Springer Proceedings in Mathematics & Statistics

Valarmathi Sigamani
John J. H. Miller
Shivaranjani Nagarajan
Parthiban Saminathan *Editors*

Differential Equations and Applications

ICABS 2019, Tiruchirappalli, India, November 19—21

Springer

Springer Proceedings in Mathematics & Statistics

Volume 368

This book series features volumes composed of selected contributions from workshops and conferences in all areas of current research in mathematics and statistics, including operation research and optimization. In addition to an overall evaluation of the interest, scientific quality, and timeliness of each proposal at the hands of the publisher, individual contributions are all refereed to the high quality standards of leading journals in the field. Thus, this series provides the research community with well-edited, authoritative reports on developments in the most exciting areas of mathematical and statistical research today.

More information about this series at https://link.springer.com/bookseries/10533

Valarmathi Sigamani · John J. H. Miller ·
Shivaranjani Nagarajan · Parthiban Saminathan
Editors

Differential Equations and Applications

ICABS 2019, Tiruchirappalli, India, November 19–21

Springer

Editors
Valarmathi Sigamani
Department of Mathematics
Bishop Heber College
Tiruchirappalli, Tamil Nadu, India

John J. H. Miller
Institute for Numerical Computation
and Analysis
Dublin, Ireland

Shivaranjani Nagarajan
National Institute of Technology
Tiruchirappalli, Tamil Nadu, India

Parthiban Saminathan
Bishop Heber College
Tiruchirappalli, Tamil Nadu, India

ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-981-16-7545-4 ISBN 978-981-16-7546-1 (eBook)
https://doi.org/10.1007/978-981-16-7546-1

Mathematics Subject Classification: 34-XX, 65-XX, 65Mxx, 14D21

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Preface

This book is the consolidation of refereed mathematical articles contributed by the participants of the International Conference on Applications of Basic Sciences (ICABS 2019) held during 19–21 November 2019 at Bishop Heber College, Tiruchirappalli, India. The aim of the conference was to give young researchers an opportunity to collaborate with researchers from other sciences and find fruitful ways of applying their research ideas to problems that arise in other fields and to promote interdisciplinary research. The research ideas presented before the pioneers who participated in the conference have been developed into research articles, refereed and are published in this book. Though the conference had participants presenting their research works in all the four basic sciences, namely mathematics, physics, chemistry and biology, this book is confined to mathematics.

These contributions are on differential equations, their areas of occurrence and the methods of solving them. The main concentration is the numerical analysis for singular perturbation problems. The solutions of these problems exhibit initial/boundary and/or interior layers, and in those regions, the solution and its derivatives are non-smooth and classical methods fail and one has to go for robust and layer resolving numerical techniques.

The most fascinating real-life example of a boundary layer is the layer that occurs on the wing of an aircraft, which is responsible for creating the drag that acts against the lift of the aircraft. Singular perturbation problems play an important role in the modelling of pupil light reflex, activation of neuronal variability, oxygen intake by red blood cells covered by membranes, blood flow in blood vessels, pattern formation of DNAs of different species, fluid mechanics, electrical networks, chemical reactions, quantum mechanics, etc.

As the conference was uniting scientists working on areas of basic sciences, the invited talks attracted all the participants with the cross-discipline research findings. There were invited talks by four eminent mathematicians: Prof. Grigorii I. Shishkin, leading research scientist, Institute of Mathematics and Mechanics, Ural Branch of Russian Academy of Sciences; S. Kovalevskaya, Yekaterinburg, Russia; Prof. Lidia P. Shishkina, leading mathematician of the same institute; Prof. Carmelo Clavero and

Prof. Jose Luis Gracia from the School of Engineering and Architecture, Department of Applied Mathematics, Campus Rio Ebro, University of Zaragoza, Spain.

Other contributory presentations in mathematics were also much interesting and well appreciated by Prof. Shishkin, one of the pioneers in the field of numerical solutions to singular perturbation problems, and the other invited speakers. A team of professors and scholars from Bishop Heber College who have been working on singular perturbation problems and many others from premier institutes of India presented their papers.

This book would serve as a good reference work for researchers as it gives a comprehensive knowledge of various classes of differential equations and in particular singular perturbation problems. The book comprises ten chapters—nine articles by researchers and one by the invited speakers.

We are grateful to Prof. John J. H. Miller, Professor at the Institute for Numerical Computation and Analysis (INCA), Dublin, Ireland, who kindly agreed to be a member of the editorial board, the invited speakers, contributors from different parts of India and the referees. We acknowledge, with sincere thanks, the support extended by the sponsors of the conference, especially the management of Bishop Heber College. Our special thanks to the Principal of Bishop Heber College and the organising committee. It is our pleasure to thank Mr. Kennet Jacob Jeyasingh, Technical Lead, Alpha Ori Technologies Pvt. Ltd., Chennai, who designed the conference brochure, logo and a dynamic website for ICABS 2019 connecting scientists all over the world with us. Finally, with pleasure, we thank our publisher, Springer Nature, and Mr. Shamim Ahmad, Senior Editor, Sciences Books, Springer Nature India Private Limited, New Delhi, India.

Tiruchirappalli, India — Valarmathi Sigamani
Dublin, Ireland — John J. H. Miller
Tiruchirappalli, India — Shivaranjani Nagarajan
Tiruchirappalli, India — Parthiban Saminathan
May 2021

Contents

Editors and Contributors

About the Editors

Valarmathi Sigamani was formerly with Bishop Heber College, Tiruchirappalli, India, for about 34 years, where she was the Head of the Department of Mathematics for 14 years and retired as the Vice-Principal of the College. She got her Ph.D. for her findings on numerical analysis of singular perturbation problems for third-order ordinary differential equations. She is an active researcher in the field of numerical analysis for singular perturbation problems for the past 23 years. She has guided eight students for their Ph.D. and published many papers in reputed international journals. She has been refereeing for a few SCI journals and doing joint research works with the great scientist Prof. John J. H. Miller, Institute for Numerical Computation and Analysis (INCA), Dublin, Ireland, for the past 16 years. She has edited books and participated in many Boundary and Interior Layers (BAIL) conferences held once in two years at different places around the globe.

John J. H. Miller was graduated with a double moderatorship in mathematics and natural sciences from Trinity College, Dublin, having first studied modern languages and literature. He obtained a Ph.D. in numerical analysis at the Massachusetts Institute of Technology, U.S.A. under the supervision of Prof. Gilbert Strang. Now, he is a Professor at the Institute for Numerical Computation and Analysis (INCA), Dublin, Ireland. Professor Miller is a pioneer in the area of constructing robust computational techniques to solve singular perturbation problems arising from the study of the Navier–Stokes equations. He worked on the well-known Fitted Operator methods and Fitted Mesh methods for certain classes of singular perturbation problems and the analysis of the convergence of these methods.

Shivaranjani Nagarajan is Assistant Professor at the Department of Mathematics at the National Institute of Technology, Tiruchirappalli, Tamil Nadu. She did her Ph.D. in constructing numerical methods for singularly perturbed delay differential equations under the guidance of Dr. Valarmathi Sigamani. She secured the first rank in her master's programme at Bharathidasan University, Tamil Nadu. During her Ph.D. studies, she received the prestigious Inspire Fellowship from the Department of Science & Technology, Government of India. This fellowship is meant for the first rank holders to do their Ph.D. if their research proposals also seem to be good. She has publications in reputed international journals. Her research interests include differential equations arising in mathematical biology, and currently, she is working on systems of singularly perturbed differential equations characterised by two singular perturbation parameters.

Parthiban Saminathan is Assistant Professor of Mathematics at Bishop Heber College, Tiruchirappalli, Tamil Nadu. He completed his Ph.D. studies in the numerical analysis of systems of parabolic delay differential equations under the guidance of Dr. V. Franklin, Assistant Professor, Department of Mathematics, Bishop Heber College. He is skilled in data science and handling data with R. He is very much interested in developing codes animating simple mathematical concepts.

Contributors

S. Chandra Sekhara Rao Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi, India

Abhay Kumar Chaturvedi Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi, India

J. L. Gracia IUMA and Department of Applied Mathematics, University of Zaragoza, Zaragoza, Spain

D. Irisarri Area of Fluid Mechanics, School of Engineering and Architecture, University of Zaragoza, Zaragoza, Spain

R. Ishwariya Department of Mathematics, Bishop Heber College, Tiruchirappalli, TN, India

Saravanasankar Kalaiselvan Department of Mathematics, Bishop Heber College, Tiruchirappalli, TN, India

S. Karthick Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur, TN, India

R. Mahendran Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur, TN, India

Manikandan Mariappan Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur, Maharashtra, India

John J. H. Miller Trinity College, Dublin, Ireland;
Institute of Numerical Computation and Analysis, Dublin, Ireland

Parthiban Saminathan Department of Mathematics, Bishop Heber College, Tiruchirappalli, Tamil Nadu, India

Valarmathi Sigamani Department of Mathematics, Bishop Heber College, Tiruchirappalli, Tamil Nadu, India

Varsha Srivastava Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas, New Delhi, India

V. Subburayan Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur, TN, India

Franklin Victor Department of Mathematics, Bishop Heber College, Tiruchirappalli, Tamil Nadu, India

Virtual Element Method for Singularly Perturbed Reaction-Diffusion Problems on Polygonal Domains

J. L. Gracia and D. Irisarri

Abstract In this paper, the virtual element method is used for the numerical approximation of singularly perturbed linear problems of reaction-diffusion type on polygonal domains. This method is defined on a special mesh of Shishkin type, and its construction is described in detail. The computed orders of convergence in the numerical experiments indicate that the method is first-order uniformly convergent in the maximum norm. The second approach combining link-cutting and post-processing techniques is used to approximate this problem class more efficiently.

Keywords Virtual element method · Singularly perturbed problems · Polygonal domains · Uniform convergence

1 Introduction

Singularly perturbed problems arise in many branches of science and are characterized by the presence of a small parameter multiplying one or more of the highest derivatives in a differential equation. The solution to these problems may exhibit layer phenomena. In this paper, we consider the following two-dimensional Dirichlet boundary value reaction-diffusion problem:

$$-\varepsilon^2 \Delta u + b(x, y)\, u = f(x, y), \quad (x, y) \in \Omega, \quad u \text{ given on } \Gamma, \tag{1a}$$

where Ω is a polygonal domain in $\mathbb{R}^2$ with boundary Γ. We assume that the reaction term satisfies $b(x, y) \geq \beta^2,\ (x, y) \in \bar{\Omega}$ with $\beta > 0$ and $0 < \varepsilon \leq 1$. The coefficient ε

J. L. Gracia (✉)
IUMA and Department of Applied Mathematics, University of Zaragoza, Zaragoza, Spain
e-mail: jlgracia@unizar.es

D. Irisarri
Area of Fluid Mechanics, School of Engineering and Architecture, University of Zaragoza, Zaragoza, Spain
e-mail: dirisarri@unizar.es

Valarmathi S et al. (eds.), *Differential Equations and Applications*,
Springer Proceedings in Mathematics & Statistics 368,
https://doi.org/10.1007/978-981-16-7546-1_1

is called the singular perturbation parameter. We also assume that the data problems are smooth functions, and then the singularities (boundary and corner layers) in the solution are only caused by the singular nature of the differential operator. Standard numerical methods typically fail to approximate this class of problems, and there is a great interest in the development of special numerical methods which provide accurate approximations to the solution for all the values of the singular perturbation parameter, i.e., they are *robust* or *uniformly convergent*. Many numerical schemes have been designed and analysed to approximate (1) in the framework of Finite Difference and Finite Element Methods [17, 23–25, 27, 28].

In the last years, Virtual Element Method (VEM) has been growing very fast [3–7, 9, 10, 12–14, 19], and it can be considered as an extension of the Finite Element Method to general polygonal and polyhedral decompositions. The main idea of VEM consists in enriching the classical polynomial space with other functions, whose explicit knowledge is not needed for the construction of the method. VEM is very flexible in handling general polygonal/polyhedral meshes (including convex or nonconvex elements), which is more suitable to discretize partial differential equations defined on complex geometric domains. An overview of VEM is provided in [1, 5, 6] and although there is an extensive bibliography on it, to the best of our knowledge, it has not been considered in the area of singularly perturbed problems.

In this paper, we approximate problem (1) with a VEM that uses linear elements $\mathbb{P}_1(K)$ in each element K of a mesh of Shishkin type condensing in the layer regions. Taking into account the singular behaviour of the solution, the domain is appropriately decomposed into several subdomains, and quadrilateral elements are used in each subdomain, although other elements can be used. In order that the method be robust, a rectangular grid, which is aligned with the singularity of the solution, is considered in the adjacent subdomains to the boundary of the domain. These adjacent subdomains, where the gradient of the solution is extremely large, are very narrow with a width of $O(\varepsilon)$. On the contrary, outside of the layer regions, where the solution is smooth, a coarse mesh is used. It is numerically shown that this scheme is first-order uniformly convergent in the maximum norm.

We shall use a second methodology in order to approximate more efficiently problem (1). This technique consists of two steps: Firstly, a stabilized solution in the whole domain is obtained using the link-cutting (LC) condition [11, 18]. This numerical solution is only accurate outside the layer regions. The approximation in the layer regions is improved in the second step by solving a boundary value problem (local problem), where the computed solution in the first step is used to set the boundary values. The computed orders of convergence with this methodology suggest an improvement in the rates of convergence compared to the previous VEM method on a Shishkin mesh.

This paper is structured as follows: In Sect. 2, a brief summary of the VEM discretization applied to the reaction-diffusion problem is given. Some information about the behaviour of the solution of the singularly perturbed problem (1) is provided in Sect. 3. This information is used in Sect. 4, where the VEM with linear elements is defined on appropriate Shishkin meshes for polygonal domains. The numerical results for an example are given in Sect. 5, and they show that the method is first-order uniformly convergent. In Sect. 6, the two-step methodology based on the link-cutting and post-processing techniques is described in detail and some numerical results are given, showing its efficiency and accuracy.

Throughout the paper, C denotes a generic positive constant that is independent of the discretization N and the singular perturbation ε parameters.

2 Virtual Element Method Discretization

An abstract framework for VEM is described in this section and, for the sake of clarity, we assume that $u(x, y) \equiv 0,\ (x, y) \in \Gamma$. The variational formulation of problem (1) reads: Find $u \in V = H_0^1(\Omega)$ such that

$$B(u, v) = (f, v), \quad \forall v \in V, \tag{2}$$

where

$$B(u, v) = a(u, v) + c(u, v), \tag{3}$$

and

$$a(u, v) = \textstyle\int_\Omega \varepsilon^2\, \nabla u \cdot \nabla v, \quad c(u, v) = \displaystyle\int_\Omega b\, u\, v, \quad (f, v) = \int_\Omega f\, v. \tag{4}$$

A detailed discussion on the discretization of the variational problem (2) with the VEM can be found in [7]. Here, we only give a brief description of the basic features of this method.

The domain Ω is first decomposed into a partition $\mathscr{P}_h$ composed of polygons K; let $\mathscr{E}_h$ be the set of edges e of $\mathscr{P}_h$. We consider on each element K, the following space for linear elements:

$$\widetilde{V}_h(K) = \left\{ v \in H^1(K) :\ v_{|e} \in \mathbb{P}_1(e)\ \forall\, e \subset \partial K,\ \Delta v \in \mathbb{P}_1(K) \right\}.$$

This is the space of the functions that are linear on each edge and, therefore, they are completely determined by their values at the vertices of K. Inside $\widetilde{V}_h(K)$, the functions are harmonic and its total dimension is equal to the number of vertices of K. For higher order elements, the degrees of freedom are different [7].

A crucial ingredient in the construction of a suitable local stiffness matrix (ensuring the consistency and stability of the method) is the projection operator $\Pi_1^\nabla : \widetilde{V}_h(K) \to \mathbb{P}_1(K)$, defined, for every $v \in \widetilde{V}_h(K)$, as the solution of

$$\int_K \nabla(\Pi_1^\nabla v - v) \cdot \nabla p = 0,\ \forall p \in \mathbb{P}_1(K) \ \text{ and } \int_{\partial K} (\Pi_1^\nabla v - v) = 0, \tag{5}$$

and, therefore, the polynomial $\Pi_1^\nabla v$ can be computed using the degrees of freedom (i.e., the values of v at the vertices of K).

We now define the local virtual element space for linear elements

$$V_h(K) = \left\{ v \in \widetilde{V}_h(K) : \int_K v\, p = \int_K \Pi_1^{\nabla} v\, p, \quad \forall p \in \mathbb{P}_1(K) \right\},$$

and the global finite-dimensional virtual element space

$$V_h = \left\{ v \in V : v_{|K} \in V_h(K)\ \forall K \in \mathscr{P}_h \right\}.$$

We denote by Π_k^0 the L^2-projection from V_h onto $\mathbb{P}_k$, which is defined locally as

$$\int_K (v - \Pi_k^0 v)\, p_k = 0 \quad \forall p_k \in \mathbb{P}_k(K).$$

The bilinear form (3) can be discretized as the sum of the bilinear forms restricted to the elements

$$B_h(u, v) = a_h(u, v) + c_h(u, v), \quad \forall u, v \in V_h, \tag{6}$$

with

$$a_h(u, v) = \sum_{K \in \mathscr{P}_h} a_h^K(u, v), \; c_h(u, v) = \sum_{K \in \mathscr{P}_h} c_h^K(u, v),$$

the elemental bilinear forms are given by

$$a_h^K(u, v) = \int_K \varepsilon^2 [\Pi_0^0 \nabla u] \cdot [\Pi_0^0 \nabla v] + S_\varepsilon^K((I - \Pi_1^{\nabla})u, (I - \Pi_1^{\nabla})v), \tag{7a}$$

$$c_h^K(u, v) = \int_K b[\Pi_1^0 u][\Pi_1^0 v] + S_b^K((I - \Pi_1^0)u, (I - \Pi_1^0)v), \tag{7b}$$

and the terms $S_\varepsilon^K(\cdot, \cdot)$ and $S_b^K(\cdot, \cdot)$ are defined later in (8). The right-hand side of (2) is approximated by

$$(f_h, v_h) = \sum_{K \in \mathscr{P}_h} (f_h, v_h)_K = \sum_{K \in \mathscr{P}_h} \int_K \Pi_1^0 f v_h.$$

As we have considered linear elements, the degrees of freedom, $\mathrm{dof}_i(\cdot)$, are the values of v_h at the vertex i for $i = 1, 2, \ldots, n$, where n is the number of vertices of $\mathscr{P}_h$. The basis functions $\varphi_i \in V_h$ are defined as the canonical basis functions, and they satisfy $\mathrm{dof}_i(\varphi_j) = \delta_{ij}$, for $i, j = 1, 2, \ldots, n$. Thus,

$$v_h = \sum_{i=1}^{n} \mathrm{dof}_i(v_h) \varphi_i, \quad v_h \in V_h.$$

The terms $S_\varepsilon^K(\cdot,\cdot)$ and $S_b^K(\cdot,\cdot)$ in (7) guarantee the stability of the method, and they are defined as (see [5, 6, 8])

$$S_\varepsilon^K((I-\Pi_1^\nabla)\varphi_i,(I-\Pi_1^\nabla)\varphi_j)=\varepsilon^2[(I-\Pi_1^\nabla)^T(I-\Pi_1^\nabla)]_{ij}\,, \tag{8a}$$
$$S_b^K((I-\Pi_1^\nabla)\varphi_i,(I-\Pi_1^\nabla)\varphi_j)=b|K|[(I-\Pi_1^0)^T(I-\Pi_1^0)]_{ij}\,. \tag{8b}$$

Then, the discrete problem can be written as follows: Find $u_h\in V_h$ such that

$$B_h(u_h,v_h)=(f_h,v_h)\quad \forall v_h\in V_h.$$

3 Asymptotic Behaviour of the Solution

The asymptotic behaviour of the solution of problem (1) on the domain $\Omega=(0,1)^2$ is analysed in [15]. The solution is decomposed into a regular component v, boundary layer components w_i associated with each side of Ω, and corner layer components z_i:

$$u=v+\sum_{i=1}^4 w_i+\sum_{i=1}^4 z_i.$$

Assume that the problem data satisfy enough regularity and compatibility conditions. Then, for $0\le m+n\le 4$, the regular component satisfies

$$|\partial_x^m\partial_y^n v(x,y)|\le C(1+\varepsilon^{1-m-n}),$$

the boundary layers w_1 and w_2 associated with the edges $y=0$ and $x=0$

$$|w_1(x,y)|\le Ce^{-\beta y/\varepsilon},\ |w_2(x,y)|\le Ce^{-\beta x/\varepsilon},\ |\partial_x^m\partial_y^n w_i(x,y)|\le C\varepsilon^{-m-n}, \tag{9}$$

for $i=1,2$, and the corner layer function associated with the corner $(0,0)$

$$|z_1(x,y)|\le Ce^{-\beta y/\varepsilon}e^{-\beta x/\varepsilon},\quad |\partial_x^m\partial_y^n z_1(x,y)|\le C\varepsilon^{-m-n}. \tag{10}$$

Similar estimates are satisfied for the other boundary and corner layer functions. These estimates prove that the solution exhibits boundary layers in Γ with a width of order $O(\varepsilon)$, and the solution is smooth away from Γ. Estimates (9) and (10) also show that the boundary and corner layer components decay exponentially away from Γ and the corner $(0,0)$, respectively. These estimates are used in [15] to construct and analyse the convergence of the standard central difference approximation on a mesh of Shishkin type [17]. This mesh is fitted to the boundary layers, and it is defined by means of a transition parameter

$$\sigma = \min\left\{\frac{1}{3}, 2\frac{\varepsilon}{\beta}\ln N\right\}, \tag{11}$$

where the positive integer N is the discretization parameter. For simplicity, the same number of grid points is used in both spatial directions. Each spatial direction is split as follows:

$$[0, \sigma] \cup [\sigma, 1-\sigma] \cup [1-\sigma, 1], \tag{12}$$

and the N space mesh points are distributed in the ratio $N/3 : N/3 : N/3$ across these three subintervals. The final mesh is constructed by taking the tensor product of the 1D meshes. In [15], it is proved that the central difference scheme on the Shishkin mesh converges uniformly at the rate $O((N^{-1}\ln N)^2)$, i.e., it is an almost second-order uniformly convergent scheme (due to the presence of the logarithm factor).

In [20, 21], using the general theory of corner singularities [16, 22, 26], some pointwise derivative bounds are proved for the solution u of problem (1) in the case of a general polygonal domain $\bar{\Omega}$. Assuming some smoothness properties on the data problem, it is proved in [21, Theorem 1, p. 786] that the tangential ∂_σ and normal ∂_ν directional derivatives to one of the sides of the domain Ω satisfy

$$|u(x,y)| \le C + Ce^{-pd_s(x,y)/\varepsilon} + Ce^{-pd_v(x,y)/\varepsilon}, \tag{13a}$$

$$|\partial_\sigma u(x,y)| \le C\varepsilon^{-2}, \; |\partial_\nu u(x,y)| \le C\varepsilon^{-1}, \tag{13b}$$

where C and p are positive constants which are independent of ε, $d_s(x,y)$ is the distance from (x,y) to the nearest side of Γ, and $d_v(x,y)$ is the distance from (x,y) to the nearest vertex of Γ. These estimates indicate the presence of a boundary layer along the sides of Ω. In addition, they reveal the dependence on ε of the derivatives in the direction normal to the boundary. The term $e^{-pd_v(x,y)/\varepsilon}$ represents the effect of the corner singularity at the vertex, showing that the effect of these corner singularities is increasingly localized as ε become small.

4 Shishkin Meshes for Polygonal Domains

In this section, we give a general description of the construction of a Shishkin mesh for a polygonal domain, which is a simple extension of the mesh used in Sect. 3 for a rectangular domain. Suppose that Ω is an n-sided polygon; from (13), the solution, in general, exhibits boundary layers along the boundary Γ of Ω with a width of $O(\varepsilon)$. Parallel segments to the n sides of the polygon are drawn at a distance σ which is defined in (11). Thus, the polygon is split into $2n+1$ subdomains which are associated with the n sides, the n vertices, and the interior part of the polygon.

In order that the computational mesh is aligned with the singularity of the solution, the adjacent subdomains to the boundary are slightly modified; this is explained for

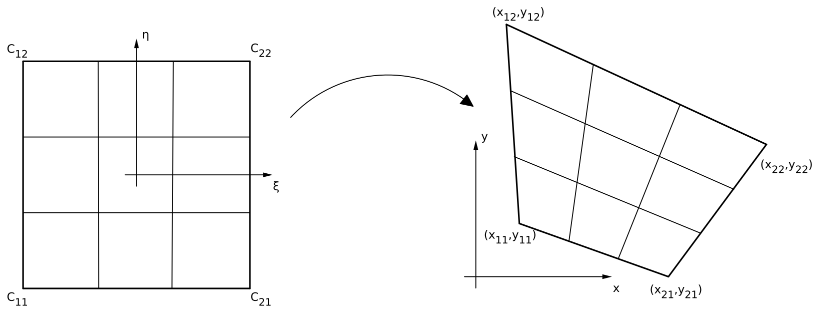


Fig. 1 Mapping of the canonical square to a quadrilateral ($N_{\mathrm{quad}} = 3$)

a particular example in Sect. 5. After this modification is made, each subdomain is split into several elements, as, for example, with triangles or quadrilaterals. In this paper, we only consider quadrilateral elements, and the associated meshes are generated using the standard bilinear transformation of the canonical 2×2 square of vertices c_{ij} to an arbitrary quadrilateral with vertices (x_{ij}, y_{ij}) (see Fig. 1). The vertices of the canonical square are $c_{ij} = ((-1)^i, (-1)^j),\ i, j = 1, 2$ and the bilinear transformation is given by

$$\begin{bmatrix} x(\xi, \eta) \\ y(\xi, \eta) \end{bmatrix} = \sum_{i,j=1}^{2} \begin{bmatrix} x_{ij} \\ y_{ij} \end{bmatrix} N_{i,j}(\xi, \eta), \tag{14}$$

where $N_{i,j}(\xi, \eta)$ are the bilinear shape functions which satisfy

$$N_{i,j}(c_{kl}) = \delta_{i,k}\delta_{j,l}, \quad \text{where } \delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

We now divide each side of the canonical square into N_{quad} equidistant subintervals and a tensor product mesh is constructed; thus, the mesh has $N_{\mathrm{quad}} \times N_{\mathrm{quad}}$ elements. Then, the transformation (14) is applied in each subdomain and the final mesh is obtained by patching these meshes. An example is considered in the next section, and the construction of the mesh is described in detail.

5 Numerical Experiments of VEM on Shishkin Meshes

Example 1 Consider the example

$$-\varepsilon^2 \Delta u + (1 + xy)u = 1, \text{ in } \Omega, \quad u(x, y) = 0, \text{ on } \Gamma,$$

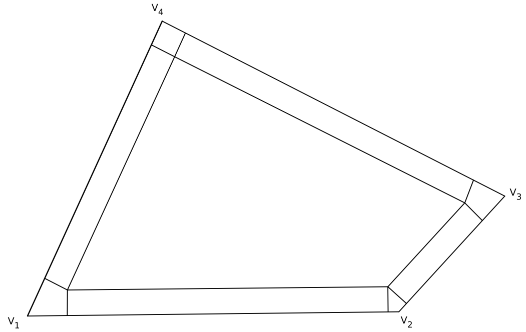


Fig. 2 Geometry of the domain Ω and its subdomains

where Ω is the polygonal domain depicted in Fig. 2. The vertices $V_i,\ 1 \le i \le 4$ are counterclockwise ordered, and their coordinates are

$$V_1(0,0),\ \ V_2(1,0),\ \ V_3\left(\frac{4l+\sqrt{3}}{1+\sqrt{3}}, \frac{4l-1}{1+\sqrt{3}}\right) \text{ and } V_4(l, \sqrt{3}l),$$

with $l = 0.45$. The angles are $\alpha_1 = \pi/3$, $\alpha_2 = 3\pi/4$, $\alpha_3 = 5\pi/12$, and $\alpha_4 = \pi/2$, where the angle of the vertex V_i is α_i, $1 \le i \le 4$.

We first construct the quadrilateral mesh to be used with the VEM by drawing parallel segments to the four sides of Ω at a distance σ. Then, the domain Ω is split into nine subdomains. In order that the mesh is aligned with the singularity in the boundary layer regions, the adjacent subdomains to the boundary of the domain are modified. This is done by drawing orthogonal segments from the four interior intersection points to the two nearest sides of the boundary of Ω. Then, a quadrilateral mesh is constructed in each subdomain by dividing both sides of the canonical square into $N/3$ equidistant subintervals and applying the transformation (14) to each one of them. The final quadrilateral mesh and the computed solution with the VEM method for $N = 12$ and $\varepsilon = 10^{-2}$ are shown in Fig. 3.

The errors are estimated using the two-mesh principle [17]; for this, each quadrilateral element is divided into four equal parts to construct the fine mesh. Thus, all

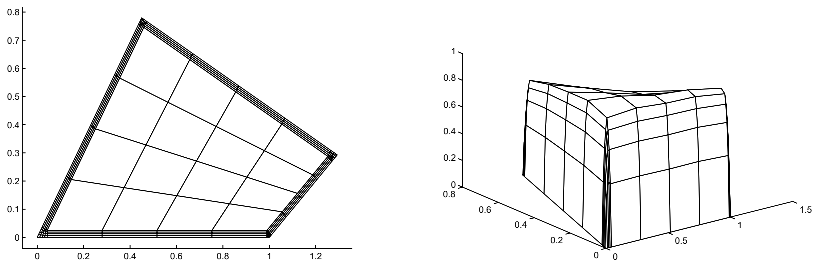
Table 1 Maximum nodal two-mesh differences and orders of convergence

	N = 24	N = 48	N = 96	N = 192
$\varepsilon = 10^{-3}$	8.943E-3	3.571E-3	1.909E-3	9.600E-4
	1.324	0.904	0.992	
$\varepsilon = 10^{-4}$	8.376E-3	3.571E-3	1.308E-3	5.394E-4
	1.230	1.449	1.278	
$\varepsilon = 10^{-5}$	8.376E-3	3.571E-3	1.308E-3	4.397E-4
	1.230	1.449	1.573	
$\varepsilon = 10^{-6}$	8.376E-3	3.571E-3	1.308E-3	4.397E-4
	1.230	1.449	1.573	

the grid points of the coarse mesh belong to the fine mesh. The maximum two-mesh differences and the corresponding orders of convergence for Example 1 are given in Table 1. They suggest that the method is first-order uniformly convergent.

6 Two-Step Methodology

In this section, we use a different methodology to approximate more accurately and efficiently the solution of (1). It is approximated in two steps: In the first step, the LC condition is applied in order to avoid the propagation of spurious oscillations in the domain. Thus, a stabilized solution is generated, but it is only accurate to a distance of order $O(\varepsilon)$ away from the boundary Γ of the domain. In the second step, the approximation in the layer regions is improved by using a post-processing technique.



(a) Shishkin mesh (b) Computed solution

Fig. 3 The patched mesh and computed solution with VEM for $\varepsilon = 10^{-2}$ and $N = 12$

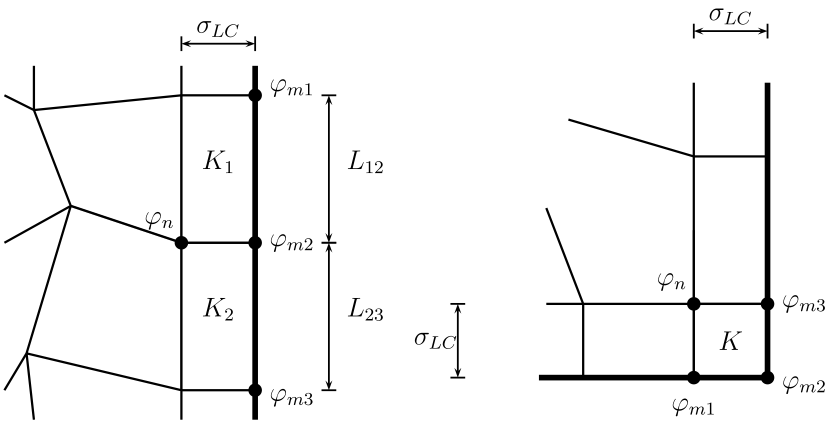


Fig. 4 LC condition on the boundary layer (left figure) and at the corners (right figure)

6.1 The Link-Cutting Condition

Following [11, 18], we obtain a stable solution with the VEM by removing the interaction between the basis functions associated with the vertices on the boundary of the domain and the neighbouring basis functions. First, we take a mesh of quadrilateral elements whose size does not depend on the singular perturbation parameter ε. In order that the LC is satisfied, this initial mesh is modified in the layer regions by patching some quadrilateral elements as follows: Let Υ_B denote the set of degrees of freedom belonging to the boundary of the domain. Then, the LC condition can be formulated as

$$\sum_{m\in\Upsilon_B} B_h(\varphi_n, \varphi_m) = 0 \;\; \forall n \notin \Upsilon_B \,, \tag{16}$$

where φ_n are the basis functions of the space $V_h(K)$. A way of fulfilling (16) is introducing a row of quadrilaterals on the boundary layer regions with a suitable width, which is called the LC distance [18] and it is denoted by σ_{LC}.

For example, the LC condition for the vertex n depicted in Fig. 4 (left) is given by

$$B_h(\varphi_n, \varphi_{m_1}) + B_h(\varphi_n, \varphi_{m_2}) + B_h(\varphi_n, \varphi_{m_3}) = 0. \tag{17}$$

From (6) and (8), we can write (17) as

$$\sum_{j=1}^{3} a_h(\varphi_n, \varphi_{m_j}) + c_h(\varphi_n, \varphi_{m_j}) + S_\varepsilon^K((I - \Pi_1^\nabla)\varphi_n, (I - \Pi_1^\nabla)\varphi_{m_j}) \\ + S_b^K((I - \Pi_1^\nabla)\varphi_n, (I - \Pi_1^\nabla)\varphi_{m_j}) = 0. \tag{18}$$

Noting that the three vertices m_1, m_2, and m_3 lie on the same side of the polygonal boundary and K_1 and K_2 are rectangles, we have

$$\sum_{j=1}^{3} S_\varepsilon^K((I-\Pi_1^\nabla)\varphi_n,(I-\Pi_1^\nabla)\varphi_{m_j})=0,\quad \sum_{j=1}^{3} S_b^K((I-\Pi_1^\nabla)\varphi_n,(I-\Pi_1^\nabla)\varphi_{m_j})=0.$$

Then, imposing these conditions on (18), the LC condition (17) is reduced to

$$\sum_{j=1}^{3} a_h(\varphi_n,\varphi_{m_j})+c_h(\varphi_n,\varphi_{m_j})=0. \tag{19}$$

Assuming that the term $b(x,y)\equiv b$ is constant with $b>0$, Eq. (19) yields

$$L_{12}\left(-\frac{\varepsilon^2}{2\sigma_{LC}}+b\frac{\sigma_{LC}}{12}\right)+L_{23}\left(-\frac{\varepsilon^2}{2\sigma_{LC}}+b\frac{\sigma_{LC}}{12}\right)=0, \tag{20}$$

where L_{12} and L_{23} are the lengths of the sides of the elements K_1 and K_2 lying on the boundary of the domain (see Fig. 4). If

$$\sigma_{LC}=\sqrt{\frac{6\varepsilon^2}{b}},$$

then (20) is satisfied. Otherwise, if m_1, m_2, and m_3 form a corner as in Fig. 4 (right), the LC condition (17) is not fulfilled.

This can be fixed with a slight modification in the VEM formulation on the element K. For this element, the LC condition (18) on the node n is

$$\sum_{j=1}^{3} a_h^K(\varphi_n,\varphi_{m_j})+c_h^K(\varphi_n,\varphi_{m_j})+C_\varepsilon S_\varepsilon^K((I-\Pi_1^\nabla)\varphi_n,(I-\Pi_1^\nabla)\varphi_{m_j})$$
$$+C_b S_b^K((I-\Pi_1^\nabla)\varphi_n,(I-\Pi_1^\nabla)\varphi_{m_j})=0, \tag{21}$$

where the stabilization terms $S_\varepsilon^K(\cdot,\cdot)$ and $S_b^K(\cdot,\cdot)$ are multiplied by the positive constants C_ε and C_b, respectively. They are chosen such that the LC condition (21) is fulfilled. For example, in the case of a right-angle corner, condition (21) for the element K gives

$$C_\varepsilon+6C_b=\frac{3}{2}.$$

If $b(x,y)$ is not constant, we have used in each element the following LC distance

$$\tilde{\sigma}_{LC} = \sqrt{\frac{6\varepsilon^2}{b(x_{m_2}, y_{m_2})}} \tag{22}$$

where (x_{m_2}, y_{m_2}) are the coordinates of the vertex m_2 depicted in Fig. 4. In this way, the LC condition (18) is almost satisfied, but we shall show in the numerical experiments that the computed solution taking (22) as the LC distance does not exhibit spurious oscillations and generates an accurate approximation to the solution of the continuous problem.

Remark 1 VEM, unlike FEM, can use different types of elements in the computational mesh. This is very important because the mesh can be modified with the row of quadrilateral elements so that the LC condition is satisfied. This modification in the mesh would not take into account how it affects the number of vertices of the neighbouring elements. Similar results to those given in Sect. 6.3 have been obtained when a triangular mesh is used instead of a quadrilateral mesh.

6.2 Post-processing the Solution

The computed numerical solution in the first step provides an accurate approximation to the problem (1) away from the boundary but not in the layer regions; i.e., it is not layer-resolving. The objective of this step is to improve the approximation in the layer regions. With this aim, a local problem is solved in the layer regions which is governed by the same differential equation, and the domain is defined by drawing parallel segments to the polygonal sides of the domain to a distance of σ, which is defined in (11). Similar to VEM on the Shishkin mesh described in Sects. 4 and 5, the mesh has to be aligned with the singularities of the solution in the vicinity of the boundary of the domain. In the local problem, the boundary conditions are obtained from the computed solution in the previous step and using bilinear interpolation. Once the solutions of both steps are obtained, they are patched to have the final approximation to the solution in the whole domain.

In order to obtain more accurate approximations to the solution, appropriate graded meshes are used in the canonical squares instead of using a uniform mesh. This type of meshes has been previously used in [2] to solve a singularly perturbed problem in an L-shaped domain. Let us denote by r, with $r \geq 1$, the grading exponent of the graded mesh. The mesh is uniform when $r = 1$ and the larger r, the more dense the grid is. In Fig. 5, graded grids in the canonical square are illustrated together with their transformed meshes to be used in the adjacent subdomains of the polygonal domain.

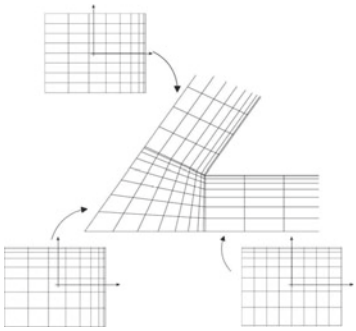
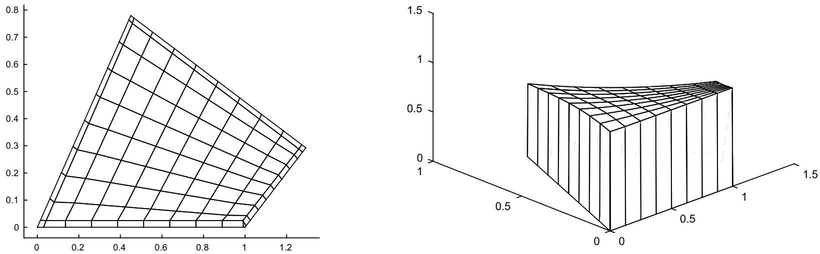


Fig. 5 Canonical and computational graded meshes for the local problem



(a) Modified mesh with LC condition

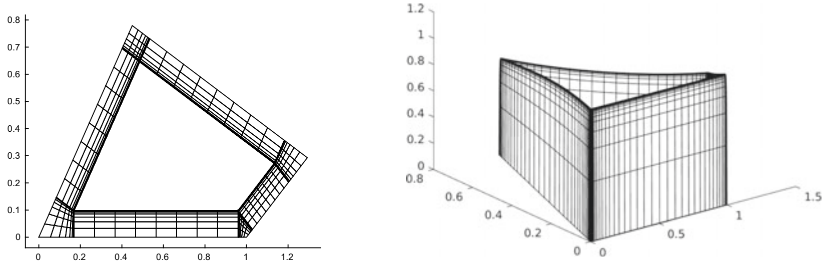
(b) Computed solution

Fig. 6 Mesh with LC condition and computed solution for $\varepsilon = 10^{-2}$ and $N = 12$

6.3 Numerical Experiments

The two-step methodology is applied to Example 1. In the first step, the example is approximated with VEM using a mesh with $N/3$ elements in each direction (see Fig. 6). The computed solution is also given in this figure and it is observed that it does not exhibit spurious oscillations. The domain and the mesh considered in the second step are displayed in Fig. 7. The domain is split into 8 subdomains, and we consider a graded mesh with $r = 2$ and $N/3$ elements in each direction of the canonical square. The numerical approximation to the solution in the local domain is given in Fig. 7.

The computed maximum two-mesh differences and the orders of convergence with the two-step methodology are given in Table 2, and it is observed that the method is uniformly convergent in the maximum norm. If we compare the numerical results from Tables 1 and 2, we see an improvement in the orders of convergence when the two-step methodology is used. Thus, the numerical results for Example 1 suggest that this scheme is more efficient than the VEM on the Shishkin mesh described in Sect. 4. If $r \geq 2$, we have obtained similar results but the mesh condenses excessively. If $r = 1$, we have observed that the computed orders of convergence are reduced to the first order.



(a) Piecewise graded mesh with $r = 2$

(b) Computed solution

Fig. 7 Mesh and computed solution for Example 1 with $\varepsilon = 10^{-2}$ and $N = 12$

Table 2 Two-step methodology: Maximum two-mesh differences and orders of convergence

	N = 24	N = 48	N = 96	N = 192
$\varepsilon = 10^{-3}$	7.206E-2	2.992E-2	8.051E-3	2.287E-3
	1.268	1.894	1.816	
$\varepsilon = 10^{-4}$	7.206E-2	2.992E-2	8.051E-3	2.287E-3
	1.268	1.894	1.816	
$\varepsilon = 10^{-5}$	7.206E-2	2.992E-2	8.051E-3	2.287E-3
	1.268	1.894	1.816	
$\varepsilon = 10^{-6}$	7.206E-2	2.992E-2	8.051E-3	2.287E-3
	1.268	1.894	1.816	

Acknowledgements The research of J. L. Gracia has been partially supported by the Institute of Mathematics and Applications (IUMA), the projects PID2019-105979GB-I00 and PGC2018-094341-B-I00, and the Diputación General de Aragón (E24-17R).

References

1. Ahmad, B., Alsaedi, A., Brezzi, F., Marini, L.D., Russo, A.: Equivalent projectors for virtual element methods. Comput. Math. Appl. **66**(3), 376–391 (2013)
2. Andreev, V.B., Kopteva, N.: Pointwise approximation of corner singularities for a singularly perturbed reaction-diffusion equation in an L-shaped domain. Math. Comp. **77**(264), 2125–2139 (2008)
3. Antonietti, P.F., Beirão da Veiga, L., Mora, D., Verani, M.: A stream virtual element formulation of the stokes problem on polygonal meshes. SIAM J. Numer. Anal. **52**(1), 386–404 (2014)
4. Beirão Da Veiga, L., Brezzi, F., Marini, L.D.: Virtual elements for linear elasticity problems. SIAM J. Numer. Anal. **51**(2), 794–812 (2013)
5. Beirão da Veiga, L., Brezzi, F., Cangiani, A., Manzini, G., Marini, L.D., Russo, A.: Basic principles of virtual element methods. Math. Mod. Meth. Appl. S. **23**(01), 199–214 (2013)
6. Beirão da Veiga, L., Brezzi, F., Marini, L.D., Russo, A.: The hitchhikers guide to the virtual element method. Math. Mod. Meth. Appl. S. **24**(08), 1541–1573 (2014)

7. Beirão da Veiga, L., Brezzi, F., Marini, L.D., Russo, A.: Mixed virtual element methods for general second order elliptic problems on polygonal meshes. M2AN **50**(3), 727–747 (2016)
8. Beirão da Veiga, L., Lovadina, C., Russo, A.: Stability analysis for the virtual element method. Math. Models Methods Appl. Sci. **27**(13), 2557–2594 (2017)
9. Beirão da Veiga, L., Lovadina, C., Vacca, G.: Virtual elements for the Navier-Stokes problem on polygonal meshes SIAM J. Numer. Anal. **56**, 1210–1242 (2018).
10. Benedetto, M.F., Berrone, S., Borio, A., Pieraccini, S., Scialo, S.: Order preserving SUPG stabilization for the virtual element formulation of advection diffusion problems. Comput. Methods Appl. Mech. Eng. **311**, 18–40 (2016)
11. Brezzi, F., Hauke, G., Marini, L.D., Sangalli, G.: Link-cutting bubbles for the stabilization of convection-diffusion-reaction problems. Math. Mod. Meth. Appl. S. **13**(03), 445–461 (2003)
12. Brezzi, F., Marini, L.D.: Virtual element methods for plate bending problems. Comput. Methods Appl. Mech. Eng. **253**, 455–462 (2013)
13. Cangiani, A., Manzini, G., Russo, A., Sukumar, N.: Hourglass stabilization and the virtual element method. Int. J. Numer. Methods Eng. **102**(3–4), 404–436 (2015)
14. Cangiani, A., Manzini, G., Sutton, O.J.: Conforming and nonconforming virtual element methods for elliptic problems. IMA J. Numer. Anal. **37**(3), 1317–1354 (2016)
15. Clavero, C., Gracia, J.L., ORiordan, E.: A parameter robust numerical method for a two dimensional reaction-diffusion problem. Math. Comp. **74**(252), 1743–1758 (2005)
16. Dauge, M.: Elliptic Boundary Value Problems on Corner Domains. Lecture Notes in Mathematics, vol. 1341. Springer, Berlin (1988) Smoothness and asymptotics of solutions
17. Farrell, P., Hegarty, A., Miller, J.J.H., ORiordan, E., Shishkin, G.I.: Robust Computational Techniques for Boundary Layers. CRC Press (2000)
18. Irisarri, D.: Virtual element method stabilization for convection-diffusion-reaction problems using the link-cutting condition. Calcolo **54**(1), 141–154 (2017)
19. Irisarri, D., Hauke, G.: Stabilized virtual element methods for the unsteady incompressible Navier-Stokes equations. Calcolo **56**(4), Paper No. 38, 21 (2019)
20. Kellogg, R.B.: Boundary layers and corner singularities for a self-adjoint problem. In: Boundary Value Problems and Integral Equations in Nonsmooth Domains (Luminy, 1993). Lecture Notes in Pure and Applied Mathematics, vol. 167, pp. 121–149. Dekker, New York (1995)
21. Kellogg, R.B.: Some simple boundary value problems with corner singularities and boundary layers. Comput. Math. Appl. **51**(5), 783–792 (2006)
22. Kozlov, V.A., Maz'ya, V.G., Rossmann, J.: Elliptic Boundary Value Problems in Domains with Point Singularities. Mathematical Surveys and Monographs, vol. 52. American Mathematical Society, Providence, RI (1997)
23. Lin, R., Stynes, M.: A balanced finite element method for singularly perturbed reaction-diffusion problems. SIAM J. Numer. Anal. **50**(5), 2729–2743 (2012)
24. Melenk, Jens M.: hp-Finite Element Methods for Singular Perturbations. Lecture Notes in Mathematics, vol. 1796. Springer, Berlin (2002)
25. Miller, J.J.H., ORiordan, E., Shishkin, G.I.: Fitted Numerical Methods for Singular Perturbation Problems, revised edn. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ (2012).Error estimates in the maximum norm for linear problems in one and two dimensions
26. Nazarov, S.A., Plamenevsky, B.A.: Elliptic Problems in Domains with Piecewise Smooth Boundaries. De Gruyter Expositions in Mathematics, vol. 13. Walter de Gruyter & Co., Berlin (1994)
27. Roos, H.-G., Stynes, M., Tobiska, L.: Robust numerical methods for singularly perturbed differential equations. Springer Series in Computational Mathematics, vol. 24, second edn. Springer, Berlin (2008). Convection-diffusion-reaction and flow problems
28. Shishkin, G.I., Shishkina, L.P.: Difference Methods for Singular Perturbation Problems. Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 140. CRC Press, Boca Raton, FL (2009)

Global Uniform Convergence of a Numerical Method for a Weakly Coupled System of Singularly Perturbed Convection-Diffusion Equations

S. Chandra Sekhara Rao, Varsha Srivastava, and Abhay Kumar Chaturvedi

Abstract We present a finite difference method for a weakly coupled system of $M(\geq 2)$ singularly perturbed convection-diffusion two point boundary value problems. The problem is discretized using a suitable combination of the second-order compact difference scheme and the second-order central difference scheme. The convergence analysis is given, and the method is shown to have almost second-order parameter-uniform convergence. Numerical experiments are conducted to demonstrate the efficiency of the method.

Keywords Weakly coupled system · Convection-diffusion equations · Shishkin mesh · Second-order compact difference scheme · Central difference scheme · Parameter-uniform convergence

1 Introduction

Consider the following coupled system of $M(\geq 2)$ singularly perturbed convection-diffusion equations

$$\boldsymbol{L}\boldsymbol{u} := -\boldsymbol{E}\boldsymbol{u}'' + \boldsymbol{A}\boldsymbol{u}' + \boldsymbol{B}\boldsymbol{u} = \boldsymbol{f}, \quad x \in \Omega = (0, 1) \tag{1a}$$

$$\boldsymbol{u}(0) = \boldsymbol{p}_1, \qquad \boldsymbol{u}(1) = \boldsymbol{p}_2, \tag{1b}$$

where $\boldsymbol{E}$ =diag$(\varepsilon, \varepsilon, \ldots, \varepsilon)$ with $0 < \varepsilon \leq 1$, $\boldsymbol{A}$ = diag$(a_1, a_2, \ldots, a_M)$, $\boldsymbol{f} = (f_1, f_2, \ldots, f_M)^T$, and $\boldsymbol{u} = (u_1, u_2, \ldots, u_M)^T$. For each $1 \leq i \leq M$ and $x \in \overline{\Omega}$, the $M \times M$ matrices $\boldsymbol{A}$ and $\boldsymbol{B} = (b_{ij})$ satisfy

S. Chandra Sekhara Rao (✉) · V. Srivastava · A. K. Chaturvedi
Department of Mathematics, Indian Institute of Technology Delhi, Hauz Khas 110 016, New Delhi, India
e-mail: scsr@maths.iitd.ac.in

Valarmathi S et al. (eds.), *Differential Equations and Applications*,
Springer Proceedings in Mathematics & Statistics 368,
https://doi.org/10.1007/978-981-16-7546-1_2

$$a_i(x) \geq \alpha_i > 0, \tag{2}$$

$$b_{ij}(x) \leq 0,\ i \neq j,\ b_{ii}(x) + \sum_{j=1, j\neq i}^{M} b_{ij}(x) > 0, \tag{3}$$

for some constant α_i. Also assume $\alpha = \min_{1\leq i\leq M} \{\alpha_i\}$, $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ are given constant column vectors.

Assume $a_i(x)$, $b_{ij}(x)$ and $f_i(x)$, $1 \leq i, j \leq M$, are sufficiently smooth functions on $\overline{\Omega} = [0, 1]$ so that the solution $\boldsymbol{u}$ of (1) is in $C(\overline{\Omega})^M \cap C^4(\Omega)^M$. The solution of (1) exhibits boundary layer of width $O(\varepsilon)$ at $x = 1$.

