-2}\gamma_{n-3}\gamma_{n-4}\gamma_{n-5}}\right\}\mathcal{S}_{n-1}(x;t)\\
&- \left\{\frac{x^4-(\gamma_{n-2}+\gamma_{n-3}+\gamma_{n-4})x+\gamma_{n-2}\gamma_{n-4}}{\gamma_{n-1}\gamma_{n-2}\gamma_{n-3}\gamma_{n-4}\gamma_{n-5}}\right\}\mathcal{S}_n(x;t).
\end{aligned} \tag{55e}$$

Substituting Eqs. (48), (54), (55b), (55d), and (55e) into Eq. (55a) yields the required result.

4 The Differential Equation

Theorem 7 *For the semiclassical weight in (9), the corresponding monic orthogonal polynomials $\mathcal{S}_n(x;t)$ obey a linear ODE (with rational coefficients) as*

$$\frac{d^2}{dx^2}\mathcal{S}_n(x;t) + \tilde{U}_n(x;t)\,\frac{d}{\mathrm{d}x}\mathcal{S}_n(x;t) + \tilde{W}_n(x;t)\,\mathcal{S}_n(x;t) = 0, \tag{56}$$

where

$$\tilde{U}_n(x;t) = -6cx^5 - t(4x^3 - 2x) + \frac{(2\sigma+1)}{x} - \left[\frac{24cx^3 + 2\left[6c(\gamma_n+\gamma_{n+1}) + 4t\right]x}{6cx^4 + 6c(\gamma_n+\gamma_{n+1})x^2 + 6c\left(\Xi_{n+1}+\Xi_n\right) - 2t + 4t\left(x^2+\gamma_n+\gamma_{n+1}\right)}\right] \tag{57a}$$

$$\begin{aligned}
\tilde{W}_n(x;t) &= 18c\gamma_n x^2 + 6c\Xi_n - \frac{(2\sigma+1)\Omega_n}{x^2} + 4t\gamma_n \\
&+ \gamma_n\Big(6cx^4 + 6c(\gamma_n+\gamma_{n+1})x^2 + 6c\left(\Xi_{n+1}+\Xi_n\right) - 2t + 4t\left(x^2+\gamma_n+\gamma_{n+1}\right)\Big) \\
&\times\Big(6cx^4 + 6c(\gamma_n+\gamma_{n-1})x^2 + 6c\left(\Xi_{n-1}+\Xi_n\right) - 2t + 4t\left(x^2+\gamma_n+\gamma_{n-1}\right)\Big) \\
&- \Bigg[\Big(6cx^5 + (6c\gamma_n+4t)x^3 - \frac{2\sigma+1}{x} + (6c\Xi_n + 4t\gamma_n - 2t)x + \frac{(2\sigma+1)\Omega_n}{x} \\
&+ \frac{24cx^3 + 2\left[6c(\gamma_n+\gamma_{n+1}) + 4t\right]x}{6cx^4 + 6c(\gamma_n+\gamma_{n+1})x^2 + 6c\left(\Xi_{n+1}+\Xi_n\right) - 2t + 4t\left(x^2+\gamma_n+\gamma_{n+1}\right)}\Big) \\
&\times\Big(6c\gamma_n x^3 + (6c\Xi_n + 4t\gamma_n)x + \frac{(2\sigma+1)\Omega_n}{x}\Big)\Bigg] \\
&\equiv -\mathcal{B}_n(x;t)\left[v'(x) + \mathcal{B}_n(x;t) + \frac{\mathcal{A}_n'(x;t)}{\mathcal{A}_n(x;t)}\right] + \gamma_n\mathcal{A}_n(x;t)A_{n-1}(x;t) + \mathcal{B}_n'(x;t),
\end{aligned} \tag{57b}$$

where Ω_n and Ξ_n are given in Eqs. (24) and (23), respectively.

Proof For the proof, consult similar ideas in [21] and [22]. □

Remark 3 One can expand Eq. (57) via symbolic packages such as Mathematica (Maple), however the resulting expression may look quite cumbersome.

5 Application of Eq. (56) for Electrostatic Zero Distribution

The authors in [14] considered a perturbation of quartic Freud weight ($w(x) = \exp(-x^4)$) by the addition of a fixed charged point of mass δ at the origin; the corresponding polynomials are Freud-type polynomials (see the recent work in [15]). For semiclassical orthogonality measure, it was shown in [14] that these polynomials obey a second-order linear differential equation of the form (7), and the electrostatic model is in sight as in [18]. Application of Eq. (56) for electrostatic zero distribution is also mentioned. Following these ideas, a similar work for the perturbed Freud-type weight in (9) is given in a recent paper [22] using the obtained differential equation in Sect. 4.

6 Conclusions

By introducing a time variable to scaled sextic Freud-type measure upon deformation (perturbation), we have found certain fresh characterizing properties: some recursive relations, moments of finite order, concise formulation and orthogonality relation, nonlinear difference equation for recurrence coefficients as well as the corresponding polynomials, and certain properties of the zeros of the corresponding polynomials. This work derived certain nonlinear difference equations, Toda-like equations, and differential equations for the recurrence coefficients of the corresponding orthogonal polynomials under consideration. Special attention, using the method of Shohat's quasi-orthogonality and ladder operators, is given to characterize the Freud-type weight (9). Such semiclassical symmetric weight in Eq. (9) follows from quadratic transformation and symmetrization as in [7]. By combining the three-term recurrence relation with the difference–recurrence relation, a second-order differential equation fulfilled by polynomials associated with the semiclassical weight Eq. (9) is obtained. Application of the resulting differential equation in Eq. (56) for electrostatic zero distribution is also noted. Following this work, investigation of these recurrence coefficients in connection with certain (discrete) integrable systems will be a prominent continuation of this study.

Acknowledgements A. S. Kelil kindly acknowledges NMU council postdoctoral fellowship programme (2020–2021). AS Kelil is also grateful to the top-up funding support of the DST-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS) towards the research. Opinions expressed and conclusions arrived at are those of the author and are not necessarily to be attributed to the CoE-MaSS.

References

1. Aptekarev, A., Branquinho A., Marcellán F.: Toda-type differential equations for the recurrence coefficients of orthogonal polynomials and Freud transformation. J. Comput. Appl. Math., Elsevier **78**, 139–160 (1997)
2. Basor, E., Chen, Y.: Painlevé V and the distribution function of a discontinuous linear statistic in the Laguerre unitary ensembles. J. Phys. A **42**, 035203 (2009)
3. Chen, Y., Ismail, M.E.H.: Ladder operators and differential equations for orthogonal polynomials. J. Phys. A **30**, 7817–7829 (1997)
4. Chen, Y., Its, A.: Painlevé III and a singular linear statistics in Hermitian random matrix ensembles. I. J. Approx. Theory **162**, 270–297 (2010)
5. Chen, Y., Ismail, M.E.H.: Ladder operators and differential equations for orthogonal polynomials. J. Phys. A **30**, 7817 (1997)
6. Chen, Y., Feigin, M.V.: Painlevé IV and degenerate Gaussian unitary ensembles. J. Phys. A **39**, 12381 (2006)
7. Chihara, T.S.: An Introduction to Orthogonal Polynomials. Gordon and Breach, New York (1978)
8. Clarkson, P.A., Jordaan, K.: The relationship between semiclassical Laguerre polynomials and the fourth Painlevé equation. Constructive Approximation, Springer **39**, 223–254 (2014)
9. Clarkson, P.A., Jordaan, K., Kelil, A.: A generalized Freud weight. Studies in Applied Mathematics, Wiley Online Library **136**, 288–320 (2016)

10. Clarkson, P.A., Jordaan, K.: A Generalized Sextic Freud Weight, arXiv preprint arXiv:2004.00260 (2020)
11. Driver, K., Jordaan, K.: Zeros of quasi-orthogonal Jacobi polynomials. SIGMA **12**, 042 (2016)
12. Filipuk, G., Van Assche, W., Zhang, L.: The recurrence coefficients of semi-classical Laguerre polynomials and the fourth Painlevé equation. J. Phys. A **45**, 205201 (2012)
13. Freud, G.: On the coefficients in the recursion formulae of orthogonal polynomials. In Math. Proc. R. Ir. Acad. Section A: Mathematical and Physical Sciences, JSTOR, 1–6 (1976)
14. Garrido, Á., Arvesu Carballo, J., Marcellán, F.: An electrostatic interpretation of the zeros of the Freud-type orthogonal polynomials, ETNA (2005)
15. Garza, L.E., Huertas, E.J., Marcellán, F.: On Freud-Sobolev type orthogonal polynomials. Afrika Matematika **30**, 505–528 (2019)
16. Han, P., Chen, Y.: The recurrence coefficients of a semi-classical Laguerre polynomials and the large n asymptotics of the associated Hankel determinant. Random Matrices Theory Appl. **6**, 1740002 (2017)
17. Hendriksen, E., van Rossum, H.: Semi-classical orthogonal polynomials. Polynômes Orthogonaux et Applications, pp. 354–361. Springer (1985)
18. Ismail, M.E.H.: Classical and Quantum Orthogonal Polynomials in One Variable, Encyclopedia of Mathematics and its Applications, 98. Cambridge University Press, Cambridge (2005)
19. Ismail, M.E.H., Mansour, Z.S.I.: q-analogues of Freud weights and non-linear difference equations. Adv. Appl. Math. **45**, 518–547 (2010)
20. Jost, J.: Postmodern Analysis. Springer (2006)
21. Kelil, A.S.: Properties of a class of generalized Freud polynomials. University of Pretoria, PhD Thesis (2018)
22. Kelil, A.S., Appadu, A.R.: On semi-classical orthogonal polynomials associated with a modified sextic freud-type weight. Mathematics **8**, 1250 (2020)
23. Lyu, S.L., Chen, Y., Fan, E.G.: Asymptotic gap probability distributions of the Gaussian unitary ensembles and Jacobi unitary ensembles. Nucl. Phys. B **926**, 639–670 (2018)
24. Maroni, P.: Prolégomènes à l'étude des polynômes orthogonaux semi-classiques. Ann. Mat. Pura Appl. **149**, 165–184 (1987)
25. Nevai, P.: Orthogonal polynomials associated with $\exp(-x^4)$, Second Edmonton Conference on Approximation Theory. In: Ditzian, Z., Meir, A., Riemenschneider, S.D., Sharma, A. (eds.) CMS Conf. Proc., vol. 3, Amer. Math. Soc., Providence, RI, pp. 263–285 (1983)
26. Nevai, P., Freud, G.: orthogonal polynomials and Christoffel functions. A case study. J. A. T., Elsevier **48**, 3–167 (1986)
27. Searcoid, M.O.: Metric Spaces. Springer (2006)
28. Shohat, J.: A differential equation for orthogonal polynomials. Duke Math. J. **5**, 401–417 (1939)
29. Szego, G.: Orthogonal Polynomials, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I. (1975)
30. Van Assche, W.: Discrete Painlevé equations for recurrence coefficients of orthogonal polynomials. In: Difference Equations, Special Functions and Orthogonal Polynomials, pp. 687–725. World Scientific (2007)
31. Van Assche, W.: Orthogonal Polynomials and Painlevé Equations, vol. 27. Cambridge University Press (2017)

Complex Chaotic Systems and Its Complexity

Ajit K. Singh

Abstract This article is deal with an attempt to study the complex chaotic system and its complexity. Chaos in the dynamical system is very complex pattern with the real variables and becomes more complex with the complex variables. But due to its real application in the physical systems, it is very useful to study its behaviour. This article starts with the Lorenz model of integer order and of real variables and in a very systematic way it explores to the fractional order to the complex variables and ends with the fractional order complex chaotic systems. Numerical algorithm and stability analysis are also presented through the simulation results.

Keywords Chaotic system · Lorenz system · Fractional calculus

1 Introduction

Chaos theory is a branch of mathematical sciences, in particular dynamics, has furnished a new system of estimating the world and is an important technique to recognise the behaviour of the approaches in the universe. Chaotic behaviours of dynamical systems have been noticed in different parts of science, engineering and technology such as physics, electronics, mechanics, biology, medicine, ecology, signal processing economy, communication and so on.

Chaotic systems are basically dynamical systems which are highly sensitive to the small perturbation in initial conditions and system parameters. Since complex chaotic systems are more efficient and feasible, it has been observed in some research articles that the complex variables are broadly used in a number of non-linear systems, for instance, secure communications, coupled map lattices, detuned laser systems and Julia sets, etc. The security of transmitted information is increased due to the state variables having fifth order. Recently, complex Lorenz system is one of the most

A. K. Singh (✉)
Department of Mathematics, Amity University, Mumbai 410206, India
e-mail: ajit.brs@gmail.com

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_10

familiar complex chaotic systems, and has been utilise to express disk dynamos, rotating fluids, etc. [1–3].

The origin of fractional calculus was close to the time of origination of differential calculus. The proposal of fractional order derivative is first struck by Leibnitz in 1695. Even though there was tiny progress in the topic which was almost theoretical. In the 19th century, after the works of Riemann, Liouville, Grunwald and Letnikov, it was realised that the fractional order differential and integral are more useful than the integer order in the application of non-linear sciences. One conceivable clarification of such dishonour might be that there are many different definitions of fractional derivatives. Then one more problem can be that fractional derivative has no evident of the geometrical interpretation because of its non-local property.

Last few decades, fractional calculus has become centre of attraction for the researcher working in the non-linear sciences. It is experienced that many, especially interdisciplinary applications can be classically expressed with aid of fractional derivatives. For instance someone may bring up research on anomalous diffusion, viscoelastic bodies, quantum evolution of complex systems, phase transitions of fractional order, polymer physics, quantitative finance and an explanation of fractional kinetics of the chaotic systems [4–6]. However, most of the aforementioned researches were based on the linear fractional differential equations. This restriction in the main results because of the dynamics of such systems may not remain chaotic. As stated by the Poincare-Bendixson theorem [7], dimensional of the integer order system must be at least 3 for chaos to happen. But this is not true in the fractional order systems. For instance, it has been proved that Chua's circuit of order 2.7 can reveal the chaotic attractors. Since discrete dynamical system reveals chaotic behaviour even in one dimension, someone should be emphasised that this theorem is applicable for the continuous time chaotic systems but not for the discrete maps. A well known mathematical model of a continuous time dynamical system which reveals chaos is the Lorenz system [8–11].

Dynamical analysis of fractional order chaotic system is a main attention of research. But it is primarily based on the bifurcation and phase diagram. After that the Lyapunov exponent is an important method to study the complexity of a chaotic system. A system with highest positive Lyapunav exponent means the system is more complex. Moreover, to choose proper parameters of systems for its practical applications, it is required to examine complexity of fractional order complex Lorenz system.

2 Fractional Calculus

2.1 Definition

Fractional order derivative has been studied by many approach. There are two primarily approach namely, frequency domain approach and time domain approach.

Since a large number of fractional derivative definitions are observed in the literature surveys [12–14], author discusses only three commonly used definitions by first considering Caputo's definition. Since this is based on the time domain approach and needs the initial conditions on integer which are readily determine. The fractional order derivative is defined by

$$\frac{d^q \phi(t)}{dt} := J^{n-q} \frac{d^n \phi(t)}{dt} = J^{n-q} \phi^{(n)}(t)$$

where $n := \lceil q \rceil$ is the first integer which is greater that or equal to q and $q > 0$, and q is an arbitrary number. $\phi^{(n)}(t)$ denotes the n-th order ordinary differential of the function $\phi(t)$ with respect to t, and J^{μ} is the μ-order the Riemann-Liouville integral operator defined in the following equation

$$J^{\mu} \psi(t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{\psi(\tau)}{(t-\tau)^{1-\mu}} d\tau$$

where $\Gamma(\mu)$ means the Gamma value of μ and $0 < \mu < 1$.

The Caputo (C) definition of the fractional derivative of q-order is written as

$$^{C}D^{q} f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{1+q-n}} d\tau. \quad (1)$$

The Grunwald-Letnikov (GL) definition of q-order is stated as

$$^{GL}D^{q} f(t) = \lim_{h \to 0} \frac{1}{h^q} \sum_{j=0}^{[\frac{t}{h}]} (-1)^j \binom{q}{j} f(t - jh),$$

where $[\cdot]$ represents the integral part.

The Riemann-Liouville (RL) definition of q-order is given as

$$^{RL}D^{q} f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{1+q-n}} d\tau.$$

Since the Caputo differential operator requires initial conditions on the integer values, and it is more efficient in the real application as compared to Riemann-Liouville operator and others. Due to its physical interpenetration, well defined and computation, the author chooses the Caputo's definition of the fractional derivatives throughout the article and writes a convenient notation D^q instead of $^{C}D^{q}$.

2.2 Numerical Algorithm

Numerical algorithm of fractional order differential equation is studied in this section. Diethelm et al. introduced the Adams-Bashforth-Moulton predictor and corrector method [15], which is numerically stable and applicable to the both linear and nonlinear fractional differential equations. This algorithm is applied for numerical calculation in this article.

The fractional differential equation with initial conditions in general form is written as

$$\begin{aligned} D^q\, y(t) &= f(t,\, y(t)),\ 0 \le t \le T \\ y^{(k)} &= y_0^k,\ k = 0,\, 1,\, 2,\, \ldots,\, \lceil q \rceil - 1. \end{aligned} \tag{2}$$

Equation (2) is analogous to Volterra integral equation

$$y(t) = \sum_{k=0}^{\lceil q \rceil - 1} \frac{y_0^k t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t \frac{f(\tau,\, y(\tau))}{(t-\tau)^{1-q}} d\tau. \tag{3}$$

Let $h = \frac{T}{N},\ \ t_n = nh,\ \ n = 0,\, 1,\, 2,\, \ldots,\, N \in \mathbb{Z}^+,$ then Eq. (3) is reduced to

$$\begin{aligned} y_h(t_{n+1}) = &\sum_{k=0}^{\lceil q \rceil - 1} \frac{y_0^k t_{n+1}^k}{k!} + \frac{h^q}{\Gamma(q+2)} f\left(t_{n+1},\, y_h^p(t_{n+1})\right) \\ &+ \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n} \alpha_{j,\,n+1} f\left(t_j,\, y_h\left(t_j\right)\right), \end{aligned} \tag{4}$$

where

$$\alpha_{j,\,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q, & j = 0 \\ (n+2-j)^{q+1} + (n-j)^{q+1} - 2(n+1-j)^{q+1}, & 1 \le j \le n \\ 1, & j = n+1 \end{cases}$$

and predicted values are calculated by the following equation

$$y_h^p(t_{n+1}) = \sum_{k=0}^{\lceil q \rceil - 1} \frac{y_0^k t_{n+1}^k}{k!} + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} \beta_{j,\,n+1} f\left(t_j,\, y_h\left(t_j\right)\right),$$

where

$$\beta_{j,\,n+1} = \frac{h^q}{q} (n+1-j)^q - (n-j)^q,\ \ 1 \le j \le n.$$

Error approximation in this method is

$$\max_{j=0,1,2,\ldots,N} \mid x\left(t_j\right) - x_h\left(t_j\right) \mid = O\left(h^p\right),$$

where $p = \min\{1+q,\ 2\}$.

2.3 Stability

Stability analysis of the equilibrium points to the fractional order chaotic system is quite complicated and is distinct to the integral order chaotic system. With the aid of following lemma, it can be answered.

Lemma 1 *Equilibrium points of the fractional order system are asymptotically stable if all the eigenvalues satisfy the following:*

$$\mid arg\,(eigen\,(J)) \mid = \mid arg\,(\lambda_i) \mid > \frac{\pi q}{2}, \quad i = 1,\ 2,\ \ldots,\ n,$$

at the equilibrium E^*. Here J represents the Jacobian matrix of the fractional order system calculated at the equilibria E^* [16].

3 Chaotic System

3.1 Lorenz System

The Lorenz system [17] was derived by the E. N. Lorenz in 1963 which is given by the following set of ordinary differential equations

$$\begin{aligned} \dot{x} &= a\,(y - x) \\ \dot{y} &= cx - y - xz \\ \dot{z} &= xy - bz, \end{aligned} \tag{5}$$

where a dot denotes the derivative with respect to time. a, b and c are the parameters.

Some basic features of the system (5) are as follows:

1. Evolution is controlled by only values of x, y, and z as the equations have the first-order time derivative.
2. This is an autonomous system as time does not explicitly emerged in right-hand side of system (5).
3. This is a non-linear system due to the presence of the xz term in second and xy term in third of system (5).

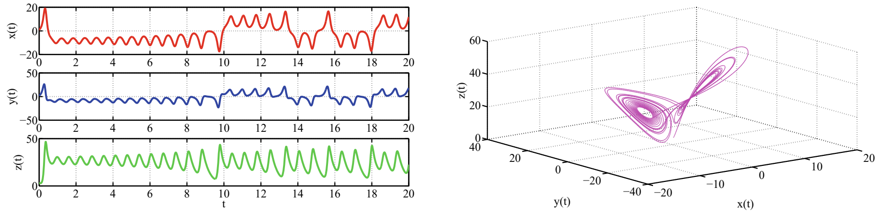


Fig. 1 Chaotic attractor of the Lorenz system

4. This is a dissipative system which means that the following inequality holds

$$\frac{d\,\dot{x}}{dx} + \frac{d\,\dot{y}}{dy} + \frac{d\,\dot{z}}{dz} = -\,(a + 1 + b) < 0$$

as the parameters a and b are positive.
5. This is an invariant system in the sense of coordinate transformation $(x, y, z) \rightarrow (-x, -y, z)$ which means that system (5) is symmetric in z-axis.

The Lorenz system (5) is chaotic in the nature for parameters value $a = 10$, $b = 8/3$, and $c = 28$. The chaotic attractor of the system (5) is shown in Fig. 1.

3.2 Fractional Order Lorenz System

Generalisation of ordinary differential equations to fractional differential equations can be practical in intentionality explanation of viscoelastic liquids such as human blood [18]. A fractional generalisation of the Lorenz system is introduced by Grigorenko and Grigorenko [19]. The fractional order Lorenz system is given as

$$\begin{aligned} D^q\,x &= a\,(y - x)\\ D^q\,y &= cx - y - xz\\ D^q\,z &= xy - bz, \end{aligned} \tag{6}$$

where q-order time fractional derivatives are in the Caputo sense and $0 < q \leq 1$. When $q = 1$ the fractional order Lorenz system (6) reduces to the standard Lorenz system (5). The chaotic attractor of the system (6) is depicted through the Fig. 2 on the fractional order $q = 0.99$ and parameters values $a = 10$, $b = 8/3$, $c = 28$.

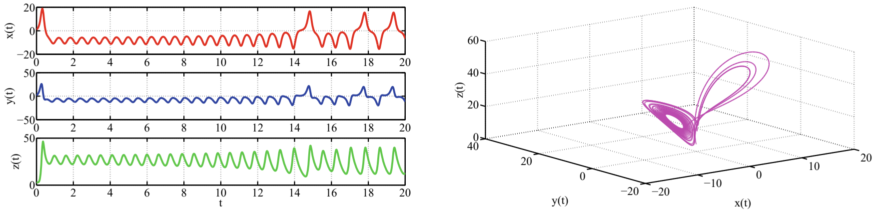


Fig. 2 Chaotic attractor of the fractional order Lorenz system

3.3 *Complex Lorenz System*

The complex Lorenz system was derived from the original Lorenz system (5) by Fowler et al. [20] which is described by the following set of differential equations:

$$\begin{aligned}\dot{x} &= a\,(y - x)\\ \dot{y} &= cx - y - xz\\ \dot{z} &= \frac{1}{2}\,(x\bar{y} + \bar{x}y) - bz,\end{aligned} \tag{7}$$

where $x = x_1 + ix_2$, $y = x_3 + ix_4$, and $z = x_5$, and bar denotes the complex conjugate. Separating the complex variables of system (7) into real and imaginary parts, the following equivalent system is obtained

$$\begin{aligned}\dot{x_1} &= a\,(x_3 - x_1)\\ \dot{x_2} &= a\,(x_4 - x_2)\\ \dot{x_3} &= cx_1 - x_3 - x_1x_5\\ \dot{x_4} &= cx_2 - x_4 - x_2x_5\\ \dot{x_5} &= x_1x_3 + x_2x_4 - bx_5.\end{aligned}$$

Basic characteristic of this complex form of the Lorenz system are similar to mentioned which means it is non-linear, autonomous, symmetric in z-axis, dissipative with bounded solutions, and appears only first order time derivative. Along with these, the complex generalisation of the real and third-order Lorenz system changed to fifth-order system. When $a = 10$, $b = 8/3$, $c = 28$, the system (7) is chaotic and phase portraits are shown through Fig. 3 in $x_1(t) - x_2(t) - x_3(t)$-axes and $x_3(t) - x_4(t) - x_5(t)$-axes.

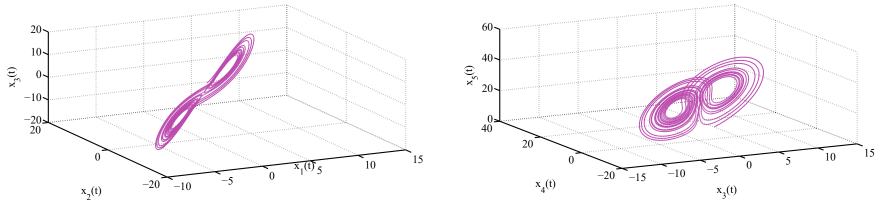


Fig. 3 Phase portrait of the complex Lorenz system

3.4 Fractional Order Complex Lorenz System

The fractional order complex Lorenz system (FOCLS) is seen as a generalisation of integer order complex Lorenz system (7). Then the FOCLS can be written in the set of fractional order differential equations as follows:

$$
\begin{aligned}
D^q\, x &= a(y - x)\\
D^q\, y &= cx - y - xz\\
D^q\, z &= \frac{1}{2}(x\bar{y} + \bar{x}y) - bz,
\end{aligned}
\tag{8}
$$

where D^q is the q-order Caputo's fractional differential operator; $x = x_1 + ix_2$, $y = x_3 + ix_4$ and $z = x_5$. When $q = 1$, system (8) is same as the complex Lorenz system (7).

4 Analysis of the FOCLS

In this section, analysis of the FOCLS is investigated, namely, real version, symmetry and invariance, equilibrium points, stability and chaotic attractors of system (8).

4.1 Real Version

Since the Caputo fractional derivative operator (1) is a linear operator, the real version of the system (8) can be written in the following form

$$
\begin{aligned}
D^q x_1 &= a(x_3 - x_1) \\
D^q x_2 &= a(x_4 - x_2) \\
D^q x_3 &= cx_1 - x_3 - x_1x_5 \\
D^q x_4 &= cx_2 - x_4 - x_2x_5 \\
D^q x_5 &= x_1x_3 + x_2x_4 - bx_5.
\end{aligned} \tag{9}
$$

4.2 Symmetry and Invariance

Under the transformation $(x_1, x_2, x_3, x_4, x_5) \to (-x_1, -x_2, -x_3, -x_4, x_5)$ system (9) remains the invariance. So the FOCLS is symmetry about x_5-axis. At the result of this, if $(x_1, x_2, x_3, x_4, x_5)$ is a solution of chaotic system (9), then $(-x_1, -x_2, -x_3, -x_4, x_5)$ is also a solution of the same system (9).

4.3 Equilibrium Points

The computation of the equilibrium points of system (9) is obtained by the calculation of the equations

$$D^q x_j = 0, \quad j = 1, 2, 3, 4, 5.$$

So, the system (9) has an isolated equilibrium point $E_0 = (0, 0, 0, 0, 0)$ and non-trivial equilibrium points $E_\theta = (r\cos\theta, r\sin\theta, r\cos\theta, r\sin\theta, x_5)$ where $r = \sqrt{bx_5}$, $\theta \in [0, 2\pi]$, $x_5 = c - 1$, It is clear that the non-trivial equilibrium point exist when $c > 1$.

4.4 Stability

Since equilibrium point E_0 is stable when $b < 1$, and unstable when $b > 1$. For E_θ, the characteristic polynomial of the Jacobian matrix for $c > 1$ is

$$z(z + a + 1) + (1 + a + b)z^2 + (ab + bc)z + 2ab(b - 1) = 0.$$

Since the Routh-Hurwitz conditions of the fractional order system [21, 22] ensure that if

$$(1 + a + b)(ab + bc) > 2ab(c - 1),$$

then E_θ will be stable.

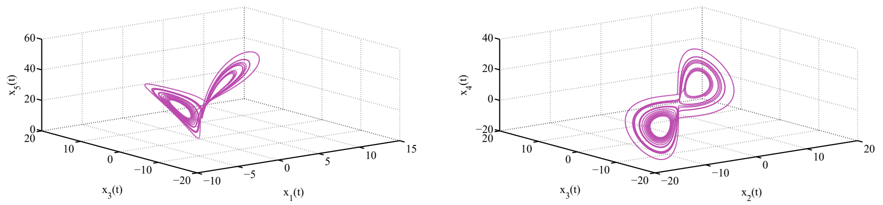


Fig. 4 Phase portrait of the complex Lorenz system

4.5 Chaotic Attractors

Consider the parameters values as before $a = 10$, $b = 8/3$, $c = 28$ and initial condition $[1 + 3i, 2 + 3i, 5]^T$ for the system (8), phase portraits are shown by Fig. 4 in $x_1(t) - x_3(t) - x_5(t)$-axes and $x_2(t) - x_3(t) - x_4(t)$-axes at fractional order $q = 0.99$.

5 Numerical Simulation

Numerical solution of the fractional differential equation is not easy as integer order. Two estimation approaches are often considered to numerical simulation of fractional differential equations. First is the modified version of the Adams-Bashforth- Moulton predictor and corrector method [15, 23, 24]. It depends on the time domain approach. Second is in the frequency domain and also known as frequency domain approach. Due to long memory effect of the fractional order systems, numerical simulation in the time domain approach takes a very long simulation time and complicated, but gives very precise result [25]. Hence, author employs the first approach for fractional order systems in this article.

6 Concluding Remarks

Dynamical behaviour of the FOCLS is investigated in this article which is the most important part of the article. It is also shown that the FOCLS of order 2.7 exhibit the chaotic attractor which is second observation. The third finding of the author is chaos can be attained with the fractional order system of order as low as 2.7 as compare to the integer order system of order at least 3. The fourth and last results of the article are the fractional order bridge oscillator that shows a limit cycle which can be generated for any fractional order with a proper value of the system parameters value.

Acknowledgements The author, Dr. A. K. Singh, is extending his gratitude to the Ph.D. thesis supervisor Prof. Subir Das, Department of Mathematical Sciences, Indian Institute of Technology (BHU), Varanasi—221005, India, for constant guidance.

References

1. Yu, Y., Li, H.X., Wang, S., Yu, J.: Dynamic analysis of a fractional-order Lorenz chaotic system. Chaos Solitons Fractals **42**(2), 1181–1189 (2009)
2. Nian, F., Wang, X., Niu, Y., Lin, D.: Module-phase synchronization in complex dynamic system. Appl. Math. Comput. **217**(6), 2481–2489 (2010)
3. Singh, A.K., Yadav, V.K., Das, S.: Synchronization between fractional order complex chaotic systems with uncertainty. Optik **133**, 98–107 (2017)
4. Hilfer, R.: Applications of Fractional Calculus in Physics, vol. 35. World Scientific Singapore (2000)
5. Dang, T.S., Palit, S.K., Mukherjee, S., Hoang, T.M., Banerjee, S.: Complexity and synchronization in stochastic chaotic systems. Eur. Phys. J. Spec. Top. **225**(1), 159–170 (2016)
6. Singh, A.K., Yadav, V.K., Das, S.: Synchronization between fractional order complex chaotic systems. Int. J. Dyn. Control **5**(3), 756–770 (2017)
7. Hirsch, M.W., Devaney, R.L., Smale, S.: Differential Equations, Dynamical Systems, and Linear Algebra, vol. 60. Academic Press (1974)
8. Singh, A.K., Yadav, V.K., Das, S.: Comparative study of synchronization methods of fractional order chaotic systems. Nonlinear Eng. **5**(3), 185–192 (2016)
9. Wen, C., Yang, J.: Complexity evolution of chaotic financial systems based on fractional calculus. Chaos Solitons Fractals **128**, 242–251 (2019)
10. Singh, A.K., Yadav, V.K., Das, S.: Nonlinear control technique for dual combination synchronization of complex chaotic systems. J. Appl. Nonlinear Dyn. **8**(2), 261–277 (2019)
11. Liao, Y., Zhou, Y., Xu, F., Shu, X.B.: A study on the complexity of a new chaotic financial system. Complexity 2020 (2020)
12. Podlubny, I.: Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Elsevier (1998)
13. Kilbas, A.A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations, vol. 204. Elsevier (2006)
14. Singh, A.K., Yadav, V.K., Das, S.: Dual combination synchronization of the fractional order complex chaotic systems. J. Comput. Nonlinear Dyn. **12**(1), 011017 (2017)
15. Diethelm, K., Ford, N.J., Freed, A.D.: A predictor-corrector approach for the numerical solution of fractional differential equations. Nonlinear Dyn. **29**, 3–22 (2002)
16. Matignon, D.: Stability results for fractional differential equations with applications to control processing. In: Computational Engineering in Systems Applications, vol. 2, pp. 963–968. Lille, France (1996)
17. Lorenz, E.N.: Deterministic nonperiodic flow. J. Atmos. Sci. **20**(2), 130–141 (1963)
18. Thurston, G.B.: Viscoelasticity of human blood. Biophys. J. **12**(9), 1205–1217 (1972)
19. Grigorenko, I., Grigorenko, E.: Chaotic dynamics of the fractional Lorenz system. Phys. Rev. Lett. **91**(3), 034101 (2003)
20. Fowler, A., Gibbon, J., McGuinness, M.: The complex Lorenz equations. Physica D: Nonlinear Phenom. **4**(2), 139–163 (1982)
21. Ahmed, E., El-Sayed, A., El-Saka, H.A.: On some Routh-Hurwitz conditions for fractional order differential equations and their applications in Lorenz, Rössler, Chua and Chen systems. Phys. Lett. A **358**(1), 1–4 (2006)
22. Wang, X.Y., He, Y.J., Wang, M.J.: Chaos control of a fractional order modified coupled dynamos system. Nonlinear Anal. Theory Methods Appl. **71**(12), 6126–6134 (2009)

23. Charef, A., Sun, H., Tsao, Y., Onaral, B.: Fractal system as represented by singularity function. IEEE Trans. Autom. Control **37**(9), 1465–1470 (1992)
24. Singh, A.K., Yadav, V.K., Das, S.: Synchronization of time-delay chaotic systems with uncertainties and external disturbances. Discontinuity Nonlinearity Complex. **8**(1), 13–21 (2019)
25. Tavazoei, M.S., Haeri, M.: Limitations of frequency domain approximation for detecting chaos in fractional order systems. Nonlinear Anal. Theory Methods Appl. **69**(4), 1299–1320 (2008)

On the Bertrand Pairs of Open Non-Uniform Rational B-Spline Curves

Muhsin Incesu, Sara Yilmaz Evren, and Osman Gursoy

Abstract B-spline curves are used basically in Computer-Aided Design (CAD), Computer-Aided Geometric Design (CAGD), and Computer-Aided Modeling (CAM). In determining the invariants of curves and surfaces at any point, there are some difficulties in expressing it analytically and calculating its invariants at the desired point. For these curves and surfaces the way to overcome these difficulties is to design them with spline curves and surfaces. In this paper the second- and third-order derivatives of open Non-Uniform Rational B-Spline (NURBS) curves at the points $t = t_d$, $t = t_{m-d}$, and arbitrary point in domain of these curves are given. In addition, the Frenet vector fields and curvatures of these open NURBS curves were expressed by their control points. The relationships between control points were expressed when given two open NURBS curves occurred as Bertrand curve pairs at the points $t = t_d$, $t = t_{m-d}$, and arbitrary point in domain of these curves.

Keywords NURBS curves · Bertrand pairs · Open spline · Frenet frame

1 Introduction

In 1850, J. Bertrand gave the feature that helix curves accept other curves with the same original normal vector field [1]. The curves that provide this feature are called Bertrand curves.

M. Incesu (✉)
Department of Mathematics, Mus Alparslan University, Mus, Turkey
e-mail: m.incesu@alparslan.edu.tr

S. Y. Evren
Mus Alparslan University Ms of Sci student, Mus, Turkey

O. Gursoy
Department of Mathematics, Maltepe University, Basibuyuk, Maltepe-Istanbul, Turkey
e-mail: osmangursoy@maltepe.edu.tr

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_11

It is possible that the Bertrand curves are defined as their principal normals are parallel [1]. It is possible to define a curve as the Bertrand curve if this curve is planar or its curvature κ and its torsion τ in R^3 satisfies the condition $\kappa + a\,\tau = b$ for nonzero constants a, b [2]. In recent years, Bertrand curves play an important role in computer-aided geometric design (CAD) and computer-aided modeling (CAM) [3–5]. Due to this importance, Bertrand curves have been studied by geometers in different spaces [6–22].

The best examples of points systems are Bezier curves and Bezier surfaces. Bezier and B-Spline curves have been studied in many different areas of CAD and CAM system. Some of these studies can be given exemplary in [23–35].

Other studies on B-spline curves and NURBS curves in [36–44, 46–52] can be given as examples.

NURBS curves are rational B-Spline curves without uniform distribution. Bezier curves, B-Spline curves, and NURBS curves are curves that are widely used in computer graphics (CAD) (CAM) systems.

The Frenet vector fields and curvatures of open Non-Uniform B-Spline (NUBS) (not rational) curves at the points $t = t_d$, $t = t_{m-d}$, and arbitrary point in domain of these curves were studied in [46]. In addition, the relationships between the control points when given two open NUBS curves occurred as Bertrand curve pairs were also studied in [46].

In this paper the second- and third-order derivatives of open Non-Uniform Rational B-Spline (NURBS) curves at the points $t = t_d$, $t = t_{m-d}$, and arbitrary point in domain of these curves have been given. In addition, the Frenet vector fields and curvatures of these open NURBS curves were expressed by their control points. Similarly the relationships between the control points have also been expressed when given two open NURBS curves occurred as Bertrand curve pairs at the points $t = t_d$, $t = t_{m-d}$ and arbitrary point in domain of these curves.

2 Preliminaries

Definition 1 The B-spline basis functions of degree d, denoted $N_{i,d}(t)$, defined by the knot vector $t_0, t_1, ..., t_m$ are defined recursively as follows:

$$N_{i,0}(t) = \begin{cases} 1, t \in [t_i, t_{i+1}) \\ 0, otherwise \end{cases}$$

and

$$N_{i,d}(t) = \frac{t - t_i}{t_{i+d} - t_i} N_{i,d-1}(t) + \frac{t_{i+d+1} - t}{t_{i+d+1} - t_{i+1}} N_{i+1,d-1}(t) \tag{1}$$

for $i = 0, ..., n$ and $d \geq 1$.

If the knot vector contains a sufficient number of repeated knot values, then a division of the form $N_{i,d-1}(t)/(t_{i+d} - t_i) = 0/0$ (for some i) may be encountered during the execution of the recursion. Whenever this occurs, it is assumed that $0/0 = 0$ [45].

Definition 2 The B-spline curve of degree d with control points $b_0, ..., b_n$ and knots $t_0, ..., t_m$ is defined on the interval $[a, b] = [t_d, t_{m-d}]$

$$B(t) = \sum_{i=0}^{n} b_i N_{i,d}(t) \tag{2}$$

where $N_{i,d}(t)$ are the B-spline basis functions of degree d. To distinguish B-spline curves from their rational form they are often referred to as integral B-splines [45].

Theorem 1 *The B-spline basis functions $N_{i,d}(t)$ satisfy the following properties [45]*

(i) ***Positivity:*** $N_{i,d}(t) > 0$ *for* $t \in (t_i, t_{i+d+1})$.
(ii) ***Local Support:*** $N_{i,d}(t) = 0$ *for* $t \notin (t_i, t_{i+d+1})$.
(iii) ***Piecewise Polynomial:*** $N_{i,d}(t)$ *are piecewise polynomial functions of degree* d.
(iv) ***Partition of Unity:*** $\sum_{i=r-d}^{r} N_{i,d}(t) = 1$ *for* $t \in [t_r, t_{r+1})$

Theorem 2 *A B-spline curve defined as (2) of degree d defined on the knot vector $t_0, ..., t_m$ satisfies the following properties [45]*

(i) ***Local Control:*** *Each segment is determined by $d + 1$ control points. If $t \in [t_r, t_{r+1}) (d \leq r \leq m - d - 1)$, then*

$$B(t) = \sum_{i=r-d}^{r} b_i N_{i,d}(t).$$

Thus to evaluate $B(t)$ it is sufficient to evaluate $N_{r-d,d}(t), ..., N_{r,d}(t)$.

(ii) ***Convex Hull:*** *If* $t \in [tr, tr + 1)(d \leq r \leq m - d - 1)$, *then* $B(t) \in CH \{b_{r-d}, ..., b_r\}$.
(iii) ***Invariance under Affine Transformations:*** *Let T be an affine transformation. Then*

$$T\left(\sum_{i=r-d}^{r} b_i N_{i,d}(t)\right) = \sum_{i=r-d}^{r} T(b_i) N_{i,d}(t)$$

Definition 3 The NURBS curve of degree d with control points $b_0, ..., b_n$ and knots $t_0, ..., t_d, ..., t_{m-d}, ..., t_m$ is defined on the interval $[a, b] = [t_d, t_{m-d}]$ by

$$B(t) = \frac{\sum_{i=0}^{n} b_i w_i N_{i,d}(t)}{\sum_{i=0}^{n} w_i N_{i,d}(t)} \tag{3}$$

where $N_{i,d}(t)$ are the B-spline basis functions of degree d and $w_0, ..., w_n$ are the weights of this curve [45].

2.1 Open B-Spline Curves

In general, B-spline curves do not interpolate the first and last control points b_0 and b_n. For any curve of degree d, endpoint interpolation and endpoint tangent conditions are obtained by open B-splines. An open B-spline curve is a B-spline curve in which exterior knot vectors are the same as the knots t_d and t_{m-d}, i.e., $t_0 = t_1 = ... = t_d$ and $t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ satisfies.

Theorem 3 *An open B-spline curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then*

$$B(t_d) = b_0 \quad \text{and} \quad B(t_{m-d}) = b_n$$

satisfies [45].

Definition 4 A B-spline curve is said to be **uniform** whenever its knots are equally spaced, and **non-uniform** otherwise. A uniform B-spline curve is said to be **open uniform** whenever its interior knots are equally spaced, and its exterior knots are same. Similarly a non-uniform B-spline curve is said to be **open non-uniform** whenever its exterior knots are same and its interior knots are not equally spaced.

Theorem 4 *An open B-spline curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then,*

$$B^{'}(t_d) = \frac{d}{t_{d+1} - t_1}(b_1 - b_0) \tag{4}$$

$$B^{'}(t_{m-d}) = \frac{d}{t_{m-1} - t_{m-d-1}}(b_n - b_{n-1}) \tag{5}$$

are satisfied [45].

Remark 1 An open B-spline curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and knot vectors $t_0 = t_1 = ... = t_d; t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. If $t_0 = t_1 = ... = t_d = 0$ and $t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m = 1$ Then,

$$B^{'}(0) = \frac{d}{t_{d+1}}(b_1 - b_0) \tag{6}$$

$$B^{'}(1) = \frac{d}{1 - t_{m-d-1}}(b_n - b_{n-1}) \tag{7}$$

are obtained.

Theorem 5 *An open B-spline curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then,*

$$B''(t_d) = \frac{d(d-1)}{(t_{d+1} - t_2)(t_{d+2} - t_2)}(b_2 - b_1) - \frac{d(d-1)}{(t_{d+1} - t_2)(t_{d+1} - t_1)}(b_1 - b_0) \tag{8}$$

$$\begin{aligned} B''(t_{m-d}) = & \frac{d(d-1)}{(t_{m-2} - t_{m-d-1})(t_{m-1} - t_{m-d-1})}(b_n - b_{n-1}) \\ & - \frac{d(d-1)}{(t_{m-2} - t_{m-d-1})(t_{m-2} - t_{m-d-2})}(b_{n-1} - b_{n-2}) \end{aligned} \tag{9}$$

are satisfied.

Proof see [46]. □

Theorem 6 *An open B-spline curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then,*

$$\begin{aligned} B'''(t_d) = & \frac{d(d-1)(d-2)}{(t_{d+1} - t_3)(t_{d+2} - t_3)(t_{d+3} - t_3)}(b_3 - b_2) \\ & - \frac{d(d-1)(d-2)(t_{d+1} - t_2 + t_{d+2} - t_3)}{(t_{d+1} - t_3)(t_{d+2} - t_2)(t_{d+2} - t_3)(t_{d+1} - t_2)}(b_2 - b_1) \\ & + \frac{d(d-1)(d-2)}{(t_{d+1} - t_3)(t_{d+1} - t_2)(t_{d+1} - t_1)}(b_1 - b_0) \end{aligned} \tag{10}$$

$$\begin{aligned} B'''(t_{m-d}) = & \frac{d(d-1)(d-2)(b_n - b_{n-1})}{(t_{m-3} - t_{m-d-1})(t_{m-2} - t_{m-d-1})(t_{m-1} - t_{m-d-1})} \\ & - \frac{d(d-1)(d-2)(t_{m-3} - t_{m-d-2} + t_{m-2} - t_{m-d-1})(b_{n-1} - b_{n-2})}{(t_{m-3} - t_{m-d-1})(t_{m-2} - t_{m-d-2})(t_{m-2} - t_{m-d-1})(t_{m-3} - t_{m-d-2})} \\ & + \frac{d(d-1)(d-2)(b_{n-2} - b_{n-3})}{(t_{m-3} - t_{m-d-1})(t_{m-3} - t_{m-d-2})(t_{m-3} - t_{m-d-3})} \end{aligned} \tag{11}$$

are satisfied.

Proof see [46]. □

Theorem 7 *An open B-spline curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then, the Frenet vector fields and curvatures of this curve at the point $t = t_d$ are as follows:*

$$\begin{array}{ll} \mathbf{T}(t_d) = \frac{b_1-b_0}{\|b_1-b_0\|} & \mathbf{B}(t_d) = \frac{(b_1-b_0)\times(b_2-b_1)}{\|(b_1-b_0)\times(b_2-b_1)\|} \\ \mathbf{N}(t_d) = \frac{(b_2-b_1)}{\|(b_2-b_1)\|} csc\Phi - \frac{(b_1-b_0)}{\|(b_1-b_0)\|}\cot\Phi & \kappa(t_d) = \frac{(d-1)(t_{d+1}-t_1)^2\|(b_2-b_1)\|}{d(t_{d+1}-t_2)(t_{d+2}-t_2)\|(b_1-b_0)\|^2}\sin\Phi \end{array} \tag{12}$$

and

$$\tau(t_d) = \frac{(d-2)(t_{d+1}-t_1)(t_{d+1}-t_2)(t_{d+2}-t_2)\|(b_3-b_2)\|\cos\varphi}{d(t_{d+1}-t_3)(t_{d+2}-t_3)(t_{d+3}-t_3)\|(b_1-b_0)\|\|(b_2-b_1)\|\sin\Phi}$$

where Φ is the angle between the vectors $b_1 - b_0$ and $b_2 - b_1$ and φ is the angle between the vectors $b_3 - b_2$ and $(b_1 - b_0) \times (b_2 - b_1)$.

Proof see [46]. □

Theorem 8 *An open B-spline curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then, the Frenet vector fields and curvatures of this curve at the point $t = t_{m-d}$ are as follows:*

$$\begin{array}{ll} \mathbf{T}(t_{m-d}) = \frac{b_n-b_{n-1}}{\|b_n-b_{n-1}\|} & \mathbf{B}(t_{m-d}) = -\frac{(b_n-b_{n-1})\times(b_{n-1}-b_{n-2})}{\|(b_n-b_{n-1})\times(b_{n-1}-b_{n-2})\|} \\ \mathbf{N}(t_{m-d}) = -\frac{(b_{n-1}-b_{n-2})}{\|b_{n-1}-b_{n-2}\|} csc\vartheta + \frac{(b_n-b_{n-1})}{\|b_n-b_{n-1}\|}\cot\vartheta & \end{array} \tag{13}$$

and

$$\kappa(t_{m-d}) = \frac{(d-1)\left(t_{m-1}-t_{m-d-1}\right)^2\left\|b_{n-1}-b_{n-2}\right\|}{d\left(t_{m-2}-t_{m-d-1}\right)\left(t_{m-2}-t_{m-d-2}\right)\left\|b_n-b_{n-1}\right\|^2}\sin\vartheta$$

$$\tau(t_{m-d}) = -\frac{d-2}{d}\frac{\left(t_{m-1}-t_{m-d-1}\right)\left(t_{m-2}-t_{m-d-1}\right)\left(t_{m-2}-t_{m-d-2}\right)}{\left(t_{m-3}-t_{m-d-1}\right)\left(t_{m-3}-t_{m-d-2}\right)\left(t_{m-3}-t_{m-d-3}\right)}\frac{\left\|(b_{n-2}-b_{n-3})\right\|\cos\sigma}{\left\|(b_n-b_{n-1})\times(b_{n-1}-b_{n-2})\right\|}$$

where ϑ is the angle between the vectors $b_n - b_{n-1}$ and $b_{n-1} - b_{n-2}$ and σ is the angle between the vectors $b_{n-3} - b_{n-2}$ and $(b_n - b_{n-1}) \times (b_{n-1} - b_{n-2})$

Proof see [46]. □

2.2 The Rational de Boor Algorithm

Let an open NURBS curve $B(t)$ of degree d with control points and weights $b_0, b_1, ..., b_n$, $w_0, w_1, ..., w_n$, respectively and knot vectors $t_0 = t_1 = ... = t_d$, $t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Suppose $t^* \in [t_r, t_{r+1})$. Then, the rational De Boor algorithm can be summarized as follows:

$$\begin{aligned} \alpha_i^j(t) &= \tfrac{t^*-t_i}{t_{i+d-j+1}-t_i} \\ w_i^j &= \left(1-\alpha_i^j(t^*)\right) w_{i-1}^{j-1} + \alpha_i^j(t^*) w_i^{j-1} \\ w_i^j b_i^j(t^*) &= \left(1-\alpha_i^j(t^*)\right) w_{i-1}^{j-1} b_{i-1}^{j-1}(t^*) + \alpha_i^j(t^*) w_i^{j-1} b_i^{j-1}(t^*) \quad \text{for } j > 0 \end{aligned} \tag{14}$$

for $j = 1, ..., d$ and $i = r - d + 1, ..., r$. Where $b_i^0 = b_i$, $b_{-1} = 0$ and $b_{m-d+1} = 0$. To summarize, for a given parameter value t, the rational de Boor algorithm (14) yields a triangular array of points such that $b_r^d = B(t^*)$ [45].

3 Main Results

3.1 The Derivatives of the Open Non-Uniform Rational B-Spline Curves

Theorem 9 *Let an open NURBS curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and weights $w_0, w_1, ..., w_n$ and the knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then, the first-order derivatives of this curve at the points $t = t_d$ and $t = t_{m-d}$ are as follows*

$$B'(t_d) = \frac{d}{t_{d+1} - t_1} \frac{w_1}{w_0} (b_1 - b_0) \tag{15}$$

$$B'(t_{m-d}) = \frac{d}{t_{m-1} - t_{m-d-1}} \frac{w_{n-1}}{w_n} (b_n - b_{n-1}) \tag{16}$$

Proof Let $\sum_{i=0}^{n} w_i b_i N_{i,d}(t)$ be denoted as f and $\sum_{i=0}^{n} w_i N_{i,d}(t)$ be denoted as g then an open NURBS curve $B(t)$ can be written as $B(t) = \frac{f}{g}$. In this case the first-order derivative of $B(t)$ can be written as $B'(t) = \frac{f'}{g} - B(t)\frac{g'}{g}$. Here the functions f and g can be thought of as two B-spline curves with control points $w_i b_i$ and w_i, respectively. So according to Theorems 3 and 4 the first-order derivatives of $B(t)$ at the points $t = t_d$ and $t = t_{m-d}$ are obtained as follows

$$B'(t_d) = \frac{\frac{d}{t_{d+1}-t_1}(w_1 b_1 - w_0 b_0)}{w_0} - b_0 \frac{\frac{d}{t_{d+1}-t_1}(w_1 - w_0)}{w_0}$$
$$= \frac{d}{t_{d+1} - t_1} \frac{w_1}{w_0}(b_1 - b_0)$$

and

$$B'(t_{m-d}) = \frac{\frac{d}{t_{m-1}-t_{m-d-1}}(w_n b_n - w_{n-1} b_{n-1})}{w_n} - b_n \frac{\frac{d}{t_{m-1}-t_{m-d-1}}(w_n - w_{n-1})}{w_n}$$
$$= \frac{d}{t_{m-1} - t_{m-d-1}} \frac{w_{n-1}}{w_n}(b_n - b_{n-1})$$

□

Theorem 10 *Let an open NURBS curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and weights $w_0, w_1, ..., w_n$ and the knot vectors $t_0 = t_1 = ... = t_d$, $t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then, the second-order derivatives of this curve at the points $t = t_d$ and $t = t_{m-d}$ are as follows*

$$B''(t_d) = \frac{d(d-1)}{\left(t_{d+1} - t_2\right)\left(t_{d+2} - t_2\right)} \frac{w_2}{w_0}(b_2 - b_0)$$
$$- \frac{d}{t_{d+1} - t_1} \frac{w_1}{w_0} \left[\frac{(d-1)\left(t_{d+1} + t_{d+2} - t_1 - t_2\right)}{\left(t_{d+1} - t_2\right)\left(t_{d+2} - t_2\right)} + \frac{2d}{\left(t_{d+1} - t_1\right)} \left(\frac{w_1}{w_0} - 1 \right) \right] (b_1 - b_0)$$

and

$$B''(t_{m-d}) = \frac{w_{n-1}}{w_n} \left[\frac{d(d-1)}{t_{m-2} - t_{m-d-1}} \left(\frac{1}{t_{m-1} - t_{m-d-1}} - \frac{1}{t_{m-1} - t_{m-d-2}} \right) - \frac{2d^2(w_n - w_{n-1})}{\left(t_{m-1} - t_{m-d-1}\right)^2} \right] (b_n - b_{n-1})$$
$$- \frac{w_{n-2}}{w_n} \frac{d(d-1)}{\left(t_{m-1} - t_{m-d-2}\right)\left(t_{m-2} - t_{m-d-1}\right)} (b_n - b_{n-2})$$

Proof Similarly as previous theorem, if the NURBS curve is written as $B = \frac{f}{g}$ then the second-order derivative of B can be written as

$$B'' = \left[\frac{f'}{g} - B \frac{g'}{g} \right]'$$
$$= \left[\frac{f''}{g} - 2B' \frac{g'}{g} - B \frac{g''}{g} \right]$$

So according to Theorems 3, 4, and 5 the second-order derivatives of B at the points $t = t_d$ and $t = t_{m-d}$ are obtained as follows

$$B''(t_d) = \frac{d(d-1)}{(t_{d+1}-t_2)(t_{d+2}-t_2)} \frac{w_2}{w_0}(b_2 - b_0)$$
$$- \frac{d}{t_{d+1}-t_1} \frac{w_1}{w_0} \left[\frac{(d-1)(t_{d+1}+t_{d+2}-t_1-t_2)}{(t_{d+1}-t_2)(t_{d+2}-t_2)} + \frac{2d}{(t_{d+1}-t_1)} \left(\frac{w_1}{w_0} - 1 \right) \right] (b_1 - b_0)$$

and

$$B''(t_{m-d}) = \frac{w_{n-1}}{w_n} \left[\frac{d(d-1)}{t_{m-2}-t_{m-d-1}} \left(\frac{1}{t_{m-1}-t_{m-d-1}} - \frac{1}{t_{m-1}-t_{m-d-2}} \right) - \frac{2d^2(w_n - w_{n-1})}{(t_{m-1}-t_{m-d-1})^2} \right] (b_n - b_{n-1})$$
$$- \frac{w_{n-2}}{w_n} \frac{d(d-1)}{(t_{m-1}-t_{m-d-2})(t_{m-2}-t_{m-d-1})} (b_n - b_{n-2})$$

□

Theorem 11 *Let an open NURBS curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and weights $w_0, w_1, ..., w_n$ and the knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then, the third-order derivatives of this curve at the points $t = t_d$ and $t = t_{m-d}$ are as follows*

$$B'''(t_d) = \frac{d(d-1)(d-2)}{zlm} \frac{w_3}{w_0}(b_3 - b_0) - d(d-1)\frac{w_2}{w_0} \left[\frac{d-2}{zlm} + \frac{d-2}{zkly} + \frac{3d(w_1 - w_0)}{zykw_0} \right] (b_2 - b_0)$$
$$+ d\frac{w_1}{w_0} \left[\begin{array}{c} \frac{(d-1)(d-2)(y+l)}{zkly} \\ + \frac{(d-1)(d-2)}{xyz} + \frac{3d(w_1-w_0)}{x^2 w_0} \left(\frac{(d-1)(x+k)}{yk} + \frac{2d(w_1-w_0)}{xw_0} \right) \\ + \frac{3d(d-1)}{xyw_0} \left(\frac{w_2-w_1}{k} + \frac{w_1-w_0}{x} \right) \end{array} \right] (b_1 - b_0)$$

and

$$B'''(t_{m-d}) = \frac{w_{n-3}}{w_n} \frac{d(d-1)(d-2)}{urh} (b_n - b_{n-3})$$
$$+ \frac{w_{n-2}}{w_n} d(d-1) \left[\frac{3d^2(w_n - w_{n-1})}{evqw_n} - \frac{(d-2)}{urh} - \frac{(d-2)(h+v)}{uqvh} \right] (b_n - b_{n-2})$$
$$+ \frac{w_{n-1}}{w_n} \left[\begin{array}{c} \frac{d(d-1)(d-2)}{uv} \left(\frac{1}{e} + \frac{h+v}{qh} \right) \\ - \frac{3d^2}{e} \left[\begin{array}{c} \frac{(d-1)(w_n-w_{n-1})\left(\frac{1}{e} - \frac{1}{p}\right)}{vw_n} + \frac{(d-1)}{vw_n} \left(\frac{w_n-w_{n-1}}{e} - \frac{w_{n-1}-w_{n-2}}{q} \right) \\ - \frac{3d}{e^2}(w_n - w_{n-1})^2 \end{array} \right] \end{array} \right] (b_n - b_{n-1})$$

where $x = t_{d+1} - t_1$, $y = t_{d+1} - t_2$, $z = t_{d+1} - t_3$, $k = t_{d+2} - t_2$, $l = t_{d+2} - t_3$, $m = t_{d+3} - t_3$ and $u = t_{m-3} - t_{m-d-1}$, $v = t_{m-2} - t_{m-d-1}$, $e = t_{m-1} - t_{m-d-1}$, $p = t_{m-1} - t_{m-d-2}$, $q = t_{m-2} - t_{m-d-2}$, $h = t_{m-3} - t_{m-d-2}$, $r = t_{m-3} - t_{m-d-3}$.

Proof Since $B = \frac{f}{g}$ then $B''' = \left[\frac{f''}{g} - 2B'\frac{g'}{g} - B\frac{g''}{g} \right]'$ and it can be written as

$$B''' = \left[\frac{f'''}{g} - 3B''\frac{g'}{g} - 3B'\frac{g''}{g} - B\frac{g'''}{g} \right]$$

So according to Theorems 3, 4, 5, and 6, the third-order derivatives of an open NURBS curve at the points $t = t_d$ and $t = t_{m-d}$ are obtained as above. □

3.2 The Frenet Frame on the Open Non-Uniform Rational B-Spline Curves

Theorem 12 *An open NURBS curve $B(t)$ of degree d with control points $b_0, b_1, ...,$ b_n and weights $w_0, w_1, ..., w_n$ and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then, the Frenet vector fields and curvatures of this curve at the point $t = t_d$ are as follows:*

$$\mathbf{T}(t_d) = \frac{b_1-b_0}{\|b_1-b_0\|} \qquad \mathbf{B}(t_d) = \frac{(b_1-b_0)\times(b_2-b_0)}{\|(b_1-b_0)\times(b_2-b_0)\|}$$
$$\mathbf{N}(t_d) = \frac{(b_2-b_0)}{\|(b_2-b_0)\|}\csc\Phi - \frac{(b_1-b_0)}{\|(b_1-b_0)\|}\cot\Phi \qquad \kappa(t_d) = \frac{d-1}{d}\frac{x^2}{yk}\frac{w_0w_2}{w_1^2}\frac{\|(b_2-b_0)\|}{\|(b_1-b_0)\|^2}\sin\Phi$$
$$\tau(t_d) = \frac{d-2}{d}\frac{xyk}{zlm}\frac{w_0w_3}{w_1w_2}\frac{\|(b_3-b_0)\|\cos\varphi}{\|(b_1-b_0)\times(b_2-b_0)\|} \tag{17}$$

where Φ is the angle between the vectors $b_1 - b_0$ and $b_2 - b_0$ and φ is the angle between the vectors $b_3 - b_0$ and $(b_1 - b_0) \times (b_2 - b_0)$. Additionally $x = t_{d+1} - t_1$, $y = t_{d+1} - t_2$, $z = t_{d+1} - t_3$, $k = t_{d+2} - t_2$, $l = t_{d+2} - t_3$, $m = t_{d+3} - t_3$.

