Chapter 15
On the Parametric Representation of Type-2 Interval and Ranking: Its Application in the Unconstrained Non-linear Programming Problem Under Type-2 Interval Uncertainty

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Abstract This chapter aims to introduce the parametric representation of the Type-2 interval and its ranking. This chapter also wishes to derive the optimality criteria of the imprecise unconstrained optimization problem in a Type-2 interval environment with these concepts. For this purpose, at first, by recapitulating the idea of Type-2 interval introduced by Rahman et al. (Rahman MS, Shaikh AA, Bhunia AK (2020d)), the parametric representation of Type-2 interval is proposed. Then an order relation on the set of Type-2 intervals is introduced. Using this order relation, the maximizer and minimizer of an unconstrained Type-2 interval-valued optimization problem are defined. Then the optimality conditions (both necessary and sufficient) for the said optimization problem are derived. Finally, the obtained optimality results are illustrated by some numerical examples.

Keywords Type-2 interval parametric form · Type-2 interval Order relation · Optimizer · Type-2 interval optimization

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1 Introduction

Due to the uncertainty and inexactness, the parameters of most real-life problems, especially optimization problems, are not precise. So, the study of optimality conditions of an imprecise optimization problem is an important research area. To tackle the imprecise optimization problems, several researchers used various approaches, viz. fuzzy, stochastic, fuzzy-stochastic and interval approaches, etc.

In the stochastic approach, the flexible parameters of an imprecise optimization problem are presented in the shape of random variables with proper probability distribution function. Whereas in the fuzzy approach, the flexible parameters are presented in fuzzy sets or fuzzy numbers with appropriate membership function. In the fuzzy-stochastic approach, some of the flexible parameters are considered in the form of fuzzy sets or fuzzy numbers, and others considered random. On the other hand, in the interval approach, the optimization problem's imprecise parameters are presented in the form of intervals. In the existing literature, a lot of research works on these approaches are available. Among those, some interesting works are reported here:

With some theoretical developments, Catoni (2004), Schneider and Kirkpatrick (2007), and Powell (2019) accomplished their works on stochastic optimization. On the other side, Heyman and Sobel (2004), Ziemba and Vickson (2014), and Tang et al. (2020) used the theory of stochastic optimization to analyse the mathematical models in operations research. In the area of fuzzy optimization, Tang et al. (2004), Wu (2004), Lodwick and Kacprzyk (2010), Heidari et al. (2016), and Anter and Ali (2020) established some applicable theories to enrich this area. Wang and Watada (2012) and Farrokh et al. (2018) used fuzzy and stochastic optimization theories to analyse some supply chain/inventory models. In the area of interval optimization, Chen and Wu (2004), Bhurjee and Panda (2016), Ghosh et al. (2019), and Rahman et al. (2020c) derived several helpful techniques to solve interval optimization problems. Rahman et al. (2020a, 2020b) solved some inventory problems in an interval environment with these concepts.

Recently, generalizing the interval approach by taking the flexibility rather than fixing the interval's bounds, Rahman et al. (2020a) introduced a new representation of intervals named Type-2 interval. In this representation, both the bounds belong to two different intervals. In this approach, a flexible parameter can be presented in the form $A = [a_L, a_U]$, where $a_L \in \left[\underline{a}_L, \overline{a}_L\right]$ and $a_U \in \left[\underline{a}_U, \overline{a}_U\right]$. Thus, in this approach, the imprecise parameter can be expressed as $A = \left[\left(\underline{a}_L, \overline{a}_L\right), \left(\underline{a}_U, \overline{a}_U\right)\right]$. In this area, till now, no theoretical development had been done.

This chapter's main objective is to introduce the concepts of parametrization of Type-2 interval and its order relation. Then using these concepts, the definition of the optimizer and Type-2 interval-valued support function is proposed. After that, the optimality conditions of an unconstrained Type-2 interval-valued optimization problem are derived. Finally, these theoretical results are illustrated with some numerical examples.

2 The Concept of Type-2 Interval

2.1 Type-2 Interval in Parametric Form:

The concept of Type-2 interval introduced by Rahman et al. (2020) is denoted by $\left[\left(\underline{a}_L, \overline{a}_L\right), \left(\underline{a}_U, \overline{a}_U\right)\right]$ and is defined by employing Type-1 intervals as follows:

$$\left[\left(\underline{a}_L, \overline{a}_L\right), \left(\underline{a}_U, \overline{a}_U\right)\right] = \left\{[a_L, a_U] : a_L \in \left[\underline{a}_L, \overline{a}_L\right] \text{and } a_U \in \left[\underline{a}_U, \overline{a}_U\right]\right\}.$$

Now, we have defined the parametric representation of the Type-2 interval in the following definitions. This representation is defined in two steps: first step parametrization and second step parametrization of Type-2 interval.

