# Modeling of Fluid-Structure Interactions with Exact Interface Tracking Methods


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**Abstract** Accuracy and numerical stability of nonlinear coupled fluid-elastic interaction simulations largely depends on the coupling and interface modeling algorithms. As part of the numerical coupling, the coupled solver requires to satisfy the kinematic and dynamic continuity conditions along the interface in addition to the fluid and structural dynamics governing equations. The interface kinematics and dynamics conditions are traditionally coupled with the governing equations that define the dynamics using either a partitioned or monolithic approaches. Irrespective of the coupling approach considered, the accuracy of the coupled numerical simulations strongly depends on the accuracy of the structural response dynamics, which in turn depends on the accuracy of the fluid dynamic forces acting on the structure. Hence, the numerical methods with exact interface become attractive when the accuracy of the coupled fluid-elastic interactions is of importance. In this paper, we present a review on the class of quasi-monolithic approaches with exact interface for numerically modeling the fluid-structure interactions involving rigid and flexible multi-body systems.

## 1 Introduction

Fluid-Structure Interaction (FSI) is a branch of multi-physics that is commonly observed in our day-to-day life, wherein the structure is considered to be elastic and it can undergo deformation/displacement due to the fluid dynamic forces acting on it which in turn would influence the fluid dynamic forces acting on them [1, 2]. As

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a result, the aerodynamic/hydrodynamic forces acting on an elastic structure would be distinctly different from that of a fixed structures [3]. Hence, it is important to investigate the FSI in the design of tall buildings, buildings housing sensitive equipment, long bridges [4], off-shore floating platforms [5], oil and gas pipelines [6, 7], wind turbines, micro-air vehicles, etc.

Numerical modeling of coupled FSI would require satisfying the interface kinematic (i.e., velocity and displacements) and dynamic (i.e., force) continuity along the fluid-structure interface along with to the fluid and structural dynamic governing equations. A FSI computational models can be categorized based on the way the interface between fluid-structure is modeled and the way in which the interface conditions are satisfied along the interface. The interface between the fluid-structure is either modeled by a conforming/non-confirming body fitted mesh for the fluid and structural domains or by using a non-body fitted Cartesian grid mesh for the combined fluid-structure domain. Methods such as level set method [8], Lagrangian multiplier, immersed boundary [9], ghost fluid, and fictitious domain [10] methods come under the later category of the interface modeling methods. On the other hand, arbitrary Lagrangian-Eulerian (ALE) is one of the popular approaches that would come under the body fitted mesh category of the interface modeling. In this coordinate system, the computational nodes can move relative to the spatial coordinate system. In an ALE approach, the mesh nodes on the fluid-structure interface behave like material points in a Lagrangian frame and the nodes inside the fluid domain can be moved arbitrarily to account for the movement of the fluid-structure interfaces so that the fluid and the structural meshes always remain two distinct non-overlapping meshes [11]. Hence, a typical body fitted FSI is a three-field problem [12].

Traditionally, in an ALE-based approach, the interface conditions are satisfied by a partitioned or monolithic approach. In a partitioned approach, the structural dynamic, the interface kinematic, dynamic continuity boundary conditions, and the fluid dynamic governing equations are solved in a sequential order [13]. As the governing equations for each of the physical fields, i.e., the fluid and structure, are solved sequentially one can use existing traditional fluid and the structural solvers as black-box solver by transferring the fluid forces acting on the solid from the fluid-solver to the structural-solver and transferring the structural kinematics, i.e., displacement and velocity, from the structural-solver to the fluid-solver. Since any fluid/structural solvers can be coupled with each other to simulate the coupled FSI phenomena, the partitioned approach offers higher levels of flexibility and modularity. Due to these traits, partitioned-based approaches are popular. However, as the effects of fluid and elastic structures on each other are transferred as boundary conditions coupling between the fluid and solid is not strong enough and can lead to numerical instability when the structural mass of the structure interacting with surrounding fluid is of the same order or lower because of the spurious energy produced due to the temporal inaccuracies [14]. This numerical instability can be solved for certain low mass cases by satisfying the interface conditions over multiple iterations till the solutions achieve a kinematic and dynamic convergence [15]. Even this strong fluid-structure coupling over multiple iterations may not be enough to sustain a numerical stability

for very low mass structures and the solutions can either suffer from non-convergence or convergence to a wrong solution-related issues.

In a monolithic approach, the governing equations that define the fluid dynamics, structural dynamics, interface continuity, and mesh dynamics are all assembled into a single large matrix and solved together [16]. These schemes provide good numerical stability even for problems with very low mass structures that experience strong inertial effects. However, monolithic approaches lack the flexibility and modularity of using an established fluid/structural solvers. In addition to the lack of flexibility, monolithic approaches can suffer from the computational resource- and convergence-related issues for solving large ill-conditioned system of linear equations. This would necessitate development of special kind of pre-conditioners for solving the matrix system of equations.

Key objective of the current work is to present a review on ALE-based FSI formulations which are numerically stable and computationally efficient for low structure to fluid-mass ratio where the inertial effects are very strong. As part of the review, we have considered two FSI coupling formulations based on improvised monolithic approach [17–19]. Unlike the traditional monolithic approaches wherein the governing equations pertaining to fluid, structure, interface, and mesh update are all solved together in a single large matrix, in the improvised monolithic approaches the equations pertaining to the mesh update algorithm are decoupled from the fluid, structure, and interface continuity equations by explicitly predicting the structural positions at the start of each time step. Explicit prediction on structural positions also enables linearization of the convective terms. Additionally, in these schemes, the interface kinematic and dynamic continuity conditions are implicitly satisfied by the construction of a single unified governing equation for combined fluid-structure system.

In the current work, we begin with a brief overview of the governing equations involved in the numerical modeling in Sect. 2. In Sect. 3, we construct the variational weak form of the combined fluid-structure governing equations. We then present in the complete second-order time-accurate quasi-monolithic with exact interface tracking formulation in Sect. 4. We then present Galerkin least square stabilized quasi-monolithic formulation in Sect. 5. We conclude the work by providing a summary of the two quasi-monolithic solvers reviewed in this work in Sect. 6.

## 2 Governing Equations

Let us consider $\boldsymbol{x}$ as a spatial point that belongs to the three-dimensional fluid domain $\Omega^{\mathrm{f}(t)}$ with boundary $\Gamma^{\mathrm{f}}(t)$ at any time $t$ and can move randomly in $\Omega^{\mathrm{f}(t)}$. The domain boundary $\Gamma^{\mathrm{f}}$ is made up of the Dirichlet ($\Gamma^{\mathrm{f}}_{\mathrm{D}}$), Neumann ($\Gamma^{\mathrm{f}}_{\mathrm{N}}$), and the fluid-structure interface ($\Gamma$) boundaries. Similarly, let us also consider a material point $\boldsymbol{Z}$ corresponding to the initial three-dimensional structural domain $\Omega^{\mathrm{s}}$ with boundary $\Gamma^{\mathrm{s}}$ that is made up of the Dirichlet ($\Gamma^{\mathrm{s}}_{\mathrm{D}}$), Neumann ($\Gamma^{\mathrm{s}}_{\mathrm{N}}$), and the interface ($\Gamma$). The

Navier–Stokes equations governing the dynamics of viscous incompressible fluid flow in an ALE reference frame are given as

$$\rho^{\mathrm{f}} \frac{\partial \boldsymbol{u}^{\mathrm{f}}}{\partial t} + \rho^{\mathrm{f}} \left(\boldsymbol{u}^{\mathrm{f}} - \boldsymbol{w}\right) \cdot \nabla \boldsymbol{u}^{\mathrm{f}} = \nabla \cdot \left[ -p\boldsymbol{I} + \mu^{\mathrm{f}} \frac{1}{2} \left( \nabla \boldsymbol{u}^{\mathrm{f}} + \left(\nabla \boldsymbol{u}^{\mathrm{f}}\right)^{T} \right) \right] + \boldsymbol{f}^{\mathrm{f}} \quad \text{in} \quad \Omega^{\mathrm{f}(t)}, \quad (1)$$

$$\nabla \cdot \boldsymbol{u}^{\mathrm{f}} = 0 \quad \text{in} \quad \Omega^{\mathrm{f}(t)}, \quad (2)$$

where $\rho^{\mathrm{f}}$ and $\mu^{\mathrm{f}}$ are the fluid density and dynamic viscosity; $\boldsymbol{u}^{\mathrm{f}} = \boldsymbol{u}^{\mathrm{f}}(\boldsymbol{x}, t)$, $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{x}, t)$, and $p = p(\boldsymbol{x}, t)$ represent the fluid velocity, fluid mesh velocity, and pressure defined at $\boldsymbol{x}$ for a time $t$; $\boldsymbol{I}$ is the second-order identity tensor; and $\boldsymbol{f}^{\mathrm{f}}$ denotes the body force.