Proof (i) $\mathbf{T}(t_d) = \frac{B'(t_d)}{\|B'(t_d)\|} = \frac{\frac{d}{t_{d+1}-t_1}\frac{w_1}{w_0}(b_1-b_0)}{\left\|\frac{d}{t_{d+1}-t_1}\frac{w_1}{w_0}(b_1-b_0)\right\|} = \frac{(b_1-b_0)}{\|(b_1-b_0)\|}$

(ii) $\mathbf{B}(t_d) = \frac{B'(t_d)\times B''(t_d)}{\|B'(t_d)\times B''(t_d)\|}$

$$= \frac{\frac{d}{t_{d+1}-t_1}\frac{w_1}{w_0}(b_1-b_0)\times\frac{d(d-1)}{(t_{d+1}-t_2)(t_{d+2}-t_2)}\frac{w_2}{w_0}(b_2-b_0)}{\left\|\frac{d}{t_{d+1}-t_1}\frac{w_1}{w_0}(b_1-b_0)\times\frac{d(d-1)}{(t_{d+1}-t_2)(t_{d+2}-t_2)}\frac{w_2}{w_0}(b_2-b_0)\right\|} = \frac{(b_1-b_0)\times(b_2-b_0)}{\|(b_1-b_0)\times(b_2-b_0)\|}$$

(iii) $\mathbf{N}(t_d) = \mathbf{B}(t_d) \times \mathbf{T}(t_d) = \frac{(b_1-b_0)\times(b_2-b_0)}{\|(b_1-b_0)\times(b_2-b_0)\|} \times \frac{(b_1-b_0)}{\|(b_1-b_0)\|}$

$$= \frac{((b_1-b_0)\times(b_2-b_0))\times(b_1-b_0)}{\|(b_1-b_0)\times(b_2-b_1)\|\|(b_1-b_0)\|} = \frac{\|(b_1-b_0)\|^2(b_2-b_0)-\langle b_1-b_0,b_2-b_0\rangle(b_1-b_0)}{\|(b_1-b_0)\times(b_2-b_1)\|\|(b_1-b_0)\|}$$
$$= \frac{(b_2-b_0)}{\|b_2-b_0\|\sin\Phi} - \frac{\cos\Phi(b_1-b_0)}{\sin\Phi\|(b_1-b_0)\|} = \frac{(b_2-b_0)}{\|(b_2-b_0)\|}\csc\Phi - \frac{(b_1-b_0)}{\|(b_1-b_0)\|}\cot\Phi$$

(iv) $\kappa(t_d) = \frac{\left\|B'(t_d)\times B''(t_d)\right\|}{\|B'(t_d)\|^3} = \frac{\left\|\frac{d}{t_{d+1}-t_1}\frac{w_1}{w_0}(b_1-b_0)\times\frac{d(d-1)}{(t_{d+1}-t_2)(t_{d+2}-t_2)}\frac{w_2}{w_0}(b_2-b_0)\right\|}{\left\|\frac{d}{t_{d+1}-t_1}\frac{w_1}{w_0}(b_1-b_0)\right\|^3}$

$$= \frac{d-1}{d}\frac{x^2}{yk}\frac{w_0w_2}{w_1^2}\frac{\|(b_2-b_0)\|}{\|(b_1-b_0)\|^2}\sin\Phi$$

(v) Let $\det(b_1 - b_0, b_2 - b_0, b_3 - b_0)$ be denoted by K. Then

$$\tau(t_d) = \frac{\det\big(B'(t_d),B''(t_d),B'''(t_d)\big)}{\|B'(t_d)\times B''(t_d)\|^2}$$
$$= \frac{\frac{d}{t_{d+1}-t_1}\frac{w_1}{w_0}\frac{d(d-1)}{(t_{d+1}-t_2)(t_{d+2}-t_2)}\frac{w_2}{w_0}\frac{d(d-1)(d-2)}{zlm}\frac{w_3}{w_0}K}{\left\|\frac{d}{t_{d+1}-t_1}\frac{w_1}{w_0}(b_1-b_0)\times\frac{d(d-1)}{(t_{d+1}-t_2)(t_{d+2}-t_2)}\frac{w_2}{w_0}(b_2-b_0)\right\|^2}$$
$$= \frac{d-2}{d}\frac{xyk}{zlm}\frac{w_0w_3}{w_1w_2}\frac{\|(b_3-b_0)\|\cos\varphi}{\|(b_1-b_0)\times(b_2-b_0)\|}$$

□

Theorem 13 *An open NURBS curve $B(t)$ of degree d with control points $b_0, b_1, ...,$ b_n and weights $w_0, w_1, ..., w_n$ and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then, the Frenet vector fields and curvatures of this curve at the point $t = t_{m-d}$ are as follows:*

$$\mathbf{T}(t_{m-d}) = \frac{b_n - b_{n-1}}{\|b_n - b_{n-1}\|} \qquad \mathbf{B}(t_{m-d}) = -\frac{(b_n - b_{n-1}) \times (b_n - b_{n-2})}{\|(b_n - b_{n-1}) \times (b_n - b_{n-2})\|}$$
$$\mathbf{N}(t_{m-d}) = \frac{(b_n - b_{n-1})}{\|b_n - b_{n-1}\|} \cot \vartheta - \frac{(b_n - b_{n-2})}{\|(b_n - b_{n-2})\|} \csc \vartheta \tag{18}$$

and

$$\kappa(t_{m-d}) = \frac{w_n w_{n-2}}{w_{n-1}^2} \frac{e^2}{vp} \frac{(d-1)}{d} \frac{\|b_n - b_{n-2}\|}{\|b_n - b_{n-1}\|^2} \sin \vartheta$$
$$\tau(t_{m-d}) = -\frac{d-2}{d} \frac{evp}{ruh} \frac{w_n w_{n-3}}{w_{n-1} w_{n-2}} \frac{\|(b_n - b_{n-3})\| \cos \sigma}{\|(b_n - b_{n-1}) \times (b_n - b_{n-2})\|}$$

where ϑ is the angle between the vectors $b_n - b_{n-1}$ and $b_n - b_{n-2}$ and σ is the angle between the vectors $b_n - b_{n-3}$ and $(b_n - b_{n-1}) \times (b_n - b_{n-2})$ and $u = t_{m-3} - t_{m-d-1}$, $v = t_{m-2} - t_{m-d-1}$, $e = t_{m-1} - t_{m-d-1}$, $p = t_{m-1} - t_{m-d-2}$, $h = t_{m-3} - t_{m-d-2}$, $r = t_{m-3} - t_{m-d-3}$.

Proof (i) $\mathbf{T}(t_{m-d}) = \frac{B'(t_{m-d})}{\|B'(t_{m-d})\|} = \frac{\frac{d}{t_{m-1}-t_{m-d-1}} \frac{w_{n-1}}{w_n}(b_n - b_{n-1})}{\left\| \frac{d}{t_{m-1}-t_{m-d-1}} \frac{w_{n-1}}{w_n}(b_n - b_{n-1}) \right\|} = \frac{b_n - b_{n-1}}{\|b_n - b_{n-1}\|}$

(ii) $\mathbf{B}(t_{m-d}) = \frac{B'(t_{m-d}) \times B''(t_{m-d})}{\|B'(t_{m-d}) \times B''(t_{m-d})\|}$

$$= -\frac{\frac{d}{t_{m-1}-t_{m-d-1}} \frac{w_{n-1}}{w_n}(b_n - b_{n-1}) \times \left[\frac{w_{n-2}}{w_n} \frac{d(d-1)}{(t_{m-1}-t_{m-d-2})(t_{m-2}-t_{m-d-1})}(b_n - b_{n-2}) \right]}{\left\| \frac{d}{t_{m-1}-t_{m-d-1}} \frac{w_{n-1}}{w_n}(b_n - b_{n-1}) \times \left[\frac{w_{n-2}}{w_n} \frac{d(d-1)}{(t_{m-1}-t_{m-d-2})(t_{m-2}-t_{m-d-1})}(b_n - b_{n-2}) \right] \right\|}$$
$$= -\frac{(b_n - b_{n-1}) \times (b_n - b_{n-2})}{\|(b_n - b_{n-1}) \times (b_n - b_{n-2})\|}$$

(iii) $\mathbf{N}(t_{m-d}) = \mathbf{B}(t_{m-d}) \times \mathbf{T}(t_{m-d}) = -\frac{(b_n - b_{n-1}) \times (b_n - b_{n-2})}{\|(b_n - b_{n-1}) \times (b_n - b_{n-2})\|} \times \frac{b_n - b_{n-1}}{\|b_n - b_{n-1}\|}$

$$= \frac{(b_n - b_{n-1})}{\|b_n - b_{n-1}\|} \cot \vartheta - \frac{(b_n - b_{n-2})}{\|(b_n - b_{n-2})\|} \csc \vartheta$$

(iv) $\kappa(t_{m-d}) = \frac{\|B'(t_{m-d}) \times B''(t_{m-d})\|}{\|B'(t_{m-d})\|^3}$

$$= \frac{\left\| \frac{d}{t_{m-1}-t_{m-d-1}} \frac{w_{n-1}}{w_n}(b_n - b_{n-1}) \times \left[\frac{w_{n-2}}{w_n} \frac{d(d-1)}{(t_{m-1}-t_{m-d-2})(t_{m-2}-t_{m-d-1})}(b_n - b_{n-2}) \right] \right\|}{\left\| \frac{d}{t_{m-1}-t_{m-d-1}} \frac{w_{n-1}}{w_n}(b_n - b_{n-1}) \right\|^3}$$
$$= \frac{w_n w_{n-2}}{w_{n-1}^2} \frac{e^2}{vp} \frac{(d-1)}{d} \frac{\|b_n - b_{n-2}\|}{\|b_n - b_{n-1}\|^2} \sin \vartheta$$

(v) Let $\det(b_n - b_{n-1}, b_n - b_{n-2}, b_n - b_{n-3})$ be denoted by J. Then

$$\tau(t_{m-d}) = \frac{\det\left(B'(t_{m-d}), B''(t_{m-d}), B'''(t_{m-d})\right)}{\|B'(t_{m-d}) \times B''(t_{m-d})\|^2}$$
$$= -\frac{\frac{d}{e} \frac{w_{n-1}}{w_n} \frac{w_{n-2}}{w_n} \frac{d(d-1)}{pv} \frac{w_{n-3}}{w_n} \frac{d(d-1)(d-2)}{urh} J}{\left\| \frac{d}{e} \frac{w_{n-1}}{w_n}(b_n - b_{n-1}) \times \left[\frac{w_{n-2}}{w_n} \frac{d(d-1)}{pv}(b_n - b_{n-2}) \right] \right\|^2}$$
$$= -\frac{d-2}{d} \frac{evp}{ruh} \frac{w_n w_{n-3}}{w_{n-1} w_{n-2}} \frac{\|(b_n - b_{n-3})\| \cos \sigma}{\|(b_n - b_{n-1}) \times (b_n - b_{n-2})\|}$$

□

In open NURBS curves, in order to express the Frenet frame of the curve $\{T, N, B\}$ and the curvatures at any point $t^* \in (t_r, t_{r+1})$, $(d \le r \le m - d - 1)$, except $t^* = t_d$ and $t^* = t_{m-d}$, the subdivision algorithm is applied to the curve by applying rational de Boor algorithm. Thus the NURBS curve is divided into two segments. The points $\left\{b_r^d, b_r^{d-1}, b_r^{d-2}, b_r^{d-3}\right\}$ obtained by the algorithm at the given point t^* will represent the first 4 control points the new NURBS curve of right side from obtained two segments. And the weights $\{w_r^d, w_r^{d-1}, w_r^{d-2}, w_r^{d-3}\}$ will also represent the weights of these control points. So the points b_0, b_1, b_2, b_3 can be considered as control points of the new NURBS curve. And the point t^* here can also be considered the point t_d of the new NURBS curve. Under the consideration if the new NURBS curve is reparametrized on the interval $[t_d, t_{m-d}]$ the knot vectors can be chosen as initial knots. Thus the differences between the new obtained NURBS curve and initial NURBS curve are only their control points and their weights. So the following theorem can be proved similarly as before.

Theorem 14 *An open NURBS curve $B(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and weights $w_0, w_1, ..., w_n$ and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. Then, the Frenet vector fields and curvatures of this curve at the point $t = t^* \in (t_r, t_{r+1})$, $(d \leq r \leq m-d-1)$ are as follows:*

$$\begin{aligned} \mathbf{T}(t^*) &= \frac{b_r^{d-1}-b_r^d}{\|b_r^{d-1}-b_r^d\|} \qquad \mathbf{B}(t^*) = \frac{(b_r^{d-1}-b_r^d)\times(b_r^{d-2}-b_r^d)}{\|(b_r^{d-1}-b_r^d)\times(b_r^{d-2}-b_r^d)\|} \\ \mathbf{N}(t^*) &= \frac{b_r^{d-2}-b_r^d}{\|b_r^{d-2}-b_r^d\|} csc\Phi - \frac{b_r^{d-1}-b_r^d}{\|(b_r^{d-1}-b_r^d)\|} \cot\Phi \end{aligned} \tag{19}$$

and

$$\begin{aligned} \kappa(t^*) &= \frac{d-1}{d}\frac{x^2}{yk}\frac{w_r^d w_r^{d-2}}{\left(w_r^{d-1}\right)^2}\frac{\left\|\left(b_r^{d-2}-b_r^d\right)\right\|}{\left\|\left(b_r^{d-1}-b_r^d\right)\right\|^2}\sin\Phi \\ \tau(t^*) &= \frac{d-2}{d}\frac{xyk}{zlm}\frac{w_r^d w_r^{d-3}}{w_r^{d-1}w_r^{d-2}}\frac{\left\|\left(b_r^{d-3}-b_r^d\right)\right\|\cos\varphi}{\left\|\left(b_r^{d-1}-b_r^d\right)\right\|\left\|\left(b_r^{d-2}-b_r^d\right)\right\|\sin\Phi} \end{aligned}$$

where Φ is the angle between the vectors $b_r^{d-1} - b_r^d$ and $b_r^{d-2} - b_r^d$ and φ is the angle between the vectors $b_r^{d-3} - b_r^d$ and $\left(b_r^{d-1} - b_r^d\right) \times \left(b_r^{d-2} - b_r^d\right)$.

3.3 The Bertrand Pairs of Open NURBS Curves

Theorem 15 *Let two open NURBS curves $\gamma_1(t)$ and $\gamma_2(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and $c_0, c_1, ..., c_n$ and the weights $w_0, w_1, ..., w_n$ and $z_0, z_1, ..., z_n$, respectively and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. These curves γ_1 and γ_2 form a Bertrand pair at the point $t = t_d$ if and only if there exist $\theta \in [0, 2\pi]$ and $k \in R$ such that*

$$\begin{aligned} c_1 &= c_0 + (b_1 - b_0)\cos\theta - (b_1 - b_0) \times (b_2 - b_0)\sin\theta \\ c_2 &= c_1 + k(b_1 - b_0) + (b_2 - b_0) \end{aligned}$$

satisfies.

Proof If these curves γ_1 and γ_2 form a Bertrand pair at the point $t = t_d$ then $\mathbf{N}_{\gamma_1}(t_d) = \mathbf{N}_{\gamma_2}(t_d)$ satisfies. Thus these vectors $((b_1 - b_0) \times (b_2 - b_0)) \times$

$(b_1 - b_0)$ and $((c_1 - c_0) \times (c_2 - c_0)) \times (c_1 - c_0)$ be parallel. So The vectors $(c_1 - c_0) \times (c_2 - c_0)$, $c_1 - c_0$, $(b_1 - b_0) \times (b_2 - b_0)$, and $b_1 - b_0$ must be coplanar. In addition since the vectors system $\{c_1 - c_0$ and $(c_1 - c_0) \times (c_2 - c_0)\}$ and $\{b_1 - b_0$ and $(b_1 - b_0) \times (b_2 - b_0)\}$ are orthogonal, these systems must be $O^+(2)-$equivalent. i.e.,

$$\{c_1 - c_0, (c_1 - c_0) \times (c_2 - c_0)\} \overset{O^+(2)}{\approx} \{b_1 - b_0, (b_1 - b_0) \times (b_2 - b_0)\}.$$

This means that there exist $\theta \in [0, 2\pi]$ such that

$$c_1 - c_0 = (b_1 - b_0)\cos\theta - (b_1 - b_0) \times (b_2 - b_0)\sin\theta$$
$$(c_1 - c_0) \times (c_2 - c_0) = (b_1 - b_0)\sin\theta + (b_1 - b_0) \times (b_2 - b_0)\cos\theta$$

can be written. From this, $c_1 = c_0 + (b_1 - b_0)\cos\theta - (b_1 - b_0) \times (b_2 - b_0)\sin\theta$ is obtained and if this is substituted to second then

$(c_1 - c_0) \times (c_2 - c_0) = [(b_1 - b_0)\cos\theta - (b_1 - b_0) \times (b_2 - b_0)\sin\theta] \times (c_2 - c_0)$

$= (b_1 - b_0) \times (c_2 - c_0)\cos\theta - [(b_1 - b_0) \times (b_2 - b_0)] \times (c_2 - c_0)\sin\theta$

$= (b_1 - b_0)\sin\theta + (b_1 - b_0) \times (b_2 - b_0)\cos\theta$ can be written. Thus, from the property of vector product "$\times$" and the linear independence of the functions sinus and cosinus,

$$(b_1 - b_0) \times (c_2 - c_0) = (b_1 - b_0) \times (b_2 - b_0)$$
$$\langle (c_2 - c_0), (b_2 - b_0) \rangle = 1$$
$$\langle (b_1 - b_0), (c_2 - c_0) \rangle = 0$$

can be obtained. So, the vectors $(c_2 - c_0) - (b_2 - b_0)$ and $(b_1 - b_0)$ must be parallel. Thus, there exist $k \in R$ such that $(c_2 - c_0) - (b_2 - b_0) = k(b_1 - b_0)$ can be written. So

$$c_2 = c_0 + k(b_1 - b_0) + (b_2 - b_0)$$

be obtained. □

Theorem 16 *Let two open NURBS curves $\gamma_1(t)$ and $\gamma_2(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and $c_0, c_1, ..., c_n$ and the weights $w_0, w_1, ..., w_n$ and $z_0, z_1, ..., z_n$, respectively and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. These curves γ_1 and γ_2 form a Bertrand pair at the point $t = t_{m-d}$ if and only if there exist $\theta \in [0, 2\pi]$ and $k \in R$ such that*

$$c_n = c_{n-1} + (b_n - b_{n-1})\cos\theta - (b_n - b_{n-1}) \times (b_n - b_{n-2})\sin\theta$$
$$c_n = c_{n-2} + (b_n - b_{n-2}) + k(b_n - b_{n-1})$$

satisfies.

Proof It is proved similarly as previous theorem. □

Theorem 17 *Let two open NURBS curves $\gamma_1(t)$ and $\gamma_2(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and $c_0, c_1, ..., c_n$ and the weights $w_0, w_1, ..., w_n$ and $z_0, z_1, ..., z_n$, respectively and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. These curves γ_1 and γ_2 form a Bertrand pair at the point $t = t^* \in (t_r, t_{r+1})$,$(d \leq r \leq m - d - 1)$ if and only if there exist $\theta \in [0, 2\pi]$ and $k \in R$ such that*

$$c_r^{d-1} = c_r^d + (b_r^{d-1} - b_r^d)\cos\theta - (b_r^{d-1} - b_r^d) \times (b_r^{d-2} - b_r^d))\sin\theta$$
$$c_r^{d-2} = c_r^d + k(b_r^{d-1} - b_r^d) + (b_r^{d-2} - b_r^d)$$

satisfies.

Proof When the rational de Boor algorithm applies to these NURBS curves $\gamma_1(t)$ and $\gamma_2(t)$ at the point $t^* \in (t_r, t_{r+1})$, the control points $\left\{b_r^d, b_r^{d-1}, b_r^{d-2}, b_r^{d-3}\right\}$ and $\left\{c_r^d, c_r^{d-1}, c_r^{d-2}, c_r^{d-3}\right\}$ can be obtained. So if these control points be written in the theorem as the point $t = t_d$, then the proof is completed. □

Theorem 18 *Let two open NURBS curves $\gamma_1(t)$ and $\gamma_2(t)$ of degree d with control points $b_0, b_1, ..., b_n$ and $c_0, c_1, ..., c_n$ and the weights $w_0, w_1, ..., w_n$ and $z_0, z_1, ..., z_n$, respectively and knot vectors $t_0 = t_1 = ... = t_d, t_{d+1}, ..., t_{m-d} = t_{m-d+1} = ... = t_{m-1} = t_m$ be given. If $c_i = b_i + p$, $(\forall i = 0, 1, ..., n,\ p \in R^3)$ satisfies then $\gamma_1(t)$ and $\gamma_2(t)$ form a Bertrand pair if and only if the weights of these NURBS curves are equal mutually or*

$$\frac{w_{i-1}^{j-1}}{z_{i-1}^{j-1}} = \frac{w_i^j}{z_i^j} = \frac{w_i^{j-1}}{z_i^{j-1}}$$

satisfies.

Proof Let $c_i = b_i + p$, $(\forall i = 0, 1, ..., n,\ p \in R^3)$ be satisfied. Firstly it is supposed that the weights of these curves are equal mutually. Then, if $t = t_d$ or $t = t_{m-d}$ it can be seen easily that $N_{\gamma_1} = N_{\gamma_2}$ (by choosing $\theta = 0$ and $k = 1$ in Theorem 15) and these curves $\gamma_1(t)$ and $\gamma_2(t)$ form a Bertrand pair. If $t = t^*$ $(t^* \in [t_r, t_{r+1})$ then it must be proved $c_i^j = b_i^j + p$ to complete the proof.

$$\alpha_i^j = \frac{t^* - t_i}{t_{i+d-j+1} - t_i}$$
$$w_i^j = \left(1 - \alpha_i^j\right) w_{i-1}^{j-1} + \alpha_i^j w_i^{j-1}$$

let's prove this by induction. For $j = 1$, (for every i)

$$
\begin{aligned}
c_i^1 &= \left(1-\alpha_i^1\right)\frac{w_{i-1}^0}{w_i^1}c_{i-1}^0+\alpha_i^1\frac{w_i^0}{w_i^1} = \left(1-\alpha_i^1\right)\frac{w_{i-1}}{w_i^1}c_{i-1}+\alpha_i^1\frac{w_i}{w_i^1}c_i \\
&= \left(1-\alpha_i^1\right)\frac{w_{i-1}}{w_i^1}\left(b_{i-1}+p\right)+\alpha_i^1\frac{w_i}{w_i^1}\left(b_i+p\right) \\
&= \left(1-\alpha_i^1\right)\frac{w_{i-1}}{w_i^1}b_{i-1}+\alpha_i^1\frac{w_i}{w_i^1}b_i+\left(\left(1-\alpha_i^1\right)\frac{w_{i-1}}{w_i^1}+\alpha_i^1\frac{w_i}{w_i^1}\right)p \\
&= b_i^1+p
\end{aligned}
$$

is obtained. Let us suppose that it is true for $j-1$. i.e., let $c_i^{j-1}=b_i^{j-1}+p$ be satisfied.

$$
\begin{aligned}
c_i^j &= \left(1-\alpha_i^j\right)\frac{w_{i-1}^{j-1}}{w_i^j}c_{i-1}^{j-1}+\alpha_i^j\frac{w_i^{j-1}}{w_i^j}c_i^{j-1} \\
&= \left(1-\alpha_i^j\right)\frac{w_{i-1}^{j-1}}{w_i^j}\left(b_{i-1}^{j-1}+p\right)+\alpha_i^j\frac{w_i^{j-1}}{w_i^j}\left(b_i^{j-1}+p\right) \\
&= \left(1-\alpha_i^j\right)\frac{w_{i-1}^{j-1}}{w_i^j}b_{i-1}^{j-1}+\alpha_i^j\frac{w_i^{j-1}}{w_i^j}b_i^{j-1}+\left(\left(1-\alpha_i^1\right)\frac{w_{i-1}}{w_i^1}+\alpha_i^1\frac{w_i}{w_i^1}\right)p \\
&= b_i^j+p
\end{aligned}
$$

is seen easily. So $c_r^d=b_r^d+p$, $c_r^{d-1}=b_r^{d-1}+p$, $c_r^{d-2}=b_r^{d-2}+p$ are obtained. Thus $N_{\gamma_1}=N_{\gamma_2}$ and $\gamma_1(t)$ and $\gamma_2(t)$ form a Bertrand pair. Secondly it is supposed that

$$
\frac{w_{i-1}^{j-1}}{z_{i-1}^{j-1}}=\frac{w_i^j}{z_i^j}=\frac{w_i^{j-1}}{z_i^{j-1}}
$$

be satisfied. So it can be seen easily that the curves $\gamma_1(t)$ and $\gamma_2(t)$ form a Bertrand pair. □

Example 1 Let $\gamma_1(t)$ be an open NURBS curve of degree 3 with control points $b_0=(4,2,2)$, $b_1=(2,1,4)$, $b_2=(3,4,1)$, $b_3=(3,5,5)$, weights $w_0=0.5$, $w_1=0.25$, $w_2=0.75$, $w_3=1$ and knot vectors $t_0=t_1=t_2=0$; $t_3=2$; $t_4=t_5=t_6=3$.

$\gamma_2(t)$ be also an open NURBS curve of degree 3 with control points $c_0=(5,6,4)$, $c_1=(3,5,6)$, $c_2=(4,8,3)$, $c_3=(4,9,7)$, weights $w_0=0.25$, $w_1=0.125$, $w_2=0.375$, $w_3=0.5$ and knot vectors $t_0=t_1=t_2=0$; $t_3=2$; $t_4=t_5=t_6=3$.

These curves are cubic NURBS curves and form a Bertrand pair (see Fig. 1). They can be obtained from the B-spline basis functions as

$$
\gamma_1(t)=\begin{cases}
\dfrac{\left(2\frac{t^2}{3}-3\frac{t}{2}+2, 31\frac{t^2}{48}-3\frac{t}{4}+1, -\frac{t^2}{24}+1\right)}{7\frac{t^2}{48}-\frac{t}{4}+\frac{1}{2}}, & t\in[0,2] \\
\dfrac{\left(\frac{t^2}{6}+\frac{t}{2}, 13\frac{t^2}{12}-5\frac{t}{2}+\frac{11}{4}, 13\frac{t^2}{3}-35\frac{t}{2}+\frac{37}{2}\right)}{\frac{t^2}{12}+\frac{1}{4}}, & t\in[2,3]
\end{cases}
$$

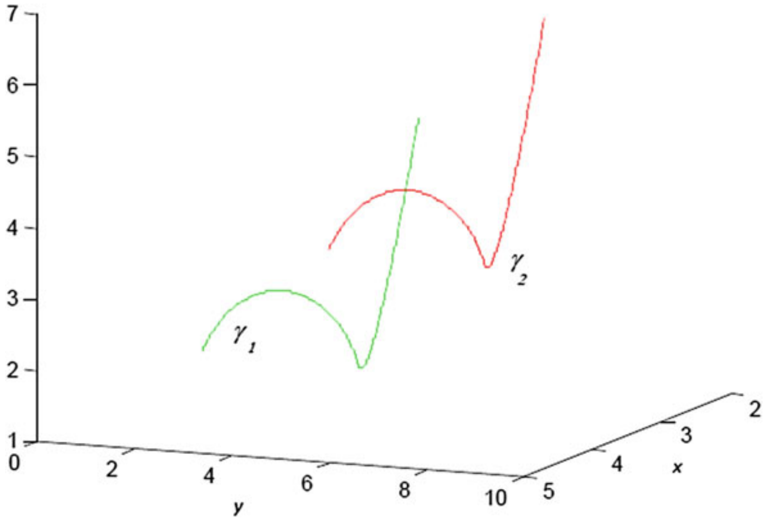


Fig. 1 Bertrand pair of open non-uniform B-spline curves γ_1 and γ_2

and

$$\gamma_2(t) = \begin{cases} \dfrac{\left(13\frac{t^2}{32} - 7\frac{t}{8} + \frac{5}{4}, 59\frac{t^2}{96} - 7\frac{t}{8} + \frac{3}{2}, \frac{t^2}{8} - \frac{t}{4} + 1\right)}{7\frac{t^2}{96} - \frac{t}{8} + \frac{1}{4}}, & t \in [0, 2] \\ \dfrac{\left(\frac{t^2}{8} + \frac{t}{4} + \frac{1}{8}, 17\frac{t^2}{24} - 5\frac{t}{4} + \frac{15}{8}, 9\frac{t^2}{4} - 35\frac{t}{4} + \frac{19}{2}\right)}{\frac{t^2}{24} + \frac{1}{8}}, & t \in [2, 3] \end{cases}$$

References

1. Bertrand, J.: Latheories de courbes a doublecourbure. Journal de Mathematiques Pures et Appliquees **15**, 332–350 (1850)
2. Do Carmo, M.P.: Differential Geometry of Curves and Surfaces. Prentice Hall, Englewood Cliffs, New Jersey (1976)
3. Neill, B.O.: Semi-Riemannian Geometry with Applications to Relativity. Academic Press, New York (1983)
4. Papaioannou, S.G., Kiritsis, D.: An application of Bertrand curves and surface to CAD/CAM. Comput. Aided Des. **17**, 348–352 (1985)
5. Ünal, D., Kişi, İ, Tosun, M.: Spinor Bishop equations of the curves in Euclidean 3- space. Adv. Appl. Clifford Algebras **23**(3), 757–765 (2013)
6. Burke, J.F.: Bertrand curves associated with a pair of curves. Math. Mag. **34**(1), 60–62 (1960)
7. Kucuk, A., Gursoy, O.: On the invariants of Bertrand trajectory surface offsets. Appl. Math. Comput. **151**, 763–773 (2004)
8. Kucuk, A.: On the geometric locus of curvature centrals of the Bertrand curve offsets. Int. J. Pure Appl. Math. **63**, 495–499 (2010)
9. Ravani, B., Ku, T.S.: Bertrand off sets of ruled and developable surfaces. Comput. Aided Des. **23**(2), 145–152 (1991)

10. Izumiya, S., Takeuchi, N.: Generic properties of helices and Bertrand curves. J. Geom. **74**, 97–109 (2002)
11. Balgetir, H., Bektaş, M., Ergüt, M.: Bertrand curves for nonnull curves in three dimensional Lorentzian space. Hadronic J. **27**, 229–236 (2004)
12. Balgetir, H., Bektas, M., Inoguchi, J.I.: Null Bertrand curves in Minkowski 3-space and the ircharacterizations. Notedi matematica **23**(1), 7–13 (2004)
13. Yilmaz, M.Y., Bektaş, M.: General properties of Bertrand curves in Riemann- Otsukispace. Nonlinear Anal. **69**(10), 3225–3231 (2008)
14. Ogrenmis, O., Oztekin, H., Ergut, M.: Bertrand curves in Galilean space and their characterizations. Kragujevac J. Math. **32**, 139–147 (2009)
15. Kazaz, M., Uğurlu, H.H., Önder, M., Oral, S.: Bertrand partner D-curves in Euclidean 3-space. Afyon Kocatepe Univ. J. Sci. Eng. **16**, 76–83 (2016)
16. Choi, J.H., Kang, T.H., Kim, Y.H.: Bertrand curves in 3-dimensional space forms. Appl. Math. Comput. **219**(3), 1040–1046 (2012)
17. Lucas, P., Ortega-Yagües, J.A.: Bertrand curves in the three-dimensional sphere. J. Geom. Phys. **62**(9), 1903–1914 (2012)
18. Tunçer, Y., Ünal, S.: New representations of Bertrand pairs in Euclide an 3-space. Appl. Math. Comput. **219**(4), 1833–1842 (2012)
19. Şenyurt, S., Özgüner, Z.: Bertrand Eğ ri Çiftinin Küresel Göstergelerinin Geodezik Eğrilikleri ve Tabii Liftleri. Ordu Univ. J. Sci. Tech. **3**(2), 58–81 (2013)
20. Yerlikaya, F., Karaahmetoglu, S., Aydemir, I.: On the Bertrand B-paircurves in 3-dimensional euclideanspace. J. Sci. Arts **36**(3), 215–224 (2016)
21. Kızıltuğ, S.: Bertrandand Mannheim Partner-curves on parallel surfaces. Boletim da Sociedade Paranaense de Matem ática **35**(2), 159–169 (2017)
22. Aksoyak, F.K., Gok, I., Ilarslan, K.: Generalized null Bertrand curves in Minkowski spacetime. Annals of the Alexandru Ioan Cuza University-Mathematics **60**(2), 489–502 (2014)
23. Farin, G.: Curvature continuity and offsets for piecewise conics. ACM T. Graph. **8**, 89–99 (1989)
24. Farouki, R.: Exact offsets procedures for simple solids. Comput. Aided. Geom. D. **2**, 257–279 (1985)
25. Farouki, R., Rajan, V.T.: On the numerical condition of polynomials in Bernstein form. Comput. Aided Geom. D. **4**, 191–216 (1987)
26. Hoschek, J.: Offset curves in the plane. Comput. Aided. Des. **17**, 77–82 (1985)
27. Tiller, W., Hanson, E.: Offsets of two-dimensional profiles. IEEE Comput. Graph. **4**, 36–46 (1984)
28. Potmann, H.: Rational curves and surfaces with rational offsets. Comput. Aided. Geom. D. **12**, 175–192 (1995)
29. Incesu, M., Gursoy, O.: Bézier Yüzeylerinde Esas Formlar ve eğrilikler. XVII Ulusal Matematik Sempozyumu 146–157 (2004)
30. Samanci, H.K., Celik, S., Incesu, M.: The Bishop frame of Bézier curves. Life Sci. J. **12**, 175–180 (2015)
31. Samanci, H.K.: Some geometric properties of the spacelike Bézier curve with a timelike principal normal in Minkowski 3-space. Cumhuriyet Sci. J. **39**, 71–79 (2018)
32. Samanci, H.K., Kalkan, O., Celik, S.: The timelike Bézier spline in Minkowski 3-space. J. Sci. Arts **19**, 357–374 (2019)
33. Samancı, H.K.: On curvatures of the Timelike rational Bezier curves in Minkowski 3-space. Bitlis Eren Üniversitesi Fen Bilimleri Dergisi **7**(2), 243–255 (2018)
34. Baydas, S., Karakas, B.: Detecting a curve as a Bézier curve. J. Taibah Univ. Sci. **13**, 522–528 (2019)
35. Incesu, M.: LS (3)-equivalence conditions of control points and application to spatial Bézier curves and surfaces. AIMS Math. **5**(2), 1216–1246 (2020). https://doi.org/10.3934/math.2020084
36. Tiller, W.: Knot-removal algorithms for NURBS curves and surfaces. Comput. Aided Des. **24**(8), 445–453 (1992)

37. Hoschek, J.: Circular splines. Comput. Aided Des. **24**(11), 611–618 (1992)
38. Meek, D.S., Walton, D.J.: Approximating quadratic NURBS curves by arc splines. Comput. Aided Des. **25**(6), 371–376 (1993)
39. Neamtu, M., Pottmann, H., Schumaker, L.L.: Designing NURBS cam profiles using trigonometric splines. J. Mech. Des. **120**(2), 175–180 (1998)
40. Piegl, L.A., ve Tiller, W.: Computing off sets of NURBS curves and surfaces. Comput. Aided Des. **31(2)**, 147–156 (1999)
41. Piegl, L.A., Tiller, W.: Biarcapproximation of NURBS curves. Comput. Aided Des. **34**(11), 807–814 (2002)
42. Liu, L., Wang, G.: Explicitmatrix representation for NURBS curves and surfaces. Comput. Aided Geom. Des. **19**(6), 409–419 (2002)
43. Selimovic, I.: Improved algorithms for the projection of points on NURBS curves and surfaces. Comput. Aided Geom. Des. **23**(5), 439–445 (2006)
44. Samancı, H.K.: Introduction to Timelike uniform B-spline curves in Minkowski 3-space. J. Math. Sci. Modell. **1**(3), 206–210 (2018)
45. Marsh, D.: Applied Geometry for Computer Graphics and CAD. Springer, London (1999)
46. Incesu, M., Evren S.Y.: The Selection of control points for two Open Non Uniform B-Spline Curves to form Bertrand pairs. will be appeared in Tblisi Math. J.
47. Mittal, R.C., Kumar, S., Jiwari, R.: A cubic B-spline quasi-interpolation method for solving two-dimensional unsteady advection diffusion equations. Int. J. Numer. Methods Heat Fluid Flow **30**(9), 4281–4306 (2020)
48. Mittal, R.C., Rohila, R.: The numerical study of advection-diffusion equations by the fourth-order cubic B-spline collocation method. Math. Sci. **14**, 409–423 (2020)
49. Lal, R., Saini, R.: On radially symmetric vibrations of functionally graded non-uniform circular plate including non-linear temperature rise. Eur. J. Mech. - A/Solids **77**, 103796 (2019). https://doi.org/10.1016/j.euromechsol.2019.103796
50. Rohila, R., Mittal, R.C.: An efficient Bi-cubic B-spline ADI method for numerical solutions of two-dimensional unsteady advection diffusion equations. Int. J. Numer. Methods Heat Fluid Flow **28**(11), 2620–2649 (2018). https://doi.org/10.1108/HFF-12-2017-0511
51. Saini, R., Lal, R.: Axisymmetric vibrations of temperature-dependent functionally graded moderately thick circular plates with two-dimensional material and temperature distribution. Eng. Comput. (2020). https://doi.org/10.1007/s00366-020-01056-1
52. Mittal, R.C., Dahiya, S.: A comparative study of modified cubic B-spline differential quadrature methods for a class of nonlinear viscous wave equations. Eng. Comput. **35**(1), 315–333 (2018). https://doi.org/10.1108/EC-06-2016-0188

Convergence Analysis of a Sixth-Order Method Under Weak Continuity Condition with First-Order Fréchet Derivative

Mona Verma, Pooja Sharma, and Neha Gupta

Abstract In this article, a semi-local convergence analysis of a well-established sixth-order method in Banach spaces is discussed. The analysis has been done under the H$\ddot{o}$lder continuity condition with the help of the recurrence relation technique. The relevance of our study lies in the fact that many examples which do not satisfy the Lipschitz continuity but satisfy the H$\ddot{o}$lder continuity. A convergence theorem has been established for the existence-uniqueness of the solution. A priori error bound expression is also derived. Finally, the convergence analysis is carried out on various examples. These examples include Fredholm, Hammerstein integral equation, and a boundary value problem that validated the theoretical development.

Keywords Banach space · Local convergence · Recurrence relation · W-continuity condition

1 Introduction

This article is concerned with the finding of approximate solution x^* of a nonlinear equation

$$P(x) = 0, \tag{1}$$

where $P\colon \Omega \subseteq X_1 \to X_2$ be a Fréchet differentiable operator in an open convex domain Ω with X_1 and X_2 be the Banach spaces. Equation (1) can be found in the form of a system of nonlinear equations such as integral equations and differential equations in various areas of science and engineering. One of the well-known

M. Verma
Govt. P.G. College, Narsinghgarh, (M.P.), India

P. Sharma
Barkatullah University, Bhopal, (M.P.), India

N. Gupta (✉)
Maulana Azad National Institute of Technology, Bhopal, (M.P.), India
e-mail: neha.gupta.mh@gmail.com

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_12

second-order method, namely, Newton's method [1] which is widely applied for solving these types of nonlinear equations. Convergence analysis of an iterative method can be categorized into two parts: local and semi-local convergence analysis. The local convergence analysis of an iterative method provides the knowledge around the solutions whereas the semi-local convergence analysis provides the information around the initial points. Many authors have studied the local and semi-local convergence analysis of the Newton's-like method in their research articles. The semi-local convergence of Newton's method in Banach spaces was initially studied by Kantorovich [2] by using the majorizing technique and also under the recurrence relation method. Later, Rall [3] has done some modifications in it. Many authors have studied various higher order iterative method under the Lipschitz continuity condition [4–6], Hölder continuity condition [7–12]. Few experiments have been performed to analyze the static/dynamic activity of temperature-independent plates and can be seen in [13, 14].

This article deals with the study of semi-local convergence analysis of the sixth-order method which is proposed by Madhu [4] and also the author has discussed the semi-local convergence analysis of this method under the Lipschitz continuity condition in [15] by using the recurrence relation technique. A step ahead, this paper discussed the same method under the weaker continuity condition namely, Hölder continuity condition. The important point to note is that the Hölder continuity condition generalizes the Lipschitz continuity condition and there are many examples for which the methods fail to converge under Lipschitz condition but converges nicely under Hölder condition. Since second-order derivatives are difficult to compute many times or may not even exist hence, we have assumed first-order Fréchet derivative of the operator.

2 Preliminary Conditions

In this segment, we shall mention some preliminary conditions which will be useful for establishing the semi-local convergence of the well-established sixth-order iterative method developed by Madhu [15]. For $x_0 \in \Omega$, the method is given as

$$y_n = x_n - \Gamma_n P(x_n), \quad z_n = y_n - \tau \Gamma_n P(y_n), \quad x_{n+1} = z_n - \tau \Gamma_n P(z_n), \tag{2}$$

where $\Gamma_n = [P'(x_n)]^{-1}$, $\tau = 2I - \Gamma_n P'(y_n)$. The semi-local convergence of the method (2) has been discussed in [4] under the following assumptions:
(A_1) $\|\Gamma_0\| \leq \Phi$,
(A_2) $\|\Gamma_0 P(x_0)\| \leq \Psi$,
(A_3) $\|P'(x) - P'(y)\| \leq K\|x - y\|, \quad x, y \in \Omega, \quad K > 0$.
But, several nonlinear equations that do not satisfy the (A_3) condition. Hence, we relax this assumption by Hölder condition which is weaker than Lipschitz condition. Thus, we have weakened the hypothesis (A_3) with:
(A_4) $\|P'(x) - P'(y)\| \leq K\|x - y\|^q, \quad x, y \in \Omega, \quad q \in (0, 1]$.

Define the auxilliary scalar functions:

$$\phi_1(x) = \frac{1}{1 - x\,(1 + \phi_3(x))^q}, \tag{3}$$

$$\phi_2(x) = \frac{1}{q+1}x + (x+1)\phi_3(x) + \frac{1}{q+1}x\phi_3(x)^{q+1}, \tag{4}$$

$$\phi_3(x) = \frac{2x + 3x^2 + x^3}{q+1} + \frac{x(1+x)}{q+1}\left(1 + \frac{x + x^2}{q+1}\right)^{q+1}. \tag{5}$$

Let $j(x) = x(1 + \phi_3(x))^q - 1$. Since, $j(0) < 0$ and $j(1) > 0$ then it follows that, there exists a positive real root of $j(x) = 0$ in $(0, 1)$, say ρ_0 and clearly, $\rho_0 < 1$.

To investigate the convergence of the sequence $\{x_n\}$, we need to prove that $\{x_n\}$ to be a Cauchy sequence. For this, we shall analyze the characteristics of the real functions $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$ described in the Eqs. (3)–(5), respectively, and some technical lemmas have been included which can be used to prove the convergence theorem.

2.1 Lemma

Let $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$ be the functions mentioned in the Eqs. (3)–(5) and ρ_0 is the minimum root of $j(x) = 0$ in $(0, 1)$. Then the following holds:
(i) ϕ_1 is an increasing function such that $\phi_1(x) > 1$ for $x \in (0, \rho_0)$;
(ii) ϕ_2 and ϕ_3 are also increasing functions for $x \in (0, \rho_0)$;
(iii) For $\theta \in (0, 1), x \in [0, 1)$ we have $\phi_1(\theta x) < \phi_1(x), \phi_3(\theta x) < \phi_3(x)$ and $\phi_2(\theta x) < \theta\phi_2(x)$. Define the sequences

$$\Psi_{n+1} = d_n \Psi_n, \tag{6}$$

$$\Phi_{n+1} = \phi_1(p_n)\Phi_n, \tag{7}$$

$$p_{n+1} = K\Phi_{n+1}\Psi_{n+1}^q, \tag{8}$$

$$d_{n+1} = \phi_1(p_{n+1})\phi_2(p_{n+1}), \tag{9}$$

where $n \geq 0$. Choose $\Psi_0 = \Psi$, $\Phi_0 = \Phi$, $p_0 = K\Phi\Psi^q$ and $d_0 = \phi_1(p_0)\phi_2(p_0)$. Then from the above-mentioned definition it follows that

$$p_{n+1} = \phi_1(p_n)d_n^q p_n = p_n\phi_1(p_n)^{q+1}\phi_2(p_n)^q. \tag{10}$$

2.2 Lemma

Consider the auxiliary function

$$m(x) = \left(\frac{x}{q+1} + (1+x)\phi_3(x) + \frac{x\phi_3(x)^{q+1}}{q+1}\right)^q - \left(1 - x(1+\phi_3(x))^q\right)^{q+1},$$

such that $m(0) < 0$, $m(\rho_0) > 0$ and $m'(x) > 0$. Thus, $m(x)$ is an increasing function and has a root ρ_1 in $(0, \rho_0)$.

2.3 Lemma

Let $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$ be the functions defined by (3)–(5), respectively. If $p_0 \in (0, \rho_1)$, then
(i) $\phi_1(p_0)^{1+q}\phi_2(p_0)^q < 1$;
(ii) $\phi_1(p_0)\phi_2(p_0) < 1$;
(iii) the sequence $\{p_n\}$ is decreasing for all $n \geq 0$;
(iv) $p_n < 1$ for all $n \geq 0$;
(v) $p_n(1 + \phi_3(p_n))^q < 1$.

Proof (i) Taking $x = p_0$ in $m(x)$, we get $\phi_1(p_0)^{q+1}\phi_2(p_0)^q < 1 \quad \forall\ p_0 \in (0, \rho_1)$.
(ii) Since, $\phi_1(p_0) > 1$, this gives $(\phi_1(p_0)\phi_2(p_0))^q < 1$, and hence, $\phi_1(p_0)\phi_2(p_0) < 1$.
(iii) This part can be proved by using mathematical induction on the Eq. (10).
For $n = 0$, we have $p_1 = p_0\phi_1(p_0)^{q+1}\phi_2(p_0)^q < p_0$.
Assume that $p_k < p_{k-1}$, for $k \leq n$. Since, ϕ_1 and ϕ_2 are increasing function, we get

$$p_{n+1} = p_n\phi_1(p_n)^{q+1}\phi_2(p_n)^q < p_{n-1}\phi_1(p_{n-1})^{q+1}\phi_2(p_{n-1})^q < p_n,$$

hence, the sequence $\{p_n\}$ is a decreasing sequence.

(iv) To see this part hold, we have $p_n < p_0 < 1$, for all $n \geq 0$ and from the fact that $\{p_n\}$ is strictly decreasing sequence and $p_0 \in (0, \rho_1)$.

(v) Since, $p_n < p_{n-1}$ and $\phi_3(x)$ is an increasing function, for all $p_0 \in (0, \rho_1)$ we get

$$p_n(1 + \phi_3(p_n))^q < p_{n-1}(1 + \phi_3(p_{n-1}))^q < p_0(1 + \phi_3(p_0))^q < 1.$$

2.4 Lemma

Let the conditions of Lemma 2.3 hold, if $p_0 \in (0, \rho_1)$ and let $\gamma = \frac{p_1}{p_0}$, then
(i) $p_n < \gamma^{[(1+q)^n-1]/q} p_0, \ \ for \ n \geq 2;$
(ii) $\phi_1(p_n)\phi_2(p_n) < \gamma^{[(1+q)^n-1]/q}\phi_1(p_0)\phi_2(p_0) = \gamma^{(1+q)^{n}/q}/\phi_1(p_0)^{1/q}, \ \ for \ \ n \geq 1;$
and if $p_0 = \rho_1$, then

$$\phi_1(p_n)\phi_2(p_n) = \phi_1(p_0)\phi_2(p_0) = 1/\phi_1(p_0)^{1/q}, \forall\, n \geq 1.$$

Proof The case when $p_0 \in (0, \rho_1)$. The proof of (i) can be done by mathematical induction. For $n = 2$ in Eq. (10) and by Lemma 2.1 (iii), we have

$$p_2 = p_1\phi_1(p_1)^{1+q}\phi_2(p_1)^q = \gamma p_0\phi_1(\gamma p_0)^{1+q}\phi_2(\gamma p_0)^q < \gamma^{[(1+q)^2-1]/q} p_0.$$

We suppose now that

$$p_{n-1} < \gamma^{[(1+q)^{n-1}-1]/q} p_0.$$

Then, by using the induction hypotheses, we have

$$\begin{aligned} p_n &= p_{n-1}\phi_1(p_{n-1})^{1+q}\phi_2(p_{n-1})^q \\ &< \gamma^{[(1+q)^{n-1}-1]/q} p_0\phi_1(\gamma^{[(1+q)^{n-1}-1]/q} p_0)^{1+q}\phi_2(\gamma^{[(1+q)^{n-1}-1]/q} p_0)^q \\ &< \gamma^{[(1+q)^n-1]/q} p_0. \end{aligned}$$

To prove (ii), we see that

$$\begin{aligned} \phi_1(p_n)\phi_2(p_n) &< \phi_1(\gamma^{[(1+q)^n-1]/q} p_0)\phi_2(\gamma^{[(1+q)^n-1]/q} p_0) \\ &< \gamma^{[(1+q)^n-1]/q}\phi_1(p_0)\phi_2(p_0) \\ &< \gamma^{(1+q)^n/q}/\phi_1(p_0)^{1/q}. \end{aligned}$$

The case when $p_0 = \rho_1$ follows by analogy.

3 Recurrence Relations

The recurrence relations for the method (2) have established under the hypotheses (A_1), (A_2) and (A_4) already mentioned in the previous section. By assuming $n = 0$ in the method (2), we get

$$\|y_0 - x_0\| \leq \Psi. \tag{11}$$

Now, from the second sub-step of the method (2) we have

$$z_0 - x_0 = -\Gamma_0 P(x_0) - \tau \Gamma_0 P(y_0). \tag{12}$$

Using Taylor's expansion of $P(y_0)$ along x_0, we get

$$P(y_0) = P(x_0) + P'(x_0)(y_0 - x_0) + \int_0^1 \left[P'(x_0 + t(y_0 - x_0)) - P'(x_0)\right] dt (y_0 - x_0),$$

and thus,

$$\Gamma_0 P(y_0) = \Gamma_0 \int_0^1 \left[P'(x_0 + t(y_0 - x_0)) - P'(x_0)\right] dt (y_0 - x_0).$$

By using the norm in the Eq. (12), we have

$$\begin{aligned}\|z_0 - x_0\| &\leq \|y_0 - x_0\| + \frac{\Gamma_0 K}{q+1}\|y_0 - x_0\|^{q+1} + \frac{\Gamma_0^2 K^2}{q+1}\|y_0 - x_0\|^{2q+1}\\ &\leq \Psi + \frac{\Gamma_0 K}{q+1}\Psi^{q+1} + \frac{\Gamma_0^2 K^2}{q+1}\Psi^{2q+1}\\ &\leq \Psi\left(1 + \frac{p_0 + {p_0}^2}{q+1}\right).\end{aligned}$$

By using triangle inequality, we obtain

$$\|z_0 - y_0\| \leq \frac{1}{q+1}\left(p_0 + p_0^2\right)\Psi.$$

Now, from the final sub-step of the Eq. (2), we obtain

$$\|x_1 - x_0\| \leq \|z_0 - y_0\| + \|y_0 - x_0\| + \tau\|\Gamma_0 P(z_0)\|. \tag{13}$$

Further,

$$\tau\Gamma_0 P(z_0) = -\tau^2\Gamma_0 P(y_0) + \tau\Gamma_0 \int_0^1 \left[P'(x_0 + t(z_0 - x_0)) - P'(x_0)\right] dt (z_0 - x_0),$$

On taking norm both sides,

$$\begin{aligned}\|\tau\Gamma_0 P(z_0)\| \ &\leq \left(1 + \Phi K\|y_0 - x_0\|^q\right)\frac{\Phi K}{q+1}\|z_0 - x_0\|^{q+1} + \left(1 + \Phi K\|y_0 - x_0\|^q\right)^2\\ &\times\left(\frac{\Phi K}{q+1}\|y_0 - x_0\|^{q+1}\right).\end{aligned}$$

On using the above equation in (13), then we have

$$\|x_1 - x_0\| \le \left[1 + \frac{p_0 + p_0^2}{q+1} + \frac{p_0 + 2p_0^2 + p_0^3}{q+1} + \frac{p_0 + p_0^2}{q+1}\left(1 + \frac{p_0 + p_0^2}{q+1}\right)^{q+1}\right]\Psi$$
$$\le \big(1 + \phi_3(p_0)\big)\Psi.$$

3.1 Lemma

Let all the hypotheses of the Lemma 2.1, 2.2 hold and the assumptions (A_1), (A_2), and (A_4) are true, then for all $n \ge 0$ the following inequalities are hold:
(I_n) $\|\Gamma_n\| \le \phi_1(p_{n-1})\|\Gamma_{n-1}\|$,
(II_n) $\|y_n - x_n\| \le \|\Gamma_n P(x_n)\| \le \phi_1(p_{n-1})\phi_2(p_{n-1})\|y_{n-1} - x_{n-1}\|$,
(III_n) $\|z_n - y_n\| \le \left(\frac{p_n + p_n^2}{q+1}\right)\|y_n - x_n\|$,
(IV_n) $K\|\Gamma_n\|\|y_n - x_n\|^q \le p_n$,
(V_n) $\|x_n - x_{n-1}\| \le \big(1 + \phi_3(p_{n-1})\big)\|y_{n-1} - x_{n-1}\|$.

Proof (I_1) : On assuming that $x_1, y_1, z_1 \in \Omega$ and $p_0 < \rho_1$ then

$$\|I - \Gamma_0 P'(x_1)\| \le \|\Gamma_0\|\|P'(x_1) - P'(x_0)\|$$
$$\le \Phi K\|x_1 - x_0\|^q$$
$$\le p_0\big(1 + \phi_3(p_0)\big)^q < 1.$$

By Banach lemma, one can deduce that Γ_1 exists and

$$\|\Gamma_1\| \le \frac{1}{1 - p_0\big(1 + \phi_3(p_o)\big)^q}\|\Gamma_0\| \le \phi_1(p_0)\|\Gamma_0\|. \tag{14}$$

(II_1): Using Taylor's expansion, we have

$$P(x_1) = \int_0^1 \big[P'(x_0 + t(y_0 - x_0) - P'(x_0)\big]dt(y_0 - x_0) + P'(y_0)(x_1 - y_0)$$
$$+ \int_0^1 \big[P'(y_0 + t(x_1 - y_0) - P'(y_0)\big]dt(x_1 - y_0).$$

On taking norm both sides, we have

$$\|P(x_1)\| \le \|P'(y_0)\|\|x_1 - y_0\| + \int_0^1 t^q K\|y_0 - x_0\|^q dt\|y_0 - x_0\| + \int_0^1 t^q K\|x_1 - y_0\|^q dt\|x_1 - y_0\|$$
$$\le \left(\frac{K}{q+1}\Psi^q + \frac{K}{q+1}[\phi_3(p_0)^{q+1}\Psi^q] + K\Psi^q\phi_3(p_0) + \frac{\phi_3(p_0)}{\Phi}\right)\|y_0 - x_0\|.$$

Therefore,

$$\begin{aligned}\|y_1 - x_1\| &\le \phi_1(p_0)\|\Gamma_0 P(x_1)\| \\ &\le \phi_1(p_0)\left[\frac{p_0}{q+1} + (1+p_0)\phi_3(p_0) + \frac{p_0}{q+1}\phi_3(p_0)^{q+1}\right]\Psi \\ &\le \phi_1(p_0)\phi_2(p_0)\|y_0 - x_0\|. \end{aligned} \tag{15}$$

(III_1): From the second sub-step of the method (2), we obtain

$$\begin{aligned}\|z_1 - y_1\| &\le \left(1 + \|\Gamma_0\| K\phi_1(p_0)\|y_1 - x_1\|^q\right)\phi_1(p_0)\|\Gamma_0\|\frac{K}{q+1}\|y_1 - x_1\|^{q+1} \\ &\le \frac{\Phi K}{q+1}(1 + \Phi K\phi_1(p_0)\|y_1 - x_1\|^q)\phi_1(p_0)\|y_1 - x_1\|^{q+1} \\ &\le \frac{p_1}{q+1}(1+p_1)\|y_1 - x_1\|. \end{aligned} \tag{16}$$

(IV_1): On using (I_1) and (II_1), we have

$$\begin{aligned}K\|\Gamma_1\|\|y_1 - x_1\|^q &\le K\phi_1(p_0)\|\Gamma_0\|\phi_1(p_0)^q\phi_2(p_0)^q\|y_0 - x_0\|^q \\ &\le p_0\phi_1(p_0)^{q+1}\phi_2(p_0)^q \le p_1. \end{aligned} \tag{17}$$

For $n = 1$, the recurrence relation $(I_1) - (IV_1)$ follows from Eqs. (14)–(17).

(V_1): From the Eq. (14) we have that

$$\|x_1 - x_0\| \le \big(1 + \phi_3(p_0)\big)\|\Gamma_0 P(x_0)\| \le \big(1 + \phi_3(p_0)\big)\|y_0 - x_0\|.$$

Assuming that (I_n)-(V_n) hold for $n = k$ and $x_k, y_k, z_k \in \Omega$. In a similar manner, it can be seen that (I_n)-(V_n) hold for $n = k + 1$. Hence, recurrence relations hold for $n \ge 1$. □

4 Semi-local Convergence

In this segment, the semi-local convergence theorem of the method (2) with a (K, q)-Hölder continuity is established by using technical lemmas and value of the norms defined in the previous section and the error bounds for it are also obtained. Let us assume, $\gamma = p_1/p_0$ and $\Delta = \frac{1}{\phi_1(p_0)^{1/q}}$, $U(x, r) = \{y \in X_1 : \|y - x\| < r\}$, $\overline{U(x, r)} = \{y \in X_1 : \|y - x\| \le r\}$. We shall now state the following theorem.