Definition 1: Let $A_2 = \left[\left(\underline{a}_L, \overline{a}_L\right), \left(\underline{a}_U, \overline{a}_U\right)\right]$ be a Type-2 interval. Then

(i) the first step parametrization of A_2 is defined by the set of Type-1 intervals as follows:

$$A_2^1 = \left\{ \begin{array}{l} [a_L(r_1), a_U(r_2)] : a_L(r_1) = \underline{a}_L + r_1\left(\bar{a}_L - \underline{a}_L\right), \\ a_U(r_2) = \underline{a}_U + r_2\left(\bar{a}_U - \underline{a}_U\right) \text{ and } r_1, r_2 \in [0, 1] \end{array} \right\}$$

(ii) the second step, parametrization of A_2 is defined as

$$A_2^2 = \left\{ \begin{array}{l} a(r_1, r_2, r_3) : \ a(r_1, r_2, r_3) = a_L(r_1) + r_2(a_U(r_3) - a_L(r_1)) \\ \text{and } r_1, r_2, r_3 \in [0, 1] \end{array} \right\}$$

Definition 2: Let $A_2 = \left[\left(\underline{a}_L, \overline{a}_L\right), \left(\underline{a}_U, \overline{a}_U\right)\right]$ and $B_2 = \left[\left(\underline{b}_L, \overline{b}_L\right), \left(\underline{b}_U, \overline{b}_U\right)\right]$ be two Type-2 intervals with their second step parametrization
$A_2^2 = \left\{a\left(r_1, r_2, r_3\right) : \ r_1, r_2, r_3 \in \left[0, 1\right]\right\}$ *and* $B_2^2 = \left\{b\left(r_1, r_2, r_3\right) : \ r_1, r_2, r_3 \in \left[0, 1\right]\right\}$, respectively. Then $A_2 = B_2$ iff $a(r_1, r_2, r_3) = b(r_1, r_2, r_3), \ \forall r_1, r_2, r_3 \in [0, 1]$.

Example 1: Let us consider a Type-2 interval $A_2 = [(2, 4), (5, 7)]$.
Then its first step, parametrization, is $A_2^1 = \{[2 + 2r_1, 5 + 2r_2] : \ r_1, r_2 \in [0, 1]\}$.
Therefore, its second step, parametrization, is $A_2^2 = \{2 + 2r_1 + r_2\left(3 + 2r_3 - 2r_1\right) : r_1, r_2, r_3 \in \left[0, 1\right]\}$.

2.2 Order Relation of Type-2 Intervals:

Here, a new type of order relation has been introduced on the set of all Type-2 intervals by using the four different centres of a Type-2 interval which is defined in the **Definition 3**.

Definition 3: Let $A_2=\left[\left(\underline{a}_L,\overline{a}_L\right),\left(\underline{a}_U,\overline{a}_U\right)\right]$ be a Type-2 interval with second step parametrization $A_2^2=\left\{a\left(r_1,r_2,r_3\right): a\left(r_1,r_2,r_3\right)=a_L\left(r_1\right)+r_2\left(a_U\left(r_3\right)-a_L\left(r_1\right)\right)\right.$ and $\left.r_1,r_2,r_3\in\left[0,1\right]\right\}$. Then a set of support of A_2 be defined by the set of four elements $\{A_{S1},A_{S2},A_{S3},A_{S4}\}$,

where

$$A_{S1}=\frac{a(1,1,1)+a(1,1,0)+a(1,0,0)+a(0,0,0)}{4}$$
$$A_{S2}=\frac{a(1,1,1)+a(1,1,0)+a(1,0,0)}{3}$$
$$A_{S3}=\frac{a(1,1,1)+a(1,1,0)}{2}$$
$$A_{S4}=a(1,1,1)$$

Definition 4: Let $A=\left[\left(\underline{a}_L,\overline{a}_L\right),\left(\underline{a}_U,\overline{a}_U\right)\right]$ and $B=\left[\left(\underline{b}_L,\overline{b}_L\right),\left(\underline{b}_U,\overline{b}_U\right)\right]$ be two Type-2 intervals with second step parametrization $\left\{a\left(r_1,r_2,r_3\right): r_1,r_2,r_3\in\left[0,1\right]\right\}$ and $\left\{b\left(r_1,r_2,r_3\right): r_1,r_2,r_3\in\left[0,1\right]\right\}$. Then A is said to be less or equal to B denoted by $A\le_2 B$ if the following conditions hold:

$$A\le_2 B\Leftrightarrow\begin{cases}A_{S1}<B_{S1}, \text{ when } A_{S1}\ne B_{S1}\\ A_{S2}<B_{S2}, \text{ when } A_{S1}=B_{S1} \text{ and } A_{S2}\ne B_{S2}\\ A_{S3}<B_{S3}, \text{ when } A_{S2}=B_{S2} \text{ and } A_{S3}\ne B_{S3}\\ A_{S4}\le B_{S4}, \text{ when } A_{S3}=B_{S3}\end{cases}$$

where $A_{S1},A_{S2},A_{S3},A_{S4}$ are defined in **Definition 3**

Definition 5: Let us consider two Type-2 intervals $A=\left[\left(\underline{a}_L,\overline{a}_L\right),\left(\underline{a}_U,\overline{a}_U\right)\right]$ and $B=\left[\left(\underline{b}_L,\overline{b}_L\right),\left(\underline{b}_U,\overline{b}_U\right)\right]$. Then $A\ge_2 B$ iff $B\le_2 A$.

Remark-1: The order relations $\le_2$ and $\ge_2$ on the set of Type-2 intervals satisfy the reflexive, anti-symmetric, and transitive properties. Thus $\le_2$ and $\ge_2$ are the partial order relations.