The coupled systems of convection-diffusion equations appear in many applications, such as optimal control problems and in resistance–capacitor electrical circuits [7]. The numerical analysis of singularly perturbed convection-diffusion problems of the single equation was well discussed in the literature [5, 6, 19, 20], while very few works had been done for the system of convection-diffusion equations. Systems of singularly perturbed reaction-diffusion in steady and unsteady states were studied in [9, 11, 13, 14, 16–18]. In [1], analysis of strongly coupled system of $M(\geq 2)$ singularly perturbed convection-diffusion problems with the same perturbation parameter ε was discussed, in which matrix $\boldsymbol{A}$ was Hermitian. The finite difference scheme on a uniform mesh was used to establish almost first-order parameter-uniform convergence under the assumption that $\varepsilon << h$, and the convergence was achieved only away from the layer. Linss [8] showed the almost first-order parameter-uniform convergence for strongly coupled system of $M(\geq 2)$ convection-diffusion equations with different perturbation parameters, where the condition (2) was replaced by: either $\min_{x\in[0,1]} a_i(x) > 0$ or $\max_{x\in[0,1]} a_i(x) < 0$ for $1 \leq i \leq M$ and also it is assumed that $b_{ii} \geq 0$ along with the condition $a_i' + b_{ii} \geq 0$ on $[0, 1]$. Bellew and O'Riordan [2] studied a system of two coupled convection-diffusion problems with different parameters and showed that the discussed method was almost first order parameter-uniform convergent. Cen [3] discussed an upwind finite difference scheme on Shishkin meshes for a system of two weakly coupled equations with different parameters and showed that scheme was almost first-order parameter-uniform convergent. O'Riordan and Stynes [10] analyzed a system of two strongly coupled convection-diffusion problems with the same parameter and proved that the proposed method was almost first-order parameter-uniform convergent.

Clavero et al. [4] constructed and analyzed two parameter-uniform convergent finite difference methods essentially of order 2 and 3 for a singularly perturbed convection-diffusion problem. Therein, the authors considered a combination of the compact finite difference scheme and the central difference scheme outside boundary layer region while the central difference scheme inside the boundary layer region. In the present work, we extend the finite difference method of order 2, which was discussed for scalar problem in [4] to the coupled system of $M(\geq 2)$ singularly perturbed problems with the same parameter and established a global parameter-uniform convergence of the method.

This paper is arranged as follows. In Sect. 2, bounds on the solution, the bounds on the regular and singular components of the solution are given. In Sect. 3, a discrete method is developed for the system of $M(\geq 2)$ coupled singularly perturbed problems. The parameter-uniform maximum pointwise error bounds and global error bounds are obtained in Sect. 4. Numerical experiments are given in Sect. 5, and conclusions are given in Sect. 6.

Notation: Throughout the paper, we use C with or without a subscript to denote a generic positive constant independent of perturbation parameter ε and the discretization parameter N. $\boldsymbol{C} = C(1, 1, \ldots, 1)^T$. We consider the maximum norm and denote it by $||.||_D$, where D is a closed subset of $\overline{\Omega}$. For a real-valued function $g \in C(D)$ and for a vector valued function $\boldsymbol{g} = (g_1, g_2, \ldots, g_M)^T \in C(D)^M$, we define $||g||_D = \max_{x \in D} |g(x)|$ and $||\boldsymbol{g}||_D = \max\{||g_1||_D, ||g_2||_D, \ldots, ||g_M||_D\}$. The analogous discrete maximum norm on the mesh D^N is denoted by $||.||_{D^N}$. For any functions $g, y \in C(\overline{\Omega})$, define $g_j = g(x_j)$. If $\boldsymbol{g} \in C(\overline{\Omega})^M$ then $\boldsymbol{g}_j = \boldsymbol{g}(x_j) = (g_{1,j}, g_{2,j}, \ldots, g_{M,j})^T$.

2 Properties of the Exact Solution

The continuous operator $\boldsymbol{L}$ satisfies the following maximum principle on $\overline{\Omega}$.

Lemma 1 (Continuous maximum principle) *Assume that* $\boldsymbol{z} \in C(\overline{\Omega})^M \cap C^2(\Omega)^M$ *with* $\boldsymbol{z}(0) \geq \boldsymbol{0}$ *and* $\boldsymbol{z}(1) \geq \boldsymbol{0}$. *Then* $\boldsymbol{L} \geq \boldsymbol{0}$ *in* Ω *implies that* $\boldsymbol{z} \geq \boldsymbol{0}$ *in* $\overline{\Omega}$.

An immediate consequence of the above lemma is the following stability estimate.

Lemma 2 *If* $\boldsymbol{Lz} = \boldsymbol{f}$ *in* Ω, *with* $\boldsymbol{z}(0) = \boldsymbol{z}_0$, $\boldsymbol{z}(1) = \boldsymbol{z}_1$. *Then* $||\boldsymbol{z}||_{\overline{\Omega}} \leq \dfrac{1}{\alpha}||\boldsymbol{f}||_{\overline{\Omega}} + \max\{\|\boldsymbol{z}_0\|, \|\boldsymbol{z}_1\|\}$, *where* $\boldsymbol{z}_0$, $\boldsymbol{z}_1$ *are some constant vectors.*

Now we give bounds on the derivatives of the exact solution $\boldsymbol{u}$ for the system (1). These bounds will be used in the error analysis in Sect. 4.

Lemma 3 *Let* $\boldsymbol{u}$ *be the solution of (1). Then*

$$|\boldsymbol{u}^{(k)}(x)| \leq \boldsymbol{C}\varepsilon^{-k}, \qquad 0 \leq k \leq 4 \text{ and } x \in \overline{\Omega}.$$

We now derive sharper bounds on the derivatives of $\boldsymbol{u}$, which show that the above estimated bounds decay rapidly as one moves away from the right boundary of the domain.

Lemma 4 *Let* $\boldsymbol{u}$ *be the solution of (1). Then*

$$|\boldsymbol{u}^{(k)}(x)| \leq \boldsymbol{C}(1 + \varepsilon^{-k}\exp(\tfrac{-\alpha(1-x)}{\varepsilon})), \qquad 0 \leq k \leq 4, \quad x \in \overline{\Omega},$$

where $\boldsymbol{C}$ *is a constant independent of* ε.

To analyze the parameter-uniform convergence of the numerical method, we decompose the exact solution $\boldsymbol{u}$ into regular component $\boldsymbol{v}$ and singular component $\boldsymbol{w}$, that is, $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$, where the regular component has three term asymptotic expansion as $\boldsymbol{v} = \boldsymbol{v}_0 + \varepsilon \boldsymbol{v}_1 + \varepsilon^2 \boldsymbol{v}_2$. Also, the smooth component $\boldsymbol{v}$ satisfies $\boldsymbol{L}\boldsymbol{v} = \boldsymbol{f}$ with the boundary conditions $v_i(0) = u_i(0)$, $v_i(1) = v_{i,0}(1) + \varepsilon v_{i,1}(1)$, for $1 \le i \le M$, where $v_{i,0}$ is the solution of the reduced problem

$$a_i(x) v'_{i,0}(x) + \sum_{m=1}^{M} b_{im}(x) v_{m,0}(x) = f_i(x), \text{ with } v_{i,0}(0) = u_i(0),$$

$v_{i,p}$, for $p = 1, 2$, are the solutions of the problems

$$-\varepsilon v''_{i,0}(x) + a_i(x)(v_{i,0}(x) + \varepsilon v_{i,1}(x))' + \sum_{m=1}^{M} b_{im}(x)(v_{m,0}(x) + \varepsilon v_{m,1}(x)) = f_i(x),$$

$$\text{with } v_{i,1}(0) = 0,$$

and $-\varepsilon(v''_{i,0}(x) + \varepsilon v''_{i,1}(x) + \varepsilon^2 v''_{i,2}(x)) + a_i(x)(v'_{i,0}(x) + \varepsilon v'_{i,1}(x) + \varepsilon^2 v'_{i,2}(x))$
$+ \sum_{m=1}^{M} b_{im}(x)(v_{m,0}(x) + \varepsilon v_{m,1}(x) + \varepsilon^2 v_{m,2}(x)) = f_i(x)$, with $v_{i,2}(0) = 0$, $v_{i,2}(1) = 0$.

The singular component $\boldsymbol{w}$ satisfies $\boldsymbol{L}\boldsymbol{w} = \boldsymbol{0}$ with the boundary conditions

$$\boldsymbol{w}(0) = \boldsymbol{0}, \quad \boldsymbol{w}(1) = \boldsymbol{u}(1) - \boldsymbol{v}(1).$$

Lemma 5 *Let $\boldsymbol{f} \in C^2(\overline{\Omega})^M$, $\boldsymbol{A} \in C^2(\overline{\Omega})^{M\times M}$, $\boldsymbol{B} \in C^2(\overline{\Omega})^{M\times M}$ satisfy the assumptions (2) and (3), respectively. Then (1) possesses unique solution $\boldsymbol{u} \in C^4(\overline{\Omega})^M$ that can be decomposed as $\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$, where*

$$|\boldsymbol{v}^{(k)}(x)| \le \boldsymbol{C}(1 + \varepsilon^{2-k}), \qquad 0 \le k \le 4 \text{ and } \forall x \in \overline{\Omega}$$

and

$$|\boldsymbol{w}^{(k)}(x)| \le \boldsymbol{C}\varepsilon^{-k} \exp\left(\tfrac{-\alpha(1-x)}{\varepsilon}\right), \qquad 0 \le k \le 4 \text{ and } \forall x \in \overline{\Omega}$$

for some constant $\boldsymbol{C}$ independent of ε.

3 Discrete Problem

Let $\xi^+ = 1 - \sigma$ be the transition point for the domain $\overline{\Omega}^N$, where $\sigma = \min\{1/2, \sigma_0 \varepsilon \ln N\}$ with σ_0 as a constant. The choice of σ_0 will be given later during the numerical experiments. The domain $\overline{\Omega}$ is divided into two subdomains, $[0, \xi^+]$ and $[\xi^+, 1]$. The mesh points in these subdomains are given by

$$x_j = \begin{cases} j\dfrac{2\xi^+}{N}, & 0 \le j \le N/2, \\ \xi^+ + (j - N/2)\dfrac{2(1-\xi^+)}{N}, & N/2 + 1 \le j \le N. \end{cases}$$

Define non-uniform mesh spacing by $h_j = x_j - x_{j-1}$, for $1 \le j \le N$.

Outside the boundary layer region for $0 \le j \le N/2$, a combination of the compact second-order difference scheme, and the second-order central difference scheme is considered. While inside the boundary layer region for $N/2 + 1 \le j \le N$, a second-order central difference scheme is considered. The corresponding discretization is

$$[\boldsymbol{L}^N \boldsymbol{U}]_j = [\boldsymbol{\Gamma f}]_j, \tag{4}$$

where $$[\boldsymbol{L}^N \boldsymbol{U}]_j := \begin{pmatrix} [L_1^N \boldsymbol{U}]_j \\ [L_2^N \boldsymbol{U}]_j \\ \vdots \\ [L_M^N \boldsymbol{U}]_j \end{pmatrix} = \begin{pmatrix} [R(U_1)]_j + \sum_{i=1, i\neq 1}^{M} [Q(b_{1i} U_i)]_j \\ [R(U_2)]_j + \sum_{i=1, i\neq 2}^{M} [Q(b_{2i} U_i)]_j \\ \vdots \\ [R(U_M)]_j + \sum_{i=1, i\neq M}^{M} [Q(b_{Mi} U_i)]_j \end{pmatrix},$$

$$[\boldsymbol{\Gamma f}]_j := \begin{pmatrix} [\Gamma_1 \boldsymbol{f}]_j \\ [\Gamma_2 \boldsymbol{f}]_j \\ \vdots \\ [\Gamma_M \boldsymbol{f}]_j \end{pmatrix} = \begin{pmatrix} [Q(f_1)]_j \\ [Q(f_2)]_j \\ \vdots \\ [Q(f_M)]_j \end{pmatrix},$$

and $[R(U_i)_j] : = r^-_{i,j} U_{i,j-1} + r^c_{i,j} U_{i,j} + r^+_{i,j} U_{i,j+1}$, $[Q(f_i)]_j := q^-_{i,j} f_{i,j-1} + q^c_{i,j} f_{i,j}$. The coefficients $r^*_{i,j}$, $1 \le i \le M$, $1 \le j \le N-1$, $* = -, c, +$ are given by

$$r^-_{i,j} = \frac{-2\varepsilon - q^c_{i,j} a_{i,j} h_{j+1} + q^-_{i,j}[-(2h_j + h_{j+1}) a_{i,j-1} + h_j (h_j + h_{j+1}) b_{ii,j-1}]}{h_j (h_j + h_{j+1})},$$

$$r^+_{i,j} = \frac{-2\varepsilon + a_{i,j} h_j - q^-_{i,j} h_j (a_{i,j} + a_{i,j-1})}{h_{j+1}(h_j + h_{j+1})},$$

$$r^c_{i,j} = q^-_{i,j} b_{ii,j-1} + q^c_{i,j} b_{ii,j} - r^-_{i,j} - r^+_{i,j}, \quad \text{and} \quad q^c_{i,j} = 1 - q^-_{i,j},$$

where $q^-_{i,j}$ is a free parameter. Here, the coefficients are determined so that the scheme is exact for polynomials up to degree 2 and satisfies the normalization condition $q^-_{i,j} + q^c_{i,j} = 1$ for $1 \le i \le M$ and $1 \le j \le N-1$. The free parameter $q^-_{i,j}$ in the subdomains depends on the relationship between h_j and ε in order to ensure that the discrete operator to be of positive type and the scheme to be second order uniformly convergent. $q^-_{i,j}$ is defined in Lemmas 6 and 7.

Lemma 6 *Let N_0 be the smallest positive integer such that*

$$\sigma_0 ||a_i||_{\overline{\Omega}} < \frac{N_0}{\ln N_0}, \qquad \frac{2(||a_i'||_{\overline{\Omega}} + ||b_{ii}||_{\overline{\Omega}})}{N_0} < \alpha$$

holds. Also, when $||a_i||_{\overline{\Omega}} h_j \ge 2\varepsilon$, the free parameter $q^-_{i,j}$ will be chosen as

$$q^-_{i,j} \ge \frac{a_{i,j}}{(a_{i,j} + a_{i,j-1})}, \qquad for \;\; 1 \le i \le M, \;\; 1 \le j \le N/2.$$

Then there exist positive constants C_1, C_2 such that for $1 \le i \le M$ and $1 \le j \le N-1$

$$0 \le r^-_{i,j} + r^c_{i,j} + r^+_{i,j} \le C_1, \;\; r^-_{i,j} < 0, \;\; r^+_{i,j} < 0.$$

Further, for any $N \ge N_0$, the operator L_i^N is of positive type and also the scheme is *uniformly stable in the maximum norm, if we have*

$$h_{j+1}r^{+}_{i,j} - h_j r^{-}_{i,j} \geq C_2 > 0, \quad 1 \leq i \leq M, \; 1 \leq j \leq N-1.$$

Lemma 7 *For all* ε, $1 \leq i \leq M$ *and* $1 \leq j \leq N-1$, *we have* $\left|\frac{h^3_{j+1}}{3!}r^{+}_{i,j} - \frac{h^3_j}{3!}r^{-}_{i,j} - q^{-}_{i,j}h_j\varepsilon\right| \leq O(N^{-2})$, *if* $q^{-}_{i,j} = \frac{h_j - h_{j+1}}{3h_j}$ *for* $||a_i||_{\overline{\Omega}} h_j < 2\varepsilon$.

Lemma 8 (Discrete Maximum Principle) *Let* $\mathbf{Z}$ *be any mesh function on* $\overline{\Omega}^N$ *with* $\mathbf{Z}_0 \geq \mathbf{0}$ *and* $\mathbf{Z}_N \geq \mathbf{0}$. *Then* $\mathbf{L}^N\mathbf{Z}_j \geq \mathbf{0}$ *for* $x_j \in \overline{\Omega}^N$ *implies that* $\mathbf{Z}_j \geq \mathbf{0}$ *for* $x_j \in \overline{\Omega}^N$.

An immediate consequence of this lemma is the following parameter-uniform stability estimate for the discrete difference operator $\mathbf{L}^N$.

Lemma 9 *Let* $\mathbf{Z}$ *be any mesh function such that* $\mathbf{L}^N\mathbf{Z} = \boldsymbol{\Gamma f}$ *in* Ω^N. *Then*

$$||\mathbf{Z}||_{\overline{\Omega}^N} \leq \frac{1}{\alpha}||\boldsymbol{f}||_{\overline{\Omega}^N} + \max\{\|\mathbf{Z}_0\|, \|\mathbf{Z}_N\|\}.$$

4 Error Analysis

We now require a special decomposition of the discrete solution $\boldsymbol{U}$ of the problem (1) into discrete regular component $\boldsymbol{V}$ and discrete singular component $\boldsymbol{W}$, that is, $\boldsymbol{U} = \boldsymbol{V} + \boldsymbol{W}$, where $\boldsymbol{V}$ is the solution of nonhomogeneous problem and $\boldsymbol{W}$ is the solution of corresponding homogeneous problem. The nodal error estimate in the regular component of the solution is given in the following lemma.

Lemma 10 *Let* $\mathbf{v}$ *and* $\mathbf{V}$ *denote the regular components of* $\mathbf{u}$ *and* $\mathbf{U}$, *respectively. Then*

$$||\boldsymbol{v} - \boldsymbol{V}||_{\overline{\Omega}^N} \leq C\sigma_0^2 N^{-2}\ln^2 N,$$

where C *and* σ_0 *are constants independent of* ε *and* N.

The following lemma is about the truncation error estimate in singular component.

Lemma 11 *Let* $\mathbf{w}$ *and* $\mathbf{W}$ *be the singular components of* $\mathbf{u}$ *and* $\mathbf{U}$, *respectively. Then for all* $1 \leq j \leq N-1$ *and* $1 \leq i \leq M$

$$|L^N_i(\boldsymbol{w} - \boldsymbol{W})_j| \leq \begin{cases} \dfrac{C}{\max\{\varepsilon, h_j\}}\exp\left(-\dfrac{\alpha(1-x_{j+1})}{\varepsilon}\right), & 1 \leq j \leq N/2, \\ \dfrac{C}{\max\{\varepsilon, h_{j+1}\}}\left(\dfrac{h_j}{\varepsilon}\right)^2\exp\left(-\dfrac{\alpha(1-x_{j+1})}{\varepsilon}\right), & N/2+1 \leq j \leq N-1, \end{cases}$$

where C *is a constant independent of* ε *and* N.

To find the nodal error estimate in the singular component on $\overline{\Omega}^N$, define the mesh functions $\boldsymbol{\Phi}_j(\beta) := \prod_{k=j+1}^{N} \boldsymbol{S}_k^{-1}(\beta)$, $0 \leq j \leq N-1$, where $\boldsymbol{S}_j(\beta) := \left(1 + \frac{\beta h_j}{\varepsilon}\right)$ for $1 \leq j \leq N$ with $\boldsymbol{\Phi}_N(\beta) = 1$ and β a positive constant.

Lemma 12 *Suppose the assumptions of Lemma 6 hold and $\beta \le \alpha/2$, then there exists $C(\beta)$ such that*

$$L_i^N \boldsymbol{\Phi}_j(\beta) \ge \frac{C(\beta)}{\max\{\varepsilon, h_j\}} \boldsymbol{\Phi}_j(\beta), \qquad 1 \le j \le N-1 \text{ and } 1 \le i \le M.$$

The nodal error estimate in the singular component of the solution is given in the following lemma.

Lemma 13 *Let $\boldsymbol{w}$ and $\boldsymbol{W}$ be the singular components of $\boldsymbol{u}$ and $\boldsymbol{U}$, respectively. Then under the hypotheses of Lemmas 6, 11 and 12, we have*

$$||\boldsymbol{w} - \boldsymbol{W}||_{\overline{\Omega}^N} \le C(N^{-\beta\sigma_0} + \sigma_0^2 N^{-2} \ln^2 N), \qquad 1 \le j \le N,$$

where C and σ_0 are constants independent of ε and N.

Proof We know that $\exp(-\alpha(1-x_j)/\varepsilon) \le \boldsymbol{\Phi}_j(\beta)$, for $\beta \le \alpha/2$ and $0 \le j \le N$. Also, using the boundary conditions, $|(\boldsymbol{w}-\boldsymbol{W})(0)| = \boldsymbol{0}$ and $|(\boldsymbol{w}-\boldsymbol{W})(1)| = \boldsymbol{0}$. Construct the barrier function as $\boldsymbol{\Psi}_j^{\pm}(\beta) = C(\beta)\boldsymbol{\Phi}_j(\beta) \pm (\boldsymbol{w}-\boldsymbol{W})(x_j)$, $0 \le j \le N$, where $C(\beta) = C\mathbf{S}_{j+1}(\beta)$. Using the expression for $\boldsymbol{\Phi}_j(\beta)$, Lemmas 11 and 12, we get

$$\boldsymbol{\Psi}_0^{\pm}(\beta) \ge \boldsymbol{0}, \boldsymbol{\Psi}_N^{\pm}(\beta) \ge \boldsymbol{0}, \text{ and } L_i^N \boldsymbol{\Psi}_j^{\pm}(\beta) \ge \boldsymbol{0}, 1 \le j \le N-1 \text{ and } 1 \le i \le M.$$

Henceforth by Lemma 7, $\boldsymbol{\Psi}_j^{\pm}(\beta) \ge \boldsymbol{0}$ and thus $|(\boldsymbol{w}-\boldsymbol{W})(x_j)| \le C\boldsymbol{\Phi}_{j+1}(\beta)$, $1 \le j \le N-1$. Therefore, for $1 \le j \le N/2$ by using bound in [21], $|(\boldsymbol{w}-\boldsymbol{W})(x_j)| \le CN^{-\beta\sigma_0}$, where $\beta \le \alpha/2$ and σ_0 is a constant.

Construct the barrier function $\boldsymbol{\Psi}_j^{\pm}(\beta) = C[(1+x_j)N^{-\beta\sigma_0} + \left(\frac{h_j}{\varepsilon}\right)^2 \boldsymbol{\Phi}_j(\beta)] \pm (\boldsymbol{w}-\boldsymbol{W})(x_j)$, for $N/2 \le j \le N$. Using the expression for $\boldsymbol{\Phi}_j(\beta)$, Lemmas 11 and 12, we get

$\boldsymbol{\Psi}_{N/2}^{\pm}(\beta) \ge \boldsymbol{0}$, $\boldsymbol{\Psi}_N^{\pm}(\beta) \ge \boldsymbol{0}$, $L_i^N \boldsymbol{\Psi}_j^{\pm}(\beta) \ge \boldsymbol{0}$, $N/2+1 \le j \le N-1$ and $1 \le i \le M$.

By the discrete maximum principle, $\boldsymbol{\Psi}_j^{\pm}(\beta) \ge \boldsymbol{0}$ and hence after substituting the value of h_2 we get

$$|(\boldsymbol{w}-\boldsymbol{W})(x_j)| \le C(N^{-\beta\sigma_0} + \sigma_0^2 N^{-2} \ln^2 N), \quad N/2 \le j \le N.$$

This is as desired and thus concludes the proof.

Now we state and prove the pointwise parameter-uniform convergence of the method using Lemmas 10 and 13.

Theorem 1 *Let $\boldsymbol{u}$ be the exact solution of the problem (1) and $\boldsymbol{U}$ be the discrete solution of the proposed method* (4). *Suppose hypotheses of Lemmas 6 and 12 hold. Then for $\beta \le \alpha/2$ and for any $N \ge N_0$,*

$$||\boldsymbol{u} - \boldsymbol{U}||_{\overline{\Omega}^N} \le C(N^{-\beta\sigma_0} + \sigma_0^2 N^{-2} \ln^2 N),$$

where C and σ_0 are constants independent of ε and N.

Proof Using triangle inequality, it follows that

$$||\boldsymbol{u} - \boldsymbol{U}||_{\overline{\Omega}^N} \le ||\boldsymbol{v} - \boldsymbol{V}||_{\overline{\Omega}^N} + ||\boldsymbol{w} - \boldsymbol{W}||_{\overline{\Omega}^N} \le C(N^{-\beta\sigma_0} + \sigma_0^2 N^{-2} \ln^2 N),$$

which gives the almost second-order parameter-uniform convergence for $\beta\sigma_0 \geq 2$.

Next we extend the pointwise parameter-uniform error estimate to the global parameter-uniform error estimate using the similar technique as in [12, 15, 18].

Theorem 2 *Let* $\boldsymbol{u}$ *be the exact solution of the problem (1) and* $\boldsymbol{U}$ *be the discrete solution of the proposed method (4). Let* $\tilde{\boldsymbol{U}}$ *be the piecewise linear interpolant of* $\boldsymbol{U}$*. Then for* $\beta \leq \alpha/2$,

$$||(\boldsymbol{u} - \tilde{\boldsymbol{U}})||_{\overline{\Omega}} \leq C(N^{-\beta\sigma_0} + \sigma_0^2 N^{-2} \ln^2 N)$$

where C, σ_0 *are constants independent of* ε *and* N.

Proof Using triangle inequality, we get

$$||\boldsymbol{u} - \tilde{\boldsymbol{U}}||_{\overline{\Omega}} \leq ||\boldsymbol{u} - \tilde{\boldsymbol{u}}||_{\overline{\Omega}} + ||\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{U}}||_{\overline{\Omega}}.$$

Using stability of the operator and Theorem 1, we have

$$||\tilde{\boldsymbol{u}} - \tilde{\boldsymbol{U}}||_{\overline{\Omega}} \leq C||\boldsymbol{u} - \boldsymbol{U}||_{\overline{\Omega}^N} \leq C(N^{-\beta\sigma_0} + N^{-2} \ln^2 N). \tag{5}$$

Now we estimate the interpolation error $||\boldsymbol{u} - \tilde{\boldsymbol{u}}||_{\overline{\Omega}}$. We have the following standard interpolation error estimates for any $\boldsymbol{u} \in C^2([x_{j-1}, x_j])^M$

$$||\boldsymbol{u} - \tilde{\boldsymbol{u}}||_{[x_{j-1},x_j]} \leq \begin{cases} Ch_j^2||\boldsymbol{u}''||_{[x_{j-1},x_j]}, \\ C||\boldsymbol{u}||_{[x_{j-1},x_j]}. \end{cases} \tag{6}$$

For regular component $\boldsymbol{v}$, use the first interpolation bound of (6),

$$||\boldsymbol{v} - \tilde{\boldsymbol{v}}||_{[x_{j-1},x_j]} \leq Ch_j^2||\boldsymbol{v}''||_{[x_{j-1},x_j]} \leq CN^{-2}.$$

To bound the interpolation error for the singular component $\boldsymbol{w}$, consider the two different cases: $\sigma = 1/2$ and $\sigma = \sigma_0\varepsilon \ln N$. In the first case, $\varepsilon^{-1} < 2\sigma_0 \ln N$ and the mesh will be uniform with mesh spacing $h_j = 1/N$. Using the first estimate of (6) and the Lemma 3, we get

$$||\boldsymbol{w} - \tilde{\boldsymbol{w}}||_{[x_{j-1},x_j]} \leq C\sigma_0^2 N^{-2} \ln^2 N.$$

In the second case, mesh will be piecewise uniform with mesh spacing $2\xi^+/N$ in $[0, \xi^+]$, and $2(1 - \xi^+)/N$ in $[\xi^+, 1]$. In $[0, \xi^+]$ use the second interpolation bound of (6) and Lemma 3 to get

$$||\boldsymbol{w} - \tilde{\boldsymbol{w}}||_{[x_{j-1},x_j]} \leq C \exp\left(-\frac{\alpha(1-x_j)}{\varepsilon}\right) \leq CN^{-\alpha\sigma_0} \leq CN^{-\beta\sigma_0},$$

where $\beta \leq \alpha/2$ and σ_0 is a constant.

To bound the interpolation error in $[\xi^+, 1]$, use first estimate of (6) and Lemma 3 with $h_j = \frac{2(1-\xi^+)}{N} = \frac{2\sigma_0\varepsilon \ln N}{N}$ to obtain

$$||\boldsymbol{w} - \tilde{\boldsymbol{w}}||_{[x_{j-1},x_j]} \leq Ch_j^2||\boldsymbol{w}''||_{[x_{j-1},x_j]} \leq C\sigma_0^2 N^{-2} \ln^2 N.$$

On combining the above interpolation error estimates, we obtain

$$||\boldsymbol{u} - \tilde{\boldsymbol{u}}||_{\overline{\Omega}} \leq C(N^{-\beta\sigma_0} + \sigma_0^2 N^{-2} \ln^2 N). \tag{7}$$

Thus from Eqs. (5) and (7), we get

$$||(\boldsymbol{u} - \tilde{\boldsymbol{U}})||_{\overline{\Omega}} \leq C(N^{-\beta\sigma_0} + \sigma_0^2 N^{-2} \ln^2 N).$$

This completes the proof.

5 Numerical Results

Numerical results for the following test example are given in this section, which confirm our theoretical findings.

Example 1 Consider the following weakly coupled system of singularly perturbed convection-diffusion equations

$-\varepsilon u_1''(x) + (1+4x)u_1'(x) + (1+3x)u_1(x) - 2xu_2(x) - xu_3(x) = 2x^2 \exp(x)$

$-\varepsilon u_2''(x) + (1+3x)u_2'(x) - (1+x)u_1(x) + (4+2x)u_2(x) - (2+x)u_3(x) = \cos(\frac{\pi x}{2}) + 3x^2$

$-\varepsilon u_3''(x) + (1+2x)u_3'(x) - 2xu_1(x) - (1+2x) + (3+4x)u_3(x) = \exp(x)$

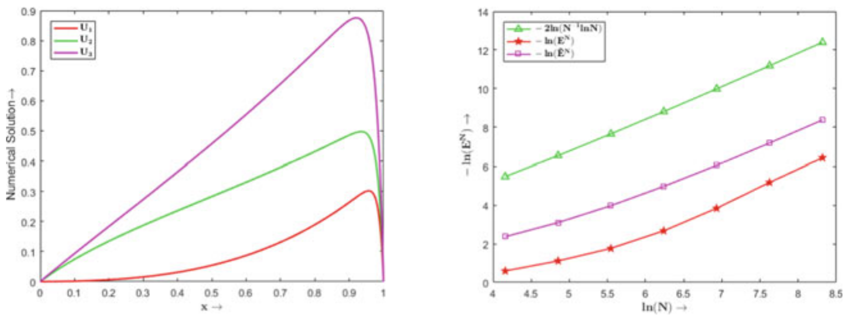
with $u_1(0) = 0, u_1(1) = 0, u_2(0) = 0, u_2(1) = 0, u_3(0) = 0, u_3(1) = 0.$

The above example is discretized and solved using the mesh and the scheme described in Sect. 3 with $\beta = \alpha/2$, $\sigma_0 = 2/\beta$ and $\alpha = 1$. Let $\boldsymbol{U}^N$ be the discrete solution of the discrete problem with N mesh intervals. Since the exact solution of the above example is not known, to compute the pointwise error in the discrete solution we bisect the each subintervals in $\overline{\Omega}^N$ to obtain the mesh $\hat{\Omega}^{2N} = \{\hat{x}_j : 0 \leq j \leq 2N\}$, that is, $\hat{x}_{2j} = x_j$ for $j = 0, \ \ldots, \ N$ and $\hat{x}_{2j+1} = (x_j + x_{j+1})/2$ for $j = 0, \ \ldots, \ N-1$. The maximum pointwise error is computed by $E_\varepsilon^N := \|\hat{\boldsymbol{U}}^{2N} - \boldsymbol{U}^N\|_{\overline{\Omega}^N}$, where $\hat{\boldsymbol{U}}^{2N}$ is the discrete solution on $\hat{\Omega}^{2N}$. We compute the numerical order of convergence by $\rho_\varepsilon^N := \frac{\ln E_\varepsilon^N - \ln E_\varepsilon^{2N}}{\ln(2\ln N) - \ln(\ln(2N))}$. The corresponding global error is computed by $\tilde{E}_\varepsilon^N := \|\tilde{\hat{\boldsymbol{U}}}^{2N} - \tilde{\boldsymbol{U}}^N\|_{\overline{\Omega}^{6N}}$, where $\tilde{\boldsymbol{U}}^N$ and $\tilde{\hat{\boldsymbol{U}}}^{2N}$ are linear interpolation of $\boldsymbol{U}^N$ and $\hat{\boldsymbol{U}}^{2N}$ over $\overline{\Omega}^N$ and $\hat{\Omega}^{2N}$ respectively. The numerical order of convergence $\tilde{\rho}_\varepsilon^N$ is computed using the similar technique as above by replacing E_ε^* with $\tilde{E}_\varepsilon^*$, where $* := N, 2N$.

For the given Example 1, for different values of ε and N, Table 1 represents the maximum pointwise errors E_ε^N and numerical order of convergence ρ_ε^N, while Table 2 represents global errors $\tilde{E}_\varepsilon^N$ and numerical order of convergence $\tilde{\rho}_\varepsilon^N$. For the pointwise error estimates in the Table 1, almost second-order parameter-uniform convergence is achieved for $N \approx 1024$, while for the global error estimates in Table 2, almost second order parameter-uniform convergence is achieved for $N \approx 4096$. A reason for achieving almost second-order global uniform convergence for large value of N in comparison with the pointwise parameter uniform convergence is that the pointwise errors are measured in discrete maximum norm, and global errors are measured in maximum norm by interpolating the numerical solution, which reduces the rate of decrease in the error. The last two rows of Table 1 represent the parameter-uniform error $E^N := \max_\varepsilon E_\varepsilon^N$ and parameter-uniform numerical order of convergence ρ^N respectively, and the last two rows of Table 2 represent the parameter-uniform global error $\tilde{E}^N := \max_\varepsilon \tilde{E}_\varepsilon^N$ and parameter-uniform numerical order of convergence $\tilde{\rho}^N$

Table 1 The maximum pointwise errors E_ε^N and numerical order of convergence ρ_ε^N for Example 1

$\varepsilon = 2^{-j}$	N = 64	N = 128	N = 256	N = 512	N = 1024	
2^0	2.23E-05	5.58E-06	1.39E-06	3.49E-07	8.72E-08	
	2.57E+00	2.48E+00	2.41E+00	2.36E+00		ρ_ε^N
2^{-4}	1.49E-02	3.16E-03	7.83E-04	1.95E-04	4.86E-05	
	2.88E+00	2.50E+00	2.42E+00	2.36E+00		ρ_ε^N
2^{-8}	6.47E-01	3.87E-01	1.72E-01	6.32E-02	1.48E-02	
	9.52E-01	1.45E+00	1.74E+00	2.47E+00		ρ_ε^N
2^{-12}	5.59E-01	3.29E-01	1.72E-01	6.94E-02	2.16E-02	
	9.82E-01	1.16E+00	1.57E+00	1.98E+00		ρ_ε^N
2^{-16}	5.53E-01	3.29E-01	1.72E-01	6.94E-02	2.16E-02	
	9.63E-01	1.16E+00	1.58E+00	1.98E+00		ρ_ε^N
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-32}	5.53E-01	3.29E-01	1.72E-01	6.94E-02	2.16E-02	
	9.63E-01	1.16E+00	1.58E+00	1.98E+00		ρ_ε^N
E^N	5.53E-01	3.29E-01	1.72E-01	6.94E-02	2.16E-02	
ρ^N	9.63E-01	1.16E+00	1.58E+00	1.98E+00		



(a) (b)

Fig. 1 The figure **a** is the plot of components of the numerical solution $\boldsymbol{U}$ with $\varepsilon = 2^{-4}$, $N = 256$, and figure **b** represents the plots of $-2\ln(N^{-1}\ln N)$, $-\ln E^N$, $-\ln \tilde{E}^N$ versus $\ln N$

respectively. The plots of the components U_1, U_2 and U_3 in the numerical solution $\boldsymbol{U}$ are given in Fig. 1, which show the layers formation in the neighborhood of $x = 1$. The lines in the Fig. 1 are with -2ln(N^{-1}ln N), -ln E^N and -ln $\tilde{E}^N$ on the y-axis and ln N on the x-axis. The first line from the top corresponds to the parameter-uniform theoretical error -2ln(N^{-1}ln N), the middle line is corresponding to parameter-uniform global error -ln $\tilde{E}^N$, and the last line corresponds to parameter-uniform pointwise errors. From Fig. 1, one can observe that parameter-uniform theoretical error is less than the parameter-uniform global error, and parameter-uniform global error is less

Table 2 Global maximum errors $\tilde{E}_{\varepsilon}^{N}$ and numerical order of convergence $\tilde{\rho}_{\varepsilon}^{N}$ for Example 1

$\varepsilon = 2^{-j}$	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$	$N = 2048$	$N = 4096$	
2^{0}	3.75E-04	9.67E-05	2.45E-05	6.18E-06	1.55E-06	3.89E-07	9.73E-08	
	2.52E+00	2.45E+00	2.40E+00	2.35E+00	2.32E+00	2.29E+00		$\tilde{\rho}_{\varepsilon}^{N}$
2^{-4}	4.18E-02	1.31E-02	3.73E-03	9.96E-04	2.58E-04	6.58E-05	1.67E-05	
	2.15E+00	2.25E+00	2.29E+00	2.30E+00	2.28E+00	2.26E+00		$\tilde{\rho}_{\varepsilon}^{N}$
2^{-8}	9.39E-02	7.94E-03	1.88E-02	2.16E-02	2.45E-03	7.66E-04	2.31E-04	
	1.36E+00	1.56E+00	1.68E+00	1.82E+00	1.94E+00	1.98E+00		$\tilde{\rho}_{\varepsilon}^{N}$
2^{-12}	9.40E-02	4.52E-02	1.89E-02	6.99E-03	2.37E-03	7.51E-04	2.28E-04	
	1.36E+00	1.56E+00	1.73E+00	1.84E+00	1.92E+00	1.96E+00		$\tilde{\rho}_{\varepsilon}^{N}$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
2^{-32}	9.40E-02	4.52E-02	1.89E-02	6.99E-03	2.37E-03	7.51E-04	2.29E-04	
	1.36E+00	1.56E+00	1.73E+00	1.84E+00	1.92E+00	1.96E+00		$\tilde{\rho}_{\varepsilon}^{N}$
$\tilde{E}^{N}$	9.40E-02	4.52E-02	1.89E-02	6.99E-03	2.37E-03	7.51E-04	2.29E-04	
$\tilde{\rho}^{N}$	1.36E+00	1.56E+00	1.73E+00	1.84E+00	1.92E+00	1.96E+00		

than the parameter-uniform pointwise error. The slope of each line represents the parameter-uniform order of convergence of respective errors.

6 Conclusion

The considered problem is discretized using a suitable combination of the compact finite difference scheme and the central difference scheme on the Shishkin mesh. For the convergence analysis, the exact solution and its numerical analog are decomposed into regular and singular components. Some improved bounds on the exact solution and its derivatives have been given. Pointwise as well as global almost second-order parameter-uniform convergence of the scheme has been obtained. The proposed scheme is implemented on a test example, which verifies the theoretical findings.

Acknowledgements The authors thank the IIT Delhi HPC for the computational resources. The research work of the second author is supported by the Council of Scientific and Industrial Research, India. The research work of the third author is supported by the University Grants Commission, India.

References

1. Abrahamsson, L.R., Keller, H.B., Kreiss, H.O.: Difference approximations for singular perturbations of systems of ordinary difference equations. Numer. Math. **22**, 367–391 (1974)
2. Bellew, S., O'Riordan, E.: A parameter robust numerical method for a system of two singularly perturbed convection-diffusion equations. Appl. Numer. Math. **51**, 171–186 (2004)
3. Cen, Z.: Parameter-uniform finite difference scheme for a system of coupled singularly perturbed convection-diffusion equations. Int. J. Comput. Math. **82**, 177–192 (2005)
4. Clavero, C., Gracia, J.L., Lisbona, F.: High order methods on Shishkin meshes for singular perturbation problems of convection-diffusion type. Numer. Algorithms **22**, 73–97 (1999)
5. Farrell, P.A., Hegarty, A.F., Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Robust Computational Techniques for Boundary Layers. Chapman & Hall, CRC Press, Boca Raton, FL (2000)
6. Kadalbajoo, M.K., Gupta, V.: Numerical solution of singularly perturbed convection-diffusion problem using parameter uniform B-spline collocation method. J. Math. Anal. Appl. **355**, 439–452 (2009)
7. Kokotovic, P.V.: Applications of singular perturbation techniques to control problems. SIAM Rev. **26**, 501–550 (1984)
8. Linß, T.: Analysis of a system of singularly perturbed convection-diffusion equations with strong coupling. SIAM J. Numer. Anal. **47**, 1847–1862 (2009)
9. Linß, T., Madden, N.: Layer-adapted meshes for a system of coupled singularly perturbed reaction-diffusion problems. IMA J. Numer. Anal. **29**, 109–125 (2009)
10. O'Riordan, E., Stynes, M.: Numerical analysis of a strongly coupled system of two singularly preturbed convection-diffusion problems. Adv. Comput. Math. **30**, 101–121 (2009)
11. Rao, S.C.S., Kumar, S.: An almost fourth order parameter-uniformlty convergent domain decomposition method for a coupled system of singularly perturbed reaction-diffusion problems. J. Comput. Appl. Math. **235**, 3342–3354 (2011)
12. Rao, S.C.S., Kumar, S.: Second order global uniformly convergent numerical method for a coupled system of singularly perturbed initial value problems. Appl. Math. Comput **219**, 3740–3753 (2012)
13. Rao, S.C.S., Chawla, S.: Interior layers in coupled system of two singularly perturbed reaction-diffusion equations with discontinuous source term, NAA 2012. LNCS **8236**, 445–453 (2013)
14. Rao, S.C.S., Kumar, M.: An almost fourth order parameter-robust numerical method for a linear system of ($M \geq 2$) coupled singularly perturbed reaction-diffusion problems. Int. J. Num. Anal. Model. **10**, 603–621 (2013)
15. Rao, S.C.S., Kumar, S.: Robust high order convergence of an overlapping Schwarz method for a singularly perturbed semilinear reaction-diffusion problem. J. Comp. Math. **31**, 509–521 (2013)
16. Rao, S.C.S., Chawla, S.: Second order uniformly convergent numerical method for a coupled system of singularly perturbed reaction-diffusion problems with discontinuous source term. LNCSE **108**, 233–244 (2015)
17. Rao, S.C.S., Chawla, S.: Numerical solution of singularly perturbed linear parabolic system with discontinuous source term. Appl. Numer. Math. **127**, 249–265 (2018)
18. Rao, S.C.S., Kumar, S., Kumar, M.: Uniform global convergence of a hybrid scheme for singularly perturbed reaction-diffusion systems. J. Optim. Theory Appl. **151**, 338–352 (2011)
19. Roos, H.G., Stynes, M., Tobiska, L.: Robust Numerical Methods for Singularly Perturbed Differential Equations, (2e). Springer, Berlin (2008)
20. Sangwan, V., Kumar, B.: Unifromly convergent 3-TGFEM Vs LSFEM for singularly perturbed convection-diffusion problems on a Shishkin based Logarithmic mesh. Int. J. Numer. Anal. Model. Ser. B **4**, 315–344 (2013)
21. Stynes, M., Tobiska, L.: A finite difference analysis of a streamline diffusion method on a Shishkin mesh. Numer. Algorithms **18**, 337–360 (1998)

A Parameter-Uniform Fitted Mesh Method for a Weakly Coupled System of Three Partially Singularly Perturbed Convection–Diffusion Equations

Valarmathi Sigamani, John J. H. Miller, and Saravanasankar Kalaiselvan

Abstract In this paper, a weakly coupled partially singularly perturbed linear system of three second-order ordinary differential equations of convection–diffusion type with given boundary conditions is considered on the interval [0, 1]. In spite of coupling, only the components whose equations are perturbed exhibit boundary layers at the origin. A numerical method composed of an upwind finite difference scheme applied on a piecewise uniform Shishkin mesh is suggested to solve the problem. The method is proved to be first-order convergent in the maximum norm uniformly in the perturbation parameters. Numerical example provided support the theory.

Keywords Partial perturbation problems · Shishkin decomposition · Boundary layers · Parameter uniform method · Shishkin mesh

1 Introduction

Singular perturbation problems of convection–diffusion type arise in applied mathematics such as control theory, fluid dynamics, elasticity, quantum mechanics, electrical networks, chemical reactor theory and many other areas. Convective heat transport problem with large Peclet number and Navier–Stokes equation with a large Reynolds number are also examples for the system of singularly perturbed convection–diffusion problems.

For a broad introduction on singularly perturbed boundary value problems of convection–diffusion type, one can refer to [1–3]. There, the authors suggest robust computational techniques to solve them. Coupled system of singularly perturbed convection–diffusion equations is studied in [4–8].

V. Sigamani (✉) · S. Kalaiselvan
Department of Mathematics, Bishop Heber College, Tiruchirappalli 620017, Tamil Nadu, India

J. J. H. Miller
Trinity College, Dublin, Ireland
e-mail: jmiller@tcd.ie

Valarmathi S et al. (eds.), *Differential Equations and Applications*,
Springer Proceedings in Mathematics & Statistics 368,
https://doi.org/10.1007/978-981-16-7546-1_3

In real-life problems, every equation of the system need not to be perturbed, for example, control problems [9]. The following system of convection–diffusion equations is considered on Ω.

$$L\mathbf{u}(x) \equiv E\mathbf{u}''(x) + A(x)\mathbf{u}'(x) - B(x)\mathbf{u}(x) = \mathbf{f}(x), \quad (1)$$

$$\mathbf{u}(0) = \mathbf{l}, \quad \mathbf{u}(1) = \mathbf{r}, \quad (2)$$

where $\Omega = (0, 1)$, $\mathbf{u}(x) = (u_1(x), u_2(x), u_3(x))^T$, $\mathbf{f}(x) = (f_1(x), f_2(x), f_3(x))^T$, $\mathbf{l} = (l_1, l_2, l_3)^T$, $\mathbf{r} = (r_1, r_2, r_3)^T$,

$$E = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A(x) = \begin{bmatrix} a_1(x) & 0 & 0 \\ 0 & a_2(x) & 0 \\ 0 & 0 & a_3(x) \end{bmatrix}, \quad B(x) = \begin{bmatrix} b_{11}(x) & b_{12}(x) & b_{13}(x) \\ b_{21}(x) & b_{22}(x) & b_{23}(x) \\ b_{31}(x) & b_{32}(x) & b_{33}(x) \end{bmatrix}.$$

Here, ε_1 and ε_2 are two distinct small positive parameters and without loss of generality, we assume that $\varepsilon_1 < \varepsilon_2$. The coefficient functions are taken to be sufficiently smooth on $\overline{\Omega}$ and $a_i(x) \geq \alpha > 0$, $\sum_{j=1}^{3} b_{ij}(x) \geq \beta > 0, i = 1, 2, 3,\ b_{ij} < 0$, for $i, j = 1, 2, 3$ and $i \neq j$.

The reduced problem corresponding to (1)–(2) is

$$L_0\mathbf{u_0}(x) \equiv E_0\mathbf{u_0}''(x) + A(x)\mathbf{u_0}'(x) - B(x)\mathbf{u_0}(x) = \mathbf{f}(x),\ x \in \Omega \quad (3)$$

$$u_{03}(0) = l_3,\ \mathbf{u_0}(1) = \mathbf{r}, \quad (4)$$

where $\mathbf{u_0}(x) = (u_{01}(x), u_{02}(x), u_{03}(x))^T$, $E_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

If $u_2(0) \neq u_{02}(0)$, then the solution components u_1 and u_2 has a boundary layer of width $O(\varepsilon_2)$ near $x = 0$ and if $u_1(0) \neq u_{01}(0)$, then the solution component u_1 has a boundary layer of width $O(\varepsilon_1)$ near $x = 0$. The solution component u_3 has no layer, as $u_3(0) = u_{03}(0)$.

This study helps to have an easy understanding of a more general partially perturbed system of equations.

2 Analytical Results

In this section, a maximum principle, a stability result and estimates of the derivatives of the solution of the system of equations (1)–(2) are presented.

Lemma 1 (Maximum Principle) *Let* $\boldsymbol{\psi} \in (C^2(\overline{\Omega}))^3$ *such that* $\boldsymbol{\psi}(0) \geq \mathbf{0}$, $\boldsymbol{\psi}(1) \geq \mathbf{0}$, $L\boldsymbol{\psi} \leq \mathbf{0}$ *on* $(0, 1)$, *then* $\boldsymbol{\psi} \geq \mathbf{0}$ *on* $[0, 1]$.

An immediate consequence of the maximum principle is the following stability result.

Lemma 2 (Stability Result) *Let* $\boldsymbol{\psi} \in (C^2(\overline{\Omega}))^3$*, then for* $x \in \overline{\Omega}$ *and* $i = 1, 2, 3$

$$|\psi_i(x)| \leq max\Big\{\|\boldsymbol{\psi}(0)\|,\ \|\boldsymbol{\psi}(1)\|,\ \frac{1}{\beta}\|L\boldsymbol{\psi}\|\Big\}.$$

Theorem 1 *Let* $\mathbf{u}$ *be the solution of (1)–(2), then for* $x \in \overline{\Omega}$ *and* $i = 1, 2$

$$|u_i(x)| \leq C \max\left\{\|\mathbf{l}\|,\ \|\mathbf{r}\|,\ \frac{1}{\beta}\|\mathbf{f}\|\right\} \tag{5}$$

$$|u_i^{(k)}(x)| \leq C\varepsilon_i^{-k}\Big(\|\mathbf{u}\| + \varepsilon_i\|\mathbf{f}\|\Big) \quad for \quad k = 1, 2 \tag{6}$$

$$|u_i^{(3)}(x)| \leq C\varepsilon_i^{-2}\varepsilon_1^{-1}\Big(\|\mathbf{u}\| + \varepsilon_i\|\mathbf{f}\|\Big) + \varepsilon_i^{-1}|f_i'(x)| \tag{7}$$

$$|u_3(x)| \leq C \max\left\{\|\mathbf{l}\|,\ \|\mathbf{r}\|,\ \frac{1}{\beta}\|\mathbf{f}\|\right\} \tag{8}$$

$$|u_3^{(k)}(x)| \leq C\Big(\|\mathbf{u}\| + \varepsilon_i\|\mathbf{f}\|\Big) \quad for \quad k = 1, 2 \tag{9}$$

$$|u_3^{(3)}(x)| \leq C\varepsilon_1^{-1}\Big(\|\mathbf{u}\| + \|\mathbf{f}\|\Big) + |f_3'(x)| \tag{10}$$

Proof It is not hard to see that the theorem follows from Lemma 2 and arguments similar to those in Theorem 1 of [8].

□

2.1 Shishkin Decomposition of the Solution

The solution $\mathbf{u}$ of the problem (1)–(2) can be decomposed into smooth and singular components $\mathbf{v}$ and $\mathbf{w}$ given by

$$\mathbf{u} = \mathbf{v} + \mathbf{w}$$

where

$$L\mathbf{v} = \mathbf{f}, \mathbf{v}(1) = \mathbf{r},\ \mathbf{v}(0) \text{ suitably chosen,} \tag{11}$$

$$L\mathbf{w} = \mathbf{0}, \mathbf{w}(0) = \mathbf{l} - \mathbf{v}(0), \mathbf{w}(1) = \mathbf{0} \tag{12}$$

with $\mathbf{v} = (v_1, v_2, v_3)^T$ and $\mathbf{w} = (w_1, w_2, w_3)^T$.