The structural dynamic of a flexible structure is typical governed by Navier's equation which is written as

$$\rho^{\mathrm{s}} \frac{\partial^2 \boldsymbol{\eta}^{\mathrm{s}}}{\partial t^2} = \nabla \cdot \boldsymbol{\sigma}^{\mathrm{s}} + \boldsymbol{f}^{\mathrm{s}} \quad \text{in} \quad \Omega^{\mathrm{s}}, \quad (3)$$

where $\rho^{\mathrm{s}}$ is the structural density and $\boldsymbol{\eta}^{\mathrm{s}}(\boldsymbol{Z}, t)$ is the displacement vector that maps the material point $\boldsymbol{Z}$ from its initial position to its position at time $t$. $\boldsymbol{\sigma}^{\mathrm{s}}$ denotes the first Piola-Kirchhoff stress tensor and $\boldsymbol{f}^{\mathrm{s}}$ represents the body force vector acting on the structure. For a linear elastic material

$$\boldsymbol{\sigma}^{\mathrm{s}}\left(\boldsymbol{\eta}^{\mathrm{s}}\right) = \mu^{\mathrm{s}} \left[ \nabla \boldsymbol{\eta}^{\mathrm{s}} + \left(\nabla \boldsymbol{\eta}^{\mathrm{s}}\right)^{T} \right] + \lambda^{\mathrm{s}} (\nabla \cdot \boldsymbol{\eta}^{\mathrm{s}}) \boldsymbol{I}, \quad (4)$$

where $\mu^{\mathrm{s}}$ and $\lambda^{\mathrm{s}}$ are the Lamé coefficients of a material satisfying [20]. Similarly, the constitutive relation for a St. Venant-Kirchhoff (SVK) material [20, 21] is

$$\boldsymbol{\sigma}^{\mathrm{s}}\left(\boldsymbol{\eta}^{\mathrm{s}}\right) = 2\mu^{\mathrm{s}} \boldsymbol{F}\boldsymbol{E} + \lambda^{\mathrm{s}} \left[\mathrm{tr}\left(\boldsymbol{E}\right)\right] \boldsymbol{F}, \quad (5)$$

where $\mathrm{tr}(\cdot)$ is the tensor trace operator, $\boldsymbol{F}$ and $\boldsymbol{E}$ represent the deformation gradient and the Green-Lagrangian strain tensors, respectively, and are given as [20]

$$\boldsymbol{F} = \left(\boldsymbol{I} + \nabla \boldsymbol{\eta}^{\mathrm{s}}\right), \qquad \boldsymbol{E} = \frac{1}{2} \left(\boldsymbol{F}^{\mathrm{T}} \boldsymbol{F} - \mathbf{I}\right). \quad (6)$$

For simplicity, one can rewrite the structural Eq. (3) as

$$\rho^{\mathrm{s}} \frac{\partial \boldsymbol{u}^{\mathrm{s}}}{\partial t} = \nabla \cdot \boldsymbol{\sigma}^{\mathrm{s}} + \boldsymbol{f}^{\mathrm{s}} \quad \text{in} \quad \Omega^{\mathrm{s}}, \quad (7)$$

considering

$$\boldsymbol{u}^{\mathrm{s}}(\mathbf{Z}, t) = \frac{\partial \boldsymbol{\varphi}^{\mathrm{s}}}{\partial t}. \quad (8)$$

The above simplification enables the implicit implementation of the kinematic continuity condition along the fluid-structure interface.

The fluid and structural dynamics governing equations presented above need to satisfy the Dirichlet and Neumann conditions along the respective non-interface domain boundaries, which can be expressed below:

$$\boldsymbol{u}(\boldsymbol{x},t)^{\mathrm{f}} = \boldsymbol{u}_{\mathrm{D}}^{\mathrm{f}} \quad \forall \boldsymbol{x} \in \Gamma_{\mathrm{D}}^{\mathrm{f}} \qquad \text{and} \qquad \boldsymbol{\sigma}^{\mathrm{f}}(\boldsymbol{x},t)\cdot\mathbf{n}^{\mathrm{f}} = \boldsymbol{\sigma}_{\mathrm{N}}^{\mathrm{f}} \quad \forall \boldsymbol{x} \in \Gamma_{\mathrm{N}}^{\mathrm{f}}. \tag{9}$$

$$\boldsymbol{u}^{\mathrm{s}}(\boldsymbol{Z},t) = \boldsymbol{u}_{\mathrm{D}}^{\mathrm{s}} \quad \forall \boldsymbol{Z} \in \Gamma_{\mathrm{D}}^{\mathrm{s}} \qquad \text{and} \qquad \boldsymbol{\sigma}^{\mathrm{s}}(\boldsymbol{\varphi}^{\mathrm{s}})\cdot\mathbf{n}^{\mathrm{s}} = \boldsymbol{\sigma}_{\mathrm{N}}^{\mathrm{s}} \quad \forall \boldsymbol{Z} \in \Gamma_{\mathrm{N}}^{\mathrm{s}}. \tag{10}$$

In addition to the above boundary conditions about the non-interface boundaries, for an FSI phenomena the fluid and structure governing equations should also satisfy the kinematic and dynamic continuity conditions along the fluid-structure interface $\Gamma$ and can be written as

$$\boldsymbol{u}^{\mathrm{f}}(\boldsymbol{\varphi}^{\mathrm{s}}(\boldsymbol{Z},t),t) = \boldsymbol{u}^{\mathrm{s}}(\boldsymbol{Z},t) \quad \forall \boldsymbol{Z} \in \Gamma, \tag{11}$$

$$\int_{\boldsymbol{\varphi}^{\mathrm{s}}(\gamma,t)} \boldsymbol{\sigma}^{\mathrm{f}}(\boldsymbol{x},t)\cdot\mathbf{n}^{\mathrm{f}} d\Gamma + \int_{\gamma} \boldsymbol{\sigma}^{\mathrm{s}}(\boldsymbol{Z},t)\cdot\mathbf{n}^{\mathrm{s}} d\Gamma = 0 \quad \forall \gamma \subset \Gamma, \tag{12}$$

where $\gamma$ is any element on the fluid-structure interface $\Gamma$ at $t = 0$ and $\boldsymbol{\varphi}^{\mathrm{s}}(\boldsymbol{Z},t)$ is the deformed position the material point $\boldsymbol{Z}$ at time $t$. In the above equation, $\mathbf{n}^{\mathrm{f}}$ and $\mathbf{n}^{\mathrm{s}}$ are the unit outward normal vectors to the fluid and structural domain boundaries.

As described earlier, in an ALE approach, the fluid nodes inside the fluid domain needs to be shifted so that the fluid and structural domain interior nodes do not overlap. The dynamics of fluid mesh motion can be modeled by considering a pseudo-elastic constitutive equation given by

$$\nabla\cdot\boldsymbol{\sigma}^{\mathrm{m}} = \mathbf{0} \qquad \text{where} \qquad \boldsymbol{\sigma}^{\mathrm{m}} = (1+\tau_{\mathrm{m}})\left[\left(\nabla\boldsymbol{\eta}^{\mathrm{f}}(\boldsymbol{\chi},t) + \left(\nabla\boldsymbol{\eta}^{\mathrm{f}}(\boldsymbol{\chi},t)\right)^{T}\right) + (\nabla\cdot\boldsymbol{\eta}^{\mathrm{f}}(\boldsymbol{\chi},t))\mathbf{I}\right], \tag{13}$$

satisfying the boundary conditions

$$\boldsymbol{\eta}^{\mathrm{f}}(\boldsymbol{Z},t) = \boldsymbol{\varphi}^{\mathrm{s}}(\boldsymbol{Z},t) - \boldsymbol{Z} \qquad \forall \quad \boldsymbol{Z} \in \Gamma, \tag{14}$$

$$\boldsymbol{\eta}^{\mathrm{f}}(\boldsymbol{\chi},t) = 0 \qquad \forall \quad \boldsymbol{\chi} \in (\partial\Omega^{\mathrm{f}}(0))\backslash\Gamma. \tag{15}$$

Here $\boldsymbol{\eta}^{\mathrm{f}}$ is the displacement of the fluid mesh node and $\tau_{\mathrm{m}}$ is the element-level stiffness to limit the distortion of the small elements.

