Fitted Mesh Methods for a Class of Weakly Coupled System of Singularly Perturbed Convection–Diffusion Equations

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Abstract In this paper, a class of singularly perturbed coupled linear systems of second-order ordinary differential equations of convection–diffusion type is considered on the interval [0, 1]. Due to the presence of different perturbation parameters multiplying the diffusion terms of the coupled system, each of the solution components exhibits multiple layers in the neighbourhood of the origin. This fact is proved in the estimates of the derivatives of the solution. A numerical method composed of an upwind finite difference scheme applied on a piecewise uniform Shishkin mesh that resolves all the layers is suggested to solve the problem. The method is proved to be almost first-order convergent in the maximum norm uniformly in all the perturbation parameters. Numerical examples are provided to support the theory.

Keywords Singular perturbation problems · System of convection-diffusion equations · Finite difference method · Shishkin Mesh · Parameter uniform method

1 Introduction

Singularly perturbed differential equations of convection–diffusion type appear in several branches of applied mathematics. Roos et al. [1] describes linear convection–diffusion equations and related non-linear flow problems. Modelling real-life problems such as fluid flow problems, control problems, heat transport problems, river networks results in singularly perturbed convection–diffusion equations. Some of those models were discussed in [2]. A form of linearized Navier Stokes equations called Oseen system of equations, which models many of the physical problems, is a system of singularly perturbed convection–diffusion equations. Also systems

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of singularly perturbed convection–diffusion equations have applications in control problems [3].

For a broad introduction to singularly perturbed boundary value problems of convection–diffusion type and robust computational techniques to solve them, one can refer to [4–6]. In [7], a coupled system of two singularly perturbed convection–diffusion equations is analysed and a parameter uniform numerical method is suggested to solve the same. Here, in this paper, the following weakly coupled system of n-singularly perturbed convection–diffusion equations is considered.

$$L\mathbf{u}(x) \equiv E\mathbf{u}''(x) + A(x)\mathbf{u}'(x) - B(x)\mathbf{u}(x) = \mathbf{f}(x),\ \ x \in \Omega = (0, 1) \tag{1}$$

$$\mathbf{u}(0) = \mathbf{l},\quad \mathbf{u}(1) = \mathbf{r}, \tag{2}$$

where $\mathbf{u}(x) = \big(u_1(x), u_2(x), \ldots, u_n(x)\big)^T$, $\mathbf{f}(x) = \big(f_1(x), f_2(x), \ldots, f_n(x)\big)^T$,

$$E = \begin{bmatrix} \varepsilon_1 & 0 & \ldots & 0 \\ 0 & \varepsilon_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \varepsilon_n \end{bmatrix}, A = \begin{bmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & a_n \end{bmatrix},\ B = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{bmatrix}.$$

Here, $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ are distinct small positive parameters and for convenience, it is assumed that $\varepsilon_i < \varepsilon_j$, for $i < j$. The functions a_i, b_{ij} and f_i, for all i and j, are taken to be sufficiently smooth on $\overline{\Omega}$. It is further assumed that, $a_i(x) \geq \alpha > 0,\ b_{ij}(x) < 0,\ i \neq j$ and $\sum_{j=1}^{n} b_{ij}(x) \geq \beta > 0$, for all $i = 1, 2, \ldots, n$. The case $a_i(x) < 0$ can be treated in a similar way with a transformation of x to $1 - x$.

In [9], Linss has analysed a broader class of weakly coupled system of singularly perturbed convection–diffusion equations and presented an estimate of the derivatives of u_i depending only on ε_i, for $i = 1, 2, \ldots, n$. He has claimed first order and almost first-order convergence if solved on Bakhvalov and Shishkin meshes, respectively, with the classical finite difference scheme.

The reduced problem corresponding to (1)–(2) is

$$\begin{aligned} L_0\mathbf{u}_0(x) &\equiv A(x)\mathbf{u}_0'(x) - B(x)\mathbf{u}_0(x) = \mathbf{f}(x),\ \ x \in \Omega \\ \mathbf{u}_0(1) &= \mathbf{r}, \end{aligned} \tag{3}$$

where $\mathbf{u}_0(x) = (u_{01}(x), u_{02}(x), ..., u_{0n}(x))^T$.

If $u_k(0) \neq u_{0k}(0)$ for any k such that $0 \leq k \leq n$, then a boundary layer of width $O(\varepsilon_k)$ is expected near $x = 0$ in each of the solution component $u_i,\ 1 \leq i \leq k$.

Notations. For any real valued function y on D, the norm of y is defined as $\|y\|_D = \sup_{x \in D}|y(x)|$. For any vector valued function $\mathbf{z}(x) = (z_1(x), z_2(x), \ldots,$

$z_n(x))^T$, $\|\mathbf{z}\|_D = \max\left\{\|z_1\|_D, \|z_2\|_D, \ldots, \|z_n\|_D\right\}$. For any mesh function Y on a mesh $D^N = \left\{x_j\right\}_{j=0}^{N}$, $\|Y\|_{D^N} = \max\limits_{0\le j\le N}|Y(x_j)|$ and for any vector valued mesh function $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)^T$, $\|\mathbf{Z}\|_{D^N} = \max\left\{\|Z_1\|_{D^N}, \|Z_2\|_{D^N}, \ldots, \|Z_n\|_{D^N}\right\}$.

Throughout this paper, C denotes a generic positive constant which is independent of the singular perturbation and discretization parameters.

2 Analytical Results

In this section, a maximum principle, a stability result and estimates of the derivatives of the solution of the system of Eqs. (1)–(2) are presented.

Lemma 1 (Maximum Principle) *Let $\boldsymbol{\psi} = (\psi_1, \psi_2, ..., \psi_n)^T$ be in the domain of L with $\boldsymbol{\psi}(0) \ge \mathbf{0}$ and $\boldsymbol{\psi}(1) \ge \mathbf{0}$. Then $L\boldsymbol{\psi} \le \mathbf{0}$ on Ω implies that $\boldsymbol{\psi} \ge \mathbf{0}$ on $\overline{\Omega}$.*

Lemma 2 (Stability Result) *Let $\boldsymbol{\psi}$ be in the domain of L, then for $x \in \overline{\Omega}$ and $1 \le i \le n$*

$$|\psi_i(x)| \le \max\left\{\|\boldsymbol{\psi}(0)\|,\ \|\boldsymbol{\psi}(1)\|,\ \frac{1}{\beta}\|L\boldsymbol{\psi}\|\right\}.$$

Theorem 1 *Let $\mathbf{u}$ be the solution of (1)–(2), then for $x \in \overline{\Omega}$ and $1 \le i \le n$, the following estimates hold.*

$$|u_i(x)| \le C\max\left\{\|\mathbf{l}\|,\ \|\mathbf{r}\|,\ \frac{1}{\beta}\|\mathbf{f}\|\right\}, \tag{4}$$

$$|u_i^{(k)}(x)| \le C\varepsilon_i^{-k}\Big(\|\mathbf{u}\| + \varepsilon_i\|\mathbf{f}\|\Big) \quad for \quad k = 1, 2, \tag{5}$$

$$|u_i^{(3)}(x)| \le C\varepsilon_i^{-2}\varepsilon_1^{-1}\Big(\|\mathbf{u}\| + \varepsilon_i\|\mathbf{f}\|\Big) + \varepsilon_i^{-1}|f_i'(x)|. \tag{6}$$

Proof The estimate (4) follows immediately from Lemma 2 and Eq. (1). Let $x \in [0, 1]$, then for each i, $1 \le i \le n$, there exists $a \in [0, 1 - \varepsilon_i]$ such that $x \in N_a = [a, a + \varepsilon_i]$. By the mean value theorem, there exists $y_i \in (a, a + \varepsilon_i)$ such that

$$u_i'(y_i) = \frac{u_i(a + \varepsilon_i) - u_i(a)}{\varepsilon_i}$$

and hence

$$|u_i'(y_i)| \le C\varepsilon_i^{-1}\|\mathbf{u}\|.$$

Also,

$$u_i'(x) = u_i'(y_i) + \int_{y_i}^{x} u_i''(s)ds.$$

Substituting for $u_i''(s)$ from (1), $|u_i'(x)| \leq C\varepsilon_i^{-1}\Big(\|\mathbf{u}\| + \varepsilon_i\|\mathbf{f}\|\Big)$. Again from (1), $|u_i''(x)| \leq C\varepsilon_i^{-2}\Big(\|\mathbf{u}\| + \varepsilon_i\|\mathbf{f}\|\Big)$. Differentiating (1) once and substituting the above bounds lead to

$$|u_i^{(3)}(x)| \leq C\varepsilon_i^{-2}\varepsilon_1^{-1}\Big(\|\mathbf{u}\| + \varepsilon_i\|\mathbf{f}\|\Big) + \varepsilon_i^{-1}|f_i'(x)|.$$

2.1 Shishkin Decomposition of the Solution

The solution $\mathbf{u}$ of the problem (1)–(2) can be decomposed into smooth $\mathbf{v} = (v_1, ..., v_n)^T$ and singular $\mathbf{w} = (w_1, ..., w_n)^T$ components given by $\mathbf{u} = \mathbf{v} + \mathbf{w}$, where

$$L\mathbf{v} = \mathbf{f}, \quad \mathbf{v}(0) = \boldsymbol{\gamma}, \quad \mathbf{v}(1) = \mathbf{r}, \tag{7}$$

$$L\mathbf{w} = \mathbf{0}, \quad \mathbf{w}(0) = \mathbf{l} - \mathbf{v}(0), \quad \mathbf{w}(1) = \mathbf{0}, \tag{8}$$

where $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T$ is to be chosen.