Theorem 1 *Let X_1, X_2 be Banach spaces and $P : \Omega \subseteq X_1 \to X_2$ be a nonlinear Fréchet differentiable operator in an open convex domain Ω. Let $\Gamma_0 = [P'(x_0)]^{-1}$ exists and (A_1), (A_2) and (A_4) assumptions hold. Let $p_0 = K\Phi\Psi^q$, suppose that $p_o < \rho_1$, $p_0(1 + \phi_3(p_0))^q < 1$ and $U(x_0, R\Psi) \subset \Omega$, where $R =$*

$\frac{p_0+p_0^2}{q+1}+\frac{1+\phi_3(p_0)}{1-\phi_1(p_0)\phi_2(p_0)}$ *then starting with* x_0, *generates the sequence* $\{x_n\}$ *defined in* (2) *converges to a solution* x^* *and it is the unique solution of* $P(x)=0$ *in* $U(x_0,r)\cap\Omega$ *with* $r=\left(\frac{1+q}{K\Phi}-(R\Psi)^q\right)^{1/q}$. *Furthermore, an error bound expression is given as*

$$\|x^*-x_n\|\le(1+\phi_3(p_0))\gamma^{[(1+q)^n-1]/q^2}\left[\frac{\Delta^n}{1-\gamma^{(1+q)^n/q}\Delta}\right]\Psi,\ n\ge 0. \tag{18}$$

Proof First of all, we will prove that y_n and z_n belong to $U(x_0,R\Psi)\subset\Omega$. By recurrence relation (V_n), it is easy to see that

$$\begin{aligned}\|x_n-x_0\| &\le \|x_n-x_{n-1}\|+\|x_{n-1}-x_{n-2}\|+\cdots+\|x_1-x_0\|\\ &\le(1+\phi_3(p_{n-1}))\|y_{n-1}-x_{n-1}\|+(1+\phi_3(p_{n-2}))\|y_{n-2}-x_{n-2}\|\\ &\quad+\cdots+(1+\phi_3(p_0))\|y_0-x_0\|.\end{aligned}$$

Since, $\phi_3(x)$ is an increasing function and p_n is a decreasing sequence, this gives

$$\|x_n-x_0\|\le(1+\phi_3(p_0))\|y_0-x_0\|\sum_{k=0}^{n-1}\big(\phi_1(p_0)\phi_2(p_0)\big)^k. \tag{19}$$

Now, using recurrence relation (II_n) and Eq. (19), we get

$$\begin{aligned}\|y_n-x_0\| &\le \|y_n-x_n\|+\|x_n-x_0\|\\ &\le \big(\phi_1(p_0)\phi_2(p_0)\big)^n\|y_0-x_0\|+(1+\phi_3(p_0))\|y_0-x_0\|\sum_{k=0}^{n-1}(\phi_1(p_0)\phi_2(p_0))^k\\ &\le \big(1+\phi_3(p_0)\big)\frac{1-\big(\phi_1(p_0)\phi_2(p_0)\big)^{n+1}}{1-\phi_1(p_0)\phi_2(p_0)}\Psi < R\Psi.\end{aligned} \tag{20}$$

By applying recurrence relations (I_n) and (II_n), we have

$$\begin{aligned}\|z_n-y_n\| &\le \|2I-\Gamma_nP'(y_n)\|\|\Gamma_n\|\|P(y_n)\|\\ &\le \frac{\Phi K}{q+1}(\phi_1(p_0))^n\big(1+\Phi K(\phi_1(p_0))^n\|y_n-x_n\|^q\big)\|y_n-x_n\|^{q+1}\\ &\le \frac{p_0}{q+1}\big[1+p_0(\phi_1(p_0)^{q+1}\phi_2(p_0)^q)^n\big]\big(\phi_1(p_0)^{q+2}\phi_2(p_0)^{q+1}\big)^n\|y_0-x_0\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|z_n - x_0\| &\le \|z_n - y_n\| + \|y_n - x_0\| \\ &\le \frac{p_0}{q+1}\left(1 + p_0(\phi_1(p_0)^{q+1}\phi_2(p_0)^q)^n\right)\big(\phi_1(p_0)^{q+2}\phi_2(p_0)^{q+1}\big)^n\|y_0 - x_0\| \\ &\quad +\big(1+\phi_3(p_0)\big)\frac{1-(\phi_1(p_0)\phi_2(p_0))^{n+1}}{1-\phi_1(p_0)\phi_2(p_0)}\|y_0 - x_0\| \\ &\le \left(\frac{p_0 + p_0^2}{q+1} + \big(1+\phi_3(p_0)\big)\frac{1-\big(\phi_1(p_0)\phi_2(p_0)\big)^{n+1}}{1-\phi_1(p_0)\phi_2(p_0)}\right)\Psi < R\Psi.\end{aligned}$$

Hence, y_n and $z_n \in U(x_0, R\Psi)$. Now,

$$\begin{aligned}\|x_{n+1} - x_n\| &\le (1+\phi_3(p_n))\|y_n - x_n\| \\ &\le \big(1+\phi_3(p_n)\big)\phi_1(p_{n-1})\phi_2(p_{n-1})\|y_{n-1} - x_{n-1}\| \\ &\;\vdots \\ &\le \big(1+\phi_3(p_n)\big)\prod_{j=0}^{n-1}\phi_1(p_j)\phi_2(p_j)\|y_0 - x_0\|. \end{aligned} \tag{21}$$

For the convergence of the sequence $\{x_n\}$, we need to prove the sequence $\{x_n\}$ to be a Cauchy sequence. For this, consider

$$\begin{aligned}\|x_{n+m} - x_n\| &\le \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\ &\le \big(1+\phi_3(p_{n+m-1})\big)\Psi\prod_{j=0}^{n+m-2}\phi_1(p_j)\phi_2(p_j) \\ &\quad + \cdots + \big(1+\phi_3(p_n)\big)\Psi\prod_{j=0}^{n-1}\phi_1(p_j)\phi_2(p_j).\end{aligned}$$

This gives

$$\begin{aligned}\|x_{n+m} - x_n\| &\le \big(1+\phi_3(p_0)\big)\Psi\left[\sum_{l=0}^{m-1}\prod_{j=0}^{n+l-1}\phi_1(p_j)\phi_2(p_j)\right] \\ &\le \big(1+\phi_3(p_0)\big)\Psi\sum_{l=0}^{m-1}(\phi_1(p_0)\phi_2(p_0))^{l+n}.\end{aligned} \tag{22}$$

Since, functions, ϕ_1 and ϕ_2 are also increasing, therefore,

$$\begin{aligned}\|x_{n+m} - x_n\| &\le \big(1+\phi_3(p_0)\big)\big(\phi_1(p_0)\phi_2(p_0)\big)^n\Psi\sum_{l=0}^{m-1}\big(\phi_1(p_0)\phi_2(p_0)\big)^l \\ &\le \big(1+\phi_3(p_0)\big)\big(\phi_1(p_0)\phi_2(p_0)\big)^n\frac{1-\big(\phi_1(p_0)\phi_2(p_0)\big)^m}{1-\phi_1(p_0)\phi_2(p_0)}\Psi.\end{aligned}$$

Therefore, the sequence $\{x_n\}$ is a Cauchy sequence if $\phi_1(p_0)\phi_2(p_0) < 1$ and hence convergent. For $n = 0$ we get

$$\|x_m - x_0\| \leq \big(1 + \phi_3(p_0)\big)\frac{1 - \big(\phi_1(p_0)\phi_2(p_0)\big)^m}{1 - \phi_1(p_0)\phi_2(p_0)}\Psi < R\Psi. \tag{23}$$

Therefore, $x_m \in \overline{U(x_0, R\Psi)}$ and taking $m \to \infty$ in (23), we get $x^* \in \overline{U(x_0, R\Psi)}$. For $n \geq 1, m \geq 1$, invoking (II_n) and Lemma 2.3 (ii), in Eq. (22), we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq (1 + \phi_3(p_0))\Psi\left[\sum_{l=0}^{m-1}\prod_{j=0}^{n+l-1}\gamma^{[(1+q)^j-1]/q}\phi_1(p_0)\phi_2(p_0)\right]\\ &\leq (1 + \phi_3(p_0))\Psi\left[\sum_{l=0}^{m-1}\prod_{j=0}^{n+l-1}\gamma^{(1+q)^j/q}\Delta\right]\\ &\leq (1 + \phi_3(p_0))\Psi\left[\sum_{l=0}^{m-1}\gamma^{[(1+q)^{n+l}-1]/q^2}\Delta^{n+l}\right].\end{aligned}$$

By using Bernoulli's inequality, we have

$$\begin{aligned}\gamma^{[(1+q)^{n+l}-1]/q^2]} &= \gamma^{[(1+q)^n-1]/q^2}\gamma^{[(1+q)^n/q^2][(1+q)^l-1]}\\ &\leq \gamma^{[(1+q)^n-1]/q^2}\gamma^{[(1+q)^n/q]l},\end{aligned}$$

and consequently,

$$\begin{aligned}\|x_{n+m} - x_n\| &< (1 + \phi_3(p_0))\Psi\Delta^n\left[\sum_{l=0}^{m-1}\gamma^{[(1+q)^n/q]l}\Delta^l\right]\gamma^{[(1+q)^n-1]/q^2}\\ &< (1 + \phi_3(p_0))\Psi\Delta^n\left[\frac{1 - (\gamma^{(1+q)^n/q}\Delta)^m}{1 - \gamma^{(1+q)^n/q}\Delta}\right]\gamma^{[(1+q)^n-1]/q^2}.\end{aligned} \tag{24}$$

Finally, let $m \to \infty$ in (24) then we get (18) for all $n \geq 0$. Now, it is to be shown, that x^* is a solution of $P(x)$. Next, from (2) we have

$$\begin{aligned}\|P(x_n)\| &\leq \|P'(x_n)\|\|y_n - x_n\|\\ &\leq \big(\phi_1(p_0)\phi_2(p_0)\big)^n\|P'(x_n)\|\|y_0 - x_0\|,\end{aligned} \tag{25}$$

and

$$\begin{aligned}\|P'(x_n)\| &\leq \|P'(x_0)\| + \|P'(x_n) - P'(x_0)\| \\ &\leq \|P'(x_0)\| + K\|x_n - x_0\|^q \\ &\leq \|P'(x_0)\| + K(R\Psi)^q. \end{aligned} \tag{26}$$

Therefore, $\|P'(x_n)\|$ is bounded and hence, by tending $n \to \infty$ in the Eq. (25) one can deduce that $\|P(x_n)\| \to 0$. By continuity of P in Ω, $P(x^*) = 0$. Now, for the uniqueness, let there exists $y^* \in U(x_0, r)$ s.t. $P(y^*) = 0$ and $y^* \neq x^*$. Then,

$$0 = P(y^*) - P(x^*) = \int_0^1 P'(x^* + t(y^* - x^*)dt(y^* - x^*) = F\big(y^* - x^*\big).$$

Now, we claim that the inverse of the operator $F = \int_0^1 P'\big(x^* + t(y^* - x^*)\big)dt$ exists which leads to the conclusion that $y^* = x^*$. Consider

$$\begin{aligned}\|I - \Gamma_0 F\| &\leq \|\Gamma_0\| \int_0^1 \|P'(x^* + t(y^* - x^*) - P'(x_0)\|dt \\ &\leq \frac{K\Phi}{q+1}\Big(\|x^* - x_0\|^q + \|y^* - x_0\|^q\Big) \\ &\leq \frac{K\Phi}{q+1}\left[R^q\Psi^q + \frac{1+q}{K\Phi} - R^q\Psi^q\right] \\ &\leq \frac{K\Phi}{q+1}\left[\frac{1+q}{K\Phi}\right] < 1,\end{aligned}$$

then by the Banach Lemma for the invertible operator F is invertible and hence, we can conclude that $y^* = x^*$. □

5 Numerical Applications

In this segment, some numerical applications have been mentioned to show the efficacy of our approach.

5.1 Example

Let the nonlinear integral equation of the form

$$P(a(x)) = a(x) - 1 - \frac{1}{3}\int_0^1 G(x, y)(a(y))^{1+q}dy, \quad x \in [0, 1], \tag{27}$$

where G is the continuous and non-negative kernel defined in $[0, 1] \times [0, 1]$ and is given as

$$G(x, y) = \begin{cases} (1-x)y & y \leq x, \\ x(1-y) & x \leq y. \end{cases}$$

The Fréchet derivative of Eq. (27) is given as

$$P'(a)b(x) = b(x) - \frac{(1+q)}{3}\int_0^1 G(x, y)(a(y))^q dy, \quad b \in [0, 1], q \in (0, 1].$$

Using max-norm, we get

$$\begin{aligned} \|P'(a) - P'(b)\| &\leq \frac{(1+q)}{3} \max_{x\in[0,1]} \left| \int_0^1 G(x, y)dy \right| \|a(y)^q - b(y)^q\| \\ &\leq \frac{(1+q)}{24}\|a-b\|^q. \end{aligned}$$

Thus, the Lipschitz condition (A_3) fails here for $q \in (0, 1]$ but the hypothesis (A_4) holds. Now,

$$\|I - P'(a_0)\| \leq \frac{1+q}{24}\|a_0\|^q.$$

If $\frac{1+q}{24}\|a_0\|^q < 1$, then by Banach Lemma, we obtain

$$\|\Gamma_0\| \leq \frac{1}{1 - \frac{1+q}{24}\|a_0\|^q} = \Phi.$$

Similarly, for $a_0 = a_0(x) = 1$, we obtain

$$\|P(a_0)\| \leq \frac{1}{24}, \quad \|\Gamma_0 P(a_0)\| \leq \frac{1}{(23-q)} = \Psi.$$

Now, for $q = 0.85$, we get $p_0 = 0.0060010926 < 0.17691956 = \rho_1$ and $p_0(1 + \phi_3(p_0))^q = 0.0060511939 < 1$. Therefore, the conditions of the Theorem 1 fulfilled. Hence, the existence of a^* is guaranteed in $\overline{U(a_0, 0.046348246)}$ and the uniqueness in $U(a_0, 38.217851) \cap \Omega$.

Notice that if we consider $q = 1$ in (27), we are getting

$$P(a(x)) = a(x) - 1 - \frac{1}{3}\int_0^1 G(x, y)\big(a(y)\big)^2 dy.$$

Therefore,

$$\|P'(a) - P'(b)\| \leq \frac{1}{12}\|a-b\|.$$

Again on assuming, $a_0 = a_0(x) = 1$, we obtain $p_0 = 0.0041322314 < 0.018598403 = \rho_1$ and $p_0(1 + \phi_3(p_0))^q = 0.0041580213 < 1$. Hence, the existence of a^* is guaranted in $\overline{U(a_0, 0.04621850)}$ and the uniqueness in $U(a_0, 21.953781) \cap \Omega$.
Thus, the existence and uniqueness domains of convergence balls have been improved under Hölder condition as compared with Lipschitz condition.

5.2 Example

Consider a nonlinear integral equation defined as

$$P(a)(x) = a(x) - 1 - \frac{1}{4}\int_0^1 \frac{x}{x+y}(a(y))^{1+q}dy, \quad x \in [0, 1], \text{ and } a \in C[0, 1]. \tag{28}$$

The Fréchet derivative of P is given as

$$P'(a)b(x) = b(x) - \frac{1+q}{4}\int_0^1 \frac{x}{x+y}(a(y))^q dy, \quad b \in [0, 1].$$

Using max-norm, we get

$$\begin{aligned}\|P'(a) - P'(b)\| &\le \frac{1+q}{4}\max_{x\in[0,1]}\left|\int_0^1 \frac{x}{x+y}dy\right| \|a(y)^q - b(y)^q\| \\ &\le \frac{1+q}{4}\log 2\|a - b\|^q.\end{aligned}$$

Thus, the Lipschitz condition (A_3) fails here for $q \in (0, 1]$ but the hypothesis (A_4) hold. Now,

$$\|I - P'(a_0)\| \le \frac{1+q}{4}\log 2\|a_0\|^q.$$

If $\frac{1+q}{4}\log 2\|a_0\|^q < 1$ then by Banach Lemma, we obtain

$$\|\Gamma_0\| \le \frac{1}{1 - \frac{1+q}{4}\log 2\|a_0\|^q} = \Phi.$$

Hence,

$$\|\Gamma_0 P(a_0)\| \le \frac{\|a_0 - 1\| + \frac{\log 2}{4}\|a_0\|^{1+q}}{1 - \frac{1+q}{4}\log 2\|a_0\|^q} = \Psi.$$

Now, for $q = 0.7$ and $a_0 = a_0(x) = 1$, we get $p_0 = 0.15631365 < 0.166432 = \rho_1$ and $p_0(1 + \phi_3(p_0))^q = 0.19340562 < 1$. Therefore, conditions of the Theorem 1 are fulfilled. Therefore, the solution exists in $\overline{U(a_0, 0.95963051)}$ and the unique in

$U(a_0, 6.4700146) \cap \Omega$.
Notice that if we consider $q = 1$ in (28), we are getting

$$P(a)(x) = a(x) - 1 - \frac{1}{4}\int_0^1 \frac{x}{x+y}(a(y))^2 dy.$$

Therefore,

$$\|P'(a) - P'(b)\| \leq \frac{1}{2}\log 2\|a - b\|.$$

Again, letting $a_0 = a_0(x) = 1$, then we get $p_0 = 0.14065901 < 0.18598403 = \rho_1$ and $p_0(1 + \phi_3(p_0))^q = 0.17798110 < 1$. Therefore, conditions of the Theorem 1 are fulfilled. Hence, the solution lies in $\overline{U(a_0, 0.64242555)}$ and it is unique in $U(a_0, 3.1283546) \cap \Omega$.
Thus, the existence and uniqueness domains of convergence balls are bigger in the case of Hölder condition.

5.3 Example

Consider the following nonlinear BVP

$$a'' + a^{1+q} = 0, \quad q \in (0, 1], \quad a(0) = a(1) = 0. \tag{29}$$

The interval [0, 1] is divided into N sub-intervals with points $t_i = ih, i = 0, 1, \ldots, N$ where $h = \frac{1}{N}$. Approximating the second derivative in (29) by central difference scheme by

$$a_i'' \approx \frac{a_{i-1} - 2a_i + a_{i+1}}{h^2}, \quad i = 1, 2, \ldots, N-1,$$

then we get

$$-a_{i-1} + 2a_i - a_{i+1} - h^2 a_i^{1+q} = 0. \tag{30}$$

This can be expressed as $P(a) = H(a) - h^2 l(a) = 0$ where $P : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$, $a = (a_1, a_2, \ldots, a_{N-1})^t$, $l(a) = ({a_1}^{1+q}, {a_2}^{1+q}, \ldots, {a_{N-1}}^{1+q})^t$ and the matrix H is given by

$$H = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}.$$

Here,

$$P'(a) = H - (1+q)h^2 M(a),$$

where

$$M(a) = diag\{a_1^q, a_2^q, \ldots\ldots, a_{N-1}^q\},$$

and

$$\|P'(a) - P'(b)\| \leq (1+q)h^2\|a-b\|^q. \tag{31}$$

Letting $q = \frac{1}{2}$, $h = \frac{1}{10}$ and on assuming the initial approximation as

$$a_0 = (33.5739, 65.2025, 91.566, 109.168, 115.363, 109.168, 91.566, 65.2025, 33.5739)^t$$

we get $K = 0.015$, $\Phi = 26.5888$ and $\Psi = 3.7570 \times 10^{-4}$. Hence, $p_0 = K\Phi\Psi^{\frac{1}{2}} = 0.0077305536 < 0.14936797 = \rho_1$ and $p_0(1+\phi_3(p_0))^q = 0.0077908536 < 1$. Therefore, the conditions of Theorem 1 are fulfilled. Hence, the existence of a^* is guaranteed in $\overline{U(a_0, 0.00039176436)}$ and the uniqueness in $U(a_0, 14.144594) \cap \Omega$. Notice that if we consider $q = 1$ in (31), we are getting

$$\|P'(a) - P'(b)\| \leq 2h^2\|a-b\|.$$

Again on assuming, $a_0 = a_0(x) = 1$, we obtain $p_0 = 0.00019978824 < 0.18598403 = \rho_1$ and $p_0(1+\phi_3(p_0))^q = 0.0009984814 < 1$. Hence, the existence of a^* is guaranted in $\overline{U(a_0, 0.00037600048)}$ and the uniqueness in $U(a_0, 3.7606061) \cap \Omega$. Thus, the existence and uniqueness domains of convergence balls are better under the Hölder condition in comparison to the Lipschitz condition.

6 Conclusion

In this paper, the semi-local convergence analysis of the sixth-order iterative method for finding the approximate root of the nonlinear equations has been studied by using the recurrence relation technique. The existence and uniqueness of the approximate solution have been calculated in the convergence theorem. Here, we also obtained an error estimate expression. Numerical applications are also included which supports our approach.

Acknowledgements The authors are grateful to the reviewers and editor for their useful comments.

References

1. Ortega, J.M., Rheinboldt, W.C.: Iterative Solution of Nonlinear Equations in Several Variables. Academic Press, New York (1970)
2. Kantorovich, L.V., Akilov, G.P.: Functional Analysis. Pergamon Press, Oxford (1982)
3. Rall, L.B.: Computational Solution of Nonlinear Operator Equations. Robert E. Krieger, New York (1979)
4. Madhu, K.: Semilocal convergence of sixth order method by using recurrence relations in Banach spaces. Appl. Math. E-Notes **18**, 197–208 (2018)
5. Cordero, A., Hernández, M.A., Romero, N., Torregrosa, J.R.: Semilocal convergence by using recurrence relations for a fifth-order method in Banach spaces. J. Comput. Appl. Math. **273**, 205–213 (2015)
6. Hernández, M.A., Martínez, E., Teruel, C.: Semilocal convergence of a k-step iterative process and its application for solving a special kind of conservative problems. Numer. Algor. **76**(2), 309–331 (2017)
7. Singh, S., Gupta, D.K., Martínez, E., Hueso, J.L.: Semilocal and local convergence of a fifth order iteration with Fréchet derivative satisfying Hölder condition. Appl. Math. Comput. **276**, 266–277 (2016)
8. Parhi, S.K., Gupta, D.K.: Semilocal convergence of Stirling's method under Hölder continuous first derivative in Banach spaces. Int. J. Comput. Math. **87**(12), 2752–2759 (2010)
9. Parhi, S.K., Gupta, D.K.: A Stirling-like method with Hölder continuous first derivative in Banach spaces. Appl. Math. Comput. **217**, 9567–9574 (2011)
10. Behl, R., Maroju, P., Motsa, S.S.: Semilocal convergence of a three step fifth order iterative method under Hölder continuity condition in Banach spaces. Int. J. Math. Comput. Sci. **10**(11), 574–578 (2016)
11. Gupta, D.K., Prashanth, M.: Semilocal convergence of a continuation method with Hölder continuous second derivative in Banach spaces. J. Comput. Appl. Math. **236**, 3174–3185 (2012)
12. Zheng, L., Gu, C.: Recurrence relations for semilocal convergence of a fifth-order method in Banach spaces. Numer. Algor. **59**, 623–638 (2012)
13. Lal, R., Saini, R.: On radially symmetric vibrations of functionally graded non-uniform circular plate including non-linear temperature rise. Eur. J. Mech. **77**, 103796 (2019)
14. Saini, R., Lal, R.: Axisymmetric vibrations of temperature-dependent functionally graded moderately thick circular plates with two-dimensional material and temperature distribution. Eng. Comput. 1–16 (2020)
15. Madhu, K.: Sixth order Newton-type method for solving system of nonlinear equations and its applications. Appl. Math. E-Notes **17**, 221–230 (2017)

(m, n)-Paranormal Composition Operators

Baljinder Kour and Sonu Ram

Abstract In this paper, we prove some characterizations for the class of (m, n)-paranormal operators acting on the complex Hilbert space $\mathcal{H}$. The class of (m, n)-paranormal operators is characterized in terms of the Radon–Nikodym derivative of the measure λT^{-1} with respect to λ. Moreover, we discuss the conditions under which the classes of composition operators, weighted composition operators, multiplication composition operators are (m, n)-paranormal.

Keywords (m, n)-paranormal operators · Composition operator · Weighted composition operator · Multiplication composition operator

1 Introduction

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $B(\mathcal{H})$ denotes the C^*-algebra of all bounded linear operators on $\mathcal{H}$. Recall that an operator T is (m, n)-paranormal if $\|Tx\|^{n+1} \leq m\|T^{n+1}x\|\|x\|^n$ for all $x \in \mathcal{H}$, where n is a positive integer and m is a positive real number [2].

Let (X, Σ, λ) be a sigma-finite measure space and T be a measurable transformation from X into itself. In this paper, $L^2(X, \Sigma, \lambda)$ is denoted by $L^2(\lambda)$. The equation $C_T f = f \circ T$, $f \in L^2(\lambda)$ is defined a composition transformation on $L^2(\lambda)$. The operator T induces a composition operator C_T on $L^2(\lambda)$ if

- The measure $\lambda \circ T^{-1}$ is absolutely continuous with respect to λ, and
- The Radon–Nikodym derivative $\frac{d(\lambda T^{-1})}{d\lambda}$ is essentially bounded.

B. Kour
Department of Mathematics, Akal University, Talwandi Sabo, Bathinda 151302, India

Department of Mathematics and Statistics, Central University of Punjab, Ghudda, Bathinda 151401, India

S. Ram (✉)
Department of Mathematics, University of Delhi, Delhi 110007, India
e-mail: ram.sonu02@gmail.com

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_13

The adjoint composition operator is given by $C_T^* f = h.E(f) \circ T^{-1}$. For the complex valued Σ-measurable u on X, the weighted composition operator $W = W_{(\lambda,T)}$ is a linear transformation acting on $L^2(\lambda)$ is defined by $Wf = u.f \circ T$ and its adjoint is $W^* f = h.E(u.f) \circ T^{-1}$.

The multiplication composite operator on $L^2(\lambda)$ is given by $M_{u,T}(f) = u \circ T.f \circ T$. Also, its adjoint is defined by $M_{u,T}^* f = u.h.E(f) \circ T^{-1}$.

Senthilkumar et al. have studied weighted composition of quasi-paranormal operator in [4]. In [8], Veluchamy et al. have studied k-quasi-P-normal composition, weighted composition, and composite multiplication on the complex Hilbert space.

In [1], Cowen et al. obtained a correlation between subnormality of composition operators on $\mathcal{H}^2$ and Denjoy–Wolff point. There are some other authors who have studied various operators for class of composition operators, weighted composition operators, and multiplication composition operators on different spaces [1, 3, 5–7].

2 (m, n)-Paranormal Composition Operators

Proposition 1 *[2, Theorem 2.1] An operator T is (m, n)-paranormal if and only if*

$$m^{\frac{2}{n+1}} T^{*n+1} T^{n+1} - (n+1)a^n T^* T + m^{\frac{2}{n+1}} n a^{n+1} \geq 0,$$

for each $a > 0$.

Theorem 1 *Let $C_T \in B(L^2)$. For each $a > 0$, the following are equivalent:*

1. *C_T is (m, n)-paranormal.*
2. $m^{\frac{1}{n+1}} \|\sqrt{h_n E(h) \circ T^{-n} f}\| \geq (n+1)^{\frac{1}{2}} a^{\frac{n}{2}} \|\sqrt{hf}\| - m^{\frac{1}{n+1}} n^{\frac{1}{2}} a^{\frac{n+1}{2}} \|f\|$.
3. $m^{\frac{2}{n+1}} h_n E(h) \circ T^{-n} \geq (n+1)a^n h - m^{\frac{2}{n+1}} n a^{n+1}$.
4. $m^{\frac{2}{n+1}} h_{n-1} h \circ T^{-(n-1)} E(h) \circ T^{-n} \geq (n+1)a^n h - m^{\frac{2}{n+1}} n a^{n+1}$.
5. $m^{\frac{2}{n+1}} h.h_{n-1} \circ T^{-1} E(h) \circ T^{-n} \geq (n+1)a^n h - m^{\frac{2}{n+1}} n a^{n+1}$.

Proof (1) $\implies$ (2): Since C_T is (m, n)-paranormal, by Proposition 1, for each $a > 0$, we have

$$m^{\frac{2}{n+1}} C_T^{*n+1} C_T^{n+1} - (n+1)a^n C_T^* C_T + m^{\frac{2}{n+1}} n a^{n+1} \geq 0,$$

that is,

$$m^{\frac{2}{n+1}} C_T^{*n+1} C_T^{n+1} \geq (n+1)a^n C_T^* C_T - m^{\frac{2}{n+1}} n a^{n+1}.$$

So, we have

$$m^{\frac{2}{n+1}} \left\langle (C_T^{*n+1} C_T^{n+1}) f, f \right\rangle \geq (n+1)a^n \left\langle (C_T^* C_T) f, f \right\rangle - m^{\frac{2}{n+1}} n a^{n+1} \langle f, f \rangle . \tag{1}$$

Consider

$$\begin{aligned}(C_T^{*n+1}C_T^{n+1})f =& C_T^{*n}(C_T^*C_T)C_T^n f\\ =& C_T^{*n}(C_T^*C_T)(f\circ T^n)\\ =& C_T^{*n}(hf\circ T^n)\\ =& h_nE(hf\circ T^n)\circ T^{-n}\\ =& h_nE(h)\circ T^{-n}.E(f)\circ T^n\circ T^{-n}\\ =& h_nE(h)\circ T^{-n}.f\end{aligned} \tag{2}$$

and

$$C_T^*C_Tf = hf. \tag{3}$$

From (1), (2) and (3), we obtain

$$m^{\frac{2}{n+1}}\left\langle h_nE(h)\circ T^{-n}f, f\right\rangle \geq (n+1)a^n\langle hf, f\rangle - m^{\frac{2}{n+1}}na^{n+1}\langle f, f\rangle,$$

that is,

$$m^{\frac{2}{n+1}}\|\sqrt{h_nE(h)\circ T^{-n}f}\|^2 \geq (n+1)a^n\|\sqrt{hf}\|^2 - m^{\frac{2}{n+1}}na^{n+1}\|f\|^2, \tag{4}$$

that is,

$$m^{\frac{1}{n+1}}\|\sqrt{h_nE(h)\circ T^{-n}f}\| \geq (n+1)^{\frac{1}{2}}a^{\frac{n}{2}}\|\sqrt{hf}\| - m^{\frac{1}{n+1}}n^{\frac{1}{2}}a^{\frac{n+1}{2}}\|f\|.$$

(2) $\implies$ (3): From (4), we have

$$m^{\frac{2}{n+1}}\left\langle h_nE(h)\circ T^{-n}f, f\right\rangle \geq (n+1)a^n\langle hf, f\rangle - m^{\frac{2}{n+1}}na^{n+1}\langle f, f\rangle,$$

that is,

$$m^{\frac{2}{n+1}}h_nE(h)\circ T^{-n} \geq (n+1)a^nh - m^{\frac{2}{n+1}}na^{n+1}.$$

(3) $\implies$ (4): Consider

$$m^{\frac{2}{n+1}}h_nE(h)\circ T^{-n} \geq (n+1)a^nh - m^{\frac{2}{n+1}}na^{n+1} \tag{5}$$

and

$$h_n = \lambda T^{-n}(B). \tag{6}$$

We have

$$\begin{aligned}\lambda T^{-n}(B) =& \lambda^{-1}(T^{-(n-1)}(B))\\ =& \int_{T^{-(n-1)}} n d\lambda\\ =& h_{n-1}h\circ T^{-(n-1)}.\end{aligned} \tag{7}$$

From (6) and (7), we get

$$h_n = h_{n-1}h\circ T^{-(n-1)}. \tag{8}$$

Now, substitute (8) in (5),

$$m^{\frac{2}{n+1}} h_{n-1} h \circ T^{-(n-1)} E(h) \circ T^{-n} \geq (n+1) a^n h - m^{\frac{2}{n+1}} n a^{n+1}.$$

(3) $\implies$ (5): Observe that

$$m^{\frac{2}{n+1}} h_n E(h) \circ T^{-n} \geq (n+1) a^n h - m^{\frac{2}{n+1}} n a^{n+1} \tag{9}$$

and

$$h_n = \lambda T^{-n}(B). \tag{10}$$

We have

$$\begin{aligned} \lambda T^{-n}(B) =& \lambda(T^{-(n-1)}(T^{-1}B)) \\ =& \int_{T^{-B}} h_{n-1} d\lambda \\ =& h.h_{n-1} \circ T^{-1}. \end{aligned} \tag{11}$$

From (10) and (11), we obtain

$$h_n = h.h_{n-1} \circ T^{-1}. \tag{12}$$

Now, substitute (12) in (9),

$$m^{\frac{2}{n+1}} h.h_{n-1} \circ T^{-1} E(h) \circ T^{-n} \geq (n+1) a^n h - m^{\frac{2}{n+1}} n a^{n+1}.$$

(5) $\implies$ (1) :
Consider that

$$m^{\frac{2}{n+1}} h.h_{n-1} \circ T^{-1} E(h) \circ T^{-n} \geq (n+1) a^n h - m^{\frac{2}{n+1}} n a^{n+1}, \tag{13}$$

holds. Since

$$h_n = h.h_{n-1} \circ T^{-1}.$$

Therefore, (13) becomes

$$\begin{aligned} & m^{\frac{2}{n+1}} h_n E(h) \circ T^{-n} \geq (n+1) a^n h - m^{\frac{2}{n+1}} n a^{n+1}, \\ & m^{\frac{2}{n+1}} \left\langle h_n E(h) \circ T^{-n} f, f \right\rangle \geq (n+1) a^n \langle hf, f \rangle - m^{\frac{2}{n+1}} n a^{n+1} \langle f, f \rangle . \end{aligned} \tag{14}$$

By definition of composition operator, we have

$$\begin{aligned} & (C_T^{*n+1} C_T^{n+1}) f = h_n E(h) \circ T^{-n}.f \\ & C_T^* C_T f = hf. \end{aligned} \tag{15}$$

Therefore, from (14) and (15), we get

$$m^{\frac{2}{n+1}} \left\langle (C_T^{*n+1} C_T^{n+1}) f, f \right\rangle \geq (n+1) a^n \left\langle (C_T^* C_T) f, f \right\rangle - m^{\frac{2}{n+1}} n a^{n+1} \langle f, f \rangle . \tag{16}$$

Hence C_T is (m, n) paranormal operator.
This proves the equivalence relations. □

Example 1 Let $\mathcal{H} = l^2(\mathbb{Z}, \mathbb{C})$ and T be a weighted shift operator on $\mathcal{H}$ defined by $Te_p = v_k e_{p+1}$ with non zero weights v_p, and the orthonormal basis e_p for all integers p, where

$$v_p = 1$$

for every p.

Equivalently, C_T is defined by

$$C_T(..., x_{-1}, x_0, x_1, ...) = (..., x_{-1}, x_0, x_1, x_2, x_3, ...).$$

By [2, Theorem 2.9], C_T is (m, n)-paranormal if and only if

$$|v_p|^{n-1} \leq m |v_{p+1}||v_{p+2}|...|v_{p+n-1}|, \tag{17}$$

for unit vectors and $n \geq 2$. Thus, for $m = 2$ and $n = 3$, C_T is $(2, 3)$-paranormal.

Let f be a function defined by $f(k) = x_k$ and $T : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $T(k) = k + 1$, then

$$C_T f(..., x_{-1}, x_0, x_1, ...) = (f \circ T)(..., x_{-1}, x_0, x_1, ...) = (\cdots, x_{-1}, x_0, x_1, \cdots) \tag{18}$$

is composition operator. Thus C_T is $(2, 3)$-paranormal composition operator.

Theorem 2 *An operator C_T^* is (m, n)-paranormal if and only if*

$$m^{\frac{2}{n+1}} \left\langle h_{n+1} \circ T^{n+1}.E(f), f \right\rangle \geq (n+1) a^n \left\langle h_n \circ T^m E(f), f \right\rangle - m^{\frac{2}{n+1}} n a^{n+1} \langle f, f \rangle , \tag{19}$$

for each $a > 0$.

Proof Let C_T^* be (m, n)-paranormal. Then by Proposition 1, for each $a > 0$, we have

$$m^{\frac{2}{n+1}} C_T^{n+1} C_T^{*n+1} - (n+1) a^n C_T C_T^* + m^{\frac{2}{n+1}} n a^{n+1} \geq 0,$$

that is,

$$m^{\frac{2}{n+1}} C_T^{n+1} C_T^{*n+1} \geq (n+1) a^n C_T C_T^* - m^{\frac{2}{n+1}} n a^{n+1},$$

that is,

$$m^{\frac{2}{n+1}} \left\langle (C_T^{n+1} C_T^{*n+1}) f, f \right\rangle \geq (n+1) a^n \left\langle (C_T C_T^*) f, f \right\rangle - m^{\frac{2}{n+1}} n a^{n+1} \langle f, f \rangle . \tag{20}$$

Observe that

$$\begin{aligned}(C_T^{n+1}C_T^{*n+1})f &= h_{n+1}\circ T^{n+1}.E(f),\\ C_TC_T^*f &= h_n\circ T^mE(f).\end{aligned}\tag{21}$$

Substitute (21) in (20),

$$m^{\frac{2}{n+1}}\left\langle h_{n+1}\circ T^{n+1}.E(f),f\right\rangle \geq (n+1)a^n\left\langle h_n\circ T^mE(f),f\right\rangle - m^{\frac{2}{n+1}}na^{n+1}\left\langle f,f\right\rangle.$$

Conversely, let (19) holds.
Consider

$$\begin{aligned}h_{n+1}\circ T^{n+1}.E(f) &=(C_T^{n+1}C_T^{*n+1})f,\\ h_n\circ T^mE(f) &=C_TC_T^*f.\end{aligned}\tag{22}$$

Substitute (22) in (19), we obtain

$$m^{\frac{2}{n+1}}\left\langle (C_T^{n+1}C_T^{*n+1})f,f\right\rangle \geq (n+1)a^n\left\langle C_TC_T^*f,f\right\rangle - m^{\frac{2}{n+1}}na^{n+1}\left\langle f,f\right\rangle,$$

that is,

$$m^{\frac{2}{n+1}}(C_T^{n+1}C_T^{*n+1}) \geq (n+1)a^nC_TC_T^* - m^{\frac{2}{n+1}}na^{n+1}.$$

Hence, C_T^* is (m,n)-paranormal. □

3 Weighted Composition Operators

Theorem 3 *An operator W is (m,n)-paranormal if and only if*

$$m^{\frac{2}{n+1}}\|u_{n+1}.f\circ T^{n+1}\|^2 \geq (n+1)a^n\|u.f\circ T\|^2 - m^{\frac{2}{n+1}}na^{n+1}\|f\|^2,\tag{23}$$

for each $f\in L^2$ and $a>0$.

Proof Assume that W is (m,n)-paranormal. Then, for each $a>0$,

$$m^{\frac{2}{n+1}}W^{*n+1}W^{n+1} - (n+1)a^nW^*W + m^{\frac{2}{n+1}}na^{n+1} \geq 0,$$

that is,

$$m^{\frac{2}{n+1}}W^{*n+1}W^{n+1} \geq (n+1)a^nW^*W - m^{\frac{2}{n+1}}na^{n+1},$$

that is,

$$m^{\frac{2}{n+1}}\left\langle (W^{*n+1}W^{n+1})f,f\right\rangle \geq (n+1)a^n\left\langle (W^*W)f,f\right\rangle - m^{\frac{2}{n+1}}na^{n+1}\left\langle f,f\right\rangle,$$

that is,

$$m^{\frac{2}{n+1}}\|W^{n+1}f\|^2 \geq (n+1)a^n\|Wf\|^2 - m^{\frac{2}{n+1}}na^{n+1}\|f\|^2. \tag{24}$$

Consider

$$\begin{aligned} W^{n+1}f &= u_{n+1}.f \circ T^{n+1}, \\ Wf &= u.f \circ T. \end{aligned} \tag{25}$$

Substitute (25) in (24), we get

$$m^{\frac{2}{n+1}}\|u_{n+1}.f \circ T^{n+1}\|^2 \geq (n+1)a^n\|u.f \circ T\|^2 - m^{\frac{2}{n+1}}na^{n+1}\|f\|^2. \tag{26}$$

Conversely, let (23) holds.
Consider

$$\begin{aligned} W^{n+1}f &= u_{n+1}.f \circ T^{n+1}, \\ Wf &= u.f \circ T. \end{aligned} \tag{27}$$

Substitute (27) in (23),

$$m^{\frac{2}{n+1}}\|W^{n+1}f\|^2 \geq (n+1)a^n\|Wf\|^2 - m^{\frac{2}{n+1}}na^{n+1}\|f\|^2,$$

that is,

$$\begin{aligned} m^{\frac{2}{n+1}}\left\langle (W^{*n+1}W^{n+1})f, f\right\rangle &\geq (n+1)a^n\left\langle (W^*W)f, f\right\rangle \\ &\quad - m^{\frac{2}{n+1}}na^{n+1}\langle f, f\rangle, \end{aligned}$$

that is,

$$m^{\frac{2}{n+1}}W^{*n+1}W^{n+1} \geq (n+1)a^nW^*W - m^{\frac{2}{n+1}}na^{n+1}.$$

This completes the converse part. □

Theorem 4 *Let $W^* \in B(L^2)$. Then W^* is (m, n)-paranormal if and only if*

$$\begin{aligned} m^{\frac{2}{n+1}}\|h_{n+1}E(u_{n+1}.f) \circ T^{-(n+1)}\|^2 &\geq (n+1)a^n\|h.E(u.f) \circ T^{-1}\|^2 \\ &\quad - m^{\frac{2}{n+1}}na^{n+1}\|f\|^2, \end{aligned} \tag{28}$$

for each $f \in L^2$ and $a > 0$.

Proof Let W^* be (m, n)-paranormal. Then for each $a > 0$,

$$m^{\frac{2}{n+1}} W^{n+1} W^{*n+1} - (n+1)a^n WW^* + m^{\frac{2}{n+1}} na^{n+1} \geq 0,$$

that is,

$$m^{\frac{2}{n+1}} W^{n+1} W^{*n+1} \geq (n+1)a^n WW^* - m^{\frac{2}{n+1}} na^{n+1},$$

that is,

$$m^{\frac{2}{n+1}} \left\langle (W^{n+1} W^{*n+1}) f, f \right\rangle \geq (n+1)a^n \left\langle (WW^*) f, f \right\rangle \\ - m^{\frac{2}{n+1}} na^{n+1} \langle f, f \rangle,$$

that is,

$$m^{\frac{2}{n+1}} \|W^{*n+1} f\|^2 \geq (n+1)a^n \|W^* f\|^2 - m^{\frac{2}{n+1}} na^{n+1} \|f\|^2. \\ . \tag{29}$$

Consider

$$W^{*n+1} f = h_{n+1} E(u_{n+1}.f) \circ T^{-(n+1)}, \\ W^* f = h.E(u.f) \circ T^{-1}. \tag{30}$$

Substitute (30) in (29),

$$m^{\frac{2}{n+1}} \|h_{n+1} E(u_{n+1}.f) \circ T^{-(n+1)}\|^2 \geq (n+1)a^n \|h.E(u.f) \circ T^{-1}\|^2 \\ - m^{\frac{2}{n+1}} na^{n+1} \|f\|^2.$$

Conversely, let (28) holds.
Since

$$W^{*n+1} f = h_{n+1} E(u_{n+1}.f) \circ T^{-(n+1)}, \\ W^* f = h.E(u.f) \circ T^{-1}. \tag{31}$$

Substitute (31) in (28),

$$m^{\frac{2}{n+1}} \|W^{*n+1} f\|^2 \geq (n+1)a^n \|W^* f\|^2 - m^{\frac{2}{n+1}} na^{n+1} \|f\|^2,$$

that is,

$$m^{\frac{2}{n+1}} \left\langle (W^{n+1} W^{*n+1}) f, f \right\rangle \geq (n+1)a^n \left\langle (WW^*) f, f \right\rangle \\ - m^{\frac{2}{n+1}} na^{n+1} \langle f, f \rangle,$$

that is,

$$m^{\frac{2}{n+1}} W^{n+1} W^{*n+1} \geq (n+1)a^n WW^* - m^{\frac{2}{n+1}} na^{n+1}.$$

Hence, W^* is (m, n)-paranormal. □

4 Multiplication Composition Operators

Theorem 5 *Let $M_{u,T}$ is (m,n)-paranormal operator if and only if*

$$m^{\frac{2}{n+1}} \| \prod_{i=1}^{n+1} (u \circ T^i). f \circ T^{n+1} \|^2 \geq (n+1)a^n \| u \circ T. f \circ T \|^2 - m^{\frac{2}{n+1}} n a^{n+1} \|f\|^2, \tag{32}$$

for each $a > 0$.

Proof Let $M_{u,T}$ be (m,n)-paranormal. Then for each $a > 0$,

$$m^{\frac{2}{n+1}} M_{u,T}^{*n+1} M_{u,T}^{n+1} - (n+1)a^n M_{u,T}^* M_{u,T} + m^{\frac{2}{n+1}} n a^{n+1} \geq 0,$$

that is,

$$m^{\frac{2}{n+1}} M_{u,T}^{*n+1} M_{u,T}^{n+1} \geq (n+1)a^n M_{u,T}^* M_{u,T} - m^{\frac{2}{n+1}} n a^{n+1},$$

that is,

$$m^{\frac{2}{n+1}} \left\langle (M_{u,T}^{*n+1} M_{u,T}^{n+1}) f, f \right\rangle \geq (n+1)a^n \left\langle (M_{u,T}^* M_{u,T}) f, f \right\rangle - m^{\frac{2}{n+1}} n a^{n+1} \langle f, f \rangle, \tag{33}$$

that is,

$$m^{\frac{2}{n+1}} \| M_{u,T}^{n+1} f \|^2 \geq (n+1)a^n \| M_{u,T} f \|^2 - m^{\frac{2}{n+1}} n a^{n+1} \|f\|^2.$$

Since

$$M_{u,T}^{n+1} f = \prod_{i=1}^{n+1} (u \circ T^i). f \circ T^{n+1}, \quad M_{u,T} f = u \circ T. f \circ T. \tag{34}$$

Substitute (34) in (33), so we get

$$m^{\frac{2}{n+1}} \| \prod_{i=1}^{n+1} (u \circ T^i). f \circ T^{n+1} \|^2 \geq (n+1)a^n \| u \circ T. f \circ T \|^2 - m^{\frac{2}{n+1}} n a^{n+1} \|f\|^2.$$

Conversely, let (32) holds.

Since

$$M_{u,T}^{n+1} f = \prod_{i=1}^{n+1} (u \circ T^i). f \circ T^{n+1}, \quad M_{u,T} f = u \circ T. f \circ T. \tag{35}$$

Substitute (35) in (32),

$$m^{\frac{2}{n+1}}\|M_{u,T}^{n+1}f\|^2 \geq (n+1)a^n\|M_{u,T}f\|^2 - m^{\frac{2}{n+1}}na^{n+1}\|f\|^2,$$

that is,

$$m^{\frac{2}{n+1}}\left\langle (M_{u,T}^{*n+1}M_{u,T}^{n+1})f, f\right\rangle \geq (n+1)a^n\left\langle (M_{u,T}^{*}M_{u,T})f, f\right\rangle - m^{\frac{2}{n+1}}na^{n+1}\langle f, f\rangle,$$

that is,

$$m^{\frac{2}{n+1}}M_{u,T}^{*n+1}M_{u,T}^{n+1} - (n+1)a^n M_{u,T}^{*}M_{u,T} + m^{\frac{2}{n+1}}na^{n+1} \geq 0,$$

for each $a > 0$. Thus, $M_{u,T}$ is (m, n)-paranormal. □

Theorem 6 *An operator $M_{u,T}^{*}$ is (m, n)-paranormal operator if and only if*

$$m^{\frac{2}{n+1}}\|u.h.\prod_{i=1}^{n+1}(E(u.h\circ T^{-i})).E(f)\circ T^{-n+1}\|^2 \geq (n+1)a^n\|u.h.E(f)\circ T^{-1}\|^2 - m^{\frac{2}{n+1}}na^{n+1}\|f\|^2, \tag{36}$$

for each $f \in L^2$ and $a > 0$.

Proof Assume that $M_{u,T}^{*}$ is (m, n) paranormal. Then for each $a > 0$,

$$m^{\frac{2}{n+1}}M_{u,T}^{n+1}M_{u,T}^{*n+1} - (n+1)a^n M_{u,T}M_{u,T}^{*} + m^{\frac{2}{n+1}}na^{n+1} \geq 0,$$

that is,

$$m^{\frac{2}{n+1}}M_{u,T}^{n+1}M_{u,T}^{*n+1} \geq (n+1)a^n M_{u,T}M_{u,T}^{*} - m^{\frac{2}{n+1}}na^{n+1},$$

that is,

$$m^{\frac{2}{n+1}}\left\langle (M_{u,T}^{n+1}M_{u,T}^{*n+1})f, f\right\rangle \geq (n+1)a^n\left\langle (M_{u,T}M_{u,T}^{*})f, f\right\rangle - m^{\frac{2}{n+1}}na^{n+1}\langle f, f\rangle, \tag{37}$$

that is,

$$m^{\frac{2}{n+1}}\|M_{u,T}^{*n+1}f\|^2 \geq (n+1)a^n\|M_{u,T}^{*}f\|^2 - m^{\frac{2}{n+1}}na^{n+1}\|f\|^2.$$

Consider

$$M_{u,T}^{*n+1}f = u.h.\prod_{i=1}^{n+1}(E(u.h\circ T^{-i})).E(f)\circ T^{-n+1}, \quad M_{u,T}^{*}f = u.h.E(f)\circ T^{-1}. \tag{38}$$

Substitute (38) in (37), so we obtain

$$\begin{aligned} & m^{\frac{2}{n+1}} \|u.h. \prod_{i=1}^{n+1} (E(u.h \circ T^{-i})).E(f) \circ T^{-n+1}\|^2 \\ & \geq (n+1)a^n \|u.h.E(f) \circ T^{-1}\|^2 - m^{\frac{2}{n+1}} n a^{n+1} \|f\|^2. \end{aligned} \tag{39}$$

Conversely, let (36) holds.

Consider

$$\begin{aligned} M_{u,T}^{*n+1} f &= u.h. \prod_{i=1}^{n+1} (E(u.h \circ T^{-i})).E(f) \circ T^{-n+1}, \\ M_{u,T}^{*} f &= u.h.E(f) \circ T^{-1}. \end{aligned} \tag{40}$$

Substitute (40) in (36),

$$m^{\frac{2}{n+1}} \|M_{u,T}^{*n+1} f\|^2 \geq (n+1)a^n \|M_{u,T}^{*} f\|^2 - m^{\frac{2}{n+1}} n a^{n+1} \|f\|^2,$$

that is,

$$\begin{aligned} m^{\frac{2}{n+1}} \left\langle (M_{u,T}^{n+1} M_{u,T}^{*n+1}) f, f \right\rangle \geq (n+1)a^n \left\langle (M_{u,T} M_{u,T}^{*}) f, f \right\rangle \\ - m^{\frac{2}{n+1}} n a^{n+1} \langle f, f \rangle, \end{aligned}$$

that is,

$$m^{\frac{2}{n+1}} M_{u,T}^{n+1} M_{u,T}^{*n+1} - (n+1)a^n M_{u,T} M_{u,T}^{*} + m^{\frac{2}{n+1}} n a^{n+1} \geq 0,$$

for each $a > 0$. Thus, $M_{u,T}^{*}$ is (m, n)-paranormal. □

5 Concluding Remarks

We have given some characterizations for the class of (m, n)-paranormal operators acting on the complex Hilbert space $\mathcal{H}$. The class of (m, n)-paranormal operators is characterized in terms of the Radon–Nikodym derivative of the measure λT^{-1} with respect to λ. Further, we have discussed the conditions for (m, n)-paranormal of the classes of composition operators, weighted composition operators, multiplication composition operators.

References

1. Cowen, C., Kriete, T.L.: Subnormality and composition operators on H^2. J. Funct. Anal. **81**, 298–319 (1988)
2. Dharmarha, P., Ram, S.: (m, n)-paranormal operators and $(m, n)^*$-paranormal operators. Commun. Korean Math. Soc. **35**, 151–159 (2020)
3. Senthilkumar, D., Thirugnanasambandam, K.: M-paranormal and * paranormal composition operators on weighted hardy space. Int. J. Contemp. Math. Sci. **5**, 2793–2799 (2010)

4. Senthilkumar, D., Naik, P.M., Santhi, R.: Weighted composition of quasi-paranormal operator. Far East. J. Math. Sci. **72**, 369–383 (2013)
5. Senthil, S., Thangaraju, P., Kumar, D.C.: k-*paranormal, k-quasi-*paranormal and (n, k)-quasi- *paranormal composite multiplication operator on L^2-spaces. Br. J. Math. Comput. Sci. **11**, 1–15 (2015)
6. Senthilkumar, D., Selvi, P.T.: Weighted composition of m-quasi k-paranormal operators. Int. J. Math. Appl. **7**, 1–8 (2019)
7. Sivamani, N.: Posinormal and * paranormal composition operators on the Fock space. Int. J. Innovative Sci. Eng. Technol. **2**(2012)
8. Veluchamy, T., Manikandan, K.M.: k-quasi-P-normal composition, weighted composition and composite multiplication on the complex Hilbert space. Int. J. Pure. Appl. Math. **119**, 14239–14266 (2018)

On the Domain of q-Euler Matrix in c_0 and c

Taja Yaying

Abstract In this study, we present the Banach spaces e_0^q and e_c^q obtained by the domain of q-analog E^q of the Euler matrix of order 1 in the spaces c_0 and c, respectively. We exhibit certain topological properties and inclusion relations of these spaces. We obtain the bases and determine the Köthe duals of the spaces e_0^q and e_c^q. We characterize certain classes of matrix mappings from the spaces e_0^q and e_c^q to the space $\mu \in \{\ell_\infty, c, c_0, \ell_1, bs, cs, cs_0\}$.

Keywords q-Euler matrix · Sequence spaces · α- · β- · γ-duals · Matrix mappings

1 Introduction

Throughout this study, the letter s stands for the set of all real-valued sequences. A sequence space is a linear subspace of s. Some examples of classical sequence spaces are ℓ_∞ (bounded sequences), c_0 (null sequences), and c (convergent sequences). A BK-space is a Banach space with continuous coordinates. The space $\lambda \in \{\ell_\infty, c_0, c\}$ is a BK-space endowed with supremum norm defined by $\|z\|_{\ell_\infty} = \sup_{r\in\mathbb{N}_0} |z_r|$. Here and in what follows, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Also, ℓ_k stands for the space of all k-absolutely summable sequences with $1 \leq k < \infty$. Further, the notations bs, cs, and cs_0 represent the spaces of all bounded, convergent, and null series, respectively.

Let λ and μ be two sequence spaces. If $\Phi = (\phi_{rv})$ is an infinite matrix of real entries, then rth row of the matrix Φ shall be denoted by Φ_r. Let $z = (z_v)$ be a sequence in λ, then the notation $\Phi z = \{(\Phi z)_r\} = \left\{\sum_{v=0}^{\infty} \phi_{rv} z_v\right\}$ is called Φ-transform of the sequence z, provided that the series $\sum_{v=0}^{\infty} \phi_{rv} z_v$ exists for each $r \in \mathbb{N}_0$. Further, the matrix Φ is said to define a mapping from λ to μ if $\Phi z \in \mu$ for every sequence $z \in \lambda$. In notation, $\Phi \in (\lambda : \mu)$ if and only if Φ is a mapping from λ to μ. Moreover, the matrix $\Phi = (\phi_{rv})$ is called a triangle if $\phi_{rr} \neq 0$ and $\phi_{rv} = 0$ for all $v > r$.

T. Yaying (✉)
Department of Mathematics, Dera Natung Government College, Itanagar 791 113, India
e-mail: tajayaying20@gmail.com

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_14

Define the set λ_Φ by

$$\lambda_\Phi = \{z \in s : \Phi z \in \lambda\}. \tag{1}$$

The set λ_Φ is a sequence space and is called the domain of matrix Φ in the space λ. The domain of a matrix plays an important role in the construction of sequence spaces. It is known that if λ is a BK-space, then the matrix domain λ_Φ is also a BK-space with the norm $\|z\|_{\lambda_\Phi} = \|\Phi z\|_\lambda$. With this concept, several researchers introduced the interesting Banach sequence spaces using the domain of special triangles. For relevant literature, we refer the papers [2, 3, 19, 24, 26, 27, 34] and textbooks/Ph.D. dissertation [7, 8, 25, 35]. For some recent publications dealing with the domain of triangles in classical spaces, we refer [1, 2, 4, 9, 12, 14–17, 21–23, 28–31, 33, 36–40].

1.1 Euler Matrix of Order **1** *and Sequence Spaces*

The Euler matrix $E = (e_{rv})$ of order 1 is defined by

$$e_{rv} = \begin{cases} \frac{\binom{r}{v}}{2^r} & 0 \le v \le r, \\ 0 & otherwise, \end{cases}$$

for all $r, v \in \mathbb{N}$. Not much studies related to sequence spaces obtained using the domain of the matrix E can be found in the literature. Recently, Başar and Braha [6] studied the space of the Euler-Cesàro bounded, convergent, and null difference sequences. It is shown that these spaces are separable BK-spaces. Further, the authors obtained certain inclusion relations, Schauder basis, and Köthe duals and characterized the certain classes of matrix transformations on these spaces. More recently, Ellidokuzoğlu and Demiriz [11] give a further generalization of the spaces defined in [6] by introducing the Euler-Riesz bounded, convergent, and null difference spaces.

1.2 q-Calculus

The q-calculus is a branch of mathematics that deals with the generalization of some well-known mathematical expressions by using the parameter q. The generalized expression so obtained is called q-analog of the original expression. Further, the q-analog returns the original expression when q approaches 1. Several researchers are engaged in the field of q-calculus due to its broad applications in mathematics, physics, and engineering sciences. It is widely used by researchers in operator theory, approximation theory, hypergeometric series, special functions, quantum algebras, combinatorics, etc. We refer the book [20] for details in q-calculus.

Now, we recall certain terminologies in q-calculus that are fundamental in our investigation:

Definition 1 Let $0 < q < 1$. Then the q-number is defined by

$$[r]_q = \begin{cases} \frac{1-q^r}{1-q} & (r = 1, 2, 3, \ldots), \\ 1 & (r = 0). \end{cases}$$

Clearly, $[r]_q$ reduces to r when $q \to 1$.

Definition 2 The q-analog of binomial coefficient or q-binomial coefficient is defined by

$$\begin{bmatrix} r \\ v \end{bmatrix}_q = \begin{cases} \frac{[r]_q!}{[r-v]_q![v]!} & (r \geq v), \\ 0 & (r < v), \end{cases}$$

where q-factorial $[r]_q!$ of r is defined by

$$[r]_q! = [r]_q[r-1]_q \ldots [2]_q[1]_q.$$

Motivated by the above studies, we construct q-analog E^q of the Euler matrix of order 1 and present new BK-spaces e_0^q and e_c^q derived by the domain of the matrix E^q in the spaces c_0 and c, respectively. We exhibit certain topological properties, inclusion relations, and bases for the spaces e_0^q and e_c^q. In Sect. 3, we determine the Köthe duals (α-, β-, and γ-duals) of the spaces e_0^q and e_c^q. In Sect. 4, we characterize certain classes of matrix mappings from the spaces e_0^q and e_c^q to the space $\mu \in \{\ell_\infty, c, c_0, \ell_1, bs, cs, cs_0\}$.

2 q-Euler Spaces e_c^q and e_0^q

Throughout this study, we shall use the notation $\pi^{(r)}(q) = \Pi_{v=0}^{r-1}(1+q^v)$ with $\pi^{(0)}(q) = 1$. Then the matrix $E^q = (e_{rv}^q)$ is defined by

$$e_{rv}^q = \begin{cases} \frac{\begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{v}{2}}}{\pi^{(r)}(q)} & (0 \leq v \leq r), \\ 0 & (v > r), \end{cases}$$

for all $r, v \in \mathbb{N}_0$. More explicitly,

$$E^q = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & \cdots \\ \frac{1}{2(1+q)} & \frac{1}{2} & \frac{q}{2(1+q)} & 0 & \cdots \\ \frac{1}{2(1+q)(1+q^2)} & \frac{1+q+q^2}{2(1+q)(1+q^2)} & \frac{q(1+q+q^2)}{2(1+q)(1+q^2)} & \frac{1}{2(1+q)(1+q^2)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

It is known from [20, p. 29] that the infinite product $\pi^{(\infty)}(q) = 2(1+q)(1+q^2)\ldots$ converges to a finite limit. Thus, we may write $\pi^{(\infty)}(q) = \pi(q)$, where $\pi(q)$ is finite. Observe that

$$\begin{aligned} \lim_{r\to\infty} e^q_{rv} = \lim_{r\to\infty} \frac{\begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{v}{2}}}{\pi^{(r)}(q)} &= q^{\binom{v}{2}} \frac{\lim_{r\to\infty} \begin{bmatrix} r \\ v \end{bmatrix}_q}{\lim_{r\to\infty} \pi^{(r)}(q)} \\ &= \frac{q^{\binom{v}{2}} \frac{1}{[v]_q!(1-q)^v}}{\pi(q)} \\ &= \frac{q^{\binom{v}{2}}}{[v]_q!(1-q)^v} \pi(q). \end{aligned}$$

This immediately yields us the following result.

Lemma 1 *The q-Euler matrix E^q is a conservative matrix. In other words, E^q maps c to c.*

Now we define the sequence spaces e^q_0 and e^q_c by

$$e^q_0 = \left\{ z = (z_r) \in s : \lim_{r\to\infty} \frac{1}{\pi^{(r)}(q)} \sum_{v=0}^{r} \begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{v}{2}} z_v = 0 \right\},$$

$$e^q_c = \left\{ z = (z_r) \in s : \lim_{r\to\infty} \frac{1}{\pi^{(r)}(q)} \sum_{v=0}^{r} \begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{v}{2}} z_v \text{ exists} \right\}.$$

Further, on using notation (1), we realize that the spaces e^q_0 and e^q_c can also be defined in the form

$$e^q_0 = (c_0)_{E^q} \text{ and } e^q_c = c_{E^q}. \tag{2}$$

The spaces e^q_0 and e^q_c reduce to the Euler spaces $e_0 = (c_0)_E$ and $e_c = c_E$, respectively, when q approaches 1.

Let us define the sequence $w = (w_v)$ in terms of the sequence $z = (z_v)$ by

$$w_r = (E^q z)_r = \frac{1}{\pi^{(r)}(q)} \sum_{v=0}^{r} \begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{v}{2}} z_v, \tag{3}$$

for each $r \in \mathbb{N}_0$. The sequence w is called the E^q-transform of the sequence z. We write on using (3)

$$z_r = \sum_{v=0}^{r} (-1)^{r-v} \frac{\pi^{(v)}(q) \left[{r \atop v}\right]_q q^{\binom{r-v}{2}}}{q^{\binom{v}{2}}} w_v, \tag{4}$$

for each $r \in \mathbb{N}_0$.