## 3 Weak Variational Form: Combined Fluid-Structure Formulation

To construct the weak form for the fluid-structure system by introducing trial function spaces $\mathcal{S}_{\boldsymbol{u}}$ and $\mathcal{S}_{\boldsymbol{p}}$ and corresponding test function spaces $\mathcal{V}_{\boldsymbol{u}}$ and $\mathcal{V}_{\boldsymbol{p}}$ for the fluid-structure velocity and fluid pressure, respectively. The definition of the trial-and-test function spaces is as follows:

$$\begin{aligned}\mathcal{S}_{\boldsymbol{u}} = \Big\{&(\boldsymbol{u}^{\mathrm{f}}, \boldsymbol{u}^{\mathrm{s}}) | (\boldsymbol{u}^{\mathrm{f}}, \boldsymbol{u}^{\mathrm{s}}) \in H^1(\Omega^{\mathrm{f}}(t)) \times H^1(\Omega^{\mathrm{s}}),\\ &\boldsymbol{u}^{\mathrm{f}}(\boldsymbol{\varphi}^{\mathrm{s}}(\boldsymbol{Z}, t)) = \boldsymbol{u}^{\mathrm{s}}(\boldsymbol{Z}, t) \quad \forall \boldsymbol{Z} \in \Gamma,\\ &\boldsymbol{u}(\boldsymbol{x}, t)^{\mathrm{f}} = \boldsymbol{u}_{\mathrm{D}}^{\mathrm{f}} \; \forall \boldsymbol{x} \in \Gamma_{\mathrm{D}}^{\mathrm{f}} \text{ and } \boldsymbol{u}^{\mathrm{s}}(\boldsymbol{Z}, t) = \boldsymbol{u}_{\mathrm{D}}^{\mathrm{s}} \; \forall \boldsymbol{Z} \in \Gamma_{\mathrm{D}}^{\mathrm{s}}\Big\},\\ \mathcal{S}_{\boldsymbol{p}} = &\{p | \; p \in L^2(\Omega^{\mathrm{f}}(t))\}.\end{aligned}$$

$$\begin{aligned}\mathcal{V}_{\boldsymbol{u}} = \Big\{&(\boldsymbol{\phi}^{\mathrm{f}}, \boldsymbol{\phi}^{\mathrm{s}}) | (\boldsymbol{\phi}^{\mathrm{f}}, \boldsymbol{\phi}^{\mathrm{s}}) \in H^1(\Omega^{\mathrm{f}}(t)) \times H^1(\Omega^{\mathrm{s}}),\\ &\boldsymbol{\phi}^{\mathrm{f}}(\boldsymbol{\varphi}^{\mathrm{s}}(\boldsymbol{Z}, t)) = \boldsymbol{\phi}^{\mathrm{s}}(\boldsymbol{Z}) \quad \forall \boldsymbol{Z} \in \Gamma,\\ &\boldsymbol{\phi}^{\mathrm{f}}(\boldsymbol{x}) = 0 \; \forall \; \boldsymbol{x} \in \Gamma_{\mathrm{D}}^{\mathrm{f}} \text{ and } \boldsymbol{\phi}^{\mathrm{s}}(\boldsymbol{Z}) = 0 \; \forall \; \boldsymbol{Z} \in \Gamma_{\mathrm{D}}^{\mathrm{s}}\Big\},\\ \mathcal{V}_{\boldsymbol{p}} = &\{q | \; q \in L^2(\Omega^{\mathrm{f}}(t))\}.\end{aligned}$$

The weak form of the Navier–Stokes Eqs. (1) and (2) can be expressed as

$$\int_{\Omega^{\mathrm{f}(t)}} \rho^{\mathrm{f}} \left( \left. \frac{\partial \boldsymbol{u}^{\mathrm{f}}}{\partial t} \right|_{\chi} + \left(\boldsymbol{u}^{\mathrm{f}} - \boldsymbol{w}\right) \cdot \boldsymbol{\nabla} \boldsymbol{u}^{\mathrm{f}} \right) \cdot \boldsymbol{\phi}^{\mathrm{f}} d\boldsymbol{x} + \int_{\Omega^{\mathrm{f}(t)}} \boldsymbol{\sigma}^{\mathrm{f}} : \boldsymbol{\nabla} \boldsymbol{\phi}^{\mathrm{f}} d\boldsymbol{x} =$$
$$\int_{\Omega^{\mathrm{f}(t)}} \boldsymbol{f}^{\mathrm{f}} \cdot \boldsymbol{\phi}^{\mathrm{f}} d\boldsymbol{x} + \int_{\Gamma_{\mathrm{H}}^{\mathrm{f}}} \boldsymbol{\sigma}_{\mathrm{H}}^{\mathrm{f}} \cdot \boldsymbol{\phi}^{\mathrm{f}} d\Gamma + \int_{\Gamma(t)} \left(\boldsymbol{\sigma}^{\mathrm{f}}(\boldsymbol{x}, t) \cdot \mathbf{n}^{\mathrm{f}}\right) \cdot \boldsymbol{\phi}^{\mathrm{f}} d\Gamma, \tag{16}$$

$$\int_{\Omega^{\mathrm{f}}(t)} \boldsymbol{\nabla} \cdot \boldsymbol{u}^{\mathrm{f}} q d\boldsymbol{x} = 0. \tag{17}$$

Similarly, weak form of the structural dynamics Eq. (7) can be written as

$$\int_{\Omega^{\mathrm{s}}} \rho^{\mathrm{s}} \frac{\partial \boldsymbol{u}^{\mathrm{s}}}{\partial t} \cdot \boldsymbol{\phi}^{\mathrm{s}} d\boldsymbol{Z} + \int_{\Omega^{\mathrm{s}}} \boldsymbol{\sigma}^{\mathrm{s}} : \boldsymbol{\nabla} \boldsymbol{\phi}^{\mathrm{s}} d\boldsymbol{Z} =$$
$$\int_{\Omega^{\mathrm{s}}} \boldsymbol{f}^{\mathrm{s}} \cdot \boldsymbol{\phi}^{\mathrm{s}} d\boldsymbol{Z} + \int_{\Gamma_{\mathrm{H}}^{\mathrm{s}}} \boldsymbol{\sigma}^{\mathrm{s}} \cdot \mathrm{H} \cdot \boldsymbol{\phi}^{\mathrm{s}} d\Gamma + \int_{\Gamma} (\boldsymbol{\sigma}^{\mathrm{s}}(\boldsymbol{Z}, t) \mathbf{n}^{\mathrm{s}}) \cdot \boldsymbol{\phi}^{\mathrm{s}}(\boldsymbol{Z}) d\Gamma. \tag{18}$$

The weak form of the dynamic traction continuity condition in Eq. (12) will be

$$\int_{\Gamma(t)} \left(\boldsymbol{\sigma}^{\mathrm{f}}(\boldsymbol{x}, t) \cdot \mathbf{n}^{\mathrm{f}}\right) \cdot \boldsymbol{\phi}^{\mathrm{f}}(\boldsymbol{x}) d\Gamma + \int_{\Gamma} (\boldsymbol{\sigma}^{\mathrm{s}}(\boldsymbol{Z}, t) \cdot \mathbf{n}^{\mathrm{s}}) \cdot \boldsymbol{\phi}^{\mathrm{s}}(\boldsymbol{Z}) d\Gamma = 0, \tag{19}$$

where $\phi^{\mathrm{f}}$ and $\phi^{\mathrm{s}}$ are required to satisfy $\phi^{\mathrm{f}}(\varphi^{\mathrm{s}}(\cdot)) = \phi^{\mathrm{s}}(\cdot)$ on $\Gamma$. A detailed derivation of the above weak form in Eq. (19) from its strong form in Eq. (12) can be found in [17]. Now we can combine Eqs. (16)–(18) using Eqs. (19) to construct a single unique relation for the combined fluid-structure domain, which is given as find $\left(\boldsymbol{u}^{\mathrm{f}}, \boldsymbol{u}^{\mathrm{s}}, p\right) \in \mathcal{S}_{\boldsymbol{u}} \times \mathcal{S}_{\boldsymbol{p}}$ such that for all $\left(\phi^{\mathrm{f}}, \phi^{\mathrm{s}}, q\right) \in \mathcal{V}_{\boldsymbol{u}} \times \mathcal{V}_{\boldsymbol{p}}$