2.1.1 Estimates for the Bounds on the Smooth Components and Their Derivatives

Theorem 2 *For a proper choice of $\boldsymbol{\gamma}$, the solution of the problem (7) satisfies for $1 \leq i \leq n$ and $0 \leq k \leq 3$,*

$$|v_i^{(k)}(x)| \leq C(1 + \varepsilon_i^{2-k}), \quad x \in \overline{\Omega}.$$

Proof Considering the layer pattern of the solution, first, the decomposition is done with ε_n, for all the components of $\mathbf{v}$. The second level decomposition with ε_{n-1} is for the first $n-1$ components of $\mathbf{v}$. Then, the decomposition continues with ε_{n-2} for the first $n-2$ components of $\mathbf{v}$ and so on. It is carried out in the following way. First, the smooth component $\mathbf{v}$ is decomposed into

$$\mathbf{v} = \mathbf{y}_n + \varepsilon_n\mathbf{z}_n + \varepsilon_n^2\mathbf{q}_n \tag{9}$$

where $\mathbf{y}_n = (y_{n1}, y_{n2}, \ldots, y_{nn})^T$ is the solution of

$$A(x)\mathbf{y}_n'(x) - B(x)\mathbf{y}_n(x) = \mathbf{f}(x), \quad \mathbf{y}_n(1) = \mathbf{r}, \tag{10}$$

$\mathbf{z}_n = (z_{n1}, z_{n2}, \ldots, z_{nn})^T$ is the solution of

$$A(x)\mathbf{z}_n'(x) - B(x)\mathbf{z}_n(x) = -\varepsilon_n^{-1}E\mathbf{y}_n''(x), \;\; \mathbf{z}_n(1) = \mathbf{0} \tag{11}$$

and $\mathbf{q}_n = (q_{n1}, q_{n2}, \ldots, q_{nn})^T$ is the solution of

$$L\mathbf{q}_n(x) = -\varepsilon_n^{-1} E\mathbf{z}_n''(x), \;\; \mathbf{q}_n(1) = \mathbf{0} \text{ and } \mathbf{q}_n(0) \text{ remains to be chosen.} \tag{12}$$

Using the fact that $\varepsilon_n^{-1}E$ is a matrix of bounded entries, and from the results in [10] for (10) and (11), it is not hard to see that

$$\|\mathbf{y}_n^{(k)}\| \le C \;\text{ and }\; \|\mathbf{z}_n^{(k)}\| \le C, \;\; 0 \le k \le 3. \tag{13}$$

Now, using Theorem 1 and (13), with the choice that $q_{nn}(0) = 0$,

$$|q_{nn}^{(k)}(x)| \le C\varepsilon_n^{-k}, \;\; 0 \le k \le 3. \tag{14}$$

Then from (9), it is clear that $v_n(0) = \gamma_n = y_{nn}(0) + \varepsilon_n z_{nn}(0)$. Also from (13) and (14),

$$|v_n^{(k)}(x)| \le C(1 + \varepsilon_n^{2-k}), \;\; 0 \le k \le 3. \tag{15}$$

Now, having found the estimates of $v_n^{(k)}$, to estimate the bounds $v_i^{(k)}$, for $1 \le i \le n-1$, the following notations are introduced, for $1 \le l \le n$,

$$E_l = \begin{bmatrix} \varepsilon_1 & 0 & \ldots & 0 \\ 0 & \varepsilon_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \varepsilon_l \end{bmatrix}, A_l = \begin{bmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & a_l \end{bmatrix}, \; B_l = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1l} \\ b_{21} & b_{22} & \ldots & b_{2l} \\ \vdots & \vdots & & \vdots \\ b_{l1} & b_{l2} & \ldots & b_{ll} \end{bmatrix},$$

$\tilde{\mathbf{q}}_l = \left(q_{l1}, q_{l2}, \ldots, q_{l(l-1)}\right)^T$, $\mathbf{g}_{(l-1)} = \left(g_{(l-1)1}, g_{(l-1)2}, \ldots, g_{(l-1)(l-1)}\right)^T$, with $g_{(l-1)j} = -\dfrac{\varepsilon_j}{\varepsilon_l} z_{lj}'' + b_{jl} q_{ll}$.

Now, considering the first $(n-1)$ equations of the system (12), it follows that

$$\tilde{L}_n\tilde{\mathbf{q}}_n \equiv E_{n-1}\tilde{\mathbf{q}}_n''(x) + A_{n-1}(x)\tilde{\mathbf{q}}_n'(x) - B_{n-1}(x)\tilde{\mathbf{q}}_n(x) = \mathbf{g}_{n-1}(x), \tag{16}$$

where $\tilde{\mathbf{q}}_n(1) = \mathbf{0}$ and $\tilde{\mathbf{q}}_n(0)$ remains to be chosen.

Furthermore, decomposing $\tilde{\mathbf{q}}_n$ in a similar way to (9), we obtain

$$\tilde{\mathbf{q}}_n = \mathbf{y}_{n-1} + \varepsilon_{n-1}\mathbf{z}_{n-1} + \varepsilon_{n-1}^2\mathbf{q}_{n-1} \tag{17}$$

where $\mathbf{y}_{n-1} = \left(y_{(n-1)1}, y_{(n-1)2}, \ldots, y_{(n-1)(n-1)}\right)^T$ is the solution of the problem

$$A_{n-1}(x)\mathbf{y}_{n-1}'(x) - B_{n-1}(x)\mathbf{y}_{n-1}(x) = \mathbf{g}_{n-1}(x), \;\; \mathbf{y}_{n-1}(1) = \mathbf{0}, \tag{18}$$

$\mathbf{z}_{n-1} = \left(z_{(n-1)1}, z_{(n-1)2}, \ldots, z_{(n-1)(n-1)}\right)^T$ is the solution of the problem

$$A_{n-1}(x)\mathbf{z}_{n-1}'(x) - B_{n-1}(x)\mathbf{z}_{n-1}(x) = -\varepsilon_{n-1}^{-1} E_{n-1}\mathbf{y}_{n-1}''(x), \ \mathbf{z}_{n-1}(1) = \mathbf{0} \tag{19}$$

and $\mathbf{q}_{n-1} = \left(q_{(n-1)1}, q_{(n-1)2}, \ldots, q_{(n-1)(n-1)}\right)^T$ is the solution of the problem

$$\tilde{L}_n\mathbf{q}_{n-1}(x) = -\varepsilon_{n-1}^{-1} E_{n-1}\mathbf{z}_{n-1}''(x), \ \mathbf{q}_{n-1}(1) = \mathbf{0} \text{ and } \mathbf{q}_{n-1}(0) \text{ remains to be chosen.} \tag{20}$$

Now choose $\mathbf{q}_{n-1}(0)$ so that its $(n-1)$th component is zero (i.e. $q_{(n-1)(n-1)}(0) = 0$). Problem (18) is similar to the problem (11). Using the estimates (13)–(14), the solution of the problem (18) satisfies the following bound for $0 \le k \le 3$.

$$\|\mathbf{y}_{n-1}^{(k)}\| \le C\left(1 + \varepsilon_n^{1-k}\right). \tag{21}$$

Using (21) and Lemma 2.2 in [10], the solution of the problem (19) satisfies

$$\|\mathbf{z}_{n-1}\| \le C\varepsilon_n^{-1}. \tag{22}$$

and from (19), for $1 \le k \le 3$,

$$\|\mathbf{z}_{n-1}^{(k)}\| \le C\varepsilon_n^{-k}. \tag{23}$$

Now, using Theorem 1 and (23), the following estimate holds:

$$|q_{(n-1)(n-1)}^{(k)}(x)| \le C\varepsilon_n^{-2}\varepsilon_{n-1}^{-k}, \quad 0 \le k \le 3. \tag{24}$$

By the choice of $q_{(n-1)(n-1)}(0)$, from (9) and (17), it is clear that $v_{n-1}(0) = \gamma_{n-1} = y_{n(n-1)}(0) + \varepsilon_n z_{n(n-1)}(0) + \varepsilon_n^2 y_{(n-1)(n-1)}(0) + \varepsilon_n^2\varepsilon_{n-1} z_{(n-1)(n-1)}(0)$. Also, the estimates (21)–(24) imply that

$$|v_{n-1}^{(k)}(x)| \le C(1 + \varepsilon_{n-1}^{2-k}). \tag{25}$$

Proceeding in a similar way, one can derive singularly perturbed systems of l equations, $l = n-2, \ n-3, \ldots, \ 2, \ 1$,

$$\tilde{L}_{l+1}\tilde{\mathbf{q}}_{l+1} \equiv E_l\tilde{\mathbf{q}}_{l+1}''(x) + A_l(x)\tilde{\mathbf{q}}_{l+1}'(x) - B_l(x)\tilde{\mathbf{q}}_{l+1}(x) = \mathbf{g}_l(x), \tag{26}$$

with $\tilde{\mathbf{q}}_{l+1}(1) = \mathbf{0}$ and $\tilde{\mathbf{q}}_{l+1}(0)$, to be chosen.
Now, decomposing $\tilde{\mathbf{q}}_{l+1}$ in a similar way to (9), we obtain

$$\tilde{\mathbf{q}}_{l+1} = \mathbf{y}_l + \varepsilon_l\mathbf{z}_l + \varepsilon_l^2\mathbf{q}_l \tag{27}$$

where $\mathbf{y}_l = (y_{l1}, y_{l2}, \ldots, y_{ll})^T$ and $\mathbf{z}_l = (z_{l1}, z_{l2}, \ldots, z_{ll})^T$ satisfy

$$A_l(x)\mathbf{y}_l'(x) - B_l(x)\mathbf{y}_l(x) = \mathbf{g}_l(x), \quad \mathbf{y}_l(1) = \mathbf{0}, \tag{28}$$

$$A_l(x)\mathbf{z}_l'(x) - B_l(x)\mathbf{z}_l(x) = -\varepsilon_l^{-1}E_l\mathbf{y}_l''(x), \ \mathbf{z}_l(1) = \mathbf{0} \tag{29}$$

respectively and $\mathbf{q}_l = (q_{l1}, q_{l2}, \ldots, q_{ll})^T$ is the solution of the problem

$$\tilde{L}_{l+1}\mathbf{q}_l(x) = -\varepsilon_l^{-1} E_l \mathbf{z}_l''(x), \ \mathbf{q}_l(1) = \mathbf{0} \text{ where } \mathbf{q}_l(0) \text{ remains to be chosen.} \tag{30}$$