Now, $\mathbf{v}$ is decomposed into $\mathbf{v} = \mathbf{y}_0 + \varepsilon_2\mathbf{y}_1 + \varepsilon_2^2\mathbf{y}_2$, where
$\mathbf{y}_0 = (y_{01}, y_{02}, y_{03})^T$ satisfies,

$$\begin{aligned} a_1 y'_{01} - b_{11} y_{01} - b_{12} y_{02} - b_{13} y_{03} &= f_1 \\ a_2 y'_{02} - b_{21} y_{01} - b_{22} y_{02} - b_{23} y_{03} &= f_2 \\ y''_{03} + a_3 y'_{03} - b_{31} y_{01} - b_{32} y_{02} - b_{33} y_{03} &= f_3 \end{aligned}$$

with $y_{03}(0) = l_3, \mathbf{y}_0(1) = \mathbf{r}$,
$\mathbf{y}_1 = (y_{11}, y_{12}, y_{13})^T$ satisfies,

$$\begin{aligned} a_1 y'_{11} - b_{11} y_{11} - b_{12} y_{12} - b_{13} y_{13} &= -\frac{\varepsilon_1}{\varepsilon_2} y''_{01} \\ a_2 y'_{12} - b_{21} y_{11} - b_{22} y_{12} - b_{23} y_{13} &= -y''_{02} \\ y''_{13} + a_3(x) y'_{13} - b_{31} y_{11} - b_{32} y_{12} - b_{33} y_{13} &= 0 \end{aligned}$$

with $y_{13}(0) = 0,\ y_{11}(1) = 0,\ y_{12}(1) = 0,\ y_{13}(1) = 0$,
$\mathbf{y}_2 = (y_{21}, y_{22}, y_{23})^T$ satisfies,

$$\varepsilon_1 y''_{21} + a_1 y'_{21} - b_{11} y_{21} - b_{12} y_{22} - b_{13} y_{23} = -\frac{\varepsilon_1}{\varepsilon_2} y''_{11} \tag{13}$$

$$\varepsilon_2 y''_{22} + a_2 y'_{22} - b_{21} y_{21} - b_{22} y_{22} - b_{23} y_{23} = -y''_{12} \tag{14}$$

$$y''_{23} + a_3 y'_{23} - b_{31} y_{21} - b_{32} y_{22} - b_{33} y_{23} = 0 \tag{15}$$

with $y_{21}(0) = p,\ y_{22}(0) = 0,\ y_{23}(0) = 0,\ y_{21}(1) = 0,\ y_{22}(1) = 0,\ y_{23}(1) = 0$, where p is a constant to be chosen such that $|p| \le C$.

Then, it is not hard to see that, for $0 \le k \le 3$,

$$\|\mathbf{y_0}^{(k)}\| \le C, \quad \|\mathbf{y_1}^{(k)}\| \le C. \tag{16}$$

Now, consider the Eqs. (13)–(15) and using Lemma 2

$$\|\mathbf{y_2}\| \le C. \tag{17}$$

Using Theorem 1, it is not hard to see that, for $k = 1, 2$,

$$|y_{22}^{(k)}(x)| \le C\varepsilon_2^{-k} \tag{18}$$

$$|y_{23}^{(k)}(x)| \le C \tag{19}$$

From (13),

$$\varepsilon_1 y''_{21} + a_1 y'_{21} - b_{11} y_{21} = -\frac{\varepsilon_1}{\varepsilon_2} y''_{11} + b_{12} y_{22} + b_{13} y_{23}. \tag{20}$$

Decompose y_{21} as $y_{21}(x) = z_0(x) + \varepsilon_1 z_1(x) + \varepsilon_1^2 z_2(x)$ with

$$a_1 z_0' - b_{11} z_0 = -\frac{\varepsilon_1}{\varepsilon_2} y_{11}'' + b_{12} y_{22} + b_{13} y_{23}, \quad z_0(1) = 0, \tag{21}$$

$$a_1 z_1' - b_{11} z_1 = -z_0'', \quad z_1(1) = 0, \tag{22}$$

$$\varepsilon_1 z_2'' + a_1 z_2' - b_{11} z_2 = -z_1'', \quad z_2(0) = 0, \ z_2(1) = 0. \tag{23}$$

Estimating z_0 and z_1 from (21) and (22) and using Chap. 8 of [1] for the problem (23), the following estimates hold for $0 \le k \le 3$:

$$|z_0^{(k)}| < C(1 + \varepsilon_2^{(1-k)}), \ |z_1^{(k)}| < C(\varepsilon_2^{-2} + \varepsilon_2^{-2}\varepsilon_1^{2-k}), \ |z_2^{(k)}| < C(\varepsilon_2^{-2} + \varepsilon_2^{-2}\varepsilon_1^{-k})$$

Then $p = z_0(0) + \varepsilon_1 z_1(0)$ and for $k = 0, 1, 2,$

$$|y_{21}^{(k)}(x)| \le C\varepsilon_2^{-2} \ , \ |y_{21}^{(3)}(x)| \le C\varepsilon_1^{-1}\varepsilon_2^{-2}. \tag{24}$$

From (15), (19) and (24), it is not hard to see that

$$|y_{23}^{(3)}(x)| \le C\varepsilon_2^{-2}.$$

Differentiating (14) once and using (18) and (24)

$$|y_{22}^{(3)}(x)| \le C\varepsilon_2^{-3}. \tag{25}$$

Hence, from the above estimates, it is not hard to see that the components $v_1, \ v_2$ and v_3 of $\mathbf{v}$ satisfies

$$|v_1^{(k)}(x)| \le C, \ \ |v_2^{(k)}(x)| \le C \text{ for } 0 \le k \le 2, \tag{26}$$

$$|v_1^{(3)}(x)| \le C\varepsilon_1^{-1}, \ \ |v_2^{(3)}(x)| \le C\varepsilon_2^{-1} \tag{27}$$

$$|v_3^{(k)}(x)| \le C, \ \text{for } 0 \le k \le 3. \tag{28}$$

2.2 Estimates for the Bounds of the Derivatives of the Singular Component

Definition 1 Let $B_i(x), \ i = 1, 2$ be the layer functions defined on $[0, 1]$ as

$$B_i(x) = \exp(-\alpha x/\varepsilon_i). \tag{29}$$

Theorem 2 *Let $\mathbf{w}(x)$ be the solution of (12), then for $x \in \overline{\Omega}$, the following estimates hold. For $i = 1, 2$*

$$|w_i(x)| \le CB_2(x) + C\varepsilon_2 + C\varepsilon_2^2(1 - B_2(x)), \tag{30}$$

$$|w_i^{(k)}(x)| \le C\Big(\varepsilon_i^{-k} B_i(x) + \varepsilon_2^{-k} B_2(x)\Big), \tag{31}$$

$$|w_i^{(3)}(x)| \le C\varepsilon_i^{-1}\Big(\sum_{q=1}^{i-1} \varepsilon_q^{-1} B_q(x) + \sum_{q=i}^{2} \varepsilon_q^{-2} B_q(x)\Big), \tag{32}$$

$$|w_3(x)| \le C\varepsilon_2 + C\varepsilon_2^2(1 - B_2(x)), \tag{33}$$

$$|w_3'(x)| \le C\varepsilon_2, \tag{34}$$

$$|w_3''(x)| \le C\varepsilon_2 + CB_2(x), \tag{35}$$

$$|w_3^{(3)}(x)| \le \Big(\varepsilon_1^{-1} B_1(x) + \varepsilon_2^{-1} B_2(x)\Big). \tag{36}$$

Proof Consider the barrier function $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)^T$ defined by

$$\phi_i(x) = C_1 B_2(x) + C_2\varepsilon_2(1 - x) + C_3\varepsilon_2^2(1 - B_2(x)),\ i = 1, 2.$$

and

$$\phi_3(x) = C_2\varepsilon_2(1 - x) + C_3\varepsilon_2^2(1 - B_2(x)).$$

Put $\boldsymbol{\psi}^{\pm}(x) = \boldsymbol{\phi}(x) \pm \mathbf{w}(x)$, then for a proper choice of C_1, C_2 and C_3, $\boldsymbol{\psi}^{\pm}(0) \ge \mathbf{0}$, $\boldsymbol{\psi}^{\pm}(1) \ge \mathbf{0}$ and $L\boldsymbol{\psi}^{\pm}(x) \le \mathbf{0}$. Using Lemma 1, it follows that, $\boldsymbol{\psi}^{\pm}(x) \ge \mathbf{0}$. Hence, estimates (30) and (33) hold. Now using the arguments similar to Theorem 2 of [8], it is not hard to see that the other estimates hold.

□

2.3 Improved Estimates for the Bounds of the Singular Components

Using the arguments similar to those used in Lemma 5 of [10], it is not hard to see that there exists point $x^{(s)} \in (0, \frac{1}{2})$ such that

$$\frac{B_1(x^{(s)})}{\varepsilon_1^s} = \frac{B_2(x^{(s)})}{\varepsilon_2^s},\ s = 1, 2, 3 \tag{37}$$

and

$$\frac{B_1(x)}{\varepsilon_1^s} > \frac{B_2(x)}{\varepsilon_2^s},\ \text{for } x \in [0, x^{(s)}),\quad \frac{B_1(x)}{\varepsilon_1^s} < \frac{B_2(x)}{\varepsilon_2^s},\ \text{for } x \in (x^{(s)}, 1]. \tag{38}$$

Now the singular components $w_1(x)$ and $w_2(x)$ are decomposed as follows:

$$w_1(x) = w_{11}(x) + w_{12}(x),\quad w_2(x) = w_{21}(x) + w_{22}(x),\quad w_3(x) = w_{31}(x) + w_{32}(x),$$

where $w_{11}, w_{12}, w_{21}, w_{22}, w_{31}$ and w_{32} are defined by

$$w_{11}(x) = \begin{cases} \sum_{k=0}^{3} \big((x - x^{(3)})^k/k!\big) w_1^{(k)}(x^{(3)}), & \text{for } x \in [0, x^{(3)}) \\ w_1(x), & \text{for } x \in [x^{(3)}, 1] \end{cases}$$

$$w_{12}(x) = w_1(x) - w_{11}(x)$$

$$w_{21}(x) = \begin{cases} \sum_{k=0}^{3} \big((x - x^{(1)})^k/k!\big) w_2^{(k)}(x^{(1)}), & \text{for } x \in [0, x^{(1)}) \\ w_2(x), & \text{for } x \in [x^{(1)}, 1] \end{cases}$$

$$w_{22}(x) = w_2(x) - w_{21}(x).$$

$$w_{31}(x) = \begin{cases} \sum_{k=0}^{3} \big((x - x^{(1)})^k/k!\big) w_3^{(k)}(x^{(1)}), & \text{for } x \in [0, x^{(1)}) \\ w_3(x), & \text{for } x \in [x^{(1)}, 1] \end{cases}$$

$$w_{32}(x) = w_3(x) - w_{31}(x).$$

Lemma 3 *Let $w_{11}, w_{12}, w_{21}, w_{22}, w_{31}$ and w_{32} are as defined above, then for $x \in \overline{\Omega}$, the following estimates hold.*

$$|w_{11}^{(3)}(x)| \le C\varepsilon_2^{-3} B_2(x), \quad |w_{12}''(x)| \le C\varepsilon_1^{-2} B_1(x), \tag{39}$$
$$|w_{21}^{(3)}(x)| \le C\varepsilon_2^{-3} B_2(x), \quad |w_{22}''(x)| \le C\varepsilon_2^{-2} B_1(x), \tag{40}$$
$$|w_{31}^{(3)}(x)| \le C\varepsilon_2^{-1} B_2(x), \quad |w_{32}''(x)| \le C B_1(x). \tag{41}$$

Proof For $x \in [0, x^{(3)})$, by the definition of $w_{11}(x)$ and using (32) and (37),

$$|w_{11}^{(3)}(x)| = |w_1^{(3)}(x^{(3)})| \le C\varepsilon_2^{-3} B_2(x^{(3)}) \le C\varepsilon_2^{-3} B_2(x).$$

For $x \in [x^{(3)}, 1]$, by the definition of $w_{11}(x)$ and using (32) and (38),

$$|w_{11}^{(3)}(x)| = |w_1^{(3)}(x)| \le C\varepsilon_2^{-3} B_2(x).$$

Hence,

$$|w_{11}^{(3)}(x)| \le C\varepsilon_2^{-3} B_2(x), \text{ on } \overline{\Omega}. \tag{42}$$

Similar arguments lead to

$$|w_{21}^{(3)}(x)| \leq C\varepsilon_2^{-3}B_2(x),$$
$$|w_{31}^{(3)}(x)| \leq C\varepsilon_2^{-1}B_2(x)$$

Using (32), (42) and (38), it is not hard to see that, for $x \in [0, x^{(3)})$,

$$|w_{12}^{(3)}(x)| \leq |w_1^{(3)}(x)| + |w_{11}^{(3)}(x)| \leq C\varepsilon_1^{-3}B_1(x).$$

Since $w''_{12}(1) = 0$, it follows that for any $x \in [0, 1]$,

$$|w''_{12}(x)| = \Big| \int_x^1 w_{12}^{(3)}(t)dt \Big| \leq C \int_x^1 \varepsilon_1^{-3}B_1(t)dt \leq C\varepsilon_1^{-2}B_1(x).$$

Hence,

$$|w''_{12}(x)| \leq C\varepsilon_1^{-2}B_1(x), \text{ on } \overline{\Omega}. \tag{43}$$

Similar arguments lead to

$$|w''_{22}(x)| \leq C\varepsilon_2^{-2}B_1(x),$$
$$|w''_{32}(x)| \leq CB_1(x)$$

□

Now consider the alternate decomposition of the singular component $w_1(x)$ as below.

$$w_1(x) = w_{11}(x) + w_{12}(x), \tag{44}$$

where w_{11} and w_{12} are defined by

$$w_{11}(x) = \begin{cases} \sum_{k=0}^{2} \big((x - x^{(2)})^k/k!\big)w_1^{(k)}(x^{(2)}), & \text{for } x \in [0, x^{(2)}) \\ w_1(x), & \text{for } x \in [x^{(2)}, 1] \end{cases} \tag{45}$$

$$w_{12}(x) = w_1(x) - w_{11}(x). \tag{46}$$

Then, arguments similar to those of Lemma 3 lead to

$$|w''_{11}(x)| \leq C\varepsilon_2^{-2}B_2(x), \quad |w'_{12}(x)| \leq C\varepsilon_1^{-1}B_1(x). \tag{47}$$

3 Numerical Method

A piecewise uniform Shishkin mesh $\overline{\Omega}^N$ is defined on [0, 1], so as to resolve the layers in the neighbourhood of $x = 0$. Let N denote the number of mesh elements which is taken to be a multiple of 4. The interval [0, 1] is divided into three subintervals $[0, \tau_1]$, $[\tau_1, \tau_2]$ and $[\tau_2, 1]$, where τ_1 and τ_2 are the transition parameters given by,

$$\tau_2 = \min\left\{\frac{1}{2}, \frac{2\varepsilon_2}{\alpha}\ln N\right\}, \qquad \tau_1 = \min\left\{\frac{\tau_2}{2}, \frac{2\varepsilon_1}{\alpha}\ln N\right\}.$$

In each of the intervals $[0, \tau_1]$, $[\tau_1, \tau_2]$, $N/4$ mesh elements are placed and $N/2$ mesh elements are placed in the interval $[\tau_2, 1]$ so that the mesh is piecewise uniform. The mesh becomes uniform when $\tau_2 = 1/2$ and $\tau_1 = \tau_2/2$.

Let H_1, H_2 and H_3 denote the step sizes in the intervals $[0, \tau_1]$, $[\tau_1, \tau_2]$ and $[\tau_2, 1]$, respectively. Thus,

$$H_1 = \frac{4\tau_1}{N}, \quad H_2 = \frac{4(\tau_2 - \tau_1)}{N} \text{ and } H_3 = \frac{2(1 - \tau_2)}{N}.$$

Therefore, the possible four Shishkin meshes are represented by $\overline{\Omega}^N = \{x_j\}_{j=0}^N$, where

$$x_j = \begin{cases} jH_1, & \text{if } 0 \le j \le \frac{N}{4} \\ \tau_1 + (j - \frac{N}{4})H_2, & \text{if } \frac{N}{4} \le j \le \frac{N}{2} \\ \tau_2 + (j - \frac{N}{2})H_3, & \text{if } \frac{N}{2} \le j \le N. \end{cases}$$

To resolve the layers, the mesh is constructed in such a way that it condenses at the inner regions where the layers are exhibited and is coarse in the outer region, away from the layers.

To solve the BVP (1)–(2) numerically the following upwind classical finite difference scheme is applied on the mesh $\overline{\Omega}^N$.

$$L^N\mathbf{U}(x_j) \equiv E\delta^2\mathbf{U}(x_j) + A(x_j)D^+\mathbf{U}(x_j) - B(x_j)\mathbf{U}(x_j) = \mathbf{f}(x_j), \tag{48}$$

$$\mathbf{U}(x_0) = \mathbf{l}, \;\; \mathbf{U}(x_N) = \mathbf{r}, \tag{49}$$

where $\mathbf{U}(x_j) = (U_1(x_j), U_2(x_j))^T$ and for $1 \le j \le N-1$,

$$D^+U(x_j) = \frac{U(x_{j+1}) - U(x_j)}{h_{j+1}}, \quad D^-U(x_j) = \frac{U(x_j) - U(x_{j-1})}{h_j},$$

$$\delta^2 U(x_j) = \frac{1}{\overline{h}_j}\Big(D^+U(x_j) - D^-U(x_j)\Big),$$

with

$$h_j = x_j - x_{j-1}, \quad \overline{h_j} = \frac{(h_j + h_{j+1})}{2}.$$

4 Error Analysis

In this section, a discrete maximum principle, a discrete stability result and the first-order convergence of the proposed numerical method are established.

Lemma 4 (Discrete Maximum Principle) *Assume that the vector valued mesh function* $\boldsymbol{\psi}(x_j) = (\psi_1(x_j), \psi_2(x_j), \psi_3(x_j))^T$ *satisfies* $\boldsymbol{\psi}(x_0) \geq \mathbf{0}$ *and* $\boldsymbol{\psi}(x_N) \geq \mathbf{0}$. *Then* $L^N \boldsymbol{\psi}(x_j) \leq \mathbf{0}$ *for* $1 \leq j \leq N-1$ *implies that* $\boldsymbol{\psi}(x_j) \geq \mathbf{0}$ *for* $0 \leq j \leq N$.

An immediate consequence of the above discrete maximum principle is the following discrete stability result.

Lemma 5 (Discrete Stability Result) *If* $\boldsymbol{\psi}(x_j) = (\psi_1(x_j), \psi_2(x_j), \psi_3(x_j))^T$ *is any vector valued mesh function defined on* $\overline{\Omega}^N$, *then for* $i = 1, 2, 3$ *and* $0 \leq j \leq N$,

$$|\psi_i(x_j)| \leq \max\left\{\|\boldsymbol{\psi}(x_0)\|, \ \|\boldsymbol{\psi}(x_N)\|, \ \frac{1}{\beta}\|L^N \boldsymbol{\psi}\|_{\Omega^N}\right\}.$$

4.1 Error Estimate

Analogous to the continuous case, the discrete solution **U** can be decomposed into **V** and **W** as defined below.

$$L^N \mathbf{V}(x_j) = \mathbf{f}(x_j), \ \text{for } 0 < j < N, \ \mathbf{V}(x_0) = \mathbf{v}(x_0), \ \mathbf{V}(x_N) = \mathbf{v}(x_N) \tag{50}$$

$$L^N \mathbf{W}(x_j) = \mathbf{0}, \ \text{for } 0 < j < N, \ \mathbf{W}(x_0) = \mathbf{w}(x_0), \ \mathbf{W}(x_N) = \mathbf{w}(x_N) \tag{51}$$

Lemma 6 *Let* $\mathbf{v}$ *be the solution of (11) and* $\mathbf{V}$ *be the solution of (50), then*

$$\|\mathbf{V} - \mathbf{v}\|_{\overline{\Omega}^N} \leq CN^{-1}.$$

Proof For $1 \leq j \leq N-1$,

$$\begin{aligned} L^N(\mathbf{V} - \mathbf{v})(x_j) &= \mathbf{f}(x_j) - L^N \mathbf{v}(x_j) \\ &= (L - L^N)\mathbf{v}(x_j) \\ &= \left(\frac{d^2}{dx^2} - \delta^2\right) E\mathbf{v}(x_j) + \left(\frac{d}{dx} - D^+\right) A(x_j)\mathbf{v}(x_j) \end{aligned}$$

$$= \begin{pmatrix} \varepsilon_1(\frac{d^2}{dx^2} - \delta^2)v_1(x_j) + a_1(x_j)(\frac{d}{dx} - D^+)v_1(x_j) \\ \varepsilon_2(\frac{d^2}{dx^2} - \delta^2)v_2(x_j) + a_2(x_j)(\frac{d}{dx} - D^+)v_2(x_j) \\ (\frac{d^2}{dx^2} - \delta^2)v_3(x_j) + a_3(x_j)(\frac{d}{dx} - D^+)v_3(x_j) \end{pmatrix}.$$

By the standard local truncation used in the Taylor expansions,

$$|\varepsilon_1(\frac{d^2}{dx^2} - \delta^2)v_1(x_j) + a_1(x_j)(\frac{d}{dx} - D^+)v_1(x_j)| \le C(x_{j+1} - x_{j-1})(\varepsilon_1\|v_1^{(3)}\| + \|v_1^{(2)}\|),$$

$$|\varepsilon_2(\frac{d^2}{dx^2} - \delta^2)v_2(x_j) + a_2(x_j)(\frac{d}{dx} - D^+)v_2(x_j)| \le C(x_{j+1} - x_{j-1})(\varepsilon_2\|v_2^{(3)}\| + \|v_2^{(2)}\|),$$

$$|(\frac{d^2}{dx^2} - \delta^2)v_3(x_j) + a_3(x_j)(\frac{d}{dx} - D^+)v_3(x_j)| \le C(x_{j+1} - x_{j-1})(\|v_3^{(3)}\| + \|v_3^{(2)}\|).$$

Since $(x_{j+1} - x_{j-1}) \le CN^{-1}$, using the estimates (26)–(28),

$$\|L^N(\mathbf{V} - \mathbf{v})\|_{\Omega^N} \le CN^{-1}.$$

Using Lemma 5,

$$\|\mathbf{V} - \mathbf{v}\|_{\overline{\Omega}^N} \le CN^{-1}. \tag{52}$$

□

Definition 2 The mesh functions $B_1^N(x_j)$ and $B_2^N(x_j)$ on $\overline{\Omega}^N$ defined by

$$B_1^N(x_j) = \prod_{i=1}^{j}\left(1 + \frac{\alpha h_i}{2\varepsilon_1}\right)^{-1} \text{ and } B_2^N(x_j) = \prod_{i=1}^{j}(1 + \frac{\alpha h_i}{2\varepsilon_2})^{-1}$$

with $B_1^N(x_0) = B_2^N(x_0) = 1$.

It is to be observed that B_1^N and B_2^N are monotonically decreasing.

Lemma 7 *Let $\tau_2 = 1/2$ and $\tau_1 = 1/4$, $\mathbf{w}$ be the solution of (12) and $\mathbf{W}$ be the solution of (51), then*

$$\|\mathbf{W} - \mathbf{w}\|_{\overline{\Omega}^N} \le CN^{-1}\ln N.$$

Proof By the standard local truncation used in the Taylor expansions,

$$|\varepsilon_1(\frac{d^2}{dx^2} - \delta^2)w_1(x_j) + a_1(x_j)(\frac{d}{dx} - D^+)w_1(x_j)| \le C(x_{j+1} - x_{j-1})(\varepsilon_1\|w_1^{(3)}\| + \|w_1^{(2)}\|)$$

$$|\varepsilon_2(\frac{d^2}{dx^2} - \delta^2)w_2(x_j) + a_2(x_j)(\frac{d}{dx} - D^+)w_2(x_j)| \le C(x_{j+1} - x_{j-1})(\varepsilon_2\|w_2^{(3)}\| + \|w_2^{(2)}\|)$$

$$|(\frac{d^2}{dx^2}-\delta^2)w_3(x_j)+a_3(x_j)(\frac{d}{dx}-D^+)w_3(x_j)| \le C(x_{j+1}-x_{j-1})(\|w_3^{(3)}\|+\|w_3^{(2)}\|)$$

where the norm is taken over the interval $[x_{j-1}, x_{j+1}]$.

For the case $\tau_2 = 1/2$ and $\tau_1 = 1/4$, the mesh is uniform, stepsize $h = N^{-1}$, $\varepsilon_1^{-1} \le C \ln N$ and $\varepsilon_2^{-1} \le C \ln N$ and thus we obtain,

$$|L^N(\mathbf{W}-\mathbf{w})(x_j)| \le \begin{pmatrix} CN^{-1}\ln N\big(\varepsilon_1^{-1}B_1(x_{j-1})+\varepsilon_2^{-1}B_2(x_{j-1})\big) \\ CN^{-1}\ln N\varepsilon_2^{-1}B_2(x_{j-1}) \\ CN^{-1}\ln N \end{pmatrix}. \quad (53)$$

Consider the following barrier function $\boldsymbol{\phi}$ given by

$$\phi_1(x_j) = \frac{CN^{-1}\ln N}{\gamma(\alpha-\gamma)}\Big(\exp(2\gamma h/\varepsilon_1)Y_j + \exp(2\gamma h/\varepsilon_2)Z_j\Big)$$
$$\phi_2(x_j) = \frac{CN^{-1}\ln N}{\gamma(\alpha-\gamma)}\Big(\exp(2\gamma h/\varepsilon_2)Z_j\Big)$$
$$\phi_3(x_j) = CN^{-1}\ln N$$

where γ is a constant such that $0 < \gamma < \alpha$,

$$Y_j = \frac{\lambda^{N-j}-1}{\lambda^N-1} \text{ with } \lambda = 1+\frac{\gamma h}{\varepsilon_1}$$

and

$$Z_j = \frac{\Lambda^{N-j}-1}{\Lambda^N-1} \text{ with } \Lambda = 1+\frac{\gamma h}{\varepsilon_2}.$$

It is not hard to see that

$$0 \le Y_j, Z_j \le 1,$$

$$(\varepsilon_1\delta^2+\gamma D^+)Y_j = 0, \quad (\varepsilon_2\delta^2+\gamma D^+)Z_j = 0,$$

$$D^+Y_j \le -\frac{\gamma}{\varepsilon_1}\exp(-\gamma x_{j+1}/\varepsilon_1), \quad D^+Z_j \le -\frac{\gamma}{\varepsilon_2}\exp(-\gamma x_{j+1}/\varepsilon_2).$$

Hence,

$$\begin{aligned}(L^N\boldsymbol{\phi})(x_j) &\le \frac{CN^{-1}}{\gamma(\alpha-\gamma)}\begin{pmatrix}\exp(2\gamma h/\varepsilon_1)D^+Y_j+\exp(2\gamma h/\varepsilon_2)D^+Z_j \\ \exp(2\gamma h/\varepsilon_2)(a_2-\gamma)D^+Z_j \\ \ln N\end{pmatrix} \\ &\le -CN^{-1}\begin{pmatrix}\varepsilon_1^{-1}B_1(x_{j-1})+\varepsilon_2^{-1}B_2(x_{j-1}) \\ \varepsilon_2^{-1}B_2(x_{j-1}) \\ \ln N\end{pmatrix}. \quad (54)\end{aligned}$$

Consider the discrete functions

$$\boldsymbol{\psi}^{\pm}(x_j) = \boldsymbol{\phi}(x_j) \pm (\mathbf{W} - \mathbf{w})(x_j),\ x_j \in \overline{\Omega}^N.$$

Then for sufficiently large C, using (53) and (54), $\boldsymbol{\psi}^{\pm}(x_0) > \mathbf{0}$, $\boldsymbol{\psi}^{\pm}(x_N) = \mathbf{0}$ and $L^N \boldsymbol{\psi}^{\pm}(x_j) \le \mathbf{0}$ on Ω^N.

Using discrete maximum principle, $\boldsymbol{\psi}^{\pm}(x_j) \ge \mathbf{0}$ on $\overline{\Omega}^N$. Hence,

$$|(\mathbf{W} - \mathbf{w})(x_j)| \le \begin{pmatrix} CN^{-1} \ln N \\ CN^{-1} \ln N \\ CN^{-1} \ln N \end{pmatrix}$$

implies that

$$\|(\mathbf{W} - \mathbf{w})\|_{\overline{\Omega}^N} \le CN^{-1} \ln N.$$

□

For any choices of τ_1 and τ_2, estimate of $\|(\mathbf{W} - \mathbf{w})\|_{\overline{\Omega}^N}$ is as follows.

Lemma 8 *Let* $\mathbf{w}$ *be the solution of (12) and* $\mathbf{W}$ *be the solution of (51), then*

$$\|\mathbf{W} - \mathbf{w}\|_{\overline{\Omega}^N} \le CN^{-1} \ln N.$$

Proof For $0 < j < N/4$, $\tau_1 \le (\varepsilon_1/\alpha) \ln N$ and hence

$$|L^N(\mathbf{W} - \mathbf{w})(x_j)| \le CN^{-1} \ln N \begin{pmatrix} \varepsilon_1^{-1} B_1(x_{j-1}) + \varepsilon_2^{-1} B_2(x_{j-1}) \\ \varepsilon_2^{-1} B_2(x_{j-1}) \\ C \end{pmatrix}.$$

For $N/4 \le j < N/2$, if $\varepsilon_2/2 \le \varepsilon_1 \le \varepsilon_2$, then $\tau_2 \le (4\varepsilon_1/\alpha) \ln N$ implies that

$$|L^N(\mathbf{W} - \mathbf{w})(x_j)| \le CN^{-1} \ln N \begin{pmatrix} \varepsilon_1^{-1} B_1(x_{j-1}) + \varepsilon_2^{-1} B_2(x_{j-1}) \\ \varepsilon_2^{-1} B_2(x_{j-1}) \\ C \end{pmatrix}. \tag{55}$$

On the other hand, if $\varepsilon_2 > 2\varepsilon_1$, then using (2.3) ,

$$\begin{pmatrix} |\varepsilon_1(\frac{d^2}{dx^2} - \delta^2)w_1(x_j)| \\ |\varepsilon_2(\frac{d^2}{dx^2} - \delta^2)w_2(x_j)| \\ |(\frac{d^2}{dx^2} - \delta^2)w_3(x_j)| \end{pmatrix} \le \begin{pmatrix} |\varepsilon_1(\frac{d^2}{dx^2} - \delta^2)w_{11}(x_j)| \\ |\varepsilon_2(\frac{d^2}{dx^2} - \delta^2)w_{21}(x_j)| \\ |(\frac{d^2}{dx^2} - \delta^2)w_{31}(x_j)| \end{pmatrix} + \begin{pmatrix} |\varepsilon_1(\frac{d^2}{dx^2} - \delta^2)w_{12}(x_j)| \\ |\varepsilon_2(\frac{d^2}{dx^2} - \delta^2)w_{22}(x_j)| \\ |(\frac{d^2}{dx^2} - \delta^2)w_{32}(x_j)| \end{pmatrix}.$$

Also, by the standard local truncation used in the Taylor expansions and using Lemma 3,

$$\begin{pmatrix} |\varepsilon_1(\frac{d^2}{dx^2}-\delta^2)w_{11}(x_j)| \\ |\varepsilon_2(\frac{d^2}{dx^2}-\delta^2)w_{21}(x_j)| \\ (\frac{d^2}{dx^2}-\delta^2)w_{31}(x_j)| \end{pmatrix} \le \begin{pmatrix} C\varepsilon_1(x_{j+1}-x_{j-1})\|w_{11}^{(3)}\|_{[x_{j-1},x_{j+1}]} \\ C\varepsilon_2(x_{j+1}-x_{j-1})\|w_{21}^{(3)}\|_{[x_{j-1},x_{j+1}]} \\ C(x_{j+1}-x_{j-1})\|w_{31}^{(3)}\|_{[x_{j-1},x_{j+1}]} \end{pmatrix}$$

$$\le CN^{-1}\ln N \begin{pmatrix} \varepsilon_2^{-1}B_2(x_{j-1}) \\ \varepsilon_2^{-1}B_2(x_{j-1}) \\ C \end{pmatrix},$$

$$\begin{pmatrix} |\varepsilon_1(\frac{d^2}{dx^2}-\delta^2)w_{12}(x_j)| \\ |\varepsilon_2(\frac{d^2}{dx^2}-\delta^2)w_{22}(x_j)| \\ (\frac{d^2}{dx^2}-\delta^2)w_{32}(x_j)| \end{pmatrix} \le C \begin{pmatrix} \varepsilon_1\|w_{12}''\|_{[x_{j-1},x_{j+1}]} \\ \varepsilon_2\|w_{22}''\|_{[x_{j-1},x_{j+1}]} \\ \|w_{32}''\|_{[x_{j-1},x_{j+1}]} \end{pmatrix} \le C \begin{pmatrix} \varepsilon_1^{-1}B_1(x_{j-1}) \\ \varepsilon_2^{-1}B_1(x_{j-1}) \\ B_1(x_{j-1}) \end{pmatrix}$$

Thus, for $N/4 \le j < N/2$,

$$\begin{pmatrix} |\varepsilon_1(\frac{d^2}{dx^2}-\delta^2)w_1(x_j)| \\ |\varepsilon_2(\frac{d^2}{dx^2}-\delta^2)w_2(x_j)| \\ |(\frac{d^2}{dx^2}-\delta^2)w_3(x_j)| \end{pmatrix} \le \begin{pmatrix} C\varepsilon_2^{-1}N^{-1}\ln N B_2(x_{j-1}) + C\varepsilon_1^{-1}B_1(x_{j-1}) \\ C\varepsilon_2^{-1}N^{-1}\ln N B_2(x_{j-1}) \\ CN^{-1}\ln N \end{pmatrix}. \quad (56)$$

Using the alternate decomposition of $w_1(x)$ given in (44) and the arguments similar to the above, it is not hard to verify that for $N/4 \le j < N/2$,

$$\begin{pmatrix} |(\frac{d}{dx}-D^+)w_1(x_j)| \\ |(\frac{d}{dx}-D^+)w_2(x_j)| \\ |(\frac{d}{dx}-D^+)w_3(x_j)| \end{pmatrix} \le \begin{pmatrix} C\varepsilon_2^{-1}N^{-1}\ln N B_2(x_{j-1}) + C\varepsilon_1^{-1}B_1(x_{j-1}) \\ C\varepsilon_2^{-1}N^{-1}\ln N B_2(x_{j-1}) \\ CN^{-1}\ln N \end{pmatrix}. \quad (57)$$

Hence, for $N/4 \le j < N/2$, expressions (56) and (57) yield

$$|L^N(\mathbf{W}-\mathbf{w})(x_j)| \le \begin{pmatrix} C\varepsilon_2^{-1}N^{-1}\ln N B_2(x_{j-1}) + C\varepsilon_1^{-1}B_1(x_{j-1}) \\ C\varepsilon_2^{-1}N^{-1}\ln N B_2(x_{j-1}) \\ CN^{-1}\ln N \end{pmatrix}.$$

For $N/2 \le j < N$, if $\tau_2 = 1/2$, then $(x_{j+1}-x_{j-1}) \le C\varepsilon_2 N^{-1}\ln N$ and hence

$$|L^N(\mathbf{W}-\mathbf{w})(x_j)| \le \begin{pmatrix} C\varepsilon_2^{-1}N^{-1}\ln N B_2(x_{j-1}) + C\varepsilon_1^{-1}B_1(x_{j-1}) \\ C\varepsilon_2^{-1}N^{-1}\ln N B_2(x_{j-1}) \\ CN^{-1}\ln N \end{pmatrix}.$$

On the other hand, if $\tau_2 = \frac{2\varepsilon_2}{\alpha}\ln N$, then it is not hard to see that

$$|L^N(\mathbf{W}-\mathbf{w})(x_j)| \le \begin{pmatrix} C\varepsilon_1^{-1}B_1(x_{j-1}) + C\varepsilon_2^{-1}B_2(x_{j-1}) \\ C\varepsilon_2^{-1}B_2(x_{j-1}) \\ CN^{-1}\ln N \end{pmatrix}.$$

Consider the following barrier functions for $0 < j < N/4$,

$$\phi_1(x_j) = CN^{-1}\ln N\big(\exp(2\alpha H_1/\varepsilon_1)B_1^N(x_j) + \exp(2\alpha H_1/\varepsilon_2)B_2^N(x_j)\big) \quad (58)$$

$$\phi_2(x_j) = CN^{-1}\ln N \exp(2\alpha H_1/\varepsilon_2)B_2^N(x_j) \quad (59)$$

$$\phi_3(x_j) = CN^{-1}\ln N \quad (60)$$

for $N/4 \le j < N/2$,

$$\phi_1(x_j) = CN^{-1}\ln N \exp(2\alpha H_2/\varepsilon_2)B_2^N(x_j) + CB_1^N(x_j) \quad (61)$$

$$\phi_2(x_j) = CN^{-1}\ln N \exp(2\alpha H_2/\varepsilon_2)B_2^N(x_j) \quad (62)$$

$$\phi_3(x_j) = CN^{-1}\ln N \quad (63)$$

and for $N/2 \le j \le N$, if $\tau_2 = 1/2$

$$\phi_1(x_j) = CN^{-1}\ln N \exp(2\alpha H_2/\varepsilon_2)B_2^N(x_j) + CB_1^N(x_j) \quad (64)$$

$$\phi_2(x_j) = CN^{-1}\ln N \exp(2\alpha H_2/\varepsilon_2)B_2^N(x_j) \quad (65)$$

$$\phi_3(x_j) = CN^{-1}\ln N \quad (66)$$

or if $\tau_2 = \frac{2\varepsilon_2}{\alpha}\ln N$

$$\phi_1(x_j) = CB_1^N(x_j) + CB_2^N(x_j) \quad (67)$$

$$\phi_2(x_j) = CB_2^N(x_j) \quad (68)$$

$$\phi_3(x_j) = CN^{-1}\ln N \quad (69)$$

Let $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)^T$ and consider the following vector-valued mesh functions, on $\overline{\Omega}^N$,

$$\boldsymbol{\psi}^{\pm}(x_j) = \boldsymbol{\phi}(x_j) \pm (\mathbf{W} - \mathbf{w})(x_j).$$

For sufficiently large C,

$$\boldsymbol{\psi}^{\pm}(x_0) \ge \mathbf{0},\ \ \boldsymbol{\psi}^{\pm}(x_N) \ge \mathbf{0} \text{ and } L^N\boldsymbol{\psi}^{\pm}(x_j) \le \mathbf{0},\ \text{ for } 0 < j < N.$$

Then by Lemma 4 $\boldsymbol{\psi}^{\pm}(x_j) \ge \mathbf{0}$ for $0 \le j \le N$. Hence,

$$\|(\mathbf{W} - \mathbf{w})\|_{\overline{\Omega}^N} \le CN^{-1}\ln N. \quad (70)$$

□

Theorem 3 *Let* **u** *be the solution of the problem (1)–(2) and* **U** *be the solution of the problem (48)–(49), then,*

$$\|(\mathbf{u} - \mathbf{U})\|_{\overline{\Omega}^N} \le CN^{-1}\ln N.$$

□

Proof The result follows by using triangle inequality, (52) and (70).

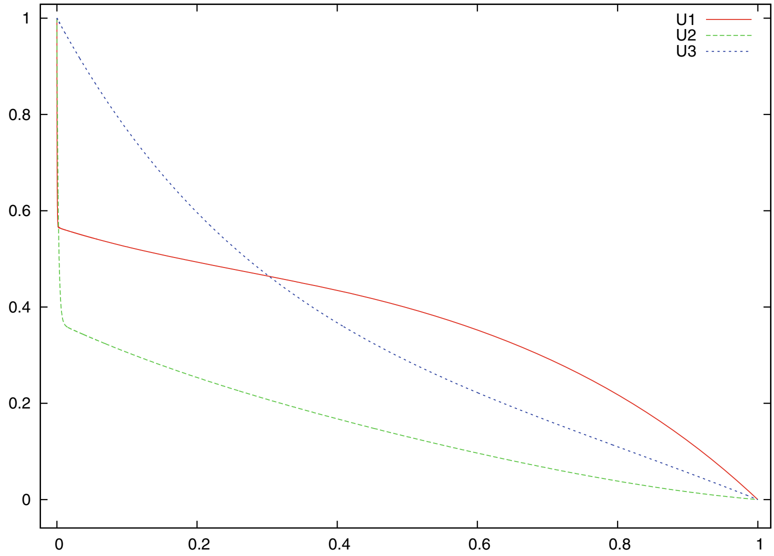


Fig. 1 Approximate solution of Example 1

Table 1 Maximum errors and order of convergence of the numerical method

ε_1	ε_2	Number of mesh elements N 128	256	512	1024
5^{-2}	3^{-2}	$0.6881E-02$	$0.3591E-02$	$0.1835E-02$	$0.9276E-03$
5^{-5}	3^{-5}	$0.1244E-01$	$0.7136E-02$	$0.3751E-02$	$0.1900E-02$
5^{-8}	3^{-8}	$0.1881E-01$	$0.1324E-01$	$0.8250E-02$	$0.4876E-02$
5^{-11}	3^{-11}	$0.1984E-01$	$0.1400E-01$	$0.8954E-02$	$0.5315E-02$
5^{-14}	3^{-14}	$0.1998E-01$	$0.1419E-01$	$0.9100E-02$	$0.5355E-02$
5^{-17}	3^{-17}	$0.2001E-01$	$0.1423E-01$	$0.9128E-02$	$0.5360E-02$
5^{-20}	3^{-20}	$0.2001E-01$	$0.1423E-01$	$0.9134E-02$	$0.5366E-02$
D^N		$0.2001E-01$	$0.1423E-01$	$0.9134E-02$	$0.5366E-02$
p^N		$0.4915E+00$	$0.6400E+00$	$0.7674E+00$	
C_p^N		$0.7526E+00$	$0.7526E+00$	$0.6790E+00$	$0.5608E+00$

Computed order of $(\varepsilon_1, \varepsilon_2)$-uniform convergence, $p^* = 0.4915$
Computed $(\varepsilon_1, \varepsilon_2)$-uniform error constant, $C_{p^*}^N = 0.7526$

5 Numerical Illustrations

Example 1 Consider the boundary value problem for the system of convection diffusion equations on (0, 1)

$$\varepsilon_1 u_1''(x) + (1+x)u_1'(x) - 4u_1(x) + 2u_2(x) + u_3(x) = -e^x,$$
$$\varepsilon_2 u_2''(x) + (2+x^2)u_2'(x) + u_1(x) - 6u_2(x) + 2u_3(x) = -e^{-x},$$
$$u_3''(x) + u_3'(x) + 3u_1(x) + 2u_2(x) - 8u_3(x) = -x^2,$$

with $u_1(0) = 1,\ u_2(0) = 1,\ u_3(0) = 1,\ u_1(1) = 0,\ u_2(1) = 0\ u_3(1) = 0.$

The above problem is solved using the suggested numerical method and plot of the approximate solution for $N = 1024$, $\varepsilon_1 = 5^{-5}$, $\varepsilon_2 = 3^{-5}$ is shown in Fig. 1. Parameter uniform error and order of convergence of the numerical method are shown in Table 1.

From Table 1, it is to be noted that the error decreases as number of mesh elements N increases. Also for each N, the error stabilizes as ε_1 and ε_2 tends to zero.

Acknowledgements The first and the third authors thank the Department of Science and Technology, Government of India, for the support to the Department through the DST-FIST Scheme to set up the Computer Lab where the computations have been carried out.

References

1. Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Fitted Numerical Methods for Singular Perturbation Problems. World Scientific Publishing Co., Singapore (1996)
2. Doolan, E.P., Miller, J.J.H., Schilders, W.H.A.: Uniform Numerical Methods for Problems with Initial and Boundary Layers. Boole press, Dublin, Ireland (1980)
3. Farrell, P.A., Hegarty, A., Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Robust Computational Techniques for Boundary Layers. Chapman and Hall/CRC Press, Boca Raton (2000)
4. Bellew, S., O'Riordan, E.: A parameter robust numerical method for a system of two singularly perturbed convection-diffusion equations. Appl. Numer. Math. **51**(2–3), 171–186 (2004)
5. O'Riordan, E., Stynes, M.: Numerical analysis of a strongly coupled system of two singularly perturbed convection-diffusion problems. Adv. Comput. Math. **30**, 101–121 (2009)
6. Cen, Z.: Parameter-uniform finite difference scheme for a system of coupled singularly perturbed convection-diffusion equations. Int. J. Comput. Math. **82**(2), 177–192 (2005)
7. Linss, T.: Analysis of an upwind finite difference scheme for a system of coupled singularly perturbed convection-diffusion equations. Computing **79**(1), 23–32 (2007)
8. Saravana Sankar, K., Miller, J.J.H., Valarmathi, S.: A parameter uniform fitted mesh method for a weakly coupled system of two singularly perturbed convection-diffusion equations. Math. Commun. **24**, 193–210 (2019)
9. Kokotovi, P.V.: Applications of singular perturbation techniques to control problems. SIAM Rev. **26**(4), 501–550 (1984)
10. Paramasivam, M., Valarmathi, S., Miller, J.J.H.: Second order parameter-uniform convergence for a finite difference method for a singularly perturbed linear reaction-diffusion system. Math. Commun. **15**(2), 587–612 (2010)

Numerical Method for a Boundary Value Problem for a Linear System of Partially Singularly Perturbed Parabolic Delay Differential Equations of Reaction-Diffusion Type

Parthiban Saminathan and Franklin Victor

Abstract The problem of a partially singularly perturbed boundary value problem for a linear system of reaction-diffusion type parabolic second-order delay differential equations is explored. The boundary value problem is partially perturbed when $\varepsilon_{m+1} = \cdots = \varepsilon_n = 1$ for some $m < n$ and the system is partial with respect to delay when the co-efficient function of delay terms $b_i(x, t) = 0$ for some $i, i = 1, 2, \ldots, n$. The components of the solution of this system exhibit boundary layers at $(x, t) = (0, t)$ and $(x, t) = (2, t)$ and interior layers at $(x, t) = (1, t)$. To approximate the solution, a numerical technique is presented that uses a classical finite difference scheme on a Shishkin piecewise-uniform mesh. For all values of singular perturbation parameters, the approach is shown to be uniformly first-order convergent. To corroborate the theoretical results, numerical illustrations are provided.

Keywords Singular perturbation problems · Shishkin meshes · Delay differential equations · Parameter uniform convergence · Parabolic differential equations

1 Introduction

Fluid dynamics, quantum physics, magnetohydrodynamics, and chemical reactor theory are all examples of applications of singularly perturbed differential equations in applied mathematics and engineering. Singularly perturbed delay differential equations are a type of singularly perturbed differential equation that has applications in population dynamics, physiology, and control theory, among other fields. Consider partially perturbed systems of Singularly Perturbed Problems to investigate the impact of these factors on the layer pattern. These systems are partial with respect to delay—that is, some of the equations do not contain delay terms—and partial with respect to the perturbation parameter—that is, some of the equations do not contain perturbation terms. The perturbation parameter has a stronger influence on

P. Saminathan · F. Victor (✉)
Department of Mathematics, Bishop Heber College, Tiruchirappalli, Tamil Nadu, India

Valarmathi S et al. (eds.), *Differential Equations and Applications*,
Springer Proceedings in Mathematics & Statistics 368,
https://doi.org/10.1007/978-981-16-7546-1_4

the layer pattern than the delay term, as evidenced by the limitations on the solution's derivatives.

A linear system of n second-order differential equations of the parabolic reaction-diffusion type with initial and boundary conditions is considered in [3]. Singular perturbation is applied to the first k equations. Each of the first m equations leading terms, $m \le k$, is multiplied by a small positive parameter, which is assumed to be distinct. The same perturbation parameter ε_m is multiplied by the leading terms of the next $k - m$ equations. Shishkin piecewise-uniform meshes are introduced, which are utilised in conjunction with a classical finite difference discretisation to develop a numerical method for solving this problem, because the components of the solution have overlapping layers. The numerical approximations obtained with this method are first-order convergent in time and essentially second-order convergent in the space variable in the maximum norm, uniformly with respect to all of the parameters. The work done in the previous study [6] is extended to a parabolic system of partially singularly perturbed delay differential equations in this paper. Related works are found in [4] and [5].

The following is the outline for the paper. The problem is defined in Sect. 2, and the existence and regularity of the solution are discussed. Section 3 establishes the maximum principle for the differential operator and, as a result, the stability result. Standard estimates of the solution's derivatives are also presented. Improved estimates for the derivatives of solution components are presented in Sect. 4. Section 5 introduces piecewise-uniform Shishkin meshes, whereas Sect. 6 defines the discrete problem and establishes the discrete maximum principle and discrete stability properties. The numerical analysis is presented in Sect. 7 together with the error bounds. Section 8 has numerical illustrations, while Sect. 9 contains the conclusion.

2 The Continuous Problem

A partially singularly perturbed boundary value problem for a system of n linear parabolic second-order delay differential equations of reaction—diffusion type is considered as follows:

$$\begin{aligned} &\mathbf{L}\mathbf{u}(x,t) = \frac{\partial \mathbf{u}}{\partial t}(x,t) - E\frac{\partial^2 \mathbf{u}}{\partial x^2}(x,t) + A(x,t)\mathbf{u}(x,t) + B(x,t)\mathbf{u}(x-1,t) = \mathbf{f}(x,t) \text{ on } \Omega, \\ &\mathbf{u} \text{ given on } \Gamma, \ \mathbf{u}(x,t) = \boldsymbol{\chi}(x,t), (x,t) \in [-1,0]\times[0,T], \end{aligned} \tag{1}$$

where $\Omega = \{(x,t) : 0 < x < 2, 0 < t \le T\}$, $\overline{\Omega} = \Omega \cup \Gamma$, $\tilde{\Omega} = \{(0,1-)\times(0,T]\} \cup \{(1+,2)\times(0,T]\}$, $\overline{\tilde{\Omega}}$=$\{[0,1-]\times[0,T]\} \cup \{[1+,2]\times[0,T]\}$, Γ=$\Gamma_L \cup \Gamma_B \cup \Gamma_R$ with $\mathbf{u}(0,t) = \boldsymbol{\chi}(0,t)$ on $\Gamma_L = \{(0,t) : 0 \le t \le T\}$, $\mathbf{u}(x,0) = \boldsymbol{\phi}_B(x)$ on $\Gamma_B = \{(x,0) : 0 \le x \le 2\}$, and $\mathbf{u}(2,t) = \boldsymbol{\phi}_R(t)$ on $\Gamma_R = \{(2,t) : 0 \le t \le T\}$. For all $(x,t) \in \overline{\Omega}$, $\mathbf{u}(x,t) = (u_1(x,t), u_2(x,t), \ldots, u_n(x,t))^T$ and $\mathbf{f}(x,t) = (f_1(x,t), f_2(x,t), \ldots, f_n(x,t))^T$. E, $A(x,t)$ and $B(x,t)$ are $n \times n$

matrices. $E = diag(\boldsymbol{\varepsilon}), \boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, $B(x,t) = diag(\mathbf{b}(x,t)), \mathbf{b}(x,t) = (b_1(x,t), b_2(x,t), \ldots, b_n(x,t))$. The function $\boldsymbol{\chi}$ is sufficiently smooth on $[-1,0] \times [0,T]$. The singular perturbation parameters satisfy

$$0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_m = \varepsilon_{m+1} = \cdots = \varepsilon_k < \varepsilon_{k+1} = \cdots = \varepsilon_n = 1.$$

For all $(x,t) \in [0,2] \times [0,T]$, it is also assumed that the entries $a_{ij}(x,t)$ of $A(x,t)$ and the components $b_i(x,t)$ of $\mathbf{b}(x,t)$ satisfy

$$b_i(x,t), a_{ij}(x,t) \le 0 \text{ for } 1 \le i \ne j \le n, a_{ii}(x,t) > \sum_{i \ne j} |a_{ij}(x,t) + b_i(x,t)| \tag{2}$$

$$\text{and } 0 < \alpha < \min_{\substack{(x,t)\in[0,2]\times[0,T]\\ i=1,2\ldots,n}} \left(\sum_{j=1}^{n} a_{ij}(x) + b_i(x) \right), \quad \text{for some } .\alpha \tag{3}$$

The functions $a_{ij}(x,t)$ and $b_i(x,t)$, $1 \le i, j \le n$ are in $C^2([0,2] \times [0,T])$. Problem (1) can be rewritten as

$$\mathbf{L}_1\mathbf{u}(x,t) = \frac{\partial \mathbf{u}}{\partial t}(x,t) - E\frac{\partial^2 \mathbf{u}}{\partial x^2}(x,t) + A(x,t)\mathbf{u}(x,t) = \mathbf{g}(x,t), \text{ on } \Omega_1 = (0,1) \times (0,T], \tag{4}$$

where $\mathbf{g}(x,t) = \mathbf{f}(x,t) - B(x,t)\boldsymbol{\chi}(x-1,t)$

$$\mathbf{L}_2\mathbf{u}(x,t) = \frac{\partial \mathbf{u}}{\partial t}(x,t) - E\frac{\partial^2 \mathbf{u}}{\partial x^2}(x,t) + A(x,t)\mathbf{u}(x,t) + B(x,t)\mathbf{u}(x-1,t) = \mathbf{f}(x,t),$$
$$\text{on } \Omega_2 = (1,2) \times (0,T] \tag{5}$$

$\mathbf{u}(0,t) = \boldsymbol{\chi}(0,t), \mathbf{u}(x,0) = \boldsymbol{\phi}_B(x)$ on $\Gamma_{B_1} = \{(x,0) : 0 \le x \le 1-\}, \mathbf{u}(1-,t) = \mathbf{u}(1+,t)$, $\frac{\partial \mathbf{u}}{\partial x}(1-,t) = \frac{\partial \mathbf{u}}{\partial x}(1+,t), \mathbf{u}(x,0) = \boldsymbol{\phi}_B(x)$ on $\Gamma_{B_2} = \{(x,0) : 1+ \le x \le 2\}, \mathbf{u}(2,t) = \boldsymbol{\phi}_R(t)$ on Γ_R.