$$\begin{aligned}\int_{\Omega^{\mathrm{f}(t)}} \rho^{\mathrm{f}} \left( \left.\frac{\partial \boldsymbol{u}^{\mathrm{f}}}{\partial t}\right|_{\chi} + (\boldsymbol{u}^{\mathrm{f}} - \boldsymbol{w}) \cdot \nabla \boldsymbol{u}^{\mathrm{f}} \right) \cdot \phi^{\mathrm{f}}(\boldsymbol{x}) d\boldsymbol{x} + \int_{\Omega^{\mathrm{f}(t)}} \boldsymbol{\sigma}^{\mathrm{f}} : \nabla \phi^{\mathrm{f}} d\boldsymbol{x} \\ - \int_{\Omega^{\mathrm{f}}(t)} \nabla \cdot \boldsymbol{u}^{\mathrm{f}} q d\boldsymbol{x} \\ + \int_{\Omega^{\mathrm{s}}} \rho^{\mathrm{s}} \frac{\partial \boldsymbol{u}^{\mathrm{s}}}{\partial t} \cdot \phi^{\mathrm{s}} d\boldsymbol{Z} + \int_{\Omega^{\mathrm{s}}} \boldsymbol{\sigma}^{\mathrm{s}} : \nabla \phi^{\mathrm{s}} d\boldsymbol{Z} = \\ \int_{\Omega^{\mathrm{f}(t)}} \boldsymbol{f}^{\mathrm{f}} \cdot \phi^{\mathrm{f}} d\boldsymbol{x} + \int_{\Gamma^{\mathrm{f}}_{\mathrm{H}}} \boldsymbol{\sigma}^{\mathrm{f}}_{\mathrm{H}} \cdot \phi^{\mathrm{f}} d\Gamma + \int_{\Omega^{\mathrm{s}}} \boldsymbol{f}^{\mathrm{s}} \cdot \phi^{\mathrm{s}} d\boldsymbol{Z} + \int_{\Gamma^{\mathrm{s}}_{\mathrm{H}}} \boldsymbol{\sigma}^{\mathrm{s}}_{\mathrm{H}} \cdot \phi^{\mathrm{s}} d\Gamma. \end{aligned} \tag{20}$$

The idea here is to solve the discrete fluid and structural domains as a single unique domain $\Omega = \Omega^{\mathrm{f}} \cup \Omega^{\mathrm{s}}$. In the above form, the velocity and displacement continuity conditions are satisfied implicitly. While the traction continuity condition is absorbed into the weak formulation.

## 4 Quasi-Monolithic Formulation

In this section, we will present a second-order time discretization of the combined fluid-structure formulation given in Sect. 3. The explicit construction of the interface at the start of each time step decouples the solid position and fluid mesh motion from the computation of fluid-structure variables $(\boldsymbol{u}^{\mathrm{f}}, p, \boldsymbol{u}^{\mathrm{s}})$. Additionally, the decoupling of the fluid mesh motion enables us to determine the convective velocity of the fluid flow explicitly and linearize the nonlinear Navier–Stokes relation. Hence, the quasi-monolithic formulation does not require nonlinear iteration per time step.

### *4.1 Second Order in Time Discretization*

Let $\mathbb{P}_2(\Omega_{\mathrm{h}})$ denote the standard second-order Lagrange finite element space on domain $\Omega_{\mathrm{h}} = \Omega^{\mathrm{f}}_{\mathrm{h}} \cup \Omega^{\mathrm{s}}_{\mathrm{h}}$. First, we employ the second-order extrapolation to describe the displacement vector $\boldsymbol{\eta}^{\mathrm{s,n}}_{\mathrm{h}}$ of the flexible structural as

$$\boldsymbol{\eta}^{\mathrm{s,n}}_{\mathrm{h}}(\boldsymbol{Z}_{\mathrm{i}}) = \boldsymbol{\eta}^{\mathrm{s,n-1}}_{\mathrm{h}}(\boldsymbol{Z}_{\mathrm{i}}) + \frac{3\Delta t}{2} \boldsymbol{u}^{\mathrm{s,n-1}}_{\mathrm{h}}(\boldsymbol{Z}_{\mathrm{i}}) - \frac{\Delta t}{2} \boldsymbol{u}^{\mathrm{s,n-2}}_{\mathrm{h}}(\boldsymbol{Z}_{\mathrm{i}}) \qquad \forall \boldsymbol{Z}_i \in \mathcal{T}^{\mathrm{s}}_{\mathrm{h}}. \tag{21}$$

Now that we have both the boundary conditions Eqs. (14) and (15) required for solving the ALE Eq. (13), we solve Eq. (13) employing $\mathbb{P}_1$ finite element space. The edges of an isoparametric element are assumed straight unless they are on the interface or on a curved boundaries. This assumption enables us to use $\mathbb{P}_1$ finite element instead of the $\mathbb{P}_2$ for updating the mesh. As a result of this, size of the system of linear equations required for solving the fluid mesh displacement, $\boldsymbol{\eta}_{\mathrm{h}}^{\mathrm{f,n}}$, on finite element space with $\mathbb{P}_1$ discretization would be smaller than the system of linear equations required for solving the $\mathbb{P}_2$ discretization space without losing the accuracy of the coupled fluid-structure solver.

We now use the solution of $\boldsymbol{\eta}_{\mathrm{h}}^{\mathrm{f,n}}$ computed on the $\mathbb{P}_1$ finite element space to update the location of triangular mesh $\mathcal{T}_{\mathrm{h},t^{\mathrm{n}}}^{\mathrm{f}}$ vertices. Since the interior edges are straight, we can position the non-vertex computational node at the center of the edge. In this way, we are able to determine the fluid mesh displacement for all the $\mathbb{P}_2$ finite element mesh $\mathcal{T}_{\mathrm{h},t^{\mathrm{n}}}^{\mathrm{f}}$ computational nodes even by solving the ALE Eq. (13) on a $\mathbb{P}_1$ finite element mesh.

The nonlinear convective term can be linearized by defining a second-order time-accurate extrapolation function given below:

$$\check{\boldsymbol{u}}_{\mathrm{h}}^{\mathrm{f}}(\boldsymbol{\Psi}_{\mathrm{h}}^{\mathrm{n}}(\boldsymbol{x},t^{\mathrm{n}})) = 2\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n-1}}(\boldsymbol{\Psi}_{\mathrm{h}}^{\mathrm{n}}(\boldsymbol{x},t^{\mathrm{n-1}})) - \boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n-2}}(\boldsymbol{\Psi}_{\mathrm{h}}^{\mathrm{n}}(\boldsymbol{x},t^{\mathrm{n-2}})), \tag{22}$$

where $\boldsymbol{\Psi}_{\mathrm{h}}^{\mathrm{n}}(\cdot,t^{\mathrm{n-j}})$ is the backward mapping function for the spatial grid points on the mesh $\mathcal{T}_{\mathrm{h},t^{\mathrm{n}}}^{\mathrm{f}}$ to $\mathcal{T}_{\mathrm{h},t^{\mathrm{n-j}}}^{\mathrm{f}}$ and mesh velocity $\boldsymbol{w}_{\mathrm{h}}^{\mathrm{n}}(\boldsymbol{x})$ is defined as

$$\begin{aligned}\boldsymbol{w}_{\mathrm{h}}^{\mathrm{n}}(\boldsymbol{x}) &= \sum_{\mathrm{i}=1}^{G}\phi_{\mathrm{i}}^{\mathrm{f,n}}(\boldsymbol{x})\frac{1}{\Delta t}\left(\left(\boldsymbol{x}_{\mathrm{i}}^{\mathrm{n}}-\boldsymbol{x}_{\mathrm{i}}^{\mathrm{n-1}}\right)+\frac{1}{2}\left(\boldsymbol{x}_{\mathrm{i}}^{\mathrm{n-1}}-\boldsymbol{x}_{\mathrm{i}}^{\mathrm{n-2}}\right)-\frac{1}{2}\left(\boldsymbol{x}_{\mathrm{i}}^{\mathrm{n-2}}-\boldsymbol{x}_{\mathrm{i}}^{\mathrm{n-3}}\right)\right)\\ &= \sum_{\mathrm{i}=1}^{G}\phi_{i}^{\mathrm{f,n}}(\boldsymbol{x})\frac{1}{\Delta t}\left(\boldsymbol{x}_{\mathrm{i}}^{\mathrm{n}}-\frac{1}{2}\boldsymbol{x}_{\mathrm{i}}^{\mathrm{n-1}}-\boldsymbol{x}_{\mathrm{i}}^{\mathrm{n-2}}+\frac{1}{2}\boldsymbol{x}_{\mathrm{i}}^{\mathrm{n-3}}\right).\end{aligned} \tag{23}$$