We choose $\mathbf{q}_l(0)$ so that its lth component is zero (i.e. $q_{ll}(0) = 0$).
From (28) it is clear that, for $0 \le k \le 3$,

$$\|\mathbf{y}_l^{(k)}\| \le C\left(1 + \varepsilon_{l+1}^{1-k}\right) \prod_{i=l+2}^{n} \varepsilon_i^{-2}. \tag{31}$$

Using (31) in (29), $\|\mathbf{z}_l\| \le C\left(1 + \varepsilon_{l+1}^{-1}\right) \prod_{i=l+2}^{n} \varepsilon_i^{-2}$ and for $1 \le k \le 3$,

$$\|\mathbf{z}_l^{(k)}\| \le C\left(1 + \varepsilon_{l+1}^{-k}\right) \prod_{i=l+2}^{n} \varepsilon_i^{-2}. \tag{32}$$

Now, using Theorem 1 for $\mathbf{q}_l$, we obtain

$$|q_{ll}^{(k)}(x)| \le C\varepsilon_l^{-k} \prod_{i=l+1}^{n} \varepsilon_i^{-2}, \quad 0 \le k \le 3. \tag{33}$$

Since $q_{ll}(0) = 0$, it is clear that

$$v_l(0) = \gamma_l = y_{nl}(0) + \varepsilon_n z_{nl}(0) + \varepsilon_n^2 y_{(n-1)l}(0) + \ldots + \left(\prod_{j=l+1}^{n} \varepsilon_j^2\right) \varepsilon_l z_{ll}(0).$$

Also, the estimates (31)–(33) imply that

$$|v_l^{(k)}(x)| \le C(1 + \varepsilon_l^{2-k}), \quad 0 \le k \le 3. \tag{34}$$

Thus, by the choice made for $\gamma_n, \ \gamma_{n-1}, \ldots, \gamma_2, \ \gamma_1$, the solution $\mathbf{v}$ of the problem (7) satisfies the following bound for $1 \le i \le n$ and $0 \le k \le 3$

$$|v_i^{(k)}(x)| \le C(1 + \varepsilon_i^{2-k}), \quad x \in \overline{\Omega}. \tag{35}$$

2.1.2 Estimates for the Bounds on the Singular Components and Their Derivatives

Let $\mathscr{B}_i(x), \ 1 \le i \le n$, be the layer functions defined on $[0, 1]$ as

$$\mathscr{B}_i(x) = \exp(-\alpha x/\varepsilon_i). \tag{36}$$

Theorem 3 *Let* $\mathbf{w}(x)$ *be the solution of (8), then for* $x \in \overline{\Omega}$ *and* $1 \leq i \leq n$ *the following estimates hold.*

$$|w_i(x)| \leq C\mathscr{B}_n(x), \tag{37}$$
$$|w_i'(x)| \leq C\Big(\varepsilon_i^{-1}\mathscr{B}_i(x) + \varepsilon_n^{-1}\mathscr{B}_n(x)\Big), \tag{38}$$
$$|w_i^{(2)}(x)| \leq C\sum_{k=i}^{n}\varepsilon_k^{-2}\mathscr{B}_k(x), \tag{39}$$
$$|w_i^{(3)}(x)| \leq C\varepsilon_i^{-1}\Big(\sum_{k=1}^{i-1}\varepsilon_k^{-1}\mathscr{B}_k(x) + \sum_{k=i}^{n}\varepsilon_k^{-2}\mathscr{B}_k(x)\Big). \tag{40}$$

Proof Consider the barrier function $\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_n)^T$ defined by $\phi_i(x) = C\mathscr{B}_n(x)$, $1 \leq i \leq n$. Put $\boldsymbol{\psi}^{\pm}(x) = \boldsymbol{\phi}(x) \pm \mathbf{w}(x)$, then for sufficiently large C, $\boldsymbol{\psi}^{\pm}(0) \geq \mathbf{0}$, $\boldsymbol{\psi}^{\pm}(1) \geq \mathbf{0}$ and $L\boldsymbol{\psi}^{\pm}(x) \leq \mathbf{0}$. Using Lemma 1, it follows that, $\boldsymbol{\psi}^{\pm}(x) \geq \mathbf{0}$. Hence, estimate (37) holds. From (8), for $1 \leq i \leq n$

$$\varepsilon_i(w_i')'(x) + a_i(x)(w_i')(x) = g_i(x) \tag{41}$$

where $g_i(x) = \sum_{j=1}^{n} b_{ij}(x)w_j(x)$. Let $\mathscr{A}_i(x) = \int_0^x a_i(s)ds$, then solving (41) leads to

$$w_i'(x) = w_i'(0)\exp\big(-\mathscr{A}_i(x)/\varepsilon_i\big) + \varepsilon_i^{-1}\int_0^x g_i(t)\exp\big(-(\mathscr{A}_i(x) - \mathscr{A}_i(t))/\varepsilon_i\big)dt.$$

Using Theorem 1 for $\mathbf{w}$, $|w_i'(0)| \leq C\varepsilon_i^{-1}$. Further from the inequalities, $\exp\big(-(\mathscr{A}_i(x) - \mathscr{A}_i(t))/\varepsilon_i\big) \leq \exp\big(-\alpha(x-t)/\varepsilon_i\big)$ for $t \leq x$ and $|g_i(t)| \leq C\mathscr{B}_n(t)$, it is clear that

$$|w_i'(x)| \leq C\varepsilon_i^{-1}\exp\big(-\alpha x/\varepsilon_i\big) + C\varepsilon_i^{-1}\int_0^x \exp\big(-\alpha t/\varepsilon_n\big)\exp\big(-\alpha(x-t)/\varepsilon_i\big)dt.$$

Using integration by parts, it is not hard to see that

$$|w_i'(x)| \leq C\varepsilon_i^{-1}\exp\big(-\alpha x/\varepsilon_i\big) + C\varepsilon_n^{-1}\exp\big(-\alpha x/\varepsilon_n\big). \tag{42}$$

Differentiating (41) once leads to

$$\varepsilon_i(w_i'')'(x) + a_i(x)(w_i'')(x) = h_i(x) \equiv g_i'(x) - a_i'(x)w_i'(x). \tag{43}$$

Then,

$$w_i''(x) = w_i''(0)\exp\big(-\mathscr{A}_i(x)/\varepsilon_i\big) + \varepsilon_i^{-1}\int_0^x h_i(t)\exp\big(-(\mathscr{A}_i(x)-\mathscr{A}_i(t))/\varepsilon_i\big)dt.$$

Using $|w_i''(0)| \le C\varepsilon_i^{-2}$, $|h_i(t)| \le C\sum_{k=1}^{n}\varepsilon_k^{-1}\mathscr{B}_k(t)$ and hence

$$|w_i''(x)| \le C\sum_{k=i}^{n}\varepsilon_k^{-2}\mathscr{B}_k(x). \tag{44}$$

Using the bounds given in (42) and (44) in (43), (40) can be derived.

As the estimates of the derivatives are to be used in the different segments of the piecewise uniform Shishkin meshes, the estimates are improved using the layer interaction points as given below.

2.1.3 Improved Estimates for the Bounds on the Singular Components and Their Derivatives

For $\mathscr{B}_i, \mathscr{B}_j$, each $i, j,\ 1 \le i < j \le n$ and each $s = 1, 2$ the point $x_{i,j}^{(s)}$ is defined by

$$\frac{\mathscr{B}_i(x_{i,j}^{(s)})}{\varepsilon_i^s} = \frac{\mathscr{B}_j(x_{i,j}^{(s)})}{\varepsilon_j^s}. \tag{45}$$

Lemma 3 *For all i, j such that $1 \le i < j \le n$ and $s = 1, 2$ the points $x_{i,j}^{(s)}$ exist, are uniquely defined and satisfy the following inequalities*

$$\frac{\mathscr{B}_i(x)}{\varepsilon_i^s} > \frac{\mathscr{B}_j(x)}{\varepsilon_j^s},\quad x \in [0, x_{i,j}^{(s)}),\quad \frac{\mathscr{B}_i(x)}{\varepsilon_i^s} < \frac{\mathscr{B}_j(x)}{\varepsilon_j^s},\ x \in (x_{i,j}^{(s)}, 1]. \tag{46}$$

In addition, the following ordering holds

$$x_{i,j}^{(s)} < x_{i+1,j}^{(s)},\ \textit{if}\ i+1 < j\ \textit{and}\ x_{i,j}^{(s)} < x_{i,j+1}^{(s)},\ \textit{if}\ i < j. \tag{47}$$

Proof Proof is similar to the Lemma 2.3.1 of [8].

