Finite Element Model of Wave Loading on a Soil Seabed Part I: Multi-layered Anisotropic Gassy Soil Conditions

Stephen D. Thomas(✉)

OGI Groundwater Specialists, Durham DH7 8TP, UK
admin@ogi.co.uk

Abstract. Ocean wave loading on a gassy soil seabed, i.e. a soft soil sediment containing biogenic methane gas bubbles, results in harmonic changes in stresses, displacements and pore pressures over depth.

For a multi-layered seabed soil with homogeneous properties within each layer, the behaviour of the dependent variables of displacement and pore pressure can be represented by a harmonic function in both horizontal space (x) and time (t).

The amplitude of the horizontal & vertical displacement and pore-water pressure will vary over depth, dependent on the particular distribution of the multi-layered properties, together with the wave loading amplitude, wave number and angular velocity, together with the boundary conditions at the seabed surface and the soil rock interface.

To simulate the behaviour of the seabed in dynamic conditions, this first paper of a two-part paper series, presents a series of governing equations based on a modified Biot theory for poro-elastic gassy soil. These equations have been further modified to take into account the harmonic behaviour using real & imaginary complex number theory.

The Finite Element Galerkin formulation combined with Green's Theorem is applied to these harmonic equations which results in six degrees of freedom per node, i.e. x-displacement, z-displacement and pore-water pressure, in both real (in-phase) & imaginary (out-of-phase) conditions.

Finally, the finite element model developed is used to present a case study of the displacement, stresses & pore pressures in a two-layer soil subject to wave loading.

Keywords: Wave loading · Marine sediment · Gassy soil · Multi-layer anisotropy

1 Introduction

This is the first of a two-part series of papers that describes a Finite Element formulation of the equations that govern the stresses, displacements and pore-water pressures induced in a multi-layered poro-elastic soft gassy soil seabed when subject to a surface wave loading.

D. V. K. Huynh et al. (Eds.): VSOE 2021, LNCE 208, pp. 105–113, 2022.
https://doi.org/10.1007/978-981-16-7735-9_9

This first paper simulates the soil behavior under the condition that the soil properties are uniform & homogeneous in the horizontal x and y directions. However, the soil structure can be anisotropic, i.e. variable in the downward z-direction as in a multi-layered soil.

Under such conditions, the behaviour of the dependent variables is harmonic in both time and x-direction, as governed by the wave period and wave length. As a consequence, only the downward z-direction requires finite element discretization, resulting in the need only for one-dimensional elements in the vertical domain, with each node having six degrees of freedom.

2 Governing Differential Equations for Multi-layer and Anisotropic Poro-Elastic Soil Conditions

Based on the Biot (1941) equations that govern the displacement, stresses and pore water pressures in a poro-elastic soil, as modified by Thomas (1987) for a Double Compressibility gassy soil, the equations of equilibrium in the x-direction, equilibrium in the z-direction and from continuity principles incorporating Darcy's Law of pore-fluid flow through a deformable poro-elastic medium, the following can be written for small strain conditions as:

$$(\lambda + 2G)\frac{\partial^2 w_x}{\partial x^2} + G\frac{\partial^2 w_x}{\partial z^2} + \lambda\frac{\partial^2 w_z}{\partial x \partial z} + G\frac{\partial^2 w_z}{\partial z \partial x} - B\frac{\partial u_w}{\partial x} = 0 \tag{1}$$

$$(\lambda + 2G)\frac{\partial^2 w_z}{\partial z^2} + G\frac{\partial^2 w_z}{\partial x^2} + \lambda\frac{\partial^2 w_x}{\partial z \partial x} + G\frac{\partial^2 w_x}{\partial x \partial z} - B\frac{\partial u_w}{\partial z} = 0 \tag{2}$$

$$\begin{aligned}\frac{\partial}{\partial x}\left[\frac{k_{xx}}{\gamma_w}\frac{\partial u_w}{\partial x} + \frac{k_{xz}}{\gamma_w}\frac{\partial u_w}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{k_{zx}}{\gamma_w}\frac{\partial u_w}{\partial x} + \frac{k_{zz}}{\gamma_w}\frac{\partial u_w}{\partial z}\right] \\ = C\frac{\partial u_w}{\partial t} + B\frac{\partial}{\partial t}\left[\frac{\partial w_x}{\partial x} + \frac{\partial w_z}{\partial z}\right]\end{aligned} \tag{3}$$