Here $\boldsymbol{w}_{\mathrm{h}}^{\mathrm{n}}$ is a second-order approximation of the fluid mesh velocity $\partial_t\boldsymbol{\Psi}_{\mathrm{h}}^{\mathrm{n}}(\boldsymbol{x},t^{\mathrm{n}})$

$$f'(t^n) = \frac{1}{\Delta t}\left(f(t^n)-\frac{1}{2}f(t^{n-1})-f(t^{n-2})+\frac{1}{2}f(t^{n-3})\right)+O(\Delta t^2).$$

We next show that

$$\boldsymbol{w}_{\mathrm{h}}^{\mathrm{n}}(\boldsymbol{x}_{\mathrm{j}}^{\mathrm{n}}) = \check{\boldsymbol{u}}_{\mathrm{h}}^{\mathrm{f}}(\boldsymbol{x}_{\mathrm{j}}^{\mathrm{n}}) \quad \text{on} \quad \Gamma_{\mathrm{h},t^{\mathrm{n}}}. \tag{24}$$

To prove the above equation, we rewrite Eq. (23) as

$$\begin{aligned}\boldsymbol{w}_{\mathrm{h}}^{\mathrm{n}}(\boldsymbol{x}_{\mathrm{j}}^{\mathrm{n}}) &= \frac{1}{\Delta t}\left(\left(\boldsymbol{x}_{\mathrm{j}}^{\mathrm{n}}-\boldsymbol{x}_{\mathrm{j}}^{\mathrm{n-1}}\right)+\frac{1}{2}\left(\boldsymbol{x}_{\mathrm{j}}^{\mathrm{n-1}}-\boldsymbol{x}_{\mathrm{j}}^{\mathrm{n-2}}\right)-\frac{1}{2}\left(\boldsymbol{x}_{\mathrm{j}}^{\mathrm{n-2}}-\boldsymbol{x}_{\mathrm{j}}^{\mathrm{n-3}}\right)\right)\\ &= \frac{\boldsymbol{\varphi}_{\mathrm{h}}^{\mathrm{s,n}}(\boldsymbol{Z}_{\mathrm{j}})-\boldsymbol{\varphi}_{\mathrm{h}}^{\mathrm{s,n-1}}(\boldsymbol{Z}_{\mathrm{j}})}{\Delta t}+\frac{\boldsymbol{\varphi}_{\mathrm{h}}^{\mathrm{s,n-1}}(\boldsymbol{Z}_{\mathrm{j}})-\boldsymbol{\varphi}_{\mathrm{h}}^{\mathrm{s,n-2}}(\boldsymbol{Z}_{\mathrm{j}})}{2\Delta t}-\frac{\boldsymbol{\varphi}_{\mathrm{h}}^{\mathrm{s,n-2}}(\boldsymbol{Z}_{\mathrm{j}})-\boldsymbol{\varphi}_{\mathrm{h}}^{\mathrm{s,n-3}}(\boldsymbol{Z}_{\mathrm{j}})}{2\Delta t}\ \forall\ \boldsymbol{Z}_{\mathrm{j}}\in\Gamma_{\mathrm{h},t^{\mathrm{n}}}.\end{aligned} \tag{25}$$

By substituting the definition of $\boldsymbol{\varphi}^{\mathrm{s,n}}(\boldsymbol{Z})$ from Eq. (21) into Eq. (25) and simplifying will yield

$$\begin{aligned}\boldsymbol{w}_{\mathrm{h}}^{\mathrm{n}}(\boldsymbol{x}_{\mathrm{j}}^{\mathrm{n}}) =& 2\left(\frac{3}{4}\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n-1}}(\boldsymbol{Z}_{\mathrm{j}}) + \frac{1}{2}\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n-2}}(\boldsymbol{Z}_{\mathrm{j}}) - \frac{1}{4}\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n-3}}(\boldsymbol{Z}_{\mathrm{j}})\right) \\ &- \left(\frac{3}{4}\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n-2}}(\boldsymbol{Z}_{\mathrm{j}}) + \frac{1}{2}\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n-3}}(\boldsymbol{Z}_{\mathrm{j}}) - \frac{1}{4}\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n-4}}(\boldsymbol{Z}_{\mathrm{j}})\right).\end{aligned}$$

## 4.2 Complete Scheme

In this subsection, we present the fully discretized finite element form of the quasi-monolithic formulation. The variational statement reads as

$$\text{find}\quad (\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}}, p_{\mathrm{h}}^{\mathrm{f,n}}, \boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n}}) \in V_{\mathrm{h}}\left(t^{\mathrm{n}}, \boldsymbol{\varphi}_{\mathrm{h}}^{\mathrm{s,n}}, \frac{3}{4}, \frac{1}{2}\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n-1}} - \frac{1}{4}\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n-2}}\right)$$

with $\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}}|_{\Gamma_{\mathrm{D}}^{\mathrm{f}}} = \boldsymbol{u}_{\mathrm{D}}^{\mathrm{f}}$ and $\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n}}|_{\Gamma_{\mathrm{D}}^{\mathrm{s}}} = \boldsymbol{u}_{\mathrm{D}}^{\mathrm{s}}$ so that for any finite element triple

$$(\boldsymbol{\phi}^{\mathrm{f}}, q^{\mathrm{f}}, \boldsymbol{\phi}^{\mathrm{s}}) \in V_{\mathrm{h}}(t^{\mathrm{n}}, \boldsymbol{\varphi}_{\mathrm{h}}^{\mathrm{s,n}}, 1, \mathbf{0}) \tag{26}$$

with $\boldsymbol{\phi}_{\mathrm{h}}^{\mathrm{f}}|_{\Gamma_{\mathrm{D}}^{\mathrm{f}}} = 0$ and $\boldsymbol{\phi}_{\mathrm{h}}^{\mathrm{s}}|_{\Gamma_{\mathrm{D}}^{\mathrm{s}}} = 0$, such that