Problem (1) is partially perturbed when $\varepsilon_{m+1} = \cdots = \varepsilon_n = 1$ for some $m < n$ and the system is partial with respect to delay when the function $b_i(x,t) = 0$ for some $i, i = 1,2,\ldots,n$. Hence the following three cases arise.

$$\begin{aligned}
\text{Case}(i):\ & 0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_m = \varepsilon_{m+1} = \cdots = \varepsilon_k < \varepsilon_{k+1} = \cdots = \varepsilon_n = 1,\\
& b_i(x,t) < 0, i = 1,2,\ldots,n, \text{ where } m \ge 2.\\
\text{Case}(ii):\ & 0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_m < \varepsilon_{m+1} < \cdots < \varepsilon_n,\\
& b_i(x,t) < 0, \text{ for } i = 1,2,\ldots,m \text{ and } b_i(x,t) = 0, \text{ for } i = m+1, m+2,\ldots,n,\\
& \text{where } m \ge 1.\\
\text{Case}(iii):\ & 0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_m = \varepsilon_{m+1} = \cdots = \varepsilon_k < \varepsilon_{k+1} = \cdots = \varepsilon_n = 1,\\
& b_i(x,t) < 0, \text{ for } i = 1,2,\ldots,k \text{ and } b_i(x,t) = 0, \text{ for } i = k+1, k+2,\ldots,n,\\
& \text{where } k \ge 1 \text{ and } m \ge 2.
\end{aligned}$$

The reduced problem corresponding to (4)–(5) for Case *(i)* is defined by

$$
\left.\begin{aligned}
&\text{for } i=1,2,\ldots,k\\
&\frac{\partial u_{0i}}{\partial t}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)u_{0j}(x,t)=f_i(x,t)-b_i(x,t)\chi_i(x-1,t),\\
&\text{for } i=k+1,k+2,\ldots,n\\
&\frac{\partial u_{0i}}{\partial t}(x,t)-\frac{\partial^2 u_{0i}}{\partial x^2}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)u_{0j}(x,t)\\
&\qquad\qquad\qquad\qquad=f_i(x,t)-b_i(x,t)\chi_i(x-1,t),
\end{aligned}\right\}\text{ on }(0,1)\times(0,T]
$$

$$
\left.\begin{aligned}
&\text{for } i=1,2,\ldots,k\\
&\frac{\partial u_{0i}}{\partial t}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)u_{0j}(x,t)+b_i(x,t)u_{0j}(x-1,t)=f_i(x,t),\\
&\text{for } i=k+1,k+2,\ldots,n\\
&\frac{\partial u_{0i}}{\partial t}(x,t)-\frac{\partial^2 u_{0i}}{\partial x^2}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)u_{0j}(x,t)\\
&\qquad\qquad\qquad+b_i(x,t)u_{0j}(x-1,t)=f_i(x,t),
\end{aligned}\right\}\text{ on }(1,2)\times(0,T]. \tag{6}
$$

The reduced problem corresponding to (4)–(5) for Case *(ii)* is defined by

$$
\left.\begin{aligned}
&\text{for } i=1,2,\ldots,m\\
&\frac{\partial u_{0i}}{\partial t}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)u_{0j}(x,t)=f_i(x,t)-b_i(x,t)\chi_i(x-1,t),\\
&\text{for } i=m+1,m+2,\ldots,n\\
&\frac{\partial u_{0i}}{\partial t}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)u_{0j}(x,t)=f_i(x,t),
\end{aligned}\right\}\text{ on }(0,1)\times(0,T]
$$

$$
\left.\begin{aligned}
&\text{for } i=1,2,\ldots,m\\
&\frac{\partial u_{0i}}{\partial t}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)u_{0j}(x,t)+b_i(x,t)u_{0j}(x-1,t)=f_i(x,t),\\
&\text{for } i=m+1,m+2,\ldots,n\\
&\frac{\partial u_{0i}}{\partial t}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)u_{0j}(x,t)=f_i(x,t),
\end{aligned}\right\}\text{ on }(1,2)\times(0,T]. \tag{7}
$$

The reduced problem corresponding to (4)–(5) for Case *(iii)* is defined by

$$
\left.\begin{aligned}
&\text{for } i=1,2,\ldots,k\\
&\frac{\partial u_{0i}}{\partial t}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)u_{0j}(x,t)=f_i(x,t)-b_i(x,t)\chi_i(x-1,t),\\
&\text{for } i=k+1,k+2,\ldots,n\\
&\frac{\partial u_{0i}}{\partial t}(x,t)-\frac{\partial^2 u_{0i}}{\partial x^2}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)u_{0j}(x,t)=f_i(x,t),
\end{aligned}\right\}\text{ on }(0,1)\times(0,T]
$$

$$\left.\begin{array}{l} \text{for } i=1,2,\ldots,k \\ \dfrac{\partial u_{0i}}{\partial t}(x,t)+\displaystyle\sum_{j=1}^{n} a_{ij}(x,t)u_{0j}(x,t)+b_i(x,t)u_{0j}(x-1,t)=f_i(x,t), \\ \text{for } i=k+1,k+2,\ldots,n \\ \dfrac{\partial u_{0i}}{\partial t}(x,t)-\dfrac{\partial^2 u_{0i}}{\partial x^2}(x,t)+\displaystyle\sum_{j=1}^{n} a_{ij}(x,t)u_{0j}(x,t)=f_i(x,t), \end{array}\right\} \text{on } (1,2)\times(0,T] \quad (8)$$

The solution $\mathbf{u}(x,t)$ has the following layer pattern. In cases *(i)* and *(iii)*, each component $u_i(x,t)$ for $i=1,2,\ldots,k$ exhibits twin boundary layers of width $O(\sqrt{\varepsilon_k})$ at $x=0$ and $x=2$ and twin interior layers of width $O(\sqrt{\varepsilon_k})$ at $x=1$, while the components u_i for $i=1,2,\ldots,k-1$ have additional twin boundary layers of width $O(\sqrt{\varepsilon_{k-1}})$ at $x=0$ and $x=2$ and twin interior layers of width $O(\sqrt{\varepsilon_{k-1}})$ at $x=1$, and so on.

In Case *(ii)*, each component $u_i(x,t)$ for $i=1,2,\ldots,m$ exhibits twin boundary layers of width $O(\sqrt{\varepsilon_m})$ at $x=0$ and $x=2$ and twin interior layers of width $O(\sqrt{\varepsilon_m})$ at $x=1$, while the components u_i for $i=1,2,\ldots,m-1$ have additional twin boundary layers of width $O(\sqrt{\varepsilon_{m-1}})$ at $x=0$ and $x=2$ and twin interior layers of width $O(\sqrt{\varepsilon_{m-1}})$ at $x=1$, and so on. Each component $u_i(x,t)$ for $i=m+1,m+2,\ldots,n$ exhibits twin boundary layers of width $O(\sqrt{\varepsilon_n})$ at $x=0$ and $x=2$, while the components u_i for $i=m+1,m+2,\ldots,n-1$ have additional twin boundary layers of width $O(\sqrt{\varepsilon_{n-1}})$ at $x=0$ and $x=2$ and so on.

The compatibility conditions for Γ corners $(0,0)$ and $(2,0)$ are derived using similar reasons as in [7] Theorem 2.1.

Then there exists a unique solution $\mathbf{u}(x,t)$ of (1) satisfying $\mathbf{u}(x,t)\in\mathscr{C}=C^0_\lambda([0,2]\times[0,T])\cap C^1_\lambda((0,2)\times(0,T])\cap C^4_\lambda(\tilde{\overline{\Omega}})$.

3 Analytical Results

This section includes maximum principle, stability result, and derivative estimations for Problem (4)–(5).

Lemma 1 *Let Conditions (2) and (3) hold. Let* $\boldsymbol{\psi}=(\psi_1,\psi_2,\ldots,\psi_n)^T$ *be any function in* $\mathscr{C}$ *such that* $\boldsymbol{\psi}(x,t)\geq\mathbf{0}$ *on* Γ. $\mathbf{L_1}\boldsymbol{\psi}(x,t)\geq\mathbf{0}$ *on* $(0,1)\times(0,T]$, $\mathbf{L_2}\boldsymbol{\psi}(x,t)\geq\ \mathbf{0}$ *on* $(1,2)\times(0,T]$ *and* $[\boldsymbol{\psi}](1,t)=\mathbf{0}$, $[\frac{\partial\boldsymbol{\psi}}{\partial x}](1,t)\leq\mathbf{0}$ *then* $\boldsymbol{\psi}(x,t)\geq\mathbf{0}$ *on* $[0,2]\times[0,T]$.

Proof It is simple to show using reasons similar to those used in Lemma 3.1 of [7]. Let $\tilde{A}(x,t)$ be any principal matrix of $A(x,t)$ and $\tilde{\mathbf{L}}$ the corresponding operator. To see that any $\tilde{\mathbf{L}}$ satisfies the same maximum principle as $\mathbf{L}$, it suffices to observe that the elements of $\tilde{A}(x,t)$ satisfy a fortiori the same inequalities as those of $A(x,t)$.

As a result of the maximal principle, the following stability result for Problem (1) has been established.

Lemma 2 *Let Conditions (2) and (3) hold. Let* $\boldsymbol{\psi}$ *be any function in* $\mathscr{C}$, *such that* $[\boldsymbol{\psi}](1,t) = \mathbf{0}$ *and* $\left[\frac{\partial \boldsymbol{\psi}}{\partial x}\right](1,t) = \mathbf{0}$, *then for each* $i = 1, 2, \ldots, n$ *and* $(x,t) \in [0,2] \times [0,T]$,

$$|\psi_i(x,t)| \leq \max\left\{\| \boldsymbol{\psi} \|_\Gamma, \frac{1}{\alpha} \| \mathbf{L_1}\boldsymbol{\psi} \|, \frac{1}{\alpha} \| \mathbf{L_2}\boldsymbol{\psi} \|\right\}.$$

Proof The proof proceeds in the same manner as [7] Lemma 3.2.

The following Lemma contains standard estimates of the solution of (1) and its derivatives.

Lemma 3 *Let Conditions (2) and (3) hold. Let* $\mathbf{u}$ *be the solution of (1). Then for all* $(x,t) \in [0,2] \times [0,T]$ *and for cases (i) and (iii) and for i = 1, 2, ..., k*

$|\frac{\partial^k u_i}{\partial t^k}(x,t)| \leq C(||\mathbf{u}||_\Gamma + \sum_{q=0}^{k} ||\frac{\partial^q \mathbf{f}}{\partial t^q}||), k = 0, 1, 2$

$|\frac{\partial^k u_i}{\partial x^k}(x,t)| \leq C\varepsilon_i^{\frac{-k}{2}}(||\mathbf{u}||_\Gamma + ||\mathbf{f}|| + ||\frac{\partial \mathbf{f}}{\partial t}||), k = 1, 2$

$|\frac{\partial^k u_i}{\partial x^k}(x,t)| \leq C\varepsilon_i^{-1}\varepsilon_1^{\frac{-(k-2)}{2}}(||\mathbf{u}||_\Gamma + ||\mathbf{f}|| + ||\frac{\partial \mathbf{f}}{\partial t}|| + ||\frac{\partial^2 \mathbf{f}}{\partial t^2}|| + \varepsilon_1^{\frac{k-2}{2}}||\frac{\partial^{k-2} \mathbf{f}}{\partial x^{k-2}}||), k = 3, 4$

$|\frac{\partial^k u_i}{\partial x^{k-1}\partial t}(x,t)| \leq C\varepsilon_i^{\frac{1-k}{2}}(||\mathbf{u}||_\Gamma + ||\mathbf{f}|| + ||\frac{\partial \mathbf{f}}{\partial t}|| + ||\frac{\partial^2 \mathbf{f}}{\partial t^2}||), k = 2, 3.$

and for i = k + 1, k + 2, ..., n,

$|\frac{\partial^k u_i}{\partial t^k}(x,t)| \leq C, k = 0, 1, 2$

$|\frac{\partial^k u_i}{\partial x^k}(x,t)| \leq C, k = 1, 2$

$|\frac{\partial^k u_i}{\partial x^k}(x,t)| \leq C\varepsilon_i^{-1}\varepsilon_1^{\frac{-(k-2)}{2}}(||\mathbf{u}||_\Gamma + ||\mathbf{f}|| + ||\frac{\partial \mathbf{f}}{\partial t}|| + ||\frac{\partial^2 \mathbf{f}}{\partial t^2}|| + \varepsilon_1^{\frac{k-2}{2}}||\frac{\partial^{k-2} \mathbf{f}}{\partial x^{k-2}}||), k = 3, 4$

$|\frac{\partial^k u_i}{\partial x^{k-1}\partial t}(x,t)| \leq C\varepsilon_i^{\frac{1-k}{2}}(||\mathbf{u}||_\Gamma + ||\mathbf{f}|| + ||\frac{\partial \mathbf{f}}{\partial t}|| + ||\frac{\partial^2 \mathbf{f}}{\partial t^2}||), k = 2, 3.$

and for Case (ii), $i = 1, 2, \ldots, n$

$|\frac{\partial^k u_i}{\partial t^k}(x,t)| \leq C(||\mathbf{u}||_\Gamma + \sum_{q=0}^{k} ||\frac{\partial^q \mathbf{f}}{\partial t^q}||), k = 0, 1, 2$

$|\frac{\partial^k u_i}{\partial x^k}(x,t)| \leq C\varepsilon_i^{\frac{-k}{2}}(||\mathbf{u}||_\Gamma + ||\mathbf{f}|| + ||\frac{\partial \mathbf{f}}{\partial t}||), k = 1, 2$

$|\frac{\partial^k u_i}{\partial x^k}(x,t)| \leq C\varepsilon_i^{-1}\varepsilon_1^{\frac{-(k-2)}{2}}(||\mathbf{u}||_\Gamma + ||\mathbf{f}|| + ||\frac{\partial \mathbf{f}}{\partial t}|| + ||\frac{\partial^2 \mathbf{f}}{\partial t^2}|| + \varepsilon_1^{\frac{k-2}{2}}||\frac{\partial^{k-2} \mathbf{f}}{\partial x^{k-2}}||), k = 3, 4$

$|\frac{\partial^k u_i}{\partial x^{k-1}\partial t}(x,t)| \leq C\varepsilon_i^{\frac{1-k}{2}}(||\mathbf{u}||_\Gamma + ||\mathbf{f}|| + ||\frac{\partial \mathbf{f}}{\partial t}|| + ||\frac{\partial^2 \mathbf{f}}{\partial t^2}||), k = 2, 3.$

Proof It is simple to show the results using reasoning similar to those in [7] Lemma 3.3.

The Shishkin decomposition of the exact solution $\mathbf{u}$ of (1) is $\mathbf{u} = \mathbf{v} + \mathbf{w}$, where the smooth component $\mathbf{v}$ and the singular component $\mathbf{w}$ are the solutions of the following systems related to the three cases.

For Case *(i)* :
For $i = 1, 2, \ldots, k$

$$\left.\begin{array}{l} \frac{\partial v_i}{\partial t}(x,t) - \varepsilon_i \frac{\partial^2 v_i}{\partial x^2}(x,t) + \sum_{j=1}^n a_{ij}(x,t)v_i(x,t) \\ \qquad\qquad = f_i(x,t) - b_i(x,t)\chi_i(x-1,t), \\ \text{with } v_i(0,t) = u_{0i}(0,t), v_i(x,0) = u_{0i}(x,0), v_i(1-,t) = u_{0i}(1-,t). \\ \frac{\partial w_i}{\partial t}(x,t) - \varepsilon_i \frac{\partial^2 w_i}{\partial x^2}(x,t) + \sum_{j=1}^n a_{ij}(x,t)w_i(x,t) = 0, \\ \text{with } w_i(0,t) = u_i(0,t) - v_i(0,t), [w_i](1,t) = -[v_i](1,t), \\ [\frac{\partial w_i}{\partial x}](1,t) = -[\frac{\partial v_i}{\partial x}](1,t). \end{array}\right\} \text{ on } (0,1)\times(0,T] \quad (9)$$

$$\left.\begin{array}{l} \frac{\partial v_i}{\partial t}(x,t) - \varepsilon_i \frac{\partial^2 v_i}{\partial x^2}(x,t) + \sum_{j=1}^n a_{ij}(x,t)v_i(x,t) + b_i(x,t)v_i(x-1,t) \\ \qquad\qquad = f_i(x,t), \\ \text{with } v_i(2,t) = u_{0i}(2,t), v_i(x,0) = u_{0i}(x,0), v_i(1+,t) = u_{0i}(1+,t). \\ \frac{\partial w_i}{\partial t}(x,t) - \varepsilon_i \frac{\partial^2 w_i}{\partial x^2}(x,t) + \sum_{i=1}^n a_{ij}(x,t)w_i(x,t) + b_i(x,t)w_i(x-1,t) = 0, \\ \text{with } [w_i](1,t) = -[v_i](1,t), [\frac{\partial w_i}{\partial x}](1,t) = -[\frac{\partial v_i}{\partial x}](1,t), \\ w_i(2,t) = u_i(2,t) - v_i(2,t). \end{array}\right\} \text{ on } (1,2)\times(0,T] \quad (10)$$

For $i = k+1, k+2, \ldots, n$

$$\left.\begin{array}{l} \frac{\partial v_i}{\partial t}(x,t) - \frac{\partial^2 v_i}{\partial x^2}(x,t) + \sum_{j=1}^n a_{ij}(x,t)v_i(x,t) \\ \qquad\qquad = f_i(x,t) - b_i(x,t)\chi_i(x-1,t), \\ \text{with } v_i(0,t) = u_{0i}(0,t), v_i(x,0) = u_{0i}(x,0), v_i(1-,t) = u_{0i}(1-,t). \\ \frac{\partial w_i}{\partial t}(x,t) - \frac{\partial^2 w_i}{\partial x^2}(x,t) + \sum_{j=1}^n a_{ij}(x,t)w_i(x,t) = 0, \\ \text{with } w_i(0,t) = u_i(0,t) - v_i(0,t), [w_i](1,t) = -[v_i](1,t), \\ [\frac{\partial w_i}{\partial x}](1,t) = -[\frac{\partial v_i}{\partial x}](1,t). \end{array}\right\} \text{ on } (0,1)\times(0,T] \quad (11)$$

$$\left.\begin{array}{l} \frac{\partial v_i}{\partial t}(x,t) - \frac{\partial^2 v_i}{\partial x^2}(x,t) + \sum_{j=1}^n a_{ij}(x,t)v_i(x,t) + b_i(x,t)v_i(x-1,t) \\ \qquad\qquad = f_i(x,t), \\ \text{with } v_i(2,t) = u_{0i}(2,t), v_i(x,0) = u_{0i}(x,0), v_i(1+,t) = u_{0i}(1+,t). \\ \frac{\partial w_i}{\partial t}(x,t) - \frac{\partial^2 w_i}{\partial x^2}(x,t) + \sum_{i=1}^n a_{ij}(x,t)w_i(x,t) + b_i(x,t)w_i(x-1,t) = 0, \\ \text{with } [w_i](1,t) = -[v_i](1,t), [\frac{\partial w_i}{\partial x}](1,t) = -[\frac{\partial v_i}{\partial x}](1,t), \\ w_i(2,t) = u_i(2,t) - v_i(2,t). \end{array}\right\} \text{ on } (1,2)\times(0,T]. \quad (12)$$

For Case *(ii)*: For $i = 1, 2, \ldots, m$

$$\left.\begin{array}{l} \frac{\partial v_i}{\partial t}(x,t) - \varepsilon_i \frac{\partial^2 v_i}{\partial x^2}(x,t) + \sum_{j=1}^n a_{ij}(x,t)v_i(x,t) \\ \qquad\qquad = f_i(x,t) - b_i(x,t)\chi_i(x-1,t), \\ \text{with } v_i(0,t) = u_{0i}(0,t), v_i(x,0) = u_{0i}(x,0), v_i(1-,t) = u_{0i}(1-,t). \\ \frac{\partial w_i}{\partial t}(x,t) - \varepsilon_i \frac{\partial^2 w_i}{\partial x^2}(x,t) + \sum_{j=1}^n a_{ij}(x,t)w_i(x,t) = 0, \\ \text{with } w_i(0,t) = u_i(0,t) - v_i(0,t), [w_i](1,t) = -[v_i](1,t), \\ [\frac{\partial w_i}{\partial x}](1,t) = -[\frac{\partial v_i}{\partial x}](1,t). \end{array}\right\} \text{ on } (0,1)\times(0,T] \quad (13)$$

$$\left.\begin{array}{l}\frac{\partial v_i}{\partial t}(x,t)-\varepsilon_i\frac{\partial^2 v_i}{\partial x^2}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)v_i(x,t)+b_i(x,t)v_i(x-1,t)\\ \qquad =f_i(x,t),\\ \text{with } v_i(2,t)=u_{0i}(2,t),v_i(x,0)=u_{0i}(x,0),v_i(1+,t)=u_{0i}(1+,t).\\ \frac{\partial w_i}{\partial t}(x,t)-\varepsilon_i\frac{\partial^2 w_i}{\partial x^2}(x,t)+\sum_{i=1}^{n}a_{ij}(x,t)w_i(x,t)+b_i(x,t)w_i(x-1,t)=0,\\ \text{with } [w_i](1,t)=-[v_i](1,t),[\frac{\partial w_i}{\partial x}](1,t)=-[\frac{\partial v_i}{\partial x}](1,t),\\ w_i(2,t)=u_i(2,t)-v_i(2,t).\end{array}\right\}\text{ on } (1,2)\times(0,T]. \tag{14}$$

For $i=m+1,m+2,\ldots,n$

$$\left.\begin{array}{l}\frac{\partial v_i}{\partial t}(x,t)-\varepsilon_i\frac{\partial^2 v_i}{\partial x^2}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)v_i(x,t)=f_i(x,t),\\ \text{with } v_i(0,t)=u_{0i}(0,t),v_i(x,0)=u_{0i}(x,0),v_i(2,t)=u_{0i}(2,t).\\ \frac{\partial w_i}{\partial t}(x,t)-\varepsilon_i\frac{\partial^2 w_i}{\partial x^2}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)w_i(x,t)=0,\\ \text{with } w_i(0,t)=u_i(0,t)-v_i(0,t),w_i(x,0)=0,\\ w_i(2,t)=u_i(2,t)-v_i(2,t).\end{array}\right\}\text{ on } (0,2)\times(0,T]. \tag{15}$$

For Case *(iii)* : For $i=1,2,\ldots,k$

$$\left.\begin{array}{l}\frac{\partial v_i}{\partial t}(x,t)-\varepsilon_i\frac{\partial^2 v_i}{\partial x^2}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)v_i(x,t)\\ \qquad =f_i(x,t)-b_i(x,t)\chi_i(x-1,t),\\ \text{with } v_i(0,t)=u_{0i}(0,t),v_i(x,0)=u_{0i}(x,0),v_i(1-,t)=u_{0i}(1-,t).\\ \frac{\partial w_i}{\partial t}(x,t)-\varepsilon_i\frac{\partial^2 w_i}{\partial x^2}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)w_i(x,t)=0,\\ \text{with } w_i(0,t)=u_i(0,t)-v_i(0,t),[w_i](1,t)=-[v_i](1,t),\\ [\frac{\partial w_i}{\partial x}](1,t)=-[\frac{\partial v_i}{\partial x}](1,t).\end{array}\right\}\text{ on } (0,1)\times(0,T] \tag{16}$$

$$\left.\begin{array}{l}\frac{\partial v_i}{\partial t}(x,t)-\varepsilon_i\frac{\partial^2 v_i}{\partial x^2}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)v_i(x,t)+b_i(x,t)v_i(x-1,t)\\ \qquad =f_i(x,t),\\ \text{with } v_i(2,t)=u_{0i}(2,t),v_i(x,0)=u_{0i}(x,0),v_i(1+,t)=u_{0i}(1+,t).\\ \frac{\partial w_i}{\partial t}(x,t)-\varepsilon_i\frac{\partial^2 w_i}{\partial x^2}(x,t)+\sum_{i=1}^{n}a_{ij}(x,t)w_i(x,t)+b_i(x,t)w_i(x-1,t)=0,\\ \text{with } [w_i](1,t)=-[v_i](1,t),[\frac{\partial w_i}{\partial x}](1,t)=-[\frac{\partial v_i}{\partial x}](1,t),\\ w_i(2,t)=u_i(2,t)-v_i(2,t).\end{array}\right\}\text{ on } (1,2)\times(0,T] \tag{17}$$

For $i=k+1,k+2,\ldots,n$

$$\left.\begin{array}{l}\frac{\partial v_i}{\partial t}(x,t)-\frac{\partial^2 v_i}{\partial x^2}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)v_i(x,t)=f_i(x,t),\\ \text{with } v_i(0,t)=u_{0i}(0,t),v_i(x,0)=u_{0i}(x,0),v_i(2,t)=u_{0i}(2,t).\\ \frac{\partial w_i}{\partial t}(x,t)-\frac{\partial^2 w_i}{\partial x^2}(x,t)+\sum_{j=1}^{n}a_{ij}(x,t)w_i(x,t)=0,\\ \text{with } w_i(0,t)=u_i(0,t)-v_i(0,t),w_i(x,0)=0,\\ w_i(2,t)=u_i(2,t)-v_i(2,t).\end{array}\right\}\text{ on } (0,2)\times(0,T] \tag{18}$$

The singular component is given a further decomposition

$$\mathbf{w}(x,t)=\tilde{\mathbf{w}}(x,t)+\hat{\mathbf{w}}(x,t), \tag{19}$$

where $\tilde{\mathbf{w}}$ is the solution of

$$\frac{\partial \tilde{w}_i}{\partial t}(x,t) - \varepsilon_i \frac{\partial^2 \tilde{w}_i}{\partial x^2}(x,t) + \sum_{j=1}^{n} a_{ij}(x,t)\tilde{w}_i(x,t) = 0, \text{ on } (0,1)\times(0,T],$$
$$\tilde{w}_i(0,t) = w_i(0,t), \tilde{w}_i(1,t) = K_1, \tilde{w}_i(x,0) = w_i(x,0), \tilde{\mathbf{w}} = \mathbf{0} \text{ on } (1,2)\times(0,T] \tag{20}$$

and $\hat{\mathbf{w}}$ is the solution of

$$\frac{\partial \hat{w}_i}{\partial t}(x,t) - \varepsilon_i \frac{\partial^2 \hat{w}_i}{\partial x^2}(x,t) + \sum_{j=1}^{n} a_{ij}(x,t)\hat{w}_i(x,t) + b_i(x,t)\hat{w}_i(x-1,t) = 0, \text{ on } (1,2)\times(0,T],$$
$$\hat{w}_i(1,t) = K_2, \hat{w}_i(2,t) = w_i(2,t), \hat{w}_i(x,0) = w_i(x,0), \hat{\mathbf{w}} = \mathbf{0} \text{ on } (0,1)\times(0,T] \tag{21}$$

for $i = 1, 2, \ldots, k,$ in Case *(i)* and $\tilde{\mathbf{w}}$ is the solution of

$$\frac{\partial \tilde{w}_i}{\partial t}(x,t) - \frac{\partial^2 \tilde{w}_i}{\partial x^2}(x,t) + \sum_{j=1}^{n} a_{ij}(x,t)\tilde{w}_i(x,t) = 0, \text{ on } (0,1)\times(0,T],$$
$$\tilde{w}_i(0,t) = w_i(0,t), \tilde{w}_i(1,t) = K_3, \tilde{w}_i(x,0) = w_i(x,0), \tilde{\mathbf{w}} = \mathbf{0} \text{ on } (1,2)\times(0,T] \tag{22}$$

and $\hat{\mathbf{w}}$ is the solution of

$$\frac{\partial \hat{w}_i}{\partial t}(x,t) - \frac{\partial^2 \hat{w}_i}{\partial x^2}(x,t) + \sum_{j=1}^{n} a_{ij}(x,t)\hat{w}_i(x,t) + b_i(x,t)\hat{w}_i(x-1,t) = 0, \text{ on } (1,2)\times(0,T],$$
$$\hat{w}_i(1,t) = K_4, \hat{w}_i(2,t) = w_i(2,t), \hat{w}_i(x,0) = w_i(x,0), \hat{\mathbf{w}} = \mathbf{0} \text{ on } (0,1)\times(0,T] \tag{23}$$

for $i = k+1, k+2, \ldots, n,$ in Case *(i)* and $\tilde{\mathbf{w}}$ is the solution of

$$\frac{\partial \tilde{w}_i}{\partial t}(x,t) - \varepsilon_i \frac{\partial^2 \tilde{w}_i}{\partial x^2}(x,t) + \sum_{j=1}^{n} a_{ij}(x,t)\tilde{w}_i(x,t) = 0, \text{ on } (0,1)\times(0,T],$$
$$\tilde{w}_i(0,t) = w_i(0,t), \tilde{w}_i(1,t) = K_5, \tilde{w}_i(x,0) = w_i(x,0), \tilde{\mathbf{w}} = \mathbf{0} \text{ on } (1,2)\times(0,T] \tag{24}$$

and $\hat{\mathbf{w}}$ is the solution of

$$\frac{\partial \hat{w}_i}{\partial t}(x,t) - \varepsilon_i \frac{\partial^2 \hat{w}_i}{\partial x^2}(x,t) + \sum_{j=1}^{n} a_{ij}(x,t)\hat{w}_i(x,t) + b_i(x,t)\hat{w}_i(x-1,t) = 0, \text{ on } (1,2)\times(0,T],$$
$$\hat{w}_i(1,t) = K_6, \hat{w}_i(2,t) = w_i(2,t), \hat{w}_i(x,0) = w_i(x,0), \hat{\mathbf{w}} = \mathbf{0} \text{ on } (0,1)\times(0,T] \tag{25}$$

for $i = 1, 2, \ldots, m,$ in Case *(ii)* and $\tilde{\mathbf{w}}$ is the solution of

$$\frac{\partial \tilde{w}_i}{\partial t}(x,t) - \varepsilon_i \frac{\partial^2 \tilde{w}_i}{\partial x^2}(x,t) + \sum_{j=1}^{n} a_{ij}(x,t)\tilde{w}_i(x,t) = 0, \text{ on } (0,1)\times(0,T],$$
$$\tilde{w}_i(0,t) = w_i(0,t), \tilde{w}_i(1,t) = K_7, \tilde{w}_i(x,0) = w_i(x,0), \tilde{\mathbf{w}} = \mathbf{0} \text{ on } (1,2)\times(0,T] \tag{26}$$

and $\hat{\mathbf{w}}$ is the solution of

$$\frac{\partial \hat{w}_i}{\partial t}(x,t) - \varepsilon_i \frac{\partial^2 \hat{w}_i}{\partial x^2}(x,t) + \sum_{j=1}^{n} a_{ij}(x,t)\hat{w}_i(x,t) = 0, \text{ on } (1,2)\times(0,T],$$
$$\hat{w}_i(1,t) = K_8, \hat{w}_i(2,t) = w_i(2,t), \hat{w}_i(x,0) = w_i(x,0), \hat{\mathbf{w}} = \mathbf{0} \text{ on } (0,1)\times(0,T] \quad (27)$$

for $i = m+1, m+2, \ldots, n$, in Case *(ii)* and $\tilde{\mathbf{w}}$ is the solution of

$$\frac{\partial \tilde{w}_i}{\partial t}(x,t) - \varepsilon_i \frac{\partial^2 \tilde{w}_i}{\partial x^2}(x,t) + \sum_{j=1}^{n} a_{ij}(x,t)\tilde{w}_i(x,t) = 0, \text{ on } (0,1)\times(0,T],$$
$$\tilde{w}_i(0,t) = w_i(0,t), \tilde{w}_i(1,t) = K_9, \tilde{w}_i(x,0) = w_i(x,0), \tilde{\mathbf{w}} = \mathbf{0} \text{ on } (1,2)\times(0,T] \quad (28)$$

and $\hat{\mathbf{w}}$ is the solution of

$$\frac{\partial \hat{w}_i}{\partial t}(x,t) - \varepsilon_i \frac{\partial^2 \hat{w}_i}{\partial x^2}(x,t) + \sum_{j=1}^{n} a_{ij}(x,t)\hat{w}_i(x,t) + b_i(x,t)\hat{w}_i(x-1,t) = 0, \text{ on } (1,2)\times(0,T],$$
$$\hat{w}_i(1,t) = K_{10}, \hat{w}_i(2,t) = w_i(2,t), \hat{w}_i(x,0) = w_i(x,0), \hat{\mathbf{w}} = \mathbf{0} \text{ on } (0,1)\times(0,T] \quad (29)$$

for $i = 1, 2, \ldots, k$, in Case *(iii)* and $\tilde{\mathbf{w}}$ is the solution of

$$\frac{\partial \tilde{w}_i}{\partial t}(x,t) - \frac{\partial^2 \tilde{w}_i}{\partial x^2}(x,t) + \sum_{j=1}^{n} a_{ij}(x,t)\tilde{w}_i(x,t) = 0, \text{ on } (0,1)\times(0,T],$$
$$\tilde{w}_i(0,t) = w_i(0,t), \tilde{w}_i(1,t) = K_{11}, \tilde{w}_i(x,0) = w_i(x,0), \tilde{\mathbf{w}} = \mathbf{0} \text{ on } (1,2)\times(0,T] \quad (30)$$

and $\hat{\mathbf{w}}$ is the solution of

$$\frac{\partial \hat{w}_i}{\partial t}(x,t) - \frac{\partial^2 \hat{w}_i}{\partial x^2}(x,t) + \sum_{j=1}^{n} a_{ij}(x,t)\hat{w}_i(x,t) = 0, \text{ on } (1,2)\times(0,T],$$
$$\hat{w}_i(1,t) = K_{12}, \hat{w}_i(2,t) = w_i(2,t), \hat{w}_i(x,0) = w_i(x,0), \hat{\mathbf{w}} = \mathbf{0} \text{ on } (0,1)\times(0,T] \quad (31)$$

for $i = k+1, k+2, \ldots, n$, in Case *(iii)*. Here, $Ki, i = 1, 2, \ldots, 12$, are constants that must be chosen in order to satisfy the jump conditions at $x = 1$. The following Lemma contains bounds on the smooth component and its derivatives.

Lemma 4 *Let Conditions (2) and (3) hold. The smooth component* $\mathbf{v}$ *and its derivatives satisfy, for each* $(x,t) \in [0,2]\times[0,T]$ *and for cases (i) and (iii) and for* $i = 1, 2, \ldots, k$,

$$|\frac{\partial^k v_i}{\partial t^k}(x,t)| \le C, \text{ for } k = 0,1,2$$
$$|\frac{\partial^k v_i}{\partial x^k}(x,t)| \le C(1+\varepsilon_i^{1-\frac{k}{2}}), \text{ for } k = 0,1,2,3,4$$
$$|\frac{\partial^k v_i}{\partial x^{k-1}\partial t}(x,t)| \le C, \text{ for } k = 2,3.$$

and for $i = k+1, k+2, \ldots, n$,

$$|\frac{\partial^k v_i}{\partial t^k}(x,t)| \le C, \text{ for } k = 0,1,2$$

$|\frac{\partial^k v_i}{\partial x^k}(x,t)| \le C,\ \text{for } k = 0, 1, 2, 3, 4$
$|\frac{\partial^k v_i}{\partial x^{k-1}\partial t}(x,t)| \le C,\ \text{for } k = 2, 3.$
and for Case (ii), i = 1, 2, ..., n
$|\frac{\partial^k v_i}{\partial t^k}(x,t)| \le C,\ \text{for } k = 0, 1, 2$
$|\frac{\partial^k v_i}{\partial x^k}(x,t)| \le C(1+\varepsilon_i^{1-\frac{k}{2}}),\ \text{for } k = 0, 1, 2, 3, 4$
$|\frac{\partial^k v_i}{\partial x^{k-1}\partial t}(x,t)| \le C,\ \text{for } k = 2, 3.$

Proof The results are obtained by applying the same arguments as in [7] Lemma 3.4.

The layer functions $B^L_{1,i}, B^R_{1,i}, B^L_{2,i}, B^R_{2,i}, B_{1,i}, B_{2,i}, i = 1, 2, \ldots, n$ associated with the solution $\mathbf{u}$, are defined by

$$B^L_{1,i}(x) = e^{-x\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},\ B^R_{1,i}(x) = e^{-(1-x)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},\ B_{1,i}(x) = B^L_{1,i}(x) + B^R_{1,i}(x),\ \text{on } [0,1]\times[0,T],$$
$$B^L_{2,i}(x) = e^{-(x-1)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},\ B^R_{2,i}(x) = e^{-(2-x)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},\ B_{2,i}(x) = B^L_{2,i}(x) + B^R_{2,i}(x),\ \text{on } [1,2]\times[0,T].$$

It has to be noted that for $i = 1, 2, \ldots, n$, $B_{1,i}(x-1) = B_{2,i}(x)$ for $x \in [1, 2]$

Definition 1 For $B^L_{1,i}, B^L_{2,j}$, let $x^{(s)}_{i,j}$, $1 \le i \ne j \le n$, $s > 0$ be the point defined by $\frac{B^L_{1,i}(x^{(s)}_{i,j})}{\varepsilon_i} = \frac{B^L_{1,j}(x^{(s)}_{i,j})}{\varepsilon_j}$. Then $\frac{B^R_{1,i}(1-x^{(s)}_{i,j})}{\varepsilon_i} = \frac{B^R_{1,j}(1-x^{(s)}_{i,j})}{\varepsilon_j}$, $\frac{B^L_{2,i}(1+x^{(s)}_{i,j})}{\varepsilon_i} = \frac{B^L_{2,j}(1+x^{(s)}_{i,j})}{\varepsilon_j}$ and $\frac{B^R_{2,i}(2-x^{(s)}_{i,j})}{\varepsilon_i} = \frac{B^R_{2,j}(2-x^{(s)}_{i,j})}{\varepsilon_j}$.

Reference [2] can be used to verify the existence, uniqueness, and properties of $x^{(s)}_{i,j}$. The following Lemma contains bounds on the singular component $\mathbf{w}$ of $\mathbf{u}(x,t)$ and its derivatives.

Lemma 5 *Let Conditions (2) and (3) hold. Then for proper choices of the constants* C_1, C_2, C_3, C_4, *for cases (i) and (iii) and for* $(x,t) \in [0,1]\times[0,T]$ *and for* $i = 1, 2, \ldots, m$
$|\frac{\partial^k w_i}{\partial t^k}(x,t)| \le C_1 B_{1,m}(x) + C_2\varepsilon_m(1 - B_{1,m}(x)),\ \text{for } k = 0, 1, 2$
$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \le C\sum_{q=i}^{m} \varepsilon_q^{\frac{-k}{2}} B_{1,q}(x),\ \text{for } k = 1, 2,$
$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \le C\sum_{q=1}^{m} \varepsilon_q^{\frac{-k}{2}} B_{1,q}(x),\ \text{for } k = 3,$
$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \le C\varepsilon_i^{-1}\sum_{q=1}^{m} \varepsilon_q^{-1} B_{1,q}(x),\ \text{for } k = 4,$
and for $i = m+1, m+2, \ldots, k$
$|\frac{\partial^k w_i}{\partial t^k}(x,t)| \le C_1 B_{1,m}(x) + C_2\varepsilon_m(1 - B_{1,m}(x)),\ \text{for } k = 0, 1, 2$

$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C\varepsilon_m^{\frac{-k}{2}} B_{1,m}(x),\ \text{for } k = 1, 2,$
$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C \sum_{q=1}^{m} \varepsilon_q^{\frac{-k}{2}} B_{1,q}(x),\ \text{for } k = 3,$
$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C\varepsilon_m^{-1} \sum_{q=1}^{m} \varepsilon_q^{-1} B_{1,q}(x),\ \text{for } k = 4,$

and for $i = k+1, k+2, \ldots, n$
$|\frac{\partial^k w_i}{\partial t^k}(x,t)| \leq C_2\varepsilon_m(1 - B_{1,m}(x)),\ \text{for } k = 0, 1, 2$
$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C_1 B_{1,m}(x) + C_2\varepsilon_m(1 - B_{1,m}(x)),\ \text{for } k = 1, 2,$
$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C \sum_{q=1}^{m} \varepsilon_q^{\frac{-1}{2}} B_{1,q}(x),\ \text{for } k = 3,$
$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C \sum_{q=1}^{m} \varepsilon_q^{-1} B_{1,q}(x),\ \text{for } k = 4,$

and for Case (ii) and for $i = 1, 2, \ldots, n$
$|\frac{\partial^k w_i}{\partial t^k}(x,t)| \leq C_1 B_{1,n}(x) + C_2\varepsilon_n(1 - B_{1,n}(x)),\ \text{for } k = 0, 1, 2$
$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C \sum_{q=i}^{n} \varepsilon_q^{\frac{-k}{2}} B_{1,q}(x),\ \text{for } k = 1, 2,$
$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C \sum_{q=1}^{n} \varepsilon_q^{\frac{-k}{2}} B_{1,q}(x),\ \text{for } k = 3,$
$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C\varepsilon_i^{-1} \sum_{q=1}^{n} \varepsilon_q^{-1} B_{1,q}(x),\ \text{for } k = 4,$

and for $(x,t) \in [1,2] \times [0,T]$ *and for* $i = 1, 2, \ldots, m$
$|\frac{\partial^k w_i}{\partial t^k}(x,t)| \leq C_3 B_{2,m}(x) + C_4\varepsilon_m(1 - B_{2,m}(x)),\ \text{for } k = 0, 1, 2$
$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C \sum_{q=i}^{m} \varepsilon_q^{\frac{-k}{2}} B_{2,q}(x),\ \text{for } k = 1, 2,$
$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C \sum_{q=1}^{m} \varepsilon_q^{\frac{-k}{2}} B_{2,q}(x),\ \text{for } k = 3,$
$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C\varepsilon_i^{-1} \sum_{q=1}^{m} \varepsilon_q^{-1} B_{2,q}(x),\ \text{for } k = 4,$

and for $i = m+1, m+2, \ldots, k$
$|\frac{\partial^k w_i}{\partial t^k}(x,t)| \leq C_3 B_{2,m}(x) + C_4\varepsilon_m(1 - B_{2,m}(x)),\ \text{for } k = 0, 1, 2$
$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C\varepsilon_m^{\frac{-k}{2}} B_{2,m}(x),\ \text{for } k = 1, 2,\ |\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq$
$C \sum_{q=1}^{m} \varepsilon_q^{\frac{-k}{2}} B_{2,q}(x),\ \text{for } k = 3,$
$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \leq C\varepsilon_m^{-1} \sum_{q=1}^{m} \varepsilon_q^{-1} B_{2,q}(x),\ \text{for } k = 4,$

and for $i = k+1, k+2, \ldots, n$
$|\frac{\partial^k w_i}{\partial t^k}(x,t)| \leq C_4\varepsilon_m(1 - B_{2,m}(x)),\ \text{for } k = 0, 1, 2$

$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \le C_3 B_{2,m}(x) + C_4 \varepsilon_m (1 - B_{2,m}(x)),$ for $k = 1, 2,$

$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \le C \sum_{q=1}^{m} \varepsilon_q^{\frac{-1}{2}} B_{2,q}(x),$ for $k = 3,$

$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \le C \sum_{q=1}^{m} \varepsilon_q^{-1} B_{2,q}(x),$ for $k = 4,$

and for Case (ii) and for $i = 1, 2, \dots, n$

$|\frac{\partial^k w_i}{\partial t^k}(x,t)| \le C_3 B_{2,n}(x) + C_4 \varepsilon_n (1 - B_{2,n}(x)),$ for $k = 0, 1, 2$

$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \le C \sum_{q=i}^{n} \varepsilon_q^{\frac{-k}{2}} B_{2,q}(x),$ for $k = 1, 2,$

$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \le C \sum_{q=1}^{n} \varepsilon_q^{\frac{-k}{2}} B_{2,q}(x),$ for $k = 3,$

$|\frac{\partial^k w_i}{\partial x^k}(x,t)| \le C \varepsilon_i^{-1} \sum_{q=1}^{n} \varepsilon_q^{-1} B_{2,q}(x),$ for $k = 4.$

Proof The proof is based on a step-by-step procedure. First, in $[0, 1] \times [0, T]$, the bounds of $\mathbf{w}$ and its derivatives are estimated. The estimates in $[1, 2] \times [0, T]$ are then obtained using these bounds of $\mathbf{w}$ and its derivatives.

Cases *(i)* and *(iii)* :

Let $(x, t) \in [0, 1] \times [0, T]$. Consider the barrier functions

$$\boldsymbol{\psi}^{\pm}(x,t) = (C_1 B_{1,1}(x) + C_2 \varepsilon_1 (1 - B_{1,1}(x)))^T \pm \mathbf{w}(x,t)$$

and the linear operator $\mathbf{L}_1$ such that

$$\mathbf{L}_1 \boldsymbol{\psi}^{\pm}(x,t) = \frac{\partial \boldsymbol{\psi}^{\pm}}{\partial t}(x,t) - E \frac{\partial^2 \boldsymbol{\psi}^{\pm}}{\partial x^2}(x,t) + A(x,t) \boldsymbol{\psi}^{\pm}(x,t)$$

For proper choices of the constants C_1 and C_2, $\boldsymbol{\psi}^{\pm}(0,t) \ge \mathbf{0}$, $\boldsymbol{\psi}^{\pm}(1,t) \ge \mathbf{0}$ and $\mathbf{L}_1 \boldsymbol{\psi}^{\pm}(x,t) \ge \mathbf{0}$ on $(0, 1) \times (0, T)$. Then using the maximum principle in [2], to the operators $\mathbf{L}_1$, the bounds on $\mathbf{w}$ follows. Using similar arguments in Lemma 3.5 of [7], the bounds on $\frac{\partial^{(k)} \mathbf{w}}{\partial x^{(k)}}$, $k = 1, 2, 3, 4$ can be obtained. The bounds on $\mathbf{w}$ and its derivatives on $[0, 1] \times [0, T]$ are obtained using the same techniques and the bounds of $\mathbf{w}$ and its derivatives on $[0, 1] \times [0, T]$. Similar arguments in Lemma 3.5 of [7] can be used to show the results in Case *(ii)*.

4 Improved Estimates

Sharper estimations of the smooth component are presented in the next Lemma.

Lemma 6 *Let Conditions (2) and (3) hold. Then the smooth component* $\mathbf{v}$ *of the solution* $\mathbf{u}$ *of (1) satisfies for cases* (i) *and* (iii) *and for all* $(x, t) \in [0, 1-] \times [0, T]$, $|\frac{\partial v_i^k}{\partial x^k}(x,t)| \le C \left(1 + \sum_{q=i}^{m} \frac{B_{1,q}(x)}{\varepsilon_q^{\frac{k}{2}-1}}\right)$, *for* $k = 0, 1, 2$ *and for* $i = 1, 2, \dots, n$

$|\frac{\partial v_i^3}{\partial x^3}(x,t)| \leq C\left(1+\sum_{q=i}^{m}\frac{B_{1,q}(x)}{\varepsilon_q^{\frac{1}{2}}}\right)$, *for* $i=1,2,\ldots,k$

$|\frac{\partial v_i^3}{\partial x^3}(x,t)| \leq C\left(1+B_{1,m}(x)\right)$, *for* $i=k+1,k+2,\ldots,n$

and for $(x,t)\in[1+,2]\times[0,T]$,

$|\frac{\partial v_i^k}{\partial x^k}(x,t)| \leq C\left(1+\sum_{q=i}^{m}\frac{B_{2,q}(x)}{\varepsilon_q^{\frac{k}{2}-1}}\right)$, *for* $k=0,1,2$ *and for* $i=1,2,\ldots,n$

$|\frac{\partial v_i^3}{\partial x^3}(x,t)| \leq C\left(1+\sum_{q=i}^{m}\frac{B_{2,q}(x)}{\varepsilon_q^{\frac{1}{2}}}\right)$, *for* $i=1,2,\ldots,k$

$|\frac{\partial v_i^3}{\partial x^3}(x,t)| \leq C\left(1+B_{2,m}(x)\right)$, *for* $i=k+1,k+2,\ldots,n$

For case (ii) *and for all* $(x,t)\in[0,1-]\times[0,T]$,

$|\frac{\partial v_i^k}{\partial x^k}(x,t)| \leq C\left(1+\sum_{q=i}^{m}\frac{B_{1,q}(x)}{\varepsilon_q^{\frac{k}{2}-1}}\right)$, *for* $k=0,1,2$ *and for* $i=1,2,\ldots,n$

$|\frac{\partial v_i^3}{\partial x^3}(x,t)| \leq C\left(1+\sum_{q=i}^{m}\frac{B_{1,q}(x)}{\varepsilon_q^{\frac{1}{2}}}\right)$, *for* $i=1,2,\ldots,n$

and for all $(x,t)\in[1+,2]\times[0,T]$,

$|\frac{\partial v_i^k}{\partial x^k}(x,t)| \leq C\left(1+\sum_{q=i}^{m}\frac{B_{2,q}(x)}{\varepsilon_q^{\frac{k}{2}-1}}\right)$, *for* $k=0,1,2$ *and for* $i=1,2,\ldots,n$

$|\frac{\partial v_i^3}{\partial x^3}(x,t)| \leq C\left(1+\sum_{q=i}^{m}\frac{B_{2,q}(x)}{\varepsilon_q^{\frac{1}{2}}}\right)$, *for* $i=1,2,\ldots,n$.

Proof The procedure of stages is used to prove this as well. Applying the Lemma 7 of [3] to cases (i) and (iii), the estimates of the derivatives of $\mathbf{v}$ on $[0,1-]\times[0,T]$ are as follows. Following that, for $(x,t)\in[1,2]\times[0,T]$, the bounds on the derivatives of $\mathbf{v}$ and the bounds on the derivatives of $\mathbf{v}$ in the interval $[0,1-]\times[0,T]$ are derived using the procedure used in the proof of Lemma 7 of [3].