In the above equations, w_x and w_z are the soil displacements in the x-horizontal and the z-vertical directions respectively, t is time, u_w the pore-water pressure within the soil matrix, k_{ij} the coefficient of permeability tensor, λ the Lame's constant and G the shear modulus. The gassy soil parameters B and C are derived from gassy soil theory (Thomas 1987) as follows:

$$B = \left(1 + K''/K^o\right)^{-1} \tag{4}$$

$$C = \left(K^o + K''\right)^{-1} + (1 - H)c_w n_w \tag{5}$$

The parameter K^o is the component of undrained bulk modulus governed by compression of the discrete gas bubbles, caused by an increase in total stress. Contributing to undrained compressibility is also water compressibility, c_w, water porosity, n_w, and Henry's constant H.

The parameter K'' is the component of drained bulk modulus governed by consolidation of the saturated soil matrix. This results in two independent strain components,

(i) caused by an increase in the total stress σ^o, and (ii) caused by an increase in the operative stress $(\sigma^o - u_w)$.

The modified equilibrium equations for a poro-elastic gassy soil are based on a Double Compressibility Model (Thomas 1987) where the compression of the discrete gas bubbles is governed by the increase in the pressure of the gas bubbles, which in turn is governed by the average total stress within the soil. The compression of the surrounding saturated soil matrix, however, is governed by the difference between the total stress and the pore water pressure.

This difference between total stress and pore water pressure $(\sigma^o - u_w)$ is normally known as *effective stress* (Terzaghi 1944). However, for a gassy soil, effective stress does not, on its own, govern the total volume change. For a gassy soil, this difference between the mean total stress on a soil sample and the water pressure has been coined the *operative stress* (Sills et al. 1991). Such a gassy soil structure is depicted in Fig. 1. according to Wheeler et al. (1990).

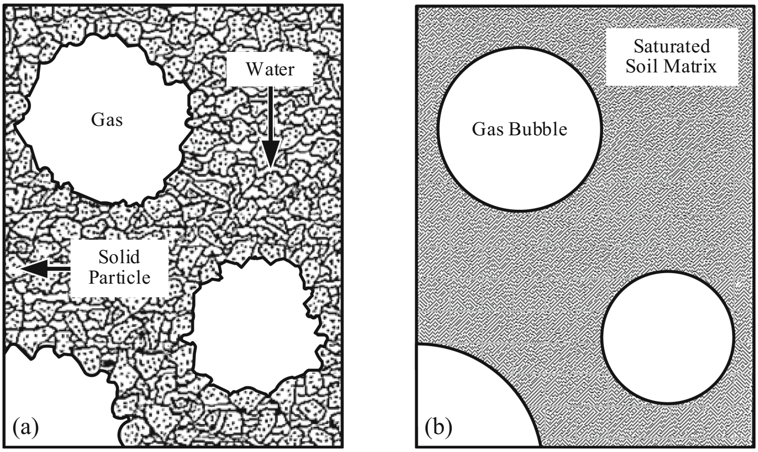

Fig. 1. Soils containing large gas bubbles: (a) Soil structure; (b) Continuum model. Wheeler et al. (1990)