Now, we can state the first result, as follows, without proof:

Theorem 1 *The spaces e_0^q and e_c^q are BK-spaces under the supremum norm defined by*

$$\|z\|_{e_0^q} = \|z\|_{e_c^q} = \sup_{r\in\mathbb{N}_0} \left| \sum_{v=0}^{r} \frac{\left[{r \atop v}\right]_q q^{\binom{v}{2}}}{\pi^{(r)}(q)} z_v \right|.$$

Proof The proof is a routine exercise and hence details are omitted. □

Theorem 2 *The q-Euler spaces e_0^q and e_c^q are linearly isomorphic to c_0 and c, respectively.*

Proof We present the proof for the space e_0^q. The proof for the other case can be obtained analogously. Define the mapping $\mathcal{T} : e_0^q \to c_0$ by $\mathcal{T}z = E^q z$ for all $z \in e_0^q$. It is easy to observe that $\mathcal{T}$ is linear and $1-1$. Let $w = (w_v) \in c_0$ and $z = (z_v)$ is as defined in (4). Then, we have

$$\lim_{r\to\infty} \sum_{v=0}^{r} \frac{\left[{r \atop v}\right]_q q^{\binom{v}{2}}}{\pi^{(r)}(q)} z_v = \lim_{r\to\infty} \sum_{v=0}^{r} \frac{\left[{r \atop v}\right]_q q^{\binom{v}{2}}}{\pi^{(r)}(q)} \left(\sum_{j=0}^{v} (-1)^{v-j} \frac{\pi^{(j)}(q) \left[{v \atop j}\right]_q q^{\binom{v-j}{2}}}{q^{\binom{v}{2}}} w_j \right)$$
$$= \lim_{r\to\infty} w_r = 0.$$

Thus, $z \in e_0^q$ and the mapping $\mathcal{T}$ is onto. Therefore, $e_0^q \cong c_0$. This completes the proof. □

Theorem 3 *The inclusion $c_0 \subset e_0^q$ does not hold. However $c \subset e_c^q$ holds.*

Proof It is known from Lemma 1 that $\lim_{r\to\infty} e_{rv}^q \neq 0$ for each $v \in \mathbf{N}_0$. That is, $E^q \notin (c_0 : c_0)$. This yields that $c_0 \not\subset e_0^q$. In a similar manner, the inclusion $c \subset e_c^q$ can be established. □

Jarrah and Malkowsky [18, Theorem 2.3, Remark 2.4] state that the domain λ_Φ of the triangle Φ in the space λ has a basis if and only if λ has a basis. In the light of this and Theorem 2, we present the following result:

Theorem 4 *Consider the sequence $x^{(v)}(q) = (x_r^{(v)}(q))$ in the space e_0^q for every fixed $v \in \mathbb{N}_0$ defined by*

$$x_r^{(v)}(q) = \begin{cases} (-1)^{r-v} \frac{\pi^{(v)}(q) \left[{r \atop v} \right]_q q^{\binom{r-v}{2}}}{q^{\binom{r}{2}}} & (v \le r), \\ 0 & (v > r). \end{cases}$$

Then we have

(a) the set $\{x^{(0)}(q), x^{(1)}(q), x^{(2)}(q), \ldots\}$ *forms the basis for the space* e_0^q *and every* $z \in e_0^q$ *has a unique representation of the form* $z = \sum_{v=0}^{\infty} w_v x^{(v)}(q)$.

(b) the set $\{e, x^{(0)}(q), x^{(1)}(q), x^{(2)}(q), \ldots\}$ *forms the basis for the space* e_c^q*, where* $e = \{1, 1, 1, \ldots\}$*, and every* $z \in e_c^q$ *can be uniquely expressed in the form* $z = \xi e + \sum_{v=0}^{\infty} (w_v - \xi) x^{(v)}(q)$*, where* $\xi = \lim_{v \to \infty} w_v = \lim_{v \to \infty} (E^q z)_v$.

3 Köthe Duals

In the current section, we determine the Köthe duals (α-, β-, and γ-duals) of the spaces e_0^q and e_c^q. We provide the proof for the space e_0^q. The proof for the space e_c^q can be obtained analogously. First, we recall the definitions of the Köthe duals.

Definition 3 The $\alpha-$, $\beta-$, and $\gamma-$duals of a subset $\lambda \subset s$ are defined by

$$\begin{aligned} \lambda^{\alpha} &= \{\varsigma = (\varsigma_v) \in s : \varsigma z = (\varsigma_v z_v) \in \ell_1 \text{ for all } z \in \lambda\}, \\ \lambda^{\beta} &= \{\varsigma = (\varsigma_v) \in s : \varsigma z = (\varsigma_v z_v) \in cs \text{ for all } z \in \lambda\}, \quad \text{and} \\ \lambda^{\gamma} &= \{\varsigma = (\varsigma_v) \in s : \varsigma z = (\varsigma_v z_v) \in bs \text{ for all } z \in \lambda\}, \end{aligned}$$

respectively.

Before proceeding to the main results, we note down celeberated lemmas due to Stieglitz and Tietz [32] that are required for computing the duals of the spaces e_0^q and e_c^q.

In the rest of the paper, $\mathcal{R}$ will represent the family of all finite subsets of $\mathbb{N}_0$.

Lemma 2 $\Phi = (\phi_{rv}) \in (c_0 : \ell_1)$ *if and only if*

$$\sup_{R \in \mathcal{R}} \left(\sum_{v=0}^{\infty} \left| \sum_{r \in R} \phi_{rv} \right| \right) < \infty.$$

Lemma 3 $\Phi = (\phi_{rv}) \in (c_0 : c)$ *if and only if*

$$\sup_{r \in \mathbb{N}_0} \sum_{v=0}^{r} |\phi_{rv}| < \infty, \tag{5}$$

$$\lim_{r \to \infty} \phi_{rv} \text{ exists for each } v \in \mathbb{N}_0. \tag{6}$$

Lemma 4 *$\Phi = (\phi_{rv}) \in (c_0 : \ell_\infty)$ if and only if (5) holds.*

Theorem 5 *The set $\delta_1(q)$ defined by*

$$\delta_1(q) = \left\{ \varsigma = (\varsigma_v) \in s : \sup_{R \in \mathcal{R}} \sum_{v=0}^{\infty} \left| \sum_{r \in R} (-1)^{r-v} \frac{\pi^{(v)}(q) \begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{r-v}{2}}}{q^{\binom{r}{2}}} \varsigma_r \right| < \infty \right\}$$

is the α-dual of the spaces e_0^q and e_c^q.

Proof Consider the following equality:

$$\begin{aligned} \varsigma_r z_r &= \sum_{v=0}^{r} (-1)^{r-v} \frac{\pi^{(v)}(q) \begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{r-v}{2}}}{q^{\binom{r}{2}}} \varsigma_r w_v \\ &= (A^q w)_r \end{aligned} \tag{7}$$

for all $r \in \mathbb{N}_0$, where the sequence $w = (w_v)$ is the E^q-transform of the sequence $z = (z_v)$ and the matrix $A^q = (a^q_{rv})$ is defined by

$$a^q_{rv} = \begin{cases} (-1)^{r-v} \frac{\pi^{(v)}(q) \begin{bmatrix} r \\ v \end{bmatrix}_q q^{\binom{r-v}{2}}}{q^{\binom{r}{2}}} \varsigma_r & (0 \le v \le r), \\ 0 & (v > r). \end{cases}$$

We realize by using (7) that $\varsigma z = (\varsigma_r z_r) \in \ell_1$ whenever $z \in e_0^q$ if and only if $A^q w \in \ell_1$ whenever $w \in c_0$. Thus, we compute that $\varsigma = (\varsigma_r)$ is a sequence in α-dual of e_0^q if and only the matrix E^q belongs to the class $(c_0 : \ell_1)$. Thus, we conclude from Lemma 2 that $\left[e_0^q\right]^\alpha = \delta_1(q)$. This completes the proof. □

Theorem 6 *Define the sets $\delta_2(q)$, $\delta_3(q)$, and $\delta_4(q)$ by*

$$\delta_2(q) = \left\{ \varsigma = (\varsigma_r) \in s : \sum_{l=v}^{\infty} (-1)^{l-v} \frac{\pi^{(v)}(q) \begin{bmatrix} l \\ v \end{bmatrix}_q q^{\binom{l-v}{2}}}{q^{\binom{l}{2}}} \varsigma_l \text{ exists for each } v \in \mathbb{N}_0 \right\},$$

$$\delta_3(q) = \left\{ \varsigma = (\varsigma_r) \in s : \sup_{r \in \mathbb{N}_0} \sum_{v=0}^{r} \left| \sum_{l=v}^{r} (-1)^{l-v} \frac{\pi^{(v)}(q) \begin{bmatrix} l \\ v \end{bmatrix}_q q^{\binom{l-v}{2}}}{q^{\binom{l}{2}}} \varsigma_l \right| < \infty \right\},$$

$$\delta_4(q) = \left\{ \varsigma = (\varsigma_r) \in s : \lim_{r \to \infty} \sum_{v=0}^{r} \sum_{l=v}^{r} (-1)^{l-v} \frac{\pi^{(v)}(q) \begin{bmatrix} l \\ v \end{bmatrix}_q q^{\binom{l-v}{2}}}{q^{\binom{l}{2}}} \varsigma_l \quad \text{exists} \right\}.$$

Then $\left[e_0^q\right]^\beta = \delta_2(q) \cap \delta_3(q)$ and $\left[e_c^q\right]^\beta = \delta_2(q) \cap \delta_3(q) \cap \delta_4(q)$.

Proof Consider the following equality:

$$\sum_{v=0}^{r} \varsigma_v z_v = \sum_{v=0}^{r} \left\{ \sum_{l=0}^{v} (-1)^{v-l} \frac{\pi^{(l)}(q) \left[{v \atop l}\right]_q q^{\binom{v-l}{2}}}{q^{\binom{v}{2}}} w_l \right\} \varsigma_v$$

$$= \sum_{v=0}^{r} \left\{ \sum_{l=v}^{r} (-1)^{l-v} \frac{\pi^{(l)}(q) \left[{l \atop v}\right]_q q^{\binom{l-v}{2}}}{q^{\binom{l}{2}}} \varsigma_l \right\} w_v \tag{8}$$

$$= (B^q w)_r \tag{9}$$

for each $r \in \mathbb{N}_0$, where the matrix $B^q = (b^q_{rv})$ is defined by

$$b^q_{rv} = \begin{cases} \sum_{l=v}^{r} (-1)^{l-v} \frac{\pi^{(v)}(q) \left[{l \atop v}\right]_q q^{\binom{l-v}{2}}}{q^{\binom{l}{2}}} \varsigma_l & (0 \le v \le r), \\ 0 & (v > r), \end{cases}$$

for all $r, v \in \mathbb{N}_0$. Thus, by using (9), we realize that $\varsigma z = (\varsigma_r z_r) \in cs$ whenever $z = (z_r) \in e^q_0$ if and only if $B^q w \in c$ whenever $w = (w_v) \in c_0$. This yields that $\varsigma = (\varsigma_r)$ is a sequence in the β-dual of e^q_0 if and only if the matrix B^q belongs to the class $(c_0 : c)$. This in turn implies on using Lemma 3 that

$$\sup_{r \in \mathbb{N}_0} \sum_{v=0}^{r} \left| b^q_{rv} \right| < \infty \text{ and } \lim_{r \to \infty} b^q_{rv} \text{ exists for each } v \in \mathbb{N}_0.$$

Thus, $e^q_0 = \delta_2(q) \cap \delta_3(q)$. This completes the proof. □

Theorem 7 *The γ-dual of the spaces e^q_0 and e^q_c is $\delta_3(q)$.*

Proof The proof is similar to the previous theorem except that Lemma 4 is employed instead of Lemma 3. □

4 Matrix Mappings

In the present section, we determine necessary and sufficient conditions for a matrix to define mapping from the spaces e^q_0 and e^q_c to the space $\mu \in \{\ell_\infty, c, c_0, \ell_1, bs, cs, cs_0\}$. The following theorem is fundamental in our investigation.

Theorem 8 *Let μ be any arbitrary subset of s. Then $\Phi = (\phi_{rv}) \in (e^q_0 : \mu)$ (or respectively $(e^q_c : \mu)$) if and only if $\Theta^{(r)} = (\theta^{(r)}_{lv}) \in (c_0 : c)$ (or respectively $(c : c)$) for each $r \in \mathbb{N}_0$, and $\Theta = (\theta_{rv}) \in (c_0 : \mu)$ (or respectively $(c : \mu)$) where*

$$\theta^{(r)}_{lv} = \begin{cases} 0 & (v > l), \\ \sum_{j=v}^{l} (-1)^{j-v} \frac{\pi^{(v)}(q) \left[{j \atop v}\right]_q q^{\binom{j-v}{2}}}{q^{\binom{j}{2}}} \phi_{rj} & (0 \le v \le l), \end{cases}$$

and

$$\theta_{rv} = \sum_{j=v}^{\infty} (-1)^{j-v} \frac{\pi^{(v)}(q) \begin{bmatrix} j \\ v \end{bmatrix}_q q^{\binom{j-v}{2}}}{q^{\binom{j}{2}}} \phi_{rj}, \tag{10}$$

for all $r, v \in \mathbb{N}_0$.

Proof The proof detailing is omitted since it is similar to the proof of Theorem 4.1 of [24]. □

Now, using the results presented in the Stieglitz and Tietz [32] together with Theorem 8, we obtain the following results:

Corollary 1 *The following statements hold:*

1. $\Phi \in (e_0^q : \ell_\infty)$ *if and only if*

$$\sup_{l \in \mathbb{N}_0} \sum_{v=0}^{\infty} \left| \theta_{lv}^{(r)} \right| < \infty, \tag{11}$$

$$\lim_{l \to \infty} \theta_{lv}^{(r)} \text{ exists for all } v \in \mathbb{N}_0 \tag{12}$$

hold and

$$\sup_{r \in \mathbb{N}_0} \sum_{v=0}^{\infty} |\theta_{rv}| < \infty \tag{13}$$

also holds.

2. $\Phi \in (e_0^q : c)$ *if and only if (11) and (12) hold, and*

$$\sup_{r \in \mathbb{N}_0} \sum_{v=0}^{\infty} |\theta_{rv}| < \infty, \tag{14}$$

$$\lim_{r \to \infty} \theta_{rv} \text{ exists for all } v \in \mathbb{N}_0 \tag{15}$$

also hold.

3. $\Phi \in (e_0^q : c_0)$ *if and only if (11) and (12) hold, and (13) and*

$$\lim_{r \to \infty} \theta_{rv} = 0 \text{ for all } v \in \mathbb{N}_0, \tag{16}$$

also hold.

4. $\Phi \in (e_0^q : \ell_1)$ *if and only if (11) and (12) hold, and*

$$\sup_{R \in \mathcal{R}} \sum_{v=0}^{\infty} \left| \sum_{r \in R} \theta_{rv} \right| < \infty \tag{17}$$

also holds.

5. $\Phi \in (e_0^q : bs)$ *if and only if (11) and (12) hold, and*

$$\sup_{r\in\mathbb{N}_0} \sum_{v=0}^{\infty} \left| \sum_{l=0}^{r} \theta_{lv} \right| < \infty \tag{18}$$

also holds.

6. $\Phi \in (e_0^q : cs)$ *if and only if (11) and (12) hold, and (18) and*

$$\sum_{r=0}^{\infty} \theta_{rv} \text{ converges for all } v \in \mathbb{N}_0 \tag{19}$$

also hold.

7. $\Phi \in (e_0^q : cs_0)$ *if and only if (11) and (12) hold, and (18) and*

$$\sum_{r=0}^{\infty} \theta_{rv} = 0 \text{ for all } v \in \mathbb{N}_0 \tag{20}$$

also hold.

Corollary 2 *The following statements hold:*

1. $\Phi \in (e_c^q : \ell_\infty)$ *if and only if (11), (12), and*

$$\lim_{l\to\infty} \sum_{v=0}^{\infty} \theta_{lv}^{(r)}, \tag{21}$$

hold, and (14) also holds.

2. $\Phi \in (e_c^q : c)$ *if and only if (11), (12), and (21) hold, and (13), (15), and*

$$\lim_{r\to\infty} \sum_{v=0}^{r} \theta_{rv} \text{ exists} \tag{22}$$

also hold.

3. $\Phi \in (e_c^q : c_0)$ *if and only if (11), (12), and (21) hold, and (13), (16), and*

$$\lim_{r\to\infty} \sum_{v=0}^{r} \theta_{rv} = 0 \tag{23}$$

also hold.

4. $\Phi \in (e_c^q : \ell_1)$ *if and only if (11), (12), and (21) hold, and (17) also holds.*
5. $\Phi \in (e_c^q : bs)$ *if and only if (11), (12), and (21) hold, and (18) also holds.*
6. $\Phi \in (e_c^q : cs)$ *if and only if (11), (12), and (21) hold, and (18), (19), and*

$$\sum_{r=0}^{\infty}\sum_{v=0}^{\infty}\theta_{rv} \text{ converges}$$

also hold.

7. $\Phi \in (e_c^q : cs_0)$ *if and only if* (11), (12), *and* (21) *hold, and* (18), (19), *and*

$$\sum_{r=0}^{\infty}\sum_{v=0}^{\infty}\theta_{rv} = 0$$

also hold.

We recall a basic lemma due to Başar and Altay [5] that will help in characterizing certain classes of matrix transformations from the spaces e_0^q and e_c^q to any arbitrary space μ.

Lemma 5 *[5] Let λ and μ be any two sequence spaces, Φ be an infinite matrix, and Θ be a triangle. Then, $\Phi \in (\lambda : \mu_\Theta)$ if and only if $\Theta\Phi \in (\lambda : \mu)$.*

Now, by combining Lemma 5 with Corollaries 1 and 2, we give the characterizations of the following classes of matrix mappings:

Corollary 3 *Let $\Phi = (\phi_{rv})$ be an infinite matrix and define the matrix $C^q = (c_{rv}^q)$ by*

$$c_{rv}^q = \sum_{l=0}^{r} \frac{q^{l-1}}{[r+1]_q}\phi_{lv}, \; (0 < q < 1)$$

for all $r, v \in \mathbb{N}$, where $[r]_q$ is the q-analog of $r \in \mathbb{N}_0$. Then, the necessary and sufficient conditions that Φ is in any one of the classes $(e_0^q : X_0^q)$, $(e_0^q : X_c^q)$, $(e_c^q : X_0^q)$, and $(e_c^q : X_c^q)$ are determined from the respective ones in Corollaries 1 and 2, by replacing the elements of the matrix Φ by those of the matrix C^q, where X_0^q and X_c^q are the q-Cesàro sequence spaces defined by Demiriz and Şahin [10].

Corollary 4 *Let $\Phi = (\phi_{rv})$ be an infinite matrix and define the matrix $\tilde{C} = (C_{rv})$ by*

$$\tilde{C}_{rv} = \sum_{l=0}^{r} \frac{C_l C_{r-l}}{C_{r+1}}\phi_{lv}, \; (r, v \in \mathbb{N}_0)$$

where (C_r) is a sequence of the Catalan numbers. Then, the necessary and sufficient conditions that Φ is in any one of the classes $(e_0^q : c_0(\tilde{C}))$, $(e_0^q : c(\tilde{C}))$, $(e_c^q : c_0(\tilde{C}))$, and $(e_c^q : c(\tilde{C}))$ are determined from the respective ones in Corollaries 1 and 2, by replacing the elements of the matrix Φ by those of matrix $\tilde{C}$, where $c(\tilde{C})$ and $c_0(\tilde{C})$ are the Catalan sequence spaces defined by İlkhan [13].

Corollary 5 *Let $\Phi = (\phi_{rv})$ be an infinite matrix and define the matrix $F = (f_{rv})$ by*

$$f_{rv} = \sum_{l=0}^{r} \frac{f_l^2}{f_r f_{r+1}} \phi_{lv}, \ (r, v \in \mathbb{N}_0)$$

where (f_r) is a sequence of the Fibonacci numbers. Then, the necessary and sufficient conditions that Φ is in any one of the classes $(e_0^q : \ell_\infty(F))$, $(e_0^q : c(F))$, $(e_0^q : c_0(F))$, $(e_c^q : \ell_\infty(F))$, $(e_c^q : c(F))$, and $(e_c^q : c_0(F))$ are determined from the respective ones in Corollaries 1 and 2, by replacing the elements of the matrix Φ by those of matrix F, where $\ell_\infty(F)$, $c(F)$, and $c_0(F)$ are the Fibonacci sequence spaces defined by Kara and Başarır [19].

Acknowledgements The research of the author is supported by the Science and Engineering Research Board (SERB), New Delhi, India, under the grant number EEQ/2019/000082.

References

1. Alp, P.Z., İlkhan, M.: On the difference sequence space $\ell_p(\hat{T}^q)$. Math. Sci. Appl. E-Notes **7**(2), 161–173 (2019)
2. Altay, B., Başar, F.: On some Euler sequence spaces of non-absolute type. Ukr. Math. J. **57**, 1–17 (2005)
3. Altay, B., Başar, F., Mursaleen, M.: On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ I. Inf. Sci. **176**, 1450–1462 (2006)
4. Altay, B., Kama, R.: On Cesàro summability of vector valued multiplier spaces and operator valued series. Positivity **22**(2), 575–586 (2018)
5. Başar, F., Altay, B.: On the spaces of p-bounded variation and related matrix mappings. Ukr. Math. J. **55**(1), 136–147 (2003)
6. Başar, F., Braha, N.L.: Euler-Cesàro difference spaces of bounded, convergent and null sequences. Tamkang J. Math. **47**(4), 405–420 (2016)
7. Başar, F.: Summability Theory and Its Applications. Bentham Science Publisher, İstanbul (2012)
8. Bekar, Ş.: q-matrix summability methods. Ph.D. Dissertation. Applied Mathematics and Computer Science, Eastern Mediterranean University (2010)
9. Demiriz, S., İlkhan, M., Kara, E.E.: Almost convergence and Euler totient matrix. Ann. Funct. Anal. **11**, 604–616 (2020)
10. Demiriz, S., Şahin, A.: q-Cesàro sequence spaces derived by q-analogue. Adv. Math. **5**(2), 97–110 (2016)
11. Ellidokuzoğlu, H.B., Demiriz, S.: Euler-Riesz difference sequence spaces. Turk. J. Math. Comput. Sci. **7**, 63–72 (2017)
12. İlkhan, M.: Matrix domain of a regular matrix derived by Euler totient function in the spaces c_0 and c. Mediterr. J. Math. **17**, 27 (2020)
13. İlkhan, M.: A new conservative matrix derived by Catalan numbers and its matrix domain in the spaces c and c_0. Linear Multilinear Algebr. **68**(2), 417–434 (2020)
14. İlkhan, M.: Certain geometric properties and matrix transformations on a newly introduced Banach space. Fundam. J. Math. Appl. **3**(1), 45–51 (2020)
15. İlkhan, M., Demiriz, S., Kara, E.E.: A new paranormed sequence space defined by Euler totient matrix. Karaelmas Sci. Eng. J. **9**(2), 277–282 (2019)
16. İlkhan, M., Kara, E.E.: A new Banach space defined by Euler totient matrix operator. Oper. Matrices **13**(2), 527–544 (2019)

17. İlkhan, M., Şimşek, N., Kara, E.E.: A new regular infinite matrix defined by Jordan totient function and its matrix domain in ℓ_p. Math. Methods Appl. Sci. (2020). https://doi.org/10.1002/mma.6501
18. Jarrah, A.M., Malkowsky, E.: Ordinary, absolute and strong summability and matrix transformations. Filomat **17**, 59–78 (2003)
19. Kara, E.E., Başarır, M.: An application of Fibonacci numbers into infinite Toeplitz matrices. Casp. J. Math. Sci. **1**(1), 43–47 (2012)
20. Kac, V., Cheung, P.: Quantum Calculus. Springer, New York (2002)
21. Kama, R., Altay, B.: On some sequence spaces related to a sequence in a normed space. Konuralp J. Math. **7**(1), 33–37 (2019)
22. Kama, R., Altay, B.: Weakly unconditionally Cauchy series and Fibonacci sequence spaces. J. Inequalities Appl. **2017**, 133 (2017)
23. Kara, E.E., İlkhan, M.: Some properties of generalized Fibonacci sequence spaces. Linear Multilinear Algebr. **64**(11), 2208–2223
24. Kirişçi, M., Başar, F.: Some new sequence spaces derived by the domain of generalized difference matrix. Comput. Math. Appl. **60**, 1299–1309 (2010)
25. Mursaleen, M., Başar, F.: Sequence spaces: topic in modern summability theory, Series: Mathematics and Its Applications. CRC Press, Taylor & Francis Group, Boca Raton, New York (2020)
26. Mursaleen, M., Başar, F., Altay, B.: On the Euler sequence spaces which include the spaces ℓ_p and ℓ_∞ II. Nonlinear Anal. **65**, 707–717 (2006)
27. Ng, P.-N., Lee, P.-Y.: Cesáro sequence spaces of non-absolute type. Comment. Math. Prace Mat. **20**(2), 429–433 (1978)
28. Roopaei, H.: Norm of Hilbert operator on sequence spaces. J. Inequalities Appl. **2020**, 117 (2020)
29. Roopei, H.: A study on Copson operator and its associated sequence space. J. Inequalities Appl. **2020**, 120 (2020)
30. Roopaei, H., Foroutannia, D., İlkhan, M., Kara, E.E.: Cesàro spaces and norm of operators on these matrix domains. Mediterr. J. Math. **17**, 121 (2020)
31. Roopaei, H., Yaying, T.: Quasi-Cesàro matrix and associated sequence spaces. Turk. J. Math. **45**, 153–166 (2021)
32. Stieglitz, M., Tietz, H.: Matrixtransformationen von Folgenräumen eine Ergebnisübersicht. Math. Z. **154**, 1–16 (1977)
33. Talebi, G.: On multipliers of matrix domains. J. Inequalities Appl. **2018**, 296 (2018)
34. Wang, C.-S.: On Nörlund sequence spaces. Tamkang J. Math. **9**, 269–274 (1978)
35. Wilansky, A.: Summability through functional analysis. North-Holland Mathematics Studies, vol. 85. Elsevier, Amsterdam (1984)
36. Yaying, T., Hazarika, B.: On sequence spaces defined by the domain of a regular Tribonacci matrix. Math. Slovaca **70**(3), 697–706 (2020)
37. Yaying, T., Hazarika, B., İlkhan, M., Mursaleen, M.: Poisson like matrix operator and its application in p-summable space. Math. Slovaca, **71**(5), 1189–1210 (2021)
38. Yaying, T., Hazarika, B.: On sequence spaces generated by binomial difference operator of fractional order. Math. Slovaca **69**(4), 901–918 (2019)
39. Yaying, T., Hazarika, B., Mohiuddine, S.A., Mursaleen, M., Ansari, K.J.: Sequence spaces derived by the triple band generalized Fibonacci difference operator. Adv. Differ. Equ. **2020**, 639 (2020)
40. Yaying, T., Hazarika, B., Mursaleen, M.: On sequence space derived by the domain of q-Cesàro matrix in ℓ_p space and the associated operator ideal. J. Math. Anal. Appl. **493**(1), 124453 (2021)

Study on Some Particular Class of Nonlinear Integral Equation with a Hybridized Approach

Nimai Sarkar and Mausumi Sen

Abstract This article deals with a particular class of integral equations involving pure delay term. The existence of a solution is described using fixed point theory. Moreover, a hybridized scheme is proposed to investigate the approximate solution. In this context, boundary element method is used with piecewise linear interpolation. Also, an algorithm is there for error estimation and in support of the considered numerical method stability analysis is done. This testimony completely demonstrates the comprehensive study of the considered class of integral equation and understanding the behaviour of the approximate solution in the presence of delay.

Keywords Nonlinear integral equation · Constant delay · Fixed point theory · Hybridized approach

1 Introduction

In this paper, our objective is to study the following nonlinear integral equation involving pure delay.

$$\mathcal{W}(x) = \Omega(x, u(x), \mathcal{W}(x)) + v(x)\mathcal{X}\Big(\int_a^{b-\tau} \mathcal{G}(x, s, \mathcal{N}(\mathcal{W}(s)))ds\Big) \qquad (1)$$

where $a, b \in \mathcal{R}$ with $a < b$, $\tau \in [0, 1)$ is the pure delay term and we denote $I^* = [a, b-\tau]$. $\mathcal{W}(x)$, $u(x)$, $v(x)$ are continuous functions from I^* to $\mathcal{R}$, which are essentially considered from $X^* = \mathcal{C}(I^*, \mathcal{R})$, $\Omega : I^* \times I^* \times I^* \to \mathcal{R}$, $\mathcal{G} : I^* \times I^* \times I^* \to \mathcal{R}$ and $\mathcal{X} : \mathcal{R} \to \mathcal{R}$ are smooth mappings in the considered space (X^*, d^*). Here, $d^*(g, h) = sup_{x \in I^*}||g(x) - h(x)||, \forall g, h \in X^*$ and $\mathcal{N}(\mathcal{W}(x))$ represents a continuously differentiable nonlinear function of $\mathcal{W}(x)$. Numerous literatures are available

N. Sarkar (✉) · M. Sen
Department of Mathematics, National Institute of Technology Silchar, Silchar 788010, Assam, India
e-mail: nimaisarkar298@gmail.com

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_15

regarding the occurrence of integral equation in various real-world modelling problems [1–4] and such cases always demand some comprehensive study related with existence uniqueness of solution, numerical treatment towards approximate solution and stability analysis. Artikis and his co researchers, Maleknejad and his co researchers, Oregan, Karoui and several other researchers have contributed in the field of integral equations from theoretical aspect, see [5–13] using different approaches. In this context, one key point is that the exact solution is not easily available in all the circumstances, therefore numerical treatment to approximate solution is an obvious requirement for the researchers to get vivid realization regarding the behaviour of solution. Over a long time, different researches have provided numerical methods [14–22] depending upon the particular class of integral equation.

In this article, the existence of unique solution has been studied by using Banach fixed point theorem. A hybridized numerical scheme along with an algorithm for error estimation is described. Moreover, we have presented stability analysis for the considered class of integral equation. The arrangement of the current manuscript is as designed as Sect. 2 covers all the fundamentals used throughout this paper, and deals with the main results, in Sect. 3, a numerical example is there in support of the proposed numerical scheme and some closing remarks are there in Sect. 4.

2 Preliminaries

Definition 1 Let (X^*, d^*) be a complete metric space. Then a map $T : X^* \to X^*$ is said to be a contraction mapping on X^* if there exists some constant $k \in [0, 1)$ such that

$$d^*(Tx, Ty) \leq kd^*(x, y), \forall x, y \in X^*$$

where k is contractivity constant.

Definition 2 ([23, 24]) Equation (1) has the Hyers-Ulam stability if there exists a constant $C \geq 0$ satisfying the following property : For each $\epsilon > 0$, $\mathcal{W} \in X^*$, if

$$|\mathcal{W}(x) - \Omega(x, u(x), \mathcal{W}(x)) - v(x)\mathcal{X}\Big(\int_a^{b-\tau} \mathcal{G}(x, s, \mathcal{N}(\mathcal{W}(s)))ds\Big)| \leq \epsilon$$

then there exists some $w \in X^*$ satisfying Eq. (1) such that

$$|w(x) - \mathcal{W}(x)| \leq C\epsilon.$$

Theorem 1 *Banach fixed point theorem: Let (X^*, d^*) be a nonempty complete metric space with a contraction mapping $T : X^* \to X^*$ and $\mathcal{L} < 1$ be the corresponding Lipschitz constant. Then T admits a unique fixed point x_0 in X^*. That is $T(x_0) = x_0$. Furthermore, for $x \in X^*$, the following propositions hold true:*

(A) The sequence $T^n x$ converges to a fixed point x_0 of T;

(B) x_0 is the unique fixed point of T in

$$X_1^* = \Big\{y \in X^* : d^*\Big(T^n x, y\Big) < \infty\Big\};$$

(C) if $y \in X_1^*$, then

$$d^*\Big(y, x_0\Big) \leq \frac{1}{1-\mathcal{L}} d^*\Big(Ty, y\Big).$$

2.1 Main Results

Before entering into the theoretical discussion, we consider the following assumptions:
(A1) $\mathcal{X} : \mathcal{R} \to \mathcal{R}$ is Lipschitz function, i.e. there exists some positive constant L such that for every $p, q \in \mathcal{R}$

$$||\mathcal{X}(p) - \mathcal{X}(q)|| \leq L||p - q||.$$

(A2) There exist integrable functions $\mathcal{Q}_1, \mathcal{Q}_2 : I^* \times I^* \to \mathcal{R}$ such that for $\alpha, \beta \in X^*$

$$0 \leq \Omega(x, u(x), \alpha) - \Omega(x, u(x), \beta) \leq \mathcal{Q}_1(x)||\alpha - \beta||$$

and

$$0 \leq \mathcal{G}(x, s, \mathcal{N}(\alpha)) - \mathcal{G}(x, s, \mathcal{N}(\beta)) \leq \mathcal{Q}_2(x, s)||\alpha - \beta||.$$

(A3) There exist $M_{\mathcal{Q}_1}, M_{\mathcal{Q}_2} \in [0, 1)$ such that

$$sup_{x \in I^*}||\mathcal{Q}_1(x)|| \leq \frac{M_{\mathcal{Q}_1}}{L}$$

$$sup_{x \in I^*} \int_a^{b-\tau} ||\mathcal{Q}_2(x, s)||ds \leq \frac{M_{\mathcal{Q}_2}}{L}.$$

(A4) υ is bounded function on I^*, i.e. there exists $M > 0$ such that for all $x \in I^*$

$$||\upsilon(x)|| \leq M.$$

Theorem 2 *Under the assumptions (A1)–(A4), the integral Eq. (1) admits a unique solution in X^* if $\Big(\frac{M_{\mathcal{Q}_1}+1}{L}\Big) < 1$ and $\Big(\frac{M_{\mathcal{Q}_1}}{L} + MM_{\mathcal{Q}_2}\Big) < 1$ hold.*

Proof Basic motivation is to apply Banach fixed point theorem on X^*. For that purpose, we consider the mapping $T^* : X^* \rightarrow X^*$ defined as

$$(T^*\mathcal{W})(x) = \Omega(x, u(x), \mathcal{W}(x)) + v(x)\mathcal{X}\Big(\int_a^{b-\tau} \mathcal{G}(x, s, \mathcal{N}(\mathcal{W}(s)))ds\Big) \quad (2)$$

and $\mathcal{B}_r = \Big\{\mathcal{W} \in X^* : ||\mathcal{W}|| \leq r\Big\}$ where $r \geq \Big(M_\Omega + MLM_\mathcal{N}M_{\mathcal{Q}_2} + MM_\mathcal{X}\Big)L$. Let $sup_{x\in I^*}||\Omega(x, u(x), 0))|| = M_\Omega$, $sup_{x\in I^*}||\mathcal{N}(\mathcal{W}(x))|| = M_\mathcal{N}$ and $sup_{x\in I^*}||\mathcal{X}\Big(\int_a^{b-\tau} \mathcal{G}(x, s, 0)ds\Big)|| = M_\mathcal{X}$. For $\mathcal{W} \in X^*$, we proceed as follows

$$\begin{aligned}
||(T^*\mathcal{W})(x)|| &= ||\Omega(x, u(x), \mathcal{W}(x)) + v(x)\mathcal{X}\Big(\textstyle\int_a^{b-\tau} \mathcal{G}(x, s, \mathcal{N}(\mathcal{W}(s)))ds\Big)|| \\
&\leq ||\Omega(x, u(x), \mathcal{W}(x)) - \Omega(x, u(x), 0)|| + ||\Omega(x, u(x), 0)|| + ||v(x)|| \\
&\quad ||\mathcal{X}\Big(\textstyle\int_a^{b-\tau} \mathcal{G}(x, s, \mathcal{N}(\mathcal{W}(s)))ds\Big) - \mathcal{X}\Big(\int_a^{b-\tau} \mathcal{G}(x, s, 0)ds\Big)||+ \\
&\quad ||v(x)||||\mathcal{X}\Big(\textstyle\int_a^{b-\tau} \mathcal{G}(x, s, 0)ds\Big)|| \\
&\leq \mathcal{Q}_1(x)||\mathcal{W}(x)|| + sup_{x\in I^*}||\Omega(x, u(x), 0)||+ \\
&\quad ML||\textstyle\int_a^{b-\tau} \mathcal{G}(x, s, \mathcal{N}(\mathcal{W}(s)))ds - \int_a^{b-\tau} \mathcal{G}(x, s, 0)ds||+ \\
&\quad Msup_{x\in I^*}||\mathcal{X}\Big(\textstyle\int_a^{b-\tau} \mathcal{G}(x, s, 0)ds\Big)|| \\
&\leq rsup_{x\in I^*}\mathcal{Q}_1(x) + M_\Omega + ML\textstyle\int_a^{b-\tau} \mathcal{Q}_2(x, s)||\mathcal{N}(\mathcal{W}(s))||ds + MM_\mathcal{X} \\
&\leq \frac{rM_{\mathcal{Q}_1}}{L} + M_\Omega + MLM_\mathcal{N}\frac{M_{\mathcal{Q}_2}}{L} + MM_\mathcal{X} \\
&\leq \frac{rM_{\mathcal{Q}_1}}{L} + M_\Omega + MLM_\mathcal{N}M_{\mathcal{Q}_2} + MM_\mathcal{X}. \\
&\leq \frac{rM_{\mathcal{Q}_1}}{L} + \frac{r}{L} \leq r
\end{aligned}$$

Thus, $T^*\mathcal{B}_r \subset \mathcal{B}_r$. Now for $\mathcal{W}_1, \mathcal{W}_2 \in X^*$ we have

$$\begin{aligned}
||(T^*\mathcal{W}_1)(x) - (T^*\mathcal{W}_2)(x)|| &\leq ||\Omega(x, u(x), \mathcal{W}_1(x)) - \Omega(x, u(x), \mathcal{W}_2(x))||+ \\
&\quad ||v(x)||||\mathcal{X}\Big(\textstyle\int_a^{b-\tau} \mathcal{G}(x, s, \mathcal{N}(\mathcal{W}_1(s)))ds\Big)- \\
&\quad \mathcal{X}\Big(\textstyle\int_a^{b-\tau} \mathcal{G}(x, s, \mathcal{N}(\mathcal{W}_1(s)))ds\Big)|| \\
&\leq \mathcal{Q}_1(x)||\mathcal{W}_1 - \mathcal{W}_2|| + ML\textstyle\int_a^{b-\tau} ||\mathcal{G}(x, s, \mathcal{N}(\mathcal{W}_1(s)))- \\
&\quad \mathcal{G}(x, s, \mathcal{N}(\mathcal{W}_2(s)))||ds \\
&\leq sup_{x\in I^*}\mathcal{Q}_1(x)||\mathcal{W}_1 - \mathcal{W}_2|| + ML\textstyle\int_a^{b-\tau} \mathcal{Q}_2(x, s)||\mathcal{W}_1 - \mathcal{W}_2||ds \\
&\leq \frac{M_{\mathcal{Q}_1}}{L}||\mathcal{W}_1 - \mathcal{W}_2|| + ML||\mathcal{W}_1 - \mathcal{W}_2||sup_{x\in I^*}\textstyle\int_a^{b-\tau} \mathcal{Q}_2(x, s)ds \\
&\leq \frac{M_{\mathcal{Q}_1}}{L}||\mathcal{W}_1 - \mathcal{W}_2|| + ML||\mathcal{W}_1 - \mathcal{W}_2||\frac{M_{\mathcal{Q}_2}}{L} \\
&\leq \Big(\frac{M_{\mathcal{Q}_1}}{L} + MM_{\mathcal{Q}_2}\Big)||\mathcal{W}_1 - \mathcal{W}_2||
\end{aligned}$$

Hence, $d^*(T^*\mathcal{W}_1, T^*\mathcal{W}_2) \leq \left(\frac{M_{\mathcal{Q}_1}}{L} + MM_{\mathcal{Q}_2}\right)||\mathcal{W}_1 - \mathcal{W}_2||$.

This implies $T^* : X^* \to X^*$ is a contraction mapping. Theorem 1 suggests that T^* has a unique fixed point.

As the exact solution or closed form solution is not always available, therefore we present a numerical scheme to obtain the approximate solution for equations of the type (1). □

2.2 Numerical Method

Applying a hybridized treatment of Boundary element method and linear interpolation, we propose the present numerical scheme on Eq. (1). Initially, the interval $I^* = [a, b - \tau]$ is splitted into n uniformly distributed linear segments by $a = \mu_0 < \mu_1 < \mu_2 < \cdots < \mu_n = b - \tau$. In each subinterval $[\mu_{j-1}, \mu_j]$, the nonlinear term of $\mathcal{W}(x)$ is approximated by using piecewise linear interpolation scheme,

$$\mathcal{N}(\mathcal{W}(x)) = \mathcal{W}(x) - \frac{\mathcal{N}(\mathcal{W}(\mu_n)) - \mathcal{N}(\mathcal{W}(\mu_{n-1}))}{\mu_n - \mu_{n-1}}\mathcal{N}(\mathcal{W}(\mu_n))(x - \mu_n) \quad (3)$$

for $\mu_{n-1} < x < \mu_n$. In the interpolation scheme, it is assumed that at the mesh points $x = \mu_j$, the nonlinear functions $\mathcal{N}(\mathcal{W}(x))$ are prescribed and we denote piecewise linear approximate function ψ_j on each sub interval $[\mu_{j-1}, \mu_j]$. i.e.

$$\psi_j = \mathcal{W}(x) - \frac{\mathcal{N}(\mathcal{W}(\mu_j)) - \mathcal{N}(\mathcal{W}(\mu_{j-1}))}{\mu_j - \mu_{j-1}}\mathcal{N}(\mathcal{W}(\mu_j))(x - \mu_j) \text{ for } j = 1, 2, \ldots, n$$

On the interval $[\mu_{j-1}, \mu_j]$, $\epsilon_j = \delta\mu_j + (1 - \delta)\mu_{j-1}$ with $\delta \in [0, 1]$ we have the following equation

$$\mathcal{W}(\mu_j) = \Omega(\mu_j, u(\mu_j), \mathcal{W}(\mu_j)) + v(\mu_1)\mathcal{X}\Big(\int_0^1 \mathcal{G}(\mu_j, \epsilon_j, \mathcal{N}(\mathcal{W}(\epsilon_j))(\mu_j - \mu_{j-1}))d\delta\Big). \quad (4)$$

For $j = 1, 2, \ldots, n$; on each subinterval, the boundary elements are assumed to be constant, see [19, 20] and then proceed with Eq. (4). Finally (4) constitutes a system of algebraic equations. Solution of the system provides values at the mesh points as $\mathcal{W}(\mu_j)$, for sufficiently large n the system generates more accurate approximate solution as output.

2.3 Error Algorithm

This section concerns an algorithm for error estimation for Eq. (1). Let the error function be $\mathcal{E}(\mu_j) = \tilde{\mathcal{W}}(\mu_j) - \mathcal{W}(\mu_j)$ at the mesh point μ_j with $\tilde{\mathcal{W}}(x)$ being exact solution and the expression for $\mathcal{W}(\mu_j)$ is given by (3). Moreover, the absolute and relative errors are to be calculated using the formula $\mathcal{E}_{abs}(\mu_j) = |\tilde{\mathcal{W}}(\mu_j) - \mathcal{W}(\mu_j)|$ and $\mathcal{E}_{rel}(\mu_j) = \frac{|\tilde{\mathcal{W}}(\mu_j) - \mathcal{W}(\mu_j)|}{|\tilde{\mathcal{W}}(\mu_j)|}$, respectively. Here, we present maximum error bound from a theoretical point of view. On the interval $[\mu_{j-1}, \mu_j]$, $\psi_j(x)$ be the linear approximation of $\mathcal{N}(\mathcal{W}(x))$ then from theory of interpolation we have

$$|\mathcal{N}(\mathcal{W}(x)) - \psi_j(x)| = |\frac{(x - \mu_{j-1})(x - \mu_j(x))}{2}\mathcal{N}''(\mathcal{W}(c_j))|$$

where $\mu_{j-1} < c_j < \mu_j$ and double prime denotes the second derivative. For $x = \epsilon_j$,

$$|\mathcal{N}(\mathcal{W}(\epsilon_j)) - \psi_j(\epsilon_j)| = |\frac{(\epsilon_j - \mu_{j-1})(\epsilon_j - \mu_j)}{2}\mathcal{N}''(\mathcal{W}(c_j))|$$

$$\leq \frac{|\epsilon_j - \mu_{j-1}||\epsilon_j - \mu_j|}{2} \max|\mathcal{N}''(\mathcal{W}(c_j))|.$$

In particular, if length of each interval is h and for $\delta = \frac{1}{2}$

$$|\mathcal{N}(\mathcal{W}(\epsilon_j)) - \psi_j(\epsilon_j)| \leq \frac{|\frac{\mu_{j-1}+\mu_j}{2} - \mu_{j-1}||\frac{\mu_{j-1}+\mu_j}{2} - \mu_j|}{2} \max|\mathcal{N}''(\mathcal{W}(c_j))|$$

$$\leq \frac{h^2}{8}\max|\mathcal{N}''(\mathcal{W}(c_j))|.$$

This concludes the error bound for Eq. (1) corresponding to the present numerical scheme.

For stability analysis, one relevant assumption is considered

(A5) $||T^{*n+1}\mathcal{W} - T^{*n}\mathcal{W}|| \leq \left(||\mathcal{X}||\frac{M_{\mathcal{Q}_2}}{L}\right)^n \frac{(b-\tau-a)^n}{n!} d^*(T^*\mathcal{W}, \mathcal{W})$.

Theorem 3 *The equation $T^*\mathcal{W} = \mathcal{W}$, where T^* is given by (2) posses Hyers-Ulam stability; For every $\mathcal{W} \in X^*$ and $\epsilon > 0$ with*

$$d^*(T^*\mathcal{W}, \mathcal{W}) \leq \epsilon$$

there exists exactly one $w \in X^*$ such that

$$T^*w = w,$$

$$d^*(\mathcal{W}, w) \leq \mathcal{A}\epsilon,$$

for some $\mathcal{A} \geq 0$.

Proof Let $\mathcal{W} \in X^*$, for $\epsilon > 0$ we have $d^*(T^*\mathcal{W}, \mathcal{W}) \leq \epsilon$. First conclusion of Theorem 1 provides

$$lim_{n\to\infty}T^{*n}\mathcal{W}(x) = w(x).$$

For $\epsilon > 0$, $d^*(T^{*n}\mathcal{W}, w) \leq \epsilon$; assuming all the conditions from (A1) to (A5) hold. We proceed as follows

$$\begin{aligned} d^*(\mathcal{W}, w) &\leq d^*(\mathcal{W}, T^{*n}\mathcal{W}) + d^*(T^{*n}\mathcal{W}, w) \\ &\leq d^*(\mathcal{W}, T^*\mathcal{W}) + d^*(T^*\mathcal{W}, T^{*2}\mathcal{W}) + d^*(T^{*2}\mathcal{W}, T^{*3}\mathcal{W}) + \cdots \\ &\qquad + d^*(T^{*n-1}\mathcal{W}, T^{*n}\mathcal{W}) + d^*(T^{*n}\mathcal{W}, w) \\ &\leq d^*(\mathcal{W}, T^*\mathcal{W}) + \tfrac{\eta}{1!}d^*(\mathcal{W}, T^*\mathcal{W}) + \tfrac{\eta^2}{2!}d^*(\mathcal{W}, T^*\mathcal{W}) + \cdots \\ &\qquad + \tfrac{\eta^{n-1}}{(n-1)!}d^*(\mathcal{W}, T^*\mathcal{W}) + d^*(T^{*n}\mathcal{W}, w) \\ &\leq d^*(\mathcal{W}, T^*\mathcal{W}) \textstyle\sum_{s=0}^{n-1} \frac{\eta^s}{s!} + d^*(T^{*n}\mathcal{W}, w) \\ &\leq \epsilon\Big(e^{\eta}\Big) + \epsilon = \Big(1 + e^{\eta}\Big)\epsilon \end{aligned}$$

where $\eta = \frac{||\mathcal{X}||M_{\mathcal{Q}_2}(b-\tau-a)}{L}$. □

3 Numerical Results

We encounter an example in support of the proposed testimony. Particularly for $\tau = 0, 0.25$ the investigation is done, which eventually leads to a quite accurate solution. The absolute and relative errors are also provided in the representative tables.

Example For $\Omega = \frac{5}{6}x$, $v = x$, $\int_a^{b-\tau} \mathcal{G}(x, s, \mathcal{N}(\mathcal{W}(s)))ds = \int_0^{1-\tau} s^2\mathcal{W}^2(s)ds$

Approximate solutions are obtained and corresponding errors are calculated by using 3.1. and 3.2., respectively. The representative tables illustrate the numerical results for considered delay (Table 1 and Table 2).

Table 1 $\tau = 0$

x	$\tilde{\mathcal{W}}(x)$	$\mathcal{W}(x)$	$\mathcal{E}_{abs}(x)$	$\mathcal{E}_{rel}(x)$
0.25	0.25	0.20858	0.04141	0.16567
0.5	0.5	0.42233	0.07766	0.15532
0.75	0.75	0.65163	0.09836	0.13115
1	1	0.85335	0.14664	0.14664

Table 2 $\tau = 0.25$

x	$\tilde{\mathcal{W}}(x)$	$\mathcal{W}(x)$	$\mathcal{E}_{abs}(x)$	$\mathcal{E}_{rel}(x)$
0.125	0.125	0.10411	0.02082	0.16660
0.250	0.250	0.20855	0.04144	0.15532
0.375	0.375	0.31406	0.06093	0.16249
0.500	0.500	0.42234	0.07765	0.15531
0.625	0.625	0.52785	0.09714	0.15543
0.750	0.750	0.64691	0.10308	0.13744

4 Conclusion

The present article demonstrates a complete study of the considered integral equation. As the existence and uniqueness are strongly supported by the hybridized numerical treatment. Error bound and stability analysis ensure that the proposed numerical scheme might be a good option to solve equations of the form (1). Moreover, the numerical method is easy to implement and less time-consuming. In this context, Eq. (1) with variable delay appears to be an exciting area for future work.

References

1. Alturk, A.: Quasimonotonicity, the regularization-homotopy method for the two-dimensional Fredholm integral equations of the first kind. Math. Comput. Appl. **21**(2), 1–10 (2016)
2. Denisov, A.M.: Approximation of quasi-solutions of Fredholm's equation of first kind with a kernel of special form. USSR Comput. Math. Math. Phys. **11**(5), 269–276 (1971)
3. Denisov, A.M.: On the approximation when solving a Fredholm equation of the first kind with a kernel of special type. USSR Comput. Math. Math. Phys. **13**(1), 255–260 (1973)
4. Artikis, C.T., Voudouri, A.P., Artikis, T.P.: Incorporating an Integral equation for characteristic functions in investigating a class of distributions. J. Stat. Manag. Syst. **19**(1), 89–97 (2016)
5. Maleknejad, K., Nouri, K., Mollapouras, R.: Existence of solutions for some non-linear integral equations. Commun. Nonlinear Sci. Numer. Simul. **14**(6), 2559–2564 (2009)
6. Oregan, D.: Existence results for nonlinear integral equations. J. Math. Anal. Appl. **192**(3), 705–726 (1995)
7. Karoui, A.: On the existence of continuous solutions of nonlinear integral equations. Appl. Math. Lett. **18**(3), 299–305 (2005)
8. Lauran, M.: Existence results for some nonlinear integral equations. Miskolc Math. Notes **13**(1), 67–74 (2012)
9. Eshaghi Gordji, M., Baghani, H., Baghani, O.: On existence and uniqueness of solutions of a nonlinear integral equation. J. Appl. Math. https://doi.org/10.1155/2011/743923
10. Tidke, H.L., Aage, C.T., Salunke, J.N.: Existence and uniqueness of continuous solution of mixed type integral equations in cone metric space. Kathmandu Univ. J. Sci. Eng. Technol. **7**(1), 48–55 (2011)
11. Moradi, S., Mohammadi Anjedani, M., Analoei, E.: On existence and uniqueness of solutions of a nonlinear Volterra-Fredholm integral equation. Int. J. Nonlinear Anal. Appl. **6**(1), 62–68 (2015)

12. Deep, A., Deepmala, T.C.: On the existence of solutions of some non-linear functional integral equations in Banach algebra with applications. Arab. J. Basic Appl. Sci. **27**(1), 279–286 (2020)
13. Saha, D., Sen, M., Sarkar, N., Saha, S.: Existence of a solution in the Holder space for a non linear functional integral equation. Armen. J. Math. **12**(7), 1–8 (2020)
14. Mandal, B.N., Bhattacharya, S.: Numerical solution of some classes of integral equations using Bernstein polynomials. Appl. Math. Comput. **190**(2), 1707–1716 (2007)
15. Bhattacharya, S., Mandal, B.N.: Numerical solution of a singular integro-differential equation. Appl. Math. Comput. **195**(1), 346–350 (2008)
16. Alturk, A.: Numerical solution of linear and nonlinear Fredholm integral equations by using weighted mean value theorem. Springer Plus **5**, 1–15 (2016)
17. Maleknejad, K., Mahmoudi, Y.: Numerical solution of linear Fredholm integral equation by using hybrid Taylor and Block-Pulse functions. Appl. Math. Comput. **149**(3), 799–806 (2004)
18. Allouch, C., Sablonniere, P., Sbibih, D.: Solving Fredholm integral equations by approximating kernels by spline quasi-interpolants. Numer. Algorithms **56**(3), 437–453 (2011)
19. Bellour, A., Rawashdeh, E.A.: Numerical solution of first kind integral equation by using Taylor polynomials. J. Inequalities Spec. Funct. **1**(2), 23–29 (2010)
20. Pozrikidis, C.: A Practical Guide to Boundary Element Methods with the Software Library. BEMLIB, CRC Press (2002)
21. Banerjea, S., Chakraborty, R., Samanta, A.: Boundary element approach of solving Fredholm and Volterra integral equations. Int. J. Math. Model. Numer. Optim. **9**(1), 1–11 (2019)
22. Sarkar, N., Sen, M., Saha, D.: Solution of non linear Fredholm integral equation involving constant delay by BEM with piecewise linear approximation. J. Interdiscip. Math. **23**(2), 537–544 (2020)
23. Wang, G., Zhou, M., Sun, L.: Hyers-Ulam stability of linear differential equations of first order. Appl. Math. Lett. **21**(10), 1024–1028 (2008)
24. Jung, S.M.: A fixed point approach to the stability of a Volterra integral equation. Fixed Point Theory Appl. Article id 57064 (2007)

Investigation of the Existence Criteria for the Solution of the Functional Integral Equation in the L^p Space

Dipankar Saha, Mausumi Sen, and Santanu Roy

Abstract This work manifests the credibility of Darbo's fixed point theory towards the solvability of nonlinear functional convolution integral equation with deviating argument. The solution space is taken to be the space of Lebesgue integrable functions defined on $\mathbb{R}_+$. The concept of measure of noncompactness in correlation with the compactness criterion, i.e., Kolmogorov–Riesz compactness theorem in $L^p(\mathbb{R}_+)$ space has been taken. Then under certain suitable hypotheses and by the assistance of Darbo's fixed point theory, sufficient conditions for the existence of the solution have been introduced. Finally, some examples have been taken to justify the result.

Keywords Fixed point · Measure of noncompactness · Integral equation

1 Introduction

Fixed point theory encompasses its domain of applications in diverse aspects of mathematical science, physical science and engineering. In a wide range of engineering problems and modelling, the existence of a solution to a real-world problem is same as the existence of a fixed point for a suitable map or operator. Specifically, it is not always easy to determine the exact solution to the problem. In such cases, fixed point theory serves as an authentic tool to develop relevant algorithms for approximating the exact solution. In the context of solving differential equation, partial differential equation and integral equation, often the existence of solution can be attained by formulating the problem to a fixed point problem [1, 2].

In recent years, the existence of solution of the nonlinear integral equations by the fixed point theory has become an emergent domain of research [3, 4]. Particularly, Darbo fixed point theory is the prominent one that is used extensively in resolving the existence of solution of the nonlinear functional integral equation (NLFIE) [5–10].

Here, we have taken nonlinear convolution integral equation involving deviating argument of the following type:

D. Saha (✉) · M. Sen · S. Roy
NIT SILCHAR, Silchar, Assam, India
e-mail: nayan0507@gmail.com

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_16

$$y(\varphi) = g(\varphi, y(\zeta(\varphi))) + \int_0^\infty k(\varphi - s)u(s, y(\eta(s)))ds, \quad (1)$$

where $g, u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are unknown functions. $\zeta, \eta : \mathbb{R}_+ \to \mathbb{R}$ are nondecreasing functions. These type of integral equations are used to model real-world problems specifically in radiation, transportation theory and kinetic gas theory. So, the existence of solution of the considered Eq. (1) is very essential. Also the motivation is extracted from the earlier literature [6] where the author has studied the existence of solution by the Darbo fixed point concept of the following linear integral equation

$$y(t) = f(\varphi, y(\varphi)) + \int_0^\infty k(\varphi - s)(Qy)(s)ds. \quad (2)$$

which is a particular case of Eq. (1). In that work as stated above, the author has introduced the new measure of noncompactness by considering Kolmogorov–Riesz compactness theorem [11].

2 Basics and Preliminaries

Here, we deal with an infinite dimensional Banach space $(S, \|\cdot\|)$, consisting of the zero element $\tilde{\theta}$. $\bar{U}$ and $ConvU$ refer to the closure and closed convex hull of a subset U of S, respectively. $B(y, \tilde{r})$ denotes the closed ball with centre y and radius $\tilde{r}$. We will denote $B_{\tilde{r}}$ for the ball $B(\tilde{\theta}, \tilde{r})$. In addition, suppose $\mathcal{A}_S$ to be the collection of all nonempty bounded subsets of S and $\mathcal{C}_S$ to be its subcollection consisting of nonempty relatively compact subsets.

Definition 1 [12] A mapping $\tilde{\nu} : \mathcal{A}_S \to \mathbb{R}_+$ is the measure of noncompactness in S under the following conditions:

(i) $ker\tilde{\nu} = \{U \in \mathcal{A}_S : \tilde{\nu}(U) = 0\}$ is nonempty and $ker\tilde{\nu} \subset \mathcal{C}_S$.
(ii) $U \subset V \Rightarrow \tilde{\nu}(U) \leq \tilde{\nu}(V)$.
(iii) $\tilde{\nu}(\bar{U}) = \tilde{\nu}(U)$.
(iv) $\tilde{\nu}(\text{Conv } U) = \tilde{\nu}(U)$.
(v) $\tilde{\nu}(\tilde{\lambda}U + (1 - \tilde{\lambda})U) \leq \tilde{\lambda}\tilde{\nu}(U) + (1 - \tilde{\lambda})\tilde{\nu}(U)$ for $\tilde{\lambda} \in [0, 1]$.
(vi) If the sequence (U_n) of closed sets from $\mathcal{A}_S$, are such that $U_{n+1} \subset U_n$ provided $n = 1, 2, \ldots$. In addition, if $\lim_{n\to\infty} \tilde{\nu}(U_n) = 0$, then the intersection $U_\infty = \bigcap_{n=1}^\infty U_n$ is nonempty. □

Lemma 1 *[6] Suppose Ω is a nonempty, convex, closed and bounded subset of the space S. Let $H : \Omega \to \Omega$ be a continuous contraction mapping with respect to the measure of noncompactness $\tilde{\nu}$, i.e., there exists a constant $0 \leq \rho < 1$ such that $\tilde{\nu}(HU) \leq \rho\tilde{\nu}(U)$ for any nonempty subset U of Ω. Then H has a fixed point in the set Ω.*

2.1 Main Results

To find the sufficient condition for the existence of the solution, the following hypotheses have been proposed, which are as follows:

i. $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions, i.e., $g(\varphi, y)$ is measureable for any $x, y \in \mathbb{R}$ and continuous for almost all $s, \varphi \in \mathbb{R}_+$. In addition, there exists $a \in L^p(\mathbb{R}_+)$ such that
$|g(\varphi, x) - g(s, y)| \leq |a(\varphi) - a(s)| + \lambda|x - y|$ also holds.
ii. $g(\cdot, 0) \in L^p(\mathbb{R}_+)$.
iii. $k \in L^1(\mathbb{R})$.
iv. $u(s, y(s))$ is a continuous function from the space $L^p(\mathbb{R}_+)$ onto itself and there exists a constant $b \in \mathbb{R}_+$ such that $\|u(s, y)\|_{L^p[\sigma,\infty)} \leq b\|y\|_{L^p[\sigma,\infty)}$ for any $y \in L^p(\mathbb{R}_+)$ and $\sigma \in \mathbb{R}_+$.
v. The inequality $\lambda + b\|k\|_{L^1(\mathbb{R})} < 1$ also holds.

Theorem 2 *[6] Suppose $1 \leq p < \infty$ and Y is a bounded subset of $L^p(\mathbb{R}_+)$. Then for $y \in Y$ and $\epsilon > 0$, let*

$$w(y, \epsilon) = \sup\{(\int_0^\infty |y(\varphi + h) - y(\varphi)|^p d\varphi)^{\frac{1}{p}} : |h| < \epsilon\}$$

$$w(Y, \epsilon) = \sup\{w(y, \epsilon) : y \in Y\}$$

$$w(Y) = lim_{\epsilon\to 0} w(Y, \epsilon)$$

and

$$d_T(Y) = \sup\{(\int_T^\infty |y(s)|^p ds)^{\frac{1}{p}} : y \in Y\}$$

$$d(Y) = lim_{T\to\infty} d_T(Y).$$

Then, $w_0(Y) = w(Y) + d(Y)$ gives the measure of noncompactness on $L^p(\mathbb{R}_+)$. Also we show that $ker\tilde{\nu} = \mathcal{C}_{L^p(\mathbb{R}_+)}$.

Remark 1 [6] Under the proposed hypotheses (iii), the linear operator, $K : L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+)$ defined as $(Ky)(\varphi) = \int_0^\infty k(\varphi - s)y(s)ds$ is a continuous operator, and $\|Ky\|_p \leq \|k\|_{L^1(\mathbb{R})}\|y\|_p$.