5 The Shishkin Mesh

For cases (i) and (iii) :

A piecewise-uniform Shishkin mesh with $M\times N$ mesh-intervals is now constructed. Let $\Omega_t^M=\{t_k\}_{k=1}^{M}$, $\overline{\Omega}_t^M=\{t_k\}_{k=0}^{M}$, $\Omega_x^N=\{x_j\}_{j=1}^{N-1}$, $\overline{\Omega}_x^N=\{x_j\}_{j=0}^{N}$, $\Omega^{M,N}=\Omega_t^M\times\Omega_x^N$, $\overline{\Omega}^{M,N}=\overline{\Omega}_t^M\times\overline{\Omega}_x^N$, $\Omega_x^{-N}=\{x_j\}_{j=1}^{\frac{N}{2}-1}$, $\overline{\Omega}_x^{-N}=\{x_j\}_{j=0}^{\frac{N}{2}}$, $\Omega_x^{+N}=\{x_j\}_{j=\frac{N}{2}+1}^{N-1}$, $\overline{\Omega}_x^{+N}=\{x_j\}_{j=\frac{N}{2}}^{N}$, $\Omega^{-M,N}=\Omega_t^M\times\Omega_x^{-N}$, $\overline{\Omega}^{-M,N}=\overline{\Omega}_t^M\times\overline{\Omega}_x^{-N}$, $\Omega^{+M,N}=\Omega_t^M\times\Omega_x^{+N}$, $\overline{\Omega}^{+M,N}=\overline{\Omega}_t^M\times\overline{\Omega}_x^{+N}$ and $\Gamma^{M,N}=\Gamma\cap\overline{\Omega}^{M,N}$. The mesh $\overline{\Omega}_t^M$ is chosen to be a uniform mesh with M mesh-intervals on $[0,T]$. The mesh $\overline{\Omega}_x^N$ is chosen to be a piecewise-uniform mesh with N mesh-intervals on $[0,2]$. The interval $[0,1]$ is divided into $2m+1$ sub-intervals as follows:

$$[0,\tau_1]\cup(\tau_1,\tau_2]\cup\cdots\cup(\tau_{m-1},\tau_m]\cup(\tau_m,1-\tau_m]\cup(1-\tau_m,1-\tau_{m-1}]\\ \cup\cdots\cup(1-\tau_2,1-\tau_1]\cup(1-\tau_1,1].$$

The parameters $\tau_r, r = 1, 2 \ldots, m$, which determine the points separating the uniform meshes, are defined by $\tau_0 = 0$, $\tau_{m+1} = \dfrac{1}{2}$,

$$\tau_m = \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon_m}}{\sqrt{\alpha}}\ln N\right\} \text{ and for } k = 1, 2 \ldots, m-1,\ \tau_k = \min\left\{\frac{k\tau_{k+1}}{k+1}, \frac{2\sqrt{\varepsilon_k}}{\sqrt{\alpha}}\ln N\right\}. \quad (32)$$

Then, on the sub-interval $(\tau_m, 1-\tau_m]$ a uniform mesh with $\dfrac{N}{4}$ mesh points is placed and on each of the sub-intervals $[0, \tau_1]$, $(1-\tau_1, 1]$, $(\tau_k, \tau_{k+1}]$, and$(1-\tau_{k+1}, 1-\tau_k]$, $k = 1, \ldots, m-1$, a uniform mesh of $\dfrac{N}{8m}$ mesh points is placed.

Similarly, the interval $(1, 2]$ is also divided into $2m+1$ sub-intervals $(1, 1+\tau_1]$, $(1+\tau_1, 1+\tau_2]$, $\ldots$, $(1+\tau_{m-1}, 1+\tau_m]$, $(1+\tau_n, 2-\tau_m]$, $(2-\tau_m, 2-\tau_{m-1}]$, $\ldots$, $(2-\tau_2, 2-\tau_1]$, $(2-\tau_1, 2]$, using the same parameters $\tau_k, k = 1, \ldots, m$. In particular, when all the parameters $\tau_k, k = 1, \ldots, m$ takes on their left hand values, the Shishkin mesh $\overline{\Omega}^N$ becomes a classical uniform mesh throughout from 0 to 2.

In practice, it is convenient to take

$$N = 8mk,\ k \geq 3. \quad (33)$$

From the above construction of $\overline{\Omega}^N$, it is clear that the transition points $\{\tau_r, 1-\tau_r, 1+\tau_r, 2-\tau_r\}, r = 1, 2, \ldots, m$ are the only points at which the mesh-size can change and that it does not necessarily change at each of these points. The following notations are introduced: $h_j = x_j - x_{j-1}$ and if $x_j = \tau$, then $h_j^- = x_j - x_{j-1}$, $h_j^+ = x_{j+1} - x_j$, $J = \{x_j : h_j^+ \neq h_j^-\}$.

For Case (ii) :

A piecewise-uniform Shishkin mesh with $M \times N$ mesh-intervals is now constructed. Let $\Omega_t^M = \{t_k\}_{k=1}^M$, $\overline{\Omega}_t^M = \{t_k\}_{k=0}^M$, $\Omega_x^N = \{x_j\}_{j=1}^{N-1}$, $\overline{\Omega}_x^N = \{x_j\}_{j=0}^N$, $\Omega^{M,N} = \Omega_t^M \times \Omega_x^N$, $\overline{\Omega}^{M,N} = \overline{\Omega}_t^M \times \overline{\Omega}_x^N$, $\Omega_x^{-N} = \{x_j\}_{j=1}^{\frac{N}{2}-1}$, $\overline{\Omega}_x^{-N} = \{x_j\}_{j=0}^{\frac{N}{2}}$, $\Omega_x^{+N} = \{x_j\}_{j=\frac{N}{2}+1}^{N-1}$, $\overline{\Omega}_x^{+N} = \{x_j\}_{j=\frac{N}{2}}^N$, $\Omega^{-M,N} = \Omega_t^M \times \Omega_x^{-N}$, $\overline{\Omega}^{-M,N} = \overline{\Omega}_t^M \times \overline{\Omega}_x^{-N}$, $\Omega^{+M,N} = \Omega_t^M \times \Omega_x^{+N}$, $\overline{\Omega}^{+M,N} = \overline{\Omega}_t^M \times \overline{\Omega}_x^{+N}$ and $\Gamma^{M,N} = \Gamma \cap \overline{\Omega}^{M,N}$. The mesh $\overline{\Omega}_t^M$ is chosen to be a uniform mesh with M mesh-intervals on $[0, T]$. The mesh $\overline{\Omega}_x^N$ is chosen to be a piecewise-uniform mesh with N mesh-intervals on $[0, 2]$. The interval $[0, 1]$ is divided into $2n+1$ sub-intervals as follows:

$$[0, \tau_1] \cup (\tau_1, \tau_2] \cup \cdots \cup (\tau_{n-1}, \tau_n] \cup (\tau_n, 1-\tau_n] \cup (1-\tau_n, 1-\tau_{n-1}] \\ \cup \cdots \cup (1-\tau_2, 1-\tau_1] \cup (1-\tau_1, 1].$$

The parameters $\tau_r, r = 1, 2 \ldots, n$, which determine the points separating the uniform meshes, are defined by $\tau_0 = 0$, $\tau_{n+1} = \dfrac{1}{2}$,

$$\tau_n = \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon_n}}{\sqrt{\alpha}}\ln N\right\} \text{ and for } k = 1, 2 \ldots, n-1,\ \tau_k = \min\left\{\frac{k\tau_{k+1}}{k+1}, \frac{2\sqrt{\varepsilon_k}}{\sqrt{\alpha}}\ln N\right\}. \quad (34)$$

Then, on the sub-interval $(\tau_n, 1-\tau_n]$ a uniform mesh with $\frac{N}{4}$ mesh points is placed and on each of the sub-intervals $[0, \tau_1]$, $(1-\tau_1, 1]$, $(\tau_k, \tau_{k+1}]$, and$(1-\tau_{k+1}, 1-\tau_k]$, $k = 1, \dots, n-1$, a uniform mesh of $\frac{N}{8n}$ mesh points is placed.

Similarly, the interval $(1, 2]$ is also divided into $2n+1$ sub-intervals $(1, 1+\tau_1]$, $(1+\tau_1, 1+\tau_2], \dots, (1+\tau_{n-1}, 1+\tau_n]$, $(1+\tau_n, 2-\tau_n]$, $(2-\tau_n, 2-\tau_{n-1}], \dots, (2-\tau_2, 2-\tau_1]$, $(2-\tau_1, 2]$, using the same parameters τ_k, $k = 1, \dots, n$. In particular, when all the parameters τ_k, $k = 1, \dots, n$ takes on their left hand values, the Shishkin mesh $\overline{\Omega}^N$ becomes a classical uniform mesh throughout from 0 to 2.

In practice, it is convenient to take

$$N = 8nk, \; k \geq 3. \tag{35}$$

From the above construction of $\overline{\Omega}^N$, it is clear that the transition points $\{\tau_r, 1-\tau_r, 1+\tau_r, 2-\tau_r\}$, $r = 1, 2, \dots, n$ are the only points at which the mesh-size can change and that it does not necessarily change at each of these points. The following notations are introduced: $h_j = x_j - x_{j-1}$ and if $x_j = \tau$, then $h_j^- = x_j - x_{j-1}$, $h_j^+ = x_{j+1} - x_j$, $J = \{x_j : h_j^+ \neq h_j^-\}$.

6 The Discrete Problem

In this section, a numerical method for (1) is constructed using a classical finite difference operator and an appropriate Shishkin mesh, which is later shown to be first-order parameter-uniform convergent in time and essentially first-order parameter-uniform convergent in the space variable.

The finite difference method can now define the discrete two-point boundary value problem on any mesh.

$$\begin{aligned} \mathbf{L}^{M,N}\mathbf{U}(x_j, t_k) &= D_t^-\mathbf{U}(x_j, t_k) - E\delta_x^2\mathbf{U}(x_j, t_k) + A(x_j, t_k)\mathbf{U}(x_j, t_k) \\ +B(x_j, t_k)\mathbf{U}(x_j - 1, t_k) &= \mathbf{f}(x_j, t_k) \;\text{ on }\; \Omega^{M,N}, 0 \leq j \leq N, 0 \leq k \leq M \end{aligned} \tag{36}$$

$$\mathbf{U} = \mathbf{u} \;\text{ on }\; \Gamma^{M,N}.$$

Problem (36) can be rewritten as

$$\mathbf{L_1}^{M,N}\mathbf{U}(x_j, t_k) = D_t^-\mathbf{U}(x_j, t_k) - E\delta_x^2\mathbf{U}(x_j, t_k) + A(x_j, t_k)\mathbf{U}(x_j, t_k) = \mathbf{g}(x_j, t_k) \;\text{ on }\; \Omega^{-M,N}, \tag{37}$$

$$\text{where } \mathbf{g}(x_j, t_k) = \mathbf{f}(x_j, t_k) - B(x_j, t_k)\boldsymbol{\chi}(x_j - 1, t_k)$$

$$\begin{aligned} \mathbf{L_2}^{M,N}\mathbf{U}(x_j, t_k) &= D_t^-\mathbf{U}(x_j, t_k) - E\delta_x^2\mathbf{U}(x_j, t_k) + A(x_j, t_k)\mathbf{U}(x_j, t_k) + B(x_j, t_k)\mathbf{U}(x_j - 1, t_k) \\ &= \mathbf{f}(x_j, t_k) \;\text{ on }\; \Omega^{+M,N} \end{aligned} \tag{38}$$

$$\mathbf{U} = \mathbf{u} \ \text{ on } \ \Gamma^{M,N}, D_x^-\mathbf{U}(x_{\frac{N}{2}}, t_k) = D_x^+\mathbf{U}(x_{\frac{N}{2}}, t_k),$$

where $D_t^-\mathbf{U}(x_j, t_k) = \frac{\mathbf{U}(x_j,t_k)-\mathbf{U}(x_j,t_{k-1})}{t_k-t_{k-1}}$, $\delta_x^2\mathbf{U}(x_j, t_k) = \frac{D_x^+\mathbf{U}(x_j,t_k)-D_x^-\mathbf{U}(x_j,t_k)}{\frac{x_{j+1}-x_{j-1}}{2}}$, $D_x^-\mathbf{U}(x_j, t_k) = \frac{\mathbf{U}(x_{j+1},t_k)-\mathbf{U}(x_j,t_k)}{x_{j+1}-x_j}$ and $D_x^+\mathbf{U}(x_j, t_k) = \frac{\mathbf{U}(x_j,t_k)-\mathbf{U}(x_{j-1},t_k)}{x_j-x_{j-1}}$.
This is used to approximate the exact solution of (1) numerically. The results for the discrete case are similar to those for the continuous case.

Lemma 7 *Let Conditions* (2) *and* (3) *hold. Then, for any mesh function* $\overrightarrow{\Psi}(x_j, t_k), 0 \le j \le N, 0 \le k \le M$, *the inequalities* $\overrightarrow{\Psi} \ge \mathbf{0}$ on $\Gamma^{M,N}$,$\mathbf{L_1}^{M,N}\overrightarrow{\Psi}(x_j, t_k) \ge \mathbf{0}$, *on* $\Omega^{-M,N}$, $\mathbf{L_2}^{M,N}\overrightarrow{\Psi}(x_j, t_k) \ge \mathbf{0}$ *on* $\Omega^{+M,N}$ *and* $D_x^+\overrightarrow{\Psi}(x_{N/2}, t_k) - D_x^-\overrightarrow{\Psi}(x_{N/2}, t_k) \le \mathbf{0}$ *imply that* $\overrightarrow{\Psi}(x_j, t_k) \ge \mathbf{0}$ *on* $\overline{\Omega}^{M,N}$.

Proof The proof proceeds by applying similar reasoning to [7] Lemma 6.1.

The following discrete stability result is an immediate result of this.

Lemma 8 *Let Conditions* (2) *and* (3) *hold. Then, for any mesh function* $\overrightarrow{\Psi}$ *satisfying* $D_x^+\overrightarrow{\Psi}(x_{N/2}, t_k) = D_x^-\overrightarrow{\Psi}(x_{N/2}, t_k)$,

$$|\Psi_i(x_j, t_k)| \le \max\left\{||\Psi_i||_{\Gamma^{M,N}}, \ \frac{1}{\alpha}||\mathbf{L_1}^{M,N}\Psi_i||_{\Omega^{-M,N}}, \ \frac{1}{\alpha}||\mathbf{L_2}^{M,N}\Psi_i||_{\Omega^{+M,N}}\right\},$$

for each $i = 1, 2, \dots, n$ *and* $0 \le j \le N, 0 \le k \le M$.

Proof Using similar arguments in [7] Lemma 6.2, one can easily obtain the result.

7 Error Estimate

The discrete solution **U** can be decomposed into **V** and **W**, which are defined as the solutions of the following discrete problems, similar to the continuous case.

$$\begin{aligned} \mathbf{L_1}^{M,N}\mathbf{V}(x_j, t_k) = \mathbf{g}(x_j, t_k), (x_j, t_k) \in \Omega^{-M,N}, 0 \le j \le \frac{N}{2} - 1, 0 \le k \le M \\ \mathbf{V}(0, t_k) = \mathbf{v}(0, t_k), \mathbf{V}(x_{N/2-1}, t_k) = \mathbf{v}(1-, t_k), \mathbf{V}(x_j, 0) = \boldsymbol{\phi}_\mathbf{B}(x_j), \end{aligned} \quad (39)$$

$$\begin{aligned} \mathbf{L_2}^{M,N}\mathbf{V}(x_j, t_k) = \mathbf{f}(x_j, t_k), (x_j, t_k) \in \Omega^{+M,N}, \frac{N}{2} + 1 \le j \le N, 0 \le k \le M \\ \mathbf{V}(x_{N/2+1}, t_k) = \mathbf{v}(1+, t_k), \mathbf{V}(2, t_k) = \mathbf{v}(2, t_k), \mathbf{V}(x_j, 0) = \boldsymbol{\phi}_\mathbf{B}(x_j) \end{aligned} \quad (40)$$

and

$$\begin{array}{c} \mathbf{L_1}^{M,N}\mathbf{W}(x_j,t_k)=\mathbf{0},\ (x_j,t_k)\in\Omega^{-M,N},\ \mathbf{W}(0,t_k)=\mathbf{w}(0,t_k),\ 0\le j\le \frac{N}{2}-1,\ 0\le k\le M \\ \mathbf{L_2}^{M,N}\mathbf{W}(x_j,t_k)=\mathbf{0},\ (x_j,t_k)\in\Omega^{+M,N},\ \mathbf{W}(2,t_k)=\mathbf{w}(2,t_k),\ \frac{N}{2}+1\le j\le N,\ 0\le k\le M \\ \mathbf{V}(x_{N/2+1},t_k)+\mathbf{W}(x_{N/2+1},t_k)=\mathbf{V}(x_{N/2-1},t_k)+\mathbf{W}(x_{N/2-1},t_k), \\ D_x^-\mathbf{W}(x_{N/2},t_k)+D_x^-\mathbf{V}(x_{N/2},t_k)=D_x^+\mathbf{W}(x_{N/2},t_k)+D_x^+\mathbf{V}(x_{N/2},t_k). \\ \mathbf{W}(x_j,0)=\mathbf{0} \end{array} \tag{41}$$

The error at each point $(x_j, t_k) \in \overline{\Omega}^{M,N}$ is denoted by $\mathbf{e}(x_j, t_k) = \mathbf{U}(x_j, t_k) - \mathbf{u}(x_j, t_k)$. Then the local truncation error $\mathbf{L}^{M,N}\mathbf{e}(x_j, t_k)$, for $j \neq N/2$, has the decomposition

$$\mathbf{L}^{M,N}\mathbf{e}(x_j,t_k) = \mathbf{L}^{M,N}(\mathbf{V}-\mathbf{v})(x_j,t_k) + \mathbf{L}^{M,N}(\mathbf{W}-\mathbf{w})(x_j,t_k).$$

The error in the smooth and singular components is bounded in the following theorem for cases *(i), (ii)*, and *(iii)*.

Lemma 9 *Let* $\mathbf{v}(x_j, t_k)$ *denote the smooth component of the exact solution from (1) and* $\mathbf{V}(x_j, t_k)$ *the smooth component of the solution from (36), then, for* $i = 1, 2$ *and* $j \neq \frac{N}{2}$

$$||(\mathbf{L_1}^{M,N}(\mathbf{V}-\mathbf{v}))_i(x_j,t_k)|| \le C(M^{-1}+(N^{-1}\ln N)^2),\ 0\le j\le \frac{N}{2}-1,\ 0\le k\le M,$$

$$||(\mathbf{L_2}^{M,N}(\mathbf{V}-\mathbf{v}))_i(x_j,t_k)|| \le C(M^{-1}+(N^{-1}\ln N)^2),\ \frac{N}{2}+1\le j\le N,\ 0\le k\le M.$$

Let $\mathbf{w}(x_j, t_k)$ *denote the singular component of the exact solution from (1) and* $\mathbf{W}(x_j, t_k)$ *the singular component of the solution from (36), then, for* $i = 1, 2$ *and* $j \neq \frac{N}{2}$

$$||(\mathbf{L_1}^{M,N}(\mathbf{W}-\mathbf{w}))_i(x_j,t_k)|| \le C(M^{-1}+(N^{-1}\ln N)^2),\ 0\le j\le \frac{N}{2}-1,\ 0\le k\le M,$$

$$||(\mathbf{L_2}^{M,N}(\mathbf{W}-\mathbf{w}))_i(x_j,t_k)|| \le C(M^{-1}+(N^{-1}\ln N)^2),\ \frac{N}{2}+1\le j\le N,\ 0\le k\le M.$$

Proof For cases *(i)* and *(iii)*, the needed bounds hold since the expressions for the local truncation error in **V** and **W**, as well as estimates for the derivatives of the smooth and singular components, are exactly in the form provided in [2].

The needed bounds for Case *(ii)* hold because the expressions for the local truncation error in **V** and **W**, as well as estimates for the derivatives of the smooth and singular components, are exactly in the form provided in [2].

At the point $x_j = x_{N/2}$,

$$(D_x^+ - D_x^-)\mathbf{e}(x_{N/2}, t_k) = (D_x^+ - D_x^-)(\mathbf{U}-\mathbf{u})(x_{N/2}, t_k),\ 0 \le k \le M.$$

Recall that $(D_x^+ - D_x^-)\mathbf{U}(x_{N/2}, t_k) = 0$.

Let $h^* = h^-_{N/2} = h^+_{N/2}$, where $h^-_{N/2} = x_{N/2} - x_{N/2-1}$ and $h^+_{N/2} = x_{N/2+1} - x_{N/2}$. Then

$$\begin{aligned}|(D_x^+ - D_x^-)\mathbf{e}(x_{N/2}, t_k)| &= |(D_x^+ - D_x^-)\mathbf{u}(x_{N/2}, t_k)| \\ &\le |(D_x^+ - \frac{\partial}{\partial x})\mathbf{u}(x_{N/2}, t_k)| + |\left(D_x^- - \frac{\partial}{\partial x}\right)\mathbf{u}(x_{N/2}, t_k)| \\ &\le \frac{1}{2}h_{N/2}^+ \max_{\eta_1\in(1,2)} |\frac{\partial^2 \mathbf{u}}{\partial x^2}(\eta_1, t_k)| + \frac{1}{2}h_{N/2}^- \max_{\eta_2\in(0,1)} |\frac{\partial^2 \mathbf{u}}{\partial x^2}(\eta_2, t_k)| \\ &\le Ch^* \max_{x\in(0,1)\cup(1,2)} \left|\frac{\partial^2 \mathbf{u}}{\partial x^2}(x, t)\right|.\end{aligned}$$

Therefore,

$$|(D_x^+ - D_x^-)\mathbf{e}(x_{N/2}, t_k)| \le C\frac{h^*}{\varepsilon}. \tag{42}$$

Define, for $i = 1, 2, \ldots, n$ and for each t_k, a set of discrete barrier functions on $\overline{\Omega}^{M,N}$ by

$$\omega_i(x_j, t_k) = \begin{cases} \dfrac{\Pi_{q=1}^{j}(1+\sqrt{\alpha}h_q/\sqrt{2\varepsilon_i})}{\Pi_{q=1}^{N/2}(1+\sqrt{\alpha}h_q/\sqrt{2\varepsilon_i})}, & 0 \le j \le N/2 \\ \dfrac{\Pi_{q=j}^{N-1}(1+\sqrt{\alpha}h_{q+1}/\sqrt{2\varepsilon_i})}{\Pi_{q=N/2}^{N-1}(1+\sqrt{\alpha}h_{q+1}/\sqrt{2\varepsilon_i})}, & N/2 \le j \le N. \end{cases} \tag{43}$$

Proceeding as in Lemma 7.1 of [7], we find that

$$\begin{aligned}(\mathbf{L_1}^{M,N}\boldsymbol{\omega})_i(x_j, t_k) &= D_t^-\omega_i(x_j, t_k) - \varepsilon_i\,\delta_x^2\omega_i(x_j, t_k) + \sum_{l=1}^{n} a_{il}(x_j, t_k)\omega_l(x_j, t_k) \\ &> -\alpha\omega_i(x_j, t_k) + \sum_{l=1}^{i} a_{il}(x_j, t_k)\omega_i(x_j, t_k) + \sum_{l=i+1}^{n} a_{il}(x_j, t_k).\end{aligned} \tag{44}$$

And

$$\begin{aligned}(\mathbf{L_2}^{M,N}\boldsymbol{\omega})_i(x_j, t_k) &= D_t^-\omega_i(x_j, t_k) - \varepsilon_i\,\delta_x^2\omega_i(x_j, t_k) \\ &\quad + \sum_{l=1}^{n} a_{il}(x_j, t_k)\omega_l(x_j, t_k) + b_i(x_j, t_k)\omega_i(x_j - 1, t_k) \\ &\ge -\alpha\omega_i(x_j, t_k) + \sum_{l=1}^{i} a_{il}(x_j, t_k)\omega_i(x_j, t_k) + \sum_{l=i+1}^{n} a_{il}(x_j, t_k) + b_i(x_j, t_k).\end{aligned} \tag{45}$$

The major theoretical result of this section is now stated and proven.

Theorem 1 *Let* $\mathbf{u}(x_j, t_k)$ *denote the exact solution of (1) and* $\mathbf{U}(x_j, t_k)$ *the solution of (36). Then, for* $0 \le j \le N, 0 \le k \le M$,

$$||\mathbf{U}(x_j, t_k) - \mathbf{u}(x_j, t_k)|| \le C(M^{-1} + N^{-1}\ln N). \tag{46}$$

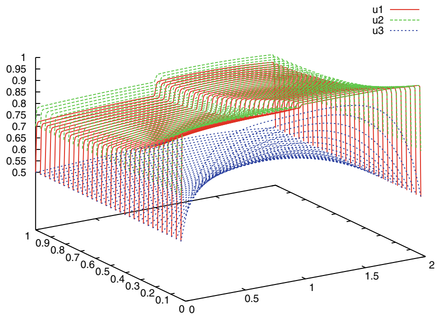


Fig. 1 The figure displays the numerical solution for Problem (47), computed for $M = 16$, $N = 96$ and $\eta = 2^{-7}$. The solution components $u_1(x, t)$ and $u_2(x, t)$ have boundary layers at $(0, t)$ and $(2, t)$ and interior layers at $(1, t)$ and the solution component $u_3(x, t)$ does not have any layers

Proof Consider the mesh function $\overrightarrow{\Psi}$ given by $\overrightarrow{\Psi}(x_j, t_k) = C_1(M^{-1} + N^{-1}\ln N) + C_2\frac{h^*}{\sqrt{\varepsilon_i}}\omega_i(x_j, t_k) \pm e_i(x_j, t_k)$, $i = 1, 2, \ldots, n$, $0 \le j \le N$, $0 \le k \le M$, where C_1 and C_2 are constants. Then the result follows by using the mesh function $\overrightarrow{\Psi}^{\pm}$ and similar arguments in Theorem 7.2 of [7].

8 Numerical Illustration

The ε-uniform convergence of the numerical method proposed in this section is illustrated in the example below using a variant of the two Mesh algorithm found in [1].

Example 1 : Consider the following problem for Case (i)

$$\frac{\partial \mathbf{u}}{\partial t}(x, t) - E\frac{\partial^2 \mathbf{u}}{\partial x^2}(x, t) + A(x, t)\mathbf{u}(x, t) + B(x, t)\mathbf{u}(x - 1, t) = \mathbf{f}(x, t),$$
$$(x, t) \in (0, 2) \times [0, T], \quad (47)$$

$\mathbf{u}(x, t) = (1, 1, 1)^T$, for $(x, t) \in [-1, 0] \times [0, T]$, $\mathbf{u}(0, t) = (0.5, 0.7, 0.5)^T$, $\mathbf{u}(x, 0) = (1, 1, 1)^T$ and $\mathbf{u}(2, t) = (0.5, 0.7, 0.5)^T$,
where

$$E = diag(\varepsilon_1, \varepsilon_2, 1),\ A(x, t) = \begin{pmatrix} 5 & -2 & -1 \\ -1 & 4+t & -1 \\ -1 & -1 & 4 \end{pmatrix},\ B(x, t) = diag(-1, -1, -1),$$

and $\mathbf{f} = (1, 1 + t, 0)^T$.

We first investigate the robustness of the temporal discretization. In results shown in Table 1 we have fixed the number of intervals in spacial (Shishkin) mesh to be $N = 96$, and present results for various M and ε. Note the fully first-order convergence as predicted in Theorem 1. In Table 2, we fix the number of time steps to be $M = 16$ and allow N to vary. Now we observe almost first-order convergence, again consistent with Theorem 1 (Fig. 1).

Table 1 Values of $D^{M,N}$, $p^{M,N}$, p^*, $C_{p^*}^{M,N}$ and $C_{p^*}^*$ for $\varepsilon_1 = \eta/128, \varepsilon_2 = \eta/64, N = 96$ and $\alpha = 0.9$

η	Number of mesh points M				
	32	64	128	256	512
2^{-7}	6.26E-03	3.34E-03	1.68E-03	8.06E-04	3.69E-04
2^{-14}	6.35E-03	3.43E-03	1.76E-03	8.76E-04	4.27E-04
2^{-21}	6.35E-03	3.43E-03	1.77E-03	8.82E-04	4.33E-04
2^{-28}	6.35E-03	3.43E-03	1.77E-03	8.83E-04	4.34E-04
2^{-35}	6.35E-03	3.43E-03	1.77E-03	8.83E-04	4.34E-04
$D^{M,N}$	6.35E-03	3.43E-03	1.77E-03	8.83E-04	4.34E-04
$p^{M,N}$	0.887	0.959	1.00	1.03	
$C_{p^*}^{M,N}$	0.299	0.299	0.285	0.263	0.239
t-order of convergence, p^*= 0.887					
The error constant, $C_{p^*}^*$= 0.299					

Where $D^{M,N}$ - the ε-uniform maximum point-wise errors, $p^{M,N}$ - the ε-uniform order of local convergence, p^* - the ε-uniform order of convergence, $C_{p^*}^{M,N} = \frac{D^{M,N} N^{p^*}}{1-2^{-p^*}}$ and $C_{p^*}^*$ - error constant

Table 2 Values of $D^{M,N}$, $p^{M,N}$, p^*, $C_{p^*}^{M,N}$ and $C_{p^*}^*$ for $\varepsilon_1 = \eta/128, \varepsilon_2 = \eta/64, M = 16$ and $\alpha = 0.9$

η	Number of mesh points N			
	96	192	384	768
2^{-7}	9.19E-03	6.86E-03	4.29E-03	2.56E-03
2^{-14}	9.38E-03	7.01E-03	4.39E-03	2.54E-03
2^{-21}	9.39E-03	7.02E-03	4.39E-03	2.55E-03
2^{-28}	9.39E-03	7.02E-03	4.39E-03	2.55E-03
2^{-35}	9.39E-03	7.02E-03	4.39E-03	2.55E-03
$D^{M,N}$	9.39E-03	7.02E-03	4.39E-03	2.56E-03
$p^{M,N}$	0.420	0.675	0.779	
$C_{p^*}^{M,N}$	0.253	0.253	0.212	0.166
x-order of convergence, p^*= 0.420				
The error constant, $C_{p^*}^*$= 0.253				

Example 2:
Consider the following problem for Case (ii)

$$\frac{\partial \mathbf{u}}{\partial t}(x,t) - E\frac{\partial^2 \mathbf{u}}{\partial x^2}(x,t) + A(x,t)\mathbf{u}(x,t) + B(x,t)\mathbf{u}(x-1,t) = \mathbf{f}(x,t),$$
$$(x,t) \in (0,2) \times [0,T], \quad (48)$$

Table 3 Values of $D^{M,N}$, $p^{M,N}$, p^*, $C_{p^*}^{M,N}$ and $C_{p^*}^*$ for $\varepsilon_1 = \eta/64$, $\varepsilon_2 = \eta/32$, $\varepsilon_3 = \eta/16$, $N = 96$ and $\alpha = 0.9$

η	Number of mesh points M				
	32	64	128	256	512
2^{-3}	5.02E-03	2.70E-03	1.41E-03	7.22E-04	3.64E-04
2^{-6}	5.11E-03	2.68E-03	1.37E-03	6.92E-04	3.48E-04
2^{-9}	5.08E-03	2.64E-03	1.35E-03	6.82E-04	3.43E-04
2^{-12}	5.08E-03	2.64E-03	1.35E-03	6.82E-04	3.43E-04
2^{-15}	5.08E-03	2.64E-03	1.35E-03	6.82E-04	3.43E-04
$D^{M,N}$	5.11E-03	2.70E-03	1.41E-03	7.22E-04	3.64E-04
$p^{M,N}$	0.919	0.937	0.969	0.986	
$C_{p^*}^{M,N}$	0.263	0.263	0.259	0.251	0.239
t-order of convergence, p^*= 0.919					
The error constant, $C_{p^*}^*$= 0.263					

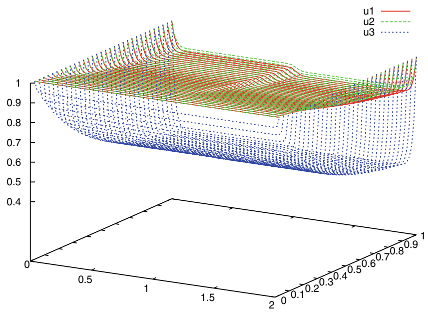


Fig. 2 The figure displays the numerical solution for Problem (48), computed for $M = 32$, $N = 96$ and $\eta = 2^{-6}$. The solution components $u_1(x,t)$ and $u_2(x,t)$ have boundary layers at $(0,t)$ and $(2,t)$ and interior layers at $(1,t)$ and the solution component $u_3(x,t)$ has boundary layers at $(0,t)$ and $(2,t)$

$\mathbf{u}(x,t) = (1,1)^T$, for $(x,t) \in [-1,0] \times [0,T]$, $\mathbf{u}(0,t) = (1,1,1)^T$, $\mathbf{u}(x,0) = (1,1,1)^T$ and $\mathbf{u}(2,t) = (1,1,1)^T$,
where

$$E = diag(\varepsilon_1, \varepsilon_2, \varepsilon_3),\ A(x,t) = \begin{pmatrix} 5 & -2 & -1 \\ -1 & 4+t & -1 \\ -1 & -1 & 4 \end{pmatrix},\ B(x,t) = diag(-1,-1,0),$$

and $\mathbf{f} = (1, 1+t, 0)^T$.

We first investigate the robustness of the temporal discretization. In results shown in Table 3 we have fixed the number of intervals in spacial (Shishkin) mesh to be $N = 96$, and present results for various M and ε. Note the fully first-order convergence as predicted in Theorem 1. In Table 4, we fix the number of time steps to be $M = 16$ and allow N to vary. Now we observe almost first-order convergence, again consistent with Theorem 1 (Fig. 2).

Table 4 Values of $D^{M,N}$, $p^{M,N}$, p^*, $C_{p^*}^{M,N}$ and $C_{p^*}^*$ for $\varepsilon_1 = \eta/64$, $\varepsilon_2 = \eta/32$, $\varepsilon_3 = \eta/16$, $M = 16$ and $\alpha = 0.9$

η	Number of mesh points N			
	96	192	384	768
2^{-3}	1.76E-02	6.74E-03	3.37E-03	1.68E-03
2^{-6}	1.62E-02	1.10E-02	6.58E-03	3.74E-03
2^{-9}	1.62E-02	1.10E-02	6.58E-03	3.74E-03
2^{-12}	1.62E-02	1.10E-02	6.58E-03	3.74E-03
2^{-15}	1.62E-02	1.10E-02	6.58E-03	3.74E-03
$D^{M,N}$	1.76E-02	1.10E-02	6.58E-03	3.74E-03
$p^{M,N}$	0.678	0.740	0.814	
$C_{p^*}^{M,N}$	1.04	1.04	0.992	0.902
x-order of convergence, p^*= 0.678				
The error constant, $C_{p^*}^*$= 1.04				

Example 3:
Consider the following problem for Case (iii)

$$\frac{\partial \mathbf{u}}{\partial t}(x,t) - E\frac{\partial^2 \mathbf{u}}{\partial x^2}(x,t) + A(x,t)\mathbf{u}(x,t) + B(x,t)\mathbf{u}(x-1,t) = \mathbf{f}(x,t),$$
$$(x,t) \in (0,2) \times [0,T], \quad (49)$$

$\mathbf{u}(x,t) = (1,1)^T$, for $(x,t) \in [-1,0] \times [0,T]$, $\mathbf{u}(0,t) = (0.5, 0.7, 0.5)^T$, $\mathbf{u}(x,0) = (1,1,1)^T$ and $\mathbf{u}(2,t) = (0.5, 0.7, 0.5)^T$,
where

$$E = diag(\varepsilon_1, \varepsilon_2, 1),\; A(x,t) = \begin{pmatrix} 5 & -2 & -1 \\ -1 & 4+t & -1 \\ -1 & -1 & 4 \end{pmatrix},\; B(x,t) = diag(-1,-1,0),$$

and $\mathbf{f} = (1, 1+t, 0)^T$.

We first investigate the robustness of the temporal discretization. In results shown in Table 5 we have fixed the number of intervals in spacial (Shishkin) mesh to be $N = 96$, and present results for various M and ε. Note the fully first-order convergence as predicted in Theorem 1. In Table 6, we fix the number of time steps to be $M = 16$ and allow N to vary. Now we observe almost first-order convergence, again consistent with Theorem 1 (Fig. 3).

Table 5 Values of $D^{M,N}$, $p^{M,N}$, p^*, $C_{p^*}^{M,N}$ and $C_{p^*}^*$ for $\varepsilon_1 = \eta/64$, $\varepsilon_2 = \eta/32$, $N = 96$ and $\alpha = 0.9$

η	Number of mesh points M				
	32	64	128	256	512
2^{-3}	4.59E-03	2.39E-03	1.18E-03	5.60E-04	2.79E-04
2^{-6}	4.46E-03	2.36E-03	1.18E-03	5.59E-04	2.79E-04
2^{-9}	4.53E-03	2.43E-03	1.23E-03	6.02E-04	2.84E-04
2^{-12}	4.53E-03	2.44E-03	1.25E-03	6.17E-04	2.98E-04
2^{-15}	4.52E-03	2.44E-03	1.25E-03	6.21E-04	3.03E-04
$D^{M,N}$	4.59E-03	2.44E-03	1.25E-03	6.21E-04	3.03E-04
$p^{M,N}$	0.910	0.966	1.01	1.04	
$C_{p^*}^{M,N}$	0.229	0.229	0.221	0.206	0.189
t-order of convergence, p^*= 0.910					
The error constant, $C_{p^*}^*$= 0.229					

Table 6 Values of $D^{M,N}$, $p^{M,N}$, p^*, $C_{p^*}^{M,N}$ and $C_{p^*}^*$ for $\varepsilon_1 = \eta/64$, $\varepsilon_2 = \eta/32$, $M = 16$ and $\alpha = 0.9$

η	Number of mesh points N			
	96	192	384	768
2^{-3}	2.75E-03	1.47E-03	7.46E-04	3.74E-04
2^{-6}	2.67E-03	1.72E-03	1.02E-03	5.85E-04
2^{-9}	2.67E-03	1.72E-03	1.02E-03	5.85E-04
2^{-12}	2.67E-03	1.72E-03	1.02E-03	5.85E-04
2^{-15}	2.67E-03	1.72E-03	1.02E-03	5.85E-04
$D^{M,N}$	2.75E-03	1.72E-03	1.02E-03	5.85E-04
$p^{M,N}$	0.674	0.751	0.807	
$C_{p^*}^{M,N}$	0.160	0.160	0.152	0.138
x-order of convergence, p^*= 0.674				
The error constant, $C_{p^*}^*$= 0.160				

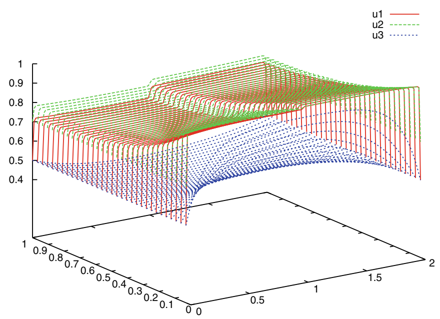


Fig. 3 The figure displays the numerical solution for Problem (49), computed for $M = 32$, $N = 96$ and $\eta = 2^{-9}$. The solution components $u_1(x,t)$ and $u_2(x,t)$ have boundary layers at $(0,t)$ and $(2,t)$ and interior layers at $(1,t)$ and the solution component $u_3(x,t)$ does not have any layers

9 Conclusion

In this study, a first-order convergent numerical technique for a parabolic system of partially singularly perturbed reaction-diffusion delay differential equations is proposed. The system is only partially complete in terms of delay and/or perturbation parameter. The effect of both the delay and the perturbation parameter on the solution profile has been investigated, and it has been determined that the perturbation parameter has a greater impact than the delay term. The parameter convergence of the proposed method is supported by numerical illustrations.

References

1. Farrell, P.A., Hegarty, A.F., Miller, J.J.H., O' Riordan, E., Shishkin, G.I.: Robust Computational Techniques for Boundary Layers. Chapman and hall, CRC, Boca Raton, Florida, USA (2000)
2. Franklin, V., Paramasivam, M., Miller, J.J.H., Valarmathi, S.: Second order parameter-uniform convergence for a finite difference method for a singularly perturbed linear parabolic system. Int. J. Numer. Anal. Model. **10**(1), 178–202
3. Franklin, V., Miller, J.J.H., Valarmathi, S.: Second order parameter-uniform convergence for a finite difference method for a partially singularly perturbed linear parabolic system. Math. Commun. **19**, 469–495 (2014)
4. Manikandan, M., Shivaranjani, N., Miller J.J.H., Valarmathi, S.: A parameter uniform first order convergent numerical method for a boundary value problem for a singularly perturbed delay differential equation. In: Advances in Applied Mathematics, Springer Proceedings in Mathematics and Statistics 87, pp. 71–88
5. Mariappan, M., Miller, J.J.H., Sigamani, V.: A parameter-uniform first order convergent numerical method for a system of singularly perturbed second order delay differential equations. In: Knobloch, P. (ed.) Boundary and Interior Layers, Computational and Asymptotic Methods - BAIL 2014. Lecture Notes in Computational Science and Engineering 108. Springer International Publishing Switzerland (2015)
6. Saminathan, P., Sigamani, V., Victor, F.: Numerical method for a singularly perturbed boundary value problem for a linear parabolic second order delay differential equation. Differential Equations and Numerical Analysis. Springer Proceedings in Mathematics and Statistics, vol. 172 (2016)
7. Saminathan, P., Sigamani, V., Victor, F.: Parameter uniform numerical method for a singularly perturbed boundary value problem for a linear system of parabolic second order delay differential equations. Malaya J. Matematik **7**(2), 147–160 (2019)

A First-Order Convergent Parameter-Uniform Numerical Method for a Singularly Perturbed Second-Order Delay-Differential Equation of Reaction–Diffusion Type with a Discontinuous Source Term

Manikandan Mariappan, John J. H. Miller, and Valarmathi Sigamani

Abstract In this paper, a boundary value problem for a second-order singularly perturbed delay differential equation of reaction–diffusion type with a discontinuous source term is considered on the interval [0, 2]. A single discontinuity in the source term is assumed to occur at a point $d \in (0, 2)$. The leading term of the equation is multiplied by a small positive parameter. The solution of this problem exhibits boundary layers at $x = 0$ and $x = 2$ and interior layers at $x = 1$ and/or at $x = d$ and $x = 1 + d$ with respect to the position of d in $(0, 2)$. A numerical method composed of a classical finite difference scheme applied on a piecewise uniform Shishkin mesh is suggested to solve the problem. The method is proved to be first-order convergent uniformly in the perturbation parameter. Numerical illustrations provided support the theory.

Keywords Singular perturbation problems · Discontinuous source term · Boundary and interior layers · Shishkin meshes · Classical finite difference schemes

1 Introduction

Differential equations with a delay are common in the mathematical modelling of various physical, biological phenomena and control theory. Singularly perturbed differential equations with a delay forms a subclass of differential equations. Inves-

M. Mariappan (✉)
Department of Mathematics, Visvesvaraya National Institute of Technology, Nagpur, Maharashtra, India

J. J. H. Miller
Trinity College, Dublin, Ireland
e-mail: jm@incaireland.org

V. Sigamani
Department of Mathematics, Bishop Heber College, Tiruchirappalli, Tamil Nadu, India

Valarmathi S et al. (eds.), *Differential Equations and Applications*,
Springer Proceedings in Mathematics & Statistics 368,
https://doi.org/10.1007/978-981-16-7546-1_5

tigation of boundary value problems for singularly perturbed linear second-order differential–difference equations was initiated by Lange and Miura [1]. Our interest lies in examining singularly perturbed delay differential equations with discontinuous source terms. This paper focus on the construction of a parameter-uniform finite difference method on a piecewise-uniform Shishkin mesh for a second-order singularly perturbed delay differential equation of reaction–diffusion type with a discontinuous source term. Some works on singularly perturbed differential equations with a discontinuous source term are reported in [2, 3].

The following two-point boundary value problem is considered for the singularly perturbed linear second-order delay differential equation with a discontinuous source term:

$$Lu(x) \;=\; -\varepsilon\, u''(x) + a(x)u(x) + b(x)u(x-1) = f(x) \;\text{ on } (0,d)\cup(d,2), \quad (1)$$

with

$$u \;=\; \phi \;\text{ on }\; [-1,0],\; u(2) = l \;\text{ and }\; f(d-) \neq f(d+) \text{ for some } d \in (0,2), \quad (2)$$

where the function ϕ is sufficiently smooth on $[-1,0]$. For all $x \in [0,2]$, the functions $a(x)$ and $b(x)$ satisfy

$$a(x) + b(x) > 2\alpha \quad (3)$$

and

$$b(x) < 0 \quad (4)$$

for some real number $\alpha > 0$. Furthermore, the functions $a(x)$ and $b(x)$ are assumed to be in $C^2([0,2])$.

Because f is discontinuous at d, the solution $u(x)$ does not necessarily have a continuous second-order derivative at the point d. Thus, $u(x) \notin C^2((0,2))$, but the first derivative of the solution exists and is continuous on $(0,2)$, as is shown in Theorem 1.

The cases $(i)\; d \in (0,1)$, $(ii)\; d \in (1,2)$ and $(iii)\; d = 1$ are considered separately. When $d = 1$, the problem (1)–(2) is same as in [4] and hence can be solved by using the same numerical method constructed in [4]. The cases $(i)\; d \in (0,1)$ and $(ii)\; d \in (1,2)$ are discussed elaborately in this paper.

The problem (1)–(2) can be rewritten as follows for Case (i) :

on $(0, d) \cup (d, 1)$,

$$L_1 u(x) = -\varepsilon u''(x) + a(x)u(x) = f(x) - b(x)\phi(x-1) = g(x) \quad (5)$$

$$L_2 u(x) = -\varepsilon u''(x) + a(x)u(x) + b(x)u(x-1) = f(x) \text{ on } (1,2) \quad (6)$$

$$u = \phi \text{ on } [-1, 0],\ u(2) = l,\ f(d-) \neq f(d+), \quad (7)$$

$$u(d-) = u(d+),\ u'(d-) = u'(d+), \quad (8)$$

$$u(1-) = u(1+) \text{ and } u'(1-) = u'(1+) \quad (9)$$

and as follows for the Case (ii):

$$L_1 u(x) = -\varepsilon u''(x) + a(x)u(x) = f(x) - b(x)\phi(x-1) = g(x) \text{ on } (0,1) \quad (10)$$

$$L_2 u(x) = -\varepsilon u''(x) + a(x)u(x) + b(x)u(x-1) = f(x) \text{ on } (1,d) \cup (d,2) \quad (11)$$

$$u = \phi \text{ on } [-1, 0],\ u(2) = l,\ f(d-) \neq f(d+), \quad (12)$$

$$u(1-) = u(1+), u'(1-) = u'(1+), u(d-) = u(d+)\ \&\ u'(d-) = u'(d+) \quad (13)$$

The reduced problem corresponding to (5)–(9) is defined by

$$a(x)u_0(x) = g(x) \text{ on } (0, d) \cup (d, 1) \quad (14)$$

$$a(x)u_0(x) + b(x)u_0(x-1) = f(x) \text{ on } (1, 1+d) \cup (1+d, 2) \quad (15)$$

and the reduced problem corresponding to (10)–(13) is defined by

$$a(x)u_0(x) = g(x) \text{ on } (0, 1) \quad (16)$$

$$a(x)u_0(x) + b(x)u_0(x-1) = f(x) \text{ on } (1, d) \cup (d, 2). \quad (17)$$

In general, as $u_0(x)$ need not satisfy $u_0(0) = u(0)$ and $u_0(2) = u(2)$, the solution $u(x)$ exhibits boundary layers of width $O(\sqrt{\varepsilon})$ at $x = 0$ and $x = 2$. In addition to that, at $x = 1$, $u_0(1-) = [f(1-) - b(1)\phi(0-)]/a(1)$, $u_0(1+) = [f(1+) - b(1)u_0(0+)]/a(1)$ and as $u_0(1-)$ need not be equal to $u_0(1+)$, the solution $u(x)$ exhibits interior layers of width $O(\sqrt{\varepsilon})$ at $x = 1$. Moreover, $f(d-) \neq f(d+)$ so that $u_0(d-)$ need not be equal to $u_0(d+)$, the solution $u(x)$ exhibits additional interior layers of width $O(\sqrt{\varepsilon})$ at $x = d$ and $x = 1 + d$ in Case (i) and at $x = d$ in Case (ii).

For any function y on a domain D, the following norm is introduced: $\| y \|_D = \sup_{x \in D} |y(x)|$. For any mesh function V, the following discrete maximum norm is introduced: $\| V \| = \max_j |V(x_j)|$. Throughout the paper, C denotes a generic positive constant, which is independent of x and singular perturbation and discretization parameters.

The plan of the paper is as follows. In Sect. 2, the analytical results of the solution are presented. Improved estimates are presented in Sect. 3. In Sect. 4, piecewise-uniform Shishkin meshes are introduced and, in Sect. 5, the discrete problem is defined and the discrete maximum principle and the discrete stability properties are established. In Sect. 6, the error bounds are established and, in Sect. 7, numerical illustrations are presented.

2 Analytical Results

Theorem 1 *The given problem* (1)–(2) *has a solution* $u \in \mathscr{C} = C^0([0,2]) \cap C^1((0,2)) \cap C^2((0,1) \cup (1,2)\backslash\{d\})$.

Proof The proof is by construction.
Case (i) : Let y_1, z_1, y_2 and z_2 be the particular solutions of

$$\begin{aligned}
-\varepsilon\, y_1''(x) + a(x)y_1(x) &= g(x), \quad x \in (0,d)\\
-\varepsilon\, z_1''(x) + a(x)z_1(x) &= g(x), \quad x \in (d,1)\\
-\varepsilon\, y_2''(x) + a(x)y_2(x) &= f(x) - b(x)y_1(x-1), \quad x \in (1,1+d)\\
-\varepsilon\, z_2''(x) + a(x)z_2(x) &= f(x) - b(x)z_1(x-1), \quad x \in (1+d,2).
\end{aligned}$$

Consider the function

$$u(x) = \begin{cases} y_1(x) + (u(0) - y_1(0))\psi_1(x) + A\,\psi_2(x), & x \in (0,d)\\ z_1(x) + B\,\psi_3(x) + C\,\psi_4(x), & x \in (d,1)\\ y_2(x) + D\,\psi_5(x) + E\,\psi_6(x), & x \in (1,1+d)\\ z_2(x) + (u(2) - z_2(2))\psi_8(x) + F\,\psi_7(x), & x \in (1+d,2) \end{cases}$$

where ψ_i's, $i = 1, \ldots, 8$ are the solutions of

$$\begin{aligned}
-\varepsilon\, \psi_1''(x) + a(x)\psi_1(x) = 0, &\quad x \in (0,d), \psi_1(0) = 1,\ \psi_1(d) = 0,\\
-\varepsilon\, \psi_2''(x) + a(x)\psi_2(x) = 0, &\quad x \in (0,d), \psi_2(0) = 0,\ \psi_2(d) = 1,\\
-\varepsilon\, \psi_3''(x) + a(x)\psi_3(x) = 0, &\quad x \in (d,1), \psi_3(d) = 1,\ \psi_3(1) = 0,\\
-\varepsilon\, \psi_4''(x) + a(x)\psi_4(x) = 0, &\quad x \in (d,1), \psi_4(d) = 0,\ \psi_4(1) = 1,
\end{aligned}$$

$$\begin{aligned}
-\varepsilon\, \psi_5''(x) + a(x)\psi_5(x) = 0, &\quad x \in (1,1+d), \psi_5(1) = 1,\ \psi_5(1+d) = 0,\\
-\varepsilon\, \psi_6''(x) + a(x)\psi_6(x) = 0, &\quad x \in (1,1+d), \psi_6(1) = 0,\ \psi_6(1+d) = 1,\\
-\varepsilon\, \psi_7''(x) + a(x)\psi_7(x) = 0, &\quad x \in (1+d,2), \psi_7(1+d) = 1,\ \psi_7(2) = 0,\\
-\varepsilon\, \psi_8''(x) + a(x)\psi_8(x) = 0, &\quad x \in (1+d,2), \psi_8(1+d) = 0,\ \psi_8(2) = 1.
\end{aligned}$$

Here, A, B, C, D, E and F are the constants determined from the conditions that u and u' are continuous at $x = d$, $x = 1$ and $x = 1 + d$.