$$\begin{aligned}&\int_{\Omega_{\mathrm{h},t^{\mathrm{n}}}^{\mathrm{f}}} \Bigg[\frac{\rho^{\mathrm{f}}}{\Delta t}\left(\frac{3}{2}\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}}(\boldsymbol{x}) - 2\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n-1}}(\boldsymbol{\Psi}_{\mathrm{h}}^{n}(\boldsymbol{x}, t^{\mathrm{n-1}})) + \frac{1}{2}\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n-2}}(\boldsymbol{\Psi}_{\mathrm{h}}^{n}(\boldsymbol{x}, t^{\mathrm{n-2}}))\right)\Bigg. \\ &\qquad + \left(\check{\boldsymbol{u}}_{\mathrm{h}}^{\mathrm{f}} - \boldsymbol{w}_{\mathrm{h}}^{\mathrm{n}}\right)\cdot\nabla\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}} + \frac{1}{2}\left(\nabla\check{\boldsymbol{u}}_{\mathrm{h}}^{\mathrm{f}}\right)\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}}\Bigg]\cdot\boldsymbol{\phi}^{\mathrm{f}}d\boldsymbol{x}\Bigg\} A \\ &\qquad + \int_{\Omega_{\mathrm{h},t^{\mathrm{n}}}^{\mathrm{f}}} \rho^{\mathrm{f}}\nu^{\mathrm{f}}\left(\nabla\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}} + (\nabla\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}})^{T}\right) : \nabla\boldsymbol{\phi}^{\mathrm{f}}d\boldsymbol{x}\Bigg\} \\ &\qquad - \int_{\Omega_{\mathrm{h},t^{\mathrm{n}}}^{\mathrm{f}}} p_{\mathrm{h}}^{\mathrm{f,n}}(\nabla\cdot\boldsymbol{\phi}^{\mathrm{f}})d\boldsymbol{x}\Bigg\} B \\ &\qquad - \int_{\Omega_{\mathrm{h},t^{\mathrm{n}}}^{\mathrm{f}}} q^{\mathrm{f}}(\nabla\cdot\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}})d\boldsymbol{x}\Bigg\} C \\ &\qquad + \int_{\Omega_{\mathrm{h}}^{\mathrm{s}}} \frac{\rho^{\mathrm{s}}}{\Delta t}\left(\frac{3}{2}\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n}} - 2\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n-1}} + \frac{1}{2}\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n-2}}\right)\cdot\boldsymbol{\phi}^{\mathrm{s}}d\boldsymbol{Z}\Bigg\} D \\ &\qquad + \frac{1}{2}\int_{\Omega_{\mathrm{h}}^{\mathrm{s}}} \left(\boldsymbol{\sigma}^{\mathrm{s}}(\boldsymbol{\varphi}_{\mathrm{h}}^{\mathrm{s,n-1}}) + \boldsymbol{\sigma}^{\mathrm{s}}(\boldsymbol{\varphi}_{\mathrm{h}}^{\mathrm{s,n+1}})\right) : \nabla\boldsymbol{\phi}^{\mathrm{s}}d\boldsymbol{Z}\Bigg\} \\ &= \int_{\Omega_{\mathrm{h},t^{\mathrm{n}}}^{\mathrm{f}}} \boldsymbol{f}^{\mathrm{f}}\cdot\boldsymbol{\phi}^{\mathrm{f}}d\Omega + \int_{(\Gamma_{\mathrm{H}}^{\mathrm{f}})_{\mathrm{h}}} \boldsymbol{\sigma}_{\mathrm{H}}^{\mathrm{f}}\cdot\boldsymbol{\phi}^{\mathrm{f}}d\Gamma + \int_{\Omega_{\mathrm{h}}^{\mathrm{s}}} \boldsymbol{f}^{\mathrm{s}}\cdot\boldsymbol{\phi}^{\mathrm{s}}d\Omega + \int_{(\Gamma_{\mathrm{H}}^{\mathrm{s}})_{\mathrm{h}}} \boldsymbol{\sigma}_{\mathrm{H}}^{\mathrm{s}}\cdot\boldsymbol{\phi}^{\mathrm{f}}d\Gamma,\Bigg\} E,\end{aligned} \tag{27}$$

where $A$ and $B$ contain the fluid velocity and pressure terms from the Navier–Stokes momentum equation, $C$ is the Navier–Stokes continuity terms, $D$ denotes Navier's equation for structural dynamics, and $E$ is the right-hand side part of the combined fluid-structure system consisting of the boundary condition and external body force terms. The $\frac{1}{2}\left(\nabla \check{\boldsymbol{u}}_{\mathrm{h}}^{\mathrm{f}}\right) \boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}}$ term in part A of Eq. (27) to stabilize the convective term is introduced and discussed in detail in [22].

### 4.3 Algorithm

The foregoing variational formulation can be expressed in the form of an algorithm. To begin with, the details of the initial setup are as follows: there is a mesh $\mathcal{T}_{\mathrm{h}}^{\mathrm{s}}$ for the solid reference domain $\Omega^{\mathrm{s}}$ which shares a part of its boundary grid points with $\mathcal{T}_{\Gamma}^{\mathrm{h}}$ along $\Gamma$. Assuming $\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n-1}}$, $\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n-1}}$ and $\boldsymbol{\eta}_{\mathrm{h}}^{\mathrm{s}}$ are known for the mesh $\mathcal{T}_{\mathrm{h,t^n-1}}^{\mathrm{f}}$ which is defined on the domain $\Omega_{\mathrm{h},t^{\mathrm{n}}-1}^{\mathrm{f}}$. Here $\Omega_{\mathrm{h},t^{\mathrm{n}}-1}^{\mathrm{f}}$ denotes the numerical approximation for the fluid domain at time $t^{n-1}$. It should be noted that we have considered $\mathbb{P}_m/\mathbb{P}_{m-1}/\mathbb{P}_m$ elements. To ensure the optimal rate of approximation on an isoparametric finite element mesh, all the constituent elements are considered as straight-edged standard Lagrangian elements. The basic steps to be performed in the quasi-monolithic combined fluid-structure formulation are summarized below:

**Algorithm 1:** Second-order quasi-monolithic formulation for fluid-structure interactions

1. Start with known $\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n-1}}$, $\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n-1}}$,$\boldsymbol{\eta}_{\mathrm{h}}^{\mathrm{s}}$ at time $t^{\mathrm{n-1}}$ and $t^{\mathrm{n-2}}$
2. Advance from $t^{\mathrm{n-1}}$ to $t^{\mathrm{n}}$
   (a) Determine the structural displacements $\boldsymbol{\eta}_{\mathrm{h}}^{\mathrm{s}}$ using Eq. (21)
   (b) Solve Eq. (13) using $P_1$ elements on $\Omega_{\mathrm{h},t^0}^{\mathrm{f}}$ to determine mesh displacements
   (c) Update the fluid mesh $\Omega_{\mathrm{h},t^{\mathrm{n}}}^{\mathrm{f}}$ using the mesh displacements from (b)
   (d) Evaluate $\check{\boldsymbol{u}}_{\mathrm{h}}^{\mathrm{f}}$ and $\boldsymbol{w}_{\mathrm{h}}^{\mathrm{n}}$ by Eqs. (22) and (23)
   (e) Solve for the updated field properties $\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}}$, $p_{\mathrm{h}}^{\mathrm{n}}$, and $\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n}}$ at current time $t^{\mathrm{n}}$ using Eq. (27)

## 5 Fully Stabilized Quasi-Monolithic Formulation

One of the primary limitations of the Galerkin finite element discretization used for discretizing $\Omega^{\mathrm{f}}$ and $\Omega^{\mathrm{s}}$ of the quasi-monolithic formulation presented in Sect. 4 is that it will experience nonphysical spurious oscillations for convection-dominant problems [23]. Traditionally, these spurious oscillations are circumvented by replacing the traditional Galerkin method with Petrov-Galerkin methods which utilize weight-

ing functions that have more weightage for the upstream part of the flow than the downstream [24, 25]. Such streamwise upwind techniques can be interpreted as a combination of traditional Galerkin method and a stabilization term calculated at the interior of an element. This elemental-level stabilization term introduces artificial numerical diffusion which stabilizes the spurious oscillations. The weak form of the combined fluid-structure formulation given in Eq. (20) can be written in the Galerkin/Least square (GLS) stabilization form as

$$\left.\begin{aligned}\int_{\Omega_{\mathrm{h}}^{\mathrm{f}}(t)} \rho^{\mathrm{f}}\left(\partial_t \boldsymbol{u}^{\mathrm{f}} + \left(\hat{\boldsymbol{u}}^{\mathrm{f}} - \boldsymbol{w}\right)\cdot\nabla\boldsymbol{u}^{\mathrm{f}}\right)\cdot\boldsymbol{\phi}^{\mathrm{f}}\mathrm{d}\Omega + \int_{\Omega^{\mathrm{f}}(t)} \boldsymbol{\sigma}^{\mathrm{f}} : \nabla\boldsymbol{\phi}^{\mathrm{f}}\mathrm{d}\Omega \\ - \int_{\Omega^{\mathrm{f}}(t)} \nabla\cdot\boldsymbol{u}^{\mathrm{f}} q\,\mathrm{d}\Omega\end{aligned}\right\} A$$

$$\left.\begin{aligned}+\sum_{e=1}^{n_{\mathrm{el}}}\int_{\Omega^{\mathrm{e}}} \tau_m\left[\rho^{\mathrm{f}}\left(\boldsymbol{u}^{\mathrm{f}} - \boldsymbol{w}\right)\cdot\nabla\boldsymbol{\phi}^{\mathrm{f}} + \nabla q\right]\cdot \\ \left[\rho^{\mathrm{f}}\partial_t\boldsymbol{u}^{\mathrm{f}} + \rho^{\mathrm{f}}\left(\boldsymbol{u}^{\mathrm{f}} - \boldsymbol{w}\right)\cdot\nabla\boldsymbol{u}^{\mathrm{f}} - \nabla\cdot\boldsymbol{\sigma}^{\mathrm{f}} - \boldsymbol{f}^{\mathrm{f}}\right]d\Omega^{\mathrm{e}}\end{aligned}\right\} B$$

$$\left.+\sum_{e=1}^{n_{\mathrm{el}}}\int_{\Omega^{\mathrm{e}}} \nabla\cdot\boldsymbol{\phi}^{\mathrm{f}}\tau_c\nabla\cdot\boldsymbol{u}^{\mathrm{f}}d\Omega^{\mathrm{e}}\right\} C$$