This results in two independent strain components caused by (i) an increase in the total stress σ^o, and (ii) an increase in the operative stress $(\sigma^o - u_w)$,

$$\epsilon^o = \Delta V^o = \Delta\sigma^o / K^o \tag{6}$$

$$\epsilon'' = \Delta V'' = \Delta\sigma'' / K'' = (\Delta\sigma^o - \Delta u_w) / K'' \tag{7}$$

where ϵ^o is the volumetric strain due to the compression of the discrete gas bubbles caused by an increase in total mean stress $\Delta\sigma^o$, with ϵ'' as the volumetric strain due to the compression of the soil matrix caused by an increase in operative stress $\Delta\sigma''$. The main principle of the Double Compressibility Model (Thomas, 1987) for a gassy soil is that these two components of volumetric strain for a gassy soil, ϵ^o & ϵ'', can then be added directly together to provide the total volumetric strain, as follows:

$$\epsilon = \epsilon^{o} + \epsilon'' = \Delta\sigma^{\mathrm{o}}/K^{\mathrm{o}} + \Delta\sigma''/K'' = \Delta\sigma^{\mathrm{o}}/\mathrm{K}^{\mathrm{o}} + (\Delta\sigma^{o} - \Delta u_{w})/K'' \tag{8}$$

from which the change in total stress can be defined in terms of strain and pore pressure as:

$$\Delta\sigma^{\mathrm{o}} = \left(\epsilon + \Delta u_{w}/K''\right)/\left(1/K^{\mathrm{o}} + 1/K''\right) \tag{9}$$

3 Harmonic Approximation of Governing Equations

On the basis that the wave loading on the seabed is travelling in the x-direction along the surface of the seabed as depicted in Fig. 2 below, under poro-elastic conditions, based on the equations by Yamamoto et al. (1978) and Madsen (1978), any particular variable ψ which is dependent on x, z and t can be defined by a variable dependent in z only, multiplied by a harmonic function dependent on x & t.

$$\psi(x, z, t) = \psi_c + i\psi_s \tag{10}$$

where

$$\psi_c = \overline{\psi}(z)\cos(ax - \omega t) \tag{11}$$

$$\psi_s = \overline{\psi}(z)\sin(ax - \omega t) \tag{12}$$

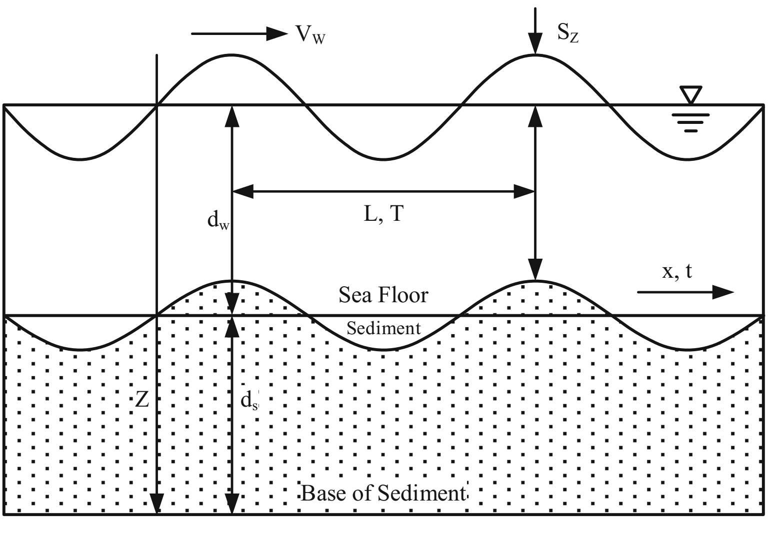

Fig. 2. Harmonic wave loading on a seabed sediment

and where $\overline{\psi}(z)$ is the peak variable function amplitude at depth z, from which:

$$\overline{\psi}(z) = \sqrt{\psi_c^2 + \psi_s^2} \tag{13}$$

$$\frac{\partial \psi}{\partial x} = +ia\psi = +ia\psi_c - a\psi_s \tag{14}$$

$$\frac{\partial \psi}{\partial t} = -i\omega\psi = -i\omega\psi_c + \omega\psi_s \tag{15}$$

In the above equations $\overline{\psi}$ is the amplitude of a particular harmonic variable at depth z, a is the wave number $2\pi/L$, and ω is the angular velocity $2\pi/T$, in which L is the wavelength and T is the angular velocity.