Consider the following decomposition of $w_i(x)$

$$w_i = \sum_{q=1}^{n} w_{i,q}, \tag{48}$$

where the components $w_{i,q}$ are defined as follows.

$$w_{i,n} = \begin{cases} \displaystyle\sum_{k=0}^{3} \frac{(x - x_{n-1,n}^{(2)})^k}{k!} w_i^{(k)}(x_{n-1,n}^{(2)}) \text{ on } [0, x_{n-1,n}^{(2)}) \\ w_i \qquad\qquad\qquad \text{otherwise} \end{cases} \tag{49}$$

and, for each $q, \quad n - 1 \geq q \geq i,$

$$w_{i,q} = \begin{cases} \displaystyle\sum_{k=0}^{3} \frac{(x - x_{q-1,q}^{(2)})^k}{k!} \, p_{i,q}^{(k)}(x_{q-1,q}^{(2)}) \text{ on } [0, x_{q-1,q}^{(2)}) \\ p_{i,q} \qquad\qquad\qquad \text{otherwise} \end{cases} \tag{50}$$

and, for each $q, \quad i - 1 \geq q \geq 2,$

$$w_{i,q} = \begin{cases} \displaystyle\sum_{k=0}^{3} \frac{(x - x_{q-1,q}^{(1)})^k}{k!} \, p_{i,q}^{(k)}(x_{q-1,q}^{(1)}) \text{ on } [0, x_{q-1,q}^{(1)}) \\ p_{i,q} \qquad\qquad\qquad \text{otherwise} \end{cases} \tag{51}$$

with $p_{i,q} = w_i - \displaystyle\sum_{k=q+1}^{n} w_{i,k}$

and

$$w_{i,1} = w_i - \sum_{k=2}^{n} w_{i,k} \text{ on } [0, 1]. \tag{52}$$

Theorem 4 *For each q and i, $1 \leq q \leq n$, $1 \leq i \leq n$ and all $x \in \overline{\Omega}$, the components in the decomposition (48) satisfy the following estimates.*

$$|w_{i,q}'''(x)| \leq C\,\varepsilon_i^{-1}\varepsilon_q^{-2}\,\mathscr{B}_q(x), \quad \textit{if } i \leq q, \quad |w_{i,q}'''(x)| \leq C\,\varepsilon_i^{-2}\varepsilon_q^{-1}\,\mathscr{B}_q(x), \quad \textit{if } i > q,$$

$$|w_{i,q}''(x)| \leq C\,\varepsilon_i^{-1}\varepsilon_q^{-1}\,\mathscr{B}_q(x), \quad \textit{if } i \leq q < n, \quad |w_{i,q}''(x)| \leq C\,\varepsilon_i^{-2}\,\mathscr{B}_q(x), \quad \textit{if } i > q,$$

$$|w_{i,q}'(x)| \leq C\,\varepsilon_i^{-1}\,\mathscr{B}_q(x), \quad \textit{if } q < n.$$

Proof Differentiating (49) thrice,

$$|w_{i,n}'''(x)| = \begin{cases} |w_i'''(x_{n-1,n}^{(2)})| \text{ on } [0, x_{n-1,n}^{(2)}) \\ |w_i'''(x)| \qquad \text{otherwise} \end{cases}.$$

Then for $x \in [0, x_{n-1,n}^{(2)})$, using Theorem 3,

$$|w'''_{i,n}(x)| \le C\varepsilon_i^{-1}\Big(\sum_{k=1}^{i-1}\varepsilon_k^{-1}\mathscr{B}_k(x^{(2)}_{n-1,n}) + \sum_{k=i}^{n}\varepsilon_k^{-2}\mathscr{B}_k(x^{(2)}_{n-1,n})\Big).$$

Since $x^{(2)}_{k,n} \le x^{(2)}_{n-1,n}$ for $k < n$, using (46) $\varepsilon_k^{-2}\mathscr{B}_k(x^{(2)}_{n-1,n}) \le \varepsilon_n^{-2}\mathscr{B}_n(x^{(2)}_{n-1,n})$ and hence

$$|w'''_{i,n}(x)| \le C\varepsilon_i^{-1}\varepsilon_n^{-2}\mathscr{B}_n(x^{(2)}_{n-1,n}) \le C\varepsilon_i^{-1}\varepsilon_n^{-2}\mathscr{B}_n(x). \tag{53}$$

For $x \in [x^{(2)}_{n-1,n}, 1]$,

$$|w'''_{i,n}(x)| = |w'''_i(x)| \le C\varepsilon_i^{-1}\Big(\sum_{k=1}^{i-1}\varepsilon_k^{-1}\mathscr{B}_k(x) + \sum_{k=i}^{n}\varepsilon_k^{-2}\mathscr{B}_k(x)\Big).$$

As $x \ge x^{(2)}_{n-1,n}$, using (46) $\varepsilon_k^{-2}\mathscr{B}_k(x) \le \varepsilon_n^{-2}\mathscr{B}_n(x)$ and hence for $x \in [x^{(2)}_{n-1,n}, 1]$

$$|w'''_{i,n}(x)| \le C\varepsilon_i^{-1}\varepsilon_n^{-2}\mathscr{B}_n(x). \tag{54}$$

From (49) and (50), it is not hard to see that for each q, $n-1 \ge q \ge i$ and $x \in [x^{(2)}_{q,q+1}, 1]$, $w_{i,q}(x) = p_{i,q}(x) = w_i(x) - \sum_{k=q+1}^{n} w_{i,k}(x) = w_i(x) - w_i(x) = 0$. Differentiating (50) thrice, on $x \in [0, x^{(2)}_{q-1,q})$

$$|w'''_{i,q}(x)| = |p'''_{i,q}(x^{(2)}_{q-1,q})| \le C\varepsilon_i^{-1}\varepsilon_q^{-2}\mathscr{B}_q(x).$$

For $x \in [x^{(2)}_{q-1,q}, x^{(2)}_{q,q+1})$, using Lemma 3,

$$|w'''_{i,q}(x)| \le C\,\varepsilon_i^{-1}\varepsilon_q^{-2}\,\mathscr{B}_q(x). \tag{55}$$

From (50) and (51), it is not hard to see that for each q, $i-1 \ge q \ge 2$ and $x \in [x^{(1)}_{q,q+1}, 1]$, $w_{i,q}(x) = 0$. Differentiating (51) thrice on $x \in [0, x^{(1)}_{q-1,q})$

$$|w'''_{i,q}(x)| = |p'''_{i,q}(x^{(1)}_{q-1,q})| \le C\varepsilon_i^{-2}\varepsilon_q^{-1}\mathscr{B}_q(x).$$

For $x \in [x^{(1)}_{q-1,q}, x^{(1)}_{q,q+1})$, using Lemma 3,

$$|w'''_{i,q}(x)| \le C\varepsilon_i^{-2}\varepsilon_q^{-1}\mathscr{B}_q(x). \tag{56}$$

From (51) and (52), it is not hard to see that $w_{i,1}(x) = 0$ for $x \in [x^{(1)}_{1,2}, 1]$ and for $x \in [0, x^{(1)}_{1,2})$, $|w'''_{i,1}(x)| \le |w'''_i(x)| \le C\varepsilon_i^{-2}\varepsilon_1^{-1}\mathscr{B}_1(x)$. Since $w''_{i,q}(1) = 0$, for $q < n$, it follows that for any $x \in [0, 1]$ and $i > q$,

$$|w''_{i,q}(x)| = \left| \int_x^1 w^{(3)}_{i,q}(t)dt \right| \le C \int_x^1 \varepsilon_i^{-2}\varepsilon_q^{-1}\mathscr{B}_q(t)dt \le C\varepsilon_i^{-2}\mathscr{B}_q(x).$$

Hence,

$$|w''_{i,q}(x)| \le C\,\varepsilon_i^{-2}\,\mathscr{B}_q(x), \quad \text{for } \; i > q. \tag{57}$$

Similar arguments lead to

$$|w''_{i,q}(x)| \le C\,\varepsilon_i^{-1}\varepsilon_q^{-1}\,\mathscr{B}_q(x), \quad \text{for } \; i \le q, \tag{58}$$

and

$$|w'_{i,q}(x)| \le C\,\varepsilon_i^{-1}\,\mathscr{B}_q(x), \;\; 1 \le i \le n, 1 \le q \le n. \tag{59}$$

3 Numerical Method

To solve the BVP (1)–(2), a numerical method comprising of a Classical Finite Difference(CFD) Scheme and a piecewise uniform Shishkin mesh fitted on the domain [0, 1] is suggested.

3.1 Shishkin Mesh

A piecewise uniform Shishkin mesh with N mesh-intervals is now constructed. The mesh $\overline{\Omega}^N$ is a piecewise uniform mesh on $[0, 1]$ obtained by dividing $[0, 1]$ into $n+1$ mesh-intervals as $[0, \tau_1] \cup [\tau_1, \tau_2] \cup \cdots \cup [\tau_{n-1}, \tau_n] \cup [\tau_n, 1]$. Transition parameters $\tau_r,\;\; 1 \le r \le n$, are defined as $\tau_n = \min\left\{\frac{1}{2},\; 2\frac{\varepsilon_n}{\alpha}\ln N\right\}$ and, for $r = n-1, \ldots 1,\; \tau_r = \min\left\{\frac{r\tau_{r+1}}{r+1},\; 2\frac{\varepsilon_r}{\alpha}\ln N\right\}$. On the sub-interval $[\tau_n, 1]$, $\frac{N}{2}+1$ mesh-points are placed uniformly and on each of the subintervals $[\tau_r, \tau_{r+1}),\;\; r = n-1, \ldots 1$, a uniform mesh of $\frac{N}{2n}$ mesh-points is placed. A uniform mesh of $\frac{N}{2n}$ mesh-points is placed on the sub-interval $[0, \tau_1)$.

The Shishkin mesh is coarse in the outer region and becomes finer and finer in the inner (layer) regions. From the above construction, it is clear that the transition points $\tau_r,\;\; r = 1, \ldots, n$, are the only points at which the mesh-size can change and that it does not necessarily change at each of these points.

If each of the transition parameters $\tau_r,\;\; r = 1, \ldots, n$, are with the left choice, the Shishkin mesh $\overline{\Omega}^N$ becomes the classical uniform mesh with $\tau_r = \frac{r}{2n},\;\; r = 1, \ldots, n$, and hence the step size is N^{-1}.