$$\left.+\int_{\Omega^{\mathrm{s}}} \rho^{\mathrm{s}}\partial_t\boldsymbol{u}^{\mathrm{s}}\cdot\boldsymbol{\phi}^{\mathrm{s}}\mathrm{d}\Omega + \int_{\Omega^{\mathrm{s}}}\boldsymbol{\sigma}^{\mathrm{s}} : \nabla\boldsymbol{\phi}^{\mathrm{s}}\mathrm{d}\Omega =\right\} D$$

$$\left.\int_{\Omega^{\mathrm{f}}_{\mathrm{h},t^n}} \boldsymbol{f}^{\mathrm{f}}\cdot\boldsymbol{\phi}^{\mathrm{f}}d\Omega + \int_{(\Gamma^{\mathrm{f}}_{\mathrm{H}})_{\mathrm{h}}}\boldsymbol{\sigma}^{\mathrm{f}}_{\mathrm{H}}\cdot\boldsymbol{\phi}^{\mathrm{f}}d\Gamma + \int_{\Omega^{\mathrm{s}}_{\mathrm{h}}}\boldsymbol{f}^{\mathrm{s}}\cdot\boldsymbol{\phi}^{\mathrm{s}}d\Omega + \int_{(\Gamma^{\mathrm{s}}_{\mathrm{H}})_{\mathrm{h}}}\boldsymbol{\sigma}^{\mathrm{s}}_{\mathrm{H}}\cdot\boldsymbol{\phi}^{\mathrm{f}}d\Gamma.\right\} E \tag{28}$$

One can observe that terms $A$, $D$, and $E$ combine to form the Galerkin weak form presented in Eq. (20). On the other hand, the term $B$ represents the GLS terms for the convective and pressure to suppress the spurious oscillations for the convection-dominant problems and to circumvent the inf-sup/LBB condition, respectively. Term C denotes the stabilization term for the incompressibility constraint to provide additional stability. Unlike $\mathbb{P}_2/\mathbb{P}_1/\mathbb{P}_2$ finite element discretization for the fluid velocity, pressure and structural velocity to justify the well-posedness. The above stabilized combined fluid-structure weak form in Eq. (28) is discretized using equal order elements for both fluid velocity and pressure to simplify the computational framework significantly.

The stabilization parameters $\tau_m$ and $\tau_c$ in the term $B$ represent the variational stabilization factors for the momentum and continuity equations [24, 26–28]. The stabilization parameter $\tau_m$ for the momentum equation is defined as [29]

$$\tau_m = \left[\left(\frac{2\rho^{\mathrm{f}}}{\Delta t}\right)^2 + \left(\rho^{\mathrm{f}}\right)^2\left(\boldsymbol{u}^{\mathrm{f}} - \boldsymbol{w}\right)\cdot\boldsymbol{G}\cdot\left(\boldsymbol{u}^{\mathrm{f}} - \boldsymbol{w}\right) + 12(\mu^{\mathrm{f}})^2\boldsymbol{G} : \boldsymbol{G}\right]^{-\frac{1}{2}}, \tag{29}$$

where $\boldsymbol{G}$ is the elemental contravariant metric tensor which is defined as

$$\boldsymbol{G} = \left(\frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}}\right)^T \frac{\partial \boldsymbol{\xi}}{\partial \boldsymbol{x}}, \tag{30}$$

where $\boldsymbol{\xi}$ is local element-level coordinate system and it depends on the element shape. $\tau_m$ in Eq. 29 consists of three parts, the first term represents the stabilization for the temporal dominant, second for advection dominant, and the last for diffusion-dominated cases. The stabilization factors are generally developed using the variational multiscale approach, where the finite element space is decomposed into coarse resolvable scales and fine non-resolvable scales. Therefore, the equation for non-resolvable scales forms the equation for error and the solution of this equation is approximated as the average of appropriate small-scale Green's function. This solution of the fine scale is used for determining the stabilization factors $\tau_m$ and $\tau_c$. For more detailed mathematical treatment refer to [30] and [31]. The stabilization parameter $\tau_c$ for the continuity equation is defined as

$$\tau_c = \frac{1}{8 \text{ tr } (\boldsymbol{G})\, \tau_m}. \tag{31}$$

The fully discretized quasi-monolithic fluid-structure formulation for multiple structures using BDF2 can be written as

$$\begin{aligned}
&\int_{\Omega^{\mathrm{f}}_{\mathrm{h},t^{\mathrm{n}}}} \left[ \frac{\rho^{\mathrm{f}}}{\Delta t} \left( \frac{3}{2}\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}}(\boldsymbol{x}) - 2\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n-1}}(\boldsymbol{\Psi}_{\mathrm{h}}^{n}(\boldsymbol{x}, t^{\mathrm{n-1}})) + \frac{1}{2}\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n-2}}(\boldsymbol{\Psi}_{\mathrm{h}}^{n}(\boldsymbol{x}, t^{\mathrm{n-2}})) \right) \right. \\
&\qquad \left. + \left(\check{\boldsymbol{u}}_{\mathrm{h}}^{\mathrm{f}} - \boldsymbol{w}_{\mathrm{h}}^{\mathrm{n}}\right) \cdot \nabla \boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}} + \frac{1}{2}\left(\nabla \check{\boldsymbol{u}}_{\mathrm{h}}^{\mathrm{f}}\right) \boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}} \right] \cdot \boldsymbol{\phi}^{\mathrm{f}} d\boldsymbol{x} \\
&\qquad + \int_{\Omega^{\mathrm{f}}_{\mathrm{h},t^{\mathrm{n}}}} \rho^{\mathrm{f}} \nu^{\mathrm{f}} \left( \nabla \boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}} + (\nabla \boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}})^T \right) : \nabla \boldsymbol{\phi}^{\mathrm{f}} d\boldsymbol{x} - \int_{\Omega^{\mathrm{f}}_{\mathrm{h},t^{\mathrm{n}}}} p_{\mathrm{h}}^{\mathrm{f,n}} (\nabla \cdot \boldsymbol{\phi}^{\mathrm{f}}) d\boldsymbol{x} \\
&\qquad - \int_{\Omega^{\mathrm{f}}_{\mathrm{h},t^{\mathrm{n}}}} q^{\mathrm{f}} (\nabla \cdot \boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}}) d\boldsymbol{x} \\
&\qquad + \sum_{e=1}^{n_{\mathrm{el}}} \int_{\Omega^{\mathrm{e}}_{\mathrm{h}}} \tau_m \left[ \rho^{\mathrm{f}} \left( \check{\boldsymbol{u}}_{\mathrm{h}}^{\mathrm{f}} - \boldsymbol{w}_{\mathrm{h}}^{\mathrm{n}} \right) \cdot \nabla \boldsymbol{\phi}^{\mathrm{f}} + \nabla q \right] \cdot \\
&\left[ \frac{\rho^{\mathrm{f}}}{\Delta t} \left( 1.5\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}} - 2\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n-1}} + 0.5\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n-2}} \right) + \rho^{\mathrm{f}} \left( \check{\boldsymbol{u}}_{\mathrm{h}}^{\mathrm{f}} - \boldsymbol{w}_{\mathrm{h}}^{\mathrm{n}} \right) \cdot \nabla \boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}} - \nabla \cdot \boldsymbol{\sigma}^{\mathrm{f,n_h}} - \boldsymbol{f}^{\mathrm{f}} \right] d\Omega^{\mathrm{e}} \\
&\qquad + \sum_{e=1}^{n_{\mathrm{el}}} \int_{\Omega^{\mathrm{e}}_{\mathrm{h}}} \nabla \cdot \boldsymbol{\phi}^{\mathrm{f}} \tau_c \nabla \cdot \boldsymbol{u}_{h}^{\mathrm{f,n}} d\Omega^{\mathrm{e}} \\
&\qquad + \int_{\Omega^{\mathrm{s}}_{\mathrm{h}}} \frac{\rho^{\mathrm{s}}}{\Delta t} \left( \frac{3}{2}\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n}} - 2\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n-1}} + \frac{1}{2}\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n-2}} \right) \cdot \boldsymbol{\phi}^{\mathrm{s}} d\boldsymbol{Z} \\
&\qquad + \frac{1}{2} \int_{\Omega^{\mathrm{s}}_{\mathrm{h}}} \left( \boldsymbol{\sigma}^{\mathrm{s}}(\boldsymbol{\varphi}_{\mathrm{h}}^{\mathrm{s,n-1}}) + \boldsymbol{\sigma}^{\mathrm{s}}(\boldsymbol{\varphi}_{\mathrm{h}}^{\mathrm{s,n+1}}) \right) : \nabla \boldsymbol{\phi}^{\mathrm{s}} d\boldsymbol{Z}
\end{aligned}$$