Example 2: Compare the following pair of Type-2 intervals using the above definitions.

$$\text{(i) } A=[(-3,-1),(2,5)],\ B=[(-1,3),(6,7)]$$
$$\text{(ii) } A=[(-1,3),(5,9)],\ B=[(1,2),(6,7)]$$

Solution:

(i) The second step parametrization of A and B are

$$\{a(r_1,r_2,r_3)=-3+2r_1+r_2(5-2r_1+3r_3):r_1,r_2,r_3\in[0,1]\},$$
$$\{b(r_1,r_2,r_3)=-1+4r_1+r_2(2-4r_1+r_3):r_1,r_2,r_3\in[0,1]\}$$

Here, $A_{S1} = \frac{a(1,1,1)+a(1,1,0)+a(1,0,0)+a(0,0,0)}{4} = \frac{3}{4}$

$$B_{S1} = \frac{b(1,1,1)+b(1,1,0)+b(1,0,0)+b(0,0,0)}{4} = \frac{15}{4}$$

Since $A_{S1} < B_{S1}$, thus, $A \leq_2 B$.

(ii) The second step parametrization of A and B are

$$\{a(r_1, r_2, r_3) = -1 + 4r_1 + r_2(6 - 4r_1 + 4r_3) : r_1, r_2, r_3 \in [0, 1]\},$$
$$\{b(r_1, r_2, r_3) = 1 + r_1 + r_2(5 - r_1 + r_3) : r_1, r_2, r_3 \in [0, 1]\}$$

Here,

$$A_{S1} = \frac{a(1,1,1)+a(1,1,0)+a(1,0,0)+a(0,0,0)}{4} = 4 = \frac{b(1,1,1)+b(1,1,0)+b(1,0,0)+b(0,0,0)}{4} = B_{S1}$$

and

$$A_{S2} = \frac{a(1,1,1)+a(1,1,0)+a(1,0,0)}{3} = \frac{17}{3} > 5 = \frac{b(1,1,1)+b(1,1,0)+b(1,0,0)}{3} = B_{S2}.$$

Hence, $A \geq_2 B$.

3 Optimality of Unconstrained Type-2 Interval-Valued Optimization Problem

3.1 Type-2 Interval-Valued Function and Its Parametrized Form

Let the set of all Type-2 intervals be denoted by $I_2(\mathbb{R})$, i.e., $I_2(\mathbb{R}) = \{[(\underline{a}_L, \overline{a}_L), (\underline{a}_U, \overline{a}_U)] : \underline{a}_L, \overline{a}_L, \underline{a}_U, \overline{a}_U \in \mathbb{R}\}$.

Now a Type-2 interval-valued function of several variables (say n variables) is a function $H_2 : S \subseteq \mathbb{R}^n \to I_2(\mathbb{R})$ given by $H_2(x) = [(\underline{h}_L(x), \overline{h}_L(x))S]$, $x \in (\underline{h}_U(x), \overline{h}_U(x))$.

Definition 6: The first parametrized representation of H_2 is defined as $H_2(x) = \{[h_L(x, r_1), h_U(x, r_2)] : r_1, r_2 \in [0, 1]\}$ where $h_L(x, r_1) = \underline{h}_L(x) + r_1\big(\overline{h}_L(x) - \underline{h}_L(x)\big)$ and $h_U(x, r_1) = \underline{h}_U(x) + r_2\big(\overline{h}_U(x) - \underline{h}_U(x)\big)$.

Definition 7: The second parameterized representation of the Type-2 interval-valued function H_2 is defined as $H_2(x) = \left\{ \begin{array}{l} h(x, r_1, r_2, r_3) : h(x, r_1, r_2, r_3) = h_L(x, r_1) \\ +r_2(h_U(x, r_3) - h_L(x, r_1)) \text{ and } r_1, r_2, r_3 \in [0, 1] \end{array} \right\}$.

where $h_L(x, r_1) = \underline{h}_L(x) + r_1\big(\overline{h}_L(x) - \underline{h}_L(x)\big)$ and $h_U(x, r_1) = \underline{h}_U(x) + r_2\big(\overline{h}_U(x) - \underline{h}_U(x)\big)$.

A special type of Type-2 interval-valued function is defined as follows:

$$H_2 : S \subseteq \mathbb{R}^n \to I_2(\mathbb{R}) \text{ given by } H_2(x) = \sum_{i=1}^{k} \big[\big(\underline{a}_{iL}, \overline{a}_{iL}\big), \big(\underline{a}_{iU}, \overline{a}_{iU}\big)\big]\, h_i(x),\ \ x \in S. \tag{1}$$

where $h_i : S \to \mathbb{R},\ i = 1, 2, ..., k$

Definition 8: The first parametrized representation of the function H_2 given in (1) is defined by $H_2(x) = \sum_{i=1}^{k} \{[a_{iL}(r_{1i}), a_{iU}(r_{2i})] : r_{1i}, r_{2i} \in [0, 1]\}\, h_i(x), a_{iL}(r_1) = \underline{a}_{iL} + r_1\big(\overline{a}_{iL} - \underline{a}_{iL}\big)$ and $a_{iU}(r_2) = \underline{a}_{iU} + r_2\big(\overline{a}_{iU} - \underline{a}_{iU}\big)$