Theorem 3 *Under the proposed assumptions (i)–(v), Eq. (1) has at least one solution in the space $L^p(\mathbb{R}_+)$.*

Proof Let us represent the integral equation in operator form $(Hy)(\varphi) = g(\varphi, y(\zeta(\varphi))) + \int\limits_0^\infty k(\varphi - s)u(s, y(\eta(s)))ds$.

Accounting the Carathéodory conditions, we found that Hy is measurable for any $y \in L^p(\mathbb{R}_+)$.

Step 1: $Hy \in L^p(\mathbb{R}_+)$ for any $y \in L^p(\mathbb{R}_+)$.

$$|H(y)(\varphi)| \leq |g(\varphi, y(\zeta(\varphi))) - g(\varphi, 0)| + |g(\varphi, 0)| + b\int\limits_0^\infty k(\varphi - s)|y(\eta s)|ds \quad (3)$$
$$\leq \lambda|y| + |g(\varphi, 0)| + b\int\limits_0^\infty k(\varphi - s)|y(\eta(s))|ds.$$

Thus,
$(\int\limits_0^\infty |H(y)(\varphi)|^p d\varphi)^{\frac{1}{p}} \leq \lambda(\int\limits_0^\infty |y(\varphi)|^p d\varphi)^{\frac{1}{p}} + (\int\limits_0^\infty |g(\varphi, 0)|^p d\varphi)^{\frac{1}{p}} + b(\int\limits_0^\infty(\int\limits_0^\infty k(\varphi - s)|y(\eta s)|ds)^p d\varphi)^{\frac{1}{p}}$ Now employing Young's inequality, we get

$$\|Hy\|_p \leq \lambda\|y\|_p + \|g(\cdot, 0)\|_p + b\|k\|_1\|y\|_p. \quad (4)$$

Hence, $Hy \in L^p(\mathbb{R}_+)$ is a well-defined map. Also, by the hypothesis (ii) (v), and the inequality given by the Eq. (4), $H(B_{r_0}) \subseteq B_{r_0}$ for $\|y\| \leq r_0$ where $r_0 = \frac{\|g(\cdot,0)\|_p}{1-\lambda b\|k\|_1}$. Furthermore, H is continuous in $L^p(\mathbb{R}_+)$ because $g(\varphi, \cdot)$, u and k are continuous for a.e $\varphi \in \mathbb{R}_+$. **Step 2:** Estimate of $w(Hy)$.

$$|(Hy)(\varphi + h) - (Hy)(\varphi)| = |g(\varphi + h, y(\zeta(\varphi + h))) + \int\limits_0^\infty k(\varphi + h - s)u(s, y(\eta s))ds$$
$$-(g(\varphi, y(\zeta(\varphi))) + \int\limits_0^\infty k(\varphi - s)u(s, y(\eta s))ds$$
$$\leq |g(\varphi + h, y(\zeta(\varphi + h))) - g(\varphi, y(\zeta(\varphi)))|$$
$$+|\int\limits_0^\infty k(\varphi + h - s)u(s, y(\eta s))ds - \int\limits_0^\infty k(\varphi - s)u(s, y(\eta s))ds|$$
$$\leq |g(\varphi + h, y(\zeta(\varphi + h))) - g(\varphi + h, y(\zeta(\varphi)))|$$
$$+|g(\varphi + h, y(\zeta(\varphi))) - g(\varphi + h, y(\zeta(\varphi)))| + \int\limits_0^\infty |k(\varphi + h - s) - k(\varphi - s)||u(s, y(\eta s))|ds$$
$$\leq |a(\varphi + h) - a(\varphi)| + \lambda|y(\zeta(\varphi + h)) - y(\zeta(\varphi + h))|$$
$$+b\int\limits_0^\infty |k(\varphi + h - s) - k(\varphi - s)||y(\eta s)|ds.$$

Thus,

$$(\int_0^\infty |(Hy)(\varphi+h)-(Hy)(\varphi)|^p d\varphi)^{\frac{1}{p}} \le (\int_0^\infty |a(\varphi+h)-a(\varphi)|^p d\varphi)^{\frac{1}{p}} + \lambda(\int_0^\infty |y(\zeta(\varphi+h))-y(\zeta(\varphi))|^p d\varphi)^{\frac{1}{p}}$$
$$+b(\int_0^\infty |\int_0^\infty |k(\varphi+h-s)-k(\varphi-s)||y(\eta s)|ds|^p d\varphi)^{\frac{1}{p}}$$
$$\le w(a,\epsilon)+\lambda w(y,\epsilon)+b\|y\|_p \int_0^\infty |k(\varphi+h-s)-k(\varphi-s)|d\varphi$$
$$\le w(a,\epsilon)+\lambda w(y,\epsilon)+b\|y\|_p\|k-\tau_h k\|_{L^1(\mathbb{R})}. \tag{5}$$

Since $\{a\}$ and $\{k\}$ are compact sets in $L^p(\mathbb{R}_+)$ and $L^1(\mathbb{R})$, respectively, we have inferred that $w(a,\epsilon)\to 0$ and $\|k-\tau_h k\|_{L^1(\mathbb{R})}\to 0$ as $\epsilon\to 0$.

Then, we obtain

$$w(HY,\epsilon)\le \lambda w(Y)\le (\lambda+b\|k\|_{L^1(\mathbb{R})})w(Y). \tag{6}$$

Step 3: Estimate of $d(Hy)$.

Let us fix an arbitrary number $\sigma>0$. Then, considering the hypothesis, for an arbitrary $y\in Y$, we get

$$(\int_\sigma^\infty |(Hy)(\varphi)|^p d\varphi)^{1/p}$$
$$\le (\int_\sigma^\infty |g(\varphi,y(\zeta(\varphi)))-g(\varphi,0)|^p d\varphi)^{1/p} + (\int_\sigma^\infty |g(\varphi,0)|^p d\varphi)^{1/p}$$
$$+(\int_\sigma^\infty |\int_0^\infty k(\varphi-s)u(s,y(\eta s))|^p d\varphi)^{1/p}$$
$$\le \lambda(\int_\sigma^\infty |y(\varphi)|^p d\varphi)^{1/p} + (\int_\sigma^\infty |g(\varphi,0)|^p d\varphi)^{1/p} + b\|k\|_{L^1(\mathbb{R})}(\int_\sigma^\infty |y(\varphi)|^p d\varphi)^{1/p}.$$

Since $g(\varphi,0)$ is compact in $L^p(\mathbb{R}_+)$, we obtain $(\int_\sigma^\infty |g(\varphi,0)|^p d\varphi)^{1/p}\to 0$ as $\sigma\to\infty$.

Thus, we have inferred the following inequality

$$d(HY)\le (\lambda+b\|k\|_{L^1(\mathbb{R})})d(Y). \tag{7}$$

Now linking (6) and (7), we have acquired that

$$w_0(HY)\le (\lambda+b\|k\|_{L^1(\mathbb{R})})w_0(Y). \tag{8}$$

Finally, by (8) and Lemma (1), the operator H has a fixed point in B_{r_0}. Thus, the NLFIE (1) has a solution in $L^p(\mathbb{R}_+)$.

3 Numerical Examples

Example 1 $y(\varphi) = \frac{\sin y(\varphi)}{2} + \frac{1}{(\varphi+5)^{\frac{3}{2}}} + \int\limits_0^\infty \frac{e^{-(\varphi-s)^2}\cos(\varphi-s)}{(\varphi-s)^2+7} |arctany(s)| ds.$

It is a particular case of Eq. (1) with $g(\varphi, y) = \frac{\sin y(\varphi)}{2} + \frac{1}{(\varphi+5)^{\frac{3}{2}}}$, $k(\varphi) = \frac{e^{-\varphi^2}\cos(\varphi)}{\varphi^2+7}$ and $u(\varphi, y(\varphi)) = |arctany(\varphi)|$. Now,

$$\begin{aligned} |g(\varphi, x(\zeta(\varphi))) - g(s, y(\zeta(\varphi)))| &= |\frac{\sin x(\varphi)}{2} + \frac{1}{(\varphi+5)^{\frac{3}{2}}} - (\frac{\sin y(s)}{2} + \frac{1}{(s+5)^{\frac{3}{2}}})| \\ &\le |\frac{\sin x(\varphi)}{2} - \frac{\sin y(s)}{2}| + |\frac{1}{(\varphi+5)^{\frac{3}{2}}} - \frac{1}{(s+5)^{\frac{3}{2}}}| \\ &\le |\frac{1}{(\varphi+5)^{\frac{3}{2}}} - \frac{1}{(s+5)^{\frac{3}{2}}}| + \frac{1}{2}|x - y|. \end{aligned} \tag{9}$$

Thus, g satisfies assumption (i) with $a(\varphi) = \frac{1}{(\varphi+5)^{\frac{3}{2}}}$ and $\lambda = \frac{1}{2}$.

Here, it is easy to interpret that $g(\cdot, 0)$ satisfies assumption (ii) for $p > \frac{2}{3}$.

Moreover, $\|k\|_{L^1} \le \frac{\sqrt{\pi}}{7}$ implies that assumption (iii) holds.

Also, $u(s, y(s)) = |arctany(s)| \le |y(s)|$ and $\zeta(\varphi) = \eta(\varphi) = \varphi$.

Thus, $u(\varphi, y(\varphi))$ satisfies hypothesis (iv) with $b = 1$.

Now, Theorem (3) guarantees that Eq. (1) has a solution in $L^p(\mathbb{R}_+)$ for $p > \frac{2}{3}$.

Example 2 $y(\varphi) = \frac{e^{-\varphi}}{\varphi^{1/2}} + \int\limits_0^\infty \frac{e^{-arctan(\varphi-s)}}{5+(\varphi-s)^2} \ln(1 + |y(s)|) ds.$

It is a particular case of Eq. (1) with $g(\varphi, y) = \frac{e^{-\varphi}}{\varphi^{1/2}}$, $k(\varphi) = \frac{e^{-arctan(\varphi)}}{5+(\varphi)^2}$, and $u(\varphi, y(\varphi)) = \ln(1 + |y(\varphi)|)$. Also, $\zeta(\varphi) = \eta(\varphi) = \varphi$.

Thus, g satisfies assumption (i) with $a(\varphi) = \frac{e^{-\varphi}}{\varphi^{1/2}}$ and $\lambda = 0$. Also $g(\cdot, 0)$ satisfies assumptions (ii) for $p < 2$. Also $k(\varphi)$ satisfies assumption (iii) as $\|k\|_{L^1} \le 1$.

In addition $u(\varphi, y(\varphi))$ satisfies assumption (iv) with $b = 1$.

Thereby, assumption (v) is satisfied.

Consequently, Theorem (3) guarantees that Eq. (2) has a solution in $L^p(\mathbb{R}_+)$ for $p < 2$.

4 Conclusion

Here, the sufficient condition for the existence of convolution integral equation with the changed argument has been derived. Finally, some examples have been added in the end to validate the result.

References

1. Agarwal, R.P., Benchohra, M., Hamani, S., Slimani, B.A.: Existence results for differential equations with fractional order and impulses. Mem. Differ. Equ. Math. Phys. **44**, 1–21 (2008)
2. Agarwal, R.P., Benchohra, M., Hamani, S.: A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. Acta Appl. Math. **109**, 973–1033 (2010)
3. Dhage, B.C.: On a fixed point theorem in Banach algebras with applications. Appl. Math. Lett. **18**, 273–280 (2005)
4. Hu, X., Yan, J.: The global attractivity and asymptotic stability of solution of a nonlinear integral equation. J. Math. Anal. Appl. **321**, 147–156 (2006)
5. Maleknejad, K., Nouri, K., Mollapourasl, R.: Existence of solutions for some nonlinear integral equations. Commun. Nonlinear Sci. Numer. Simul. **14**, 153–168 (2009)
6. Khosravi, H., Allahyari, R., Haghighi, A.S.: Existence of solutions of functional integral equations of convolution type using a new construction of a measure of noncompactness on $L^p(\mathbb{R}_+)$. Appl. Math. Comput. **260**, 140–147 (2015)
7. Mollapourasl, R., Ostadi, A.: On solution of functional integral equation of fractional order. Appl. Math. Comput. **270**, 631–643 (2015)
8. Rabbani, M., Arab, R., Hazarika, B.: Solvability of nonlinear quadratic integral equation by using simulation type condensing operator and measure of noncompactness. Appl. Math. Comput. **349**, 102–117 (2019)
9. Sen, M., Saha, D., Agarwal, R.P.: A Darbo fixed point theory approach towards the existence of a functional integral equation in a Banach algebra. Appl. Math. Comput. **358**, 111–118 (2019)
10. Saha, D., Sen, M., Sarkar, N., Saha, S.: Existence of solutions of functional integral equations of convolution type using a new construction of a measure of noncompactness on $L^p(\mathbb{R}_+)$. Armen. J. Math. **12**, 1–8 (2020)
11. Hanche-Olsen, H., Holden, H.: The Kolmogorov-Riesz compactness theorem. Expo. Math. **28**, 385–394 (2010)
12. Banas, J.: On measures of noncompactness in Banach spaces. Comment. Math. Univ. Carol. **21**, 131–143 (1980)

Functional Inequalities for the Generalized Wright Functions

Sourav Das and Khaled Mehrez

Abstract In this work, our aim is to obtain some mean value inequalities for the generalized Wright function. Mainly, we establish Turán, Redheffer, Wilker and Lazarević-type inequalities for the generalized Wright function. Furthermore, the monotonicity properties of ratios for partial sums of the series of these functions are discussed. Finally, some other related inequalities are also derived as a consequence.

Keywords Wright functions · Turán-type inequalities · Monotonicity properties

2010 Mathematics Subject Classification 33B15 · 33C10 · 33C15

1 Introduction

Special functions play a vital role in mathematical physics, quantum physics, theoretical physics, approximation theory, number theory and in many branches of science. A vital role of the Wright function can be found [6, 16] in the complex systems. These functions are also involved in the solution of the fractional order linear partial differential equations.

Special functions and the theory of inequalities are related to each other, and several open problems were solved with the help of various inequalities. One of the special kind of inequalities is Turán's inequality. Let $R_n(x)$ be polynomial of degree n. Then Turán's determinant is defined as $\Delta_n(x) = \left[R_{n+1}(x)\right]^2 - R_{n+2}(x)R_n(x)$. If $\Delta_n(x) \geq 0$, then $R_n(x)$ is said to satisfy Turán's inequality, which was introduced

S. Das (✉)
Department of Mathematics, National Institute of Technology Jamshedpur, Jamshedpur 831014, Jharkhand, India
e-mail: souravdasmath@gmail.com; souravdas.math@nitjsr.ac.in

K. Mehrez
Department of Mathematics, Faculty of Sciences of Tunis, University of Tunis El Manar, Tunis, Tunisia

Department of Mathematics, Kairouan Preparatory Institute for Engineering Studies, University of Kairouan, Kairouan 3100, Tunisia

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_17

by P. Turán in 1950 [18]. This inequality has been attracted the attention of several mathematicians and researchers and has been proved for various special functions such as Hypergeometric functions [1, 2, 17], Wright functions [9, 10], Mittag-Leffler function [11, 12], Fox-Wright functions [13–15] and so forth due to its several applications in information theory [7] and in modelling credit risk, as discussed below. Turán's inequality has applications in various areas of science. In [4], a model has been considered where the bank has an option to foreclose upon the borrower at any time. Using the results of [3, 8], it can be verified that geometric Brownian motion is followed by the firm's assets [9]. Recently, K. Mehrez [9] derived several functional inequalities for the Wright function $W_{\alpha,\beta}(z)$, defined [16, 19] as

$$W_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{m!\Gamma(m\alpha + \beta)}, \quad \beta \in \mathbb{C}, \quad \alpha > -1. \tag{1}$$

It is well known that $W_{\alpha,\beta}(z)$ is an holomorphic function and the order is $(1+\alpha)^{-1}$. $W_{\alpha,\beta}(z)$ is also called as the generalized Bessel function [6, 16].

The above results inspire us to consider the generalized Wright function, defined as

$$W^k_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma_k(i+1)\Gamma_k(\beta + i\alpha)}, \qquad k > 0,\ \alpha > -1,\ \beta, z \in \mathbb{C}, \tag{2}$$

where $\Gamma_k(x)$ is defined [20] for $\Re(z) > 0$, $k > 0$ as

$$\Gamma_k(z) = \int_0^{\infty} t^{z-1} e^{-\frac{t^k}{k}} dt = \lim_{m\to\infty} \frac{(mk)^{\frac{z}{k}-1} m! k^m}{z(z+k)\cdots(z+(m-1)k)}. \tag{3}$$

It can be noted that $\lim_{k\to 1} \Gamma_k(z) = \Gamma(z)$. The k-digamma and k-polygamma functions are defined [20] as

$$\psi_k(z) = \frac{d}{dz} \ln \Gamma_k(z), \quad \psi_k^{(n)}(z) = \frac{d^n}{dz^n} \psi_k(z), \quad n \in \mathbb{N}.$$

It is well known that $\Gamma_k(z)$ and $\psi_k(z)$ satisfy the following relations [20]:

$$\Gamma_k(z+k) = z\Gamma_k(z), \quad \Re(z) > 0,$$

$$\psi_k(z) = \frac{\ln k - \gamma}{k} - \frac{1}{z} + \sum_{m=1}^{\infty} \frac{z}{mk(mk+z)},$$

$$\psi_k^{(n)}(z) = (-1)^{n+1} n! \sum_{m=0}^{\infty} \frac{1}{(mk+z)^{n+1}}, \quad n \in \mathbb{N}.$$

Before proceeding with the main results, let us discuss about convex functions, which will be helpful to obtain the main results. A function $g(x) : [c, d] \subset \mathbb{R} \to \mathbb{R}$ is called convex if for each $u, v \in [c, d]$ and $\mu \in [0, 1]$, we have

$$g(\mu u + (1-\mu)v) \leq \mu g(u) + (1-\mu)g(v). \tag{4}$$

$g(x)$ is called concave function if the inequality (4) is reversed. A function $g(x)$ defined on $[c, d]$ is called logarithmically convex or log-convex (log-concave) if $\log g(x)$ is convex (concave). A differentiable function $f(x)$ on $[0, 1]$ is convex (concave) if and only if $f'(x)$ is increasing (decreasing) [5]. From Bohr–Mollerup theorem [5], we can see that the gamma function $\Gamma(x)$ is logarithmically convex and the psi function (digamma function) $\psi(x)$ is concave for any positive real x.

We have organized this paper as follows. In Sect. 2, we derive some Turán type inequalities for $W^k_{\alpha,\beta}(z)$. In addition, monotonicity criterion of ratios for sections of series of $W^k_{\alpha,\beta}(z)$ is established. In Sect. 3, Lazarević and Wilker type inequalities for this function are derived. In Sect. 4, sharpened Redheffer type inequalities associated to $W^k_{\alpha,\beta}(z)$ are proved. Finally, some other related inequalities are also proved in this section.

Let us state the following lemmas [9] which will be helpful to derive the main results.

Lemma 1 *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be real numbers. If $b_n > 0$ and $\{a_n/b_n\}_{n=0}^{\infty}$ is monotonically increasing (decreasing), then $\left\{\frac{\sum_{i=0}^{n} a_i}{\sum_{i=0}^{n} b_i}\right\}_{n=0}^{\infty}$ is monotonically increasing (decreasing).*

Lemma 2 *Let the region of convergence of $F(x) = \sum_{n=0}^{\infty} a_n x^n$ and $G(x) = \sum_{n=0}^{\infty} b_n x^n$ be $(-r, r)$. If $b_n > 0$ and $\{a_n/b_n\}_{n=0}^{\infty}$ is (strictly) monotonically increasing (decreasing), then the ratio $\frac{F(x)}{G(x)}$ is (strictly) monotonically increasing on $(0, r)$.*

Lemma 3 *Let the two continuous functions $g, h : [c, d] \to \mathbb{R}$, be differentiable on (c, d) and $h'(x)$ do not vanish on (c, d). If g'/h' is increasing (or decreasing) on (c, d), then the ratios*

$$\frac{g(x) - g(c)}{h(x) - h(c)} \quad \text{and} \quad \frac{g(x) - g(d)}{h(x) - h(d)},$$

are increasing (or decreasing) on (c, d).

2 Turán-Type Inequalities

We consider normalized Wright functions $\mathcal{W}^k_{\alpha,\beta}(z)$ defined as

$$\mathcal{W}^k_{\alpha,\beta}(z) = \Gamma_k(\beta) W^k_{\alpha,\beta}(z), \tag{5}$$

and for modified k-Wright function $\mathbb{W}^k_{\alpha,\beta}(z)$ defined as

$$\mathbb{W}^k_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i+1)\Gamma_k(\beta+i\alpha)}, \qquad k>0,\ \alpha>-1,\ \beta, z \in \mathbb{C}. \tag{6}$$

Clearly,

$$\left(\mathbb{W}^k_{\alpha,\beta}(z)\right)' = \mathbb{W}^k_{\alpha,\beta+\alpha}(z).$$

Let us state the results.

Theorem 1 *$\mathcal{W}^k_{\alpha,\beta}(z)$ satisfies the following inequality for $z \in (0,\infty)$:*

$$\mathcal{W}^k_{\alpha,\beta}(z)\mathcal{W}^k_{\alpha,\beta+2k}(z) \geq \left(\mathcal{W}^k_{\alpha,\beta+k}(z)\right)^2, \quad \forall \alpha, \beta, k > 0. \tag{7}$$

Proof We have

$$\mathcal{W}^k_{\alpha,\beta}(z) = \sum_{i=0}^{\infty} a_i(\alpha,\beta,k) z^i,$$

where $a_i(\alpha,\beta,k) = \frac{\Gamma_k(\beta)}{\Gamma_k(i+1)\Gamma_k(\beta+i\alpha)}$.

Now, $\frac{\partial^2}{\partial\beta^2}\log(a_i(\alpha,\beta,k)) = \psi_k'(\beta) - \psi_k'(\beta+i\alpha)$, where $\psi_k = \frac{\Gamma_k'(x)}{\Gamma_k(x)}$ is concave. Hence, $a_i(\alpha,\beta,k)$ is log convex on $(0,\infty)$. Therefore, $\mathcal{W}^k_{\alpha,\beta}(z)$ is logarithmically convex on the positive real line. Hence,

$$\mathcal{W}^k_{\alpha,t\beta_1+(1-t)\beta_2}(z) \leq \left(\mathcal{W}^k_{\alpha,\beta_1}(z)\right)^t \left(\mathcal{W}^k_{\alpha,\beta_2}(z)\right)^{1-t}, \quad \forall \alpha, \beta_1, \beta_2 > 0, t \in [0,1].$$

Putting $t = 1/2$, $\beta_1 = \beta$ and $\beta_2 = \beta + 2k$, we claim that (7) holds. Now, the Cauchy product helps us to obtain

$$\begin{aligned}
&\mathcal{W}^k_{\alpha,\beta}(z)\mathcal{W}^k_{\alpha,\beta+2k}(z)\\
&= \Gamma_k(\beta)\Gamma_k(\beta+2k)\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\frac{1}{\Gamma_k(j+1)\Gamma_k(i-j+1)\Gamma_k(\beta+j\alpha)\Gamma_k(\beta+(i-j)\alpha+2k)}\right)z^i
\end{aligned}$$

$$\begin{aligned}
&\left(\mathcal{W}^k_{\alpha,\beta+k}(z)\right)^2\\
&= \Gamma_k^2(\beta+k)\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}\frac{1}{\Gamma_k(j+1)\Gamma_k(i-j+1)\Gamma_k(\beta+j\alpha+k)\Gamma_k(\beta+(i-j)\alpha+k)}\right)z^i
\end{aligned}$$

Now,

$$\begin{aligned}
&\mathcal{W}^k_{\alpha,\beta}(z)\mathcal{W}^k_{\alpha,\beta+2k}(z) - \left(\mathcal{W}^k_{\alpha,\beta+k}(z)\right)^2\\
&= \Gamma_k(\beta)\Gamma_k(\beta+p)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{(\beta+k)(\beta+j\alpha)-\beta(\beta+(i-j)\alpha)+}{\Gamma_k(j+1)\Gamma_k(i-j+1)\Gamma_k(\beta+j\alpha+k)\Gamma_k(\beta+(i-j)\alpha+2k)}z^i\\
&= \Gamma_k(\beta)\Gamma_k(\beta+p)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{\beta^2+j\alpha\beta+k\beta+kj\alpha-\beta^2-(i-j)\alpha\beta-k\beta}{\Gamma_k(j+1)\Gamma_k(i-j+1)\Gamma_k(\beta+j\alpha+k)\Gamma_k(\beta+(i-j)\alpha+2k)}z^i\\
&= \Gamma_k(\beta)\Gamma_k(\beta+p)\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{\alpha\beta(2j-i)+\alpha jk}{\Gamma_k(j+1)\Gamma_k(i-j+1)\Gamma_k(\beta+j\alpha+k)\Gamma_k(\beta+(i-j)\alpha+2k)}z^i\\
&= \Gamma_k(\beta)\Gamma_k(\beta+p)\sum_{i=0}^{\infty}\sum_{j=0}^{i}A_{i,j}(\alpha,\beta)z^i,
\end{aligned}$$

where

$$A_{i,j}(\alpha,\beta) = \frac{\alpha\beta(2j-i)+\alpha jk}{\Gamma_k(j+1)\Gamma_k(i-j+1)\Gamma_k(\beta+j\alpha+k)\Gamma_k(\beta+(i-j)\alpha+2k)}.$$

If i is even, then we obtain

$$\begin{aligned}
&\sum_{j=0}^{i}A_{i,j}(\alpha,\beta)\\
&= \sum_{j=0}^{i/2-1}A_{i,j}(\alpha,\beta) + \sum_{j=i/2+1}^{i}A_{i,j}(\alpha,\beta) + A_{i,i/2}(\alpha,\beta)\\
&= \sum_{j=0}^{i/2-1}A_{i,i-j}(\alpha,\beta) + \sum_{j=0}^{i/2-1}A_{i,j}(\alpha,\beta) + \frac{\frac{k\alpha i}{2}}{\Gamma_k^2(i/2+1)\Gamma_k(\beta+\frac{i\alpha}{2}+k)\Gamma_k(\beta+\frac{i\alpha}{2}+2k)}\\
&= \sum_{j=0}^{[(i-1)/2]}(A_{i,j}(\alpha,\beta)+A_{i,i-j}(\alpha,\beta)) + \frac{\frac{k\alpha i}{2}}{\Gamma_k^2(i/2+1)\Gamma_k(\beta+\frac{i\alpha}{2}+k)\Gamma_k(\beta+\frac{i\alpha}{2}+2k)}.
\end{aligned}$$

Similarly, if i is odd, then

$$\begin{aligned}
\sum_{j=0}^{i}A_{i,j}(\alpha,\beta) = &\sum_{j=0}^{[(i-1)/2]}(A_{i,j}(\alpha,\beta)+A_{i,i-j}(\alpha,\beta))\\
&+ \frac{\frac{k\alpha i}{2}}{\Gamma_k^2(i/2+1)\Gamma_k(\beta+\frac{i\alpha}{2}+k)\Gamma_k(\beta+\frac{i\alpha}{2}+2k)}.
\end{aligned}$$

Simplifying the above expression, we obtain

$$A_{i,j}(\alpha,\beta)+A_{i,i-j}(\alpha,\beta)=\frac{\alpha^2\beta(i-2j)^2+\alpha^2j^2k+\alpha^2k^2(i-j)^2+\alpha ik(\beta+k)}{\Gamma_k(j+1)\Gamma_k(i-j+1)\Gamma_k(\beta+j\alpha+2k)\Gamma_k(\beta+(i-j)\alpha+2k)}.$$

Consequently for any $z>0$, we have

$$\mathcal{W}^k_{\alpha,\beta}(z)\mathcal{W}^k_{\alpha,\beta+2k}(z)-\left(\mathcal{W}^k_{\alpha,\beta+k}(z)\right)^2>0\quad\forall\alpha,\beta>0.$$

Remark 1 It can be observed from the proof of above Theorem 1, that $\log\mathcal{W}^k_{\alpha,\beta}(z)$ is a convex function on $(0,\infty)$. Therefore, $\log\mathcal{W}^k_{\alpha,\beta+\epsilon}(z)-\log\mathcal{W}^k_{\alpha,\beta}(z)$ is monotonically increasing for each $\epsilon>0$. By choosing $\epsilon=\alpha>0$, we claim that $\mathcal{W}^k_{\alpha,\beta+\alpha}(z)/\mathcal{W}^k_{\alpha,\beta}(z)$ is monotonically increasing for each $z\in(0,\infty)$.

Theorem 2 *Let α, β_1, β_2 and k be positive. If $\beta_1<\beta_2$, (or $\beta_2<\beta_1$), then $\mathbb{W}^k_{\alpha,\beta_1}(z)/\mathbb{W}^k_{\alpha,\beta_2}(z)$ is monotonically increasing (or decreasing) on $(0,\infty)$. Furthermore, the following Turán-type inequality holds :*

$$\mathbb{W}^k_{\alpha,\beta_2}(z)\mathbb{W}^k_{\alpha,\beta_1+\alpha}(z)-\mathbb{W}^k_{\alpha,\beta_1}(z)\mathbb{W}^k_{\alpha,\beta_2+\alpha}(z),\quad\forall\beta_2>\beta_1>0,\ \alpha>0.\tag{8}$$

In particular,

$$\left(\mathbb{W}^k_{\alpha,\beta+\alpha}(z)\right)^2-\mathbb{W}^k_{\alpha,\beta}(z)\mathbb{W}^k_{\alpha,\beta+2\alpha}(z)\geq0,\quad\forall\alpha,\beta,z>0.\tag{9}$$

Proof Using the definition of modified k-Wright function (6), we have

$$\mathbb{W}^k_{\alpha,\beta_1}(z)/\mathbb{W}^k_{\alpha,\beta_2}(z)=\sum_{i=0}^{\infty}\frac{z^i}{\Gamma(i+1)\Gamma_k(\beta_1+i\alpha)}\Bigg/\sum_{i=0}^{\infty}\frac{z^i}{\Gamma(i+1)\Gamma_k(\beta_2+i\alpha)}.$$

Let

$$u_i=\Gamma_k(\beta_2+i\alpha)/\Gamma_k(\beta_1+i\alpha),\qquad i\geq0.$$

Then

$$\frac{u_{i+1}}{u_i}=\frac{\Gamma_k(\beta_2+\alpha+i\alpha)\Gamma_k(\beta_1+i\alpha)}{\Gamma_k(\beta_2+i\alpha)\Gamma_k(\beta_1+\alpha+i\alpha)}.$$

Since $\Gamma_k(z)$ is logarithmically convex, the ratio $\Gamma_k(z+a)/\Gamma_k(z)$ is monotonically increasing on the positive real line for any positive a. Hence, for any $a,b,z>0$, we have

$$\frac{\Gamma_k(z+a)}{\Gamma_k(z)}\leq\frac{\Gamma_k(z+a+b)}{\Gamma_k(z+b)}.\tag{10}$$

Case I: When $\beta_1>\beta_2$.

Putting $z=\beta_2+i\alpha$, $a=\alpha$ and $b=\beta_1-\beta_2>0$ in (10), we obtain

$$\frac{u_{i+1}}{u_i} = \frac{\Gamma_k(\beta_2 + \alpha + i\alpha)\Gamma_k(\beta_1 + i\alpha)}{\Gamma_k(\beta_2 + i\alpha)\Gamma_k(\beta_1 + \alpha + i\alpha)} \le 1.$$

Now combining Lemma 2 and the above inequality, we claim that $u_{i+1} \le u_i \forall i \ge 0$ iff $\beta_1 > \beta_2$, and $\mathbb{W}^k_{\alpha,\beta_1}(z) \Big/ \mathbb{W}^k_{\alpha,\beta_2}(z)$ is monotonically decreasing on the positive real line if $\beta_1 > \beta_2$.

Case II: When $\beta_2 > \beta_1$.

Putting $z = \beta_1 + i\alpha, a = \alpha$ and $b = \beta_2 - \beta_1 > 0$ in (10), we have $u_{i+1} \ge u_i \forall i \ge 0$. Hence, $\mathbb{W}^k_{\alpha,\beta_1}(z) \Big/ \mathbb{W}^k_{\alpha,\beta_2}(z)$ is monotonically increasing on $(0, \infty)$ if $\beta_2 > \beta_1$, by Lemma 2. Since $\mathbb{W}^k_{\alpha,\beta_1}(z) \Big/ \mathbb{W}^k_{\alpha,\beta_2}(z)$ is monotonically increasing on $(0, \infty)$ if $\beta_2 > \beta_1$, we get

$$\left[\frac{\mathbb{W}^k_{\alpha,\beta_1}(z)}{\mathbb{W}^k_{\alpha,\beta_2}(z)}\right]' = \frac{\mathbb{W}^k_{\alpha,\beta_2}(z)\mathbb{W}^k_{\alpha,\beta_1+\alpha}(z) - \mathbb{W}^k_{\alpha,\beta_1}(z)\mathbb{W}^k_{\alpha,\beta_2+\alpha}(z)}{\left(\mathbb{W}^k_{\alpha,\beta_2}(z)\right)^2}.$$

Consequently, (8) holds. Now putting $\beta_1 = \beta$ and $\beta_2 = \beta + \alpha$ in (8), we obtain (9). □

Theorem 3 *Consider the function* $W^{k,n}_{\alpha,\beta}(z)$ *defined as*

$$W^{k,n}_{\alpha,\beta}(z) = W^k_{\alpha,\beta}(z) - \sum_{i=0}^{n} \frac{z^i}{\Gamma_k(i+1)\Gamma_k(\beta + i\alpha)} = \sum_{i=n+1}^{\infty} \frac{z^i}{\Gamma_k(i+1)\Gamma_k(\beta + i\alpha)}, n \in \mathbb{N}. \quad (11)$$

Then

$$\left[W^{k,n+1}_{\alpha,\beta}(z)\right]^2 - W^{k,n}_{\alpha,\beta}(z)W^{k,n+2}_{\alpha,\beta}(z) \ge 0, \quad \forall \alpha, \beta, k, z > 0, 0 < k \le 1.$$

Proof Using the definition of $W^{k,n}_{\alpha,\beta}(z)$, we obtain

$$W^{k,n}_{\alpha,\beta}(z) = W^{k,n+1}_{\alpha,\beta}(z) + \frac{z^{n+1}}{\Gamma_k(n+1)\Gamma_k(\beta + (n+1)\alpha)}$$
$$W^{k,n+2}_{\alpha,\beta}(z) = W^{k,n+1}_{\alpha,\beta}(z) - \frac{z^{n+2}}{\Gamma_k(n+2)\Gamma_k(\beta + (n+2)\alpha)}.$$

Now,

$$\left[W_{\alpha,\beta}^{k,n+1}(z)\right]^2 - W_{\alpha,\beta}^{k,n}(z)W_{\alpha,\beta}^{k,n+2}(z)$$
$$= W_{\alpha,\beta}^{k,n+1}(z)\left[\frac{z^{n+2}}{\Gamma_k(n+2)\Gamma_k(\beta+(n+2)\alpha)} - \frac{z^{n+1}}{\Gamma_k(n+1)\Gamma_k(\beta+(n+1)\alpha)}\right]$$
$$+ \frac{z^{2n+3}}{\Gamma_k(n+1)\Gamma_k(n+2)\Gamma_k(\beta+(n+1)\alpha)\Gamma_k(\beta+(n+2)\alpha)}$$
$$\geq \sum_{i=n+3}^{\infty}\left[\frac{1}{\Gamma_k(n+2)\Gamma_k(\beta+(n+2)\alpha)\Gamma_k(i)\Gamma_k(\beta+(i-1)\alpha)}\right.$$
$$\left. - \frac{1}{\Gamma_k(n+1)\Gamma_k(\beta+(n+1)\alpha)\Gamma_k(i+1)\Gamma_k(\beta+i\alpha)}\right]z^{i+n+1}$$
$$= \sum_{i=n+3}^{\infty}\left[\frac{\Gamma_k(n+1)\Gamma_k(i+1)\Gamma_k(\beta+(n+1)\alpha)\Gamma_k(\beta+i\alpha) - \Gamma_k(n+2)\Gamma_k(\beta+(n+2)\alpha)\Gamma_k(i)\Gamma_k(\beta+(i-1)\alpha)}{\Gamma_k(n+1)\Gamma_k(n+2)\Gamma_k(\beta+(n+1)\Gamma_k(\beta+(n+2)\alpha)\,\Gamma_k(i)\Gamma_k(i+1)\Gamma_k(\beta+(i-1)\alpha)\Gamma_k(\beta+i\alpha)}\right]z^{i+n+1}.$$

Since $i \geq n+3$, therefore

$$\Gamma_k(n+1)\Gamma_k(i+1) \geq \Gamma_k(n+2). \tag{12}$$

Putting $z = \beta + (n+1)\alpha$, $a = \alpha$ and $b = \alpha(i-(n+2))$ in (10)

$$\Gamma_k(\beta+i\alpha)\Gamma_k(\beta+(n+1)\alpha) \geq \Gamma_k(\beta+(n+2)\alpha)\Gamma_k(\beta+(i-1)\alpha) \tag{13}$$

Combining (12) and (13), we have

$$\Gamma_k(n+1)\Gamma_k(i+1)\Gamma_k(\beta+(n+1)\alpha)\Gamma_k(\beta+i\alpha) \geq \Gamma_k(n+2)\Gamma_k(\beta+(n+2)\alpha)\Gamma_k(i)\Gamma_k(\beta+(i-1)\alpha),$$

which proves the theorem. □

Theorem 4 *Let $\alpha, \beta, k > 0$ and $n \in \mathbb{N}$. Then the function $H_n^k(\alpha,\beta,z)$ defined by*

$$H_n^k(\alpha,\beta,z) = \frac{W_{\alpha,\beta}^{k,n}(z)W_{\alpha,\beta}^{k,n+2}(z)}{\left[W_{\alpha,\beta}^{k,n+1}(z)\right]^2}$$

is monotonically increasing on the positive real line. Furthermore, for any positive α, β and z and $n \in \mathbb{N}$, the following inequality holds:

$$\frac{\Gamma_k^2(n+3)}{\Gamma_k(n+2)\Gamma_k(n+2)} \cdot \frac{\Gamma_k^2(\beta+(n+2)\alpha)}{\Gamma_k(\beta+(n+1)\alpha)\Gamma_k(\beta+(n+3)\alpha)}\left[\mathbf{W}_{\alpha,\beta}^{k,n+1}(z)\right]^2 \leq \mathbf{W}_{\alpha,\beta}^{k,n}(z)\mathbf{W}_{\alpha,\beta}^{k,n+2}(z) \tag{14}$$

The constant in left-hand side of the inequality (14) *is sharp.*

Proof Using Cauchy product, we have

$$H_n^k(\alpha,\beta,z)=\frac{\sum_{i=0}^{\infty}\sum_{j=0}^{i}A_j(\alpha,\beta)z^{2n+2+i}}{\sum_{i=0}^{\infty}\sum_{j=0}^{i}B_j(\alpha,\beta)z^{2n+2+i}}$$

where

$$A_j(\alpha,\beta)=\frac{1}{\Gamma_k(j+n+2)\Gamma_k(i-j+n+4)\Gamma_k(\beta+(n+1+j)\alpha)\Gamma_k(\beta+(n+3+i-j)\alpha)}$$

and

$$B_j(\alpha,\beta)=\frac{1}{\Gamma_k(j+n+3)\Gamma_k(i-j+n+3)\Gamma_k(\beta+(n+2+j)\alpha)\Gamma_k(\beta+(n+2+i-j)\alpha)}.$$

Now consider the following sequence $(V_j)_{j\geq 0}$ defined by

$$\begin{aligned}V_j(\alpha,\beta)&=\frac{A_j(\alpha,\beta)}{B_j(\alpha,\beta)}\\&=\frac{\Gamma_k(j+n+3)\Gamma_k(i-j+n+3)\Gamma_k(\beta+(n+2+j)\alpha)\Gamma_k(\beta+(n+2+i-j)\alpha)}{\Gamma_k(j+n+2)\Gamma_k(i-j+n+4)\Gamma_k(\beta+(n+1+j)\alpha)\Gamma_k(\beta+(n+3+i-j)\alpha)}.\end{aligned}$$

Then

$$\frac{V_{j+1}(\alpha,\beta)}{C}=\left[\frac{\Gamma_k(j+n+4)\Gamma_k(j+n+2)}{\Gamma_k^2(j+n+3)}\right]\left[\frac{\Gamma_k(i-j+n+4)\Gamma_k(i-j+n+2)}{\Gamma_k^2(i-j+n+3)}\right]T_{i,j}(\alpha,\beta),$$

with

$$\begin{aligned}T_{i,j}(\alpha,\beta)&=\frac{\Gamma_k(\beta+(n+3+j)\alpha)\Gamma_k(\beta+(n+1+j)\alpha)}{\Gamma_k^2(\beta+(n+2+j)\alpha)}\\&\quad\cdot\frac{\Gamma_k(\beta+(n+1+i-j)\alpha)\Gamma_k(\beta+(n+3+i-j)\alpha)}{\Gamma_k^2(\beta+(n+2+i-j)\alpha)}\\&=\frac{\Gamma_k(\beta_1+(n+3)\alpha)\Gamma_k(\beta_1+(n+1)\alpha)}{\Gamma_k^2(\beta_1+(n+2)\alpha)}\cdot\frac{\Gamma_k(\beta_2+(n+1)\alpha)\Gamma_k(\beta_2+(n+3)\alpha)}{\Gamma_k^2(\beta_2+(n+2)\alpha)},\end{aligned}$$

where $\beta_1=\beta+j\alpha$ and $\beta_2=\beta+(i-j)\alpha$. Again with the help of (10), we have $T_{i,j}(\alpha,\beta)\geq 1\quad\forall\alpha,\beta>0$, which proves that $\big(V_j(\alpha,\beta)\big)_j$ is monotonically increasing. Hence, by Lemma 1 we have, $(\sum_{j=0}^{\infty}A_j(\alpha,\beta)/\sum_{j=0}^{\infty}B_j(\alpha,\beta))_i$ is monotonically increasing. Again by Lemma 3, $H_n^k(\alpha,\beta,z)$ is monotonically increasing on $(0,\infty)$. Now,

$$\lim_{z\to 0} H_n^k(\alpha,\beta,z) = \frac{\Gamma_k^2(n+3)}{\Gamma_k(n+2)\Gamma_k(n+2)} \cdot \frac{\Gamma_k^2(\beta+(n+2)\alpha)}{\Gamma_k(\beta+(n+1)\alpha)\Gamma_k(\beta+(n+3)\alpha)},$$

which shows that

$$\frac{\Gamma_k^2(n+3)}{\Gamma_k(n+2)\Gamma_k(n+2)} \cdot \frac{\Gamma_k^2(\beta+(n+2)\alpha)}{\Gamma_k(\beta+(n+1)\alpha)\Gamma_k(\beta+(n+3)\alpha)}, \quad \forall \alpha,\beta,z>0, n\in\mathbb{N}$$

is the best possible for the inequality (14). □

3 Lazarević- and Wilker-Type Inequalities

In this section, our aim is to obtain some Lazarević- and Wilker-type inequalities for modified normalized k-Wright functions defined as

$$\mathbf{W}_{\alpha,\beta}^k(z) = \Gamma_k(\beta)\sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i+1)\Gamma_k(\beta+i\alpha)}, \qquad k>0,\ \alpha>-1,\ \beta,z\in\mathbb{C}. \tag{15}$$

Clearly,

$$\left(\mathbf{W}_{\alpha,\beta}^k(z)\right)' = \frac{\Gamma_k(\beta)}{\Gamma_k(\beta+\alpha)}\mathbf{W}_{\alpha,\beta+\alpha}^k(z). \tag{16}$$

Theorem 5 *Let $\beta_1 \geq \beta_2$ and α be positive real numbers. Then*

$$\left[\mathbf{W}_{\alpha,\beta_2}^k(z)\right]^{\Gamma_k(\beta_2+\alpha)/\Gamma_k(\beta_2)} \leq \left[\mathbf{W}_{\alpha,\beta_1}^k(z)\right]^{\Gamma_k(\beta_1+\alpha)/\Gamma_k(\beta_1)} \quad \forall z\in(0,\infty). \tag{17}$$

In particular,

$$\left[\mathbf{W}_{\alpha,\beta}^k(z)\right]^{\beta} \leq \left[\mathbf{W}_{\alpha,\beta+1}^k(z)\right]^{\beta+\alpha}, \quad z,\beta>0. \tag{18}$$

Proof Let $f:(0,\infty)\to\mathbb{R}$ be defined as

$$f(z) = \frac{\Gamma_k(\beta_2)\Gamma_k(\beta_1+\alpha)}{\Gamma_k(\beta_1)\Gamma_k(\beta_2+\alpha)}\log[\mathbf{W}_{\alpha,\beta_1}^k(z)] - \log[\mathbf{W}_{\alpha,\beta_2}^k(z)].$$

Using (16), we obtain

$$f'(z) = \frac{\Gamma_k(\beta_2)}{\Gamma_k(\beta_2+\alpha)}\left[\frac{\mathbf{W}_{\alpha,\beta_1+\alpha}^k(z)}{\mathbf{W}_{\alpha,\beta_1}^k(z)} - \frac{\mathbf{W}_{\alpha,\beta_2+\alpha}^k(z)}{\mathbf{W}_{\alpha,\beta_2}^k(z)}\right]. \tag{19}$$

Again using Remark 1, we claim that $f(x)$ is monotonically increasing on $(0, \infty)$ if $\beta_1 \geq \beta_2$. Hence, $f(x) \geq f(0) = 0$, which proves the inequality (17). Finally, putting $\beta_1 = \beta$ and $\beta_2 = \beta + 1$ in (17), the inequality (18) can be obtained, which completes the proof. □

Corollary 1 *Let $\beta_1 \geq \beta_2$, α and z be any positive real numbers. Then*

$$\frac{\mathbf{W}^k_{\alpha,\beta_1}(z)}{\mathbf{W}^k_{\alpha,\beta_2}(z)} + [\mathbf{W}^k_{\alpha,\beta_1}(z)]^{\dfrac{\Gamma_k(\beta_2)\Gamma_k(\beta_1+\alpha) - \Gamma_k(\beta_1)\Gamma_k(\beta_2+\alpha)}{\Gamma_k(\beta_1)\Gamma_k(\beta_2+\alpha)}} \geq 2. \tag{20}$$

In particular,

$$\frac{\mathbf{W}^k_{\alpha,\beta+1}(z)}{\mathbf{W}^k_{\alpha,\beta}(z)} + [\mathbf{W}^k_{\alpha,\beta+1}(z)]^{\alpha/\beta} \geq 2, \quad \forall z, \beta > 0. \tag{21}$$

Proof Withy the help of the inequality (17), we have

$$\frac{[\mathbf{W}^k_{\alpha,\beta_1}(z)]^{\frac{\Gamma_k(\beta_2)\Gamma_k(\beta_1+\alpha)}{\Gamma_k(\beta_1)\Gamma_k(\beta_2+\alpha)}}}{\mathbf{W}^k_{\alpha,\beta_2}(z)} = \left[\frac{\mathbf{W}^k_{\alpha,\beta_1}(z)}{\mathbf{W}^k_{\alpha,\beta_2}(z)}\right] \cdot [\mathbf{W}^k_{\alpha,\beta_1}(z)]^{\frac{\Gamma_k(\beta_2)\Gamma_k(\beta_1+\alpha)-\Gamma_k(\beta_1)\Gamma_k(\beta_2+\alpha)}{\Gamma_k(\beta_1)\Gamma_k(\beta_2+\alpha)}} \geq 1.$$

Using the arithmetic-geometric mean inequality with the above expression, we obtain

$$\frac{1}{2}\left[\frac{\mathbf{W}^k_{\alpha,\beta_1}(z)}{\mathbf{W}^k_{\alpha,\beta_2}(z)} + [\mathbf{W}^k_{\alpha,\beta_1}(z)]^{\frac{\Gamma_k(\beta_2)\Gamma_k(\beta_1+\alpha)-\Gamma_k(\beta_1)\Gamma_k(\beta_2+\alpha)}{\Gamma_k(\beta_1)\Gamma_k(\beta_2+\alpha)}}\right] \geq \sqrt{\frac{[\mathbf{W}^k_{\alpha,\beta_1}(z)]^{\frac{\Gamma_k(\beta_2)\Gamma_k(\beta_1+\alpha)}{\Gamma_k(\beta_1)\Gamma_k(\beta_2+\alpha)}}}{\mathbf{W}^k_{\alpha,\beta_2}(z)}} \geq 1.$$

Hence, the inequality (20) holds true. Now, putting $\beta_1 = \beta$ and $\beta_2 = \beta + 1$ in (20), the inequality (21) can be derived. □

4 Redheffer-Type Inequalities

Theorem 6 *Let $r, \alpha, \beta, k > 0$. Then the following inequality holds:*

$$\left(\frac{r+z}{r-z}\right)^{\sigma_{\alpha,\beta,k}} \leq \mathbf{W}^k_{\alpha,\beta}(z) \leq \left(\frac{r+z}{r-z}\right)^{\gamma_{\alpha,\beta,k}}, \quad \forall 0 < z < r, \tag{22}$$

where $\sigma_{\alpha,\beta,k} = 0$ and $\gamma_{\alpha,\beta,k} = \frac{r\Gamma_k(\beta)}{2\Gamma_k(\beta+\alpha)}$ are the best possible constants.

Proof Let

$$H(z) = \frac{\log \mathbf{W}^k_{\alpha,\beta}(z)}{\log\left(\dfrac{r+z}{r-z}\right)} = \frac{f(z)}{g(z)},$$

where $f(z) = \log \mathbf{W}^k_{\alpha,\beta}(z)$ and $g(z) = \log((r+z)/(r-z))$. Then

$$\frac{f'(z)}{g'(z)} = \frac{(r^2 - z^2)(\mathbf{W}^k_{\alpha,\beta}(z))'}{2r\mathbf{W}^k_{\alpha,\beta}(z)} = \frac{A(z)}{2rB(z)}$$

with $A(z) = (r^2 - z^2)(\mathbf{W}^k_{\alpha,\beta}(z))'$ and $B(z) = \mathbf{W}^k_{\alpha,\beta}(z)$. Using (15), we obtain

$$\begin{aligned} A(z) &= (r^2 - z^2)(\mathbf{W}^k_{\alpha,\beta}(z))' \\ &= \frac{r^2\Gamma_k(\beta)}{\Gamma_k(\beta+\alpha)} + \frac{r^2\Gamma_k(\beta)}{\Gamma_k(\beta+2\alpha)}z \\ &+ \sum_{i=2}^{\infty}\left(\frac{r^2\Gamma_k(\beta)}{\Gamma_k(i+1)\Gamma_k(\beta+(i+1)\alpha)} - \frac{\Gamma_k(\beta)}{\Gamma_k(i-1)\Gamma_k(\beta+(i-1)\alpha)}\right)z^i \\ &= \sum_{i=0}^{\infty} a_i z^i, \end{aligned}$$

with $a_0 = r^2\Gamma_k(\beta)/\Gamma_k(\beta+\alpha)$, $a_1 = r^2\Gamma_k(\beta)/\Gamma_k(\beta+2\alpha)$ and $a_i = r^2\Gamma_k(\beta)/\Gamma_k(i+1)\Gamma_k(\beta+(i+1)\alpha) - \Gamma_k(\beta)/\Gamma_k(i-1)\Gamma_k(\beta+(i-1)\alpha)$, $i \geq 2$. Similarly, $B(z)$
can be expressed as

$$B(z) = \sum_{i=0}^{\infty} b_i z^i,$$

with $b_0 = 1$, $b_1 = \Gamma_k(\beta)/\Gamma_k(\beta+\alpha)$ and $b_i = \Gamma_k(\beta)/\Gamma_k(i+1)\Gamma_k(\beta+i\alpha)$, $i \geq 2$. Now consider $u_i = a_i/b_i$, $i \in \mathbb{N}_0$, which satisfies $u_0 = r^2\Gamma_k(\beta)/\Gamma_k(\beta+\alpha)$, $u_1 = r^2\Gamma_k(\beta+\alpha)/\Gamma_k(\beta+2\alpha)$, and

$$u_i = \frac{r^2\Gamma_k(\beta+i\alpha)}{\Gamma_k(\beta+(i+1)\alpha)} - \frac{\Gamma_k(i+1)\Gamma_k(\beta+i\alpha)}{\Gamma_k(i-1)\Gamma_k(\beta+(i-1)\alpha)}, \quad i \geq 2.$$

Putting $a = b = \alpha$, $x = b$ in (10) we have $u_1 \leq u_0$. Again,

$$\begin{aligned} u_{i+1} - u_i = r^2 &\left[\frac{\Gamma_k(\beta+(i+1)\alpha)}{\Gamma_k(\beta+(i+2)\alpha)} - \frac{\Gamma_k(\beta+i\alpha)}{\Gamma_k(\beta+(i+1)\alpha)}\right] \\ &+ \frac{\Gamma_k(i+1)\Gamma_k(\beta+i\alpha)}{\Gamma_k(i-1)\Gamma_k(\beta+(i-1)\alpha)} - \frac{\Gamma_k(i+2)\Gamma_k(\beta+(i+1)\alpha)}{\Gamma_k(i)\Gamma_k(\beta+i\alpha)}, \quad \forall i \geq 2. \end{aligned}$$

Using (10) with $z = i - 1$, $a = 2$ and $b = 1$, we have

$$u_{i+1}-u_i \le r^2\left[\frac{\Gamma_k(\beta+(i+1)\alpha)}{\Gamma_k(\beta+(i+2)\alpha)}-\frac{\Gamma_k(\beta+i\alpha)}{\Gamma_k(\beta+(i+1)\alpha)}\right] + \frac{\Gamma_k(i+1)}{\Gamma_k(i-1)}\left[\frac{\Gamma_k(\beta+i\alpha)}{\Gamma_k(\beta+(i-1)\alpha)}-\frac{\Gamma_k(\beta+(i+1)\alpha)}{\Gamma_k(\beta+i\alpha)}\right]. \tag{23}$$

Again putting $x=\beta+(i-1)\alpha$ and $a=b=\alpha$ in (10), we obtain

$$\frac{\Gamma_k(\beta+i\alpha)}{\Gamma_k(\beta+(i-1)\alpha)} \le \frac{\Gamma_k(\beta+(i+1)\alpha)}{\Gamma_k(\beta+i\alpha)}. \tag{24}$$

Replacing i by $i+1$ in (24), we get

$$\frac{\Gamma_k(\beta+i\alpha)}{\Gamma_k(\beta+(i-1)\alpha)} \le \frac{\Gamma_k(\beta+(i+1)\alpha)}{\Gamma_k(\beta+i\alpha)}. \tag{25}$$

Combining (23)–(25), we can conclude that for all $i\ge 2$, u_i is monotonically decreasing sequence. Hence, u_i is monotonically decreasing for all $i\ge 0$. Furthermore, with the help of Lemma 2, it can be proved that f'/g' is monotonically decreasing on $(0,r)$. Therefore, using Lemma 3, we claim that $F(z)=\frac{f(z)-f(0)}{g(z)-g(0)}$ is monotonically decreasing on $(0,r)$. Again,

$$\lim_{x\to 0} F(z)=\frac{u_0}{2r}=\frac{r\Gamma_k(\beta)}{2\Gamma_k(\beta+\alpha)}, \quad \text{and } \lim_{x\to r} F(x)=0.$$

Hence,

$$\frac{\log \mathbf{W}^k_{\alpha,\beta}(z)}{\log\left(\dfrac{r+z}{r-z}\right)} \le \frac{r\Gamma_k(\beta)}{2\Gamma_k(\beta+\alpha)},$$

which proves the theorem. □

Now we will derive some other inequalities for $\mathbf{W}^k_{\alpha,\beta}(z)$.

Theorem 7 *Let $\alpha,\beta,k>$ be positive real numbers. Then the following statements hold:*

1. *$\mathbf{W}^k_{\alpha,\beta}(z)$ is logarithmically concave on the positive real line.*
2. *The following inequalities hold:*

$$\mathbf{W}^k_{\alpha,\beta}(z_1)\mathbf{W}^k_{\alpha,\beta}(z_2) \le \left[\mathbf{W}^k_{\alpha,\beta}\left(\frac{z_1+z_2}{2}\right)\right]^2, \quad z_1,z_2>0. \tag{26}$$

$$\mathbf{W}^k_{\alpha,\beta}(z)\mathbf{W}^k_{\alpha,\beta+2\alpha}(z) \le \frac{\Gamma_k(\beta)+\Gamma_k(\beta+2\alpha)}{\Gamma_k^2(\beta+\alpha)}\left[\mathbf{W}^k_{\alpha,\beta+\alpha}(z)\right]^2, \quad z>0. \tag{27}$$

$$\mathbf{W}^k_{\alpha,\beta}(z) \le \exp\left(\frac{\Gamma_k(\beta)}{\Gamma_k(\beta+\alpha)}z\right), \quad z>0. \tag{28}$$

$$\mathbf{W}^k_{\alpha,\beta+\alpha}(z) \leq \mathbf{W}^k_{\alpha,\beta}(z), \quad z > 0. \tag{29}$$

$$\exp\left(\frac{\Gamma_k(\beta)(x-y)}{\Gamma_k(\beta+\alpha)}\right) \leq \frac{\mathbf{W}^k_{\alpha,\beta}(x)}{\mathbf{W}^k_{\alpha,\beta}(y)}, \quad 0 < x < y. \tag{30}$$

Proof 1. To prove this part, it is enough to show that $\left[\mathbf{W}^k_{\alpha,\beta}(z)\right]' / \mathbf{W}^k_{\alpha,\beta}(z)$ is monotonically decreasing on the positive real line. Now, the power series representation for $\mathbf{W}^k_{\alpha,\beta}(z)$ gives us the following

$$\frac{\left[\mathbf{W}^k_{\alpha,\beta}(z)\right]'}{\mathbf{W}^k_{\alpha,\beta}(z)} = \frac{\displaystyle\sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i+1)\Gamma_k(\beta+(i+1)\alpha)}}{\Gamma_k(\beta+\alpha)\displaystyle\sum_{i=0}^{\infty} \frac{z^i}{\Gamma(i+1)\Gamma_k(\beta+i\alpha)}}.$$

Using the Lemma 2, we have

$$u_i = \frac{a_i}{b_i} = \frac{\Gamma_k(\beta+i\alpha)}{\Gamma_k(\beta+(i+1)\alpha)}, \quad i \geq 0.$$

Then,

$$\frac{u_{i+1}}{u_i} = \frac{\Gamma_k^2(\beta+(i+1)\alpha)}{\Gamma_k(\beta+i\alpha)\Gamma_k(\beta+(i+2\alpha))}, \quad i \geq 2.$$

Using (24), we obtain $u_{i+1} \leq u_i \forall i \geq 0$, which proves the first part of the theorem.

2. Since $\mathbf{W}^k_{\alpha,\beta}(z)$ is logarithmically concave. Therefore,

$$[\mathbf{W}^k_{\alpha,\beta}(z_1)]^t [\mathbf{W}^k_{\alpha,\beta}(z_2)]^{1-t} \leq \mathbf{W}^k_{\alpha,\beta}(tz_1 + (1-t)z_2), \quad \forall t \in [0,1] \text{ and } \alpha, \beta, z_1, z_2 > 0.$$

Substituting $t = 1/2$ in the above inequality, the inequality (26) can be derived.

Now we proceed to prove the inequality (27). Since, $\mathbf{W}^k_{\alpha,\beta}(z)$ is logarithmically concave on the positive real line. Therefore, $\left[\mathbf{W}^k_{\alpha,\beta}(z)\right]' / \mathbf{W}^k_{\alpha,\beta}(z)$ is monotonically decreasing on the positive real line. Using (16), we have

$$\begin{aligned}
&\left[\frac{\left(\mathbf{W}^k_{\alpha,\beta}(z)\right)'}{\mathbf{W}^k_{\alpha,\beta}(z)}\right]' \\
&= \frac{\Gamma_k(\beta)}{\Gamma_k(\beta+\alpha)\left(\mathbf{W}^k_{\alpha,\beta}(z)\right)^2}\left[\frac{\Gamma_k(\beta+\alpha)}{\Gamma_k(\beta+2\alpha)}\mathbf{W}^k_{\alpha,\beta}(z)\mathbf{W}^k_{\alpha,\beta+2\alpha}(z) - \frac{\Gamma_k(\beta)}{\Gamma_k(\beta+\alpha)}\left(\mathbf{W}^k_{\alpha,\beta+\alpha}(z)\right)^2\right] \\
&\leq 0.
\end{aligned}$$

This proves the inequality (27).