Case (ii) : Let y, z_1 and z_2 be the particular solutions of

$$\begin{aligned}
-\varepsilon\, y''(x) + a(x)y(x) &= g(x), \quad x \in (0,1)\\
-\varepsilon\, z_1''(x) + a(x)z_1(x) &= f(x) - b(x)y(x-1), \quad x \in (1,d)\\
-\varepsilon\, z_2''(x) + a(x)z_2(x) &= f(x) - b(x)y(x-1), \quad x \in (d,2).
\end{aligned}$$

Consider the function

$$u(x) = \begin{cases} y(x) + (u(0) - y(0))\psi_1(x) + A\,\psi_2(x), & x \in (0,1)\\ z_1(x) + B\,\psi_3(x) + C\,\psi_4(x), & x \in (1,d)\\ z_2(x) + (u(2) - z_2(2))\psi_6(x) + D\,\psi_5(x), & x \in (d,2) \end{cases}$$

where ψ_i's, $i = 1, \ldots, 6$, are the solutions of

$$\begin{aligned}
-\varepsilon\, \psi_1''(x) + a(x)\psi_1(x) = 0, &\quad x \in (0,1), \psi_1(0) = 1,\ \psi_1(1) = 0,\\
-\varepsilon\, \psi_2''(x) + a(x)\psi_2(x) = 0, &\quad x \in (0,1), \psi_2(0) = 0,\ \psi_2(1) = 1,\\
-\varepsilon\, \psi_3''(x) + a(x)\psi_3(x) = 0, &\quad x \in (1,d), \psi_3(1) = 1,\ \psi_3(d) = 0,\\
-\varepsilon\, \psi_4''(x) + a(x)\psi_4(x) = 0, &\quad x \in (1,d), \psi_4(1) = 0,\ \psi_4(d) = 1,\\
-\varepsilon\, \psi_5''(x) + a(x)\psi_5(x) = 0, &\quad x \in (d,2), \psi_5(d) = 1,\ \psi_5(2) = 0,\\
-\varepsilon\, \psi_6''(x) + a(x)\psi_6(x) = 0, &\quad x \in (d,2), \psi_6(d) = 0,\ \psi_6(2) = 1.
\end{aligned}$$

Here, A, B, C and D are constants determined from the conditions that u and u' are continuous at $x = 1$ and $x = d$.

Case (iii) : Let y and z be the particular solutions of

$$\begin{aligned}
-\varepsilon\, y''(x) + a(x)y(x) &= g(x), \quad x \in (0,1)\\
-\varepsilon\, z''(x) + a(x)z(x) &= f(x) - b(x)y(x-1), \quad x \in (1,2).
\end{aligned}$$

Consider the function

$$u(x) = \begin{cases} y(x) + (u(0) - y(0))\psi_1(x) + A\,\psi_2(x), & x \in (0,1)\\ z(x) + (u(2) - z(2))\psi_4(x) + B\,\psi_4(x), & x \in (1,2) \end{cases}$$

where ψ_i's, $i = 1, \ldots, 4$, are the solutions of

$$-\varepsilon\,\psi_1''(x)+a(x)\psi_1(x)=0,\quad x\in(0,1),\,\psi_1(0)=1,\ \psi_1(1)=0,$$
$$-\varepsilon\,\psi_2''(x)+a(x)\psi_2(x)=0,\quad x\in(0,1),\,\psi_2(0)=0,\ \psi_2(1)=1,$$
$$-\varepsilon\,\psi_3''(x)+a(x)\psi_3(x)=0,\quad x\in(1,2),\,\psi_3(1)=1,\ \psi_3(2)=0,$$
$$-\varepsilon\,\psi_4''(x)+a(x)\psi_4(x)=0,\quad x\in(1,2),\,\psi_4(1)=0,\ \psi_4(2)=1.$$

Here, A and B are constants determined from the conditions that u and u' are continuous at $x=1$.

Let $\Omega=(0,2)$, Ω_1=$[0,d)\cup(d,1)\cup(1,1+d)\cup(1+d,2]$ and Ω_2=$[0,1)\cup(1,d)\cup(d,2]$.

The operator L satisfies the following maximum principle.

Lemma 1 *Let conditions* (3) *and* (4) *hold. Let* ψ *be any function in the domain of* L *such that* $\psi(0)\geq 0,\ \ \psi(2)\geq 0,\ L_1\psi\geq 0$ *on* $(0,d)\cup(d,1)$, $L_2\psi\geq 0$ *on* $(1,2)$ *in Case* (i) *and* $L_1\psi\geq 0$ *on* $(0,1)$, $L_2\psi\geq 0$ *on* $(1,d)\cup(d,2)$ *in Case* (ii) *and* $[\psi](d)=0$, $[\psi](1)=0$, $[\psi'](d)\leq 0$ *and* $[\psi'](1)\leq 0$ *then* $\psi\geq 0$ *on* $[0,2]$.

Proof The result follows by using similar arguments as in Lemma 1 of [4].

As a consequence of the maximum principle, there is established the stability result for the problem (1)–(2) in the following.

Lemma 2 *Let conditions* (3) *and* (4) *hold. Let* ψ *be any function in the domain of* L *such that* $[\psi](1)=0$, $[\psi](d)=0$, $[\psi'](1)=0$ *and* $[\psi'](d)=0$ *in Cases* (i) *and* (ii), *then for* $x\in[0,2]$,

$$|\psi(x)|\leq \max\left\{|\psi(0)|,|\psi(2)|,\tfrac{1}{\alpha}\parallel f\parallel_{\Omega\setminus\{d\}}\right\}+|[f](d)|.$$

Proof By using similar arguments as in Lemma 2 of [4], it is not hard to prove the result.

Standard estimates of the solution of (1)–(2) and its derivatives are contained in the following.

Lemma 3 *Let conditions* (3) *and* (4) *hold and let* u *be the solution of* (1)–(2). *Then, in Case* (i), *for all* $x\in[0,2]\backslash\{d,1+d\}$,

$$|u^{(k)}(x)|\leq C\,\varepsilon^{-\frac{k}{2}}\left(||u||+||f||_{\Omega_1}\right),\ \textit{for } k=0,1$$

and

$$|u^{(k)}(x)|\leq C\,\varepsilon^{-\frac{k}{2}}\left(||u||+||f||_{\Omega_1}+\varepsilon^{\frac{(k-2)}{2}}||f^{(k-2)}||_{\Omega_1}\right),\ \ \textit{for}\ \ k=2,3,4$$

and in Case (ii), *for all* $x\in[0,2]\backslash\{d\}$,

$$|u^{(k)}(x)|\leq C\,\varepsilon^{-\frac{k}{2}}\left(||u||+||f||_{\Omega_2}\right),\ \textit{for } k=0,1$$

and

$$|u^{(k)}(x)| \le C\,\varepsilon^{-\frac{k}{2}}\left(||u|| + ||f||_{\Omega_2} + \varepsilon^{\frac{(k-2)}{2}}||f^{(k-2)}||_{\Omega_2}\right),\quad \textit{for}\ \ k = 2, 3, 4.$$

Proof It is not hard to derive the bounds by using similar arguments as in Lemma 3 of [4].

The Shishkin decomposition of the solution u of (1)–(2) is

$$u = v + w$$

where the smooth component v is the solution of

$$\begin{aligned} &L_1 v = g \ \text{ on } (0, d-),\\ &v(0) = u_0(0),\ \ v(d-) = (a(d))^{-1}(f(d-) - b(1)\phi(d-1)) \end{aligned} \tag{18}$$

$$\begin{aligned} &L_1 v = g \ \text{ on } (d+, 1-),\\ &v(d+) = (a(d))^{-1}(f(d+) - b(d)\phi(d-1)),\ \ v(1-) = (a(1))^{-1}(f(1) - b(1)\phi(0)), \end{aligned} \tag{19}$$

$$\begin{aligned} &L_2 v = f \ \text{ on } (1+, 2),\\ &v(1+) = (a(1))^{-1}(f(1) - b(1)u_{01}(0)),\ v(2) = u_0(2) \end{aligned} \tag{20}$$

and the singular component w is the solution of

$$\begin{aligned} &L_1 w = 0 \ \text{ on } (0, d) \cup (d, 1),\ \ L_2 w = 0 \ \text{ on } (1, 2)\\ &\text{with } \ w(0) = u(0) - v(0),\ \ [w](d) = -[v](d),\ \ [w](1) = -[v](1),\\ &[w'](d) = -[v'](d),\ \ [w'](1) = -[v'](1),\ \ w(2) = u(2) - v(2), \end{aligned} \tag{21}$$

for Case (i) and the smooth component v is the solution of

$$L_1 v = g \ \text{ on } (0, 1-),\ \ v(0) = u_0(0),\ \ v(1-) = (a(1))^{-1}(f(1) - b(1)\phi(0)) \tag{22}$$

$$\begin{aligned} &L_1 v = f \ \text{ on } (1+, d-),\\ &v(1+) = (a(1))^{-1}(f(1) - b(1)u_0(0)),\ \ v(d-) = (a(d))^{-1}(f(d-) - b(d)u_0(d-1)), \end{aligned} \tag{23}$$

$$L_2 v = f \ \text{ on } (d+, 2),\ \ v(d+) = (a(d))^{-1}(f(d+) - b(d)u_0(d-1)),\ v(2) = u_0(2) \tag{24}$$

and the singular component w is the solution of

$$\begin{aligned} &L_1 w = 0 \ \text{ on } (0, 1),\ \ L_2 w = 0 \ \text{ on } (1, d) \cup (d, 2)\\ &\text{with } \ w(0) = u(0) - v(0),\ [w](1) = -[v](1),\ [w](d) = -[v](d),\\ &[w'](1) = -[v'](1),\ [w'](d) = -[v'](d),\ w(2) = u(2) - v(2), \end{aligned} \tag{25}$$

for Case (ii).

The singular component is given a further decomposition, for case(i),

$$w(x) = \tilde{w}_1(x) + \tilde{w}_2(x) + \hat{w}(x), \tag{26}$$

where $\tilde{w}_1$ is the solution of

$$-\varepsilon\, \tilde{w}_1''(x) + a(x)\tilde{w}_1(x) = 0 \text{ on } (0, d),\ \ \tilde{w}_1(0) = w(0), \tilde{w}_1(d) = K_1, \tilde{w}_1 = 0 \text{ on } (d, 2],$$

$\tilde{w}_2$ is the solution of

$$-\varepsilon\, \tilde{w}_2''(x) + a(x)\tilde{w}_2(x) = 0 \text{ on } (d, 1),\ \ \tilde{w}_2(d) = K_2, \tilde{w}_2(1) = K_3, \tilde{w}_2 = 0 \text{ on } [0, d) \cup (1, 2]$$

and on $[0, 1)$, $\hat{w}$ is the solution of

$$-\varepsilon\, \hat{w}''(x) + a(x)\hat{w}(x) + b(x)\hat{w}(x-1) = 0 \text{ on } (1, 2), \hat{w}(1) = K_4,\ \ \hat{w}(2) = w(2), \hat{w} = 0.$$

Here, K_1, K_2, K_3 and K_4 are constants to be chosen in such a way that the jump conditions at $x = d$ and $x = 1$ are satisfied.

For case (ii),

$$w(x) = \tilde{w}(x) + \hat{w}_1(x) + \hat{w}_2(x), \tag{27}$$

where $\tilde{w}$ is the solution of

$$-\varepsilon\, \tilde{w}''(x) + a(x)\tilde{w}(x) = 0 \text{ on } (0, 1),\ \ \tilde{w}(0) = w(0), \tilde{w}(1) = K_5, \tilde{w} = 0 \text{ on } (1, 2],$$

$\hat{w}_1$ is the solution of

$$\begin{aligned} -\varepsilon\, \hat{w}_1''(x) + a(x)\hat{w}_1(x) + b(x)\hat{w}_1(x-1) &= 0 \text{ on } (1, d), \\ \hat{w}_1(1) = K_6,\ \ \hat{w}_1(d) = K_7, \hat{w}_1 &= 0 \text{ on } [0, 1) \cup (d, 2] \end{aligned}$$

and $\hat{w}_2$ is the solution of

$$\begin{aligned} -\varepsilon\, \hat{w}_2''(x) + a(x)\hat{w}_2(x) + b(x)\hat{w}_2(x-1) &= 0 \text{ on } (d, 2), \\ \hat{w}_2(d) = K_8,\ \ \hat{w}_2(2) = w(2), \hat{w}_2 &= 0 \text{ on } [0, d). \end{aligned}$$

Here, K_5, K_6, K_7 and K_8 are constants to be chosen in such a way that the jump conditions at $x = 1$ and $x = d$ are satisfied.

In Cases (i) and (ii), the bounds on the smooth component v of u and its derivatives are contained in the following:

Lemma 4 *Let conditions* (3) *and* (4) *hold. Then the smooth component* v *and its derivatives satisfy, for all* $x \in [0, 2]\backslash\{d, 1 + d\}$ *for Case* (i) *and* $x \in [0, 2]\backslash\{d\}$ *for Case* (ii),

$$|v^{(k)}(x)| \le C\,(1+\varepsilon^{1-\frac{k}{2}}),\ \ for\ \ k=0,1,2,3.$$

Proof The result follows by using similar arguments as in Lemma 4 of [4].

The layer functions $B_1^l, B_1^r, B_2^l, B_2^r, B_3^l, B_3^r, B_4^l, B_4^r, B_1, B_2, B_3, B_4,$ associated with the solution u of Case (i), are defined by

$$B_1^l(x)=e^{-x\sqrt{\alpha}/\sqrt{\varepsilon}},\ B_1^r(x)=e^{-(d-x)\sqrt{\alpha}/\sqrt{\varepsilon}},\ B_1(x)=B_1^l(x)+B_1^r(x),\ \text{on}\ [0,d],$$

$$B_2^l(x)=e^{-(x-d)\sqrt{\alpha}/\sqrt{\varepsilon}},\ B_2^r(x)=e^{-(1-x)\sqrt{\alpha}/\sqrt{\varepsilon}},\ B_2(x)=B_2^l(x)+B_2^r(x),\ \text{on}\ [d,1],$$

$$B_3^l(x)=e^{-(x-1)\sqrt{\alpha}/\sqrt{\varepsilon}},\ B_3^r(x)=e^{-(1+d-x)\sqrt{\alpha}/\sqrt{\varepsilon}},$$

$$B_3(x)=B_3^l(x)+B_3^r(x),\ \text{on}\ [1,1+d],$$

$$B_4^l(x)=e^{-(x-(1+d))\sqrt{\alpha}/\sqrt{\varepsilon}},\ B_4^r(x)=e^{-(2-x)\sqrt{\alpha}/\sqrt{\varepsilon}},\ B_4(x)=B_4^l(x)+B_4^r(x),\ \text{on}\ [1,2].$$

The layer functions $B_1^l, B_1^r, B_2^l, B_2^r, B_3^l, B_3^r, B_1, B_2, B_3$, associated with the solution u of Case (ii), are defined by

$$B_1^l(x)=e^{-x\sqrt{\alpha}/\sqrt{\varepsilon}},\ B_1^r(x)=e^{-(1-x)\sqrt{\alpha}/\sqrt{\varepsilon}},\ B_1(x)=B_1^l(x)+B_1^r(x),\ \text{on}\ [0,1],$$

$$B_2^l(x)=e^{-(x-1)\sqrt{\alpha}/\sqrt{\varepsilon}},\ B_2^r(x)=e^{-(d-x)\sqrt{\alpha}/\sqrt{\varepsilon}},\ B_2(x)=B_2^l(x)+B_2^r(x),\ \text{on}\ [1,d],$$

$$B_3^l(x)=e^{-(x-d)\sqrt{\alpha}/\sqrt{\varepsilon}},\ B_3^r(x)=e^{-(2-x)\sqrt{\alpha}/\sqrt{\varepsilon}},\ B_3(x)=B_3^l(x)+B_3^r(x),\ \text{on}\ [d,2].$$

In Cases (i) and (ii), the bounds on the singular component w of u and its derivatives are contained in the following:

Lemma 5 *Let conditions* (3) *and* (4) *hold. Then there exists a constant* C, *such that, for* $k=0,1,2,3,$

$$\left|w^{(k)}(x)\right| \le C\,\frac{B_1(x)}{\varepsilon^{k/2}},\ for\ x\in[0,d),\ \left|w^{(k)}(x)\right| \le C\,\frac{B_2(x)}{\varepsilon^{k/2}},\ \ for\ x\in(d,1],$$

$$\left|w^{(k)}(x)\right| \le C\,\frac{B_3(x)}{\varepsilon^{k/2}},\ for\ x\in[1,1+d),\ \left|w^{(k)}(x)\right| \le C\,\frac{B_4(x)}{\varepsilon^{k/2}},\ \ for\ x\in(1+d,2]$$

in Case (i) *and*

$$\left|w^{(k)}(x)\right| \le C\,\frac{B_1(x)}{\varepsilon^{k/2}},\ for\ x\in[0,1],\ \left|w^{(k)}(x)\right| \le C\,\frac{B_2(x)}{\varepsilon^{k/2}},\ \ for\ x\in[1,d),$$

$$\left|w^{(k)}(x)\right| \le C\,\frac{B_3(x)}{\varepsilon^{k/2}},\ for\ x\in(d,2],$$

in Case (ii).

Proof By using similar arguments as in Lemma 5 of [4], it is not hard to prove the results.

3 Improved Estimates

In the following lemma, sharper estimates of the smooth component are presented.

Lemma 6 *Let conditions* (3) *and* (4) *hold. Then the smooth component* v *of the solution* u *of* (1)–(2) *satisfies,*

$$|v^{(k)}(x)| \le C\,(1 + B_1(x)),\ \ for\ \ k = 0, 1, 2\ \text{ and }\ |v'''(x)|\ \le\ C\ \left(1 + \frac{B_1(x)}{\sqrt{\varepsilon}}\right),\ on\ (0, d),$$

$$|v^{(k)}(x)| \le C\,(1 + B_2(x)),\ \ for\ \ k = 0, 1, 2\ \text{ and }\ |v'''(x)|\ \le\ C\ \left(1 + \frac{B_2(x)}{\sqrt{\varepsilon}}\right),\ on\ (d, 1),$$

$$|v^{(k)}(x)| \le C\,(1 + B_3(x)),\ \ for\ \ k = 0, 1, 2\ \text{ and }\ |v'''(x)|\ \le\ C\ \left(1 + \frac{B_3(x)}{\sqrt{\varepsilon}}\right),\ on\ (1, 1 + d),$$

$$|v^{(k)}(x)| \le C\,(1 + B_4(x)),\ \ for\ \ k = 0, 1, 2\ \text{ and }\ |v'''(x)|\ \le\ C\ \left(1 + \frac{B_4(x)}{\sqrt{\varepsilon}}\right),\ on\ (1 + d, 2),$$

in Case (i) *and*

$$|v^{(k)}(x)| \le C\,(1 + B_1(x)),\ \ for\ \ k = 0, 1, 2\ \text{ and }\ |v'''(x)|\ \le\ C\ \left(1 + \frac{B_1(x)}{\sqrt{\varepsilon}}\right),\ on\ (0, 1),$$

$$|v^{(k)}(x)| \le C\,(1 + B_2(x)),\ \ for\ \ k = 0, 1, 2\ \text{ and }\ |v'''(x)|\ \le\ C\ \left(1 + \frac{B_2(x)}{\sqrt{\varepsilon}}\right),\ on\ (1, d),$$

$$|v^{(k)}(x)| \le C\,(1 + B_3(x)),\ \ for\ \ k = 0, 1, 2\ \text{ and }\ |v'''(x)|\ \le\ C\ \left(1 + \frac{B_3(x)}{\sqrt{\varepsilon}}\right),\ on\ (d, 2),$$

in Case (ii).

Proof The results follow by using similar arguments as in Lemma 6 of [4].

4 The Shishkin Mesh

4.1 The Shishkin Mesh for Case (i)

A piecewise uniform Shishkin mesh with N mesh intervals is now constructed on $[0, 2]$ as follows. Let $\Omega^N = {\Omega_1}^N \cup {\Omega_2}^N \cup {\Omega_3}^N$, where ${\Omega_1}^N = \{x_j\}_{j=1}^{\frac{N}{4}-1}$, ${\Omega_2}^N = \{x_j\}_{j=\frac{N}{4}+1}^{\frac{N}{2}-1}$, ${\Omega_3}^N = \{x_j\}_{j=\frac{N}{2}+1}^{N-1}$, $x_{\frac{N}{4}} = d$ and $x_{\frac{N}{2}} = 1$. Then $\overline{\Omega_1}^N = \{x_j\}_{j=0}^{\frac{N}{4}}$, $\overline{\Omega_2}^N = \{x_j\}_{j=\frac{N}{4}}^{\frac{N}{2}}$, $\overline{\Omega_3}^N = \{x_j\}_{j=\frac{N}{2}}^{N}$, $\overline{\Omega_1}^N \cup \overline{\Omega_2}^N \cup \overline{\Omega_3}^N = \overline{\Omega}^N = \{x_j\}_{j=0}^{N}$ and $\Gamma^N = \{0, 2\}$.

The interval $[0, d]$ is divided into 3 sub-intervals $[0, \tau]$, $(\tau, d - \tau]$ and $(d - \tau, d]$.

The parameter τ, which determines the points separating the uniform meshes, is defined by

$$\tau = \min\left\{\frac{d}{4}, \frac{\sqrt{\varepsilon}}{\sqrt{\alpha}}\ln N\right\}. \tag{28}$$

Then, on the sub-interval $(\tau, d-\tau]$ a uniform mesh with $\frac{N}{8}$ mesh points is placed and on each of the sub-intervals $[0, \tau]$ and $(d-\tau, d]$, a uniform mesh of $\frac{N}{16}$ mesh points is placed.

Similarly, the interval $(d, 1]$ is divided into three sub-intervals $(d, d+\eta]$, $(d+\eta, 1-\eta]$ and $(1-\eta, 1]$, where

$$\eta = \min\left\{\frac{1-d}{4}, \frac{\sqrt{\varepsilon}}{\sqrt{\alpha}}\ln N\right\}, \tag{29}$$

the interval $(1, 1+d]$ is divided into three sub-intervals $(1, 1+\tau]$, $(1+\tau, 1+d-\tau]$ and $(1+d-\tau, 1+d]$ and the interval $(1+d, 2]$ is divided into three sub-intervals $(1+d, 1+d+\eta]$, $(1+d+\eta, 2-\eta]$ and $(2-\eta, 2]$ having the same mesh pattern as in $[0, 1]$.

In practice, it is convenient to take $N = 16k,\ \ k \geq 2$.

From the above construction of $\overline{\Omega}^N$, it is clear that the transition points $\{\tau, d-\tau, d+\eta, 1-\eta, 1+\tau, 1+d-\tau, 1+d+\eta, 2-\eta\}$ are the only points at which the mesh-size can change and that it does not necessarily change at each of these points.

4.2 The Shishkin Mesh for Case (ii)

In this case, a piecewise uniform Shishkin mesh with N mesh intervals is now constructed on $[0, 2]$ as follows. Let $\Omega^N = {\Omega_1}^N \cup {\Omega_2}^N \cup {\Omega_3}^N$, where ${\Omega_1}^N = \{x_j\}_{j=1}^{\frac{N}{3}-1}$, ${\Omega_2}^N = \{x_j\}_{j=\frac{N}{3}+1}^{\frac{2N}{3}-1}$, ${\Omega_3}^N = \{x_j\}_{j=\frac{2N}{3}+1}^{N-1}$, $x_{\frac{N}{3}} = 1$ and $x_{\frac{2N}{3}} = d$. Then $\overline{\Omega_1}^N = \{x_j\}_{j=0}^{\frac{N}{3}}$, $\overline{\Omega_2}^N = \{x_j\}_{j=\frac{N}{3}}^{\frac{2N}{3}}$, $\overline{\Omega_3}^N = \{x_j\}_{j=\frac{2N}{3}}^{N}$, $\overline{\Omega_1}^N \cup \overline{\Omega_2}^N \cup \overline{\Omega_3}^N = \overline{\Omega}^N = \{x_j\}_{j=0}^{N}$ and $\Gamma^N = \{0, 2\}$.

The interval $[0, 1]$ is divided into three sub-intervals $[0, \tau]$, $(\tau, 1-\tau]$ and $(1-\tau, 1]$. The parameter τ, which determine the points separating the uniform meshes, is defined by

$$\tau = \min\left\{\frac{1}{4}, \frac{\sqrt{\varepsilon}}{\sqrt{\alpha}}\ln N\right\}. \tag{30}$$

Then, on the sub-interval $(\tau, 1-\tau]$, a uniform mesh with $\frac{N}{6}$ mesh points is placed and on each of the sub-intervals $[0, \tau]$ and $(1-\tau, 1]$, a uniform mesh of $\frac{N}{12}$ mesh points is placed

Similarly, the interval $(1, d]$ is divided into three sub-intervals $(1, 1+\eta]$, $(1+\eta, d-\eta]$ and $(d-\eta, d]$, where

$$\eta = \min\left\{\frac{d-1}{4}, \frac{\sqrt{\varepsilon}}{\sqrt{\alpha}}\ln N\right\}, \tag{31}$$

and the interval $(d, 2]$ is divided into three sub-intervals $(d, d+\gamma]$, $(d+\gamma, 2-\gamma]$ and $(2-\gamma, 2]$, where

$$\gamma = \min\left\{\frac{2-d}{4}, \frac{\sqrt{\varepsilon}}{\sqrt{\alpha}}\ln N\right\}. \tag{32}$$

In practice, it is convenient to take $N = 12k, \ \ k \geq 2$.

From the above construction of $\overline{\Omega}^N$, it is clear that the transition points $\{\tau, d-\tau, 1+\eta, d-\eta, d+\gamma, 2-\gamma\}$ are the only points at which the mesh-size can change and that it does not necessarily change at each of these points.

5 The Discrete Problem

In this section, a classical finite difference operator with an appropriate Shishkin mesh is used to construct a numerical method for (1)–(2) which is shown later to be essentially first-order parameter-uniform convergent.

The discrete two-point boundary value problem is now defined to be

$$\begin{array}{l} L^N U(x_j) = -\varepsilon\delta^2 U(x_j) + a(x_j)U(x_j) + b(x_j)U(x_j-1) = f(x_j) \ \text{ on } \Omega^N, \\ U = u \ \text{ on } \Gamma^N \text{ and } U(x_j-1) = \phi(x_j-1) \text{ for } x_j \in {\Omega_1}^N \cup {\Omega_2}^N \text{ in Case } (i) \\ \text{and for } x_j \in {\Omega_1}^N \text{ in Case } (ii) \end{array} \tag{33}$$

Here, $\delta^2 V(x_j) = \dfrac{(D^+ - D^-)V(x_j)}{\overline{h}_j}, \quad D^+V(x_j) = \dfrac{V(x_{j+1}) - V(x_j)}{h_{j+1}}$, $D^-V(x_j) = \dfrac{V(x_j) - V(x_{j-1})}{h_j}$, $h_j = x_j - x_{j-1}$, $\overline{h}_j = \dfrac{h_{j+1}+h_j}{2}$, $\overline{h}_0 = \dfrac{h_1}{2}$, $\overline{h}_N = \dfrac{h_N}{2}$.

The problem (33) can be rewritten as

$$
\begin{aligned}
L_1^N U(x_j) &= -\varepsilon\delta^2 U(x_j) + a(x_j)U(x_j) = g(x_j) \ \text{ on } {\Omega_1}^N \cup {\Omega_2}^N \\
L_2^N U(x_j) &= -\varepsilon\delta^2 U(x_j) + a(x_j)U(x_j) + b(x_j)U(x_j - 1) = f(x_j) \ \text{ on } {\Omega_3}^N \\
U &= u \ \text{ on } \Gamma^N,\ D^- U(x_{N/4}) = D^+ U(x_{N/4}), D^- U(x_{N/2}) = D^+ U(x_{N/2}),
\end{aligned}
\tag{34}
$$

in Case (i) and

$$
\begin{aligned}
L_1^N U(x_j) &= -\varepsilon\delta^2 U(x_j) + a(x_j)U(x_j) = g(x_j) \ \text{ on } {\Omega_1}^N \\
L_2^N U(x_j) &= -\varepsilon\delta^2 U(x_j) + a(x_j)U(x_j) + b(x_j)U(x_j - 1) = f(x_j) \ \text{ on } {\Omega_2}^N \cup {\Omega_3}^N \\
U &= u \ \text{ on } \Gamma^N,\ D^- U(x_{N/3}) = D^+ U(x_{N/3}),\ D^- U(x_{2N/3}) = D^+ U(x_{2N/3})
\end{aligned}
\tag{35}
$$

in Case (ii).

The following discrete results are analogous to those for the continuous case.

Lemma 7 *Let conditions* (3) *and* (4) *hold. Then, for any mesh function Ψ, the inequalities $\Psi \geq 0$ on Γ^N, in Case (i)*

$$L_1^N \Psi \geq 0 \text{ on } \Omega_1^N \cup \Omega_2^N, L_2^N \Psi \geq 0 \text{ on } \Omega_3^N \text{ and } D^+\Psi(x_j) - D^-\Psi(x_j) \leq 0, j = N/4, N/2,$$

$$\text{and } \ L_1^N \Psi \geq 0 \text{ on } \Omega_1^N, L_2^N \Psi \geq 0 \text{ on } \Omega_1^N \cup \Omega_2^N \text{ and } D^+\Psi(x_j) - D^-\Psi(x_j) \leq 0,$$

$j = N/3, 2N/3$ in Case (ii) imply that $\Psi \geq 0$ on $\overline{\Omega}^N$.

Proof The result follows by using similar arguments as in Lemma 7 of [4].

An immediate consequence of this is the following discrete stability result.

Lemma 8 *Let conditions* (3) *and* (4) *hold. Then, for any mesh function Ψ satisfying*

$$D^+\Psi(x_j) - D^-\Psi(x_j) = 0,\ j = N/4, N/2 \text{ in Case } (i),\ \text{ then for } 0 \leq j \leq N,$$

$$|\Psi(x_j)| \leq \max\left\{|\Psi(x_0)|, |\Psi(x_N)|,\ \frac{1}{\alpha}||L_1^N\Psi||_{\Omega_1^N \cup \Omega_2^N},\ \frac{1}{\alpha}||L_2^N\Psi||_{\Omega_3^N}\right\},$$

$$\text{and } D^+\Psi(x_j) - D^-\Psi(x_j) = 0,\ j = N/3, 2N/3 \text{ in Case } (ii),\ \text{ then for } 0 \leq j \leq N,$$

$$|\Psi(x_j)| \leq \max\left\{|\Psi(x_0)|, |\Psi(x_N)|,\ \frac{1}{\alpha}||L_1^N\Psi||_{\Omega_1^N},\ \frac{1}{\alpha}||L_2^N\Psi||_{\Omega_2^N \cup \Omega_3^N}\right\}.$$

Proof By using similar arguments as in Lemma 8 of [4], it is not hard to derive the results.

6 Error Estimate

Analogous to the continuous case, the discrete solution U can be decomposed into V and W which are defined to be the solutions of the following discrete problems:

$$\begin{gathered} L_1^N V(x_j) = g(x_j),\ x_j \in {\Omega_1}^N \cup {\Omega_2}^N, \\ V(0) = v(0),\ V(x_{N/4-1}) = v(d-), V(x_{N/4+1}) = v(d+),\ V(x_{N/2-1}) = v(1-) \\ L_2^N V(x_j) = f(x_j),\ x_j \in {\Omega_3}^N,\ V(x_{N/2+1}) = v(1+),\ V(2) = v(2) \end{gathered}$$

and

$$\begin{gathered} L_1^N W(x_j) = 0,\ x_j \in {\Omega_1}^N \cup {\Omega_2}^N,\ L_2^N W(x_j) = 0,\ x_j \in {\Omega_3}^N, \\ W(0) = w(0), \quad W(2) = w(2), \\ D^- W(x_j) + D^- V(x_j) = D^+ W(x_j) + D^+ V(x_j), j = N/4, N/2, \end{gathered}$$

in Case (i) and

$$\begin{gathered} L_1^N V(x_j) = g(x_j),\ x_j \in {\Omega_1}^N,\ V(0) = v(0),\ V(x_{N/3-1}) = v(1-) \\ L_2^N V(x_j) = f(x_j),\ x_j \in {\Omega_2}^N \cup {\Omega_3}^N, \\ V(x_{N/3+1}) = v(1+),\ V(x_{2N/3-1}) = v(d-),\ V(x_{2N/3+1}) = v(d+),\ V(2) = v(2) \end{gathered}$$

and

$$\begin{gathered} L_1^N W(x_j) = 0,\ x_j \in {\Omega_1}^N,\ L_2^N W(x_j) = 0,\ x_j \in {\Omega_2}^N \cup {\Omega_3}^N, \\ W(0) = w(0),\ W(2) = w(2), \\ D^- W(x_j) + D^- V(x_j) = D^+ W(x_j) + D^+ V(x_j), j = N/3, 2N/3, \end{gathered}$$

in Case (ii).

The error at each point $x_j \in \overline{\Omega}^N$ is denoted by $e(x_j) = U(x_j) - u(x_j)$. Then the local truncation error $L^N e(x_j)$, for $j \neq N/4, N/2$ in Case (i) and $j \neq N/3, 2N/3$ in Case (ii), has the decomposition

$$L^N e(x_j) = L^N (V - v)(x_j) + L^N (W - w)(x_j).$$

The error in the smooth and singular components are bounded in the following

Theorem 2 *Let conditions* (3) *and* (4) *hold. If* v *denotes the smooth component of the solution of* (1)–(2) *and* V *the smooth component of the solution of the problem* (33)*, then*

$$|L_i^N(V-v)(x_j)| \leq C\,N^{-1},\ \textit{for } i=1,2 \textit{ and } j \neq N/4, N/2,\ \textit{ in Case } (i),$$

and

$$|L_i^N(V-v)(x_j)| \leq C\,N^{-1}\,\textit{for } i=1,2 \textit{ and } j \neq N/3, 2N/3,\ \textit{ in Case } (ii).$$

If w denotes the singular component of the solution of (1)–(2) *and W the singular component of the solution of the problem* (33)*, then*

$$|L_i^N(W-w)(x_j)| \leq C\,N^{-1}\ln N,\ \textit{for } i=1,2 \textit{ and } j \neq N/4, N/2,\ \textit{ in Case } (i),$$

and

$$|L_i^N(W-w)(x_j)| \leq C\,N^{-1}\ln N,\ \textit{for } i=1,2 \textit{ and } j \neq N/3, 2N/3,\ \textit{ in Case } (ii).$$

Proof As the expression derived for the local truncation error in V and W and estimates for the derivatives of the smooth and singular components are exactly in the form found in Chap. 6 of [5], the required bounds hold good.

Note: The following arguments are applicable only to Case (i), for Case (ii) separate arguments are given.

At the points $x_j,\ j = N/4, N/2,$

$$\begin{aligned}(D^+ - D^-)e(x_j) &= (D^+ - D^-)(U-u)(x_j)\\ &= (D^+ - D^-)U(x_j) - (D^+ - D^-)u(x_j).\end{aligned}$$

Recall that $(D^+ - D^-)U(x_j) = 0$ for $j = N/4, N/2$. Let $h^* = \max\{h_{N/4}, h_{N/2}\}$, where $h_j = h_j^- = h_j^+$, $h_j^- = x_j - x_{j-1}$ and $h_j^+ = x_{j+1} - x_j$ for $j = N/4, N/2$.

Then

$$|(D^+ - D^-)e(x_j)| \ \leq \ C\,\frac{h^*}{\varepsilon},\ \text{ for } j = N/4, N/2. \tag{36}$$

Define a set of discrete barrier functions on $\overline{\Omega}^N$ by

$$\omega(x_j) = \begin{cases} \dfrac{\Pi_{k=1}^{j}(1+\sqrt{\alpha}h_k/\sqrt{\varepsilon})}{\Pi_{k=1}^{N/4}(1+\sqrt{\alpha}h_k/\sqrt{\varepsilon})}, & 0 \le j \le \frac{N}{4} \\ \dfrac{\Pi_{k=j}^{(3N/8)-1}(1+\sqrt{\alpha}h_{k+1}/\sqrt{\varepsilon})}{\Pi_{k=N/4}^{(3N/8)-1}(1+\sqrt{\alpha}h_{k+1}/\sqrt{\varepsilon})}, & \frac{N}{4} \le j \le \frac{3N}{8} \\ \dfrac{\Pi_{k=3N/8}^{j-1}(1+\sqrt{\alpha}h_k/\sqrt{\varepsilon})}{\Pi_{k=3N/8}^{(N/2)-1}(1+\sqrt{\alpha}h_k/\sqrt{\varepsilon})}, & \frac{3N}{8} \le j \le \frac{N}{2} \\ \dfrac{\Pi_{k=j}^{(5N/8)-1}(1+\sqrt{\alpha}h_{k+1}/\sqrt{\varepsilon})}{\Pi_{k=N/2}^{(5N/8)-1}(1+\sqrt{\alpha}h_{k+1}/\sqrt{\varepsilon})}, & \frac{N}{2} \le j \le \frac{5N}{8} \\ \dfrac{\Pi_{k=5N/8}^{j-1}(1+\sqrt{\alpha}h_k/\sqrt{\varepsilon})}{\Pi_{k=5N/8}^{(3N/4)-1}(1+\sqrt{\alpha}h_k/\sqrt{\varepsilon})}, & \frac{5N}{8} \le j \le \frac{3N}{4} \\ \dfrac{\Pi_{k=j}^{N-1}(1+\sqrt{\alpha}h_{k+1}/\sqrt{\varepsilon})}{\Pi_{k=3N/4}^{N-1}(1+\sqrt{\alpha}h_{k+1}/\sqrt{\varepsilon})}, & \frac{3N}{4} \le j \le N. \end{cases} \tag{37}$$

Note that

$$\omega(0)=0,\ \ \omega(d)=1,\ \ \omega(1)=1,\ \ \omega(1+d)=1,\ \ \omega(2)=0 \tag{38}$$

and from (37), for $0 \le j \le N$,

$$0 \le \omega(x_j) \le 1. \tag{39}$$

Let $\overline{\Omega}_i^N = \{x_j\}_{j=0}^{\frac{N}{4}}$, $\overline{\Omega}_{ii}^N = \{x_j\}_{j=\frac{N}{4}}^{\frac{3N}{8}}$, $\overline{\Omega}_{iii}^N = \{x_j\}_{j=\frac{3N}{8}}^{\frac{N}{2}}$, $\overline{\Omega}_{iv}^N = \{x_j\}_{j=\frac{N}{2}}^{\frac{5N}{8}}$, $\overline{\Omega}_{v}^N = \{x_j\}_{j=\frac{5N}{8}}^{\frac{3N}{4}}$ and $\overline{\Omega}_{vi}^N = \{x_j\}_{j=\frac{3N}{4}}^{N}$.

Proceeding as in [4], we find that, for $x_j \in \overline{\Omega}_i^N$, $\overline{\Omega}_{iii}^N$ and $\overline{\Omega}_v^N$,

$$D^+\omega(x_j) = \frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}\omega(x_j) \text{ and } D^-\omega(x_j) = \frac{\sqrt{\alpha}}{\sqrt{\varepsilon}(1+\sqrt{\alpha}h_j/\sqrt{\varepsilon})}\omega(x_j)$$

and for $x_j \in \overline{\Omega}_{ii}^N$, $\overline{\Omega}_{iv}^N$ and $\overline{\Omega}_{vi}^N$,

$$D^+\omega(x_j) = -\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}(1+\sqrt{\alpha}h_{j+1}/\sqrt{\varepsilon})}\omega(x_j) \text{ and } D^-\omega(x_j) = -\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}\omega(x_j).$$

Thus for $x_j \in \overline{\Omega}_l^N, l = i, \ldots, vi,$

$$\delta^2 \omega(x_j) \leq \frac{2\alpha}{\varepsilon}\, \omega(x_j).$$

In particular, at the points x_j, $j = N/4, N/2$ and $3N/4$,

$$\begin{aligned}(D^+ - D^-)\omega(x_j) &= -\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}(1+\sqrt{\alpha}h^+_{N/2}/\sqrt{\varepsilon})} - \frac{\sqrt{\alpha}}{\sqrt{\varepsilon}(1+\sqrt{\alpha}h^-_{N/2}/\sqrt{\varepsilon})} \\ &\leq -\frac{C}{\sqrt{\varepsilon}}.\end{aligned} \tag{40}$$

Proceeding as in [4], it is not hard to see that

$$L_1^N \omega(x_j) \geq (a(x_j) - 2\alpha)\, \omega(x_j),$$
$$\text{and } L_2^N \omega(x_j) \geq (a(x_j) - 2\alpha)\, \omega(x_j) + b(x_j).$$

For Case (ii), define a set of discrete barrier functions on $\overline{\Omega}^N$ by

$$\omega(x_j) = \begin{cases} \dfrac{\Pi_{k=1}^{j}(1+\sqrt{\alpha}h_k/\sqrt{\varepsilon})}{\Pi_{k=1}^{N/3}(1+\sqrt{\alpha}h_k/\sqrt{\varepsilon})}, & 0 \leq j \leq \frac{N}{3} \\[2ex] \dfrac{\Pi_{k=j}^{(N/2)-1}(1+\sqrt{\alpha}h_{k+1}/\sqrt{\varepsilon})}{\Pi_{k=N/3}^{(N/2)-1}(1+\sqrt{\alpha}h_{k+1}/\sqrt{\varepsilon})}, & \frac{N}{3} \leq j \leq \frac{N}{2} \\[2ex] \dfrac{\Pi_{k=N/2}^{j-1}(1+\sqrt{\alpha}h_k/\sqrt{\varepsilon})}{\Pi_{k=N/2}^{(2N/3)-1}(1+\sqrt{\alpha}h_k/\sqrt{\varepsilon})}, & \frac{N}{2} \leq j \leq \frac{2N}{3} \\[2ex] \dfrac{\Pi_{k=j}^{N-1}(1+\sqrt{\alpha}h_{k+1}/\sqrt{\varepsilon})}{\Pi_{k=2N/3}^{N-1}(1+\sqrt{\alpha}h_{k+1}/\sqrt{\varepsilon})}, & \frac{2N}{3} \leq j \leq N. \end{cases} \tag{41}$$

Note that

$$\omega(0) = 0,\ \ \omega(1) = 1,\ \ \omega(d) = 1,\ \ \omega(2) = 0 \tag{42}$$

and from (41), for $0 \leq j \leq N$,

$$0 \leq \omega(x_j) \leq 1. \tag{43}$$

Using the above discrete barrier functions (41) and the procedure adopted in Case (i) of this section, it is not hard to see that

$$L_1^N \omega(x_j) \geq (a(x_j) - 2\alpha)\ \omega(x_j)$$
$$\text{and } L_2^N \omega(x_j) \geq (a(x_j) - 2\alpha)\ \omega(x_j) + b(x_j).$$

The following theorem gives the required essentially first order parameter-uniform error estimate.

Theorem 3 *Let $u(x_j)$ be the solution of the problem (1)–(2) and $U(x_j)$ be the solution of the discrete problem (33). Then,*

$$|U(x_j) - u(x_j)| \leq C\, N^{-1} \ln N, \quad 0 \leq j \leq N.$$

Proof Consider the mesh function $\Psi^{\pm}$ given by

$$\Psi^{\pm}(x_j) = C_1 N^{-1} \ln N + C_2 \frac{\sqrt{\alpha}\, h^*}{\sqrt{\varepsilon}} \omega(x_j) \pm e(x_j), \quad 0 \leq j \leq N,$$

where C_1 and C_2 are constants. Then the result follows by using the mesh function $\Psi^{\pm}$, Theorem 2, Lemma 7 and the procedure adopted in Theorem 2 of [4].

7 Numerical Illustrations

In order to show the efficiency and accuracy of the proposed methods for the singularly perturbed linear second-order delay differential equations with a discontinuous source term, three numerical examples are presented in this section.

The point of discontinuity in the source term is assumed to occur in the interval $(0, 1)$ in Example 1, at the point $x = 1$ in Example 2 and in the interval $(1, 2)$ in Example 3. The ε-uniform order of convergence and the ε-uniform error constant are computed using the general methodology from [6]. The notations $D_\varepsilon^N,\ D^N,\ p^N,\ p^*$ and $C_{p^*}^N$ bear the same meaning as in [6].

Example 1 Consider the BVP

$$-\varepsilon\, u''(x) + a(x)u(x) + b(x)u(x-1) = 1, \text{ for } x \in (0, 0.5),$$
$$-\varepsilon\, u''(x) + a(x)u(x) + b(x)u(x-1) = 1 + x, \text{ for } x \in (0.5, 2)$$
$$u(x) = 1 \text{ on } [-1, 0],\ u(2) = 1$$

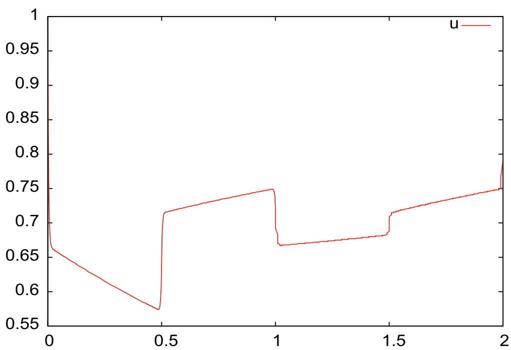
where $a(x) = 3 + x$ and $b(x) = -1$ for $x \in [0, 2]$.

The maximum pointwise errors and the rate of convergence for this BVP are presented in Table 1. The solution of this problem for $\varepsilon = 2^{-15}$ and $N = 512$ is portrayed in Fig. 1.

Table 1 Values of D_ε^N, D^N, p^N, p^* and $C_{p^*}^N$ for $\alpha = 0.9$

ε	Number of mesh points N			
	512	1024	2048	4096
2^{-3}	0.244E-03	0.121E-03	0.603E-04	0.302E-04
2^{-6}	0.326E-03	0.163E-03	0.812E-04	0.406E-04
2^{-9}	0.881E-03	0.440E-03	0.220E-03	0.110E-03
2^{-12}	0.239E-02	0.122E-02	0.612E-03	0.306E-03
2^{-15}	0.324E-02	0.196E-02	0.111E-02	0.611E-03
2^{-18}	0.324E-02	0.195E-02	0.111E-02	0.610E-03
2^{-21}	0.323E-02	0.195E-02	0.110E-02	0.609E-03
2^{-24}	0.323E-02	0.195E-02	0.110E-02	0.609E-03
2^{-27}	0.323E-02	0.195E-02	0.110E-02	0.609E-03
2^{-30}	0.323E-02	0.195E-02	0.110E-02	0.609E-03
D^N	0.324E-02	0.196E-02	0.111E-02	0.611E-03
p^N	0.728E+00	0.822E+00	0.858E+00	
C_p^N	0.769E+00	0.769E+00	0.720E+00	0.658E+00
Computed order of ε-uniform convergence, $p^* = 0.728106$				
Computed ε-uniform error constant, $C_{p^*}^N = 0.7687576$				

Fig. 1 Solution profile



Example 2 Consider the BVP

$$
\begin{aligned}
-\varepsilon\, u''(x) + a(x)u(x) + b(x)u(x-1) &= 1, \text{ for } x \in (0,1),\\
-\varepsilon\, u''(x) + a(x)u(x) + b(x)u(x-1) &= 1 + x, \text{ for } x \in (1,2)\\
u(x) \;=\; 1 - 0.3x \text{ on } [-1,0],\;\; u(2) &= 1
\end{aligned}
$$

where $a(x) = 3 + x$ and $b(x) = -1$ for $x \in [0, 2]$.

Table 2 Values of D_ε^N, D^N, p^N, p^* and $C_{p^*}^N$ for $\alpha = 0.9$

ε	Number of mesh points N			
	128	256	512	1024
2^{-3}	0.218E-03	0.104E-03	0.507E-04	0.250E-04
2^{-5}	0.158E-03	0.855E-04	0.442E-04	0.225E-04
2^{-8}	0.120E-02	0.633E-03	0.322E-03	0.162E-03
2^{-11}	0.158E-02	0.975E-03	0.566E-03	0.320E-03
2^{-14}	0.170E-02	0.104E-02	0.601E-03	0.339E-03
2^{-17}	0.174E-02	0.106E-02	0.613E-03	0.345E-03
2^{-20}	0.176E-02	0.107E-02	0.617E-03	0.348E-03
2^{-23}	0.176E-02	0.107E-02	0.619E-03	0.348E-03
2^{-26}	0.176E-02	0.107E-02	0.619E-03	0.349E-03
2^{-29}	0.177E-02	0.107E-02	0.619E-03	0.349E-03
2^{-32}	0.177E-02	0.107E-02	0.619E-03	0.349E-03
2^{-35}	0.177E-02	0.107E-02	0.619E-03	0.349E-03
D^N	0.177E-02	0.107E-02	0.619E-03	0.349E-03
p^N	0.718E+00	0.792E+00	0.828E+00	
C_p^N	0.147E+00	0.147E+00	0.140E+00	0.129E+00
Computed order of ε-uniform convergence, $p^* = 0.7184946$				
Computed ε-uniform error constant, $C_{p^*}^N = 0.1469943$				

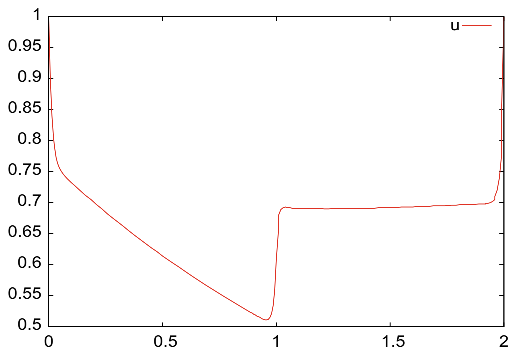


Fig. 2 Solution profile

The maximum pointwise errors and the rate of convergence for this BVP are presented in Table 2. The solution of this problem for $\varepsilon = 2^{-11}$ and $N = 256$ is portrayed in Fig. 2.

Table 3 Values of D_ε^N, D^N, p^N, p^* and $C_{p^*}^N$ for $\alpha = 0.9$

ε	Number of mesh points N			
	128	256	512	1024
2^{-3}	0.463E-03	0.236E-03	0.119E-03	0.598E-04
2^{-6}	0.198E-02	0.985E-03	0.490E-03	0.244E-03
2^{-9}	0.553E-02	0.282E-02	0.140E-02	0.697E-03
2^{-12}	0.112E-01	0.749E-02	0.398E-02	0.199E-02
2^{-15}	0.618E-02	0.656E-02	0.492E-02	0.306E-02
2^{-18}	0.618E-02	0.656E-02	0.492E-02	0.306E-02
D^N	0.112E-01	0.749E-02	0.492E-02	0.306E-02
p^N	0.586E+00	0.605E+00	0.685E+00	
C_p^N	0.578E+00	0.578E+00	0.571E+00	0.533E+00
Computed order of ε-uniform convergence, $p^* = 0.5860379$				
Computed ε-uniform error constant, $C_{p^*}^N = 0.5782413$				

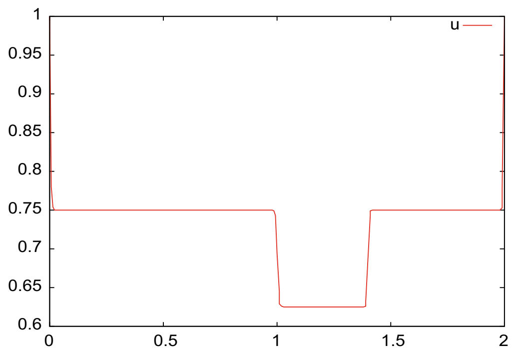


Fig. 3 Solution profile

Example 3 Consider the BVP

$$
\begin{aligned}
-\varepsilon u''(x) + a(x)u(x) + b(x)u(x-1) &= 1, \text{ for } x \in (0, 1.4),\\
-\varepsilon u''(x) + a(x)u(x) + b(x)u(x-1) &= 1.5, \text{ for } x \in (1.4, 2)\\
u(x) = 1 \text{ on } [-1, 0], \quad u(2) &= 1
\end{aligned}
$$

where $a(x) = 4$ and $b(x) = -2$ for $x \in [0, 2]$.