Definition 9: The second parameterized representation of the function H_2 given in (1) is defined by $H_2\big(x\big) = \sum_{i=1}^{k} \big\{a_i\big(r_{1i}, r_{2i}, r_{3i}\big) : r_{1i}, r_{2i}, r_{3i} \in \big[0, 1\big]\big\}\, h_i\big(x\big)$, $a_i\big(r_{1i}, r_{2i}, r_{3i}\big) = a_{iL}\big(r_{1i}\big) + r_{2i}\big(a_{iU}\big(r_{3i}\big) - a_{iL}\big(r_{1i}\big)\big)$

3.2 Standard Form Type-2 Interval-Valued Optimization Problem

The standard form of unconstrained Type-2 interval maximization problem is as follows:

$$Maximize/Minimize\ \ H_2(x) \tag{2}$$

subject to $x \in S \subseteq \mathbb{R}^n$

where $H_2(x)$ may be represented by either

$H_2(x) = \left[\left(\underline{h}_L(x), \bar{h}_L(x)\right), \left(\underline{h}_U(x), \bar{h}_U(x)\right)\right]$ or

$$H_2(x) = \sum_{i=1}^{n} \left[\left(\underline{a}_{iL}, \bar{a}_{iL}\right), \left(\underline{a}_{iU}, \bar{a}_{iU}\right)\right] g_i(x)$$

and $\underline{h}_L, \bar{h}_L, \underline{h}_U, \bar{h}_U, g_i : S \to \mathbb{R}, i = 1, \ldots, n.$

The corresponding second step parametric form of (2) is as follows:

$$Maximize/Minimize\ h(r_1, r_2, r_3, x) \tag{3}$$

subject to $x \in S \subseteq \mathbb{R}^n, \quad r_i \in [0, 1]$

where

either, $h(r_1, r_2, r_3, x) = h_L(x) + p(h_U(x) - h_L(x))$

$$or\ h(r_1, r_2, r_3, x) = \sum_{i=1}^{n} a_i(r_1, r_2, r_3) g_i(x) \text{ and}$$

and $a_i(r_1, r_2, r_3) = a_{iL}(r_1) + r_2(a_{iU}(r_3) - a_{iL}(r_1))$,

$a_{iL}(r_1) = \underline{a}_{iL} + r_1\left(\overline{a}_{iL} - \underline{a}_{iL}\right), \ a_{iU}(r_2) = \underline{a}_{iU} + r_2\left(\overline{a}_{iU} - \underline{a}_{iU}\right)$

Definition 10: The support of the optimization problem (3) is the set of four real-valued functions $\{H_{S1}, H_{S2}, H_{S3}, H_{S4}\}$ such that.

$$H_{S1}(x) = \frac{h(0,0,0,x) + h(0,0,1,x) + h(0,1,1,x) + h(1,1,1,x)}{4}$$
$$H_{S2}(x) = \frac{h(0,0,1,x) + h(0,1,1,x) + h(1,1,1,x)}{3}$$
$$H_{S3}(x) = \frac{h(0,1,1,x) + h(1,1,1,x)}{2}$$
$$H_{S4}(x) = h(1,1,1,x)$$

Definition 11: The point $x^* \in S$ will be a local maximizer of maximization problem (2) if $\exists$ a $\delta > 0$ such that $H_2(x^*) \geq_2 H_2(x), \ \forall x \in N(x^*, d) \cap S$,

where $N(x^*, d)$ is a neighbourhood with a centre at x^* and radius d.

Definition 12: The point $x^* \in S$ will be a global maximizer of the maximization problem (3) if $H_2(x^*) \geq_2 H_2(x), \ \forall x \in S$.

Note 1: Similarly, the local and global minimizer of the problem (3) can be defined.

Note 2: The inequality $\geq_2$ used in **Definition 11 and 12** can be written explicitly as follows:

$$H_2(x^*) \geq_2 H_2(x) \Leftrightarrow \begin{cases} H_{S1}(x^*) > H_{S1}(x) \text{ when } H_{S1}(x^*) \neq H_{S1}(x) \\ H_{S2}(x^*) > H_{S2}(x) \text{ when } H_{S1}(x^*) = H_{S1}(x) \\ \text{and when } H_{S2}(x^*) \neq H_{S2}(x) \\ H_{S3}(x^*) > H_{S3}(x) \text{ when } H_{S2}(x^*) = H_{S2}(x) \\ \text{and when } H_{S3}(x^*) \neq H_{S3}(x) \\ H_{S4}(x^*) \geq H_{S4}(x) \text{ when } H_{S3}(x^*) = H_{S3}(x) \end{cases}$$

4 Optimality Conditions of Unconstrained Type-2 Interval-Valued Optimization Problem

4.1 Necessary Condition

Theorem 1: If $x^* \in S$ be a local optimizer (maximizer or minimizer) of the unconstrained Type-2 interval-valued optimization problem (2) or (3), then the following conditions are satisfied:

$$\begin{cases} \nabla H_{S1}(x^*) = 0 \text{ when } H_{S1}(x) \text{ is nonconstant} \\ \nabla H_{S2}(x^*) = 0 \text{ when } H_{S1}(x) \text{ is constant and } H_{S2}(x) \text{is nonconstant} \\ \nabla H_{S3}(x^*) = 0 \text{ when } H_{S2}(x) \text{ is constant and } H_{S3}(x) \text{ is nonconstant} \\ \nabla H_{S4}(x^*) = 0 \text{ when } H_{S3}(x) \text{ is constant} \end{cases}$$

Proof: Here, this theorem has been proved for the minimization case only. The proof is also similar to the maximization case.