Let us now prove the inequality (28). To do so, let us assume that

$$F(z) = \log \mathbf{W}^k_{\alpha,\beta}(z) \quad \text{and} \quad G(z) = z.$$

Since $\left(\mathbf{W}^k_{\alpha,\beta}(z)\right)' / \mathbf{W}^k_{\alpha,\beta}(z)$ is monotonically decreasing on $(0, \infty)$. Therefore, using Lemma 3, we have

$$\frac{F(z)}{G(z)} = \frac{F(z) - F(0)}{G(z) - G(0)}$$

is monotonically decreasing on the positive real line. Now, using the Bernoulli-l'Hopital's rule and (16), we obtain

$$\lim_{z\to 0} \frac{F(z)}{G(z)} = \lim_{z\to 0} \frac{\left(\mathbf{W}^k_{\alpha,\beta}(z)\right)'}{\mathbf{W}^k_{\alpha,\beta}(z)} = \frac{\Gamma_k(\beta)}{\Gamma_k(\beta+\alpha)}.$$

Since, $\left(\mathbf{W}^k_{\alpha,\beta}(z)\right)' / \mathbf{W}^k_{\alpha,\beta}(z)$ is monotonically decreasing for $z \in (0, \infty)$, we obtain

$$\left[\mathbf{W}^k_{\alpha,\beta}(z)\right]' \le \frac{\Gamma_k(\beta)}{\Gamma_k(\beta+\alpha)} \mathbf{W}^k_{\alpha,\beta}(z). \tag{31}$$

Again using (16) and the inequality (31), the inequality (29) can be obtained. Now, using (31), we have

$$\exp\left(\frac{\Gamma_k(\beta)(x-y)}{\Gamma_k(\beta+\alpha)}\right) \le \frac{\mathbf{W}^k_{\alpha,\beta}(x)}{\mathbf{W}^k_{\alpha,\beta}(y)}.$$

Acknowledgements The authors would like to thank the reviewers for their valuable comments that help to improve the paper.

References

1. Baricz, Á.: Functional inequalities involving Bessel and modified Bessel functions of the first kind. Expo. Math. **26**(3), 279–293 (2008)
2. Baricz, Á.: Turán type inequalities for hypergeometric functions. Proc. Am. Math. Soc. **136**(9), 3223–3229 (2008)
3. Black, F., Cox, J.C.: Valuing corporate securities: some effects of bond indenture provisions. J. Financ. **31**(2), 351–367
4. Carey, M., Gordy, M.B.: The bank as grim reaper: debt composition and recoveries on defaulted debt, Preprint (2007)
5. Conway, J.B.: Functions of one complex variable, 2nd edn. Graduate Texts in Mathematics, vol. 11. Springer, New York (1978)

6. Kiryakova, V.: Some special functions related to fractional calculus and fractional (non-integer) order control systems and equations. Facta Univ. Ser. Autom. Control Robot. **7**(1), 79–98 (2008)
7. McEliece, R.J., Reznick, B., Shearer, J.B.: A Turán inequality arising in information theory. SIAM J. Math. Anal. **12**(6), 931–934 (1981)
8. Merton, R.C.: On the pricing of corporate debt: the risk structure of interest rates. J. Financ. **29**(2), 449–470 (1974)
9. Mehrez, K.: Functional inequalities for the Wright functions. Integr. Transform. Spec. Funct. **28**(2), 130–144 (2017)
10. Mehrez, K.: Monotonicity patterns and functional inequalities for classical and generalized Wright functions. Math. Inequalities Appl. **22**(3), 901–916 (2019)
11. Mehrez, K., Sitnik, S.M.: Functional inequalities for the Mittag-Leffler functions. Results Math. **72**(1–2), 703–714 (2017)
12. Mehrez, K., Sitnik, S.M.: Turán type inequalities for classical and generalized Mittag-Leffler functions. Anal. Math. **44**(4), 521–541 (2018)
13. Mehrez, K.: New integral representations for the Fox-Wright functions and its applications. J. Math. Anal. Appl. **468**(2), 650–673 (2018)
14. Mehrez, K., Sitnik, S.M.: Functional inequalities for the Fox-Wright functions. Ramanujan J. **50**(2), 263–287 (2019)
15. Mehrez, K.: New properties for several classes of functions related to the Fox-Wright functions. J. Comput. Appl. Math. **362**, 161–171 (2019)
16. Podlubny, I.: Fractional differential equations, Mathematics in Science and Engineering, vol. 198. Academic Press Inc., San Diego (1999)
17. Szász, O.: Inequalities concerning ultraspherical polynomials and Bessel functions. Proc. Am. Math. Soc. **1**, 256–267 (1950)
18. Turán, P.: On the zeros of the polynomials of Legendre. Časopis Pěst. Mat. Fys. **75**, 113–122 (1950)
19. Wright, E.M.: On the coefficients of power series having exponential singularities. J. Lond. Math. Soc. **8**(1), 71–79 (1933)
20. Yin, L., Huang, L.-G., Lin, X.-L., Wang, Y.-L.: Monotonicity, concavity, and inequalities related to the generalized digamma function. Adv. Differ. Equ. **2018**, paper no. 246, 9

An Information-Theoretic Entropy Related to Ihara ζ Function and Billiard Dynamics

Supriyo Dutta and Partha Guha

Abstract This article aims to establish a connection between the dynamical billiards and information theory. We propose two generalized information-theoretic entropies based on the Ihara zeta functions associated with a combinatorial graph representing a billiard dynamical system, rigorously discussed in [5, 6].

Keywords Ihara zeta function · Billiard dynamics · Entropy · Information theory

1 Introduction: Dynamical Billiard and Ihara Zeta Function

The dynamical billiards are mathematical models for describing different physical phenomena, where one or more particles travel in a container and collide with the walls and with each other. Its dynamical properties depend on the shape of the walls of the container. Physicists started studying these models in the early nineteenth century [8]. Ya. Sinai initiated the mathematical studies of chaotic dynamical billiards in 1970 [17]. During the last 50 years, it is extensively investigated within the modern theory of dynamical systems and statistical mechanics [3, 13, 14].

We assume a billiard system consists of a moving particle on a plane and a set of reflectors placed on a bounded region. The boundary of the region is also a combination of reflectors. We assume that the particle will not be reflected between any two reflectors consecutively. We depict a combination of reflectors in Fig. 1.

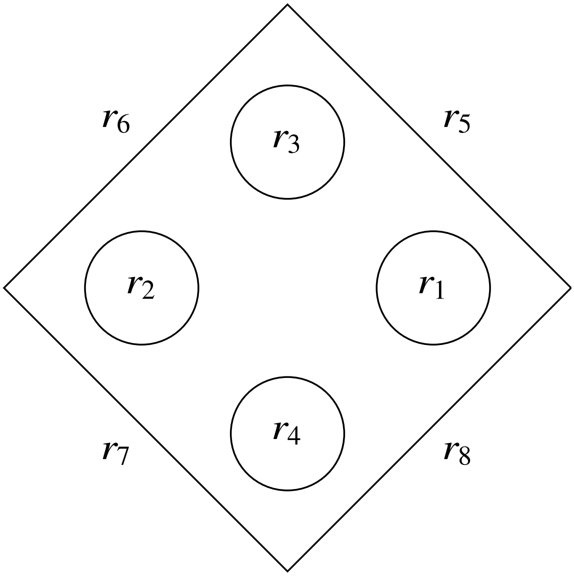
S. Dutta (✉)
School of Sciences, SR University, Ananthasagar, Hasanparthy, Warangal Urban 506371, Telangana, India
e-mail: dosupriyo@gmail.com

Centre for Theoretical Studies, Indian Institute of Technology Kharagpur, Kharagpur 721302, West Bengal, India

P. Guha
Department of Mathematics, Khalifa University, Al Saada St - Zone 1, Abu Dhabi, United Arab Emirates
e-mail: partha.guha@ku.ac.ae

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_18

Fig. 1 This billiard system consists of 8 reflectors $\{r_1, r_2, \ldots, r_8\}$, where r_5, r_6, r_7, r_8 form the boundary of the region and others lie inside it



In combinatorics, a graph $G = (V(G), E(G))$ is a set of vertices $V(G)$ and a set of edges $E(G) \subset V(G) \times V(G) - \{[u, u] : u \in V(G)\}$. There is a graph G associated with the system of billiard under consideration, whose vertices correspond to the reflectors. Also, there is an edge $[u, v]$ if the particle can be reflected between the reflectors u and v. The graph in Fig. 2 represents the billiard system in Fig. 1. The following characteristics of G are easy to observe:

1. The particle moves in any direction, between two reflectors u and v. Therefore, an edge $[u, v]$ has two opposite orientations $e = (u, v)$ and $e^{-1} = (v, u)$.
2. As the particle cannot be reflected on a reflector consecutively, there is no loop on the vertices.
3. As the region is bounded by reflectors, G is neither a path graph nor a cycle.
4. In the neighborhood of a reflector, there is more than one reflector. Hence, there is no vertex with degree 1. Recall that the degree of a vertex in a graph is the number of vertices adjacent to it.

These graph-theoretic properties are crucial for defining the Ihara zeta function. A simple graph without a single-degree vertex which is not a cycle or a path graph is called an admissible graph.

Let $e = (u, v)$ be an oriented edge with the initiating and terminating vertices $u = i(e)$ and $v = t(e)$, respectively. The moving particle generates a bi-infinite-directed walk in G, which is a sequence of oriented edges $\ldots e_{-2}e_{-1}e_0e_1e_2\ldots$ such that $t(e_i) = i(e_{i+1})$ for $i \in \mathbb{Z}$. As we assume that the particle cannot be reflected consecutively between two reflectors, hence e and e^{-1} cannot appear consecutively. A cycle $W = e_1e_2\ldots e_k$ of length k is a finite walk such that $i(e_1) = t(e_k)$. Two cycles $W_1 = e_{1,1}e_{1,2}\ldots e_{1,k}$ and $W_2 = e_{2,1}e_{2,2}\ldots e_{2,k}$ are equiv-

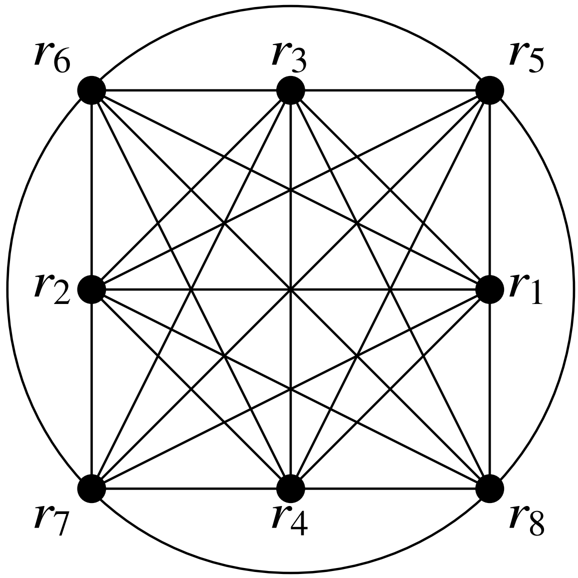


Fig. 2 Graph representing the system of billiards in Fig. 1. The vertices correspond to the reflectors in Fig. 1. Edges indicate possible movement of the particle between the corresponding reflectors

alent if $e_{2,1} = e_{1,r}, e_{2,2} = e_{1,(r+1)}, \ldots e_{2,(k-r+1)} = e_{1,k}, e_{2,(k-r+2)} = e_{1,1}, \ldots, e_{2,k} = e_{1,(r-1)}$ for some $r \in \{1, 2, \ldots, k\}$. The set of equivalence classes of cycles are called prime cycles. The length of a prime cycle P is $\gamma(P)$.

Definition 1 The Ihara zeta function [10, 20] $\zeta_G(z) : \{|z| < R\} \to \mathbb{C}$ of a combinatorial graph G is defined by

$$\zeta_G(z) = \prod_P \left(1 - z^{\gamma(P)}\right)^{-1}, \tag{1}$$

where R is the radius of convergence of the infinite product.

The series representation of $\zeta_G(z)$ is stimulated by the oriented line graph $\overline{G} = (V(\overline{G}), E(\overline{G}))$ of G, where $V(\overline{G}) = \cup_{e \in E(G)} \{e, e^{-1}\}$. An edge $(e, f) \in E(\overline{G})$ if $t(e) = i(f)$ and $i(e) \neq t(f)$. The adjacency matrix $T = (t_{(e,f)})_{2m \times 2m}$ is defined by

$$t_{(e,f)} = \begin{cases} 1 & \text{if } (e, f) \in E(\overline{G}); \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Here m is the number of edges in the graph G. It can be proved that the following power series

$$\zeta_G(z) = \exp\left(\sum_{k=1}^{\infty} \frac{\text{trace}(T^k)}{k} z^k\right), \tag{3}$$

where $|z| < \frac{1}{\lambda}$ [12] represents $\zeta_G(z)$. Here, $\lambda > 0$ is the greatest eigenvalue of T.

In information theory an entropy $S(\mathcal{P})$ is the measure of information induced by a probability distribution $\mathcal{P}$. For simplicity, we assume a discrete probability distribution $\mathcal{P} = \{p_i : i = 1, 2, \ldots, W, 0 \leq p_i \leq 1, \sum_i p_i = 1\}$ throughout this article. As all the probability values are real, we restrict the domain of definition of the function $\zeta_G(z)$ to $\zeta_G(x) : [0, \frac{1}{\lambda}) \rightarrow \mathbb{R}^+$.

Expanding Eq. (3) we find the following power series for $\zeta_G(x)$:

$$\zeta_G(x) = 1 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots, \tag{4}$$

where $c_1 = \text{trace}(T) = 0$, $c_2 = \frac{\text{trace}(T^2)}{2}$, $c_3 = \frac{\text{trace}\, T^3}{3}$, $c_4 = \frac{\text{trace}(T^4)}{4} + \frac{(\text{trace}(T^2))^2}{8}$, $c_5 = \frac{\text{trace}(T^5)}{5} + \frac{\text{trace}(T^2)\,\text{trace}(T^3)}{6}, \ldots$. The following lemma is easy to prove by applying the fact that trace T^k is positive for all $k > 1$.

Lemma 1 $\zeta_G(x)$ *is a monotone increasing function in* $[0, \frac{1}{\lambda})$.

In this article, we propose an information-theoretic entropy based on Ihara ζ function. The preliminary concepts of generalized entropy are discussed in Sect. 2. Equation (4) along with the Lemma 1 suggest that the compositional inverse $\zeta_G(x)$ does not exist. Hence, to define a universal group entropy, we construct a number of invertible power series in terms of ζ_G. These constructions are mentioned in Sect. 3. In Sect. 4, we define two new generalized entropies and discuss some of their important properties. Then we conclude this article.

2 Preliminary Concepts on the Generalized Entropy

In his seminal work, Shannon proposed an entropy function

$$S(\mathcal{P}) = \sum_{i=1}^{W} p_i \ln\left(\frac{1}{p_i}\right), \tag{5}$$

for any given probability distribution $\mathcal{P}$ [15]. It fulfills all four Shannon–Khinchin axioms (SK axioms) [11, 16], mentioned below:

1. $S(\mathcal{P})$ is continuous with respect to p_i for $i = 1, 2, \ldots, W$;
2. $S(\mathcal{P}_1) = S(\mathcal{P})$ where $\mathcal{P}_1 = \mathcal{P} \cup \{0\}$;
3. $S(\mathcal{P})$ is maximum when $p_i = \frac{1}{W}$ for all i;
4. $S(X, Y) = S(X) + S(Y|X)$, where $S(X, Y)$ and $S(Y|X)$ are the entropy of joint probability distribution and the conditional probability distribution of the random variables (X, Y) and $Y|X$, respectively.

There are different proposals for generalizing the Shannon entropy. A few well-known attempts include the Min entropy, Hartley entropy, Rényi entropy, Tsallis entropy, Sharma–Mittal entropy, etc. Addressing a number of parameters in the entropy function is a procedure of generalization. For instance, the Tsallis entropy is a q-deformed version of Shannon entropy. Here, q acts as a parameter. Similarly, there is only one parameter in the Rényi entropy. The Sharma–Mittal entropy consists of two parameters. A review on two-parameter deformed entropy is available in [7]. Most of this entropy satisfies the first three SK axioms. The fourth axiom is generalized depending on the generalized entropy function. A family of entropy functions is proposed in terms of formal power series, which is the fundamental theme of this article.

A formal power series is a generalization of a polynomial with infinitely many terms. Given two formal power series $F = \sum_{i=1}^{\infty} b_i x^i$ and $G = \sum_{i=1}^{\infty} a_i x^i$ the composition $F \circ G(x)$ is defined by $F \circ G(x) = F(G(x))$ [2]. The power series G is called invertible if it has a compositional inverse F which is a power series such that $F(G(x)) = x$. The following Lemma [1, 4, 9] is useful for further derivations:

Lemma 2 *A formal power series $G = \sum_{i=1}^{\infty} a_i x^i$ is invertible if and only if $a_1 = 1$.*

We are in a position to define the formal group entropy which is as follows:

Definition 2 The universal group entropy [18, 19] of the discrete probability distribution $\mathcal{P}$ is defined by

$$S(\mathcal{P}) = \sum_{i=0}^{W} p_i G\left(\ln \frac{1}{p_i}\right), \tag{6}$$

where $G(t) = \sum_{i=1}^{\infty} a_i x^i$ is an invertible power series, that is $a_1 = 1$.

It can be proved that $S(\mathcal{P})$ fulfills the first three Shannon–Khinchin axioms. The fourth axiom is generalized in terms of the Lazard formal group law.

Definition 3 The Lazard formal group law is a bi-variate formal power series $\Phi(s_1, s_2) \in \mathbb{Z}\{s_1, s_2\}$ such that

$$\Phi(s_1, s_2) = G(F(s_1) + F(s_2)), \tag{7}$$

where F is the compositional inverse of G.

3 Invertible Formal Power Series Related to Ihara Zeta Function

Recall that an invertible formal power series is essential for defining the universal group entropy. But, Eq. (4) along with Lemma 2 suggest that $\zeta_G(x)$ is not invertible. In this section, we propose two invertible formal power series in terms of $\zeta_G(x)$ satisfying Lemma 2.

Theorem 1 *Let T be the adjacency matrix of the oriented line graph of an admissible graph G, and λ be the largest eigenvalue of T. Then the following formal power series*

$$\mathcal{G}_a(t) = \frac{\zeta_G(ae^{-t}) - \zeta_G(a) + a(e^{-t} - 1)}{-a(1 + \zeta_G'(a))} \tag{8}$$

is invertible, where $t \geq 0$ and $0 \leq a < \frac{1}{\lambda}$.

Proof For proving $\mathcal{G}_a(t)$ is invertible we need to justify that it has constant term zero and coefficient of t is 1. Now, Eq. (4) indicates

$$\zeta_G(ae^{-t}) = 1 + c_2(ae^{-t})^2 + c_3(ae^{-t})^3 + c_4(ae^{-t})^4 + \cdots . \tag{9}$$

Hence,

$$\zeta_G(ae^{-t}) - \zeta_G(a) = c_2a^2(e^{-2t} - 1) + c_3a^3(e^{-3t} - 1) + c_4a^4(e^{-4t} - 1) + \cdots \tag{10}$$

Observe that the power series

$$\begin{aligned} &\zeta_G(ae^{-t}) - \zeta_G(a) + a(e^{-t} - 1) \\ &= a(e^{-t} - 1) + c_2a^2(e^{-2t} - 1) + c_3a^3(e^{-3t} - 1) + c_4a^4(e^{-4t} - 1) + \cdots \end{aligned} \tag{11}$$

has no constant term. Also, the coefficient of t in $\zeta_G(ae^{-t}) - \zeta_G(a) + a(e^{-t} - 1)$ is

$$\frac{d}{dt}\left[\zeta_G(ae^{-t}) - \zeta_G(a) + a(e^{-t} - 1)\right]|_{t=0} = -a\left[1 + \zeta_G'(a)\right]. \tag{12}$$

We obtain $\mathcal{G}_a(t)$ by dividing $\zeta_G(ae^{-t}) - \zeta_G(a) + a(e^{-t} - 1)$ by $-a\left[1 + \zeta_G'(a)\right]$. Hence, the formal power series $\mathcal{G}_a(t)$ has zero constant coefficient as well as the coefficient for t is 1. □

In Theorem 1 a acts as a parameter in $\mathcal{G}_a(t)$. In the next theorem, we derive a power series with two parameters.

Theorem 2 *Let λ be the largest eigenvalue of T, where T is the adjacency matrix of the oriented line graph $\overline{G}$ of an admissible graph G. Then, the formal power series*

$$\mathcal{G}_{a,\sigma}(t) = \frac{\zeta_G(ae^{-t\sigma}) - \zeta_G(a) + a(e^{-t\sigma} - 1)}{-a\sigma\left(1 + \zeta_G'(a)\right)} \tag{13}$$

is invertible, where $t > 0$, $0 \leq a < \frac{1}{\lambda}$ and $\sigma > 0$.

Proof Following the proof of Theorem 1 we find

$$\begin{aligned}&\zeta_G(ae^{-t\sigma}) - \zeta_G(a) + a(e^{-t\sigma} - 1)\\ =&a(e^{-t\sigma} - 1) + c_2a^2(e^{-2t\sigma} - 1) + c_3a^3(e^{-3t\sigma} - 1) + c_4a^4(e^{-4t\sigma} - 1) + \cdots\end{aligned} \tag{14}$$

Clearly, $\zeta_G(ae^{-t\sigma}) - \zeta_G(a) + a(e^{-t\sigma} - 1)$ has no constant term. If we divide $\zeta_G(ae^{-t\sigma}) - \zeta_G(a) + a(e^{-t\sigma} - 1)$ with

$$\frac{d}{dt}\left[\zeta_G(ae^{-t\sigma}) - \zeta_G(a) + a(e^{-t\sigma} - 1)\right]|_{t=0} = -a\sigma\left[1 + \zeta_G'(a)\right] \tag{15}$$

then we observe that the coefficient of t in the resultant formal power series $\mathcal{G}_{a,\sigma}(t)$ is 1. Therefore, $\mathcal{G}_{a,\sigma}(t)$ is invertible. □

It can be trivially checked that

$$\mathcal{G}_{a,\sigma}(t) = \mathcal{G}_a(t), \text{ for } \sigma = 1. \tag{16}$$

4 New Information-Theoretic Entropy

In this section, we propose two information-theoretic entropies based on the observations in Sect. 3. We also mention a few of their characteristics.

For any non-zero probability value p define $t = \log(\frac{1}{p})$, which corresponds to $p = e^{-t}$. Setting $t = \log(\frac{1}{p})$ in Eq. (8) and simplifying we get

$$\mathcal{G}_a\left(\log\left(\frac{1}{p}\right)\right) = \frac{\zeta_G(a) - \zeta_G(ap) + a(1-p)}{a(1+\zeta_G'(a))}. \tag{17}$$

Now Lemma 1 suggests that $\mathcal{G}\left(\log\left(\frac{1}{p}\right)\right) \geq 0$. It leads us to define the single-parameter Ihara entropy as follows:

Definition 4 Given an admissible graph G, the one-parameter Ihara entropy of a probability distribution $\mathcal{P}$ is defined by

$$S_G^{(a)}(\mathcal{P}) = \sum_{i=1}^{W} p_i\mathcal{G}_a\left(\log\left(\frac{1}{p_i}\right)\right) = \sum_{i=1}^{W} p_i\frac{\zeta_G(a) - \zeta_G(ap_i) + a(1-p_i)}{a(1+\zeta_G'(a))}, \tag{18}$$

where $0 < a < \frac{1}{\lambda}$.

Similarly, setting $t = \log(\frac{1}{p})$ in Eq. (13) and simplifying we get

$$\mathcal{G}_{a,\sigma}\left(\log\left(\frac{1}{p}\right)\right) = \frac{\zeta_G(a) - \zeta_G(ap^\sigma) + a(1-p^\sigma)}{a\sigma(1+\zeta_G'(a))}, \tag{19}$$

which leads us to the definition of a two-parameter Ihara entropy as follows:

Definition 5 Given an admissible graph G, the two-parameter Ihara entropy of a probability distribution $\mathcal{P}$ is defined by

$$S_G^{(a,\sigma)}(\mathcal{P}) = \sum_{i=1}^{W} p_i \mathcal{G}_{a,\sigma}\left(\log\left(\frac{1}{p_i}\right)\right) = \sum_{i=1}^{W} p_i \left(\frac{\zeta_G(a) - \zeta_G(ap_i^\sigma) + a(1-p_i^\sigma)}{a\sigma(1+\zeta_G'(a))}\right), \tag{20}$$

where $0 \leq a < \frac{1}{\lambda}$ and $\sigma > 0$ are two real parameters.

From our construction it is clear that

$$S_G^{(a,\sigma)}(\mathcal{P}) = S_G^{(a)}(\mathcal{P}), \text{ for } \sigma = 1. \tag{21}$$

Therefore we discuss the properties of $S_G^{(a,\sigma)}(\mathcal{P})$ which holds in the special case for $\sigma = 1$.

Define a function $s_{a,\sigma} : [0, 1] \to \mathbb{R}^+$ such that

$$s_{a,\sigma}(p) = p \times \left(\frac{\zeta_G(a) - \zeta_G(ap^\sigma) + a(1-p^\sigma)}{a\sigma(1+\zeta_G'(a))}\right). \tag{22}$$

Therefore, the Ihara entropy mentioned in Definition 5 can be expressed as $S_G^{(a,\sigma)}(\mathcal{P}) = \sum_{i=1}^{W} s_{a,\sigma}(p_i)$. Now we have the following observations:

Corollary 1 *As $s_{a,\sigma}(p)$ is a continuous function, $S_G^{(a,\sigma)}(\mathcal{P})$ is also continuous with respect to all its arguments p_i for $i = 1, 2, \ldots, W$. Thus, $S_G^{(a,\sigma)}(\mathcal{P})$ satisfies the SK axiom 1.*

Corollary 2 *The SK axiom 2 is also trivially satisfied as $s_{a,\sigma}(0) = 0$.*

To prove the SK axiom 3 we need the following theorem from [6], which we mention without proof.

Theorem 3 *The function $s_{a,\sigma}(p)$ has a global maxima in* (0, 1).

We can conclude from Theorem 3 that $S_G^{(a,\sigma)}(\mathcal{P})$ attains the maximum value if $s(p_i)$ is maximum for all $p_i \in \mathcal{P}$. Therefore, to maximize $S_G^{(a,\sigma)}(\mathcal{P})$ we need $p_i = c$ for all i, which is the uniform distribution after a normalization. Hence, $S_G^{(a,\sigma)}(\mathcal{P})$ follows the SK axiom 3.

The entropy $S_G^{(a,\sigma)}(\mathcal{P})$ follows a generalized version of the SK axiom 4 induced by the Lazard formal group law. The generalized version is mentioned in the theorem below [6]:

Theorem 4 *Let* $\mathcal{P}_A = \left\{p_i^{(A)}\right\}_{i=1}^{W_A}$ *and* $\mathcal{P}_B = \left\{p_j^{(B)}\right\}_{j=1}^{W_B}$ *be two independent probability distributions. Then*

$$S_G^{(a,\sigma)}(\mathcal{P}_A\mathcal{P}_B) = \sum_{i=1}^{W_A}\sum_{j=1}^{W_B} p_i^{(A)} p_j^{(B)} \Phi\left(\mathcal{G}\left(\log\left(\frac{1}{p_i^{(A)}}\right)\right), \mathcal{G}\left(\log\left(\frac{1}{p_j^{(B)}}\right)\right)\right), \tag{23}$$

where Φ *is Lazard formal group law given by* $\Phi(s_1, s_2) = \mathcal{G}(\mathcal{F}(s_1) + \mathcal{F}(s_2))$ *as well as* $\mathcal{G} = \mathcal{G}_a$ *or* $\mathcal{G} = \mathcal{G}_{a,\sigma}$.

5 Conclusion

The universal group entropy generated by an invertible formal power series is proposed in the literature [18, 19]. This article considers the power series representation of the Ihara zeta function for defining universal group entropy, which we call Ihara entropy. An interesting connection between the billiard dynamical system and the Ihara zeta function is illustrated in this article. The interesting facet of our proposed entropy is that the different arrangement of reflectors in the billiard system induces different entropy functions. We justify that the new entropy functions fulfill the generalized version of the Shannon–Khinchin axioms.

Disclaimer

This article provides a short overview on the authors' articles [5, 6].

References

1. Bochner, S.: Formal Lie groups. Ann. Math. 192–201 (1946)
2. Brewer, T.S.: Algebraic properties of formal power series composition (2014)
3. Chernov, N., Markarian, R.: Chaotic Billiards, vol. 127. American Mathematical Society, Providence (2006)
4. Dieudonne, J.A.: Introduction to the Theory of Formal Groups, vol. 20. CRC Press, Boca Raton (1973)
5. Dutta, S., Guha, P.: Ihara zeta entropy (2019). arXiv:190602514
6. Dutta, S., Guha, P.: A system of billiard and its application to information-theoretic entropy (2020). arXiv:200403444
7. Dutta, S., Furuichi, S., Guha, P.: Elements of generalized Tsallis relative entropy in classical information theory (2019). arXiv:190801696
8. Gibbs, J.W.: Elementary Principles in Statistical Mechanics: Developed with Especial Reference to the Rational Foundations of Thermodynamics. C. Scribner's Sons, New York (1902)
9. Hazewinkel, M.: Formal Groups and Applications, vol. 78. Elsevier, Amsterdam (1978)

10. Ihara, Y.: On discrete subgroups of the two by two projective linear group over p-adic fields. J. Math. Soc. Jpn. **18**(3), 219–235 (1966)
11. Khinchin, A.Y.: Mathematical Foundations of Information Theory. Courier Corporation (2013)
12. Kotani, M., Sunada, T.: 2.-zeta functions of finite graphs. J. Math. Sci.-Univ. Tokyo **7**(1), 7–26 (2000)
13. Kozlov, V.V., Kozlov, V.V., Treshchev, D.V., Treshchëv, D.V.: Billiards: a Genetic Introduction to the Dynamics of Systems with Impacts, vol. 89. American Mathematical Society, Providence (1991)
14. Rozikov, U.A.: An Introduction to Mathematical Billiards. World Scientific, Singapore (2018)
15. Shannon, C.E.: A mathematical theory of communication. Bell Syst. Tech. J. **27**(3), 379–423 (1948)
16. Shannon, C.E., Weaver, W.: The Mathematical Theory of Communication. 1949. University of Illinois Press, Urbana (1963)
17. Sinai, Y.G.: Dynamical systems with elastic reflections. Russ. Math. Surv. **25**(2), 137 (1970)
18. Tempesta, P.: A theorem on the existence of trace-form generalized entropies. Proc. R. Soc. A **471**(2183), 20150165 (2015)
19. Tempesta, P.: Beyond the Shannon-Khinchin formulation: the composability axiom and the universal-group entropy. Ann. Phys. **365**, 180–197 (2016)
20. Terras, A.: Zeta Functions of Graphs: a Stroll Through the Garden, vol. 128. Cambridge University Press, Cambridge (2010)

On a New Subclass of Sakaguchi Type Functions Using $(\mathfrak{p}, \mathfrak{q})$-Derivative Operator

S. Baskaran, G. Saravanan, and K. Muthunagai

Abstract The authors have introduced a new subclass of bi-univalent functions consisting of Sakaguchi type functions involving $(\mathfrak{p}, \mathfrak{q})$-derivative operator. Further, the estimation of bounds for $|a_2|$ and $|a_3|$ has been obtained. The authors have stated a few examples in this paper.

Keywords Analytic function · Bi-univalent function · $(\mathfrak{p}, \mathfrak{q})$-derivative operator · Sakaguchi type function · Subordination

2010 Subject Classification 30C45 · 30C15

1 Introduction and Preliminaries

A function of one or more complex variables which is complex-valued is said to be analytic if it is differentiable at every point of the domain. Every normalized analytic function can be expressed as a series of the form

$$\mathfrak{f}(z) = z + \sum_{t=2}^{\infty} a_t z^t \tag{1}$$

S. Baskaran
Department of Mathematics, Agurchand Manmull Jain College, Meenambakkam, Chennai 600114, Tamil Nadu, India

G. Saravanan (✉)
Department of Mathematics, Patrician College of Arts and Science, Adyar, Chennai 600020, Tamil Nadu, India
e-mail: gsaran825@yahoo.com

K. Muthunagai
School of Advanced Sciences, VIT Chennai, Chennai 600 127, Tamil Nadu, India
e-mail: muthunagai@vit.ac.in

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_19

in the complex variable z that is convergent in $\mathfrak{U} = \{z : z \in \mathbb{C}, |z| < 1\}$. Let A consist of every such function. A subclass $\mathscr{S}$ of A be defined by $\mathscr{S} = \{\mathfrak{f}(z) \in A : \mathfrak{f}(z_1) = \mathfrak{f}(z_2) \Rightarrow z_1 = z_2\}$ (i.e.) $\mathscr{S}$ consists of all univalent functions.

A function $\mathfrak{f}(z) \in A$ is called bi-univalent in $\mathfrak{U}$, if $\mathfrak{f}(z) \in \mathscr{S}$ and its inverse function has an analytic continuation to $|w| < 1$. Let $\sigma = \{\mathfrak{f} \in \mathscr{S} : \mathfrak{f}$ is bi-univalent$\}$.

Though Lewin [7] introduced the class of bi-univalent functions, the passion on the bounds for the coefficients of these classes was upraised by Netanyahu, Clunie, Brannan and many others [3, 8, 12–14, 16, 17]. This has been a field of fascination for young researchers to date.

If, for $\mathfrak{f}_1(z)$ and $\mathfrak{f}_2(z)$ analytic in $\mathfrak{U}$, there exists a Schwarz function $\mathfrak{w}(z)$ with $\mathfrak{w}(0) = 0$ and $|\mathfrak{w}(z)| < 1$ in $\mathfrak{U}$ such that $\mathfrak{f}_1(z) = \mathfrak{f}_2(\mathfrak{w}(z))$, then we say that $\mathfrak{f}_1(z) \prec \mathfrak{f}_2(z)$.

A subclass consisting of functions satisfying the analytic criterion $Re\left(\frac{z\mathfrak{f}'(z)}{\mathfrak{f}(z)-\mathfrak{f}(-z)}\right) > \alpha$ was introduced by Sakaguchi [11] and these functions were named after him as Sakaguchi type functions [9, 10, 15]. Sakaguchi type functions are starlike with respect to symmetric points. Frasin [6] generalized Sakaguchi type class which had functions of the form (1) given by $Re\left(\frac{(\mathsf{s}_1-\mathsf{s}_2)z\mathfrak{f}'(z)}{\mathfrak{f}(\mathsf{s}_1 z)-\mathfrak{f}(\mathsf{s}_2 z)}\right) > \alpha,\ 0 \le \alpha < 1, \mathsf{s}_1, \mathsf{s}_2 \in \mathbb{C}$ with $\mathsf{s}_1 \neq \mathsf{s}_2, |\mathsf{s}_2| \le 1, \forall z \in \mathfrak{U}$.

Definition 1 For $\mathfrak{q}, \mathfrak{p} \in (0, 1]$ and $\mathfrak{q} < \mathfrak{p}$, the $(\mathfrak{p}, \mathfrak{q})$-derivative operator $\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z))$ [1, 4] is defined as

$$\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z)) = \frac{\mathfrak{f}(\mathfrak{p}z) - \mathfrak{f}(\mathfrak{q}z)}{(\mathfrak{p} - \mathfrak{q})(z)}, z \neq 0 \tag{2}$$

and $\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(0)) = \mathfrak{f}'(0)$ provided $\mathfrak{f}'(0)$ exists. It can be easily deduced that

$$\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z)) = 1 + \sum_{t=2}^{\infty}[t]_{\mathfrak{p},\mathfrak{q}} a_t z^{t-1},$$

where $[t]_{\mathfrak{p},\mathfrak{q}} = \frac{\mathfrak{p}^t - \mathfrak{q}^t}{\mathfrak{p}-\mathfrak{q}}$, the $(\mathfrak{p}, \mathfrak{q})$ bracket of t. It is also called a twin-basic number. It is to be noted that $\mathfrak{D}_{p,q}(z^t) = [t]_{\mathfrak{p},\mathfrak{q}} z^{t-1}$. Also for $\mathfrak{p} = 1$, the$(\mathfrak{p}, \mathfrak{q})$-derivative operator $\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}$ reduces to the $\mathfrak{q}$-derivative operator $\mathfrak{D}_{\mathfrak{q}}$.

The inverse series of (2) is given by

$$\begin{aligned}(\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{g}))(w) &= \frac{\mathfrak{g}(\mathfrak{p}w) - \mathfrak{g}(\mathfrak{q}w)}{(\mathfrak{p} - \mathfrak{q})w} \\ &= 1 - [2]_{\mathfrak{p},\mathfrak{q}} a_2 w + [3]_{\mathfrak{p},\mathfrak{q}}(2a_2^2 - a_3)w^2 \\ &\quad -[4]_{\mathfrak{p},\mathfrak{q}}(5a_2^3 - 5a_2 a_3 + a_4)w^3 + \cdots .\end{aligned}$$

Consider an analytic function with $Re(\Psi(z)) > 0$ in $\mathfrak{U}$, $\Psi(0) = 1$ and $\Psi'(0) > 0$. Also $\Psi(\mathfrak{U})$ be starlike with respect to 1 and symmetric with respect to the real axis. Thus, $\Psi(z)$ has the Taylor series expansion

$$\Psi(z) = 1 + \mathfrak{B}_1 z + \mathfrak{B}_2 z^2 + \mathfrak{B}_3 z^3 + \cdots \qquad (\mathfrak{B}_1 > 0). \tag{3}$$

Consider two analytic functions $u(z)$ and $v(z)$ in $\mathfrak{U}$. Let $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(z)| < 1$.

Assume

$$u(z) = p_1 z + \sum_{t=2}^{\infty} p_t z^t, \quad v(w) = q_1 w + \sum_{t=2}^{\infty} q_t w^t \quad (|z| < 1, |w| < 1). \tag{4}$$

It is to be noted that

$$|p_1| \le 1, \quad |p_2| \le 1 - |p_1|^2, \quad |q_1| \le 1, \quad |q_2| \le 1 - |q_1|^2. \tag{5}$$

Equations (3) and (4) take the form

$$\Psi(u(z)) = 1 + \mathfrak{B}_1 p_1 z + \left(p_1^2 \mathfrak{B}_2 + p_2 \mathfrak{B}_1\right) z^2 + \cdots, \quad |z| < 1 \tag{6}$$

and

$$\Psi(v(w)) = 1 + \mathfrak{B}_1 q_1 w + \left(q_1^2 \mathfrak{B}_2 + q_2 \mathfrak{B}_1\right) w^2 + \cdots, \quad |w| < 1. \tag{7}$$

In this paper we have introduced a new class using $(\mathfrak{p}, \mathfrak{q})$-derivative operator and subordination. We have obtained bounds for $|a_2|$ and $|a_3|$.

2 Main Results

Definition 2 A function $\mathfrak{f} \in \sigma$ is said to be in the class $\mathscr{S}_{\sigma}^{\mathfrak{p},\mathfrak{q}}(\Psi, \mathsf{s}_1, \mathsf{s}_2)$, if the following subordination relations hold

$$\frac{(\mathsf{s}_1 - \mathsf{s}_2) z \mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z))}{\mathfrak{f}(\mathsf{s}_1 z) - \mathfrak{f}(\mathsf{s}_2 z)} \prec \Psi(z)$$

and

$$\frac{(\mathsf{s}_1 - \mathsf{s}_2) w \mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{g}(w))}{\mathfrak{g}(\mathsf{s}_1 w) - \mathfrak{g}(\mathsf{s}_2 w)} \prec \Psi(w),$$

where $\mathfrak{g}(w) = \mathfrak{f}^{-1}(w)$, $\mathsf{s}_1, \mathsf{s}_2 \in \mathbb{C}$ with $\mathsf{s}_1 \neq \mathsf{s}_2$, $|\mathsf{s}_2| \le 1$.

Theorem 1 *Let $\mathfrak{f}$ given by (1) be in the class $\mathscr{S}_{\sigma}^{\mathfrak{p},\mathfrak{q}}(\Psi, \mathsf{s}_1, \mathsf{s}_2)$. Then*

$$|a_2| \le \frac{\mathfrak{B}_1 \sqrt{\mathfrak{B}_1}}{\sqrt{|\mathfrak{B}_1^2 \left[[3]_{\mathfrak{p}\,\mathfrak{q}} - [2]_{\mathfrak{p}\,\mathfrak{q}} \mathsf{s}_1 - [2]_{\mathfrak{p}\,\mathfrak{q}} \mathsf{s}_2 + \mathsf{s}_1 \mathsf{s}_2\right] - \mathfrak{B}_2 \left[[2]_{\mathfrak{p}\,\mathfrak{q}} - \mathsf{s}_1 - \mathsf{s}_2\right]^2| + \mathfrak{B}_1 |[2]_{\mathfrak{p}\,\mathfrak{q}} - \mathfrak{s}_1 - \mathsf{s}_2|^2}}$$

and

$$|a_3| \leq \begin{cases} \frac{\mathfrak{B}_1}{|[3]_{\mathfrak{p}\,\mathfrak{q}} - \mathsf{s}_1{}^2 - \mathsf{s}_2{}^2 - \mathsf{s}_1\mathsf{s}_2|}, & \mathfrak{B}_1 \leq \mathfrak{S}_1(\mathfrak{p},\mathfrak{q},\mathsf{s}_1,\mathsf{s}_2) \\ \frac{|\mathfrak{M}_1(\mathfrak{p},\mathfrak{q},\mathsf{s}_1,\mathsf{s}_2,\mathfrak{B}_1,\mathfrak{B}_2)|\mathfrak{B}_1 + |[3]_{\mathfrak{p}\,\mathfrak{q}} - \mathsf{s}_1^2 - \mathsf{s}_2{}^2 - \mathsf{s}_1\mathsf{s}_2|\mathfrak{B}_1^3}{|[3]_{\mathfrak{p}\,\mathfrak{q}} - \mathsf{s}_1{}^2 - \mathsf{s}_2{}^2 - \mathsf{s}_1\mathsf{s}_2|\mathfrak{M}_2(\mathfrak{p},\mathfrak{q},\mathsf{s}_1,\mathsf{s}_2,\mathfrak{B}_1,\mathfrak{B}_2)}, & \mathfrak{B}_1 > \mathfrak{S}_1(\mathfrak{p},\mathfrak{q},\mathsf{s}_1,\mathsf{s}_2) \end{cases}$$

where

$$\begin{aligned} \mathfrak{M}_1(\mathfrak{p},\mathfrak{q},\mathsf{s}_1,\mathsf{s}_2,\mathfrak{B}_1,\mathfrak{B}_2) &= ([3]_{\mathfrak{p}\mathfrak{q}} - [2]_{pq}\mathsf{s}_1 - [2]_{\mathfrak{p}\mathfrak{q}}\mathsf{s}_2 + \mathsf{s}_1\mathsf{s}_2)\mathfrak{B}_1^2 \\ &\quad -([2]_{\mathfrak{p}\mathfrak{q}} - \mathsf{s}_1 - \mathsf{s}_2)^2\mathfrak{B}_2 \\ \mathfrak{M}_2(\mathfrak{p},\mathfrak{q},\mathsf{s}_1,\mathsf{s}_2,\mathfrak{B}_1,\mathfrak{B}_2) &= \big[|([3]_{\mathfrak{p}\mathfrak{q}} - [2]_{\mathfrak{p}\mathfrak{q}}\mathsf{s}_1 - [2]_{\mathfrak{p}\mathfrak{q}}\mathsf{s}_2 + \mathsf{s}_1\mathsf{s}_2)\mathfrak{B}_1^2 \\ &\quad -([2]_{\mathfrak{p}\mathfrak{q}} - \mathsf{s}_1 - \mathsf{s}_2)^2\mathfrak{B}_2| + |[2]_{\mathfrak{p}\mathfrak{q}} - \mathsf{s}_1 - \mathsf{s}_2|^2\mathfrak{B}_1\big] \\ \mathfrak{S}_1(\mathfrak{p},\mathfrak{q},\mathsf{s}_1,\mathsf{s}_2) &= \frac{|[2]_{\mathfrak{p}\mathfrak{q}} - \mathsf{s}_1 - \mathsf{s}_2|^2}{|[3]_{\mathfrak{p}\mathfrak{q}} - \mathsf{s}_1{}^2 - \mathsf{s}_2^2 - \mathsf{s}_1\mathsf{s}_2|}. \end{aligned}$$

Proof Let $\mathfrak{f} \in \mathscr{S}_\sigma^{\mathfrak{p},\mathfrak{q}}(\Psi,\mathsf{s}_1,\mathsf{s}_2)$. Then, there exist analytic functions $u, v : \mathfrak{U} \to \mathfrak{U}$ given by (4) such that

$$\frac{(\mathsf{s}_1 - \mathsf{s}_2)z\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z))}{\mathfrak{f}(\mathsf{s}_1 z) - \mathfrak{f}(\mathsf{s}_2 z)} = \Psi(u(z)) \tag{8}$$

and

$$\frac{(\mathsf{s}_1 - \mathsf{s}_2)w\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{g}(w))}{\mathfrak{g}(\mathsf{s}_1 w) - \mathfrak{g}(\mathsf{s}_2 w)} = \Psi(v(w)). \tag{9}$$

Since

$$\begin{aligned} \frac{(\mathsf{s}_1 - \mathsf{s}_2)z\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{f}(z))}{\mathfrak{f}(\mathsf{s}_1 z) - \mathfrak{f}(\mathsf{s}_2 z)} =& 1 + \left([2]_{\mathfrak{p}\mathfrak{q}} - \mathsf{s}_1 - \mathsf{s}_2\right)a_2 z + \Big\{\Big([3]_{\mathfrak{p}\mathfrak{q}} - \mathsf{s}_1{}^2 - \mathsf{s}_2{}^2 - \mathsf{s}_1\mathsf{s}_2\Big)a_3 \\ &- \Big([2]_{\mathfrak{p}\mathfrak{q}}\mathsf{s}_1 + [2]_{\mathfrak{p}\mathfrak{q}}\mathsf{s}_2 - \mathsf{s}_1{}^2 - \mathsf{s}_2{}^2 - 2\mathsf{s}_1\mathsf{s}_2\Big)a_2^2\Big\} \times z^2 + \cdots. \end{aligned} \tag{10}$$

$$\begin{aligned} \frac{(\mathsf{s}_1 - \mathsf{s}_2)w\mathfrak{D}_{\mathfrak{p},\mathfrak{q}}(\mathfrak{g}(w))}{\mathfrak{g}(\mathsf{s}_1 w) - \mathfrak{g}(\mathsf{s}_2 w)} =& 1 - \left([2]_{\mathfrak{p}\mathfrak{q}} - \mathsf{s}_1 - \mathsf{s}_2\right)a_2 w - \{\Big([3]_{\mathfrak{p}\mathfrak{q}} - \mathsf{s}_1{}^2 - \mathsf{s}_2{}^2 - \mathsf{s}_1\mathsf{s}_2\Big)a_3 \\ &- \Big(2[3]_{\mathfrak{p}\mathfrak{q}} - \mathsf{s}_1{}^2 - \mathsf{s}_2{}^2 - [2]_{\mathfrak{p}\mathfrak{q}}\mathsf{s}_1 - [2]_{\mathfrak{p}\mathfrak{q}}\mathsf{s}_2\Big)a_2^2\} \times w^2 + \cdots. \end{aligned} \tag{11}$$

It follows from (6), (7), (10) and (11) that

$$\big[[2]_{pq} - \mathsf{s}_1 - \mathsf{s}_2\big]a_2 = \mathfrak{B}_1 p_1, \tag{12}$$

$$\begin{aligned} \big[[3]_{pq} - \mathsf{s}_1{}^2 - \mathsf{s}_2{}^2 - \mathsf{s}_1\mathsf{s}_2\big]a_3 - \big[[2]_{pq}\mathsf{s}_1 + [2]_{pq}\mathsf{s}_2 - \mathsf{s}_1{}^2 - \mathsf{s}_2{}^2 - 2\mathsf{s}_1\mathsf{s}_2\big]a_2^2 \\ = \mathfrak{B}_1 p_2 + \mathfrak{B}_2 p_1^2, \end{aligned} \tag{13}$$

$$-[[2]_{pq}-\mathsf{s}_1-\mathsf{s}_2]a_2=\mathfrak{B}_1q_1, \tag{14}$$

$$\left[2[3]_{pq}-{\mathsf{s}_1}^2-{\mathsf{s}_2}^2-[2]_{pq}\mathsf{s}_1-[2]_{pq}\mathsf{s}_2\right]a_2^2-\left[[3]_{pq}-{\mathsf{s}_1}^2-{\mathsf{s}_2}^2-\mathsf{s}_1\mathsf{s}_2\right]a_3 \\ =\mathfrak{B}_1q_2+\mathfrak{B}_2q_1^2. \tag{15}$$

From (12) and (14)

$$p_1=-q_1. \tag{16}$$

Further computation using (12), (13), (15) and (16) leads to

$$\left[2\left([3]_{pq}-[2]_{pq}\mathsf{s}_1-[2]_{pq}\mathsf{s}_2+\mathsf{s}_1\mathsf{s}_2\right)\mathfrak{B}_1^2\right]-2\left[([2]_{pq}-\mathsf{s}_1-\mathsf{s}_2)^2\mathfrak{B}_2\right]a_2^2 \\ =\mathfrak{B}_1^3(p_2+q_2). \tag{17}$$

Equations (16) and (17), together with (5), result in

$$|\left[[3]_{pq}-[2]_{pq}\mathsf{s}_1-[2]_{pq}\mathsf{s}_2+\mathsf{s}_1\mathsf{s}_2\right]\mathfrak{B}_1^2-\left[[2]_{pq}-\mathsf{s}_1-\mathsf{s}_2\right]^2\mathfrak{B}_2||a_2|^2 \\ \leq|\mathfrak{B}_1^3|(1-|p_1|^2), \tag{18}$$

the desired estimate for $|a_2|$.

Next, in order to obtain the bound for $|a_3|$, subtracting (15) from (13), we have

$$2\left[[3]_{pq}-{\mathsf{s}_1}^2-{\mathsf{s}_2}^2-\mathsf{s}_1\mathsf{s}_2\right]a_3+2\left[\mathsf{s}_1^2+{\mathsf{s}_2}^2+\mathsf{s}_1\mathsf{s}_2-[3]_{pq}\right]a_2^2 \\ =\mathfrak{B}_1(p_2-q_2)+\mathfrak{B}_2\left(p_1^2-q_1^2\right). \tag{19}$$

Then, in view of (5) and (16), we have

$$|[3]_{pq}-{\mathsf{s}_1}^2-{\mathsf{s}_2}^2-\mathsf{s}_1\mathsf{s}_2||a_3|\mathfrak{B}_1\leq\left[\mathfrak{B}_1|[3]_{pq}-{\mathsf{s}_1}^2-{\mathsf{s}_2}^2-\mathsf{s}_1\mathsf{s}_2| \\ -|[2]_{pq}-\mathsf{s}_1-\mathsf{s}_2|\right]|a_2|^2+\mathfrak{B}_1^2.$$

Substituting for $|a_2|$, we get the desired estimate for $|a_3|$.

Remark 1 Let $\mathfrak{p}=1$ and $\mathfrak{q}\to1^-$. The above theorem reduces to Altinkaya et al. [2]

Corollary 1 *Suppose* $\mathfrak{f}$, *given by* (1) *is in the class* $\mathscr{S}_\sigma^{\mathfrak{p},\mathfrak{q}}(\Psi,1,-1)$, *then*

$$|a_2|\leq\frac{\mathfrak{B}_1\sqrt{\mathfrak{B}_1}}{\sqrt{|\mathfrak{B}_1^2\left[[3]_{\mathfrak{p}\,\mathfrak{q}}-1\right]-\mathfrak{B}_2\left[[2]_{\mathfrak{p}\,\mathfrak{q}}\right]^2|+\mathfrak{B}_1|[2]_{\mathfrak{p}\,\mathfrak{q}}|^2}}$$

and

$$|a_3|\leq\begin{cases}\frac{\mathfrak{B}_1}{|[3]_{\mathfrak{p}\,\mathfrak{q}}-1|}, & \mathfrak{B}_1\leq\mathfrak{S}_2(\mathfrak{p},\mathfrak{q})\\ \frac{|([3]_{\mathfrak{p}\,\mathfrak{q}}-1)\mathfrak{B}_1^2-[2]_{\mathfrak{p}\,\mathfrak{q}}^2\mathfrak{B}_2|\mathfrak{B}_1+|[3]_{\mathfrak{p}\,\mathfrak{q}}-1|\mathfrak{B}_1^3}{|[3]_{\mathfrak{p}\,\mathfrak{q}}-1|\left[|([3]_{\mathfrak{p}\,\mathfrak{q}}-1)\mathfrak{B}_1^2-([2]_{\mathfrak{p}\,\mathfrak{q}})^2\mathfrak{B}_2|+|[2]_{\mathfrak{p}\,\mathfrak{q}}|^2\mathfrak{B}_1\right]}, & \mathfrak{B}_1>\mathfrak{S}_2(\mathfrak{p},\mathfrak{q})\end{cases}$$

where $\mathfrak{S}_2(\mathfrak{p},\mathfrak{q}) = \frac{|[2]_{\mathfrak{p}\,\mathfrak{q}}|^2}{|[3]_{\mathfrak{p}\,\mathfrak{q}}-1|}$.

Remark 2 Let $\mathfrak{p} = 1$ and $\mathfrak{q} \to 1^-$. The above corollary reduces to Emeka et al. [5]

Corollary 2 *Suppose* $\mathfrak{f}$*, given by* (1) *is in the class* $\mathscr{S}_{\sigma}^{\mathfrak{p},\mathfrak{q}}(\Psi, 1, 0)$*, then*

$$|a_2| \le \frac{\mathfrak{B}_1\sqrt{\mathfrak{B}_1}}{\sqrt{|\mathfrak{B}_1^2\left[[3]_{\mathfrak{p}\,\mathfrak{q}}-[2]_{\mathfrak{p}\,\mathfrak{q}}\right]-\mathfrak{B}_2\left[[2]_{\mathfrak{p}\,\mathfrak{q}}-1\right]^2|+\mathfrak{B}_1|[2]_{\mathfrak{p}\,\mathfrak{q}}-1|^2}}$$

and

$$|a_3| \le \begin{cases} \frac{\mathfrak{B}_1}{|[3]_{\mathfrak{p}\,\mathfrak{q}}-1|}, & \mathfrak{B}_1 \le \mathfrak{S}_3(\mathfrak{p},\mathfrak{q}) \\ \frac{|([3]_{\mathfrak{p}\,\mathfrak{q}}-[2]_{pq})\mathfrak{B}_1^2-([2]_{\mathfrak{p}\,\mathfrak{q}}-1)^2\mathfrak{B}_2|\mathfrak{B}_1+|[3]_{\mathfrak{p}\,\mathfrak{q}}-1|\mathfrak{B}_1^3}{|[3]_{\mathfrak{p}\,\mathfrak{q}}-1|\left[|([3]_{\mathfrak{p}\,\mathfrak{q}}-[2]_{\mathfrak{p}\,\mathfrak{q}})\mathfrak{B}_1^2-([2]_{\mathfrak{p}\,\mathfrak{q}}-1)^2\mathfrak{B}_2|+|[2]_{\mathfrak{p}\,\mathfrak{q}}-1|^2\mathfrak{B}_1\right]}, & \mathfrak{B}_1 > \mathfrak{S}_3(\mathfrak{p},\mathfrak{q}) \end{cases}$$

where $\mathfrak{S}_3(\mathfrak{p},\mathfrak{q}) = \frac{|[2]_{\mathfrak{p}\,\mathfrak{q}}-1|^2}{|[3]_{\mathfrak{p}\,\mathfrak{q}}-1|}$.

Remark 3 Let $\mathfrak{p} = 1$ and $\mathfrak{q} \to 1^-$. The above corollary reduces to Emeka et al. [5]

Corollary 3 *Let*

$$\Psi(z) = \left(\frac{1+z}{1-z}\right)^{\beta} = 1 + 2\beta z + 2\beta^2 z^2 + \cdots, \quad (0 < \beta \le 1).$$

We have

$$|a_2| \le \frac{2\beta}{\sqrt{\left|2\left[[3]_{pq}-[2]_{pq}\mathsf{s}_1-[2]_{pq}\mathsf{s}_2+\mathsf{s}_1\mathsf{s}_2\right]-\left[[2]_{pq}-\mathsf{s}_1-\mathsf{s}_2\right]^2\right|\beta+|[2]_{pq}-\mathsf{s}_1-\mathsf{s}_2|^2}}$$

and

$$|a_3| \le \begin{cases} \frac{2\beta}{|[3]_{pq}-\mathsf{s}_1{}^2-\mathsf{s}_2{}^2-\mathsf{s}_1\mathsf{s}_2|}, & if\ 0 < \beta \le \mathfrak{S}_4(p,q,\mathsf{s}_1,\mathsf{s}_2) \\ \frac{2\left[|\mathfrak{M}_3(p,q,\mathsf{s}_1,\mathsf{s}_2)|+2|[3]_{pq}-\mathsf{s}_1^2-\mathsf{s}_2{}^2-\mathsf{s}_1\mathsf{s}_2|\right]\beta^2}{|[3]_{pq}-\mathsf{s}_1{}^2-\mathsf{s}_2{}^2-\mathsf{s}_1\mathsf{s}_2|\mathfrak{M}_4(p,q,\mathsf{s}_1,\mathsf{s}_2)}, & if\ \mathfrak{S}_4(p,q,\mathsf{s}_1,\mathsf{s}_2) < \beta \le 1 \end{cases}$$

where

$$\mathfrak{M}_3(p,q,\mathsf{s}_1,\mathsf{s}_2) = 2([3]_{pq} - [2]_{pq}\mathsf{s}_1 - [2]_{pq}\mathsf{s}_2 + \mathsf{s}_1\mathsf{s}_2) - ([2]_{pq} - \mathsf{s}_1 - \mathsf{s}_2)^2$$
$$\mathfrak{M}_4(p,q,\mathsf{s}_1,\mathsf{s}_2) = \Big[|2([3]_{pq} - [2]_{pq}\mathsf{s}_1 - [2]_{pq}\mathsf{s}_2 + \mathsf{s}_1\mathsf{s}_2) - ([2]_{pq} - \mathsf{s}_1 - \mathsf{s}_2)^2|\beta + |[2]_{pq} - \mathsf{s}_1 - \mathsf{s}_2|^2\Big]$$
$$\mathfrak{S}_4(p,q,\mathsf{s}_1,\mathsf{s}_2) = \frac{|[2]_{pq} - \mathsf{s}_1 - \mathsf{s}_2|^2}{2|[3]_{pq} - \mathsf{s}_1{}^2 - \mathsf{s}_2^2 - \mathsf{s}_1\mathsf{s}_2|}.$$

Corollary 4 *Let*

$$\Psi(z) = \frac{1+(1-2\beta)z}{1-z} = 1 + 2(1-\beta)z + 2(1-\beta)z^2 + \cdots, \quad (0 \le \beta < 1).$$

We have

$$|a_2| \le \frac{2(1-\beta)}{\sqrt{\left|2\big([3]_{pq} - [2]_{pq}\mathsf{s}_1 - [2]_{pq}\mathsf{s}_2 + \mathsf{s}_1\mathsf{s}_2\big)(1-\beta) - \big([2]_{pq} - \mathsf{s}_1 - \mathsf{s}_2\big)^2\right| + \left|[2]_{pq} - \mathsf{s}_1 - \mathsf{s}_2\right|^2}}$$

and

$$|a_3| \le \begin{cases} \dfrac{2(1-\beta)}{\left|[3]_{pq} - \mathsf{s}_1^2 - \mathsf{s}_2^2 - \mathsf{s}_1\mathsf{s}_2\right|}, & if\ \mathfrak{S}_5(p,q,\mathsf{s}_1,\mathsf{s}_2) \le \beta < 1 \\ \dfrac{2\left[|\mathfrak{M}_5(p,q,\mathsf{s}_1,\mathsf{s}_2)| + 2|[3]_{pq} - \mathsf{s}_1^2 - \mathsf{s}_2{}^2 - \mathsf{s}_1\mathsf{s}_2|(1-\beta)\right](1-\beta)}{|[3]_{pq} - \mathsf{s}_1{}^2 - \mathsf{s}_2{}^2 - \mathsf{s}_1\mathsf{s}_2|\mathfrak{M}_6(p,q,\mathsf{s}_1,\mathsf{s}_2)}, & if\ 0 \le \beta < \mathfrak{S}_5(p,q,\mathsf{s}_1,\mathsf{s}_2) \end{cases}$$

where

$$\begin{aligned} \mathfrak{M}_5(p,q,\mathsf{s}_1,\mathsf{s}_2) &= 2([3]_{pq} - [2]_{pq}\mathsf{s}_1 - [2]_{pq}\mathsf{s}_2 + \mathsf{s}_1\mathsf{s}_2)(1-\beta) - ([2]_{pq} - \mathsf{s}_1 - \mathsf{s}_2)^2 \\ \mathfrak{M}_6(p,q,\mathsf{s}_1,\mathsf{s}_2) &= \Big[|2([3]_{pq} - [2]_{pq}\mathsf{s}_1 - [2]_{pq}\mathsf{s}_2 + \mathsf{s}_1\mathsf{s}_2)(1-\beta) - ([2]_{pq} - \mathsf{s}_1 - \mathsf{s}_2)^2| \\ &\quad + |[2]_{pq} - \mathsf{s}_1 - \mathsf{s}_2|^2\Big] \\ \mathfrak{S}_5(p,q,\mathsf{s}_1,\mathsf{s}_2) &= \frac{2|[3]_{pq} - \mathsf{s}_1^2 - \mathsf{s}_2^2 - \mathsf{s}_1\mathsf{s}_2| - |2 - \mathsf{s}_1 - \mathsf{s}_2|^2}{2|[3]_{pq} - \mathsf{s}_1^2 - \mathsf{s}_2^2 - \mathsf{s}_1\mathsf{s}_2|}. \end{aligned}$$

3 Conclusion

We have estimated the bounds for $|a_2|$ and $|a_3|$ for functions belonging to the new class defined by us in this paper. We will extend our work by finding the bounds for $|a_4|$ and $|a_5|$. Though it is too difficult to find sharpness for our class as it is defined using the pq-derivative operator, we will try to extend our work by finding sharpness.