The maximum pointwise errors and the rate of convergence for this BVP are presented in Table 3. The solution of this problem for $\varepsilon = 2^{-15}$ and $N = 256$ is portrayed in Fig. 3.

References

1. Lange, C.G., Miura, R.M.: Singular perturbation analysis of boundary-value problems for differential-difference equations. SIAM J. Appl. Math. **42**(3), 502–530 (1982)
2. Farrell, P.A., Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Singularly perturbed differential equations with discontinuous source term. In: Proceedings of the Workshop on Analytical and Numerical Methods For Convection-Dominated and Singularly Perturbed Problems, Bulgaria, NOVA Science Publishers, New York, pp. 23–30 (1998)
3. Roos, H.G., Zarin, H.: A second-order scheme for singularly perturbed differential equations with discontinuous source term. J. Numer. Math. **10**(4), 275–289 (2002)
4. Manikandan, M., Shivaranjani, N., Miller, J.J.H., Valarmathi, S.: A parameter uniform first order convergent numerical method for a singularly perturbed delay differential equation. Adv. Appl. Math., Springer Proceedings in Mathematics and Statistics **87**, 71–88 (2014)
5. Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Fitted Numerical Methods for Singular Perturbation Problems. World Scientific Publishing Co., Singapore, New Jersey, London (1996)
6. Farrell, P.A., Hegarty, A.F., Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Robust Computational Techniques for Boundary Layers. CHAPMAN & HALL/CRC, Boca Raton, Florida, USA (2000)

Fitted Numerical Method with Linear Interpolation for Third-Order Singularly Perturbed Delay Problems

R. Mahendran and V. Subburayan

Abstract Singularly perturbed third-order ordinary delay differential equation with a discontinuous source term and discontinuous convection coefficient is considered in this article. To obtain a numerical approximate solution, a layer adapted mesh called the Shishkin mesh is constructed. On this mesh, a fitted finite difference method with piecewise linear interpolation is applied. Also, we present some classes of nonlinear problems. An error estimate is derived and is found to be of almost first-order convergence. Numerical results are given to validate the theoretical results

1 Introduction

Third-order singularly perturbed differential equations appear in various fields of applied sciences. For example, Howes [1] studied the boundary and interior layer phenomena exhibited by solutions of singularly perturbed third-order boundary value problems which govern the motion of thin liquid films subject to viscous, capillary and gravitational forces. The precise conditions specifying where and when the third-order derivative terms in the differential equations that can be neglected were derived, and improved estimates for the actual solutions in terms of solutions of the lower order models were constructed. He also presented a technique for replacing a third-order problem with an asymptotically equivalent second-order one that may have wider applications.

To analyze the analytical behavior of the solution of the third-order singularly perturbed differential equations, some of the researchers obtained an asymptotic expansion of the solution. For example, Nayfeh [2] presented perturbation techniques to obtain a asymptotic expansion for the third-order problem considered by Howes [1]. Zhao Weili [3] proved the existence, uniqueness and asymptotic estimates of

R. Mahendran · V. Subburayan (✉)
Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur 603203, TN, India
e-mail: suburayan123@gmail.com

Valarmathi S et al. (eds.), *Differential Equations and Applications*,
Springer Proceedings in Mathematics & Statistics 368,
https://doi.org/10.1007/978-981-16-7546-1_6

the solution of singularly perturbed Boundary Value Problems (BVPs) for a class of third-order nonlinear differential equations.

Numerical techniques are indispensable tools to obtain an approximate solution of the problems. Various numerical methods for third-order singularly perturbed differential equations without delay are available in the literature. To mention a few here, in [4], Valarmathi and Ramanujam applied asymptotic numerical Method; the authors in [5] applied asymptotic initial value Method; in [6], the authors applied the shooting method and Mahalik and Mohapatra [7] applied Newton's divided difference method for third-order singularly perturbed differential equations without delay.

Motivated by the works of [1, 4–6, 8–10], we consider the following boundary value problem (1). For this problem, a fitted finite difference method combined with piecewise linear interpolation is presented. It is proved that the present method is of almost first-order convergence. Further, we consider a nonlinear problem. The nonlinear problem is linearized as a sequence of linear problems using Newton's method of linearization. The present paper is organized as follows: The problem under investigation is stated in Sect. 2. Section 3 presents the existence and stability of the solution of the problem. Section 4 presents the derivative estimates of the solution. The present numerical method is discussed in Sect. 5. Section 6 deals with the error estimates of the present numerical method. Section 7 deals with a nonlinear problem. Section 8 presents numerical examples to illustrate the theoretical results. Finally, We conclude this article with remarks. The following notations are used in the rest of the article:

- ε is small parameter such that $0 < \varepsilon \ll 1$.
- The set $(0, 2)$ is denoted as Λ and its closure is $\bar{\Lambda}$. Further, $\Lambda^* = \Lambda^- \cup \Lambda^+$, where $\Lambda^- = (0, 1)$ and $\Lambda^+ = (1, 2)$.
- $\bar{\Lambda}^N$ denotes the set of mesh points $\{x_0,\ x_1,\ \ldots, x_N\}$.
- The norm $\| \star \|$ denotes the supremum norm $\| \psi \|_\Lambda = \sup_{x\in\Lambda} |\psi(x)|$.
- The collections $\mathbb{Y}$, $\mathbb{Y}_1$ and $\mathbb{Y}_2$, respectively, denote $C^1(\bar{\Lambda}) \cap C^2(\Lambda) \cap C^3(\Lambda^*)$, $C^0(\bar{\Lambda}) \cap C^1(\Lambda \cup \{2\})$ and $C^0(\bar{\Lambda}) \cap C^1(\Lambda) \cap C^2(\Lambda^*)$.

2 Continuous Problem

Find $u \in \mathbb{Y}$ such that

$$\begin{cases} -\varepsilon u'''(x) + a(x)u''(x) + b(x)u'(x) + c(x)u(x) + d(x)u'(x-1) = f(x),\ x \in \Lambda^*, \\ u(x) = \phi(x), x \in [-1, 0],\ u'(0) = \phi'(0),\ u'(2) = \ell, \end{cases} \tag{1}$$

where

$$a(x) = \begin{cases} a_1(x) > 0,\ x \in [0, 1], \\ a_2(x) < 0,\ x \in (1, 2], \end{cases} \quad f(x) = \begin{cases} f_1(x),\ x \in [0, 1], \\ f_2(x),\ x \in (1, 2], \end{cases}$$

$a_1(x) \geq \alpha_1 > \alpha > 0,\ 0 > -\alpha \geq -\alpha_2 \geq a_2(x) \geq -\alpha_2^*,\ b(x) \geq \beta_0 \geq 0,\ \gamma_0 \leq c(x) \leq 0, \eta_0 \leq d(x) \leq 0, 1 < \alpha \leq \min\{\alpha_1, \alpha_2\}, \alpha + \beta_0 + 7\gamma_0 + 3\eta_0 > 0$ and b, c, d are sufficiently differentiable on $\bar{\Lambda}$, ϕ is sufficiently differentiable on $[-1, 0]$ and a and f are sufficiently differentiable and bounded on Λ^*.

The above BVP (1) is transformed into the following:

Find $\bar{u} = (u_1, u_2),\ u_1 \in \mathbb{Y}_1,\ u_2 \in \mathbb{Y}_2$ such that

$$P_1\bar{u} = u_1'(x) - u_2(x) = 0,\ x \in (0, 2], \tag{2}$$

$$P_2\bar{u} = \begin{cases} -\varepsilon u_2''(x) + a_1(x)u_2'(x) + b(x)u_2(x) + c(x)u_1(x) \\ \qquad\qquad = f_1(x) - d(x)\phi'(x-1), x \in \Lambda^-, \\ -\varepsilon u_2''(x) + a_2(x)u_2'(x) + b(x)u_2(x) + c(x)u_1(x) \\ \qquad\qquad +d(x)u_2(x-1) = f_2(x), x \in \Lambda^+, \end{cases} \tag{3}$$

$u_1(0) = \phi(0),\ u_2(0) = \phi'(0),\ u_2(1-) = u_2(1+),\ u_2'(1-) = u_2'(1+),\ u_2(2) = \ell$, where $u_2(1-)$ and $u_2(1+)$ represent left and right limits of u_2 at $x = 1$, respectively.

3 Existence and Stability Results

This section presents the existence and stability results for the problem (1) stated above.

Theorem 1 *The problem* (1) *has a solution $\bar{u} = (u_1, u_2)$, where $u_1 \in \mathbb{Y}_1$ and $u_2 \in \mathbb{Y}_2$.*

Proof Refer [11, Theorem 2.1].

Theorem 2 (Maximum principle) *Suppose that $\bar{w} = (w_1,\ w_2)$ satisfies $w_1(0) \geq 0$ and $w_2(0) \geq 0,\ w_2(2) \geq 0,\ P_1\bar{w}(x) \geq 0, \forall x \in \Lambda \cup \{2\},\ P_2\bar{w}(x) \geq 0, \forall x \in \Lambda^*$ and $w_2'(1+) - w_2'(1-) = [w_2'](1) \leq 0$. Then $w_i(x) \geq 0,\ \forall x \in \bar{\Lambda},\ i = 1, 2$. Here $w_1 \in C^1(\Lambda)$ and $w_2 \in C^0(\bar{\Lambda}) \cap C^2(\Lambda^*)$.*

Proof Using the following barrier function $\bar{s}(x) = (s_1(x), s_2(x))$ and the procedure given in the proof of [12, Theorem 3.1], one can prove the theorem. Here,

$$s_1(x) = 1 + 3x,\ x \in \bar{\Lambda},\ s_2(x) = \begin{cases} \frac{1}{2} + \frac{3x}{2}, x \in [0, 1], \\ 3 - x, x \in [1, 2]. \end{cases}$$

An immediate consequence of the above theorem is the following stability result.

Corollary 1 *For any $\bar{u} = (u_1, u_2),\ u_1 \in \mathbb{Y}_1,\ u_2 \in \mathbb{Y}_2$, we have*

$$|u_i(x)| \le C \max\Big\{|u_1(0)|, |u_2(0)|, |u_2(2)|, \sup_{\varsigma_1 \in \Lambda \cup \{2\}} |P_1\bar{u}(\varsigma_1)|, \sup_{\varsigma_2 \in \Lambda^*} |P_2\bar{u}(\varsigma_2)|\Big\}, \forall x \in \bar{\Lambda},\ i = 1, 2.$$

Note: Using the above result, one can prove that the solution of the above problem (2)–(3) is unique, if it exists.

4 Derivative Estimates

Lemma 1 *Let $\bar{u}$ be the solution of the problem (2)–(3). Then, for $k = 0(1)3$, we have the following bounds $\| u_j^{(k)} \|_{\Lambda^*} \le C(1 + \varepsilon^{2-j-k}),\ j = 1, 2$.*

Proof Using the lines of proofs of [13, Lemma 3.2] and [14, Theorem 4.1], one can prove the lemma.

We use the following decomposition of the solution into regular and singular components for obtaining the uniform error estimates : $\bar{u}(x) = \bar{v}(x) + \bar{w}(x)$ where $\bar{v} = \bar{v}_0 + \varepsilon\bar{v}_1 + \varepsilon^2\bar{v}_2$ and $\bar{v}_0$, $\bar{v}_1$ and $\bar{v}_2$ are in turn defined, respectively, to be the solutions of the following problems:

Find $\bar{v}_0 = (v_{0,1}, v_{0,2})$, $v_{0,1} \in C^0(\bar{\Lambda}) \cap C^1(\Lambda^* \cup \{2\}), v_{0,2} \in C^0(\Lambda^* \cup \{0, 2\}) \cap C^1(\Lambda^*)$ such that

$$\begin{cases} v_{0,1}'(x) = v_{0,2}(x),\ x \in \Lambda^* \cup \{2\}, \\ a(x)v_{0,2}'(x) + b(x)v_{0,2}(x) + c(x)v_{0,1}(x) + d(x)v_{0,2}(x-1) = f(x),\ x \in \Lambda^*, \\ v_{0,1}(0) = \phi(0),\ v_{0,2}(x) = \phi'(x),\ x \in [-1, 0],\ v_{0,2}(2) = l, \end{cases} \tag{4}$$

$\bar{v}_1 = (v_{1,1}, v_{1,2})$, $v_{1,1} \in C^0(\bar{\Lambda}) \cap C^1(\Lambda^* \cup \{2\})$, $v_{1,2} \in C^0(\Lambda^* \cup \{0, 2\}) \cap C^1(\Lambda^*)$ such that

$$\begin{cases} v_{1,1}'(x) = v_{1,2}(x),\ x \in \Lambda^* \cup \{2\}, \\ a(x)v_{1,2}'(x) + b(x)v_{1,2}(x) + c(x)v_{1,1}(x) + d(x)v_{1,2}(x-1) = v_{0,2}''(x),\ x \in \Lambda^*, \\ v_{1,1}(0) = 0,\ v_{1,2}(x) = 0,\ x \in [-1, 0],\ v_{1,2}(2) = 0, \end{cases} \tag{5}$$

and $\bar{v}_2 = (v_{2,1}, v_{2,2})$, $v_{2,1} \in C^0(\bar{\Lambda}) \cap C^1(\Lambda^* \cup \{2\})$, $v_{2,2} \in C^0(\bar{\Lambda}) \cap C^1(\Lambda) \cap C^2(\Lambda^*)$ such that

$$\begin{cases} P_1\bar{v}_2 = 0,\ x \in \Lambda^* \cup \{2\}, \\ P_2\bar{v}_2 = v_{1,2}''(x),\ x \in \Lambda^*, \\ v_{2,1}(0) = 0,\ v_{2,2}(x) = 0,\ x \in [-1, 0],\ v_{2,2}(2) = 0. \end{cases} \tag{6}$$

Thus, the component $\bar{v}$ satisfies the following boundary value problem:

find $\bar{v} = (v_1, v_2),\ v_1 \in C^0(\bar{\Lambda}) \cap C^1(\Lambda^* \cup \{2\}),\ v_2 \in C^0(\Lambda^* \cup \{0, 2\}) \cap C^2(\Lambda^*)$ such that

$$\begin{cases} P_1\bar{v}(x) = 0,\ x \in \Lambda^* \cup \{2\}, \\ P_2\bar{v}(x) = f(x),\ x \in \Lambda^*, \\ v_1(0) = \phi(0),\ v_2(x) = \phi'(x),\ x \in [-1, 0], v_2(2) = \ell, \\ v_2(1) = v_{0,2}(1) + \varepsilon v_{1,2}(1) + \varepsilon^2 v_{2,2}(1). \end{cases} \tag{7}$$

Further, the component $\bar{w}$ satisfies the following boundary value problem:

find $\bar{w} = (w_1, w_2), w_1 \in C^0(\bar{\Lambda}) \cap C^1(\Lambda^* \cup \{2\}),\quad w_2 \in C^0(\Lambda^* \cup \{0, 2\}) \cap C^2(\Lambda^*)$ such that

$$\begin{cases} P_1\bar{w}(x) = 0,\ x \in \Lambda^* \cup \{2\}, \\ P_2\bar{w}(x) = 0,\ x \in \Lambda^*, \\ w_1(0) = 0,\ w_2(x) = 0,\ x \in [-1, 0], \\ [w_2](1) = -[v_2](1),\ [w_2'](1) = -[v_2'](1),\ w_2(2) = 0. \end{cases} \tag{8}$$

Note: It is observed that $\mid w_1(1) \mid = O(\varepsilon)$ and $\mid w_2(1) \mid = O(1)$.

Theorem 3 *Let $\bar{v}$ and $\bar{w}$ be the solution of the regular and singular components of the solution $\bar{u}$. Then, for $r = 0(1)3$, we have*

$$\| v_k^{(r)} \|_{\Lambda^*} \le C(1 + \varepsilon^{2-r}),\ k = 1, 2, \tag{9}$$

$$\mid w_k^{(r)}(x) \mid \le C\, \varepsilon^{2-k-r} \begin{cases} \exp(\frac{-\alpha(1-x)}{\varepsilon}), x \in \Lambda^-,\ k = 1, 2, \\ \exp(\frac{-\alpha(x-1)}{\varepsilon}) + \varepsilon \exp(\frac{-\alpha(2-x)}{\varepsilon}),\ x \in \Lambda^+. \end{cases} \tag{10}$$

Proof Applying the procedure given in [13, Lemma 3.2], [15, Lemma 3] and [14, Theorem 4.2], one can prove that

$$\| v_k^{(r)} \|_{\Lambda^*} \le C(1 + \varepsilon^{2-r}),\ r = 1, 2, 3.$$

To prove the second part of the theorem, we consider the following barrier functions, $\bar{\phi}^{\pm} = (\phi_1^{\pm}, \phi_2^{\pm})$ and $\bar{\psi}^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm})$, defined, respectively, in [0, 1] and [1, 2].

Let $x \in [0, 1]$. Then define $\bar{\phi}^{\pm} = (\phi_1^{\pm}, \phi_2^{\pm})$, where

$$\phi_k^{\pm}(x) = C\varepsilon^{2-k} \exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right) \pm w_k(x),\ x \in [0, 1],\ k = 1, 2.$$

Note that $\phi_1^{\pm}(0) \ge 0$, $\phi_2^{\pm}(0) \ge 0$, $\phi_2^{\pm}(1) \ge 0$ and

$$
\begin{aligned}
P_1^*\bar{\phi}^{\pm}(x) =& C\exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right)[\alpha-1] \pm P_1^*\bar{w}(x)\\
=& C\exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right)[\alpha-1] \pm 0 \geq 0,\\
P_2^*\bar{\phi}^{\pm}(x) =& C\exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right)[\frac{\alpha}{\varepsilon}(a_1-\alpha)+b(x)+c(x)\varepsilon] \pm P_2^*\bar{w}(x)\\
=& C\exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right)[\frac{\alpha}{\varepsilon}(a_1-\alpha)+b(x)+c(x)\varepsilon] \pm 0 \geq 0,
\end{aligned}
$$

where $P_1^*\bar{y} := y_1'(x) - y_2(x)$, $P_2^*\bar{y} := -\varepsilon y_2''(x) + a(x)y_2'(x) + b(x)y_2(x) + c(x)$ $y_1(x)$. Then by [4, Theorem 2.1], we have

$$
|w_k(x)| \leq C\varepsilon^{2-k}\exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right),\ x \in [0,1].
$$

Let $x \in [1,2]$, then define $\bar{\psi}^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm})$ where

$$
\begin{aligned}
\psi_1^{\pm}(x) &= C(2\alpha x\varepsilon - \frac{\varepsilon}{\alpha}\exp\left(\frac{-\alpha(x-1)}{\varepsilon}\right) - \frac{\varepsilon^2}{\alpha}\exp(\frac{-\alpha(2-x)}{\varepsilon})) \pm w_1(x),\ x \in [1,2],\\
\psi_2^{\pm}(x) &= C\left(2\varepsilon + \exp\left(\frac{-\alpha(x-1)}{\varepsilon}\right) - \varepsilon\exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right)\right) \pm w_2(x),\ x \in [1,2].
\end{aligned}
$$

Also note that $\psi_1^{\pm}(1) \geq 0$, $\psi_1^{\pm}(0) \geq 0$, $\psi_2^{\pm}(2) \geq 0$ and

$$
\begin{aligned}
P_1^*\bar{\psi}^{\pm}(x) =& C\,[2\varepsilon(\alpha-1)] \pm P_1^*\bar{w}(x)\\
=& C[2\varepsilon(\alpha-1)] \pm 0 \geq 0,\\
P_2^*\bar{\psi}^{\pm}(x) =& C\{\exp\left(\frac{-\alpha(x-1)}{\varepsilon}\right)\left[\frac{-\alpha}{\varepsilon}(\alpha+a_2(x)) + b(x) - \frac{\varepsilon}{\alpha}c(x)\right]\\
&+ \exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right)\left[\alpha(\alpha-a_2(x)) - b(x)\varepsilon - c(x)\frac{\varepsilon^2}{\alpha}\right]\\
&+ b(x)2\varepsilon + c(x)2\alpha x\varepsilon\} \pm P_2^*\bar{w}(x)\\
=& C\{\exp\left(\frac{-\alpha(x-1)}{\varepsilon}\right)\left[\frac{-\alpha}{\varepsilon}(\alpha+a_2(x)) + b(x) - \frac{\varepsilon}{\alpha}c(x)\right]\\
&+ \exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right)\left[\alpha(\alpha-a_2(x)) - b(x)\varepsilon - c(x)\frac{\varepsilon^2}{\alpha}\right]\\
&+ b(x)2\varepsilon + c(x)2\alpha x\varepsilon\} \mp C\exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right) \geq 0.
\end{aligned}
$$

Again, by using [4, Theorem 2.1], we have

$$
|\, w_k(x)\,| \leq C\,\varepsilon^{2-k}\left[\exp\left(\frac{-\alpha(x-1)}{\varepsilon}\right) + \varepsilon\exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right)\right],\ x \in \Lambda^+,\ k = 1,2.
$$

Successive differentiation of (7) and (8) yields the results (10).

Note: From the above theorem, it is easy to see that

$$| u_k(x) - v_k(x) | \leq C\varepsilon^{2-k} \begin{cases} \exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right), & x \in \Lambda^-, k = 1, 2, \\ \exp\left(\frac{-\alpha(x-1)}{\varepsilon}\right) + \varepsilon \exp\left(\frac{-\alpha(2-x)}{\varepsilon}\right), & x \in \Lambda^+. \end{cases} \tag{11}$$

5 Discrete Problem

In this section, mesh selection strategy namely piecewise uniform mesh (Shishkin mesh) is given and an upwind finite difference scheme with piecewise linear interpolation on the Shishkin mesh for the problem (2)–(3) is also presented.

5.1 Shishkin Mesh

The following Shishkin mesh $\bar{\Lambda}^N = \{x_k\}_{k=0}^N$ defined in [16, Sect. 6.1] is used in this article, where

$$\begin{aligned} x_0 = 0,\ x_k &= x_0 + k \star h_1,\ k = 1(1)\frac{N}{4},\ x_{k+\frac{N}{4}} = x_{\frac{N}{4}} + k \star h_2,\ k = 1(1)\frac{N}{4}, \\ x_{k+\frac{N}{2}} &= x_{\frac{N}{2}} + k \star h_3,\ k = 1(1)\frac{N}{8},\ x_{k+\frac{5N}{8}} = x_{\frac{5N}{8}} + k \star h_4, k = 1(1)\frac{N}{4}, \\ x_{k+\frac{7N}{8}} &= x_{\frac{7N}{8}} + k \star h_3,\ k = 1(1)\frac{N}{8}, \end{aligned}$$

$\rho_1 = \min\{0.5, \frac{2\varepsilon \log N}{\alpha}\}$ and $\rho_2 = \min\{0.25, \frac{2\varepsilon \log N}{\alpha}\}$, $h_1 = 4N^{-1}(1-\rho_1)$, $h_2 = 4N^{-1}\rho_1$, $h_3 = 8N^{-1}\rho_2$, $h_4 = 4N^{-1}(1-2\rho_2)$.

5.2 Finite Difference Scheme

On the Shishkin mesh $\bar{\Lambda}^N$, we define a fitted finite difference scheme to problem (2)–(3):

$$P_1^N \bar{U}(x_i) := D^- U_1(x_i) - U_2(x_i) = 0, \tag{12}$$

$$\begin{aligned} P_2^N \bar{U}(x_i) :=& -\varepsilon\delta^2 U_2(x_i) + a(x_i)(x_i) D^0 U_2(x_i) + b(x_i) U_2(x_i) + c(x_i) U_1(x_i) \\ &+ d(x_i) U_2^I(x_i) = f^*(x_i), \end{aligned} \tag{13}$$

$$U_1(x_0) = \phi(0), U_2(x_0) = \phi'(0),\ D^- U_2(x_{N/2}) = D^+ U_2(x_{N/2}),\ U_2(x_N) = l, \tag{14}$$

where

$$\delta^2 U_2(x_i) = \frac{2\left[D^+U_2(x_i) - D^-U_2(x_i)\right]}{h_i + h_{i+1}},\ D^0U_2(x_i) = \begin{cases} D^-U_2(x_i),\ i < N/2, \\ D^+U_2(x_i),\ i > N/2, \end{cases}$$

$$D^-U_2(x_i) = \frac{U_2(x_i) - U_2(x_{i-1})}{h_i},\ D^+U_2(x_i) = \frac{U_2(x_{i+1}) - U_2(x_i)}{h_{i+1}},$$

$$U_2^I(x_i) = \begin{cases} 0,\ x_i \in \Lambda^- \cap \bar{\Lambda}^N, \\ U_2(x_j)\frac{x_{j+1}-(x_i-1)}{h_{j+1}} + U_2(x_{j+1})\frac{(x_i-1)-x_j}{h_{j+1}},\ x_i \in \Lambda^+ \cap \bar{\Lambda}^N, \\ \qquad x_j \le x_i - 1 \le x_{j+1}, \end{cases}$$

$$h_i = x_i - x_{i-1},\ i = 1(1)N,$$

$$f^*(x_i) = \begin{cases} f(x_i) - d(x_i)\phi'(x_i - 1),\ x_i \in \Lambda^- \cap \bar{\Lambda}^N, \\ f(x_i),\ x_i \in \Lambda^+ \cap \bar{\Lambda}^N. \end{cases}$$

5.3 Discrete Stability Result

Lemma 2 (Discrete maximum principle) *Let* $\bar{Z}(x_i) = (Z_1(x_i), Z_2(x_i))$ *be mesh function satisfying* $Z_1(x_0) \ge 0$, $Z_2(x_0) \ge 0$, $Z_2(x_N) \ge 0$, $P_1^N\bar{Z}(x_i) \ge 0,\ x_i \in (0,2] \cap \bar{\Lambda}^N$, $P_2^N\bar{Z}(x_i) \ge 0,\ x_i \in \Lambda^* \cap \bar{\Lambda}^N$ *and* $[D]Z_2(x_{N/2}) \le 0$. *Then,* $Z_1(x_i) \ge 0$ *and* $Z_2(x_i) \ge 0$, $x_i \in \bar{\Lambda}^N$.

Proof Using the following mesh function $\bar{s}(x_i) = (s_1(x_i), s_2(x_i))$, where

$$s_1(x_i) = 1 + 3x_i,\ x_i \in \bar{\Lambda}^N \quad \text{and } s_2(x_i) = \begin{cases} \frac{1}{2} + \frac{3x_i}{2}, & x_i \in [0,1] \cap \bar{\Lambda}^N, \\ 3 - x_i, & x_i \in [1,2] \cap \bar{\Lambda}^N, \end{cases}$$

and the line of proof of [14, Lemma 5.1], one can easily prove the theorem.

A consequence of the above lemma is the following result.

Lemma 3 (Discrete stability result) *Let* $\bar{U}(x_i) = (U_1(x_i), U_2(x_i))$ *be any mesh function. Then,*

$$|\ U_k(x_i)\ | \le C \max\Big\{\ |\ U_1(x_0)\ |, |\ U_2(x_0)\ |, |\ U_2(x_N)\ |, \max_{j\in I_N} |\ P_1^N\bar{U}(x_j)\ |, \max_{j\in I_N\setminus\{0,N/2,N\}} |\ P_2^N\bar{U}(x_j)\ |\ \Big\}, i \in I_N,\ k = 1, 2.$$

Proof Applying the above Lemma 2 to the following mesh function $\bar{\psi}^{\pm} = (\psi_1^{\pm}, \psi_2^{\pm})$, where $\psi_k^{\pm}(x_i) = CC_1 s_k(x_i) \pm U_k(x_i)$, $x \in \bar{\Lambda}^N,\ k = 1, 2$, then we get result

$$| U_k(x_i) | \leq C \max\Big\{ | U_1(x_0) |, | U_2(x_0) |, | U_2(x_N) |, \max_{j \in I_N} | P_1^N \bar{U}(x_j) |, \max_{j \in I_N \setminus \{0, N/2, N\}} | P_2^N \bar{U}(x_j) | \Big\}, i \in I_N,\ k = 1, 2,$$

where $I_N = \{0, 1, \dots, N\}$.

Similar to the continuous function $\bar{u}$, the numerical solution $\bar{U}(x_i)$ defined by (12)–(13) is decomposed as $\bar{U}(x_i) = \bar{V}(x_i) + \bar{W}(x_i)$, where $\bar{V}(x_i)$ and $\bar{W}(x_i)$ satisfy the following:

$$\begin{cases} P_1^N \bar{V}(x_i) = 0,\ i \in I_N \setminus \{0\}, \\ P_2^N \bar{V}(x_i) = f^*(x_i),\ i \in I_N \setminus \{0, N/2, N\}, \\ V_j(x_0) = v_j(0),\ [D]V_2(x_{N/2}) = [v_2'](1),\ V_2(x_N) = v_2(2),\ j = 1, 2 \end{cases} \tag{15}$$

and

$$\begin{cases} P_1^N \bar{W}(x_i) = 0,\ i \in I_N \setminus \{0\}, \\ P_2^N \bar{W}(x_i) = 0,\ i \in I_N \setminus \{0, N/2, N\}, \\ W_j(x_0) = w_j(0),\ [D]W_2(x_{N/2}) = -[D]V_2(x_{N/2}),\ W_2(x_N) = w_2(2),\ j = 1, 2. \end{cases} \tag{16}$$

In the following, an estimate for the difference of $\bar{U}$ and $\bar{V}$ is given.

Theorem 4 *Let $\bar{U}(x_i)$ and $\bar{V}(x_i)$ be two mesh functions defined by* (12), (13) *and* (15)*, respectively. Then, for $k = 1, 2$, we have*

$$| U_k(x_i) - V_k(x_i) | \leq C \begin{cases} N^{-1}, & i \in 0(1)\frac{N}{4}, \frac{5N}{8}(1)N, \\ N^{-1} + \zeta, & i \in \frac{N}{4} + 1(1)\frac{5N}{8} - 1, \end{cases}$$

where $\zeta = \max\{|U_1(x_{\frac{N}{2}}) - V_1(x_{\frac{N}{2}})|, |U_2(x_{\frac{N}{2}}) - V_2(x_{\frac{N}{2}})|\}$.

Proof Consider a mesh function $\bar{\varphi}^{\pm}(x_i) = C_1(N^{-1}\bar{s}(x_i) + \bar{\psi}(x_i)) \pm (\bar{U}(x_i) - \bar{V}(x_i))$ where

$$\psi_1(x_i) = \begin{cases} 0, & i \in 0(1)\frac{N}{4}, \frac{5N}{8} + 1(1)N, \\ 3x_i\zeta, & i \in \frac{N}{4} + 1(1)\frac{5N}{8}, \end{cases}$$

$$\psi_2(x_i) = \begin{cases} 0, & i \in 0(1)\frac{N}{4}, \frac{5N}{8} + 1(1)N, \\ [1 + x_i]\zeta, & i \in \frac{N}{4} + 1(1)\frac{N}{2}, \\ [3 - x_i]\zeta, & i \in \frac{N}{2} + 1(1)\frac{5N}{8}, \end{cases}$$

and $s_1(x_i) = 1 + 3x_i,\ x_i \in \bar{\Lambda}^N$, $s_2(x_i) = \begin{cases} \frac{1}{2} + \frac{3x_i}{2},\ x_i \in \Lambda^- \cap \bar{\Lambda}^N, \\ 3 - x_i,\ x_i \in \Lambda^+ \cap \bar{\Lambda}^N. \end{cases}$ It is easy to see that $\varphi^{\pm}(x_0) \geq 0$ and $\varphi^{\pm}(x_N) \geq 0$ for a suitable choice of $C_1 > 0$.

When $x_i \in \Lambda^- \cap \bar{\Lambda}^N$, we have

$$\begin{aligned} P_1^N \bar{\varphi}^{\pm}(x_i) &= C_1(N^{-1} P_1^N \bar{s}(x_i) + P_1^N \bar{\psi}(x_i)) \pm P_1^N (\bar{U}(x_i) - \bar{V}(x_i)) \\ &= C_1(N^{-1} P_1^N \bar{s}(x_i) + P_1^N \bar{\psi}(x_i)) \pm 0 \geq 0, \\ P_2^N \bar{\varphi}^{\pm}(x_i) &= C_1(N^{-1} P_2^N \bar{s}(x_i) + P_2^N \bar{\psi}(x_i)) \pm P_2^N (\bar{U}(x_i) - \bar{V}(x_i)) \\ &= C_1(N^{-1} P_2^N \bar{s}(x_i) + P_2^N \bar{\psi}(x_i)) \pm 0 \geq 0. \end{aligned}$$

Similarly, one can prove that $P_1^N \bar{\varphi}^{\pm}(x_i) \geq 0$ and $P_2^N \bar{\varphi}^{\pm}(x_i) \geq 0$ on $\Lambda^+ \cap \bar{\Lambda}^N$.

6 Error Estimates

Theorem 5 *Let $\bar{V}(x_i)$ be a numerical solution of (7) defined by (15). Then*

$$| v_k(x_i) - V_k(x_i) | \leq CN^{-1}, \ i \in I_N, \ k = 1, 2.$$

Proof Now,

$$P_1^N(\bar{v}(x_i) - \bar{V}(x_i)) = P_1^N \bar{v}(x_i) - P_1^N \bar{V}(x_i) = (D^- - \frac{d}{dx}) v_1(x_i),$$

$$P_2^N(\bar{v}(x_i) - \bar{V}(x_i)) = \begin{cases} -\varepsilon\Big(\delta^2 - \frac{d^2}{dx^2}\Big) v_2(x_i) + a(x_i)\Big(D^- - \frac{d}{dx}\Big) v_2(x_i), \ i < N/2, \\ -\varepsilon\Big(\delta^2 - \frac{d^2}{dx^2}\Big) v_2(x_i) + a(x_i)\Big(D^+ - \frac{d}{dx}\Big) v_2(x_i) \\ \qquad + d(x_i)[v_2^I(x_i) - v_2(x_i - 1)], \ i > N/2. \end{cases}$$

Therefore, $| P_k^N(\bar{v}(x_i) - \bar{V}(x_i)) | \leq CN^{-1}, \ i \in I_N \setminus \{0, N/2, N\}, \ k = 1, 2$. Then by Lemma 3, we have $| v_k(x_i) - V_k(x_i) | \leq CN^{-1}, \ i \in I_N, \ k = 1, 2$, which concludes the proof.

Theorem 6 *Let $\bar{w}(x_i)$ be the solution of the problem (8) and let $\bar{W}(x_i)$ be its numerical solution defined by (16). If $\varepsilon \leq CN^{-1}$, then we have $| w_k(x_i) - W_k(x_i) | \leq CN^{-1}(\log N)^2, i \in I_N, k = 1, 2$.*

Proof Let $\bar{Z} = (Z_1, Z_2)$ where $Z_k(x_i) = w_k(x_i) - W_k(x_i)$. Note that

$$\begin{aligned} | U_k(x_i) - u_k(x_i) | &\leq | U_k(x_i) - V_k(x_i) | + | V_k(x_i) - v_k(x_i) | + | v_k(x_i) - u_k(x_i) | \\ &\leq C \begin{cases} N^{-1}, \ i \in 1(1)\frac{N}{4}, \frac{5N}{8}(1)N, \\ N^{-1} + \zeta, \ i \in \frac{N}{4} + 1(1)\frac{5N}{8} - 1, \end{cases} + CN^{-1} \\ &\quad + C\varepsilon^{2-k} \begin{cases} \exp(\frac{-\alpha(1-x_i)}{\varepsilon}), x_i \in \Lambda^-, \\ \exp(\frac{-\alpha(x_i-1)}{\varepsilon}) + \varepsilon \exp(\frac{-\alpha(2-x_i)}{\varepsilon}), x_i \in \Lambda^+, \end{cases} \end{aligned}$$

where $\zeta = \max\{|U_1(x_{\frac{N}{2}}) - V_1(x_{\frac{N}{2}})|, |U_2(x_{\frac{N}{2}}) - V_2(x_{\frac{N}{2}})|\}$. Also, note that

$$
\begin{aligned}
| Z_k(x_i) | \le & | U_k(x_i) - V_k(x_i) | + | (u_k(x_i)) - v_k(x_i) | \\
& \le C \begin{cases} N^{-1}, \; i \in 0(1)\frac{N}{4}, \frac{5N}{8}(1)N, \\ N^{-1} + \zeta, \; i \in \frac{N}{4} + 1(1)\frac{5N}{8} - 1, \end{cases} \\
& + C\varepsilon^{2-k} \begin{cases} \exp\left(\frac{-\alpha(1-x_i)}{\varepsilon}\right), \; x_i \in \Lambda^-, \\ \exp\left(\frac{-\alpha(x_i-1)}{\varepsilon}\right) + \varepsilon \exp\left(\frac{-\alpha(2-x_i)}{\varepsilon}\right), \; x_i \in \Lambda^+. \end{cases}
\end{aligned}
$$

So that $| Z_k(x_i) | \le CN^{-1}, i \in 0(1)\frac{N}{4}, \frac{5N}{8}(1)N, k = 1, 2$. To prove the remaining part, consider the mesh function $\bar{\varphi}^{\pm} = (\varphi_1^{\pm}, \varphi_2^{\pm})$, where

$$
\varphi_1^{\pm}(x_i) = \begin{cases} C_1 N^{-1}[1 + 3x_i + \frac{\rho}{\varepsilon^2}(x_i - 1 + \rho_1)] \pm Z_1(x_i), \; x_i \in [1 - \rho_1, 1] \cap \bar{\Lambda}^N, \\ C_1 N^{-1}[1 + 3x_i + \frac{\rho}{\varepsilon^2}(x_i - 1 + \rho_2)] \pm Z_1(x_i), \; x_i \in [1, 1 + \rho_2] \cap \bar{\Lambda}^N, \end{cases}
$$

$$
\varphi_2^{\pm}(x_i) = \begin{cases} C_1 N^{-1}[(\frac{1}{2} + \frac{3x_i}{2}) + \frac{\rho}{\varepsilon^2}(x_i - 1 + \rho_1)] \pm Z_2(x_i), \; x_i \in [1 - \rho_1, 1] \cap \bar{\Lambda}^N, \\ C_1 N^{-1}[(3 - x_i) + \frac{\rho}{\varepsilon^2}(1 + \rho_2 - x_i)] \pm Z_2(x_i), \; x_i \in [1, 1 + \rho_2] \cap \bar{\Lambda}^N, \end{cases}
$$

and $\varphi_k^{\pm}(x_i) = 0, \; x_i \notin [1 - \rho_1, 1 + \rho_2]$ and $\rho = \min\{\rho_1, \rho_2\}$, by the above result $\varphi_k^{\pm}(x_{\frac{N}{4}}) \ge 0, k = 1, 2$ and $\varphi_k^{\pm}(x_{\frac{5N}{8}}) \ge 0$, for suitable choice of $C_1 > 0$.

When $x_i \in (1 - \rho_1, 1)$,

$$
\begin{aligned}
P_1^N \bar{\varphi}(x_i) &\ge C_1 N^{-1}\left\{1 + \frac{\rho}{\varepsilon^2}\right\} \pm P_1^N \bar{Z}(x_i) = C_1 N^{-1}\left\{1 + \frac{\rho}{\varepsilon^2}\right\} \pm P_1^N \bar{w}(x_i) - P_1 \bar{w}(x_i) \\
&= C_1 N^{-1}\left\{1 + \frac{\rho}{\varepsilon^2}\right\} \pm (P_1^N - P_1)\bar{w}(x_i) \ge 0, \\
P_2^N \bar{\varphi}(x_i) &\ge C_1 N^{-1}\left\{\frac{3\alpha}{2} + \frac{5\beta_0}{4} + 4\gamma_0 + \frac{\rho}{\varepsilon^2}(\alpha + \frac{\gamma_0}{2})\right\} \pm P_2^N \bar{Z}(x_i) \ge 0.
\end{aligned}
$$

Similarly, one can prove that $P_1^N \bar{\varphi}(x_i) \ge 0$ and $P_2^N \bar{\varphi}(x_i) \ge 0$, when $x_i \in [1, 1 + \rho_2]$. At the point $x_{N/2}$, we have $[D]\varphi_2^{\pm} < 0$. Then by Lemma 2, we have $\varphi_k^{\pm}(x_i) \ge 0$ on $[1 - \rho_1, 1 + \rho_2]$, which concludes the proof.

Theorem 7 *Let $\bar{u}$ be the solution of* (2)–(3) *and its numerical solution $\bar{U}(x_i)$ is given by* (12)–(13)*. If $\varepsilon \le CN^{-1}$, then we have* $|u_k(x_i) - U_k(x_i)| \le CN^{-1}(\log N)^2$, $i = 0(1)N, k = 1, 2$.

Proof Using Theorems 5 and 6, one can prove the theorem.

7 Nonlinear Problem

Consider the nonlinear BVP

$$-\varepsilon u'''(x) = F(x, u(x), u'(x), u''(x), \tilde{u}'(x)),\ x \in \Lambda^*, \tag{17}$$

$$u(x) = \phi(x),\ u'(x) = \phi'(x),\ x \in [-1, 0],\ u'(2) = \ell, \tag{18}$$

where $\tilde{u}'(x) = u'(x-1)$,

$$|F_{u''}(x, u, u', u'', \tilde{u}')| \geq \alpha > 0,\ F_{u'}(x, u, u', u'', \tilde{u}') \geq \beta \geq 0,$$
$$F_u(x, u, u', u'', \tilde{u}') \leq \gamma \leq 0,\ F_{\tilde{u}'}(x, u, u', u'', \tilde{u}') \leq \eta \leq 0.$$

Assume that the reduced problem

$$F(x, u_0(x), u_0'(x), u_0''(x), \tilde{u}_0'(x)) = 0,$$
$$u_0'(x) = \phi'(x),\ x \in [-1, 0],\ u_0'(2) = \ell$$

has a solution. The Newton method of linearization discussed in [17, Part II, Sect. 14] is applied to (17)–(18). This method yields the sequence $\{u^{[k+1]}\}_{k=0}^{\infty}$ of successive approximations with a proper choice of initial guess. For each fixed non-negative integer k, $\bar{u}^{[k+1]}(x) = (u_1^{[k+1]}, u_2^{[k+1]})$ is the solution of the following linear problem:

$$P_1^{[k]}\bar{u}^{[k+1]} = u_1'^{[k+1]}(x) - u_2^{[k+1]}(x) = 0, \tag{19}$$

$$\begin{aligned} P_2^{[k]}\bar{u}^{[k+1]} &= -\varepsilon u_2''^{[k+1]}(x) + a^k(x)u_2'^{[k+1]}(x) + b^k(x)u_2^{[k+1]}(x) \\ &\quad + c^k(x)u_1^{[k+1]}(x) + d^k(x)\tilde{u}_2^{[k+1]}(x) = F^k(x),\ x \in \Lambda^*, \end{aligned} \tag{20}$$

where

$$\begin{aligned} &a^k(x) = F_{u_2'}(x, u_1^{[k]}, u_2^{[k]}, u_2'^{[k]}, \tilde{u}_2^{[k]}),\ b^k(x) = F_{u_2}(x, u_1^{[k]}, u_2^{[k]}, u_2'^{[k]}, \tilde{u}_2^{[k]}), \\ &c^k(x) = F_{u_1}(x, u_1^{[k]}, u_2^{[k]}, u_2'^{[k]}, \tilde{u}_2^{[k]}),\ d^k(x) = F_{\tilde{u}_2}(x, u_1^{[k]}, u_2^{[k]}, u_2'^{[k]}, \tilde{u}_2^{[k]}), \\ &F^k(x) = F(x, u_1^{[k]}(x), u_2^{[k]}(x), u_2'^{[k]}(x), \tilde{u}_2^{[k]}(x)) - a^k(x)u_2'^{[k]} - b^k(x)u_2^{[k]} \\ &- c^k(x)u_1^{[k]} - d^k(x)\tilde{u}_2^{[k]}. \end{aligned}$$

For convenience, respectively, we denote $F(x, u_1(x), u_2(x), u_2'(x), \tilde{u}_2(x))$, $F(x, u_1^{[k]}(x), u_2^{[k]}(x), u_2'^{[k]}(x), \tilde{u}_2^{[k]}(x))$, $F_{u_1}(x, u_1^{[k]}(x), u_2^{[k]}(x), u_2'^{[k]}(x), \tilde{u}_2^{[k]}(x))$, $F_{u_2}(x, u_1^{[k]}(x), u_2^{[k]}(x), u_2'^{[k]}(x), \tilde{u}_2^{[k]}(x))$, $F_{u_2'}(x, u_1^{[k]}(x), u_2^{[k]}(x), u_2'^{[k]}(x), \tilde{u}_2^{[k]}(x))$, and $F^k_{\tilde{u}_2}(x, u_1^{[k]}(x), u_2^{[k]}(x), u_2'^{[k]}(x), \tilde{u}_2^{[k]}(x))$ by F, F^k, $F^k_{u_1}$, $F^k_{u_2}$, $F^k_{u_2'}$ and $F^k_{\tilde{u}_2}$. To prove the convergence of the successive iterations, the following theorem is established.

Theorem 8 *Suppose* $|F_{u_1u_1}|$, $|F_{u_2u_2}|$, $|F_{u_1u_2}|$, $|F_{u_2\tilde{u}_2}|$, $|F_{\tilde{u}_2u_1}|$, $|F_{\tilde{u}_2\tilde{u}_2}|$, $|F_{u_2'u_2'}|$, $|F_{u_2u_2'}|$, $|F_{u_1u_2'}|$ *and* $|F_{u_2'\tilde{u}_2}|$ *are bounded above by* M. *Let* $\{\bar{u}^{[k]}\}_0^{\infty}$ *be the Newton sequence defined by* (19)–(20). *Then, for all* $x \in \bar{\Lambda}$, *we have*

$$\| \bar{u}^{[k+1]} - \bar{u} \| \leq M \| \bar{u}^{[k]} - \bar{u} \|^2 .$$

Proof It is easy to see that

$$\begin{aligned}
P_1^{[k]}(\bar{u}^{[k+1]} - \bar{u}) &= 0,\\
P_2^{[k]}(\bar{u}^{[k+1]} - \bar{u}) &= F^k - u_1^{[k]}F_{u_1}^k - u_2^{[k]}F_{u_2}^k - u_2^{'[k]}F_{u_2}^k - \tilde{u}_2^{[k]}F_{\tilde{u}_2}^k\\
&\quad - (F - u_1F_{u_1}^k - u_2F_{u_2}^k - u_2'F_{u_2'}^k - \tilde{u}_2F_{\tilde{u}_2}^k)\\
&= F^k - F + (u_1 - u_1^{[k]})F_{u_1}^k + (u_2 - u_2^{[k]})F_{u_2}^k + (u_2' - u_2^{'[k]})F_{u_2'}^k\\
&\quad + (\tilde{u}_2 - \tilde{u}_2^{[k]})F_{\tilde{u}_2}^k\\
&= F^k - \Big\{(F^k + (u_1 - u_1^{[k]})F_{u_1}^k + (u_2 - u_2^{[k]})F_{u_2}^k + (u_2' - u_2^{'[k]})F_{u_2'}^k\\
&\quad + (\tilde{u}_2 - \tilde{u}_2^{[k]})F_{\tilde{u}_2}^k) + \frac{1}{2}\Big[((u_1 - u_1^{[k]})^2F_{u_1u_1}(\bar{\theta}) + (u_2 - u_2^{[k]})^2F_{u_2u_2}(\bar{\theta})\\
&\quad + (u_2' - u_2^{'[k]})^2F_{u_2'u_2'}(\bar{\theta}) + (\tilde{u}_2 - \tilde{u}_2^{[k]})^2F_{\tilde{u}_2\tilde{u}_2}(\bar{\theta}))\\
&\quad + 2(u_1 - u_1^{[k]})(u_2 - u_2^{[k]})F_{u_1u_2}(\bar{\theta}) + 2(u_2 - u_2^{[k]})(u_2' - u_2^{'[k]})F_{u_2u_2'}(\bar{\theta})\\
&\quad + 2(u_2 - u_2^{[k]})(\tilde{u}_2 - \tilde{u}_2^{[k]})F_{u_2\tilde{u}_2}(\bar{\theta}) + 2(u_2' - u_2^{'[k]})(\tilde{u}_2 - \tilde{u}_2^{[k]})F_{u_2'\tilde{u}_2}(\bar{\theta})\\
&\quad + 2(u_1 - u_1^{[k]})(u_2' - u_2^{'[k]})F_{u_1u_2'}(\bar{\theta}) + 2(\tilde{u}_2 - \tilde{u}_2^{[k]})(u_1 - u_1^{[k]})F_{\tilde{u}_2u_1}(\bar{\theta})\Big]\Big\}\\
&\quad + (u_1 - u_1^{[k]})F_{u_1}^k + (u_2 - u_2^{[k]})F_{u_2}^k + (u_2' - u_2^{'[k]})F_{u_2'}^k + (\tilde{u}_2 - \tilde{u}_2^{[k]})F_{\tilde{u}_2}^k,
\end{aligned}$$

where $\bar{\theta} = (x, \theta, \theta', \theta'', \tilde{\theta}')$ is such that $(x, u_1, u_2, u_2', \tilde{u}_2) > \bar{\theta} > (x, u_1^{[k]}, u_2^{[k]}, u_2^{'[k]}, \tilde{u}_2^{[k]})$.

$$\begin{aligned}
P_2^{[k]}(\bar{u}^{[k+1]} - \bar{u}) &= -\frac{1}{2}\Big\{((u_1 - u_1^{[k]})^2F_{u_1u_1}(\bar{\theta}) + (u_2 - u_2^{[k]})^2F_{u_2u_2}(\bar{\theta})\\
&\quad + (u_2' - u_2^{'[k]})^2F_{u_2'u_2'}(\bar{\theta}) + (\tilde{u}_2 - \tilde{u}_2^{[k]})^2F_{\tilde{u}_2\tilde{u}_2}(\bar{\theta}))\\
&\quad + 2(u_1 - u_1^{[k]})(u_2 - u_2^{[k]})F_{u_1u_2}(\bar{\theta}) + 2(u_2 - u_2^{[k]})(u_2' - u_2^{'[k]})F_{u_2u_2'}(\bar{\theta})\\
&\quad + 2(u_2 - u_2^{[k]})(\tilde{u}_2 - \tilde{u}_2^{[k]})F_{u_2\tilde{u}_2}(\bar{\theta}) + 2(u_2' - u_2^{'[k]})(\tilde{u}_2 - \tilde{u}_2^{[k]})F_{u_2'\tilde{u}_2}(\bar{\theta})\\
&\quad + 2(u_1 - u_1^{[k]})(u_2' - u_2^{'[k]})F_{u_1u_2'}(\bar{\theta}) + 2(\tilde{u}_2 - \tilde{u}_2^{[k]})(u_1 - u_1^{[k]})F_{\tilde{u}_2u_1}(\bar{\theta})\Big\}.
\end{aligned}$$

Then, we have

$$\begin{aligned} | P_2^{[k]}(\bar{u}^{[k+1]} - \bar{u}) | \leq M \Big\{ & | u_1^{[k]} - u_1 |^2 + | u_2^{[k]} - u_2 |^2 + | u_2^{'[k]} - u_2 |^2 \\ & + | \tilde{u}_2^{[k]} - \tilde{u}_2 |^2 + | u_1^{[k]} - u_1 || u_2^{[k]} - u_2 | \\ & + | u_2^{[k]} - u_2 || u_2^{'[k]} - u_2^{'} | + | u_2^{'[k]} - u_2^{'} || \tilde{u}_2^{[k]} - \tilde{u}_2 | \\ & + | \tilde{u}_2^{[k]} - \tilde{u}_2 || u_1^{[k]} - u_1 | + | u_2^{[k]} - u_2 || \tilde{u}_2^{[k]} - \tilde{u}_2 | \\ & + | u_1^{[k]} - u_1 || u_2^{'[k]} - u_2^{'} | \Big\} \\ & \leq M \parallel \bar{u}^{[k]} - \bar{u} \parallel^2 . \end{aligned}$$

Then by Corollary 1, we have the desired result.