The following notations are introduced: $h_j = x_j - x_{j-1}$ and if $x_j = \tau_r$, then $h_r^- = x_j - x_{j-1}$, $h_r^+ = x_{j+1} - x_j$, $J = \{\tau_r : h_r^+ \neq h_r^-\}$. Let $H_r = 2n\,N^{-1}(\tau_r - \tau_{r-1})$, $2 \leq r \leq n$ denote the step size in the mesh interval $(\tau_{r-1}, \tau_r]$. Also, $H_1 = 2\,nN^{-1}\tau_1$ and $H_{n+1} = 2\,N^{-1}(1 - \tau_n)$. Thus, for $1 \leq r \leq n-1$, the change in the step size at the point $x_j = \tau_r$ is

$$h_r^+ - h_r^- = 2\,nN^{-1}\Big(\frac{(r+1)}{r} d_r - d_{r-1}\Big), \tag{60}$$

where $d_r = \frac{r\tau_{r+1}}{r+1} - \tau_r$ with the convention $d_n = 0$, when $\tau_n = 1/2$. The mesh $\overline{\Omega}^N$ becomes a classical uniform mesh when $d_r = 0$ for all $r = 1, \ldots, n$ and $\tau_r \leq C\,\varepsilon_r \ln N$, $1 \leq r \leq n$. Also $\tau_r = \frac{r}{s}\tau_s$ when $d_r = \cdots = d_s = 0$, $1 \leq r \leq s \leq n$.

3.2 Discrete Problem

To solve the BVP (1)–(2) numerically the following upwind classical finite difference scheme is applied on the mesh $\overline{\Omega}^N$.

$$L^N \mathbf{U}(x_j) \equiv E\delta^2 \mathbf{U}(x_j) + A(x_j)D^+\mathbf{U}(x_j) - B(x_j)\mathbf{U}(x_j) = \mathbf{f}(x_j), \tag{61}$$

$$\mathbf{U}(x_0) = \mathbf{l},\ \mathbf{U}(x_N) = \mathbf{r}, \tag{62}$$

where $\mathbf{U}(x_j) = (U_1(x_j), U_2(x_j), \ldots, U_n(x_j))^T$ and for $1 \leq j \leq N-1$,

$$D^+U(x_j) = \frac{U(x_{j+1}) - U(x_j)}{h_{j+1}},\quad D^-U(x_j) = \frac{U(x_j) - U(x_{j-1})}{h_j},$$

$$\delta^2 U(x_j) = \frac{1}{\overline{h_j}}\Big(D^+U(x_j) - D^-U(x_j)\Big),$$

with

$$\overline{h_j} = \frac{(h_j + h_{j+1})}{2}.$$

4 Numerical Results

In this section a discrete maximum principle, a discrete stability result and the first-order convergence of the proposed numerical method are established.

Lemma 4 (Discrete Maximum Principle) *Assume that the vector valued mesh function* $\boldsymbol{\psi}(x_j) = (\psi_1(x_j), \psi_2(x_j), \ldots, \psi_n(x_j))^T$ *satisfies* $\boldsymbol{\psi}(x_0) \geq \mathbf{0}$ *and* $\boldsymbol{\psi}(x_N) \geq \mathbf{0}$. *Then* $L^N\boldsymbol{\psi}(x_j) \leq \mathbf{0}$ *for* $1 \leq j \leq N-1$ *implies that* $\boldsymbol{\psi}(x_j) \geq \mathbf{0}$ *for* $0 \leq j \leq N$.

Lemma 5 (Discrete Stability Result) *If* $\boldsymbol{\psi}(x_j) = (\psi_1(x_j), \psi_2(x_j), \ldots, \psi_n(x_j))^T$ *is any vector valued mesh function defined on* $\overline{\Omega}^N$, *then for* $1 \leq i \leq n$ *and* $0 \leq j \leq N$,

$$|\psi_i(x_j)| \leq \max\left\{\|\boldsymbol{\psi}(x_0)\|,\ \|\boldsymbol{\psi}(x_N)\|,\ \frac{1}{\beta}\ \|L^N\boldsymbol{\psi}\|_{\Omega^N}\right\}.$$

4.1 Error Estimate

Analogous to the continuous case, the discrete solution $\mathbf{U}$ can be decomposed into $\mathbf{V}$ and $\mathbf{W}$ as defined below.

$$L^N\mathbf{V}(x_j) = \mathbf{f}(x_j),\ \text{for } 0 < j < N,\ \mathbf{V}(x_0) = \mathbf{v}(x_0),\ \mathbf{V}(x_N) = \mathbf{v}(x_N) \quad (63)$$

$$L^N\mathbf{W}(x_j) = \mathbf{0},\ \text{for } 0 < j < N,\ \mathbf{W}(x_0) = \mathbf{w}(x_0),\ \mathbf{W}(x_N) = \mathbf{w}(x_N) \quad (64)$$

Lemma 6 *Let* $\mathbf{v}$ *be the solution of (7) and* $\mathbf{V}$ *be the solution of (63), then*

$$\|\mathbf{V} - \mathbf{v}\|_{\overline{\Omega}^N} \leq CN^{-1}.$$

Proof For $1 \leq j \leq N-1$,

$$L^N(\mathbf{V} - \mathbf{v})(x_j) = \begin{pmatrix} \varepsilon_1(\frac{d^2}{dx^2} - \delta^2)v_1(x_j) + a_1(x_j)(\frac{d}{dx} - D^+)v_1(x_j) \\ \varepsilon_2(\frac{d^2}{dx^2} - \delta^2)v_2(x_j) + a_2(x_j)(\frac{d}{dx} - D^+)v_2(x_j) \\ \vdots \\ \varepsilon_n(\frac{d^2}{dx^2} - \delta^2)v_n(x_j) + a_n(x_j)(\frac{d}{dx} - D^+)v_n(x_j) \end{pmatrix}.$$

By the standard local truncation used in the Taylor expansions,

$$|\varepsilon_i\left(\frac{d^2}{dx^2} - \delta^2\right)v_i(x_j) + a_i(x_j)\left(\frac{d}{dx} - D^+\right)v_i(x_j)| \leq C(x_{j+1} - x_{j-1})(\varepsilon_i\|v_i^{(3)}\| + \|v_i^{(2)}\|).$$

Since $(x_{j+1} - x_{j-1}) \leq CN^{-1}$, by using (35),

$$\|L^N(\mathbf{V} - \mathbf{v})\|_{\Omega^N} \leq CN^{-1}.$$

As v and V agree at the boundary points, using Lemma 5,

$$\|\mathbf{V} - \mathbf{v}\|_{\overline{\Omega}^N} \leq CN^{-1}. \quad (65)$$

To estimate the error in the singular component $(\mathbf{W} - \mathbf{w})$, the mesh functions $B_i^N(x_j)$ for $1 \leq i \leq n$ on $\overline{\Omega}^N$ are defined by

$$B_i^N(x_j) = \prod_{k=1}^{j}\left(1 + \frac{\alpha h_k}{2\varepsilon_i}\right)^{-1}$$

with $B_i^N(x_0) = 1$. It is to be observed that B_i^N are monotonically decreasing.

Lemma 7 *The singular components W_i, $1 \le i \le n$ satisfy the following bound on $\overline{\Omega}^N$;*

$$|W_i(x_j)| \le C B_n^N(x_j).$$

Proof Consider the following vector valued mesh functions on $\overline{\Omega}^N$,

$$\boldsymbol{\psi}^{\pm}(x_j) = C B_n^N(x_j)\mathbf{e} \pm \mathbf{W}(x_j)$$

where $\mathbf{e}$ is the n- vector $\mathbf{e} = (1, 1, \ldots, 1)^T$.
Then for sufficiently large C, $\boldsymbol{\psi}^{\pm}(x_0) \ge \mathbf{0}$, $\boldsymbol{\psi}^{\pm}(x_N) \ge \mathbf{0}$ and $L^N\boldsymbol{\psi}^{\pm}(x_j) \le \mathbf{0}$, for $1 \le j \le N-1$. Using Lemma 4, $\boldsymbol{\psi}^{\pm}(x_j) \ge \mathbf{0}$ on $\overline{\Omega}^N$, which implies that

$$|W_i(x_j)| \le C B_n^N(x_j).$$

Lemma 8 *Assume that $d_r = 0$, for $r = 1, 2, \ldots, n$. Let $\mathbf{w}$ be the solution of (8) and $\mathbf{W}$ be the solution of (64). Then*

$$\|\mathbf{W} - \mathbf{w}\|_{\overline{\Omega}^N} \le CN^{-1}\ln N.$$

Proof By the standard local truncation used in the Taylor expansions,

$$\left|\varepsilon_i(\frac{d^2}{dx^2} - \delta^2)w_i(x_j) + a_i(x_j)(\frac{d}{dx} - D^+)w_i(x_j)\right| \le C(x_{j+1} - x_{j-1})(\varepsilon_i\|w_i^{(3)}\| + \|w_i^{(2)}\|)$$

where the norm is taken over the interval $[x_{j-1}, x_{j+1}]$.
Since $d_r = 0$, the mesh is uniform, $h = N^{-1}$ and $\varepsilon_k^{-1} \le C\ln N$. Then,

$$|(\mathbf{L}^N(\mathbf{W} - \mathbf{w}))_i(x_j)| \le C\,N^{-1}\Big(\sum_{k=1}^{i-1}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + \sum_{k=i}^{n}\varepsilon_k^{-2}\mathscr{B}_k(x_{j-1})\Big) \quad (66)$$

$$\le CN^{-1}\ln N + CN^{-1}\ln N\sum_{k=i}^{n}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}). \quad (67)$$

Consider the barrier function $\boldsymbol{\phi} = (\phi_1(x_j), \phi_2(x_j), \ldots, \phi_n(x_j))^T$ given by

$$\phi_i(x_j) = CN^{-1}\ln N + \frac{CN^{-1}\ln N}{\gamma(\alpha - \gamma)}\Big(\sum_{k=i}^{n}\exp(2\gamma h/\varepsilon_k)Y_k(x_j)\Big),\ \text{on } \overline{\Omega}^N,$$

where γ is a constant such that $0 < \gamma < \alpha$,

$$Y_k(x_j) = \frac{\lambda_k^{N-j} - 1}{\lambda_k^N - 1} \text{ with } \lambda_k = 1 + \frac{\gamma h}{\varepsilon_k}.$$