This results in the harmonic behaviour of displacements and pore-water pressure in a homogeneous, anisotropic soil sediment being represented as follows:

$$w_x(x, z, t) = w_{xc} + iw_{xs} = \overline{w}_x(z)[\cos(ax - \omega t) + i\,\sin(ax - \omega t)] \tag{16}$$

$$w_z(x, z, t) = w_{zc} + iw_{zs} = \overline{w}_z(z)[\cos(ax - \omega t) + i\,\sin(ax - \omega t)] \tag{17}$$

$$u_w(x, z, t) = u_{wc} + iu_{ws} = \overline{u}_w(z)[\cos(ax - \omega t) + i\,\sin(ax - \omega t)] \tag{18}$$

Substituting the above three harmonic functions of displacements and pore pressure into the governing equilibrium and continuity Eqs. (1), (2) and (3) produces three differential equations which can be written as a mixture of real & imaginary harmonic variables:

$$-a^2(\lambda + 2G)w_x + G\frac{\partial^2 w_x}{\partial z^2} + ia(\lambda + G)\frac{\partial w_z}{\partial z} - iaBu_w = 0 \tag{19}$$

$$+(\lambda + 2G)\frac{\partial^2 w_z}{\partial z^2} - a^2 G w_z + ia(\lambda + G)\frac{\partial w_x}{\partial z} - B\frac{\partial u_w}{\partial z} = 0 \tag{20}$$

$$\begin{aligned} -a^2\frac{k_{xx}}{\gamma_w}u_w + \frac{k_{zz}}{\gamma_w}\frac{\partial^2 u_w}{\partial z^2} + ia\frac{k_{xz}}{\gamma_w}\frac{\partial u_w}{\partial z} + ia\frac{k_{zx}}{\gamma_w}\frac{\partial u_w}{\partial z} \\ = -i\omega C u_w + a\omega B w_x - i\omega B\frac{\partial w_z}{\partial z} \end{aligned} \tag{21}$$

4 Galerkin Finite Element Formulation

Applying the Galerkin Finite Element formulation to the above equations in the vertical dimension, combined with Green's Theorem results in:

$$\begin{aligned} \int \left[a^2(\lambda + 2G)N_I N_J w_{xJ} + G\frac{\partial N_I}{\partial z}\frac{\partial N_J}{\partial z}w_{xJ} + iaG\frac{\partial N_I}{\partial z}N_J w_{zJ} - ia\lambda N_I\frac{\partial N_J}{\partial z}w_{zJ}\right] dz \\ + \int iaBN_I N_J u_{wJ}\,dz = S_{xI} \end{aligned} \tag{22}$$

from equilibrium in the x-direction

$$\int\left[(\lambda+2G)\frac{\partial N_I}{\partial z}\frac{\partial N_J}{\partial z}w_{zJ}+a^2GN_IN_Jw_{zJ}+ia\lambda\frac{\partial N_I}{\partial z}N_Jw_{xJ}-iaGN_I\frac{\partial N_J}{\partial z}w_{xJ}\right]dz$$
$$-\int B\frac{\partial N_I}{\partial z}N_Ju_{wJ}dz=S_{zI} \quad (23)$$

from equilibrium in the z-direction, and

$$\int\left[\frac{k_{zz}}{\gamma_w}\frac{\partial N_I}{\partial z}\frac{\partial N_J}{\partial z}-ia\frac{k_{xz}}{\gamma_w}N_I\frac{\partial N_J}{\partial z}+ia\frac{k_{zx}}{\gamma_w}\frac{\partial N_I}{\partial z}N_J+a^2\frac{k_{xx}}{\gamma_w}N_IN_J-i\omega CN_IN_J\right]u_{wJ}dz$$
$$+\int\left[a\omega BN_IN_Jw_{xJ}-i\omega BN_I\frac{\partial N_J}{\partial z}w_{zJ}\right]dz=Q_{wI} \quad (24)$$

from continuity principles incorporating Darcy's Law.