$$= \int_{\Omega^{\mathrm{f}}_{\mathrm{h},t^{\mathrm{n}}}} \boldsymbol{f}^{\mathrm{f}} \cdot \boldsymbol{\phi}^{\mathrm{f}} d\Omega + \int_{(\Gamma^{\mathrm{f}}_{\mathrm{H}})_{\mathrm{h}}} \boldsymbol{\sigma}^{\mathrm{f}}_{\mathrm{H}} \cdot \boldsymbol{\phi}^{\mathrm{f}} d\Gamma + \int_{\Omega^{\mathrm{s}}_{\mathrm{h}}} \boldsymbol{f}^{\mathrm{s}} \cdot \boldsymbol{\phi}^{\mathrm{s}} d\Omega + \int_{(\Gamma^{\mathrm{s}}_{\mathrm{H}})_{\mathrm{h}}} \boldsymbol{\sigma}^{\mathrm{s}}_{\mathrm{H}} \cdot \boldsymbol{\phi}^{\mathrm{f}} d\Gamma. \tag{32}$$

The implementation of the above fully stabilized quasi-monolithic combined fluid-structure formulation differs slightly from the implementation in Sect. 4. Instead of Eq. (22) we define an alternative second-order time-accurate explicit function given by

$$\begin{aligned} \check{\boldsymbol{u}}^{\mathrm{f}}_{\mathrm{h}}(\boldsymbol{\Psi}^{\mathrm{n}}_{\mathrm{h}}(\boldsymbol{x}, t^{\mathrm{n}})) = 2.25\boldsymbol{u}^{\mathrm{f},\mathrm{n}-1}_{\mathrm{h}}(\boldsymbol{\Psi}^{\mathrm{n}}_{\mathrm{h}}(\boldsymbol{x}, t^{\mathrm{n}-1})) - 1.5\boldsymbol{u}^{\mathrm{f},\mathrm{n}-2}_{\mathrm{h}}(\boldsymbol{\Psi}^{\mathrm{n}}_{\mathrm{h}}(\boldsymbol{x}, t^{\mathrm{n}-2})) \\ + 0.25\boldsymbol{u}^{\mathrm{f},\mathrm{n}-3}_{\mathrm{h}}(\boldsymbol{\Psi}^{\mathrm{n}}_{\mathrm{h}}(\boldsymbol{x}, t^{\mathrm{n}-3})). \end{aligned} \tag{33}$$

Similarly, we also define an alternate function for $\boldsymbol{w}^{\mathrm{n}}_{\mathrm{h}}$ as

$$\boldsymbol{w}^{\mathrm{n}}_{\mathrm{h}}(\boldsymbol{x}) = \sum_{\mathrm{i}=1}^{G} \phi^{\mathrm{f},\mathrm{n}}_{i}(\boldsymbol{x}) \frac{1}{\Delta t} \left( \frac{3}{2}\boldsymbol{x}^{\mathrm{n}}_{\mathrm{i}} - 2\boldsymbol{x}^{\mathrm{n}-1}_{\mathrm{i}} + \frac{1}{2}\boldsymbol{x}^{\mathrm{n}-2}_{\mathrm{i}} \right). \tag{34}$$

The main reason behind redefining $\check{\boldsymbol{u}}^{\mathrm{f}}_{\mathrm{h}}$ and $\boldsymbol{w}_{\mathrm{h}}$ is that Eqs. (33)–(34) enable us to implement the exact interface continuity, i.e.,

$$\boldsymbol{u}^{\mathrm{f},\mathrm{n}}_{\mathrm{h}} = \boldsymbol{u}^{\mathrm{s},\mathrm{n}}_{\mathrm{h}} \tag{35}$$

instead of the second-order approximation in time given by Eq. (22). Unlike the velocity continuity in Eq. (11) which requires us to enforce the condition explicitly, we can satisfy the velocity continuity in Eq. (35) implicitly by treating the fluid and its corresponding solid node on the interface as a single unique node. Thereby, we can decrease the size of the algebraic system of equations required per time step compared to the implementation presented in Sect. 4.

## 5.1 Algorithm

Unlike the quasi-monolithic combined fluid-structure formulation in Sect. 4 where we have considered $\mathbb{P}_m/\mathbb{P}_{m-1}/\mathbb{P}_m$ elements to satisfy the inf-sup or LBB condition, here we use equal order elements for both fluid pressure and velocity. The basic steps to be performed in the fully stabilized quasi-monolithic combined fluid-structure formulation are summarized below:

Similar to the quasi-monolithic formulation presented in Sect. 4, the fully stabilized quasi-monolithic formulation also solves the combined fluid-structure system only once per time step. A matrix-free version of Krylov subspace-based iterative solvers is utilized to solve the system of equations that arise from both pseudo-elastic

**Algorithm 2:** Second-order fully stabilized quasi-monolithic formulation for fluid-structure interactions

1. Start with known solutions $\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n-1}}$, $\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n-1}}$,$\boldsymbol{\eta}_{\mathrm{h}}^{\mathrm{s,n-1}}$ at times $t^{\mathrm{n-1}}$ and $t^{\mathrm{n-1}}$
2. Advance from $t^{\mathrm{n-1}}$ to $t^{\mathrm{n}}$
   (a) Determine the structural displacements $\boldsymbol{\eta}_{\mathrm{h}}^{\mathrm{s,n}}$ using Eq. (21)
   (b) Solve Eq. (13) on $\Omega_{\mathrm{h},t^0}^{\mathrm{f}}$ to determine mesh displacements
   (c) Update the fluid mesh $\Omega_{\mathrm{h},t^{\mathrm{n}}}^{\mathrm{f}}$ using the mesh displacement from (b)
   (d) Evaluate $\check{\boldsymbol{u}}_{\mathrm{h}}^{\mathrm{f}}$ and $\boldsymbol{w}_{\mathrm{h}}^{\mathrm{n}}$ by Eqs. (33) and (34)
   (e) Determine the element level stabilization parameters $\tau_m$ and $\tau_c$ using Eqs. (29) and (31) respectively.
   (f) Solve for the updated field properties $\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}}$, $p_{\mathrm{h}}^{\mathrm{n}}$, and $\boldsymbol{u}_{\mathrm{h}}^{\mathrm{s,n}}$ at current time $t^{\mathrm{n}}$ using Eq. (32)

mesh motion and combined fluid-structure equations. To scale the fluid-structure solver for large-scale computations using distributed memory parallel cluster, we next present the parallel finite element implementation of three-dimensional incompressible flow interacting with generic elastic structures for high $Re$ flow.

## 6 Conclusions

In this work, we have discussed about two different ALE-based improvised monolithics, i.e., quasi-monolithic, FSI formulations that are computationally efficient and numerical stable for low mass ratios. In both these formulations, the fluid mesh motion has been decoupled from the monolithic matrix consisting of governing equations that describe the fluid flow, structural dynamics, interface conditions, and the mesh motion. As a result, the size of the matrix which needs to be solved reduces by a maximum of 20% in the case of two-dimensional simulations and a maximum of 40% for the three-dimensional simulations. The decoupling of mesh motion has been made possible by predicting the structural displacements at the start of each time step based on the previous time step velocities. Additionally, both these methods linearlize the convective velocities by using a second-order explicit approximation based on previous time step information. The first quasi-monolothic approach discussed uses an extra stabilization term, $\frac{1}{2}\left(\nabla\check{\boldsymbol{u}}_{\mathrm{h}}^{\mathrm{f}}\right)\boldsymbol{u}_{\mathrm{h}}^{\mathrm{f,n}}$, which has been proposed by Temam to provide numerical stability. This term plays a role in proving the unconditional stability of the method theoretically. The formulation is stable for any mixed finite element discretization for the velocity and pressure. On the other hand, the second approach discussed in this paper considers a Galerkin least square-based stabilization to provide convective stabilization for convectively dominant problems. Additionally, this method requires equal order finite element for the fluid velocity and pressure. As a result, this approach would require lower memory requirements compared to the first approach.

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