It is not hard to see that, $0 \leq Y_k(x_j) \leq 1, \quad D^+ Y_k(x_j) \leq -\frac{\gamma}{\varepsilon_k} \exp(-\gamma x_{j+1}/\varepsilon_k)$ and $(\varepsilon_k \delta^2 + \gamma D^+) Y_k(x_j) = 0$. Hence,

$$(L^N \boldsymbol{\phi})_i(x_j) \leq -CN^{-1} \ln N - CN^{-1} \ln N \sum_{k=i}^{n} \varepsilon_k^{-1} \mathscr{B}_k(x_{j-1}).$$

Consider the discrete functions

$$\boldsymbol{\psi}^{\pm}(x_j) = \boldsymbol{\phi}(x_j) \pm (\mathbf{W} - \mathbf{w})(x_j),\ x_j \in \overline{\Omega}^N.$$

Then for sufficiently large C, $\boldsymbol{\psi}^{\pm}(x_0) > \mathbf{0}$, $\boldsymbol{\psi}^{\pm}(x_N) \geq \mathbf{0}$ and $L^N \boldsymbol{\psi}^{\pm}(x_j) \leq \mathbf{0}$ on Ω^N. Using Lemma 4, $\boldsymbol{\psi}^{\pm}(x_j) \geq \mathbf{0}$ on $\overline{\Omega}^N$. Hence, $|(\mathbf{W} - \mathbf{w})_i(x_j)| \leq CN^{-1} \ln N$ for $1 \leq i \leq n$, implies that

$$\|(\mathbf{W} - \mathbf{w})\|_{\overline{\Omega}^N} \leq CN^{-1} \ln N. \tag{68}$$

Lemma 9 *Let* $\mathbf{w}$ *be the solution of (8) and* $\mathbf{W}$ *be the solution of (64); then*

$$\|\mathbf{W} - \mathbf{w}\|_{\overline{\Omega}^N} \leq CN^{-1} \ln N.$$

Proof This is proved for each mesh point $x_j \in (0, 1)$ by dividing $(0, 1)$ into $n + 1$ subintervals (a) $(0, \tau_1)$, (b) $[\tau_1, \tau_2)$, (c) $[\tau_m, \tau_{m+1})$ for some m, $2 \leq m \leq n - 1$ and (d) $[\tau_n, 1)$.

For each of these cases, an estimate for the local truncation error is derived and a barrier function is defined. Lastly, using these barrier functions, the required estimate is established.

Case (a): $x_j \in (0, \tau_1)$.
Clearly $x_{j+1} - x_{j-1} \leq C\varepsilon_1 N^{-1} \ln N$. Then, by standard local truncation used in Taylor expansions, the following estimates hold for $x_j \in (0, \tau_1)$ and $1 \leq i \leq n$.

$$\begin{aligned}|(\mathbf{L}^N(\mathbf{W} - \mathbf{w}))_i(x_j)| &\leq C\,(x_{j+1} - x_{j-1})(\varepsilon_i \|w_i^{(3)}\| + \|w_i^{(2)}\|) \\ &\leq C\,N^{-1} \ln N \sum_{k=i}^{n} \varepsilon_k^{-1} \mathscr{B}_k(x_{j-1}).\end{aligned}$$

Consider the following barrier functions for $x_j \in (0, \tau_1)$ and $1 \leq i \leq n$.

$$\phi_i(x_j) = CN^{-1}\ln N\sum_{k=i}^{n}\exp(2\alpha H_1/\varepsilon_k)B_k^N(x_j) + \sum_{k=1}^{n}B_k^N(\tau_k). \tag{69}$$

Case (b): $x_j \in [\tau_1, \tau_2)$.
There are 2 possibilities: **Case (b1):** $\mathbf{d_1 = 0}$ and **Case (b2):** $\mathbf{d_1 > 0}$.
Case (b1): $\mathbf{d_1 = 0}$
Since the mesh is uniform in $(0, \tau_2)$, it follows that $x_{j+1} - x_{j-1} \le C\,\varepsilon_1 N^{-1}\ln N$, for $x_j \in [\tau_1, \tau_2)$. Then,

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,N^{-1}\ln N\sum_{k=i}^{n}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}). \tag{70}$$

Now for $x_j \in [\tau_1, \tau_2)$ and $1 \le i \le n$, define,

$$\phi_i(x_j) = CN^{-1}\ln N\sum_{k=i}^{n}\exp(2\alpha H_2/\varepsilon_k)B_k^N(x_j) + \sum_{k=2}^{n}B_k^N(\tau_k). \tag{71}$$

Case (b2): $\mathbf{d_1 > 0}$.
For this case, $x_{j+1} - x_{j-1} \le C\,\varepsilon_2 N^{-1}\ln N$, and hence for $x_j \in [\tau_1, \tau_2)$

$$\begin{aligned}\left|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)\right| &\le \left|\varepsilon_i(\frac{d^2}{dx^2}-\delta^2)w_i(x_j)\right| + C\left|(\frac{d}{dx}-D^+)w_i(x_j)\right| \\ &\le \left|\varepsilon_i(\frac{d^2}{dx^2}-\delta^2)\sum_{k=1}^{n}w_{i,k}\right| + C\left|(\frac{d}{dx}-D^+)\sum_{k=1}^{n}w_{i,k}\right|.\end{aligned}$$

By the standard local truncation used in Taylor expansions

$$\begin{aligned}|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,\varepsilon_i|w_{i,1}^{(2)}(x_{j-1})| + C\,(x_{j+1}-x_{j-1})\varepsilon_i\sum_{k=2}^{n}|w_{i,k}^{(3)}(x_{j-1})| \\ +C\,|w_{i,1}^{(1)}(x_{j-1})| + C\,(x_{j+1}-x_{j-1})\sum_{k=2}^{n}|w_{i,k}^{(2)}(x_{j-1})|.\end{aligned} \tag{72}$$

Now using Theorem 4, it is not hard to derive that

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_1(x_j)| \le C\,N^{-1}\ln N\sum_{k=2}^{n}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + C\,\varepsilon_1^{-1}\mathscr{B}_1(x_{j-1}) \tag{73}$$

and for $2 \le i \le n$,

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,N^{-1}\ln N \sum_{k=i}^{n} \varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + C\,\varepsilon_i^{-1}\mathscr{B}_1(x_{j-1}). \tag{74}$$

Define

$$\phi_1(x_j) = CN^{-1}\ln N \sum_{k=2}^{n} \exp(2\alpha H_2/\varepsilon_k) B_k^N(x_j) + C\,B_1^N(x_j) + C\,\sum_{k=2}^{n} B_k^N(\tau_k)$$

and for $2 \le i \le n$,

$$\phi_i(x_j) = CN^{-1}\ln N \sum_{k=i}^{n} \exp(2\alpha H_2/\varepsilon_k) B_k^N(x_j) + C\,B_1^N(x_j) + C\,\sum_{k=2}^{n} B_k^N(\tau_k).$$

Case (c): $x_j \in (\tau_m, \tau_{m+1}]$**.** There are 3 possibilities:

Case (c1): $d_1 = d_2 = \cdots = d_m = 0$,
Case (c2): $d_r > 0$ and $d_{r+1} = \ \ldots \ = d_m = 0$ for some r, $1 \le r \le m-1$ and
Case (c3): $d_m > 0$.

Case (c1): $d_1 = d_2 = \cdots = d_m = 0$,
Since $\tau_1 = C\tau_{m+1}$ and the mesh is uniform in $(0, \tau_{m+1})$, it follows that, for $x_j \in (\tau_m, \tau_{m+1}]$, $x_{j+1} - x_{j-1} \le C\,\varepsilon_1 N^{-1}\ln N$ and hence

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \ \le\ C\,N^{-1}\ln N \sum_{k=i}^{n} \varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}). \tag{75}$$

For $1 \le i \le n$,

$$\phi_i(x_j) = CN^{-1}\ln N \sum_{k=i}^{n} \exp(2\alpha H_{m+1}/\varepsilon_k) B_k^N(x_j) + C \sum_{k=m+1}^{n} B_k^N(\tau_k). \tag{76}$$

Case (c2): $d_r > 0$ and $d_{r+1} = \ \ldots \ = d_m = 0$ for some r, $1 \le r \le m-1$
Since, $\tau_{r+1} = C\tau_{m+1}$, the mesh is uniform in (τ_r, τ_{m+1}), it follows that $x_{j+1} - x_{j-1} \le C\,\varepsilon_{r+1} N^{-1}\ln N$, for $x_j \in (\tau_m, \tau_{m+1}]$.
By the standard local truncation used in Taylor expansions

$$\begin{aligned}|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,\varepsilon_i \sum_{k=1}^{r} |w_{i,k}^{(2)}(x_{j-1})| + C\,(x_{j+1}-x_{j-1})\varepsilon_i \sum_{k=r+1}^{n} |w_{i,k}^{(3)}(x_{j-1})| \\ +C \sum_{k=1}^{r} |w_{i,k}^{(1)}(x_{j-1})| + C\,(x_{j+1}-x_{j-1}) \sum_{k=r+1}^{n} |w_{i,k}^{(2)}(x_{j-1})|.\end{aligned} \tag{77}$$

Now using Theorem 4, it is not hard to derive that for $i \le r$

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,N^{-1}\ln N\sum_{k=r+1}^{n}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + C\sum_{k=i}^{r}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1})$$

and for $i > r$

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,N^{-1}\ln N\sum_{k=i}^{n}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + C\varepsilon_i^{-1}\mathscr{B}_r(x_{j-1}).$$