Substituting Eqs. 16, 17, 18 into the above three mixed real & imaginary differential equations, then by separating the real & imaginary terms, results in six differential equations, each with nodal harmonic dependent variables as follows:

$$\left(a^2(\lambda+2G)\bar{\Gamma}_{\mathrm{IJ}}+G\bar{A}^{\mathrm{zz}}_{\mathrm{IJ}}\right)\bar{w}_{xcJ}-\left(aG\bar{B}^{\mathrm{zN}}_{\mathrm{IJ}}-a\lambda\bar{B}^{\mathrm{Nx}}_{\mathrm{IJ}}\right)\bar{w}_{zsJ}-aB\bar{\Gamma}_{\mathrm{IJ}}\bar{u}_{wsJ}=\bar{S}_{xcI} \quad (25)$$

$$\left(a^2(\lambda+2G)\bar{\Gamma}_{\mathrm{IJ}}+G\bar{A}^{\mathrm{zz}}_{\mathrm{IJ}}\right)\bar{w}_{xsJ}+\left(aG\bar{B}^{\mathrm{zN}}_{\mathrm{IJ}}-a\lambda\bar{B}^{\mathrm{Nx}}_{\mathrm{IJ}}\right)\bar{w}_{zcJ}+aB\bar{\Gamma}_{\mathrm{IJ}}\bar{u}_{wcJ}=\bar{S}_{xsI} \quad (26)$$

$$\left((\lambda+2G)\bar{A}^{\mathrm{zz}}_{\mathrm{IJ}}+a^2G\bar{\Gamma}_{\mathrm{IJ}}\right)\bar{w}_{zsJ}+\left(a\lambda\bar{B}^{\mathrm{zN}}_{\mathrm{IJ}}-aG\bar{B}^{\mathrm{Nx}}_{\mathrm{IJ}}\right)\bar{w}_{xcJ}+B\bar{B}^{\mathrm{zN}}_{\mathrm{IJ}}\bar{u}_{wsJ}=\bar{S}_{zsI} \quad (27)$$

$$\left((\lambda+2G)\bar{A}^{\mathrm{zz}}_{\mathrm{IJ}}+a^2G\bar{\Gamma}_{\mathrm{IJ}}\right)\bar{w}_{zsJ}+\left(a\lambda\bar{B}^{\mathrm{zN}}_{\mathrm{IJ}}-aG\bar{B}^{\mathrm{Nx}}_{\mathrm{IJ}}\right)\bar{w}_{xcJ}+B\bar{B}^{\mathrm{zN}}_{\mathrm{IJ}}\bar{u}_{wsJ}=\bar{S}_{zsI} \quad (28)$$

$$\left(a^2\frac{k_{xx}}{\gamma_w}\bar{\Gamma}_{\mathrm{IJ}}+\frac{k_{zz}}{\gamma_w}\bar{A}^{\mathrm{zz}}_{\mathrm{IJ}}\right)\bar{u}_{wcJ}-\left(a\frac{k_{zx}}{\gamma_w}\bar{B}^{\mathrm{zN}}_{\mathrm{IJ}}-a\frac{k_{xz}}{\gamma_w}\bar{B}^{\mathrm{Nz}}_{\mathrm{IJ}}-\omega C\bar{\Gamma}_{\mathrm{IJ}}\right)\bar{u}_{wsJ}$$
$$+a\omega B\bar{\Gamma}_{\mathrm{IJ}}\bar{w}_{xcJ}+\omega B\bar{B}^{\mathrm{Nz}}_{\mathrm{IJ}}\bar{w}_{zsJ}=\bar{Q}_{wcI} \quad (29)$$