Let $x^* \in T$ be a local minimizer of $H_2(x)$.
From the definition of the local minimizer, $H_2(x^*) \leq_2 H_2(x), \forall x \in S \cap N(x^*, d)$.
Then by **Definition 4**, it follows that $\forall x \in S \cap N(x^*, d)$,

$$\begin{cases} H_{S1}(x^*) < H_{S1}(x), \text{ when } H_{S1}(x^*) \neq H_{S1}(x) \\ H_{S2}(x^*) < H_{S2}(x), \text{ when } H_{S1}(x^*) = H_{S1}(x) \\ H_{S3}(x^*) < H_{S3}(x), \text{ when } H_{S1}(x^*) = H_{S1}(x) \text{ and } H_{S2}(x^*) = H_{S2}(x) \\ H_{S4}(x^*) < H_{S4}(x), \text{ when } H_{S2}(x^*) = H_{S2}(x) \text{ and } H_{S3}(x^*) = H_{S3}(x) \end{cases}$$

It follows that, $\forall x \in S \cap N(x^*, d)$

$$\begin{cases} H_{S1}(x^*) \le H_{S1}(x), & \text{when } H_{S1}(x) \text{ is non - constant} \\ H_{S2}(x^*) \le H_{S2}(x), & \text{when } H_{S1}(x) \text{ is constant and } H_{S2}(x) \text{ is non - constant} \\ H_{S3}(x^*) \le H_{S3}(x), & \text{when } H_{S2}(x) \text{ is constant and } H_{S3}(x) \text{ is constant} \\ H_{S4}(x^*) \le H_{S4}(x), & \text{when } H_{S3}(x) \text{ is constant} \end{cases}$$

Then by the necessary conditions of optimality $H_{S1}(x), H_{S2}(x), H_{S3}(x)$, and $H_{S4}(x)$, we have

$$\begin{cases} \nabla H_{S1}(x^*) = 0 \text{ when } H_{S1}(x) \text{ is nonconstant} \\ \nabla H_{S2}(x^*) = 0 \text{ when } H_{S1}(x) \text{is constant and } H_{S2}(x) \text{is nonconstant} \\ \nabla H_{S3}(x^*) = 0 \text{ when } H_{S2}(x) \text{is constant and } H_{S3}(x) \text{ is nonconstant} \\ \nabla H_{S4}(x^*) = 0 \text{ when } H_{S3}(x) \text{ is constant.} \end{cases}$$

Definition 13: Let us consider a twice differentiable real-valued function $f : S \to \mathbb{R}$ on a nonempty open set $S \subseteq \mathbb{R}^n$. Then the Hessian matrix of f is denoted by $\nabla^2 f(x)$ and is defined by the $n \times n$ matrix of second-order partial derivatives of f, i.e.$\nabla^2 f(x) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{n \times n}$.

4.2 Sufficient Conditions

Theorem 2: Suppose each member of the support of the unconstrained Type-2 interval-valued optimization problem (2) is continuously differentiable up to second-order, i.e. $\nabla^2 H_{S1}(x)$, $\nabla^2 H_{S2}(x)$, $\nabla^2 H_{S3}(x)$ and $\nabla^2 H_{S4}(x)$ exist. Further, assume that $x^* \in S$ satisfies the conditions of **Theorem 1**.

Then (i) $x^* \in S$ be a local minimizer of the optimization problem (2), if

$$\begin{cases} \nabla^2 H_{S1}(x^*) \text{ is positive definite when } H_{S1}(x) \text{ is nonconstant} \\ \nabla^2 H_{S2}(x^*) \text{ is positive definite when } H_{S1}(x) \text{ is constant} \\ \text{and } H_{S2}(x) \text{ is nonconstant} \\ \nabla^2 H_{S3}(x^*) \text{ is positive definite when } H_{S2}(x) \text{ is constant} \\ \text{and } H_{S3}(x) \text{ is nonconstant} \\ \nabla^2 H_{S4}(x^*) \text{ is positive definite when } H_{S3}(x) \text{ is constant} \end{cases}$$

(ii) $x^* \in S$ be a local maximizer of the optimization problem (2), if

$$\begin{cases} \nabla^2 H_{S1}(x^*) \text{ is negetive definite when } H_{S1}(x) \text{ is nonconstant} \\ \nabla^2 H_{S2}(x^*) \text{ is negetive definite when } H_{S1}(x) \text{ is constant} \\ \text{and } H_{S2}(x) \text{ is nonconstant} \\ \nabla^2 H_{S3}(x^*) \text{ is negetive definite when } H_{S2}(x) \text{ is constant} \\ \text{and } H_{S3}(x) \text{ is nonconstant} \\ \nabla^2 H_{S4}(x^*) \text{ is negetive definite when } H_{S3}(x) \text{ is constant} \end{cases}$$

Proof: (i) When $H_{S1}(x) \neq$ constant,.

$\nabla H_{S1}(x^*) = 0$ and $\nabla^2 H_{S1}(x^*)$ is positive definite, then $x^* \in S$ will be a local minimizer of $H_{S1}(x)$.