Now define, for $i \le r$

$$\phi_i(x_j) = CN^{-1}\ln N\sum_{k=r+1}^{n}\exp\left(\frac{2\alpha H_{m+1}}{\varepsilon_k}\right)B_k^N(x_j) + C\sum_{k=i}^{r}B_k^N(x_j) + C\sum_{k=m+1}^{n}B_k^N(\tau_k)$$

and for $i > r$

$$\phi_i(x_j) = CN^{-1}\ln N\sum_{k=i}^{n}\exp\left(\frac{2\alpha H_{m+1}}{\varepsilon_k}\right)B_k^N(x_j) + CB_r^N(x_j) + C\sum_{k=m+1}^{n}B_k^N(\tau_k).$$

Case (c3): $d_m > 0$
Replacing r by m in the arguments of the previous case **Case(c2)** and using $x_{j+1} - x_{j-1} \le C\varepsilon_{m+1}N^{-1}\ln N$, the following estimates hold for $x_j \in (\tau_m, \tau_{m+1}]$.
For $i \le m$,

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,N^{-1}\ln N\sum_{k=m+1}^{n}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + C\sum_{k=i}^{m}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) \tag{78}$$

and for $i > m$

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,N^{-1}\ln N\sum_{k=i}^{n}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + C\,\varepsilon_i^{-1}\mathscr{B}_m(x_{j-1}). \tag{79}$$

For $i \le m$, define,

$$\phi_i(x_j) = CN^{-1}\ln N\sum_{k=m+1}^{n}\exp\left(\frac{2\alpha H_{m+1}}{\varepsilon_k}\right)B_k^N(x_j) + C\sum_{k=i}^{m}B_k(x_j) + C\sum_{k=m+1}^{n}B_k^N(\tau_k)$$

and for $i > m$

$$\phi_i(x_j) = CN^{-1}\ln N\sum_{k=i}^{n}\exp\left(\frac{2\alpha H_{m+1}}{\varepsilon_k}\right)B_k^N(x_j) + CB_m(x_j) + C\sum_{k=m+1}^{n}B_k^N(\tau_k).$$

Case (d): There are 3 possibilities.

Case (d1): $d_1 = \ \dots\ = d_n = 0$,
Case (d2): $d_r > 0$ and $d_{r+1} = \ \dots\ = d_n = 0$ for some $r,\ 1 \leq r \leq n-1$ and
Case (d3): $d_n > 0$.

Case (d1): $d_1 = \ \dots\ = d_n = 0$,
The mesh is uniform in [0, 1] and the result is established in the Lemma 8.
Case (d2): $d_r > 0$ and $d_{r+1} = \ \dots\ = d_n = 0$ for some $r,\ 1 \leq r \leq n-1$
In this case from the definition of τ_n it follows that $x_{j+1} - x_{j-1} \ \leq\ C\,\varepsilon_{r+1}N^{-1}\ln N$ and arguments similar to the **Case(c2)** lead to the following estimates for $x_j \in (\tau_n, 1]$. For $i \leq r$,

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \leq C\,N^{-1}\ln N \sum_{k=r+1}^{n} \varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + C\sum_{k=i}^{r}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) \tag{80}$$

and for $i > r$

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \leq C\,N^{-1}\ln N \sum_{k=i}^{n}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + C\,\varepsilon_i^{-1}\mathscr{B}_r(x_{j-1}). \tag{81}$$

Define the barrier functions ϕ_i for $i \leq r$ by

$$\phi_i(x_j) = CN^{-1}\ln N \sum_{k=r+1}^{n} \exp(2\alpha H_{n+1}/\varepsilon_k)B_k^N(x_j) + C\sum_{k=i}^{r} B_k^N(x_j) \tag{82}$$

and for $i > r$

$$\phi_i(x_j) = CN^{-1}\ln N \sum_{k=i}^{n}\exp(2\alpha H_{n+1}/\varepsilon_k)B_k^N(x_j) + CB_r^N(x_j). \tag{83}$$

Case (d3): $d_n > 0$
Now $\tau_n = 2\dfrac{\varepsilon_n}{\alpha}\ln N$. Then on $(\tau_n, 1]$,

$$\begin{aligned}|(W_i - w_i)(x_j)| &\leq |W_i(x_j)| + |w_i(x_j)| \\ &\leq CB_n^N(x_j) + C\mathscr{B}_n(x_j),\ \text{using Lemma 7 and Theorem 3}\end{aligned}$$

Hence,

$$|(W_i - w_i)(x_j)| \leq CN^{-1}, \quad \text{on } [\tau_n, 1]. \tag{84}$$

Now using the estimates derived and the barrier functions $\phi_i,\ 1 \leq i \leq n$, defined for all the four cases, the main proof is split into two cases

Case 1: $d_n > 0$. Consider the following discrete functions for $0 \leq j \leq N/2$,

$$\boldsymbol{\psi}^{\pm}(x_j) = \boldsymbol{\phi}(x_j) \pm (\mathbf{W} - \mathbf{w})(x_j) \tag{85}$$

where $\boldsymbol{\phi}(x_j) = (\phi_1(x_j), \phi_2(x_j), \ldots, \phi_n(x_j))^T$.
For sufficiently large C, it is not hard to see that

$$\boldsymbol{\psi}^{\pm}(x_0) \geq \mathbf{0},\ \ \boldsymbol{\psi}^{\pm}(x_{\frac{N}{2}}) \geq \mathbf{0} \text{ and } L^N \boldsymbol{\psi}^{\pm}(x_j) \leq \mathbf{0}, \text{ for } 0 < j < N/2.$$

Then by Lemma 4, $\boldsymbol{\psi}^{\pm}(x_j) \geq \mathbf{0}$ for $0 \leq j \leq N/2$. Consequently,

$$|(W_i - w_i)(x_j)| \leq CN^{-1}, \quad \text{on } [0, \tau_n]. \tag{86}$$

Hence, (84) and (86) imply that, for $d_n > 0$

$$\|(\mathbf{W} - \mathbf{w})\|_{\overline{\Omega}^N} \leq CN^{-1} \ln N. \tag{87}$$

Case 2: $d_n = 0$. Consider the following discrete functions for $0 \leq j \leq N$,

$$\boldsymbol{\psi}^{\pm}(x_j) = \boldsymbol{\phi}(x_j) \pm (\mathbf{W} - \mathbf{w})(x_j). \tag{88}$$

For sufficiently large C, it is not hard to see that

$$\boldsymbol{\psi}^{\pm}(x_0) \geq \mathbf{0},\ \ \boldsymbol{\psi}^{\pm}(x_N) \geq \mathbf{0} \text{ and } L^N \boldsymbol{\psi}^{\pm}(x_j) \leq \mathbf{0}, \text{ for } 0 < j < N.$$

Then by Lemma 4, $\boldsymbol{\psi}^{\pm}(x_j) \geq \mathbf{0}$ for $0 \leq j \leq N$. Hence, for $d_n = 0$,

$$\|(\mathbf{W} - \mathbf{w})\|_{\overline{\Omega}^N} \leq CN^{-1} \ln N.$$

Theorem 5 *Let u be the solution of the problem (1)–(2) and U be the solution of the problem (61)–(62), then,*

$$\|(\mathbf{u} - \mathbf{U})\|_{\overline{\Omega}^N} \leq CN^{-1} \ln N.$$

Proof From the Eqs. (7), (8), (63) and (64), we have

$$\begin{aligned}\|(\mathbf{u} - \mathbf{U})\|_{\overline{\Omega}^N} &= \|(\mathbf{v} + \mathbf{w} - \mathbf{V} + \mathbf{W})\|_{\overline{\Omega}^N} \\ &\leq \|(\mathbf{v} - \mathbf{V}\|_{\overline{\Omega}^N} + \|(\mathbf{w} - \mathbf{W})\|_{\overline{\Omega}^N}\end{aligned}$$

Then the result follows from Lemmas 6 and 9.

5 Numerical Illustrations

Example 1 Consider the following boundary value problem for the system of convection–diffusion equations on (0, 1)

$$\begin{aligned} &\varepsilon_1 u_1''(x) + (1+x)u_1'(x) - 4u_1(x) + 2u_2(x) + u_3(x) = -e^x, \\ &\varepsilon_2 u_2''(x) + (2+x^2)u_2'(x) + u_1(x) - 6u_2(x) + 2u_3(x) = -\sin x, \\ &\varepsilon_3 u_3''(x) + (e^x)u_3'(x) + 3u_1(x) + 2u_2(x) - 8u_3(x) = -\cos x, \\ \text{with}\quad &u_1(0) = 1,\ u_2(0) = 1,\ u_3(0) = 1\ u_1(1) = 0,\ u_2(1) = 0\ u_3(1) = 0. \end{aligned}$$

The above problem is solved using the suggested numerical method and plot of the approximate solution for $N = 1536$, $\varepsilon_1 = 5^{-4}$, $\varepsilon_2 = 3^{-4}$, $\varepsilon_3 = 2^{-5}$ is shown in Fig. 1.

Parameter uniform error constant and the order of convergence of the numerical method for $\varepsilon_1 = \eta/625$, $\varepsilon_2 = \eta/81$ and $\varepsilon_3 = \eta/32$ are computed using a variant of the two mesh algorithm suggested in [6] and are shown in Table 1.

It is found that the parameter ε_i for any i, influences the components $u_1, u_2, \ldots, u_i$ and causes multiple layers for these components, in the neighbourhood of the origin and has no significant influence on $u_{i+1}, u_{i+2}, \ldots, u_n$. The following examples illustrate this.