$$\left(a^2\frac{k_{xx}}{\gamma_w}\bar{\Gamma}_{\mathrm{IJ}}+\frac{k_{zz}}{\gamma_w}\bar{A}^{\mathrm{zz}}_{\mathrm{IJ}}\right)\bar{u}_{wsJ}+\left(a\frac{k_{zx}}{\gamma_w}\bar{B}^{\mathrm{zN}}_{\mathrm{IJ}}-a\frac{k_{xz}}{\gamma_w}\bar{B}^{\mathrm{Nz}}_{\mathrm{IJ}}-\omega C\bar{\Gamma}_{\mathrm{IJ}}\right)\bar{u}_{wcJ}$$
$$+a\omega B\bar{\Gamma}_{\mathrm{IJ}}\bar{w}_{xsJ}-\omega B\bar{B}^{\mathrm{Nz}}_{\mathrm{IJ}}\bar{w}_{zcJ}=\bar{Q}_{wsI} \quad (30)$$

Where

$$\overline{A}^{\mathrm{zz}}_{\mathrm{IJ}}=\int\frac{\partial N_I}{\partial z}\frac{\partial N_J}{\partial z}dz \quad (31)$$

$$\overline{\Gamma}_{\mathrm{IJ}}=\int N_IN_Jdz \quad (32)$$

$$\overline{B}^{\mathrm{xN}}_{\mathrm{IJ}}=\int\frac{\partial N_I}{\partial x}N_Jdz \quad (33)$$

$$\overline{B}^{\mathrm{Nx}}_{\mathrm{IJ}}=\int N_J\frac{\partial N_I}{\partial x}dz \quad (34)$$

5 Incorporation into a Finite Element Model

Using a simple two-noded linear finite element shape function is sufficient to approximate the above matrix equations (Thomas 1989). This provides the ability to simulate accurately the displacement, stress and pore water pressure, both in-phase and out-of-phase of the harmonic loading wave (Thomas 1995).

The above Finite Element formulation has been programmed in to the software WISPP (Wave Induced Stresses & Pore Pressures) and is freely available at http://ogi.co.uk/software.

For selected input of soil conditions, together with applied boundary conditions of surface loading, wavelength and angular velocity, this software produces a graphical output of the in-phase and out-of-phase x & z-displacements and pore-pressure at each of the nodes over the depth of the seabed, and the x-x, z-z & x-z operative stresses at the centroid of each element.

6 Example of Wave Loading on a Two-Layered Soil Seabed

As an example, Fig. 3 below represents the finite element modelling of a multi-layered saturated soil with a 5m upper layer with shear modulus $G_1 = 750$ kPa & drained bulk modulus $K_1 = 1$ MPa, overlying a second 10m layer with higher stiffness of $G_2 = 3$ MPa & $K_2 = 4$ MPa.

A harmonic wave load of 20 kPa amplitude is applied to the soil surface with a wavelength of 20 m and wave period of 20 s. The vertical domain has been discretized into 300 finite elements, with 100 elements in the upper layer and 200 elements in the lower layer.

The coefficient of permeability is 1×10^{-5} m/s for both layers. The output demonstrates that the heterogeneity resulting from two layers, with the stiffness of the underlying soil only four times that of the upper soil, produces a variation in behaviour as depicted below.