Then $\exists\, d_1 > 0,$ such that $H_{S1}(x^*) \leq H_{S1}(x)$, $\forall x \in S \cap N(x^*, d)$.

where $N(x^*, d_1)$ is an open ball with centre at x^* and radius d_1.

when $H_{S1}(x)$ = constant and $H_{S2}(x) \neq$ constant.

$\nabla H_{S2}(x^*) = 0$ and $\nabla^2 H_{S2}(x^*)$ is positive definite, then $x^* \in S$ will be a local minimizer of $H_{S2}(x)$.

$$\text{Then } \exists\ d_2 > 0,\ \text{such that } H_{S2}(x^*) \leq H_{S2}(x),\ \forall x \in S \cap N(x^*, d_2)$$

where $N(x^*, d_2)$ is an open ball with centre at x^* and radius d_2.

when $H_{S1}(x)$ = constant, $H_{S2}(x)$ = constant and $H_{S3}(x) \neq$ constant,.

$\nabla H_{S3}(x^*) = 0$ and $\nabla^2 H_{S3}(x^*)$ is positive definite, then $x^* \in S$ will be a local minimizer of $H_{S3}(x)$.

$$\text{Then } \exists\ d_3 > 0,\ \text{such that } H_{S3}(x^*) \leq H_{S3}(x),\ \forall x \in S \cap N(x^*, d_3)$$

where $B(x^*, d_3)$ is an open ball with centre at x^* and radius d_3.

When $H_{S1}(x)$ = constant, $H_{S2}(x)$ = constant and $H_{S3}(x)$ = constant,.

$\nabla H_{S4}(x^*) = 0$ and $\nabla^2 H_{S4}(x^*)$ is positive definite, then $x^* \in S$ will be a local minimizer of $H_{S4}(x)$.

$$\text{Then } \exists\, d_4 > 0,\ \text{such that } H_{S4}(x^*) \leq H_{S4}(x),\ \forall x \in S \cap N(x^*, d_4)$$

where $N(x^*, d_4)$ is an open ball with centre at x^* and radius d_4.

$$\text{Let us take } d = \min\{d_1, d_2, d_3, d_4\}$$

Then combining the above conditions, we obtain

$$\forall x \in S \cap N(x^*, d)$$

$$\begin{cases} H_{S1}(x^*) \le H_{S1}(x), & \text{when } H_{S1}(x) \text{ is nonconstant} \\ H_{S2}(x^*) \le H_{S2}(x), & \text{when } H_{S1}(x) \text{ is constant } H_{S2}(x) \text{ is nonconstant} \\ H_{S3}(x^*) \le H_{S3}(x), & \text{when } H_{S2}(x) \text{ is constant and } H_{S3}(x) \text{ is nonconstant} \\ H_{S4}(x^*) \le H_{S4}(x), & \text{when } H_{S3}(x) \text{ is constant.} \end{cases}$$

So, by **Definition 4**, we get

$$H_2(x^*) \le_2 H_2(x),\ \forall x \in S \cap N(x^*, d).$$

Therefore $x^* \in S$ is the local minimizer of the Type-2 interval-valued function H_2.

(ii) The proof of maximization case can be derived similarly.

Example 3: Let us consider the following function for optimization.

$$H_2(x_1, x_2) = \left[\left(-6\left(x_1^2 + x_2^2\right) - 2, \left(x_1^2 + x_2^2\right)\right), \left(\left(x_1^2 + x_2^2\right) + 3, 6\left(x_1^2 + x_2^2\right) + 6\right)\right] \tag{4}$$

Solution: Here $H_{S1}(x_1, x_2) = \frac{x_1^2 + x_2^2}{2} + \frac{7}{4} \ne$ constant.

So, if (x_1^*, x_2^*) be an optimizer of H_2, then from the necessary condition 4.1, it follows that $\nabla H_{S1}(x_1^*, x_2^*) = 0$ which gives $(x_1^*, x_2^*) \equiv (0, 0)$.

$$\text{Now, } \nabla^2 H_{S1}(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ which is a positive definite matrix.}$$

Therefore, from sufficient condition 4.2, it follows that $(x_1^*, x_2^*) \equiv (0, 0)$ is a minimizer of (4).

Example 4: Let us take the following interval optimization problem.

$$\begin{aligned} \text{Minimize } H_2(x_1, x_2) =& [(1, 2), (4, 5)]x_1^2 - [(1, 3), (5, 6)]x_1 x_2 \\ &+ [(2, 3), (6, 7)]x_2^2 + [(-1, 1), (3, 4)] \\ \text{subject to } (x_1, x_2) \in \mathbb{R}^2 & \end{aligned} \tag{5}$$

Solution: The corresponding second step parametrization of the objective function is given by

$h(p_1, p_2, p_3, x_1, x_2) = A(p_1, p_2, p_3)x_1^2 - B(p_1, p_2, p_3)x_1x_2 + C(p_1, p_2, p_3)x_2^2 + D(p_1, p_2, p_3)$
where
$A(p_1, p_2, p_3) = (1 + p_1 + 3p_2 + p_2p_3 - p_1p_2)$
$B(p_1, p_2, p_3) = (1 + 2p_1 + 4p_2 + p_2p_3 - 2p_1p_2)$
$C(p_1, p_2, p_3) = (2 + p_1 + 4p_2 + p_2p_3 - p_1p_2)$
$D(p_1, p_2, p_3) = (-1 + 2p_1 + 4p_2 + p_2p_3 - 2p_1p_2)$
$p_1, p_2, p_3 \in [0, 1]$

Now,

$$H_{S1}(x_1, x_2) = \frac{h(1, 1, 1, x_1, x_2) + h(1, 1, 0, x_1, x_2) + h(1, 0, 0, x_1, x_2) + h(0, 00, x_1, x_2)}{4}$$
$$= \frac{(13x_1^2 - 15x_1x_2 + 15x_2^2 + 7)}{4} \neq \text{constant},$$

Thus, from the optimality conditions of $H_{S1}(x_1, x_2)$ we get

$$\nabla H_{S1}(x_1, x_2) = (0, 0)$$

which implies $(x_1, x_2) = (0, 0)$.