References

1. Ahuja, O.P., Çetinkaya, A.: Use of quantum calculus approach in mathematical sciences and its role in geometric function theory. AIP Conf. Proc. **2095**(1), AIP Publishing LLC (2019)
2. Altinkaya, S., Yalçin, S.: On a new subclass of bi-univalent functions of Sakaguchi type satisfying subordinate conditions. Malaya J. Math. **5**(2), 305–309 (2017)
3. Brannan, D.A., Clunie, J.: Aspects of Contemporary Complex Analysis. Academic, New York (1980)

4. Chakrabarti, R., Jagannathan, R.: A (p, q)-oscillator realization of two-parameter quantum algebras. J. Phys. A: Math. Gen. **24**(13), 7–11 (1991)
5. Emeka, M.A.Z.I., Altinkaya, S.: On a new subclass of bi-univalent functions satisfying subordinate conditions. Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **68**(1), 724–733 (2019)
6. Frasin, B.A.: Coefficient inequalities for certain classes of Sakaguchi type functions. Int. J. Nonlinear Sci. **10**(2), 206–211 (2010)
7. Lewin, M.: On a coefficient problem for bi-univalent functions. Proc. Am. Math. Soc. **18**(1), 63–68 (1967)
8. Netanyahu, E.: The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z|<1$. Arch. Ration. Mech. Anal. **32**(2), 100–112 (1969)
9. Owa, S., Sekine, T., Yamakawa, R.: Notes on Sakaguchi functions (Coefficient inequalities in univalent function theory and related topics) (2005)
10. Owa, S., Sekine, T., Yamakawa, R.: On Sakaguchi type functions. Appl. Math. Comput. **187**, 356–361 (2007)
11. Sakaguchi, K.: On a certain univalent mapping. J. Math. Soc. Jpn. **11**(1), 72–75 (1959)
12. Saravanan, G., Muthunagai, K.: Coefficient estimates and Fekete-Szegö inequality for a subclass of bi-univalent functions defined by symmetric Q-derivative operator by using Faber polynomial techniques. Period. Eng. Nat. Sci. **6**(1), 241–250 (2018)
13. Saravanan, G., Muthunagai, K.: Estimation of Upper Bounds for Initial Coefficients and Fekete-Szegö Inequality for a Subclass of Analytic Bi-univalent Functions. Applied Mathematics and Scientific Computing, pp. 57–65. Birkhäuser, Cham (2019)
14. Srivastava, H.M., Mishra, A.K., Gochhayat, P.: Certain subclasses of analytic and bi-univalent functions. Appl. Math. Lett. **23**(10), 1188–1192 (2010)
15. Vijayalakshmi, S.P., Sudharsan, T.V.: Coefficient estimates for bi-univalent Sakaguchi type functions. Asia Pac. J. Math. **4**(1), 32–37 (2017)
16. Xu, Q.H., Gui, Y.C., Srivastava, H.M.: Coefficient estimates for a certain subclass of analytic and bi-univalent functions. Appl. Math. Lett. **25**(6), 990–994 (2012)
17. Xu, Q.H., Xiao, H.G., Srivastava, H.M.: A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems. Appl. Math. Comput. **218**(23), 11461–11465 (2012)

Some Double Integral Formulae Associated with Q Function and Galue-Type Struve Function

Nirmal Kumar Jangid, Sunil Joshi, and Sunil Dutt Purohit

Abstract In this study, with the aid of Edward's double integral formula, we establish some double integral formula; our results are associated with Q function and Galue-type Struve function. We often examine their special cases in the form of recognized functions such as the generalized Mittag–Leffler function and the generalized Struve function. The findings of our present paper would be both useful and helpful in the study of applied science and engineering problems.

Keywords Mittag-Leffler function · Galue-type Struve function · Generalized Struve function · Q-function

1 Introduction and Preliminary

In the field of science and engineering, integral and transform formulas are very useful [see [6, 12]] as we understand that the Mittag–Leffler function and Struve function and their specific generalizations are very useful. Due to the application of the related problems, double integral formulae are very helpful. Several integral mechanisms have already been developed, but we still need our contribution to enhancing new double integral formulae connected to Q capacity and Galue-type Struve work. We also study the accompanying interesting and beneficial result characterized by Edward [4] for our present review as follows:

N. K. Jangid · S. Joshi (✉)
Department of Mathematics and Statistics, Manipal University Jaipur, Jaipur, India
e-mail: sunil.joshi@jaipur.manipal.edu

S. D. Purohit
Department of HEAS (Mathematics), Rajasthan Technical University Kota, Kota, India
e-mail: sdpurohit@rtu.ac.in

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_20

$$\int_0^1 \int_o^1 t_2^{\eta}(1-t_1)^{\eta-1}(1-t_2)^{\xi-1}(1-t_1t_2)^{1-\xi-\eta}dt_1dt_2$$
$$= \frac{\Gamma(\eta)\Gamma(\xi)}{\Gamma(\eta+\xi)}, \qquad 0 < \Re(\xi) < \Re(\eta). \tag{1}$$

Many researchers have used this double integral formula because of its large applications in the field of science and engineering and have found many substantial findings that are used to solve many relevant problems. Recently, this Edward's formula was used by Haq et al. [5] and the result for generalized Lommel–Wright function in the form of a Wright hypergeometric function is found. Ali et al. [1] also extended this formula to the generalized Bessel–Maitland function and the results obtained are very useful for solving many problems that have been applied. Kim et al. [8] further described an extension of Edward's double integral formula due to its additional applications. In this sequel we also want to develop some double integral formulae associated with Q function and Galue-type Struve function.

For our current analysis, we need descriptions of the Mittag–Leffler function, Struve function and their generalizations, which many researchers have already described as follows.

The well-known Mittag–Leffler functions $E_u(z)$ and $E_{u,v}(z)$ were introduced by Mittag–Leffler [11] and Wiman [20], respectively, which are defined as follows:

$$E_u(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(ul+1)}, \quad z \in \mathbb{C}, \Re(u) > 0 \tag{2}$$

and

$$E_{u,v}(z) = \sum_{l=0}^{\infty} \frac{z^l}{\Gamma(ul+v)}, \quad z, u, v \in \mathbb{C}, \Re(u) > 0, \Re(v) > 0. \tag{3}$$

Prabhakar [16] defined the generalization of Mittag–Leffler function (3) as follows:

$$E_{u,v}^{w}(z) = \sum_{l=0}^{\infty} \frac{(w)_l z^l}{\Gamma(ul+v)l!}, \quad z, u, v, w \in \mathbb{C}, \Re(u) > 0, \Re(v) > 0, \Re(w) = 0, \tag{4}$$

where

$$(w)_l = \begin{cases} 1, & \text{if } w \in \mathbb{C}, l = 0 \\ w(w+1)(w+2)\ldots(w+l-1), & \text{if } l \in \mathbb{N}, w \in \mathbb{C}. \end{cases} \tag{5}$$

Many researchers also developed the Mittag–Leffler function in various forms of generalization. Shukla et al. [18] and Chouhan et al. [3] established the following generalization formulae as defined in (6) and (7), respectively.

$$E_{u,v}^{w,q}(z) = \sum_{l=0}^{\infty} \frac{(w)_{ql} z^l}{\Gamma(ul+v) l!}, \tag{6}$$

$$u, v, w \in \mathbb{C}, \Re(u) > 0, \Re(v) > 0, \Re(w) > 0 \text{ and } q \in (0,1) \cup N,$$

where $(w)_{ql} = \frac{\Gamma(w+ql)}{\Gamma(w)}$ is the generalized Pochhammer symbol.

And

$$E_{u,v}^{w,t,q}(z) = \sum_{l=0}^{\infty} \frac{(w)_{ql} z^l}{\Gamma(ul+v)(f)_{ql}}, \tag{7}$$

where $u, v, w, f \in \mathbb{C}, \Re(u) > 0, \Re(v) > 0, \Re(w), \Re(f) > 0, q \in (0,1) \cup N$, and $(w)_{ql} = \frac{\Gamma(w+ql)}{\Gamma(w)}$. $(f)_{ql} = \frac{\Gamma(f+ql)}{\Gamma(f)}$ denotes the generalized Pochhammer symbols. In this sequel, Khan et al. [7] investigated the generalized Mittag–Leffler function $E_{u,v,\sigma,\rho,t,p}^{\lambda,\mu,w,q}(z)$ defined as follows:

$$E_{u,v,\sigma,\rho,f,p}^{\lambda,\mu,w,q}(z) = \sum_{l=0}^{\infty} \frac{(\lambda)_{\mu l}(w)_{ql} z^l}{\Gamma(ul+v)(f)_{pl}(\sigma)_{\rho l}}, \tag{8}$$

where $u, v, w, \sigma, \rho, \mu, f, \lambda \in \mathbb{C}$; $p, q > 0$ and $q \le \Re(u) + p$, and $min\{\Re(u) > 0, \Re(v) > 0, \Re(w), \Re(\sigma) > 0\Re(\rho) > 0, \Re(\mu) > 0, \Re(f), \Re(\lambda) > 0\}$.

Furthermore, Mazhar-ul-Haque et al. [10] investigated the further generalization of Mittag–Leffler function $Q_{u,v,f}^{w,q,r}(x)$ which is described by

$$Q_{u,v,f}^{w,q,r}(x) = Q_{u,v,f}^{w,q,r}(c_1, c_2, \ldots, c_r, d_1, d_2, \ldots, d_r, x),$$

where

$$Q_{u,v,f}^{w,q,r}(x) = \sum_{l=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(d_n, l)(w)_{ql}\, x^l}{\prod_{n=1}^{r} \beta(c_n, l)(f)_{ql}\, \Gamma(ul+v)}, \tag{9}$$

for $u, v, w, f, c, d \in \mathbb{C}, min\{\Re(u) > 0, \Re(v) > 0, \Re(w) > 0\}$ and $q \in (0,1) \cup N$, and $(w)_{ql} = \frac{\Gamma(w+ql)}{\Gamma(w)}, (f)_{ql} = \frac{\Gamma(f+ql)}{\Gamma(f)}$ denotes the generalized Pochhammer symbols.

Bhatnagar et al. [2] have recently developed the novel generalized Q function using the generalized Mittag–Leffler function described as follows:
$Q_{u,v,\sigma,\rho,f,p}^{\lambda,\mu,w,q,r}(x) = Q_{u,v,\sigma,\rho,f,p}^{\lambda,\mu,w,q,r}(c_1, c_2, \ldots, c_r, d_1, d_2, \ldots, d_r, x)$,

$$Q_{u,v,\sigma,\rho,f,p}^{\lambda,\mu,w,q,r}(x) = \sum_{l=0}^{\infty} \frac{\prod_{n=1}^{r} \beta(d_n, l)\, (\lambda)_{\mu l}(w)_{ql}\, x^l}{\prod_{n=1}^{r} \beta(c_n, l)\, (\sigma)_{\rho l}(f)_{pl}\, \Gamma(ul+v)}, \tag{10}$$

where $u, v, w, f, c, d \in \mathbb{C}, q \in (0,1) \cup N, min\{\Re(u) > 0, \Re(v) > 0, \Re(w), \Re(\sigma) > 0\Re(\rho) > 0, \Re(\mu) > 0, \Re(\lambda) > 0\}$, and $(w)_{ql} = \frac{\Gamma(w+ql)}{\Gamma(w)}$, $(f)_{pl} = \frac{\Gamma(f+pl)}{\Gamma(f)}$, $(\lambda)_{\mu l} = \frac{\Gamma(\lambda+\mu l)}{\Gamma(\lambda)}$, $(\sigma)_{\rho l} = \frac{\Gamma(\sigma+\rho l)}{\Gamma(\sigma)}$ denotes the generalized Pochhammer symbols.

We also recognize in our present study the generalized Galue-type Struve function established by Nisar et al. [12] as follows:

$$W^{\rho,\sigma}_{q,g,h,\delta}(z) = \sum_{l=0}^{\infty} \frac{(-h)^l}{\Gamma(\rho l + \sigma)\,\Gamma\left(fl + \frac{q}{\delta} + \frac{g+2}{2}\right)} \left(\frac{z}{2}\right)^{2l+q+1}, \quad f \in \mathbb{N}, q, g, h \in \mathbb{C}, \tag{11}$$

where $\rho > 0, \delta > 0$ and σ stand for arbitrary parameter.

Also Orhan et al. [13, 14] demarcated generalization of Struve function $H_{q,g,h}$ which is the special case of (11) for $\rho = f = \delta = 1, \sigma = \frac{3}{2}$ defined as follows:

$$H_{q,g,h}(z) == \sum_{l=0}^{\infty} \frac{(-h)^l}{\Gamma\left(l + \frac{3}{2}\right)\Gamma\left(l + q + \frac{g+2}{2}\right)} \left(\frac{z}{2}\right)^{2l+q+1}, \quad q, g, h \in \mathbb{C}. \tag{12}$$

In our present paper we need Hadamard product of two analytic functions, which facilitate us to split the emerged function into two eminent functions. If we let $h(t)$ and $g(t)$ to be two power series defined as
$h(t) = \sum_{l=o}^{\infty} a_l t^l$, $(|t| < R_h)$ and $g(t) = \sum_{l=o}^{\infty} b_l t^l$, $(|t| < R_g)$,
where R_h and R_g are radii of convergence, respectively, then their Hadamard product [15, 17] is defined as follows:

$$(h * g)(t) = \sum_{l=0}^{\infty} a_l b_l t^l = (g * h)(t)\ (|t| < R), \tag{13}$$

where

$$R = \lim_{l\to\infty} \left|\frac{a_l b_l}{a_{l+1} b_{l+1}}\right| = \left(\lim_{l\to\infty} \left|\frac{a_l}{a_{l+1}}\right|\right)\left(\lim_{l\to\infty} \left|\frac{b_l}{b_{l+1}}\right|\right) = R_h R_g,$$

in general $R \geq R_h R_g$.

We also recall the generalized hypergeometric function (see [19], Sect. 1.5) and generalized Wright hypergeometric function (see for details Mathai et al. ([9], pp. 23)) defined in the following Eqs. (14) and (15), respectively

$${}_pF_q\left[\begin{matrix} u_1, \ldots, u_p; \\ v_1, \ldots, v_q; \end{matrix}\, z\right] = \sum_{l=0}^{\infty} \frac{(u_1)_l, \ldots, (u_p)_l}{(v_1)_l, \ldots, (v_q)_l\ l!} z^l, \tag{14}$$

where $z, u_i, v_j \in \mathbb{C}, i = 1, 2, \ldots, p; j = 1, 2, \ldots, q$ and v_j are non-zero, non-negative integers.

And $(u)_l$ is the Pochhammer symbol defined above in (5).

$$
{}_p\psi_q \left[\begin{matrix} (u_1, U_1), \ldots, (u_p, U_p); \\ (v_1, V_1), \ldots, (v_q, V_q); \end{matrix} z \right] = \sum_{l=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(u_i + U_i l) z^l}{\prod_{j=1}^{q} \Gamma(v_j + V_j l) l!}, \tag{15}
$$

where $z, u_i, v_j \in \mathbb{C}, U_i, V_j \in R, i = 1, 2, \ldots, p; j = 1, 2, \ldots, q$.

2 Main Results

In this section we determine some double integral formulae related to Q function and Galue-type Struve work with the assistance of Edward's double integral formula. Since our outcomes appeared underneath in Theorems 1 and 2, we additionally examine about the diversity in Theorem 2.

Theorem 1 *For* $u, v, w, f, c_i, d_i \in \mathbb{C}; \Re(u) > 0, \Re(v) > 0, \Re(w) > 0, \Re(\sigma) > 0, \Re(\lambda) > 0, \Re(\rho) > 0, \Re(\mu) > 0; 0 < \Re(\xi) < \Re(\eta)$ *and* $q \in (0, 1) \cup N$, *then we have*

$$
\int_0^1 \int_o^1 t_2^{\eta} (1 - t_1)^{\eta - 1} (1 - t_2)^{\xi - 1} (1 - t_1 t_2)^{1 - \xi - \eta}
\times Q_{u,v,\sigma,\rho,f,p}^{\lambda,\mu,w,q,r} \left(\frac{k t_2 (1 - t_1)(1 - t_2)}{(1 - t_1 t_2)^2} \right) dt_1 dt_2 = B(\eta, \xi)
$$

$$
\times Q_{u,v,\sigma,\rho,f,p}^{\lambda,\mu,w,q,r} \left(\frac{k}{4} \right) * {}_3F_2 \left[\begin{matrix} u, v, 1; \\ \frac{u+v}{2}, \frac{u+v+1}{2}; \end{matrix} \frac{k}{4} \right]. \tag{16}
$$

Proof Let us assume L.H.S. to be denoted by I_1, then we have

$$
I_1 = \int_0^1 \int_o^1 t_2^{\eta} (1 - t_1)^{\eta - 1} (1 - t_2)^{\xi - 1} (1 - t_1 t_2)^{1 - \xi - \eta}
\times Q_{u,v,\sigma,\rho,f,p}^{\lambda,\mu,w,q,r} \left(\frac{k t_2 (1 - t_1)(1 - t_2)}{(1 - t_1 t_2)^2} \right) dt_1 dt_2,
$$

now using Eq. (10) in the above integral, we have

$$I_1 = \int_0^1 \int_o^1 t_2^\eta (1-t_1)^{\eta-1}(1-t_2)^{\xi-1}(1-t_1t_2)^{1-\xi-\eta}$$
$$\times \sum_{l=0}^{\infty} \frac{\prod_{n=1}^r \beta(d_n, l)\,(\lambda)_{\mu l}(w)_{ql}\, z^l}{\prod_{n=1}^r \beta(c_n, l)\,(\sigma)_{\rho l}(f)_{pl}\,\Gamma(ul+v)} \left(\frac{t_2(1-t_1)(1-t_2)}{(1-t_1t_2)^2}\right)^l dt_1dt_2.$$

By altering the order of integration and summation, and in view of Eq. (1), we have

$$I_1 = \sum_{l=0}^{\infty} \frac{\prod_{n=1}^r \beta(d_n, l)\,(\lambda)_{\mu l}(w)_{ql}\, z^l}{\prod_{n=1}^r \beta(c_n, l)\,(\sigma)_{\rho l}(f)_{pl}\,\Gamma(ul+v)} \frac{\Gamma(\eta+l)\Gamma(\xi+l)}{\Gamma(\xi+\eta+2l)}.$$

Simplifying the above equation, we have

$$I_1 = \frac{\Gamma(\eta+l)\Gamma(\xi+l)}{\Gamma(\xi+\eta+2l)} \sum_{l=0}^{\infty} \frac{\prod_{n=1}^r \beta(d_n, l)\,(\lambda)_{\mu l}(w)_{ql}\, z^l}{\prod_{n=1}^r \beta(c_n, l)\,(\sigma)_{\rho l}(f)_{pl}\,\Gamma(ul+v)2^{2l}\left(\frac{\eta+\xi}{2}\right)_l\left(\frac{\eta+\xi+1}{2}\right)_l}. \tag{17}$$

Finally, using Hadamard product (13) in (17), and making use of (10) and (14), we obtain the desired result.

Theorem 2 *For* $q, g, h \in \mathbb{C}$; $f \in \mathbb{N}$; $\rho > 0, \delta > 0$; $0 < \Re(\xi) < \Re(\eta)$; $\Re(\eta+q) > -1$, $\Re(\xi+q) > -1$, $\Re(\eta+\xi+2q) > -2$, $\Re\left(\frac{q}{\delta}+\frac{g}{2}\right) > -1$ *and* $\Re(\sigma) > 0$*, then we have*

$$\int_0^1 \int_o^1 t_2^\eta (1-t_1)^{\eta-1}(1-t_2)^{\xi-1}(1-t_1t_2)^{1-\xi-\eta}$$
$$\times\, w_{q,g,h,\delta}^{\rho,\sigma}\left(\frac{kt_2(1-t_1)(1-t_2)}{(1-t_1t_2)^2}\right)dt_1dt_2$$

$$= \left(\frac{k}{2}\right)^{q+1} {}_3\psi_3\left[\begin{matrix}(\eta+q+1,2),(\xi+q+1,2),(1,1);\\(\sigma,\rho),\left(\frac{q}{\delta}+\frac{g}{2}+1,f\right),(\eta+\xi+2q+2,4)\end{matrix};\frac{-hk^2}{4}\right]. \tag{18}$$

Proof Let us assume L.H.S. be denoted by I_2, then we have

$$I_2 = \int_0^1 \int_o^1 t_2^\eta (1-t_1)^{\eta-1}(1-t_2)^{\xi-1}(1-t_1t_2)^{1-\xi-\eta}$$
$$\times\, w_{q,g,h,\delta}^{\rho,\sigma}\left(\frac{kt_2(1-t_1)(1-t_2)}{(1-t_1t_2)^2}\right)dt_1dt_2.$$

Using Eq. (11) in the above integral, we have

$$I_2 = \int_0^1 \int_o^1 t_2^{\eta}(1-t_1)^{\eta-1}(1-t_2)^{\xi-1}(1-t_1t_2)^{1-\xi-\eta}$$
$$\times \sum_{l=0}^{\infty} \frac{(-h)^l}{\Gamma(\rho l+\sigma)\Gamma(fl+\frac{q}{\delta}+\frac{g+2}{2})}\left(\frac{k}{2}\right)^{2l+q+1}\left(\frac{kt_2(1-t_1)(1-t_2)}{(1-t_1t_2)^2}\right)^{2l+q+1} dt_1dt_2.$$

Now by altering the order of integration and summation, and in view of Eq. (1), we have

$$I_2 = \sum_{l=0}^{\infty} \frac{(-h)^l}{\Gamma(\rho l+\sigma)\Gamma(fl+\frac{q}{\delta}+\frac{g+2}{2})}\left(\frac{k}{2}\right)^{2l+q+1} \frac{\Gamma(\eta+q+1+2l)\Gamma(\xi+q+1+2l)}{\Gamma(\eta+\xi+2q+2+4l)}.$$

Simplifying the above equation, we have

$$I_2 = \left(\frac{k}{2}\right)^{q+1} \sum_{l=0}^{\infty} \frac{\Gamma(\eta+q+1+2l)\Gamma(\xi+q+1+2l)}{\Gamma(\rho l+\sigma)\Gamma(fl+\frac{q}{\delta}+\frac{g+2}{2})\Gamma(\eta+\xi+2q+2+4l)}\left(\frac{-hk^2}{4}\right)^l. \tag{19}$$

Finally using Eq. (15) in (19) and further simplification we obtain the desired result.

Variation of (18): If the conditions of above Theorem 2 to be fulfilled, then we can find the variation of Eq. (18) in the following integral formula which holds true

$$\int_0^1 \int_o^1 t_2^{\eta}(1-t_1)^{\eta-1}(1-t_2)^{\xi-1}(1-t_1t_2)^{1-\xi-\eta}$$

$$\times\, w_{q,g,h,\delta}^{\rho,\sigma}\left(\frac{kt_2(1-t_1)(1-t_2)}{(1-t_1t_2)^2}\right)dt_1dt_2 = \left(\frac{k}{2}\right)^{q+1} \frac{B(\eta+q+1,\xi+q+1)}{\Gamma(\sigma)\Gamma(\frac{q}{\delta}+\frac{g+2}{2})}$$

$$\times\, {}_5F_{\rho+f+4}\left[\begin{matrix}\Delta(2,\eta+q+1),\Delta(2,\xi+q+1),1; \\ \Delta(\rho,\sigma),\Delta\Gamma(f+\frac{q}{\delta}+\frac{g+2}{2}),\Delta(4,\eta+\xi+2q+2);\end{matrix}\ \frac{-hk^2}{64\rho^{\rho}f^{f}}\right]. \tag{20}$$

Proof For the proof of result (20), we use the following results:

$$\Gamma(\eta+l) = \Gamma(\eta)(\eta)_l,$$

and Gauss multiplication theorem $(\zeta)_{kl} = k^{kl}\left(\frac{\zeta}{k}\right)_l\left(\frac{\zeta+1}{k}\right)_l \cdots \left(\frac{\zeta+k-1}{k}\right)_l$, in (19) and making use of Eq. (14), we get our required result (20).

3 Special Cases

In this segment we define some special cases by substituting particular values as if we put $r = 0$ in Eq. (16), then we have our result in view of Hadamard product of generalized Mittag–Leffler function (8) defined by Khan et al. [7] with hypergeometric function.

Corollary 1 *For* $u, v, w, f, c_i, d_i \in \mathbb{C}; \Re(u) > 0, \Re(v) > 0, \Re(w) > 0, \Re(\sigma) > 0, \Re(\lambda) > 0, \Re(\rho) > 0, \Re(\mu) > 0; 0 < \Re(\xi) < \Re(\eta)$ *and* $q \in (0, 1) \cup N,$ *then we have*

$$\int_0^1 \int_o^1 t_2^{\eta}(1-t_1)^{\eta-1}(1-t_2)^{\xi-1}(1-t_1t_2)^{1-\xi-\eta} \times E^{\lambda,\mu,w,q}_{u,v,\sigma,\rho,f,p}\left(\frac{kt_2(1-t_1)(1-t_2)}{(1-t_1t_2)^2}\right)dt_1dt_2$$

$$= B(\eta,\xi).E^{\lambda,\mu,w,q}_{u,v,\sigma,\rho,f,p}\left(\frac{k}{4}\right) * {}_3F_2\left[\begin{matrix} u, v, 1; \\ \frac{u+v}{2}, \frac{u+v+1}{2}; \end{matrix} \frac{k}{4}\right]. \tag{21}$$

Corollary 2 *Let the condition of Theorem 1 be satisfied and for* $\lambda = \mu = \sigma = \rho = p = 1$*, then we have our result in view of Hadamard product of generalized Mittag–Leffler function defined by Mazhar-ul-Haque et al. [10] with hypergeometric function as follows:*

$$\int_0^1 \int_o^1 t_2^{\eta}(1-t_1)^{\eta-1}(1-t_2)^{\xi-1}(1-t_1t_2)^{1-\xi-\eta}$$

$$\times Q^{w,q,r}_{u,v,f}\left(\frac{kt_2(1-t_1)(1-t_2)}{(1-t_1t_2)^2}\right)dt_1dt_2 = B(\eta,\xi).Q^{w,q,r}_{u,v,f}\left(\frac{k}{4}\right) * {}_3F_2\left[\begin{matrix} u, v, 1; \\ \frac{u+v}{2}, \frac{u+v+1}{2}; \end{matrix} \frac{k}{4}\right]. \tag{22}$$

Corollary 3 *Let the condition of Theorem 1 be satisfied and if* $\lambda = \mu = \sigma = \rho = p = 1, r = 0$*, then we have our result in view of Hadamard product of generalized Mittag–Leffler function defined by Chouhan et al. [3] with hypergeometric function as follows:*

$$\int_0^1 \int_o^1 t_2^{\eta}(1-t_1)^{\eta-1}(1-t_2)^{\xi-1}(1-t_1t_2)^{1-\xi-\eta}$$

$$\times E^{w,f,q}_{u,v}\left(\frac{kt_2(1-t_1)(1-t_2)}{(1-t_1t_2)^2}\right)dt_1dt_2 = B(\eta,\xi).E^{w,f,q}_{u,v}\left(\frac{k}{4}\right) * {}_3F_2\left[\begin{matrix} u, v, 1; \\ \frac{u+v}{2}, \frac{u+v+1}{2}; \end{matrix} \frac{k}{4}\right]. \tag{23}$$

Corollary 4 *If we put $f = 1$, in the above Eq. (23), then we have our result in view of Hadamard product of generalized Mittag–Leffler function investigated by Shukla et al. [18] with hypergeometric function as follows:*

$$\int_0^1 \int_o^1 t_2^{\eta}(1-t_1)^{\eta-1}(1-t_2)^{\xi-1}(1-t_1t_2)^{1-\xi-\eta} E_{u,v}^{w,q}\left(\frac{kt_2(1-t_1)(1-t_2)}{(1-t_1t_2)^2}\right)dt_1dt_2$$

$$= B(\eta,\xi).E_{u,v}^{w,q}\left(\frac{k}{4}\right) * {}_3F_2\left[\begin{matrix} u, v, 1; \\ \frac{u+v}{2}, \frac{u+v+1}{2}; \end{matrix} \frac{k}{4}\right]. \tag{24}$$

Corollary 5 *If setting $q = 1$ in the above Eq. (24), then we have our result in view of Hadamard product of generalized Mittag–Leffler function defined by Prabhakar [16] with hypergeometric function as follows:*

$$\int_0^1 \int_o^1 t_2^{\eta}(1-t_1)^{\eta-1}(1-t_2)^{\xi-1}(1-t_1t_2)^{1-\xi-\eta} E_{u,v}^{w}\left(\frac{kt_2(1-t_1)(1-t_2)}{(1-t_1t_2)^2}\right)dt_1dt_2$$

$$= B(\eta,\xi).E_{u,v}^{w}\left(\frac{k}{4}\right) * {}_3F_2\left[\begin{matrix} u, v, 1; \\ \frac{u+v}{2}, \frac{u+v+1}{2}; \end{matrix} \frac{k}{4}\right]. \tag{25}$$

Similarly, if we put $w = 1$ and $v = 1$, respectively, in Eq. (25), then we have our result in view of Hadamard product of generalized Mittag–Leffler function defined by Wiman [20] and Mittag–Leffler function established by Mittag–Leffler [11] with hypergeometric function, respectively.

Corollary 6 *If we put $\rho = f = 1, \sigma = \frac{3}{2}$ and $\delta = 1$ in Theorem 2 under the assumption of given conditions then it will reduce to the following form:*

$$\int_0^1 \int_o^1 t_2^{\eta}(1-t_1)^{\eta-1}(1-t_2)^{\xi-1}(1-t_1t_2)^{1-\xi-\eta} \times H_{q,g,h}\left(\frac{kt_2(1-t_1)(1-t_2)}{(1-t_1t_2)^2}\right)dt_1dt_2$$

$$= \left(\frac{k}{2}\right)^{q+1} {}_3\psi_3\left[\begin{matrix} (\eta+q+1,2), (\xi+q+1,2), (1,1); \\ (\frac{3}{2},1), \left(\frac{2q+g+2}{2},1\right), (\eta+\xi+2q+2,4); \end{matrix} \frac{-hk^2}{4}\right]. \tag{26}$$

Corollary 7 *Let the restrictions of Theorem 2 be fulfilled and for $\rho = f = 1, \sigma = \frac{3}{2}$ and $\delta = 1$ then Eq. (25) will reduce to the following form:*

$$\int_0^1 \int_o^1 t_2^{\eta}(1-t_1)^{\eta-1}(1-t_2)^{\xi-1}(1-t_1t_2)^{1-\xi-\eta}$$

$$\times H_{q,g,h}\left(\frac{kt_2(1-t_1)(1-t_2)}{(1-t_1t_2)^2}\right)dt_1dt_2 = \left(\frac{k}{2}\right)^{q+1}\frac{2B(\eta+q+1,\xi+q+1)}{(\phi)^{\frac{1}{2}}\Gamma(\frac{2q+g+2}{2})}$$

$$\times\ {}_5F_6\left[\begin{array}{c}\Delta(2,\eta+q+1),\ \Delta(2,\xi+q+1),\ 1;\\ \Delta(1,\frac{3}{2}),\ \Delta\Gamma\left(1,\ \frac{2q+g+2}{2}\right),\ \Delta(4,\eta+\xi+2q+2);\end{array}\ \frac{-hk^2}{64}\right]. \tag{27}$$

4 Conclusions

In the present paper we obtained double integral formulae that are associated with Q function and Galue-type Struve function. Further, we frequently examined their special cases in the form of recognized functions such as the generalized Mittag–Leffler function and the generalized Struve function. The findings of our present paper would be both useful and helpful in the study of applied science and engineering problems.

Acknowledgements The authors would like to thank the editor and referees for their valuable comments and recommendations on this paper.

References

1. Ali, M., Khan, W.A., Khan, I.A.: Study on double integral operator associated with generalized Bessel-Maitland function. Palest. J. Math. **9**(2), 991–998 (2020)
2. Bhatnagar, D., Pandey, R.M.: A study of some integral transforms on Q function. South East Asian J. Math. Math. Sci. **16**(1), 99–110 (2020)
3. Chouhan, A., Saraswat, S.: Some remarks on generalized Mittag-Leffler function and fractional operators. Adv. Appl. Math. Anal. **6**(2), 131–139 (2011)
4. Edward, J.: A Treatise on the Integral Calculus, vol. II. Chelsea Publication Company, New York (1922)
5. Haq, S., Khan, A.H., Nisar, K.S.: Certain unified double integrals associated with the generalized Lommel-Wright function. Palest. J. Math. **9**(1), 420–426 (2020)
6. Haubold, H.J., Mathai, A.M., Saxena, R.K.: Mittag-Leffler functions and their applications. J. Appl. Math. (2011). https://doi.org/10.1155/2011/298628
7. Khan, M.A., Ahmed, S.: On some properties of the generalized Mittag-Leffler function. Springerplus **337**(2) (2013)
8. Kim, I., Jun, S., Vyas, Y., Rathie, A.K.: On an extension of Edward's double integral with applications. Aust. J. Math. Anal. Appl. **16**(2), 1–13 (2019)
9. Mathai, A.M., Saxena, R.K., Haubold, H.J.: The H-Function: Theory and Applications. Springer Science, New York (2010)

10. Mazhar-ul-Haque, M., Holambe, T.L.: A Q function in fractional calculus. J. Basic Appl. Res. Int. International Knowledge Press **6**(4), 248–252 (2015)
11. Mittag-Leffler, G.: Sur laNouvelle Fonction E(x). Comptes Rendus de 1A-cademie des Sciences Paris **137**, 554–558 (1903)
12. Nisar, K.S., Baleanu, D., Qurashi, M.M.A.: Fractional calculus and application of generalized Struve function. Springerplus (2016). https://doi.org/10.1186/s40064-016-2560-3
13. Orhan, H., Yagmur, N.: Starlike and convexity of generalized Struve function. Abstr. Appl. Anal. (2013) Art. ID 954513:6
14. Orhan, H., Yagmur, N.: Geometric properties of generalized Struve functions. Analele stiintifice ale universitatii Al1Cuza din lasi- Matematica (2014). https://doi.org/10.2478/aicu-2014-0007
15. Pohlen, T.: The Hadamard product and universal power series. Ph.D. thesis, Universitat Trier, Trier, Germany (2009)
16. Prabhakar, T.R.: A singular integral equation with a generalize Mittag-Leffler function in the Kernel. Yokohama Math. J. **19**, 7–15 (1971)
17. Saxena, R.K., Parmar, R.K.: Fractional integration and differentiation of the generalized Mathieu series. Axioms, MDPI (2017)
18. Shukla, A.K., Prajapati, J.C.: On a generalization of Mittag-Leffler function and its properties. J. Math. Anal. Appl. **2**, 797–811 (2007)
19. Srivastava, H.M., Choi, J.: Zeta and q-Zeta Functions and Associated Series and Integrals. Elsevier Science Publishers, Amsterdam (2012)
20. Wiman, A.: Uber den Fundamental Satz in der Theorie der Funktionen E(x). Acta Math. **29**, 191–201 (1905)

Time-Dependent Analytical and Computational Study of an M/M/1 Queue with Disaster Failure and Multiple Working Vacations

Madhu Jain, Mayank Singh, and Rakesh Kumar Meena

Abstract An M/M/1 working vacation (WV) queueing model with disaster failure is considered to examine time-dependent behavior. When the system is in busy mode, it can fail such that all the customers in the system are flushed out and never returns; such type of failure is known as disaster failure. The server is allowed to go for a WV after each busy period for a random duration of time. In the duration of WV, the server reduces the service rate rather than halting the service. After completing the vacation period, the server can take any number of vacation until he found some customers waiting in the queue; this vacation policy is known as multiple vacation policy. The transient analytical formulae for the queue size distributions are formulated by solving Chapman–Kolmogorov equations using continued fractions, modified Bessel function and probability generating function methods. Moreover, various queueing performance measures are given, and real-time performance is evaluated by computing the performance measures numerically.

Keywords Transient queue · System disaster · Working vacation · Repair · Continued fraction · Modified Bessel function

1 Introduction

In some instances, servers are always accessible in the case of a classical queueing model. However, for some time, the server may be inaccessible in real life due to various reasons. This server's absence time might indicate the server's work on some additional jobs, being examined for maintenance, or merely taking breaks. We refer

M. Jain · M. Singh (✉)
Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee 247667, India
e-mail: mayank@ma.iitr.ac.in

M. Jain
e-mail: madhufma@iitr.ac.in

R. K. Meena
Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi, India
e-mail: meena.rk@bhu.ac.in

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_21

the readers to [5, 23, 24] in order to understand the background and applications of vacationing server queueing systems. Also, the concept of vacation plays a vital role in machining environment (cf. [7, 11–13, 15]). To examine such systems and signify the period of temporary absence of the server, the concept of server vacation in queueing models has been introduced. Vacation occurs when a server is not available to serve the next customer or has gone for some recreation or leisure activities. Permitting servers to take vacations makes queueing systems more accurate and adaptable in the world of waiting lines. There may be a loss of profit in the case when jobs accumulate during the vacation period. To avoid such loss of profit, the server will provide services to the customers at a different pace. The server provides service at a slower rate while on vacation instead of completely stopping the service, and it will be called on to working vacation (WV), see [17].

Since the inclusion of WV in queueing models makes the model more economical and closer to real-life situations, many queueing theorists included the WV scenarios in their queueing models. Tian et al. [25] studied a single-server Markov queueing model with single WV and framed the queue length, busy period, etc. in various formulae. Vijayashree and Janani [26] presented an analytical study on the multi-server Markov queueing model with WV and used matrix geometric method to formulate various queueing probability distributions. Kannadasan and Sathiyamoorth [10] fuzzified the parameters in an M/M/1 WV queue and obtained queue length distribution under the steady-state condition. Recently, Ameur et al. [1] established some explicit results for a Markovian retrial queue with WV and vacation interruption.

Losing all the jobs due to a single fault is known as disaster failure, and can be commonly seen in various queueing systems. For example, in computer networking, all the commands are flushed out as soon as a single fault occurs in the server computer; in the telecommunication systems, the entire call request can be dropped due to sudden network fluctuations and so on. Many queueing theorists included disaster failure in various queueing situations. Chen and Renshaw [4] studied an M/M/1 queue with disaster failure and obtained factorial moments and various system state probabilities. Jain and Sigman [6] obtained Pollaczek–Khintchine formulae for an M/G/1 disaster queue. Bocharov et al. [3]; Kim et al. [14] and Shin [18] included the concept of disaster in multi-server queueing systems. The arriving customers may get discouraged upon arrival on finding the failed server and can decide not to queue in the system. The behavior of discouragement of the arriving customers shows a significant impact on the operation of the queueing system (cf. [16, 19]). Recently, Jain and Singh [8] included feedback, balking, and reneging in an M/M/1 disaster queue and formulated system size distribution analytically. They also used these explicit formulae to construct a performance matrix for the model.

The time-dependent analysis of the queueing model is a tedious task as the governing differential-difference equations are highly nonlinear in nature. However, there are some methods to deal with this situation; one is the continued fraction technique. It is a powerful tool to handle the complexity of the nonlinear differential-difference equations and can be used to obtain a closed-form solution for the transient queueing model. This technique is applicable when we found the governing differential-difference equation is in three-term recurrence relation. Sudhesh [19] implemented

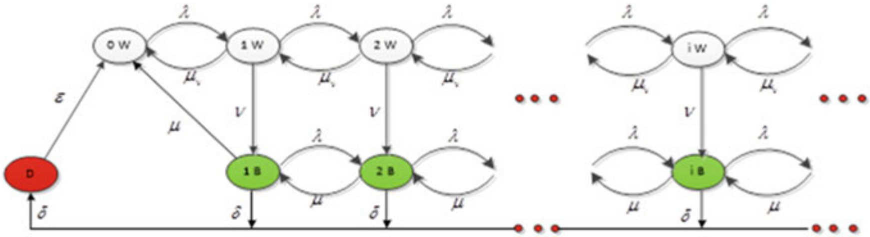


Fig. 1 State-transition diagram of M/M/1 WV queueing model with disaster

the continued fraction technique for an M/M/1 disaster queue to obtain probability distribution for a number of jobs in the system. Many researchers, including Sudhesh and Raj [20], Ammar [2], Sudhesh et al. [21], Suranga Sampath and Liu [22], and Jain et al. [9] used the technique of continued fraction on various queueing models to solve Chapman–Kolmogorov differential-difference equation.

In this paper, a single-server Markov queueing model is considered with disaster failure and WV to formulate the time-dependent analytical results. In Sect. 2, the model is illustrated by making some assumptions, and a state transition diagram (Fig. 1) is given to understand the birth–death transition of the system to various states. The governing equations are formulated in Sect. 3. Transient analytical solutions are obtained in Sect. 4. Section 5 is devoted to formulate the various performance measures for the transient queueing model, such as mean system size, throughput of the system, and various system state probabilities. In Sect. 6, a numerical solution is presented to look at the performance of the queueing model. Finally, in Sect. 7, the conclusion of the present study is given along with the future scope of the model.

2 Model Description

Consider an M/M/1 WV queue with disaster failure. The jobs arrive into the system according to a Poisson distribution with rate λ and the service is rendered by a single server according to first-come first-serve (FCFS) discipline with an exponentially distributed (Exp-D) time, having the rate μ. The server can go for a vacation of random duration, after completing each busy period. The length of vacation follows Exp-D with rate v. If no customers are waiting after the termination of vacation, the server avails another vacation. Moreover, the system can suffer disaster breakdown during the busy period, and consequently, all the customers are lost. As soon as the system fails, a dedicated repairman is assigned to repair the system. The occurrence of disaster and repair time both are supposed to follow Exp-D with rates δ and β, respectively. Let $\{N(t), t \geq 0\}$ and $Y(t)$ denote the number of customers and status of the server at time t, respectively. Here

$\chi(t) = \{N(t), Y(t); t \geq 0\}$ is a two-dimensional continuous-time Markov chain, with state-space $\Omega = \{(i, j), i = 0, 1, 2, \ldots; j = W, B\} \cup \{j = D\}$.

$$Y(t) = \begin{cases} D: \text{ When system is in downstate,} \\ W: \text{ When system is in working vacation state,} \\ B: \text{ When system is in busy state.} \end{cases}$$

Then various system state probabilities for $i \geq 0$ associated with down state, working state and WV state are denoted and defined as follows:
(a) The probability of system in down state is denoted by

$$Q_D(t) = \text{Prob}[Y(t) = D].$$

(b) The probability of system in busy state is denoted by

$$Q_{i,B}(t) = \text{Prob}[N(t) = i, Y(t) = B,] \text{ for } i \geq 0.$$

(c) The probability of system in working vacation state is denoted by:

$$Q_{i,W}(t) = \text{Prob}[N(t) = i, Y(t) = W] \text{ for } i \geq 0.$$

Denote Laplace transform of $Q_D(t)$, $Q_{i,W}(t)$ and $Q_{i,B}(t)$ by $Q^*_D(s)$, $Q^*_{i,W}(s)$ and $Q^*_{i,B}(s)$; $i \geq 0$, respectively.

3 Governing Equations

The balance equations for the queueing model are formulated by following the birth–death rule as follows:
(i) The governing equation for $(Y(t) = D)$ down state:

$$\frac{d}{dt} Q_D(t) = -\varepsilon Q_D(t) + \delta \sum_{i=1}^{\infty} Q_{i,B}(t) \tag{1}$$

(ii) The governing equations for $(Y(t) = B)$ busy state:

$$\frac{d}{dt} Q_{1,B}(t) = -(\lambda + \mu + \delta) Q_{1,B}(t) + \mu Q_{2,B}(t) + v Q_{1,W}(t) \tag{2}$$

$$\frac{d}{dt} Q_{i,B}(t) = -(\lambda + \mu + \delta) Q_{i,B}(t) + \lambda Q_{i-1,B}(t) + \mu Q_{i+1,B}(t) + v Q_{i,W}(t), i \geq 2 \tag{3}$$

(iii) The governing equations for $(Y(t) = W)$ working state:

$$\frac{d}{dt} Q_{0,W}(t) = -\lambda Q_{0,W}(t) + \mu_v Q_{1,W}(t) + \mu Q_{1,B}(t) + \varepsilon Q_D(t) \tag{4}$$

$$\frac{d}{dt} Q_{i,W}(t) = -(\lambda + \mu_v + v)\, Q_{i,W}(t) + \lambda Q_{i-1,W}(t) + \mu_v Q_{i+1,W}(t),\, i \geq 1 \quad (5)$$

with initial condition $Q_{0,W}(0) = 1$.

3.1 Transient Analysis

In this section, the equations obtained in Sect. 3 are solved using continued fractions and probability generation functions to obtain transient solutions for the queueing model.

Evaluation of $Q_D(t)$: Laplace transforms and some algebra on (1) yield

$$Q_D^*(s) = \frac{\delta}{s+\varepsilon} \sum_{i=1}^{\infty} Q_{i,B}^*(s) \quad (6)$$

Taking inverse Laplace transform of (6)

$$Q_D(t) = \delta e^{-\varepsilon t} * \sum_{i=1}^{\infty} Q_{i,B}(t) \quad (7)$$

Clearly $Q_D(t)$ is expressed in terms of $Q_{i,B}(t)$.

Evaluation of $Q_{i,W}(t)$: Laplace transform and some algebraic on (5) yield

$$\frac{Q_{i,W}^*(s)}{Q_{i-1,W}^*(s)} = \frac{\lambda}{(s+\lambda+\mu_v+\nu) - \mu_v \frac{Q_{i+1,W}^*(s)}{Q_{i,W}^*(s)}} \quad (8)$$

which gives

$$Q_{i,W}^*(s) = \beta_v^i [\phi(s)]^i\, Q_{0,W}^*(s) \quad (9)$$

where $\phi(s) = \left(\frac{p_v - \sqrt{p_v^2 - \alpha_v^2}}{\alpha_v} \right)^i$, $p_v = s + \lambda + \mu_v + v,\ \alpha_v = 2\sqrt{\mu_v \lambda}$ and $\beta_v = \sqrt{\frac{\lambda}{\mu_v}}$.

Inverse Laplace transform of (9) gives

$$Q_{i,W}(t) = \lambda \beta^{i-1} \left[e^{-(\lambda+\mu_v+v)t} \left\{ I_{i-1}(\alpha t) - I_{i+1}(\alpha t) \right\} \right]^{*_i} * Q_{0,W}(t) \quad (10)$$

where '$*$' and '$i*$', respectively, stand for convolution and i-fold convolution and $I_i(\alpha t)$ denotes the modified Bessel's function of first kind (MBF-I) of order i.

Clearly $Q_{i,W}(t)$ is expressed in terms of $Q_{0,W}(t)$.

Evaluation of $Q_{i,B}(t)$: Consider $z \in C(C$ is the set of complex numbers) such that $|z| \le 1$, Define

$$Q(z,t) = \sum_{i=1}^{\infty} Q_{i,B}(t) z^i, \;\; Q(z,0) = 0$$

Multiplying (2) by z and (3) by z^i, after some algebraic manipulations we get

$$\frac{\partial}{\partial t} Q(z,t) + \left[\delta + (1-z)\lambda + \left(1 - z^{-1}\right)\mu\right] Q(z,t) = \mu Q_{1,B}(t) + v \sum_{i=1}^{\infty} Q_{i,W}(t) z^i \tag{11}$$

Solving (11), we get

$$\begin{aligned} Q(z,t) =& v \int_0^t \sum_{m=1}^{\infty} z^m Q_{m,W}(y) \times e^{-(\lambda+\mu+\delta)(t-y)} e^{\left(\lambda z + \mu z^{-1}\right)(t-y)} dy \\ & - \mu \int_0^t Q_{1,B}(y) \times e^{-(\lambda+\mu+\delta)(t-y)} e^{\left(\lambda z + \mu z^{-1}\right)(t-y)} dy \end{aligned} \tag{12}$$

Let $I_i(t)$ be MBF-I of order i. It is well known that

$$e^{\left(\lambda + \mu z^{-1}\right)(t-y)} = \sum_{i=-\infty}^{\infty} (\beta z) I_i[\alpha(t-y)] \tag{13}$$

where $\alpha = 2\sqrt{\lambda\mu}$ and $\beta = \sqrt{\frac{\lambda}{\mu}}$.

For $i = 0, 1, 2, \ldots$, comparing the coefficient of z^i on both sides of Eq. (12) using (13), we get

$$\begin{aligned} Q_{i,B}(t) =& v \int_0^t \sum_{m=1}^{\infty} Q_{m,W}(y) \times \beta^{i-m} I_{i-m}(\alpha(t-y)) e^{-(\lambda+\mu+\delta)(t-y)} dy \\ & - \mu \int_0^t Q_{1,B}(y) \times \beta^i I_i(\alpha(t-y)) e^{-(\lambda+\mu+\delta)(t-y)} dy \end{aligned} \tag{14}$$

For negative values of i, i.e. $i = -1, -2, -3, \ldots$, Eq. (14) yields

$$\begin{aligned} 0 =& v \int_0^t \sum_{m=1}^{\infty} Q_{m,W}(y) \times \beta^{i-m} I_{i+m}(\alpha(t-y)) e^{-(\lambda+\mu+\delta)(t-y)} dy \\ & - \mu \int_0^t Q_{1,B}(y) \times \beta^i I_i(\alpha(t-y)) e^{-(\lambda+\mu+\delta)(t-y)} dy \end{aligned} \tag{15}$$

Subtracting (15) from (14), we get

$$Q_{i,B}(t) = v\int_0^t \sum_{m=1}^{\infty} Q_{m,W}(y) \times \beta^{i-m}\left[I_{i-m}(\alpha(t-y)) - I_{i+m}(\alpha(t-y))\right] e^{-(\lambda+\mu+\delta)(t-y)} dy \quad (16)$$

Thus $Q_{i,B}(t)$ is expressed in terms of $Q_{i,W}(t)$.

Evaluation of $Q_{0,W}(t)$: Laplace transform of (4) gives

$$(s+\lambda)Q^*_{0,W}(s) = 1 + \mu_v Q^*_{1,W}(s) + \mu Q^*_{1,B}(s) + \varepsilon Q^*_D(s) \quad (17)$$

Taking Laplace transform of (16) at $i = 1$ and using (9), we get

$$Q^*_{1,B}(s) = 2v\sum_{m=1}^{\infty} \beta^{1-m}\beta_v^m[\phi(s)]^m Q^*_{0,J}(s)\frac{\alpha^{m-1}}{\left(p+\sqrt{p^2-\alpha^2}\right)^m} \quad (18)$$

where $p = s + \lambda + \mu + \delta$.

Again taking Laplace transform of (16) and using (9), we get

$$Q^*_{1,B}(s) = v\sum_{m=1}^{\infty} \beta^{i-m}\beta_v^m[\phi(s)]^m Q^*_{0,T}(s)\left[\frac{\left(p+\sqrt{p^2-\alpha^2}\right)^{i-m}}{\alpha^{i-m}} - \frac{\left(p+\sqrt{p^2-\alpha^2}\right)^{i+m}}{\alpha^{i+m}}\right] \quad (19)$$

Using (19) in (6)

$$Q^*_D(s) = \frac{\delta v}{s+\varepsilon}\sum_{m=1}^{\infty} \beta^{i-m}\beta_v^m[\phi(s)]^m Q^*_{0,W}(s)\left[\frac{\left(p+\sqrt{p^2-\alpha^2}\right)^{i-m}}{\alpha^{i-m}} - \frac{\left(p+\sqrt{p^2-\alpha^2}\right)^{i+m}}{\alpha^{i+m}}\right] \quad (20)$$

Using (9) for $i = 1$, (18) and (20) in (17), and some algebra yields

$$Q^*_{0,W}(s) = \sum_{n=0}^{\infty}\frac{1}{(s+\lambda)^{n+1}}\left[\mu_v\beta_v\phi(s) + 2\mu v\sum_{m=1}^{\infty}\beta^{1-m}\beta_v^m\{\phi(s)\}^m\frac{\alpha^{m-1}}{\left(p+\sqrt{p^2-\alpha^2}\right)^m}\right.$$
$$\left. + \frac{\varepsilon\delta v}{s+\varepsilon}\sum_{i=1}^{\infty}\sum_{m=1}^{\infty}\frac{\beta^{1-m}\beta_v^m}{\sqrt{p^2-\alpha^2}}\{\phi(s)\}^m\left\{\frac{\left(p+\sqrt{p^2-\alpha^2}\right)^{i-m}}{\alpha^{i-m}} - \frac{\left(p+\sqrt{p^2-\alpha^2}\right)^{i+m}}{\alpha^{i+m}}\right\}\right]^n \quad (21)$$

On inversion, (21) gives

$$Q_{0,W}(t) = \sum_{n=0}^{\infty} e^{-\lambda t}\frac{t^n}{n!}\left[\mu_v\beta_v\phi(t) + 2\mu v\sum_{m=1}^{\infty}\beta^{1-m}\beta_v^m\{\phi(t)\}^{*m}\{I_{m-1}(\alpha t) - I_{m+1}(\alpha t)\}e^{-(\lambda+\mu+\delta)t}\right.$$
$$\left. + \varepsilon\delta v e^{-\varepsilon t} * \sum_{i=1}^{\infty}\sum_{m=1}^{\infty}\beta^{1-m}\beta_v^m\{\phi(s)\}^{*m} * \{I_{i-m}(\alpha t) - I_{i+m}(\alpha t)\}e^{-(\lambda+\mu+\delta)t}\right]^{*n} \quad (22)$$

4 System Performance Measures

In this section, we used the transient formulae obtained in Sect. 4 to formulate the performance measures for the queueing model. Mean system size, system's throughput and various system state probabilities are established as follows:
(i) Let $L_S(t) = E\{N(t)\}$ be the mean system size at time t. Then

$$L_S(t) = E\{N(t)\} = \left[\sum_{i=1}^{\infty} i \left\{Q_{i,B}(t) + Q_{i,W}(t)\right\}\right] \tag{23}$$

where $Q_{i,W}(t)$ and $Q_{i,B}(t)$ are given in Eqs. (10) and (16), respectively.
(ii) If $TH(t)$ is the throughput (effective service rate) of the system at time t, then

$$TH(t) = \sum_{m=0}^{\infty} \mu Q_{i,B}(t) \tag{24}$$

where $Q_{i,B}(t)$ is given by Eqs. (16).
(iii) Let $PB(t)$, $PW(t)$ and $PD(t)$ be the transient probabilities when system is in busy, WV and in down states, respectively. Then

$$PB(t) = \sum_{i=0}^{\infty} Q_{i,B}(t),\ \ PW(t) = \sum_{i=0}^{\infty} Q_{i,D}(t) \text{ and } PD(t) = \sum_{i=0}^{\infty} Q_D(t) \tag{25}$$

where $Q_{i,B}(t)$, $Q_{i,W}(t)$ and $Q_D(t)$ are given by Eqs. (16), (10) and (7), respectively.

5 Numerical Simulation

In this section, numerical simulation results are given in the form of various figures and tables. MATLAB's ode45 function, i.e., Runge–Kutta 4th-order method is used for calculating numerical results. The computation default parameters are set as $\lambda = 2.4$, $\mu = 2, 8$, $\mu_v = 2.6$, $v = 1$, $\delta = 0.05 and \beta = 1$. In Fig. 2(i), the mean system size $L_S(t)$ is plotted against time t for various values of λ. It is noticed that initially $L_S(t)$ grows sharply and after some time settles down to a specific value. Also, it is evident that the higher the value of λ, the larger the system size. Form Fig. 2 (ii), it is clearly observable that if the server serves the customers with a faster rate, the mean system size $L_S(t)$ will decrease. In Fig. 3 (i)–(ii), the transient throughput of the system $TH(t)$ is plotted for varying values of μ and μ_v. From both of the figures we see that $TH(t)$ initially grows rapidly and after some time obtains equilibrium state and shows steady-state behavior; also $TH(t)$ goes up with both the parameters μ and μ_v.

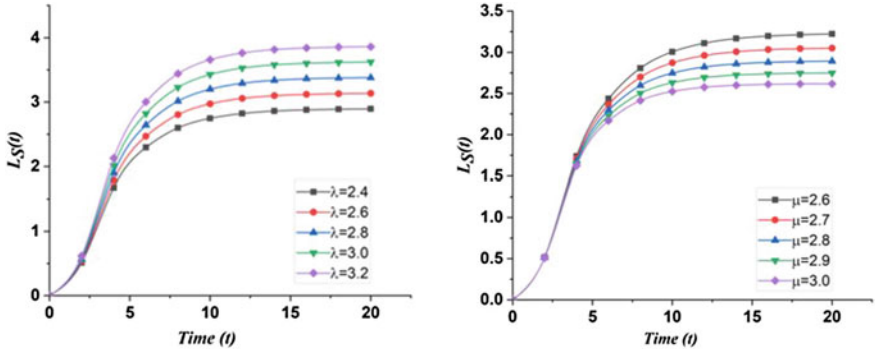


Fig. 2 $L_s(t)$ versus t for various values of (i) λ (ii) μ

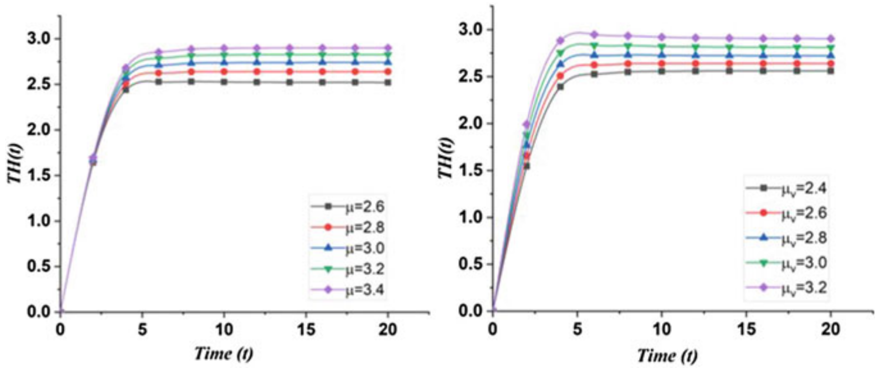


Fig. 3 $TH(t)$ versus t for various values of (i) μ (ii) μ_v

In Tables 1, 2 and 3, it is noticeable that $L_S(t)$, $TH(t)$, $PB(t)$ increase with respect to time while $PD(t)$ decreases, but $PW(t)$ starts from a lower value and then attains a peak and finally starts decreasing continuously. Moreover, all the tabulated system indices except $PD(t)$ are decreasing but as disaster rate (δ) increases (Table 1). Entirely reverse relation of these system indices with respect to repair rate (ε) is seen from Table 2; i.e. $PD(t)$ is decreasing but rest of the other tabulated system indices are increasing as repair rate (ε) goes high. Also, $L_S(t)$, $PB(t)$ and $PD(t)$ are decreasing but $TH(t)$ and $PW(t)$ increase with respect to working vacation rate μ_v (Table 3).