Example 2 Consider the following boundary value problem for the system of convection–diffusion equations on (0, 1)

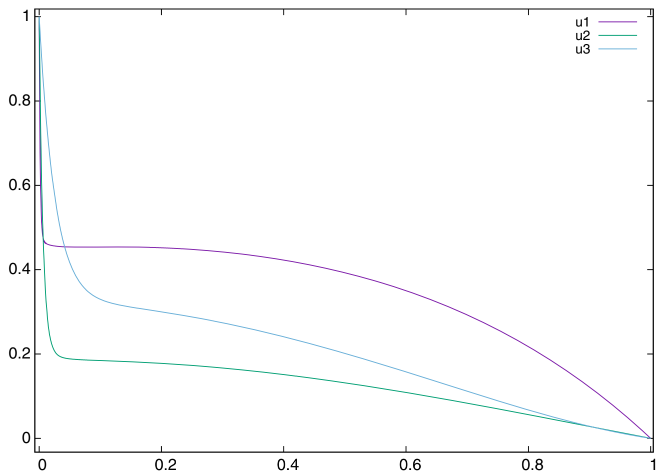

Fig. 1 Approximate solution of Example 1

Table 1 Maximum errors and order of convergence

η	Number of mesh elements N				
	96	192	384	768	1536
2^{0}	$0.1604E-01$	$0.9767E-02$	$0.5495E-02$	$0.2860E-02$	$0.1430E-02$
2^{-1}	$0.1626E-01$	$0.9895E-02$	$0.5560E-02$	$0.2893E-02$	$0.1446E-02$
2^{-2}	$0.1637E-01$	$0.9955E-02$	$0.5587E-02$	$0.2905E-02$	$0.1451E-02$
2^{-3}	$0.1643E-01$	$0.9983E-02$	$0.5598E-02$	$0.2910E-02$	$0.1452E-02$
2^{-4}	$0.1645E-01$	$0.9995E-02$	$0.5603E-02$	$0.2911E-02$	$0.1453E-02$
2^{-5}	$0.1647E-01$	$0.1000E-01$	$0.5604E-02$	$0.2911E-02$	$0.1453E-02$
2^{-6}	$0.1647E-01$	$0.1000E-01$	$0.5604E-02$	$0.2911E-02$	$0.1453E-02$
2^{-7}	$0.1648E-01$	$0.1000E-01$	$0.5604E-02$	$0.2911E-02$	$0.1453E-02$
2^{-8}	$0.1648E-01$	$0.1000E-01$	$0.5604E-02$	$0.2911E-02$	$0.1453E-02$
D^N	$0.1648E-01$	$0.1000E-01$	$0.5604E-02$	$0.2911E-02$	$0.1453E-02$
P^N	$0.7203E+00$	$0.8358E+00$	$0.9447E+00$	$0.1002E+01$	
C_p^N	$0.1123E+01$	$0.1123E+01$	$0.1037E+01$	$0.8877E+00$	$0.7300E+00$

The computed order of ε_i-uniform convergence, $p^* = 0.7203$.
The computed ε_i-uniform error constant, $C_{p^*}^N = 1.1235$.
From Table 1, it is to be noted that the error decreases as the number of mesh elements N increases. Also for each N, the error stabilises as η tends to zero

$$\begin{aligned}
&\varepsilon_1 u_1''(x) + (1+x)u_1'(x) - 4u_1(x) + 2u_2(x) + u_3(x) = 1 - x,\\
&\varepsilon_2 u_2''(x) + (2+x^2)u_2'(x) + 2u_1(x) - 6u_2(x) + 3u_3(x) = 3 - 3x,\\
&\varepsilon_3 u_3''(x) + u_3'(x) + 3u_1(x) + 3u_2(x) - 7u_3(x) = 7x - 8,\\
\text{with}\quad & u_1(0) = 0,\ u_2(0) = 1,\ u_3(0) = 1\ u_1(1) = 0,\ u_2(1) = 0\ u_3(1) = 0
\end{aligned}$$

The above problem is solved using the suggested numerical method. As $u_2(0) \neq u_{02}(0)$ and $u_i(0) = u_{0i}(0),\ i = 1, 3$ for this problem, a layer of width $O(\varepsilon_2)$ is expected to occur in the neighbourhood of the origin for u_1 and u_2 but not for u_3. Further u_1 cannot have ε_1 layer or ε_3 layer. The plot of an approximate solution of this problem for $N = 384$, $\varepsilon_1 = 5^{-4}$, $\varepsilon_2 = 3^{-4}$, $\varepsilon_3 = 2^{-5}$ is shown in Fig. 2a–d.

Example 3 Consider the following boundary value problem for the system of convection–diffusion equations on (0, 1)

$$\begin{aligned}
&\varepsilon_1 u_1''(x) + (1+x)u_1'(x) - 4u_1(x) + 2u_2(x) + u_3(x) = x,\\
&\varepsilon_2 u_2''(x) + (2+x^2)u_2'(x) + 2u_1(x) - 6u_2(x) + 3u_3(x) = 3x,\\
&\varepsilon_3 u_3''(x) + u_3'(x) + 3u_1(x) + 3u_2(x) - 7u_3(x) = 1 - 7x,\\
\text{with}\quad & u_1(0) = 0,\ u_2(0) = 0,\ u_3(0) = 1\ u_1(1) = 0,\ u_2(1) = 0\ u_3(1) = 1.
\end{aligned}$$

The above problem is solved using the suggested numerical method. As $u_3(0) \neq u_{03}(0)$ and $u_i(0) = u_{0i}(0),\ i = 1, 2$ for this problem, a layer of width $O(\varepsilon_3)$ is expected to occur in the neighbourhood of the origin for u_1, u_2 and u_3. Further

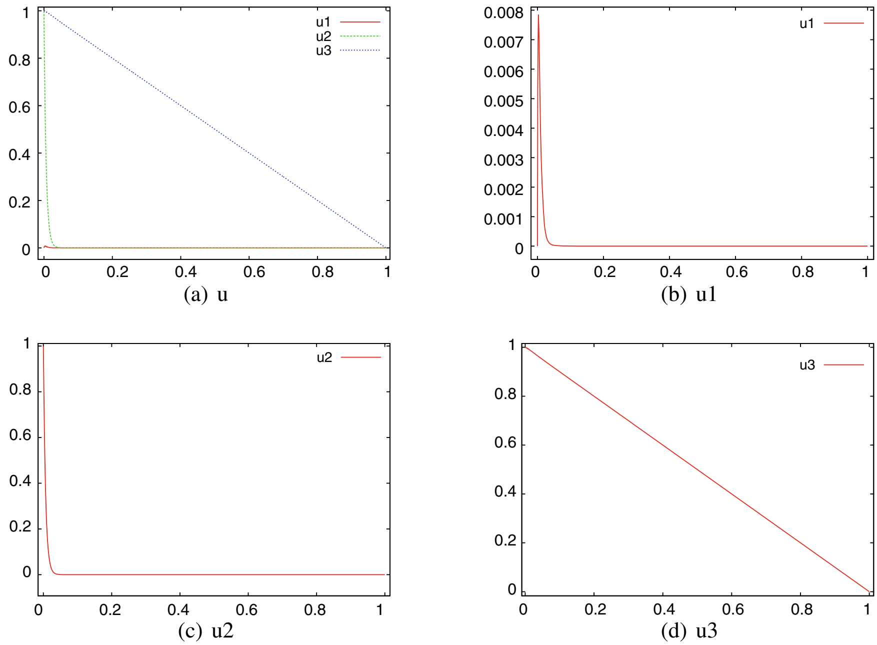

Fig. 2 Approximation of solution components of Example 2

u_1 will not have ε_1 layer or ε_2 layer. Similarly u_2 will not have ε_2 layer. The plot of an approximate solution of this problem for $N = 384$, $\varepsilon_1 = 5^{-4}$, $\varepsilon_2 = 3^{-4}$, $\varepsilon_3 = 2^{-5}$ is shown in Fig. 3a–d.

6 Conclusions

The method presented in this paper is the extension of the work done for the scalar problem in [4]. The novel estimates of derivatives of the solution help us to establish the desired error bound for the Classical Finite Difference Scheme when applied on any of the 2^n Shishkin meshes.

The examples given are to facilitate the reader to note the effect of coupling with the assumed order of the perturbation parameters.

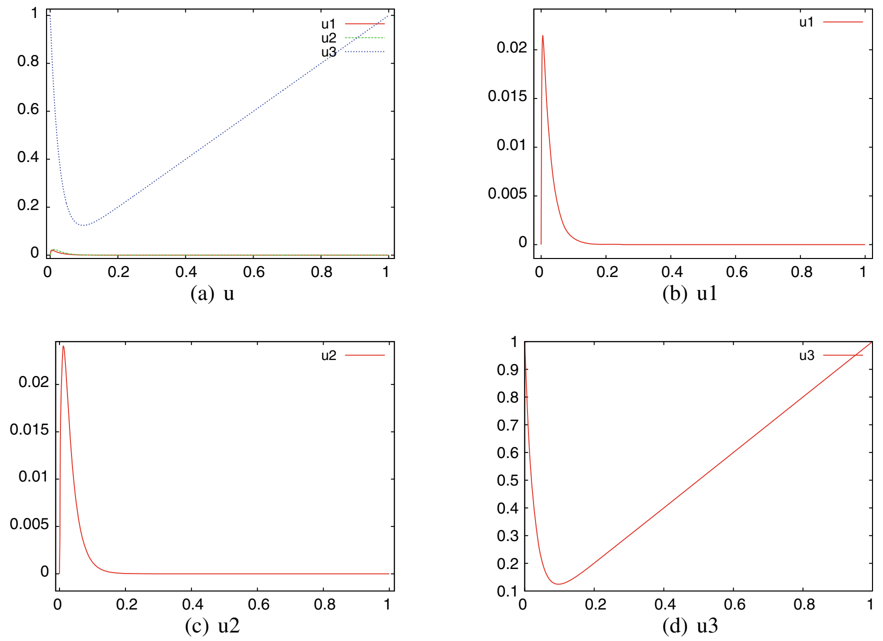

Fig. 3 Approximation of solution components of Example 3

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