And $\nabla^2 H_{S1}(0, 0) = \begin{pmatrix} 26 & -15 \\ -15 & 30 \end{pmatrix}$ which is positive definite.

Hence, the optimization problem (5) has a minimum value at $(x_1, x_2) = (0, 0)$, and the minimum value is $H_2(0, 0) = [(-1, 1), (3, 4)]$.

5 Application

In this section, the optimal policy of the classical EOQ model with Type-2 interval-valued inventory costs are derived as an application of the optimality theory of Type-2 interval-valued function that are derived in this chapter. To formulate the model, some essential notations and assumptions are given below:

5.1 Notations

Notation	Description
$q(t)$	Inventory level at time t
Q	Initial inventory level
D	Demand rate

(continued)

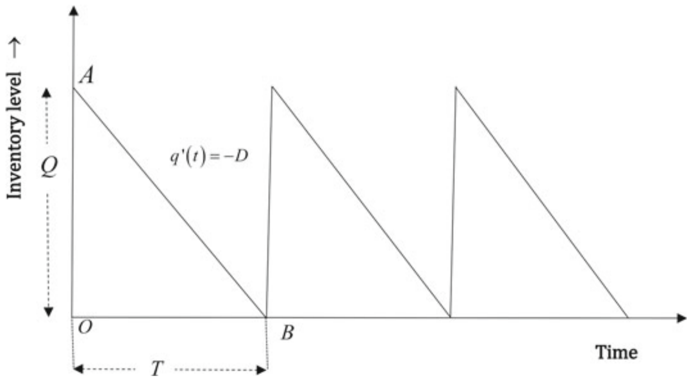

Fig. 1 Inventory level

(continued)

Notation	Description
T	Cycle length
$O_2 = \left[\left(\underline{O}_L, \overline{O}_L\right), \left(\underline{O}_U, \overline{O}_U\right)\right]$	Type-2 interval-valued ordering cost
$H_2 = \left[\left(\underline{h}_L, \overline{h}_L\right), \left(\underline{h}_U, \overline{h}_U\right)\right]$	Type-2 interval-valued holding cost

5.2 Assumptions

(i). Single item is being delivered during per order.
(ii). A known constant demand rate of D units per unit time.
(iii). The order quantity (Q) to replenish inventory arrives all at once just when desired, namely, when the inventory level drops to 0.
(iv). Taking uncertainty under consideration, all the cost components (namely, ordering cost, holding cost) are taken Type-2 interval-valued.
(v). System deals with a constant lead time with planned shortages are not allowed (Fig. 1).

5.3 Model Formulation

The Rate of Change of Inventory Level is Governed by the Differential Equation

$$\frac{dq(t)}{dt} = -D \tag{6}$$

With the conditions

$$q(0) = Q \text{ and } q(T) = 0. \tag{7}$$

Solving (6) and using (7), we get

$$q(t) = D(T - t),\ \ 0 \leq t \leq T \tag{8}$$

Type-2 interval-valued ordering cost: $O_2 = \left[\left(\underline{O}_L, \overline{O}_L\right), \left(\underline{O}_U, \overline{O}_U\right)\right]$.

Tpye-2 interval-valued holding cost: $HC_2 = \int_0^T \left[\left(\underline{h}_L, \overline{h}_L\right), \left(\underline{h}_U, \overline{h}_U\right)\right] q(t)\, dt = \frac{1}{2}\left[\left(\underline{h}_L, \overline{h}_L\right), \left(\underline{h}_U, \overline{h}_U\right)\right] DT^2$.

Therefore Type-2 interval-valued average cost is given by

$$\begin{aligned}
AC_2(T) &= \frac{1}{T}\left[\left(T\underline{C}(T), T\overline{C}(T)\right), \left(T\underline{C}(T), T\overline{C}(T)\right)\right] \\
&= \frac{1}{2T}\left[\left(\underline{h}_L, \overline{h}_L\right), \left(\underline{h}_U, \overline{h}_U\right)\right] DT^2 + \left[\left(\underline{O}_L, \overline{O}_L\right), \left(\underline{O}_U, \overline{O}_U\right)\right] \\
&= \left[\left(\frac{\underline{h}_L DT}{2} + \frac{\underline{O}_L}{T}, \frac{\overline{h}_L DT}{2} + \frac{\overline{O}_L}{T}\right), \left(\frac{\underline{h}_U DT}{2} + \frac{\underline{O}_U}{T}, \frac{\overline{h}_U DT}{2} + \frac{\overline{O}_U}{T}\right)\right]
\end{aligned}$$