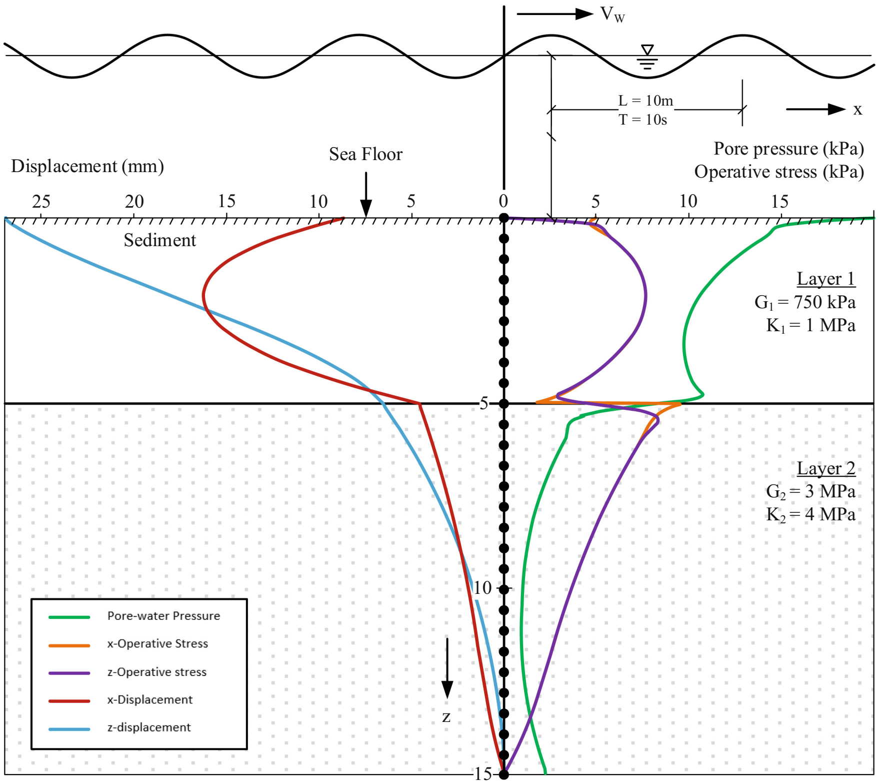

Fig. 3. Wave induced x & z displacements, x & z operative stresses and pore-water pressures in a two-layer soil, simulated using the one-dimensional harmonic finite element WISSP model (Thomas 1988)

7 Conclusions

Ocean wave loading on waves on a multi-layered seabed of varying properties can result in unexpected displacements, stresses and pore-pressure behaviour. This is especially the case for a sediment containing gassy soil where significant undrained soil compression and shear strain can occur.

As a consequence, such deformation, stress and pore-water pressure variations can result in unforeseen behaviour. For this reason, a Finite Element Model has been developed to simulate such behaviour under harmonic wave loading conditions. This model is freely available for public use.

Acknowledgements. My gratitude to colleagues George French, who regularly incorporates my original FORTRAN algorithms into more modern software, and Megan Thomas who converts my text and drawings into a more professional presentation. Finally, recognition always to then my supervisor, now colleague, Professor Gilliane Sills, who initiated at the University of Oxford her research into soft soil behaviour, including gassy soils and wave loading on the seabed.

References

Biot, M.A.: General theory of three-dimensional consolidation. J. Appl. Phys. **12**, 155–164 (1941)

Madsen, O.S.: Wave-induced pore pressures and effective stresses in a porous bed. Geotechnique **28**(4), 377–393 (1978)

Sills, G.C., Wheeler, S.J., Thomas, S.D., Gardner, T.N.: Behaviour of offshore soils containing gas bubbles. Geotechnique **41**(2), 227–241 (1991)

Terzaghi, K.: Theoretical Soil Mechanics. Wiley, Hoboken (1944)

Thomas, S.D.: The Consolidation Behaviour of Gassy Soil. D. Phil. Thesis, University of Oxford (1987)

Thomas, S.D.: WISPP: A finite element model for the analysis of wave induced stresses, and pore pressures in a variably saturated seabed. University of Oxford, Department of Engineering Science, Report OUEL 1750/88 (1988)

Thomas, S.D.: A finite element model for the analysis of wave induced stresses, displacements and pore pressures in an unsaturated seabed. I: theory. Comput. Geotech. **8**, 1–38 (1989)

Thomas, S.D.: A finite element model for the analysis of wave induced stresses, displacements and pore pressures in an unsaturated seabed. II: model verification. Comput. Geotech. **17**, 107–132 (1995)

Wheeler, S.J., Sham, W.K., Thomas, S.D.: Gas pressure in unsaturated offshore soils. Can. Geotech. J. **27**, 79–89 (1990)

Yamamoto, T., Koning, H.L., Sellmeiher, H., Hijum, E.V.: On the response of a poro-elastic bed to water waves. J. Fluid Mech. **87**(1), 193–206 (1978)

Zienkiewicz, O.C.: The Finite Element Method, 3rd edn. McGraw-Hill, New York (1977)