Table 1 Various system performance measures for varying values of δ with respect to time

δ	Time (t)	$L_S(t)$	$TH(t)$	$PB(t)$	$PW(t)$	$PD(t)$
0.05	3	0.51375	1.65609	0.07607	0.55501	0.36889
	5	1.67699	2.50884	0.3371	0.60191	0.06099
	7	2.29466	2.62342	0.46653	0.50659	0.02688
0.1	3	0.51163	1.65331	0.0751	0.55501	0.36989
	5	1.62949	2.47886	0.3232	0.60535	0.07145
	7	2.16651	2.5708	0.43554	0.51973	0.04473
0.2	3	0.50753	1.64792	0.07322	0.55496	0.37182
	5	1.54413	2.42459	0.29805	0.61156	0.09039
	7	1.95365	2.48213	0.38323	0.54196	0.07481

Table 2 Various system performance measures for varying values of ε with respect to time

ε	Time (t)	$L_S(t)$	$TH(t)$	$PB(t)$	$PW(t)$	$PD(t)$
0.5	3	0.30203	1.030121	0.043692	0.349147	0.607161
	5	1.244142	2.041886	0.245008	0.521486	0.233506
	7	1.939998	2.3997	0.39253	0.500237	0.107233
1.0	3	0.513752	1.65609	0.076071	0.555035	0.368894
	5	1.67699	2.508842	0.337098	0.601911	0.060991
	7	2.294662	2.623415	0.466533	0.506586	0.026881
1.5	3	0.664195	2.036716	0.100371	0.675261	0.224369
	5	1.851207	2.620441	0.37625	0.602669	0.02108
	7	2.396506	2.656511	0.487092	0.497174	0.015734

Table 3 Various system performance measures for varying values of μ_v with respect to time

μ_v	Time (t)	$L_S(t)$	$TH(t)$	$PB(t)$	$PW(t)$	$PD(t)$
2.4	3	0.529794	1.545708	0.077752	0.553334	0.368914
	5	1.748995	2.391705	0.347137	0.591551	0.061312
	7	2.389926	2.526112	0.480134	0.49239	0.027476
2.6	3	0.513752	1.65609	0.076071	0.555035	0.368894
	5	1.67699	2.508842	0.337098	0.601911	0.060991
	7	2.294662	2.623415	0.466533	0.506586	0.026881
2.8	3	0.498389	1.767151	0.074439	0.556686	0.368875
	5	1.608991	2.630096	0.327355	0.611965	0.06068
	7	2.20426	2.726356	0.453235	0.520463	0.026301

6 Conclusion and Future Scope

The time-dependent results for an M/M/1 queue with working vacation (WV) and disaster failure are established by implementing the methods of continued fraction and probability generation function. The time-dependent results are useful when the system does not acquire equilibrium conditions. The internet server, cloud computing, telecommunication systems are some examples of such system. Numerical results are also presented to test the sensitiveness of parameters on various system descriptors. This model can be extended by including complete vacation.

Acknowledgements The author (Rakesh Kumar Meena) is thankful to the institute of Eminence (IoE) cell, Banaras Hindu University for providing a reserach grant with project reference number IoE (6031) to carry out the present research wok.

References

1. Ameur, L., Berdjoudj, L., Abbas, K.: Sensitivity analysis of the M/M/1 retrial queue with working vacations and vacation interruption. Int. J. Manag. Sci. Eng. Manag. **14**, 293–303 (2019). https://doi.org/10.1080/17509653.2019.1566034
2. Ammar, S.I.: Transient solution of an M/M/1 vacation queue with a waiting server and impatient customers. J. Eqypt. Math. Soc. **25**, 337–342 (2017). https://doi.org/10.1016/j.joems.2016.09.002
3. Bocharov, P.P., d'Apice, C., Manzo, R., Pechinkin, A.V.: Analysis of the multi-server Markov queuing system with unlimited buffer and negative customers. Autom. Remote Control **68**, 85–94 (2007). https://doi.org/10.1134/S0005117907010080
4. Chen, A., Renshaw, E.: The M/M/1 queue with mass exodus and mass arrivals when empty. J. Appl. Probab. **34**, 192–207 (1997). https://doi.org/10.2307/3215186
5. Doshi, B.T.: Queueing systems with vacations - A survey. Queueing Syst. **1**, 29–66 (1986). https://doi.org/10.1007/BF01149327
6. Jain, G., Sigman, K.: A Pollaczek-Khintchine formula for M/G/1 queues with disasters. J. Appl. Probab. **33**, 1191–1200 (1996). https://doi.org/10.2307/3214996
7. Jain, M., Meena, R.K.: Fault tolerant system with imperfect coverage, reboot and server vacation. J. Ind. Eng. Int. **13**, 171–180. https://doi.org/10.1007/s40092-016-0180-8
8. Jain, M., Singh, M.: Transient analysis of a Markov queueing model with feedback, discouragement and disaster. Int. J. Appl. Comput. Math. **6**, 31 (2020). https://doi.org/10.1007/s40819-020-0777-x
9. Jain, M., Rani, S., Singh, M.: Transient analysis of Markov feedback queue with working vacation and discouragement. In: Deep, K., Jain, M., Salhi, S. (eds.), Performance Prediction and Analytics of Fuzzy, Reliability and Queuing Models: Theory and Applications, pp. 235–250. Springer, Singapore (2019). https://doi.org/10.1007/978981-13-0857-418
10. Kannadasan, G., Sathiyamoorth, N.: The analysis of M/M/1 queue with working vacation in fuzzy environment. Appl. Appl. Math. **13**, 566–577 (2018)
11. Ke, J.C., Wang, K.H.: Vacation policies for machine repair problem with two type spares. Appl. Math. Model. **31**, 880–894 (2007). https://doi.org/10.1016/j.apm.2006.02.009
12. Ke, J.C., Wu, C.H.: Multi-server machine repair model with standbys and synchronous multiple vacation. Comput. Ind. Eng. **62**, 296–305 (2012). https://doi.org/10.1016/j.cie.2011.09.017
13. Ke, J.C., Wu, C.H., Liou, C.H., Wang, T.Y.: Cost analysis of a vacation repair model. Procedia - Soc. Behav. Sci. **25**, 246–256 (2011). https://doi.org/10.1016/j.sbspro.2011.10.545

14. Kim, C.S., Klimenok, V.I., Orlovskii, D.S.: Multi-server queueing system with a batch Markovian arrival process and negative customers. Autom. Remote Control **67**, 1958–1973 (2006). https://doi.org/10.1134/S0005117906120083
15. Meena, R.K., Jain, M., Sanga, S.S., Assad, A.: Fuzzy modeling and harmony search optimization for machining system with general repair, standby support and vacation. Appl. Math. Comput. **361**, 858–873 (2019). https://doi.org/10.1016/j.amc.2019.05.053
16. Sanga, S.S., Jain, M.: Cost optimization and ANFIS computing for admission control of M/M/1/K queue with general retrial times and discouragement. Appl. Math. Comput. **363**, 124624 (2019). https://doi.org/10.1016/j.amc.2019.124624
17. Servi, L.D., Finn, S.G.: M/M/1 queues with working vacations (M/M/1/WV). Perform. Eval. **50**, 41–52 (2002). https://doi.org/10.1016/S0166-5316(02)00057-3
18. Shin, Y.W.: Multi-server retrial queue with negative customers and disasters. Queueing Syst. **55**, 223–237 (2007). https://doi.org/10.1007/s11134-007-9018-9
19. Sudhesh, R.: Transient analysis of a queue with system disasters and customer impatience. Queueing Syst. **66**, 95–105 (2010). https://doi.org/10.1007/s11134-010-9186-x
20. Sudhesh, R., Raj, L.F.: Computational analysis of stationary and transient distribution of single server queue with working vacation BT - Global trends in computing and communication systems. In: Krishna, P.V., Babu, M.R., Ariwa, E. (eds.), pp. 480–489. Springer, Berlin (2012). https://doi.org/10.1007/978-3-642-29219-4_55
21. Sudhesh, R., Azhagappan, A., Dharmaraja, S.: Transient analysis of M/M/1 queue with working vacation, heterogeneous service and customers' impatience. RAIRO - Oper. Res. **51**, 591–606 (2017). https://doi.org/10.1051/ro/2016046
22. Suranga Sampath, M.I.G., Liu, J.: Impact of customers' impatience on an M/M/1 queueing system subject to differentiated vacations with a waiting server. Qual. Technol. Quant. Manag. **17**, 125–148 (2020). https://doi.org/10.1080/16843703.2018.1555877
23. Teghem, J.: Control of the service process in a queueing system. Eur. J. Oper. Res. **23**, 141–158 (1986). https://doi.org/10.1016/0377-2217(86)90234-1
24. Tian, N., Zhang, Z.G.: Vacation Queueing Models Theory and Applications, 1st edn. Springer US, Springer, Berlin (2006). https://doi.org/10.1007/978-0-387-33723-4
25. Tian, N., Zhao, X., Wang, K.: The M/M/1 queue with single working vacation. Int. J. Inf. Manag. Sci. **19**, 621–634 (2008). https://doi.org/10.1504/IJOR.2009.026941
26. Vijayashree, K.V., Janani, B.: Transient analysis of an M/M/c queue subject to multiple exponential vacation. Adv. Intell. Syst. Comput. **412**, 551–563 (2016). https://doi.org/10.1007/978981-10-0251-951

Usual Stochastic Ordering Results for Series and Parallel Systems with Components Having Exponentiated Chen Distribution

Madhurima Datta and Nitin Gupta

Abstract In this paper, we have considered two n-component series systems and two n-component parallel systems. The random variables corresponding to each of these components are assumed to be independent and non-identically distributed. When the random variables followed Exponentiated Chen distribution (denoted as $ECD(\alpha, \beta, \lambda)$ where α, β, λ are the 3 parameters), the systems can be compared based on the usual stochastic ordering. Some counterexamples were constructed to show that the hazard rate and reversed hazard rate orderings cannot be obtained under certain conditions.

Keywords Exponentiated Chen distribution · Majorization · Parallel system · Series system · Usual stochastic order

1 Introduction

Exponentiated Chen distribution is an extension of [1] family of distributions obtained by using Lehman alternatives. It is used for modeling survival data. The family of distributions obtained by using Lehman alternatives is known as exponentiated type family. The resultant cumulative distribution function is obtained as follows

$$F(x, \alpha) = (F_0(x))^{\alpha}, \ x > 0, \ \alpha > 0,$$

where $F_0(x)$ is the baseline distribution and $F(x, \alpha)$ is the generalization of $F_0(x)$ and α is a parameter. This model is referred to as the Proportional reversed hazard rate (PRHR) model where α is a proportionality constant. The Chen distribution function introduced by [1] is

M. Datta (✉) · N. Gupta
Indian Institute of Technology Kharagpur, West Midnapore 721302, India
e-mail: madhurima92.datta@gmail.com

N. Gupta
e-mail: nitin.gupta@maths.iitkgp.ac.in

O. Chadli et al. (eds.), *Mathematical Analysis and Applications*, Springer Proceedings in Mathematics & Statistics 381, https://doi.org/10.1007/978-981-16-8177-6_22

$$F(x;\beta,\lambda) = (1 - e^{\lambda(1-e^{x^{\beta}})}),\ x > 0, \beta > 0, \lambda > 0. \tag{1}$$

Applying the transformation $T = (e^{X^{\beta}} - 1)^{1/\beta}$ in Eq. (1), we observe that T follows Weibull distribution with scale parameter λ and shape parameter β. Also [2] extended Chen distribution by adding a parameter and the survival function of the resultant Extended Weibull distribution is

$$\overline{F}(x;\alpha,\beta,\lambda) = (e^{\lambda(1-e^{(\frac{x}{\alpha})^{\beta}})})^{\alpha},\ x > 0, \alpha > 0, \beta > 0, \lambda > 0. \tag{2}$$

[3] introduced another shape parameter to the Extended Weibull distribution and obtained a four parameter modified Weibull extension distribution using the Marshall-Olkin technique.

Later [4] introduced the generalization of Chen distribution given by [1] by introducing a new parameter α. The new distribution function with parameters α, β, λ is

$$F(x;\alpha,\beta,\lambda) = (1 - e^{\lambda(1-e^{x^{\beta}})})^{\alpha},\ x > 0, \alpha > 0, \beta > 0, \lambda > 0. \tag{3}$$

[5] discussed various important properties of Exponentiated Chen distribution such as the density function can be either decreasing or unimodal depending on the parameters α and β. Also the hazard rate function can be bathtub shaped or increasing depending on α and β. The reversed hazard rate function of Exponentiated Chen distribution is

$$\tilde{r}(x;\alpha,\beta,\lambda) = \frac{\alpha\beta\lambda x^{\beta-1}e^{x^{\beta}}e^{\lambda(1-e^{x^{\beta}})}}{1 - e^{\lambda(1-e^{x^{\beta}})}},\ x > 0, \alpha > 0, \beta > 0, \lambda > 0. \tag{4}$$

In this paper we have studied the usual stochastic ordering relations for the minimum and maximum ordered statistics (series and parallel systems respectively) for two different samples whose components follow the Exponentiated Chen distribution (ECD). The order statistics are extremely important in reliability theory. Consider a set of random variables $X_1, \ldots, X_n$, these random variables can be arranged as $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ such that $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$, here $X_{k:n}$ is the kth minimum of the set and is known as the kth order statistic. $X_{k:n}$ corresponds to the lifetime of a (n − k + 1)-out-of-n system. Details on order statistics are available in the book [6]. Pledger and Proschan [7] pioneered the field of stochastic ordering and developed usual stochastic ordering results for proportional hazard rate models which implied similar results for exponential distribution, gamma distribution, Weibull distribution, etc. Studies of stochastic ordering of ordered statistics are extremely popular nowadays and a wide variety of stochastic ordering results are available in the literature [8]. The paper has been organized as follows: Sect. 2 includes the definitions of various stochastic orders followed by some important lemmas. Section 3 contains the usual stochastic ordering results for series and parallel systems along with few examples and counterexamples. Lastly Sect. 4 summaries the results obtained in the paper.

2 Definitions

Let the random variables X and Y be absolutely continuous with distribution functions $F(x)$ and $G(x)$; survival functions as $\overline{F}(x)$ and $\overline{G}(x)$; probability density functions as $f(x)$ and $g(x)$; hazard rate functions as $r(x) = \dfrac{f(x)}{\overline{F}(x)}$ and $s(x) = \dfrac{g(x)}{\overline{G}(x)}$; reversed hazard rate functions as $\tilde{r}(x) = \dfrac{f(x)}{F(x)}$ and $\tilde{s}(x) = \dfrac{g(x)}{G(x)}$, where F^{-1} and G^{-1} are the right continuous quantiles respectively. Barlow and Proschan [9] contains detailed explanation of the above terms. We shall now explain the stochastic ordering between the random variables X and Y with the help of above mentioned terms. These relations are mentioned in the book [10].

(a) X is smaller than Y in the usual stochastic order ($X \leq_{\text{st}} Y$) if and only if $\overline{F}(x) \leq \overline{G}(x)\ \forall\, x \in \mathbb{R}$.
(b) X is smaller than Y in hazard rate order ($X \leq_{\text{hr}} Y$) if $r(x) \geq s(x),\ x \in \mathbb{R}$. Equivalently, if $\dfrac{\overline{G}(x)}{\overline{F}(x)}$ is increasing in x over the union of the supports of X and Y.
(c) X is smaller than Y in reversed hazard rate order ($X \leq_{\text{rh}} Y$) if $\tilde{r}(x) \leq \tilde{s}(x),\ x \in \mathbb{R}$. Equivalently, if $\dfrac{G(x)}{F(x)}$ is increasing in x over the union of the supports of X and Y.
(d) X is smaller than Y in likelihood ratio order ($X \leq_{\text{lr}} Y$) if $\dfrac{g(x)}{f(x)}$ is increasing in x over the union of the supports of X and Y.

The likelihood ratio ordering implies both hazard rate and reversed hazard rate ordering which again implies the usual stochastic ordering.

Definition 1 Majorization (*see* [11] *for further details*)
Consider $\underline{a} = (a_1, \ldots, a_n)$ and $\underline{b} = (b_1, \ldots, b_n)$ as two real valued vectors then $\underline{a}$ is majorized by $\underline{b}$ ($\underline{a} \prec \underline{b}$) if

$$\sum_{i=1}^{n} a_{i:n} = \sum_{i=1}^{n} b_{i:n} \text{ and } \sum_{i=1}^{k} a_{i:n} \geq \sum_{i=1}^{k} b_{i:n}\ \forall\, k = 1, \ldots, n-1, \tag{5}$$

where $a_{1:n} \leq \cdots \leq a_{n:n}$ $(b_{1:n} \leq \cdots \leq b_{n:n})$ is the increasing arrangement of $a_1, \ldots, a_n (b_1, \ldots, b_n)$.

In general for two matrices $A = \{a_{ij}\}_{m\times n}$ and $B = \{b_{ij}\}_{m\times n}$, A is majorized by B ($A \prec B$) if $A = BP$, where $P = \{p_{ij}\}_{n\times n}$ is a doubly stochastic matrix (this matrix need not be unique but the existence of atleast one such matrix ensures majorization). $\underline{a}$ is weakly submajorized by $\underline{b}$ ($\underline{a} \prec_{\text{w}} \underline{b}$) if

$$\sum_{i=1}^{k} a_{n-i+1:n} \leq \sum_{i=1}^{k} b_{n-i+1:n}\ \forall\, k = 1, \ldots, n \tag{6}$$

and $\underline{a}$ is weakly supermajorized by $\underline{b}$ ($\underline{a} \prec^{w} \underline{b}$) if

$$\sum_{i=1}^{k} a_{i:n} \geq \sum_{i=1}^{k} b_{i:n} \; \forall \, k = 1, \ldots, n. \tag{7}$$

Note that for $\underline{a}, \underline{b} \in \mathbb{R}$

$$\underline{a} \prec^{w} \underline{b} \Leftarrow \underline{a} \prec \underline{b} \Rightarrow \underline{a} \prec_{w} \underline{b}$$

Definition 2 Schur-convexity (Schur-concavity) [11] A real valued function ψ defined on a subset of $\mathbb{R}^n$ is *Schur-convex (Schur-concave)* if

$$\underline{a} \prec \underline{b} \Rightarrow \psi(\underline{a}) \;\leq (\geq)\; \psi(\underline{b}), \tag{8}$$

where $\underline{a} = (a_1, \ldots, a_n)$ and $\underline{b} = (b_1, \ldots, b_n)$ are two real valued vectors.

Throughout the paper, the notation $a \overset{\text{sgn}}{=} b$ has been used to represent sign of a is same as b. The following lemmas are useful for obtaining results in the next section.

Lemma 1 (Theorem 3.A.4, see [11])

For an open interval $\mathbb{A} \subset \mathbb{R}$, a continuously differentiable function $\psi : \mathbb{A}^n \to \mathbb{R}$ is Schur-convex (Schur-concave) if and only if it is symmetric on $\mathbb{A}^n$ and for all $i \neq j$, let $\Delta = (a_i - a_j)\left(\frac{\partial \psi(\underline{a})}{\partial a_i} - \frac{\partial \psi(\underline{a})}{\partial a_j}\right)$. Then $\Delta \geq (\leq) 0$.

Lemma 2 (Theorem 3.A.8, see [11])

Let $S \subset \mathbb{R}^n$, a function $f : S \to \mathbb{R}$ satisfying

$$\underline{a} \prec_{w} \underline{b} \; (\underline{a} \prec^{w} \underline{b}) \text{ on } S \;\Rightarrow\; f(\underline{a}) \leq f(\underline{b})$$

if and only if f is increasing (decreasing) and Schur-convex on S.

Lemma 3 *Let $\psi_1 : (0, \infty) \times (0, 1) \to (0, \infty)$ be defined as*

$$\psi_1(\alpha, y) = \frac{y(1-y)^{\alpha-1}}{1-(1-y)^{\alpha}}. \tag{9}$$

Then

(i) $\psi_1(\alpha, y)$ increases with respect to y for $0 < \alpha < 1$,
(ii) $\psi_1(\alpha, y)$ decreases with respect to y for $\alpha > 1$.

Proof Differentiating $\psi_1(\alpha, y)$ partially with respect to y we obtain,

$$\frac{\partial \psi_1(\alpha, y)}{\partial y} \overset{\text{sgn}}{=} -(1-y)^{\alpha}(\alpha y + (1-y)^{\alpha} - 1).$$

Let $f_1(y) = \alpha y + (1-y)^\alpha - 1,\ 0 < y < 1$ and $f_1(0) = 0$. The derivative of $f_1(y)$ is

$$\begin{aligned} f_1'(y) &= \alpha(1-(1-y)^{\alpha-1}) \\ &= g_1(y) \text{ (say)}. \end{aligned}$$

Again the derivative of $g_1(y)$ is

$$g_1'(y) = \alpha(\alpha-1)(1-y)^{\alpha-2}.$$

The two possible cases are

(i) for $0 < \alpha < 1$, $g_1'(y) < 0$ and $g_1(0) = 0 \Rightarrow g_1(y) < 0$, i.e., $f_1'(y) < 0$. And $f_1(0) = 0 \Rightarrow f_1(y) < 0$ which implies $\dfrac{\partial \psi_1(\alpha, y)}{\partial y} > 0$,

(ii) for $\alpha > 1, g_1'(y) > 0$ and $g_1(0) = 0 \Rightarrow g_1(y) > 0$, i.e., $f_1'(y) > 0$. And $f_1(0) = 0 \Rightarrow f_1(y) > 0$ which implies $\dfrac{\partial \psi_1(\alpha, y)}{\partial y} < 0$. □

3 Results

In this section we shall compare two systems (series and parallel) by using the usual stochastic ordering relations. We shall henceforth denote that the random variable X follows Exponentiated Chen distribution as $X \sim ECD(\alpha, \beta, \lambda)$. Along with the results we shall present few examples and counterexamples to support the results. The first theorem shows the usual stochastic ordering between the sample minimum or between two series system when only the parameter λ is varying and all the other parameters remain constant.

Theorem 1 *Let $X_1, X_2, \ldots, X_n$ be a set of n independent random variables such that $X_i \sim ECD(\alpha, \beta, \lambda_i)$, $\alpha > 0, \beta > 0, \lambda_i > 0$ for $i = 1, 2, \ldots, n$. Consider another set of n independent random variables $Y_1, Y_2, \ldots, Y_n$ and the distribution function of each random variable is $ECD(\alpha, \beta, \mu_i)$ for $i = 1, 2, \ldots, n$. Then*

$$\begin{aligned} \underline{\lambda} \prec^{\mathrm{w}} \underline{\mu} &\Rightarrow X_{1:n} \le_{\mathrm{st}} Y_{1:n} (0 < \alpha < 1) \\ \underline{\lambda} \prec_{\mathrm{w}} \underline{\mu} &\Rightarrow X_{1:n} \ge_{\mathrm{st}} Y_{1:n} (\alpha > 1), \end{aligned}$$

where $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$.

Proof The survival function of the series system $X_{1:n}$ is

$$\overline{F}_{X_{1:n}}(x) = \prod_{k=1}^{n} \left[1 - \left(1 - e^{\lambda_k(1-e^{x^\beta})}\right)^\alpha\right],\ \alpha > 0,\ \beta > 0,\ \lambda_k > 0 \,\forall\, k = 1, 2, \ldots, n. \tag{10}$$

$\overline{F}_{X_{1:n}}(x)$ is symmetric with respect to the vector $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Partially differentiating (10) with respect to λ_i,

$$\frac{\partial \overline{F}_{X_{1:n}}(x)}{\partial \lambda_i} = \overline{F}_{X_{1:n}}(x) \frac{\alpha(1-e^{x^\beta})e^{\lambda_i(1-e^{x^\beta})}\left(1-e^{\lambda_i(1-e^{x^\beta})}\right)^{\alpha-1}}{\left(1-\left(1-e^{\lambda_i(1-e^{x^\beta})}\right)^{\alpha}\right)}$$
$$\leq 0 \;\; \forall\, \lambda_i > 0, \;\; i = 1, 2, \ldots, n.$$

Assume $\lambda_i \neq \lambda_j$ for $i, \;\; j = 1, 2, \ldots, n$ and consider

$$\Delta_1 = (\lambda_i - \lambda_j)\left(\frac{\partial \overline{F}_{X_{1:n}}(x)}{\partial \lambda_i} - \frac{\partial \overline{F}_{X_{1:n}}(x)}{\partial \lambda_j}\right)$$
$$= \alpha(\lambda_i - \lambda_j)\overline{F}_{X_{1:n}}(x)(1-e^{x^\beta})\left(\psi_1(\alpha, y_i) - \psi_1(\alpha, y_j)\right),$$

where $y_i = e^{\lambda_i(1-e^{x^\beta})}$ and the function $\psi_1(\alpha, y_i)$ is same as in Eq. (9).

Thus using Lemma 3, when $0 < \alpha < 1$, $\Delta_1 > 0 \Rightarrow \overline{F}_{X_{1:n}}(x)$ is Schur-convex with respect to $\underline{\lambda}$. And for $\alpha > 1$, $\Delta_1 < 0 \Rightarrow \overline{F}_{X_{1:n}}(x)$ is Schur-concave with respect to $\underline{\lambda}$.

Finally using Lemmas 1 and 2, we obtain the following two cases:

(i) for $0 < \alpha < 1$, $\underline{\lambda} \prec^{w} \underline{\mu} \Rightarrow \overline{F}_{X_{1:n}}(x) \leq \overline{F}_{Y_{1:n}}(x)$,

(ii) for $\alpha > 1$, $\underline{\lambda} \prec_{w} \underline{\mu} \Rightarrow \overline{F}_{X_{1:n}}(x) \geq \overline{F}_{Y_{1:n}}(x)$. □

Hence the result follows.

Since majorization implies weak majorization, a realization of the above result can be observed with the help of the following example.

Example 1 Let us consider two series systems with 4 components each, such that the parameter vectors are taken as $\underline{\lambda} = (0.8, 1.2, 1.3, 1.9)$ and $\underline{\mu} = (0.5, 0.7, 1.5, 2.5)$, $\underline{\lambda} \prec \underline{\mu}$, $\beta = 2$. The first plot shows $\overline{F}_{Y_{1:4}}(x) - \overline{F}_{X_{1:4}}(x)$ for $\alpha = 0.7$ and the next plot for $\alpha = 1.5$ (Figs. 1 and 2).

It has also been observed that the above result cannot be extended to hazard rate ordering under the same conditions. For instance, we observe that the ratio $\dfrac{\overline{F}_{Y_{1:4}}(x)}{\overline{F}_{X_{1:4}}(x)}$ is non-monotone for the above example (Fig. 3 and 4).

The next result shows the usual stochastic ordering between two maximum order statistic when the parameters α and β are constant but only the parameter λ is varying.

Theorem 2 *Let $X_1, X_2, \ldots, X_n$ be a set of n independent random variables where each $X_i \sim ECD(\alpha, \beta, \lambda_i)$, $\alpha > 0, \beta > 0, \lambda_i > 0$ for $i = 1, 2, \ldots, n$. Another set $Y_1, Y_2, \ldots, Y_n$ be n independent random variables such that $Y_i \sim ECD(\alpha, \beta, \mu_i)$ for $i = 1, 2, \ldots, n$. Then*

$$\underline{\lambda} \prec^{w} \underline{\mu} \Rightarrow X_{n:n} \leq_{\text{st}} Y_{n:n},$$

Fig. 1 $\overline{F}_{Y_{1:4}}(x) - \overline{F}_{X_{1:4}}(x) > 0$ when $\alpha = 0.7$

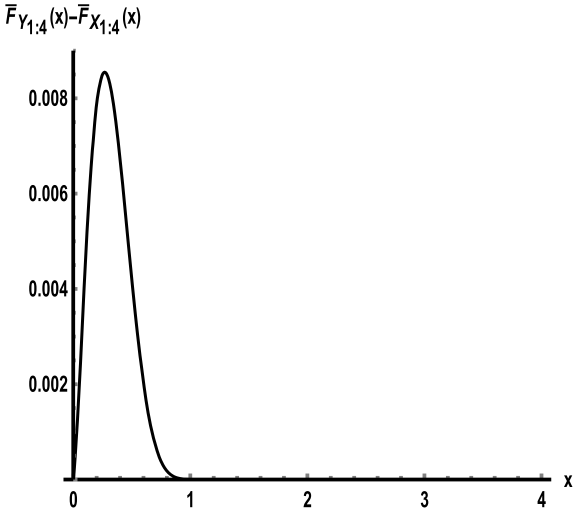
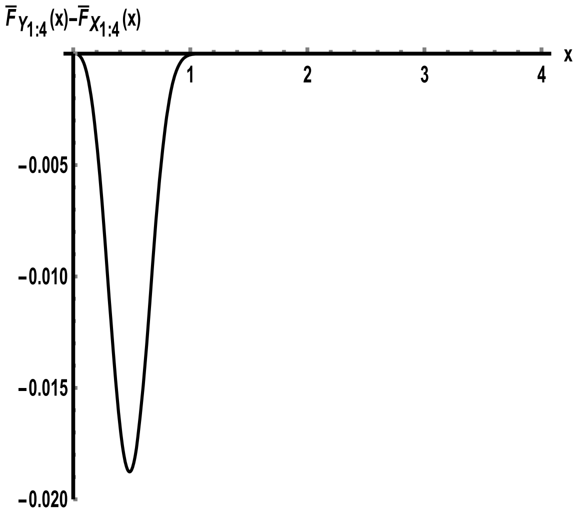


Fig. 2 $\overline{F}_{Y_{1:4}}(x) - \overline{F}_{X_{1:4}}(x) < 0$ when $\alpha = 1.5$



where $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ *and* $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)$.

Proof The distribution function corresponding to the maximum order statistic is

$$F_{X_{n:n}}(x) = \prod_{k=1}^{n} \left(1 - e^{\lambda_k (1 - e^{x^\beta})}\right)^\alpha, \ \alpha > 0, \ \beta > 0, \ \lambda_k > 0 \, \forall \, k = 1, 2, \ldots, n. \quad (11)$$

$F_{X_{n:n}}(x)$ is symmetric with respect to the parameter vector $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$. Differentiating (11) partially with respect to the parameter λ_i, we obtain

Fig. 3 $\frac{\overline{F}_{Y_{1:4}}(x)}{\overline{F}_{X_{1:4}}(x)}$ is non-monotone for $\alpha = 0.7$

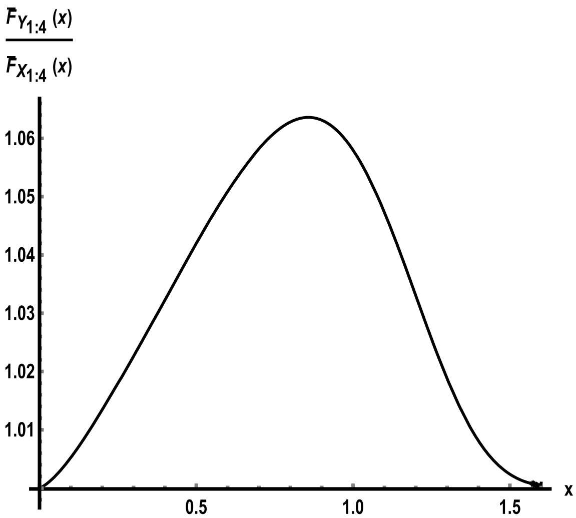
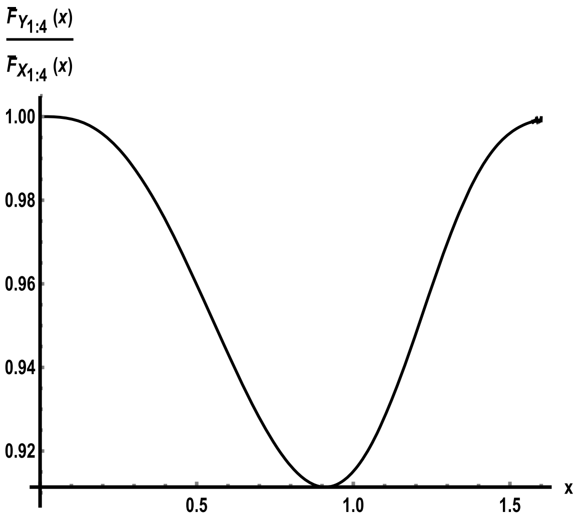


Fig. 4 $\frac{\overline{F}_{Y_{1:4}}(x)}{\overline{F}_{X_{1:4}}(x)}$ is non-monotone for $\alpha = 1.5$



$$\frac{\partial F_{X_{n:n}}(x)}{\partial \lambda_i} = -\alpha(1 - e^{x^\beta}) F_{X_{n:n}}(x) \frac{e^{\lambda_i(1-e^{x^\beta})}}{1 - e^{\lambda_i(1-e^{x^\beta})}}$$
$$\geq 0 \ \ \forall \lambda_i, \ i = 1, 2, \ldots, n.$$

$F_{X_{n:n}}(x)$ is increasing in each λ_i for $i = 1, 2, \ldots, n$.

Consider for $\lambda_i \neq \lambda_j$, $i \neq j$,

$$\Delta_2 = (\lambda_i - \lambda_j)\left(\frac{\partial F_{X_{n:n}}(x)}{\partial \lambda_i} - \frac{\partial F_{X_{n:n}}(x)}{\partial \lambda_j}\right)$$
$$= \alpha(1 - e^{x^\beta}) F_{X_{n:n}}(x)(\lambda_i - \lambda_j)(\phi_1(\lambda_i) - \phi_1(\lambda_j)),$$

where $\phi_1(\lambda) = 1 - \dfrac{1}{1 - e^{\lambda(1-e^{x^\beta})}}$. In order to determine the sign of Δ_2, we evaluate the derivative of $\phi_1(\lambda)$,

$$\phi_1'(\lambda) = \frac{(e^{x^\beta} - 1)e^{\lambda(1-e^{x^\beta})}}{(1 - e^{\lambda(1-e^{x^\beta})})^2}.$$

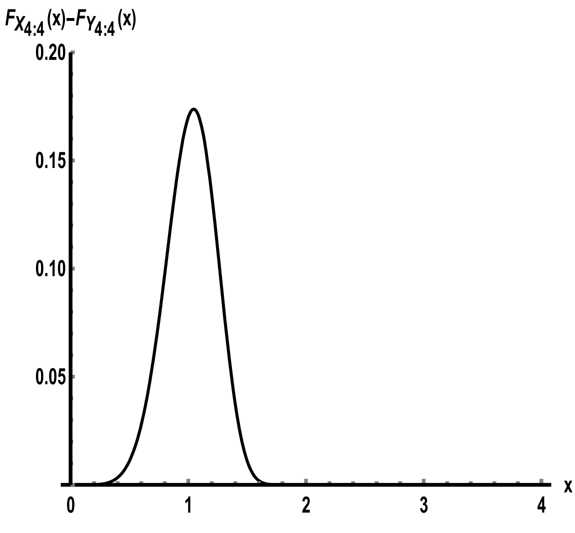
We observe that $\phi_1'(\lambda) > 0$, i.e., $\phi_1(\lambda)$ is an increasing function of λ. Thus $\Delta_2 \le 0$ $\Rightarrow F_{X_{n:n}}(x)$ is Schur-concave. Clearly, $-F_{X_{n:n}}(x)$ is decreasing and Schur-convex w.r.t $\underline{\lambda}$. Hence using Lemma 2,

$$\underline{\lambda} \prec^{\mathrm{w}} \underline{\mu} \Rightarrow F_{X_{n:n}}(x) \ge F_{Y_{n:n}}(x)$$
$$\Rightarrow X_{n:n} \le_{\mathrm{st}} Y_{n:n}.$$

Example 2 Consider Example 1, the same set of 4 components forms the parallel system and we plot the difference between $F_{X_{4:4}}(x) - F_{Y_{4:4}}(x)$, the ratio $\dfrac{F_{Y_{4:4}}(x)}{F_{X_{4:4}}(x)}$ and the ratio $\dfrac{\overline{F}_{Y_{4:4}}(x)}{\overline{F}_{X_{4:4}}(x)}$ for $\alpha = 0.7$ and $\alpha = 1.5$ (Figs. 5 and 6).

Further the plot for the ratio $\dfrac{F_{Y_{4:4}}(x)}{F_{X_{4:4}}(x)}$ is Figs. 7 and 8.

Fig. 5 $F_{X_{4:4}}(x) - F_{Y_{4:4}}(x) > 0$ when $\alpha = 0.7$



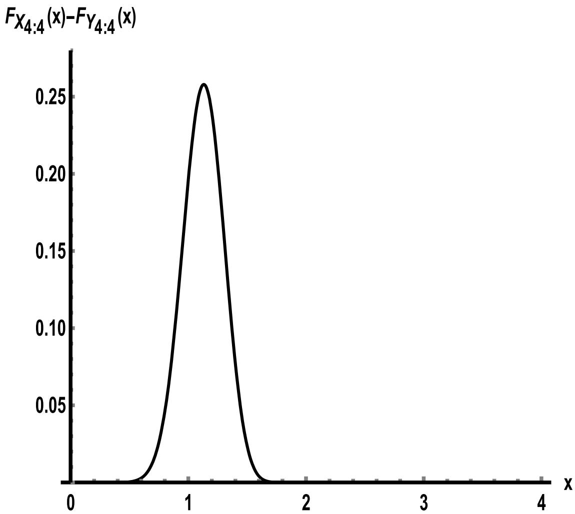


Fig. 6 $F_{X_{4:4}}(x) - F_{Y_{4:4}}(x) > 0$ when $\alpha = 1.5$

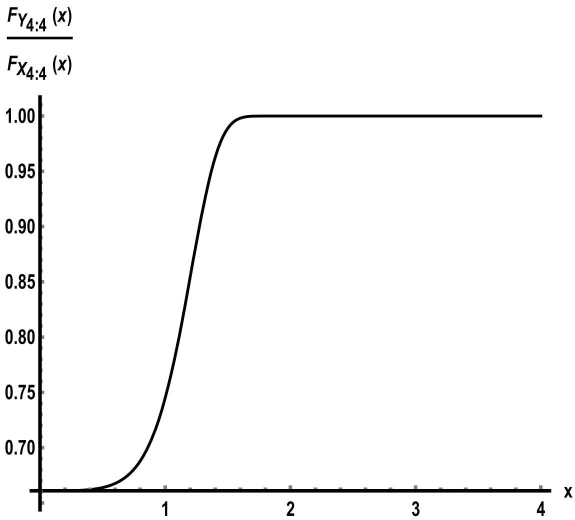


Fig. 7 $\frac{F_{Y_{4:4}}(x)}{F_{X_{4:4}}(x)}$ is increasing in x when $\alpha = 0.7$

Both the plot Figs. 7 and 8 shows that the ratio of $F_{Y_{4:4}}(x)$ and $F_{X_{4:4}}(x)$ is increasing irrespective of the value of α, but the analytical proof includes rigorous calculations. Hence it might be possible that the reversed hazard rate ordering exists for the parallel system.

Next we plot the ratio of $\frac{\overline{F}_{Y_{4:4}}(x)}{\overline{F}_{X_{4:4}}(x)}$ Figs. 9 and 10.

The ratio of $\overline{F}_{Y_{4:4}}(x)$ and $\overline{F}_{X_{4:4}}(x)$ shows that the plot is non-monotone and surely this implies that the hazard rate ordering for parallel system does not exist under these circumstances.

Fig. 8 $\frac{F_{Y_{4:4}}(x)}{F_{X_{4:4}}(x)}$ is increasing in x when $\alpha = 1.5$

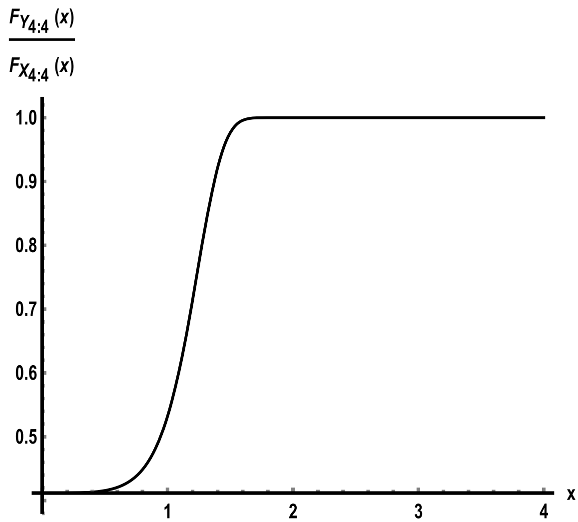
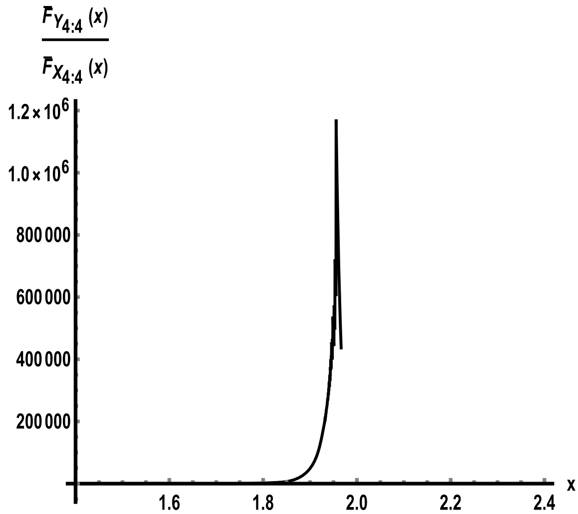


Fig. 9 $\frac{\overline{F}_{Y_{4:4}}(x)}{\overline{F}_{X_{4:4}}(x)}$ is non-monotone when $\alpha = 0.7$



The next theorem proves that there exists usual stochastic ordering relation between two parallel systems when the parameter β is varied and all the other parameters remain constant.

Theorem 3 *Consider two parallel systems consisting of n components, the components of one system corresponds to the set of n independent random variables* $X_1, X_2, \ldots, X_n$ *such that* $X_i \sim ECD(\alpha, \beta_i, \lambda)$ *for* $i = 1, 2, \ldots, n$. *Let* $Y_1, Y_2, \ldots, Y_n$ *be the random variables corresponding to the components of the other system, and*

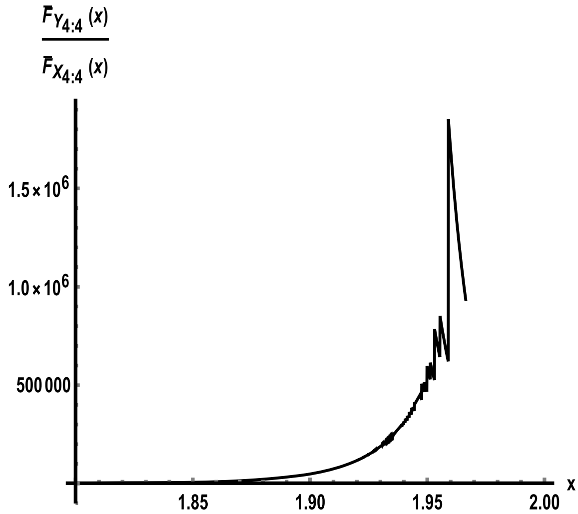


Fig. 10 $\frac{\overline{F}_{Y_{4:4}}(x)}{\overline{F}_{X_{4:4}}(x)}$ is non-monotone when $\alpha = 1.5$

$Y_i \sim ECD(\alpha, \beta_i^*, \lambda)$ for $i = 1, 2, \ldots, n$. Then $\lambda > 1$ and $\underline{\beta} \prec \underline{\beta}^* \Rightarrow X_{n:n} \leq_{\text{st}} Y_{n:n}$, where $\underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$ and $\underline{\beta}^* = (\beta_1^*, \beta_2^*, \ldots, \beta_n^*)$.

Proof The distribution function corresponding to $X_{n:n}$ is

$$F_{X_{n:n}}(x) = \prod_{k=1}^{n} \left(1 - e^{\lambda(1-e^{x^{\beta_k}})}\right)^{\alpha}, \ \alpha > 0, \ \beta_k > 0, \ \lambda > 0 \ \forall \, k = 1, 2, \ldots, n. \quad (12)$$

$F_{X_{n:n}}(x)$ is symmetric with respect to the parameter vector $\underline{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$. Differentiating (12) partially with respect to β_i we have

$$\frac{\partial F_{X_{n:n}}(x)}{\partial \beta_i} = F_{X_{n:n}}(x) \frac{\alpha \lambda x^{\beta_i} e^{\lambda(1-e^{x^{\beta_i}})+x^{\beta_i}} \ln x}{1 - e^{\lambda(1-e^{x^{\beta_i}})}}$$

Consider for $\beta_i \neq \beta_j$, $i \neq j$

$$\begin{aligned}\Delta_3 &= (\beta_i - \beta_j) \left(\frac{\partial F_{X_{n:n}}(x)}{\partial \beta_i} - \frac{\partial F_{X_{n:n}}(x)}{\partial \beta_j} \right) \\ &= \alpha \lambda \ln x \, F_{X_{n:n}}(x) (\beta_i - \beta_j)(\phi_2(x^{\beta_i}) - \phi_2(x^{\beta_j})),\end{aligned}$$

where $\phi_2(t) = \frac{t e^{\lambda(1-e^t)+t}}{1 - e^{\lambda(1-e^t)}}$. Computing $\phi_2'(t)$, we observe

$$\phi_2'(t) = -\frac{e^{\lambda(1-e^t)+t}((t+1)e^{\lambda(1-e^t)} + \lambda t e^t - t - 1)}{(e^{\lambda(1-e^t)} - 1)^2}.$$

Let $g(t) = (t+1)e^{\lambda(1-e^t)} + \lambda t e^t - t - 1$ and $g(0) = 0$.

Also, $g'(t) = (1 - e^{\lambda(1-e^t)})(\lambda(t+1)e^t - 1)$. In order to determine the sign of $g'(t)$, let $h(t) = \lambda(t+1)e^t - 1$ and $h(0) = \lambda - 1$. Again differentiating $h(t)$, we obtain

$$\begin{aligned} h'(t) &= \lambda e^t(t+2) \\ &> 0 \; \forall \; t > 0, \end{aligned}$$

i.e., $h(t)$ is an increasing function of t. Thus, $h(t) > 0$ when $\lambda > 1$, and this implies $g(t)$ is an increasing function of t. Hence $g(t) > 0$. Finally we conclude that $\Delta_3 \leq 0$, or in other words $F_{X_{n:n}}(x)$ is Schur-concave with respect to the parameter vector $\underline{\beta}$. Using Lemma 1, $\underline{\beta} \prec \underline{\beta}^* \Rightarrow F_{X_{n:n}}(x) \geq F_{Y_{n:n}}(x)$, i.e., $X_{n:n} \leq_{\text{st}} Y_{n:n}$ for $\lambda > 1$.

We can realize the above theorem with the help of the following example.

Example 3 Let us consider a 4 component parallel system, such that the random variables corresponding to each component follows $ECD(0.6, \beta_i, 2)$, $i = 1, \ldots, 4$ as described in the above theorem. Let $\underline{\beta} = (0.4, 0.9, 2, 7.5)$ and $\underline{\beta}^* = (0.2, 1, 1.9, 7.7)$, here $\underline{\beta} \prec \underline{\beta}^*$. We shall now plot the difference between $F_{X_{4:4}}(x)$ and $F_{Y_{4:4}}(x)$, Fig. 11 also we can observe the plot for $\dfrac{F_{Y_{4:4}}(x)}{F_{X_{4:4}}(x)}$ and $\dfrac{\overline{F}_{Y_{4:4}}(x)}{\overline{F}_{X_{4:4}}(x)}$ here Figs. 12 and 13.

Thus the reversed hazard rate ordering is not possible but the hazard rate ordering may exist. Now if the same set of components form a series system, in that case the plot for $\overline{F}_{X_{1:4}}(x) - \overline{F}_{Y_{1:4}}(x)$ is Fig. 14.

This shows that there exists a possibility of usual stochastic ordering, but the analytical proof is quite complicated, and this can be considered as a future problem Fig. 14.

The next result describes the usual stochastic order relation for the minimum order statistic with the parameters β, λ being constant and only the parameter α varies.

Theorem 4 *Let $X_1, X_2, \ldots, X_n$ be n independent random variables corresponding to the components of a series system such that each $X_i \sim ECD(\alpha_i, \beta, \lambda)$ for $i = 1, 2, \ldots, n$. Let $Y_1, Y_2, \ldots, Y_n$ be the set of n independent random variables corresponding to another series system where the random variables $Y_i \sim ECD(\alpha_i^*, \beta, \lambda)$ for $i = 1, 2, \ldots, n$. Then as*

(a) $\underline{\alpha} \prec_{\text{w}} \underline{\alpha}^* \Rightarrow X_{1:n} \geq_{\text{st}} Y_{1:n}$,

(b) $\sum_{k=1}^{n} \alpha_k \leq \sum_{k=1}^{n} \alpha_k^* \Rightarrow X_{n:n} \leq_{\text{lr}} Y_{n:n}$,

where $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\underline{\alpha}^* = (\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*)$

(a) ***Proof*** The survival function of $X_{1:n}$ is

Fig. 11 shows $F_{X_{4:4}}(x) - F_{Y_{4:4}}(x) > 0$

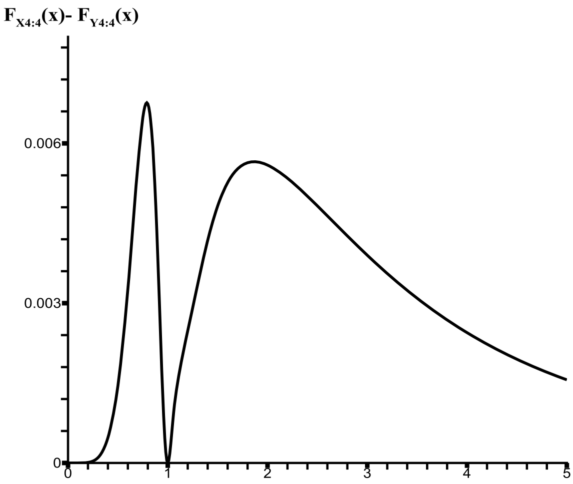
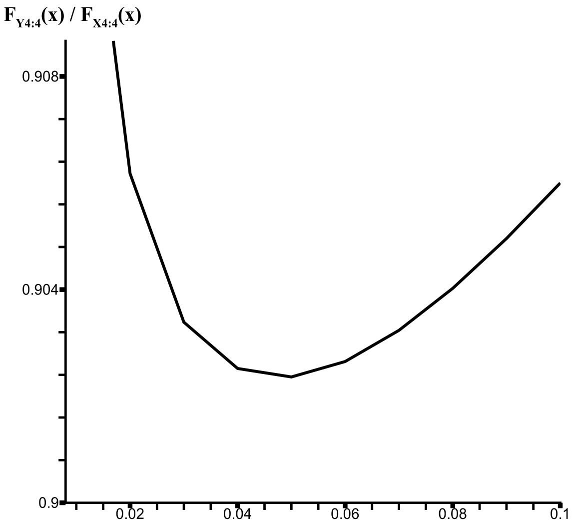


Fig. 12 shows the plot $\frac{F_{Y_{4:4}}(x)}{F_{X_{4:4}}(x)}$ is not monotone



$$\overline{F}_{X_{1:n}}(x) = \prod_{k=1}^{n} \left(1 - (1 - e^{\lambda(1-e^{x^{\beta}})})^{\alpha_k}\right), \ \alpha_k > 0, \ \beta > 0, \ \lambda > 0 \, \forall \, k = 1, 2, \ldots, n \tag{13}$$

$\overline{F}_{X_{1:n}}(x)$ is symmetric with respect to the parameter vector $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, now differentiating (13) partially with respect to α_i we obtain

$$\begin{aligned} \frac{\partial \overline{F}_{X_{1:n}}(x)}{\partial \alpha_i} &= -\overline{F}_{X_{1:n}}(x) \ln(1 - e^{\lambda(1-e^{x^{\beta}})}) \frac{(1 - e^{\lambda(1-e^{x^{\beta}})})^{\alpha_i}}{1 - (1 - e^{\lambda(1-e^{x^{\beta}})})^{\alpha_i}} \\ &\geq 0, \ \forall \alpha_i, \ i = 1, 2, \ldots, n. \end{aligned}$$

Fig. 13 shows the plot $\dfrac{\overline{F}_{Y_{4:4}}(x)}{\overline{F}_{X_{4:4}}(x)}$ is increasing

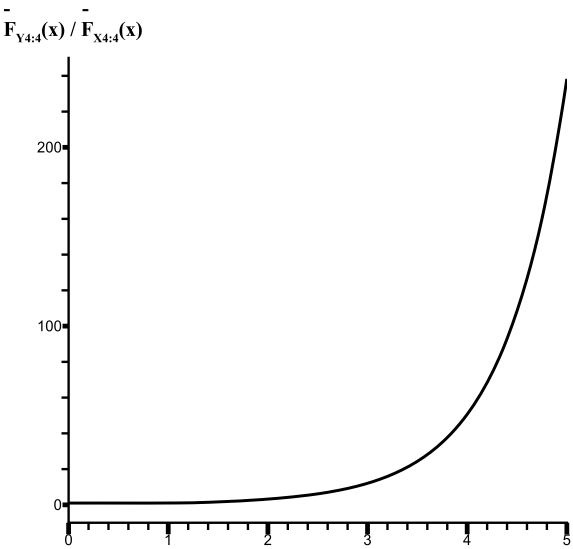
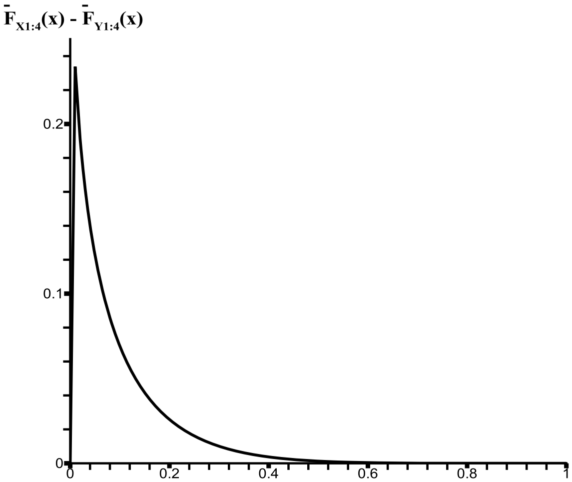


Fig. 14 shows $\overline{F}_{X_{1:4}}(x) - \overline{F}_{Y_{1:4}}(x) > 0$



For $\alpha_i \neq \alpha_j, i \neq j$, consider

$$\begin{aligned}\Delta_4 &= (\alpha_i - \alpha_j)\left(\frac{\partial \overline{F}_{X_{1:n}}(x)}{\partial \alpha_i} - \frac{\partial \overline{F}_{X_{1:n}}(x)}{\partial \alpha_j}\right)\\ &= (\alpha_i - \alpha_j)\overline{F}_{X_{1:n}}(x)\ln(1 - e^{\lambda(1-e^{x^\beta})})(\psi_2(\alpha_i) - \psi_2(\alpha_j))\end{aligned}$$

where $\psi_2(\alpha) = 1 - \dfrac{1}{1-(1-e^{\lambda(1-e^{x^\beta})})^\alpha}$.

It has been observed that $\psi_2'(\alpha) = -\ln(1-e^{\lambda(1-e^{x^\beta})})\dfrac{(1-e^{\lambda(1-e^{x^\beta})})^\alpha}{(1-(1-e^{\lambda(1-e^{x^\beta})})^\alpha)^2}$. Therefore, $\Delta_4 \le 0$, i.e., $\overline{F}_{X_{1:n}}(x)$ is Schur-concave. Using Lemma 2, $\underline{\alpha} \prec_{\mathrm{w}} \underline{\alpha}^*$ $\Rightarrow \overline{F}_{X_{1:n}}(x) \ge \overline{F}_{Y_{1:n}}(x)$ and the result follows. □

(b) ***Proof*** The distribution function of the parallel system represented as $X_{n:n}$ is

$$F_{X_{n:n}}(x) = \left(1-e^{\lambda(1-e^{x^\beta})}\right)^{\sum_{k=1}^{n}\alpha_k}, \ \alpha_k > 0,\ \beta > 0,\ \lambda > 0,\ \forall\, k = 1, 2, \ldots, n,$$

and the corresponding probability density function is

$$f_{X_{n:n}}(x) = \sum_{k=1}^{n}\alpha_k \left(1-e^{\lambda(1-e^{x^\beta})}\right)^{(\sum_{k=1}^{n}\alpha_k - 1)} \beta\lambda x^{\beta-1}e^{\lambda(1-e^{x^\beta})+x^\beta}.$$

It is enough to prove that the ratio $\dfrac{f_{Y_{n:n}}(x)}{f_{X_{n:n}}(x)}$ is increasing in x. And

$$\frac{f_{Y_{n:n}}(x)}{f_{X_{n:n}}(x)} = \frac{\sum_{k=1}^{n}\alpha_k^*}{\sum_{k=1}^{n}\alpha_k}\left(1-e^{\lambda(1-e^{x^\beta})}\right)^{(\sum_{k=1}^{n}\alpha_k^* - \sum_{k=1}^{n}\alpha_k)}.$$

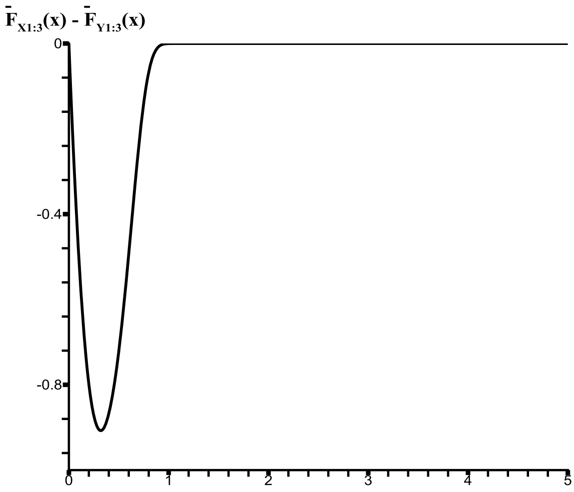
Differentiating with respect to x, we find that

$$\frac{d}{dx}\left(\frac{f_{Y_{n:n}}(x)}{f_{X_{n:n}}(x)}\right)$$
$$= \beta\lambda x^{\beta-1}e^{(\lambda(1-e^{x^\beta})+x^\beta)}\frac{\sum_{k=1}^{n}\alpha_k^*}{\sum_{k=1}^{n}\alpha_k}\left(\sum_{k=1}^{n}\alpha_k^* - \sum_{k=1}^{n}\alpha_k\right)\left(1-e^{\lambda(1-e^{x^\beta})}\right)^{(\sum_{k=1}^{n}\alpha_k^* - \sum_{k=1}^{n}\alpha_k - 1)}$$
$$\ge 0 \text{ for } \sum_{k=1}^{n}\alpha_k \le \sum_{k=1}^{n}\alpha_k^*.$$

Hence the result follows. □

Example 4 Consider two series systems each having 3 components. For system 1, the random variables corresponding to each component are X_1, X_2, X_3 such that $X_i \sim ECD(\alpha_i, \beta, \lambda)$, for $i = 1, 2, 3$ and for system 2, the random variables corresponding to each component are Y_1, Y_2, Y_3 and each $Y_i \sim ECD(\alpha_i^*, \beta, \lambda)$, for $i = 1, 2, 3$. The parameter values are $\beta = 3$, $\lambda = 2$, and $\underline{\alpha} = (1, 1.6, 2.8)$, $\underline{\alpha}^* =$

Fig. 15 shows $\overline{F}_{X_{1:3}}(x) - \overline{F}_{Y_{1:3}}(x)$ for varying α



$(1.2, 1.5, 3)$. The plot for the difference of their survival function, $\overline{F}_{X_{1:3}}(x) - \overline{F}_{Y_{1:3}}(x)$ See Fig. 15.

The above graph shows that the difference is negative and hence there exists an usual stochastic ordering between $X_{1:3}$ and $Y_{1:3}$. In other words Theorem 4 holds true for a 3 component system.

4 Inference

The results discussed in this paper include the usual stochastic ordering between $X_{1:n}$ and $Y_{1:n}$ (two series system) when the parameter λ is varied, or when only the parameter α has been varied. Whereas for two parallel systems, usual stochastic ordering exists when only the parameter λ is varied, or only the parameter β is varied. A likelihood ratio ordering has been possible for $X_{n:n}$ and $Y_{n:n}$ (two parallel systems) when the parameter α is varying.

Acknowledgements The authors are immensely thankful to the anonymous reviewers for their suggestions. The first author M.D. acknowledges the research fellowship received from the Indian Institute of Technology Kharagpur.

References

1. Chen, Z.: A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function. Stat. Probab. Lett. **49**(2), 155–161 (2000)
2. Xie, M., Tang, Y., Goh, T.N.: A modified Weibull extension with bathtub-shaped failure rate function. Reliab. Eng. Syst. Saf. **76**(3), 279–285 (2002)
3. Pappas, V., Adamidis, K., Loukas, S.: A family of lifetime distributions. Int. J. Qual. Stat. Reliab. 1–6 (2012)
4. Chaubey, Y.P., Zhang, R.: An extension of Chen's family of survival distributions with bathtub shape or increasing hazard rate function. Commun. Stat.-Theory Methods **44**(19), 4049–4064 (2015)
5. Dey, S., Kumar, D., Ramos, P.L., Louzada, F.: Exponentiated Chen distribution: properties and estimation. Commun. Stat.-Simul. Comput. **46**(10), 8118–8139 (2017)
6. David, H.A., Nagaraja, H.N.: Order Statistics. Wiley, Hoboken (2003)
7. Pledger, G., Proschan, F.: Comparisons of order statistics and of spacings from heterogeneous distributions. In: Rustagi, J.S. (ed.) Optimizing Methods in Statistic, 89–113 (1971)
8. Gupta, N., Patra, L.K., Kumar, S.: Stochastic comparisons in systems with Frèchet distributed components. Oper. Res. Lett. **43**(6), 612–615 (2015)
9. Barlow, R.E., Proschan, F.: Statistical Theory of Reliability and Life Testing: Probability Models. Holt, Rinehart and Winston Inc., New York (1975)
10. Shaked, M., Shanthikumar, J.G.: Stochastic Orders. Springer Science & Business Media, Berkeley (2007)
11. Marshall, A.W., Olkin, I.: Inequalities: Theory of Majorization and Its Applications. Springer, New York (2011)