Hence, we obtain a Type-2 interval-valued unconstrained optimization problem

$$Minimize\ AC_2(T)$$

Now, the second step parametric form of $AC_2(T)$ is given by $AC_2(T, r_1, r_2, r_3)$

$$\begin{aligned}
&= \left(\frac{\underline{h}_L DT}{2} + \frac{\underline{O}_L}{T} + r_1\left(\frac{\bar{h}_L DT}{2} + \frac{\bar{O}_L}{T} - \frac{\underline{h}_L DT}{2} - \frac{\underline{O}_L}{T}\right)\right) \\
&+ \ r_2 \left(\begin{array}{l} \frac{\underline{h}_U DT}{2} + \frac{\underline{O}_U}{T} + r_3\left(\frac{\bar{h}_U DT}{2} + \frac{\bar{O}_U}{T} - \frac{\underline{h}_U DT}{2} - \frac{\underline{O}_U}{T}\right) \\ -\frac{\underline{h}_L DT}{2} - \frac{\underline{O}_L}{T} - r_1\left(\frac{\bar{h}_L DT}{2} + \frac{\bar{O}_L}{T} - \frac{\underline{h}_L DT}{2} - \frac{\underline{O}_L}{T}\right) \end{array}\right)
\end{aligned}$$

$$\begin{aligned}
(AC_2(T))_{s1} &= \frac{AC_2(T, 0, 0, 0) + AC_2(T, 1, 0, 0) + AC_2(T, 1, 1, 0) + AC_2(T, 1, 1, 1)}{4} \\
&= \frac{\left(\underline{h}_L + \overline{h}_L + \underline{h}_U + \overline{h}_U\right) DT^2 + 2\left(\underline{O}_L + \overline{O}_L + \underline{O}_U + \overline{O}_U\right)}{8T}
\end{aligned}$$

Now, using **Theorem 1**, we obtain

$$\begin{aligned}
&\nabla (AC_2(T))_{s1} = 0 \\
&\Rightarrow T^* = \sqrt{\frac{2\left(\underline{O}_L + \overline{O}_L + \underline{O}_U + \overline{O}_U\right)}{D\left(\underline{h}_L + \overline{h}_L + \underline{h}_U + \overline{h}_U\right)}}
\end{aligned} \tag{9}$$

Now, $\nabla^2(AC_2(T))_{s1} = \frac{(\underline{O}_L+\overline{O}_L+\overline{O}_U+\underline{O}_U)}{2T^3} > 0$.
Therefore, using **Theorem 2**, we get the minimum value of average cost,

$$\begin{aligned} AC_2(T^*)_{\min} &= \frac{1}{T}\left[\left(T\underline{C}(T^*), T\overline{C}(T^*)\right), \left(T\underline{C}(T^*), T\overline{C}(T^*)\right)\right] \\ &= \frac{1}{2T}\left[\left(\underline{h}_L, \overline{h}_L\right), \left(\underline{h}_U, \overline{h}_U\right)\right]DT^2 + \left[\left(\underline{O}_L, \overline{O}_L\right), \left(\underline{O}_U, \overline{O}_U\right)\right] \\ &= \left[\left(\frac{\underline{h}_L DT^*}{2} + \frac{\underline{O}_L}{T^*}, \frac{\overline{h}_L DT^*}{2} + \frac{\overline{O}_L}{T^*}\right), \left(\frac{\underline{h}_U DT^*}{2} + \frac{\underline{O}_U}{T^*}, \frac{\overline{h}_U DT^*}{2} + \frac{\overline{O}_U}{T^*}\right)\right] \end{aligned} \tag{10}$$

And the optimal initial inventory level is given by

$$Q^* = DT^* = \sqrt{\frac{2D\left(\underline{O}_L + \overline{O}_L + \underline{O}_U + \overline{O}_U\right)}{\left(\underline{h}_L + \overline{h}_L + \underline{h}_U + \overline{h}_U\right)}} \tag{11}$$

5.4 *Numerical Illustration*

To illustrate the optimal policy of the proposed model, a numerical example is considered as follows:

Example 5: The values of parameters for these examples are given below:

$$D = 350, \left[\left(\underline{O}_L, \bar{O}_L\right), \left(\underline{O}_U, \bar{O}_U\right)\right] = [(250, 260), (290, 300)],$$
$$\left[\left(\underline{h}_L, \bar{h}_L\right), \left(\underline{h}_U, \bar{h}_U\right)\right] = [(3, 4), (7, 8)].$$

Solution:

The optimal cycle, lot-size, and average cost are obtained by using Eqs. (9)-(11). The optimal values of these inventory parameters are

$$T^* = 0.142857, \ \ Q^* = 17500.$$
$$\left[\left(A\underline{C}_L^*, A\overline{C}_L^*\right), \left(A\underline{C}_U^*, A\overline{C}_U^*\right)\right] = [(1825, 1920), (2205, 2300)]$$

6 Conclusion

In this chapter, the idea of the parametric representation of the Type-2 interval has been introduced. Then, using this concept, a new definition of Type-2 interval order relation has been introduced. Then, optimality conditions (necessary and sufficient) of an unconstrained Type-2 interval-valued optimization have been derived as an application of Type-2 interval ranking.

Future research may derive the optimality conditions of non-linear constrained optimization problems of Type-2 interval-valued functions. Besides one may apply the concept of this work in the economical modelling, bio-economical modelling under imprecise circumstances.

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