To observe the nature of the solution of the linearized singularly perturbed third-order delay differential equations, we consider the following:

$$-\varepsilon u'''(x) + a(x)u''(x) - u(x-1) = \begin{cases} 1, & x \in (0, 1], \\ -1, & x \in (1, 2), \end{cases} \tag{21}$$

$$u(x) = 1, \ x \in [-1, 0], \ u'(x) = 0, \ x \in [-1, 0], \ u'(2) = 2, \tag{22}$$

where $a(x) = 1, \ x \in [0, 1]$ and $a(x) = -1, \ x \in (1, 2]$. The reduced problem solution of (21)–(22) is

$$\begin{aligned} u_0(x) &= 1 + \frac{x^2}{2}, \ x \in [0, 1], \\ u_0(x) &= \frac{-2}{3} + 2x - \frac{(x-2)^3}{6}, \ x \in (1, 2]. \end{aligned}$$

The solution of (21)–(22) is

$$\begin{aligned} u(x) &= (1 - C_1) - \frac{C_1 x}{\varepsilon} + C_1 \exp(\frac{x}{\varepsilon}) + \frac{x^2}{2}, \ x \in [0, 1], \\ u(x) &= D_1 + x\Big[\frac{1}{2} - \varepsilon + \frac{F_1}{\varepsilon}\exp(\frac{-2}{\varepsilon}) - \frac{2C_1}{\varepsilon} + \frac{C_1}{2}\exp(\frac{1}{\varepsilon})\Big] + F_1 \exp(\frac{-x}{\varepsilon}) \\ & \quad + \Big(1 + \frac{C_1}{\varepsilon}\Big)\frac{x^2}{2} - \frac{(x-1)^3}{6} + \varepsilon\frac{(x-1)^2}{2} - \frac{C_1 \varepsilon}{2}\exp(\frac{(x-1)}{\varepsilon}), \ x \in [1, 2], \end{aligned}$$

where

$$\begin{aligned} C_1 &= \frac{\exp(\frac{-1}{\varepsilon})[\varepsilon^3 + F_1 \exp(\frac{-1}{\varepsilon})]}{1 - \frac{\varepsilon}{2}\exp(\frac{-1}{\varepsilon})} = O(\varepsilon \exp(-1/\varepsilon)), \\ F_1 &= \varepsilon \exp(\frac{1}{\varepsilon}) \frac{[\frac{1}{2} - \varepsilon(1 + \frac{1}{4}\exp(\frac{-1}{\varepsilon})) + \varepsilon^2(\frac{1}{2}\exp(\frac{-1}{\varepsilon}) - \exp(\frac{-2}{\varepsilon})) - \frac{\varepsilon^3}{2}(\exp(\frac{-1}{\varepsilon}) + 1)]}{1 - \frac{\varepsilon}{2} - \exp(\frac{-1}{\varepsilon}) + \exp(\frac{-2}{\varepsilon})(1 + \frac{\varepsilon}{2})} \\ &= O(\varepsilon \exp(1/\varepsilon)), \end{aligned}$$

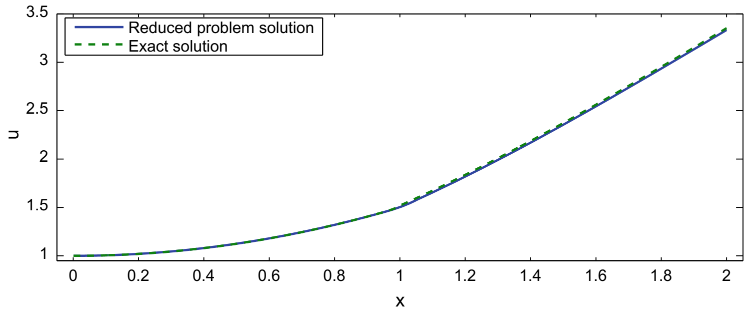


Fig. 1 The exact solution and reduced problem solution of the problem (21)–(22) for $\varepsilon = 2^{-6}$ and $N = 2^{10}$

$$D_1 = \frac{1}{2} + \varepsilon + C_1\frac{1}{2\varepsilon}(1 - 2\varepsilon + \varepsilon\exp(\frac{1}{\varepsilon}) + \varepsilon^2) - F_1\frac{1}{\varepsilon}(\exp(\frac{-2}{\varepsilon}) - \varepsilon\exp(\frac{-1}{\varepsilon})) = O(1).$$

It is observed that

$$u(x) \sim u_0(x) + O(\varepsilon)\exp\left(\frac{-(1-x)}{\varepsilon}\right) + O(\varepsilon),\ \ x \in [0, 1] \text{ and}$$
$$u(x) \sim u_0(x) + O(\varepsilon)\exp\left(\frac{-(x-1)}{\varepsilon}\right) + O(\varepsilon^2)\exp\left(\frac{-(2-x)}{\varepsilon}\right) + O(\varepsilon),\ \ x \in (1, 2].$$

In [9], the authors considered convection diffusion problem with discontinuous convection coefficient. They analyzed that the solution exhibits strong interior twin layers and no weak boundary layer at the boundary point(s). In the above model problem (21)–(22), we considered discontinuous convection coefficient with different signs in different sub domains and discontinuous source term. But in the above problem (21)–(22), the solution exhibits twin weak interior layers at $x = 1$ and a weak boundary layer at $x = 2$. Figure 1 presents the exact solution and reduced problem solution to (21)–(22).

From the above observation, one can see that the reduced problem solution is a reasonable approximate solution to the original problem. Therefore, choose the initial approximation as the reduced problem solution, that is, $\bar{u}^{[0]} = (u_1^{[0]}, u_2^{[0]})$ and $u_1^{[0]} = u_0$, $u_2^{[0]} = u_0'$.

Remark 1 The reduced problem of (2)–(3) is stated in (4).

8 Numerical Illustration

In this section, three examples are presented to illustrate the theory discussed in this paper. We use the double mesh principle to estimate the error and compute the experiment rate of convergence in our computed solution. For this, we put

$$D_\varepsilon^M = \max_{0 \le i \le M} \mid U_i^M - U_{2i}^{2M} \mid,$$

where U_i^M and U_{2i}^{2M} are the ith components of the numerical solutions on meshes of M and $2M$ points, respectively. We compute the uniform error and rate of convergence as

$$D^M = \max_\varepsilon D_\varepsilon^M \text{ and } \quad p^M = \log_2\left(\frac{D^M}{D^{2M}}\right).$$

For the following examples, the numerical results are presented for the values of perturbation parameter $\varepsilon \in \{2^{-6}, \dots, 2^{-25}\}$.

Example 1 Consider the following third-order equation

$$\begin{cases} -\varepsilon u'''(x) + a(x)u''(x) + b(x)u'(x) + c(x)u(x) + d(x)u'(x-1) = f(x), x \in \Lambda^*, \\ u(x) = \phi(x), x \in [-1, 0], u'(2) = 2, \phi \in C^1([-1, 0]), \end{cases}$$

where $a_1(x) = 16,\ a_2(x) = -17,\ b(x) = 1 + x,\ c(x) = \frac{-x}{4},\ d(x) = -x,\ f_1(x) = x,\ f_2(x) = 1 - x,\ \phi_1(x) = 1,\ \phi_2(x) = 0$.

Table 1 presents the values of D_k^M and $p_k^M,\ k = 1, 2$ corresponding to the solution components u_1 and u_2, respectively, of this example. Figure 2 represents the numerical solution, and Figs. 4 and 5 present the loglog plots for u_1 and for u_2.

Example 2 In this example, we consider $a_1(x) = 3\exp(x),\ a_2(x) = -3(x^2 + 1)$, $b(x) = |\sin(x)|,\ c(x) = \frac{-x}{20},\ d(x) = \frac{-x^2}{10},\ f_1(x) = x^3,\ f_2(x) = 1 - x^3,\ \phi_1(x) = 1$, $\phi_2(x) = 0$.

Table 2 presents the values of D_k^M and $p_k^M,\ k = 1, 2$ corresponding to the solution components u_1 and u_2, respectively, of this example. Figure 3 represents the numerical solution.

Example 3 Consider the nonlinear BVP

$$\begin{aligned} -\varepsilon u'''(x) &= -a(x)[u''(x)]^2 - 0.5[\tilde{u}'(x)]^2, \\ a(x) &= -1,\ x \in [0, 1],\ a(x) = 1,\ x \in (1, 2], \\ u(x) &= 1,\ x \in [-1, 0],\ u'(x) = 0,\ u'(2) = 1. \end{aligned}$$

Table 1 Maximum pointwise error estimates and convergence rates for various N of u_1 and u_2 of Example 1

	N (Number of mesh points)						
ε	32	64	128	256	512	1024	2048
D_1^M	1.6271e-2	6.9523e-3	3.3909e-3	1.7064e-3	8.6288e-4	4.3664e-4	2.2097e-4
p_1^M	1.2267e-0	1.0358e-0	9.9074e-1	9.8369e-1	9.8272e-1	9.8258e-1	–
D_2^M	4.5658e-2	3.5418e-2	2.6971e-2	1.9276e-2	1.2830e-2	7.5094e-3	4.3557e-3
p_2^M	3.6639e-1	3.9308e-1	4.8459e-1	5.8725e-1	7.7280e-1	7.8578e-1	–

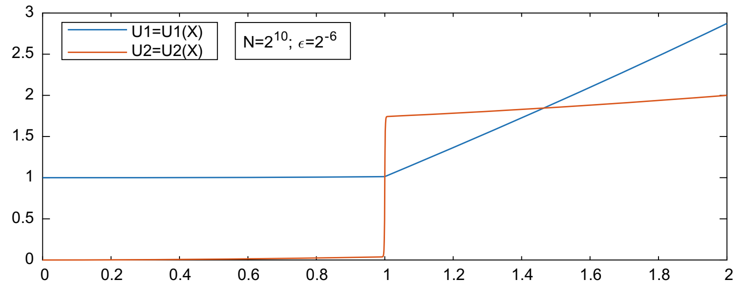


Fig. 2 Numerical solution of the above Example 1

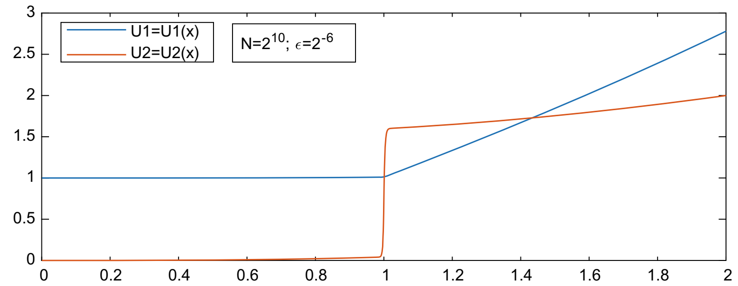


Fig. 3 Numerical solution of the above Example 2

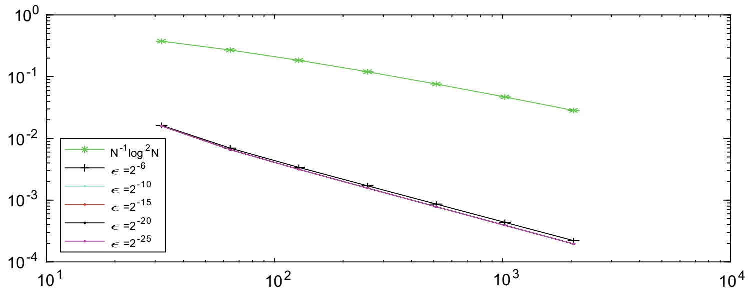


Fig. 4 Loglog plot for the component u_1 of Example 1

Tables 5 and 6 present the iterations of u_1 and u_2, respectively. Figures 6 and 7 represent the iterations of u_1 and u_2 for fixed $\varepsilon = 2^{-6}$ and $N = 32$.

Remark 2 Error tolarence of successive iterations is 10^{-3}.

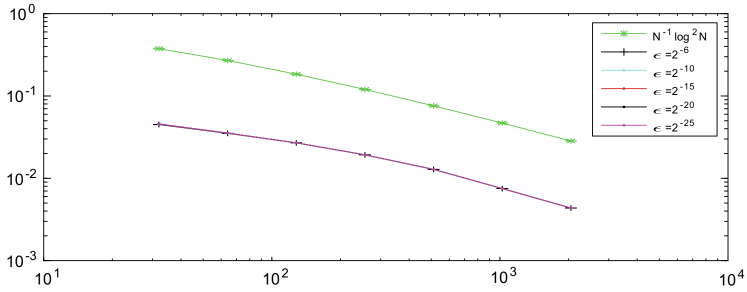


Fig. 5 Loglog plot for the component u_2 of Example 1

Table 2 Maximum pointwise error estimates and convergence rates for various N of u_1 and u_2 of Example 2

	N (Number of mesh points)						
ε	32	64	128	256	512	1024	2048
D_1^M	1.1306e-2	5.8553e-3	3.0512e-3	1.5975e-3	8.3800e-4	4.3939e-4	2.3006e-4
p_1^M	9.4934e-1	9.4039e-1	9.3354e-1	9.3080e-1	9.3145e-1	9.3349e-1	–
D_2^M	4.3856e-2	4.3080e-2	3.4874e-2	2.5771e-2	1.7173e-2	1.0721e-2	6.2195e-3
p_2^M	2.5783e-2	3.0487e-1	4.3640e-1	5.8563e-1	6.7963e-1	7.8560e-1	–

Table 3 Numerical results of Example 1 on uniform mesh

	N (Number of mesh points)						
	32	64	128	256	512	1024	2048
D_1^M	3.3389e-2	1.6646e-2	8.3124e-3	4.1554e-3	2.0790e-3	1.0405e-3	5.2030e-4
p_1^M	1.0042	1.0018	1.0003	9.9907e-1	9.9868e-1	9.9983e-1	–
D_2^M	7.5829e-2	7.5829e-2	7.5829e-2	7.5829e-2	7.5895e-2	7.5851e-2	7.5829e-2
p_2^M	0	0	-1.2580e-3	8.3811e-4	4.1991e-4	0	–

Table 4 Maximum pointwise error estimates and convergence rates

	N (Number of mesh points)						
ε	32	64	128	256	512	1024	2048
D_1^M	2.2960e-2	1.1518e-2	5.8358e-3	2.9442e-3	1.4792e-3	7.4147e-4	3.7118e-4
p_1^M	9.9516e-1	9.8093e-1	9.8707e-1	9.9310e-1	9.9630e-1	9.9829e-1	–
D_2^M	1.5622e-2	1.1728e-2	9.0401e-3	6.3420e-3	4.0544e-3	2.3786e-3	1.4228e-3
p_2^M	4.1368e-1	3.7549e-1	5.1140e-1	6.4546e-1	7.6938e-1	7.4138e-1	–

Table 5 Iterations of u_1 of the Example 3

x_i	$u_1^{[0]}$	$u_1^{[1]}$	$u_1^{[2]}$	$u_1^{[3]}$	$u_1^{[4]}$	$u_1^{[5]}$	$u_1^{[6]}$	$u_1^{[7]}$	$u_1^{[8]}$	$u_1^{[9]}$
0.0135	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.0271	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.0406	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.0542	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1656	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.2771	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.3885	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.5000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.6115	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.7229	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.8344	1.0000	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001	1.0001
0.9458	1.0000	1.0016	1.0009	1.0013	1.0011	1.0012	1.0012	1.0012	1.0012	1.0012
0.9594	1.0000	1.0020	1.0010	1.0016	1.0014	1.0015	1.0014	1.0015	1.0015	1.0015
0.9729	1.0000	1.0029	1.0014	1.0023	1.0019	1.0021	1.0020	1.0021	1.0020	1.0020
0.9865	1.0000	1.0052	1.0023	1.0040	1.0032	1.0036	1.0035	1.0035	1.0035	1.0035
1.0000	1.0000	1.0113	1.0047	1.0086	1.0067	1.0076	1.0072	1.0074	1.0073	1.0073
1.0135	1.0001	1.0213	1.0084	1.0160	1.0123	1.0141	1.0133	1.0136	1.0135	1.0136
1.0271	1.0004	1.0327	1.0127	1.0245	1.0187	1.0214	1.0203	1.0208	1.0205	1.0206
1.0406	1.0008	1.0444	1.0173	1.0333	1.0255	1.0292	1.0275	1.0282	1.0279	1.0281
1.0542	1.0015	1.0563	1.0221	1.0422	1.0324	1.0370	1.0350	1.0359	1.0355	1.0356
1.1656	1.0137	1.1501	1.0715	1.1155	1.0946	1.1042	1.1001	1.1018	1.1011	1.1014
1.2771	1.0384	1.2399	1.1296	1.1893	1.1615	1.1741	1.1687	1.1710	1.1700	1.1704
1.3885	1.0755	1.3273	1.1956	1.2650	1.2332	1.2474	1.2414	1.2439	1.2428	1.2433
1.5000	1.1250	1.4141	1.2689	1.3439	1.3099	1.3250	1.3186	1.3213	1.3202	1.3207
1.6115	1.1869	1.5024	1.3496	1.4274	1.3923	1.4078	1.4012	1.4040	1.4028	1.4033
1.7229	1.2613	1.5941	1.4377	1.5168	1.4812	1.4969	1.4902	1.4931	1.4919	1.4924
1.8344	1.3481	1.6918	1.5340	1.6136	1.5778	1.5936	1.5869	1.5897	1.5885	1.5890
1.9458	1.4473	1.7977	1.6396	1.7194	1.6835	1.6994	1.6926	1.6955	1.6943	1.6948
1.9594	1.4602	1.8108	1.6526	1.7324	1.6965	1.7124	1.7056	1.7085	1.7073	1.7078
1.9729	1.4733	1.8240	1.6658	1.7456	1.7097	1.7256	1.7188	1.7217	1.7205	1.7210
1.9865	1.4866	1.8373	1.6791	1.7589	1.7230	1.7389	1.7322	1.7350	1.7338	1.7343
2.0000	1.5000	1.8509	1.6927	1.7725	1.7366	1.7524	1.7457	1.7486	1.7473	1.7479

Table 6 Iterations of u_2 of the Example 3

x_i	$u_2^{[0]}$	$u_2^{[1]}$	$u_2^{[2]}$	$u_2^{[3]}$	$u_2^{[4]}$	$u_2^{[5]}$	$u_2^{[6]}$	$u_2^{[7]}$	$u_2^{[8]}$	$u_2^{[9]}$
0.0135	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0271	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0406	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.0542	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1656	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.2771	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.3885	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.5000	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.6115	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.7229	0	0.0001	0.0000	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001	0.0001
0.8344	0	0.0009	0.0006	0.0008	0.0007	0.0007	0.0007	0.0007	0.0007	0.0007
0.9458	0	0.0136	0.0072	0.0112	0.0094	0.0102	0.0099	0.0100	0.0099	0.0100
0.9594	0	0.0276	0.0130	0.0218	0.0176	0.0196	0.0187	0.0191	0.0189	0.0190
0.9729	0	0.0656	0.0281	0.0503	0.0396	0.0447	0.0425	0.0434	0.0430	0.0432
0.9865	0	0.1696	0.0678	0.1278	0.0985	0.1124	0.1063	0.1089	0.1078	0.1083
1.0000	0	0.4538	0.1719	0.3378	0.2563	0.2950	0.2780	0.2853	0.2822	0.2835
1.0135	0.0135	0.7380	0.2760	0.5478	0.4142	0.4775	0.4496	0.4617	0.4566	0.4587
1.0271	0.0271	0.8378	0.3185	0.6230	0.4736	0.5443	0.5132	0.5267	0.5210	0.5234
1.0406	0.0406	0.8703	0.3400	0.6494	0.4981	0.5695	0.5381	0.5517	0.5459	0.5484
1.0542	0.0542	0.8782	0.3542	0.6582	0.5101	0.5798	0.5492	0.5624	0.5568	0.5592
1.1656	0.1656	0.8414	0.4430	0.6580	0.5583	0.6031	0.5840	0.5921	0.5887	0.5901
1.2771	0.2771	0.8056	0.5212	0.6623	0.6005	0.6269	0.6159	0.6205	0.6186	0.6194
1.3885	0.3885	0.7841	0.5920	0.6789	0.6430	0.6577	0.6517	0.6542	0.6531	0.6536
1.5000	0.5000	0.7789	0.6582	0.7079	0.6884	0.6962	0.6930	0.6943	0.6938	0.6940
1.6115	0.6115	0.7916	0.7232	0.7488	0.7391	0.7430	0.7414	0.7421	0.7418	0.7419
1.7229	0.7229	0.8235	0.7905	0.8020	0.7976	0.7994	0.7986	0.7990	0.7988	0.7989
1.8344	0.8344	0.8760	0.8639	0.8683	0.8664	0.8672	0.8668	0.8670	0.8669	0.8670
1.9458	0.9458	0.9505	0.9479	0.9494	0.9485	0.9490	0.9487	0.9488	0.9488	0.9488
1.9594	0.9594	0.9623	0.9600	0.9613	0.9606	0.9610	0.9608	0.9608	0.9608	0.9608
1.9729	0.9729	0.9745	0.9726	0.9737	0.9730	0.9734	0.9732	0.9733	0.9732	0.9733
1.9865	0.9865	0.9870	0.9857	0.9865	0.9861	0.9863	0.9862	0.9862	0.9862	0.9862
2.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

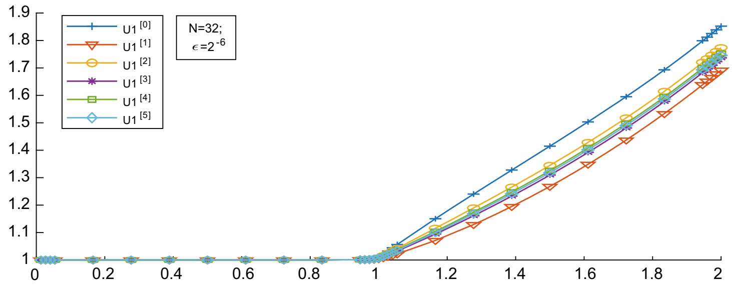


Fig. 6 Iterative numerical solutions of u_1 stated in Example 3

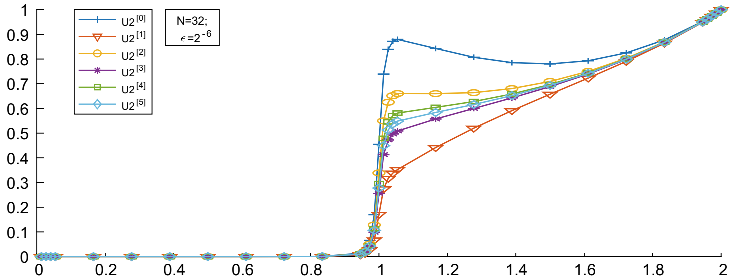


Fig. 7 Iterative numerical solutions of u_2 stated in Example 3

9 Conclusions

An uniformly valid numerical method for solving Third-order singularly perturbed delay differential equations is discussed in this article. We considered the problem with discontinuous source term and discontinuous convection coefficient. From Theorem 3, we observed that the solution component u_2 exhibits strong interior twin layers at $x = 1$ and a weak boundary layer at $x = 2$. This weak boundary layer occurs due to the presence of the delay term. But this will not happen in the case of the nondelay differential equations [9]. Since it exhibits interior and boundary layers, respectively, at $x = 1$ and $x = 2$, we divide the domain into five subdomains. On each subdomain, we define the mesh points with different mesh sizes. On this mesh, a fitted finite difference method with piecewise linear interpolation is applied. It has been observed that when $i > N/2$, the point $x_i - 1$ need not be a mesh point. Therefore, we are forced to apply the interpolation to approximate $u_2(x_i - 1)$. Further, it has been proved that the present method with piecewise linear interpolation gives almost linear convergence of order $O(N^{-1}(\log N)^2)$. Tables 1 and 2 validate Theorem 7. Further, the same finite difference method is applied on uniform mesh, but the numerical results are not satisfactory (See Table 3). This table presents the numerical results for Example 1 on a uniform mesh. From Figs. 2 and 3, we see that the component u_2 exhibits the interior twin layers at $x = 1$ and a weak boundary layer at $x = 2$. Figures 4 and 5, respectively, provide the loglog plot for the components u_1 and u_2 of Example 1. In Sect. 8 a nonlinear problem is considered and Newton's linearization method is applied. To illustrate the method, Example 3 is presented. The maximum pointwise error and computed rate of convergence for Example 3 is presented in Table 4. Figures 6 and 7 present the iterations of u_1 and u_2 for fixed $\varepsilon = 2^{-6}$ and $N = 32$. Figures 6 and 7 indicate that iterations converge.

References

1. Howes, F.A.: The asymptotic solution of a class of third-order boundary value problem arising in the theory of thin flow. SIAM J. Appl. Math. **43**, 993–1004 (1983)
2. Nayfeh, A.H.: Problems in Perturbation Methods. Wiley Inter Science, New York (1985)
3. Weili, Zhao: Singular perturbations of boundary value problems for a class of third order nonlinear ordinary differential equation. J. Differ. Equ. **82**, 265–278 (1990)
4. Valarmathi, S., Ramanujam, N.: An asymptotic numerical method for singularly perturbed third order ordinary differential equations of convection diffusion type. Comput. Math. Appl. **44**, 693–710 (2002)
5. Valanarasu, T., Ramanujam, N.: An asymptotic numerical method for singularly perturbed third-order ordinary differential equations with a weak interior layer. Int. J. Comput. Math. **84**, 333–346 (2007)
6. Christy Roja, J., Tamilselvan, A.: Numerical method for singularly perturbed third order ordinary differential equations of convection-diffusion type. Numer. Math. **7**, 265–287 (2014)
7. Mahalik, M.K., Mohapatra, J.: A robust numerical method for singularly perturbed third order boundary value problem with layer behaviour. Procedia. Eng. **127**, 258–262 (2015)
8. Lange, G.C., Miura, M.R.: Singular perturbation analysis of boundary-value problems for differential-difference equations. SIAM J. Appl. Math. **42**, 502–531 (1982)
9. Farrell, P.A., Hegarty, A.F., Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Global maximum norm parameter-uniform numerical method for a singularly perturbed convection-diffusion problems with discontinuous convection coefficient. Math. Comput. Model. **40**, 1375–1392 (2004)
10. Cen, Z.: A hybrid difference scheme for a singularly perturbed convection-diffusion problem with discontinuous convection coefficient. Appl. Math. Comput. **169**, 689–699 (2005)
11. Subburayan, V., Mahendran, R.: An ε-uniform numerical method for third order singularly perturbed delay differential equations with discontinuous convection coefficient and source term. Appl. Math. Comput. **331**, 404–415 (2018)
12. Subburayan, V., Ramanujam, N.: An initial value method for singularly perturbed third order delay differential equations. In: Proceedings of the International Conference on Mathematical Sciences, pp. 221–229. Elsevier (2014). (ISBN:978-93-5107-261-4)
13. Farrell, P.A., Hegarty, A.F., Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Robust Computational Techniques for Boundary Layers. Chapman Hall/CRC, Boca Raton (2000)
14. Subburayan, V.: A robust computational method for system of singularly perturbed differential difference equations with discontinuous convection coefficients. Int. J. Comput. Methods **13**, 1641008-1–1641008-17 (2016)
15. Mythili Priyadharshini, R., Ramanujam, N., Shanthi, V.: Approximation of derivative in a system of singularly perturbed convection diffusion equations. J. Appl. Math. Comput. **29**, 137–151 (2009)
16. Subburayan, V.: A parameter uniform numerical method for singularly perturbed delay problems with discontinuous convection coefficients. Arab J. Math. Sci. **22**, 191–206 (2016)
17. Doolan, E.P., Miller, J.J.H., Schilders, W.H.A.: Uniform Numerical Methods for Problems with Initial and Boundary Layers. Boole Press, Dublin (1980)

A Parameter-Uniform Essentially First-Order Convergence of a Fitted Mesh Method for a Class of Parabolic Singularly Perturbed System of Robin Problems

R. Ishwariya, John J. H. Miller, and Valarmathi Sigamani

Abstract In this paper, a class of linear parabolic systems of singularly perturbed Robin problems is considered. The components of the solution $\vec{v}$ of this system exhibit parabolic boundary layers with sublayers. The numerical method suggested in this paper is composed of a classical finite difference scheme on a piecewise-uniform Shishkin mesh. This method is proved to be first-order convergent in time and essentially first-order convergent in the space variable in the maximum norm uniformly in the perturbation parameters.

Keywords Singular perturbation problems · Boundary layers · Linear parabolic differential equations · Robin boundary conditions · Finite difference schemes · Shishkin meshes · Parameter-uniform convergence.

1 Introduction

In this paper, a class of linear parabolic systems of singularly perturbed second-order differential equations of reaction-diffusion type with initial and Robin boundary conditions is considered.

In [1, 2, 10, 11], a general introduction to singular perturbation problems and parameter-uniform numerical methods to solve the problems are established. Franklin et al. [3] constructed a parameter-uniform numerical method to solve a linear system of singularly perturbed second-order parabolic partial differential equations of reaction-diffusion type with given initial and Dirichlet boundary conditions. In [6], a linear parabolic singularly perturbed Dirichlet boundary value problem is considered, and a uniformly convergent numerical method with respect to the small

R. Ishwariya (✉) · V. Sigamani
Department of Mathematics, Bishop Heber College, Tiruchirappalli, TN, India

J. J. H. Miller
Institute of Numerical Computation and Analysis, Dublin, Ireland
e-mail: jm@incaireland.org

Valarmathi S et al. (eds.), *Differential Equations and Applications*,
Springer Proceedings in Mathematics & Statistics 368,
https://doi.org/10.1007/978-981-16-7546-1_7

parameter in the maximum norm comprising a standard finite difference operator on a fitted piecewise-uniform mesh is established.

In [7], J. L. Gracia et.al. suggested a uniformly convergent numerical method with layer-adapted piecewise-uniform mesh to solve a parabolic coupled system of singularly perturbed reaction-diffusion equations. In [8], a linear system of second-order singularly perturbed differential equations of reaction-diffusion type with Dirichlet boundary conditions is considered, and essentially second-order convergent numerical approximations are constructed.

Consider the following parabolic singularly perturbed linear system of second-order differential equations with initial and Robin boundary conditions.

$$\frac{\partial \vec{v}}{\partial t} - E\frac{\partial^2 \vec{v}}{\partial x^2} + A\vec{v} = \vec{f}, \text{ on } \Omega, \tag{1}$$

$$\begin{gathered} \vec{v}(0,t) - E_*\frac{\partial \vec{v}}{\partial x}(0,t) = \vec{\xi}_L(t),\ \vec{v}(1,t) + E_*\frac{\partial \vec{v}}{\partial x}(1,t) = \vec{\xi}_R(t),\ 0 \le t \le T, \\ \vec{v}(x,0) = \vec{\xi}_B(x),\ 0 \le x \le 1, \end{gathered} \tag{2}$$

where $\Omega = \{(x,t) : 0 < x < 1,\ 0 < t \le T\}$, $\bar{\Omega} = \Omega \cup \Gamma$, $\Gamma = \Gamma_L \cup \Gamma_B \cup \Gamma_R$ with $\Gamma_L = \{(0,t) : 0 \le t \le T\}$, $\Gamma_R = \{(1,t) : 0 \le t \le T\}$ and $\Gamma_B = \{(x,0) : 0 < x < 1\}$. Here, for all $(x,t) \in \bar{\Omega}$, $\vec{v}(x,t)$ and $\vec{f}(x,t)$ are column $n-$vectors, E, E_* and A are $n \times n$ matrices, $E = diag(\vec{\varepsilon})$, $\vec{\varepsilon} = (\varepsilon_1, ..., \varepsilon_n)$, $E_* = diag(\overrightarrow{\sqrt{\varepsilon}})$, $\overrightarrow{\sqrt{\varepsilon}} = (\sqrt{\varepsilon}_1, ..., \sqrt{\varepsilon}_n)$ with $0 < \varepsilon_i \le 1$ for all $i = 1, ..., n$. The parameters ε_i are assumed to be distinct and for convenience, to have the ordering $\varepsilon_1 < \cdots < \varepsilon_n$.

The operator form of problem (1), (2) is

$$\vec{H}\vec{v} = \vec{f} \text{ on } \Omega,$$

$$\vec{b}_0\vec{v}(0,t) = \vec{\xi}_L(t),\ \vec{b}_1\vec{v}(1,t) = \vec{\xi}_R(t),\ \vec{v}(x,0) = \vec{\xi}_B(x),$$

where the operators $\vec{H}, \vec{b}_0, \vec{b}_1$ are defined by

$$\vec{H} = I\frac{\partial}{\partial t} - E\frac{\partial^2}{\partial x^2} + A,\ \vec{b}_0 = I - E_*\frac{\partial}{\partial x},\ \vec{b}_1 = I + E_*\frac{\partial}{\partial x}$$

where I is the identity operator. The reduced problem corresponding to (1), (2) is defined by

$$\frac{\partial \vec{v}_0}{\partial t} + A\vec{v}_0 = \vec{f}, \text{ on } \Omega,\ \vec{v}_0 = \vec{v} \text{ on } \Gamma_B. \tag{3}$$

The problem (1), (2) is said to be singularly perturbed in the following sense.

Each component $v_i,\ i = 1, ..., n$ of the solution $\vec{v}$ of (1), (2) is expected to exhibit twin layers of width $O(\sqrt{\varepsilon_n})$ at $x = 0$ and $x = 1$ while the components $v_i,\ i = 1, ..., n-1$ have additional twin sublayers of width $O(\sqrt{\varepsilon_{n-1}})$, the components $v_i,\ i = 1, ..., n-2$ have additional twin sublayers of width $O(\sqrt{\varepsilon_{n-2}})$ and so on.

2 Solution to the Continuous Problem

Standard theoretical results on the existence of the solution of (1), (2) are stated, without proof, in this section. See [4, 5] for more details. For all $(x, t) \in \bar{\Omega}$, it is assumed that the components $a_{ij}(x, t)$ of $A(x, t)$ satisfy the inequalities

$$a_{ii}(x,t) > \sum_{\substack{j\neq i\\ j=1}}^{n} |a_{ij}(x,t)| \ \text{for}\ 1 \le i \le n, \ \text{and}\ a_{ij}(x,t) \le 0 \ \text{for}\, i \neq j \tag{4}$$

and for some α,

$$0 < \alpha < \min_{\substack{(x,t)\in\bar{\Omega}\\ 1\le i\le n}} \left(\sum_{j=1}^{n} a_{ij}(x,t) \right). \tag{5}$$

It is also assumed, without loss of generality, that

$$\sqrt{\varepsilon_n} \le \frac{\sqrt{\alpha}}{6}. \tag{6}$$

The norms, $\parallel y \parallel_D = \sup\{|y(x,t)| : (x,t) \in D\}$ for any scalar-valued function y and domain D, and $\parallel \vec{y} \parallel_D = \max\limits_{1\le k\le n} \parallel y_k \parallel_D$ for any vector-valued function $\vec{y} = (y_1, ..., y_n)^T$, are introduced. When $D = \bar{\Omega}$ or Ω, the subscript D is usually dropped. In a compact domain D, a function u is said to be Hölder continuous of degree $\lambda,\ 0 < \lambda \le 1$, if, for all $(x_1, t_1), (x_2, t_2) \in D$,

$$|v(x_1,t_1) - v(x_2,t_2)| \le C(|x_1 - x_2|^2 + |t_1 - t_2|)^{\lambda/2}.$$

The set of Hölder continuous functions forms a normed linear space $C^0_\lambda(D)$ with the norm

$$||v||_{\lambda,D} = ||v||_D + \sup_{(x_1,t_1),(x_2,t_2)\in D} \frac{|v(x_1,t_1) - v(x_2,t_2)|}{(|x_1 - x_2|^2 + |t_1 - t_2|)^{\lambda/2}},$$

where $||v||_D = \sup\limits_{(x,t)\in D} |u(x,t)|$. For each integer $k \ge 1$, the subspaces $C^k_\lambda(D)$ of $C^0_\lambda(D)$, which contain functions having Hölder continuous derivatives, are defined as follows:

$$C^k_\lambda(D) = \left\{u : \frac{\partial^{l+m} v}{\partial x^l \partial t^m} \in C^0_\lambda(D) \text{ for } l, m \geq 0 \text{ and } 0 \leq l + 2m \leq k\right\}.$$

The norm on $C^k_\lambda(D)$ is taken to be $||v||_{\lambda,k,D} = \max_{0\leq l+2m\leq k} ||\frac{\partial^{l+m} v}{\partial x^l \partial t^m}||_{\lambda,D}$. For a vector function $\vec{\vartheta} = (v_1, v_2, ..., v_n)$, the norm is defined by $||\vec{\vartheta}||_{\lambda,k,D} = \max_{1\leq i\leq n} ||v_i||_{\lambda,k,D}$.

Assume that A, $\vec{f}$ are sufficiently smooth. Also assume that $\vec{\xi}_L \in C^2(\Gamma_L)$, $\vec{\xi}_B \in C^5(\Gamma_B)$, $\vec{\xi}_R \in C^2(\Gamma_R)$ and that the compatibility conditions are fulfilled at the corners $(0, 0)$ and $(1, 0)$ of Γ. Then there exists a unique solution $\vec{v}$ of (1), (2) satisfying $v_i \in C^4_\lambda(\bar{\Omega})$.

The assumptions (4)–(6) are assumed throughout the paper. Furthermore, C denotes a generic positive constant, which is independent of x, t, all singular perturbation and discretization parameters. Inequalities between vectors are understood in the componentwise sense.

3 Analytical Results

The operator $\vec{H}$ satisfies the following maximum principle:

Lemma 1 *Let $\xi\vec{\psi}$ be any vector-valued function in the domain of $\vec{H}$ such that $\vec{b}_0\vec{\psi}(0, t) \geq \vec{0}$, $\vec{b}_1\vec{\psi}(1, t) \geq \vec{0}$, $\vec{\psi}(x, 0) \geq \vec{0}$. Then $\vec{H}\vec{\psi}(x, t) \geq \vec{0}$ on Ω implies that $\vec{\psi}(x, t) \geq \vec{0}$ on $\bar{\Omega}$.*

Proof Let i^*, x^*, t^* be such that $\psi_{i^*}(x^*, t^*) = \min_i \min_{\bar{\Omega}} \psi_i(x, t)$ and assume that the lemma is false. Then $\psi_{i^*}(x^*, t^*) < 0$. For $x^* = 0$, $(\vec{b}_0\vec{\psi})_{i^*}(0, t^*) = \psi_{i^*}(0, t^*) - \sqrt{\varepsilon_{i^*}}\frac{\partial \psi_{i^*}}{\partial x}(0, t^*) < 0$, for $x^* = 1$, $(\vec{b}_1\vec{\psi})_{i^*}(1, t^*) = \psi_{i^*}(1, t^*) + \sqrt{\varepsilon_{i^*}}\frac{\partial \psi_{i^*}}{\partial x}(1, t^*) < 0$ and for $t^* = 0$, $\psi_{i^*}(x^*, 0) < 0$, contradicting the hypothesis. Therefore, $(x^*, t^*) \notin \Gamma$ and $\frac{\partial^2 \psi_{i^*}}{\partial x^2}(x^*, t^*) \geq 0$. Also

$$(\vec{H}\vec{\psi})_{i^*}(x^*, t^*) = \frac{\partial \psi_{i^*}}{\partial t}(x^*, t^*) - \varepsilon_{i^*}\frac{\partial^2 \psi_{i^*}}{\partial x^2}(x^*, t^*) + \sum_{j=1}^{n} a_{i^*j}(x^*, t^*)\psi_j(x^*, t^*) < 0,$$

which contradicts the assumption and proves the result for $\vec{H}$.

Lemma 2 *If $\vec{\psi}$ is any vector-valued function in the domain of $\vec{H}$, then, for each i, $1 \leq i \leq n$ and $(x, t) \in \bar{\Omega}$,*

$$|\psi_i(x, t)| \leq \max\left\{\| \vec{b}_0\vec{\psi}(0, t) \|, \| \vec{b}_1\vec{\psi}(1, t) \|, \| \vec{\psi}(x, 0) \|, \frac{1}{\alpha} \| \vec{H}\vec{\psi} \|\right\}.$$

Proof Define the two functions

$$\vec{\theta}^{\pm}(x,t) = \max\left\{\| \vec{b}_0\vec{\psi}(0,t) \|, \| \vec{b}_1\vec{\psi}(1,t) \|, \| \vec{\psi}(x,0) \|, \frac{1}{\alpha} \| \vec{H}\vec{\psi} \|\right\}\vec{e} \pm \vec{\psi}(x,t), \ (x,t) \in \bar{\Omega}$$

and $\vec{e} = (1, ..., 1)^T$. Using the properties of A, it is not hard to verify that $\vec{b}_0\vec{\theta}^{\pm}(0,t) \geq \vec{0}$, $\vec{b}_1\vec{\theta}^{\pm}(1,t) \geq \vec{0}$, $\vec{\theta}^{\pm}(x,0) \geq \vec{0}$ and $\vec{H}\vec{\theta}^{\pm} \geq \vec{0}$ on Ω. It follows from Lemma 1 that $\vec{\theta}^{\pm} \geq \vec{0}$ on $\bar{\Omega}$ as required.

A standard estimate of the solution $\vec{v}$ to the problem (1), (2) and its derivatives is contained in the following lemma.

Lemma 3 *Let $\vec{v}$ be the solution of (1), (2). Then, for all $(x,t) \in \bar{\Omega}$ and each $i = 1, ..., n$,*

$$|v_i(x,t)| \leq C(\| \vec{\xi}_L(t) \| + \| \vec{\xi}_R(t) \| + \| \vec{\xi}_B(x) \| + \| \vec{f} \|),$$

$$|\frac{\partial^l v_i}{\partial t^l}(x,t)| \leq C(\| \vec{v} \| + \sum_{q=0}^{l} \| \frac{\partial^q \vec{f}}{\partial t^q} \|), \ l = 1, 2,$$

$$|\frac{\partial^l v_i}{\partial x^l}(x,t)| \leq C\varepsilon_i^{\frac{-l}{2}}(\| \vec{v} \| + \| \vec{f} \| + \| \frac{\partial \vec{f}}{\partial t} \|), \ l = 1, 2,$$

$$|\frac{\partial^l v_i}{\partial x^l}(x,t)| \leq C\varepsilon_i^{-1}\varepsilon_1^{\frac{-(l-2)}{2}}(\| \vec{v} \| + \| \vec{f} \| + \| \frac{\partial \vec{f}}{\partial t} \| + \| \frac{\partial^2 \vec{f}}{\partial t^2} \| + \varepsilon_1^{\frac{l-2}{2}} \| \frac{\partial^{l-2} \vec{f}}{\partial x^{l-2}} \|), l = 3, 4,$$

$$|\frac{\partial^l v_i}{\partial x^{l-1}\partial t}(x,t)| \leq C\varepsilon_i^{\frac{-(l-1)}{2}}(\| \vec{v} \| + \| \vec{f} \| + \| \frac{\partial \vec{f}}{\partial t} \| + \| \frac{\partial^2 \vec{f}}{\partial t^2} \|), \ l = 2, 3.$$

Proof The bound on $\vec{v}$ is an immediate consequence of Lemma 2.

Differentiating (1) partially with respect to 't' once and twice, respectively, and applying Lemma 2, the bounds on $\frac{\partial \vec{v}}{\partial t}$ and $\frac{\partial^2 \vec{u}}{\partial t^2}$, respectively, are derived.

By using mean-value theorem, the bound on $\frac{\partial v_i}{\partial x}$, for each (x,t), is determined as follows:

$$|\frac{\partial v_i}{\partial x}(x,t)| \leq C\varepsilon_i^{\frac{-1}{2}}(||\vec{v}|| + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}||).$$

Rearranging (1), the following bound is derived:

$$|\frac{\partial^2 v_i}{\partial x^2}(x,t)| \leq C\varepsilon_i^{-1}(||\vec{v}|| + ||\vec{f}|| + ||\frac{\partial \vec{f}}{\partial t}||).$$

Following steps similar to those used to bound $\frac{\partial v}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$, the bounds of the mixed derivatives are also obtained.

Differentiating (1) once and twice partially with respect to 'x' and rearranging the equation, the bounds on $\frac{\partial^3 v_i}{\partial x^3}$ and $\frac{\partial^4 v_i}{\partial x^4}$, respectively, are obtained.

The Shishkin decomposition of the solution $\vec{v}$ of the problem (1), (2) is

$$\vec{v} = \vec{\vartheta} + \vec{\omega} \tag{7}$$

where $\vec{\vartheta}$ and $\vec{\omega}$ are the smooth and singular components of the solution $\vec{v}$, respectively.

Taking into consideration the sublayers that appear for the components, the smooth component $\vec{\vartheta}$ is subjected to further decomposition.

$$\begin{aligned} \vartheta_n &= \vartheta_{0,n} + \varepsilon_n \phi_{n,n}, \\ \phi_{n-1} &= \vartheta_{0,n-1} + \varepsilon_n \phi^1_{n-1,n}, \\ &\vdots \\ \vartheta_1 &= \vartheta_{0,1} + \varepsilon_n \phi^1_{1,n}, \end{aligned} \tag{8}$$

as all the components have ε_n layers. Since components except ϑ_n have ε_{n-1} sublayers, the components ϕ_{n-1}, ..., ϑ_1 take the form:

$$\begin{aligned} \phi_{n-1} &= \vartheta_{0,n-1} + \varepsilon_n(\phi_{n-1,n} + \varepsilon_{n-1}\phi_{n-1,n-1}), \\ \phi_{n-2} &= \vartheta_{0,n-2} + \varepsilon_n(\phi_{n-2,n} + \varepsilon_{n-1}\phi^1_{n-2,n-1}), \\ &\vdots \\ \vartheta_1 &= \vartheta_{0,1} + \varepsilon_n(\phi_{1,n} + \varepsilon_{n-1}\phi^1_{1,n-1}). \end{aligned} \tag{9}$$

Further, $\vartheta_{n-2},\ \vartheta_{n-3}, ..., \vartheta_2, \vartheta_1$ have ε_{n-2} sublayers and hence that leads to the decomposition:

$$\begin{aligned} \phi_{n-2} &= \vartheta_{0,n-2} + \varepsilon_n(\phi_{n-2,n} + \varepsilon_{n-1}(\phi_{n-2,n-1} + \varepsilon_{n-2}\phi_{n-2,n-2})), \\ \phi_{n-3} &= \vartheta_{0,n-3} + \varepsilon_n(\phi_{n-3,n} + \varepsilon_{n-1}(\phi_{n-3,n-1} + \varepsilon_{n-2}\phi^1_{n-3,n-2})), \\ &\vdots \\ \vartheta_1 &= \vartheta_{0,1} + \varepsilon_n(\phi_{1,n} + \varepsilon_{n-1}(\phi_{1,n-1} + \varepsilon_{n-2}\phi^1_{1,n-2})). \end{aligned} \tag{10}$$

Proceeding like this, it is not hard to see that

$$\begin{pmatrix} \vartheta_1 \\ \vartheta_2 \\ \vdots \\ \vartheta_n \end{pmatrix} = \begin{pmatrix} \vartheta_{0,1} \\ \vartheta_{0,2} \\ \vdots \\ \vartheta_{0,n} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}$$

i.e.

$$\vec{\vartheta}(x,t) = \vec{v}_0(x,t) + \vec{\gamma}(x,t) \tag{11}$$

where

$$\gamma_j = \vec{\varepsilon}^{\,n}(\vec{v}^{\,j}_j)^T, \tag{12}$$

$$\vec{\varepsilon}^{\,n} = (\varepsilon_1\varepsilon_2...\varepsilon_n,\ \varepsilon_2\varepsilon_3...\varepsilon_n, ...,\ \varepsilon_{n-1}\varepsilon_n,\ \varepsilon_n),\ \ \vec{v}^{\,i}_i = (0,\ 0, ...,\ \phi_{i,i},\ \phi_{i,i+1}, ...,\ \phi_{i,n}).$$

Then using (7) and (11) in (1) and (2), it is found that the smooth component $\vec{\vartheta}$ of the solution $\vec{v}$ satisfies

$$\vec{H}\vec{\vartheta} = \vec{f}, \text{ on } \Omega \tag{13}$$

with

$$\vec{b}_0\vec{\vartheta}(0,t) = \vec{b}_0(\vec{v}_0 + \vec{\gamma})(0,t), \; \vec{b}_1\vec{\vartheta}(1,t) = \vec{b}_1(\vec{v}_0 + \vec{\gamma})(1,t), \; \vec{\vartheta}(x,0) = (\vec{v}_0 + \vec{\gamma})(x,0), \tag{14}$$

and the singular component $\vec{\omega}$ of the solution $\vec{v}$ satisfies

$$\vec{H}\vec{\omega} = \vec{0}, \text{ on } \Omega \tag{15}$$

with

$$\vec{b}_0\vec{\omega}(0,t) = \vec{b}_0(\vec{v} - \vec{\vartheta})(0,t), \; \vec{b}_1\vec{\omega}(1,t) = \vec{b}_1(\vec{v} - \vec{\vartheta})(1,t), \; \vec{\omega}(x,0) = \vec{0}. \tag{16}$$

Consider the following parabolic initial-boundary value problem for a singularly perturbed linear system of second-order differential equations:

$$\frac{\partial \hat{\vec{v}}}{\partial t}(x,t) - \hat{E}\frac{\partial^2 \hat{\vec{v}}}{\partial x^2}(x,t) + \hat{A}(x,t)\hat{\vec{v}}(x,t) = \hat{\vec{f}}(x,t), \text{ on } \Omega, \tag{17}$$

with

$$\hat{u}_2(0,t) - \sqrt{\varepsilon_n}\frac{\partial \hat{u}_n}{\partial x}(0,t) = \alpha(t), \; \hat{u}_2(1,t) + \sqrt{\varepsilon_n}\frac{\partial \hat{u}_n}{\partial x}(1,t) = \beta(t), \; 0 \le t \le T,$$
$$\hat{\vec{v}}(x,0) = \vec{\delta}(x), \; 0 \le x \le 1, \tag{18}$$

where $\hat{E}$ is an $n \times n$ matrix, $\hat{E} = diag(0, 0, ..., 0, \varepsilon_n)$ with $0 < \varepsilon_n < 1$.

The problem (17), (18) can also be written in the operator form

$$\hat{L}\hat{\vec{v}} = \hat{\vec{f}} \text{ on } \Omega,$$

$$\mu_0\hat{u}_n(0,t) = \alpha(t), \; \mu_1\hat{u}_n(1,t) = \beta(t), \; \hat{\vec{v}}(x,0) = \vec{\delta}(x),$$

where the operators $\hat{L}$, μ_0, μ_1 are defined by

$$\hat{L} = I\frac{\partial}{\partial t} - \hat{E}\frac{\partial^2}{\partial x^2} + \hat{A}, \; \mu_0 = I - \sqrt{\varepsilon_n}\frac{\partial}{\partial x}, \; \mu_1 = I + \sqrt{\varepsilon_n}\frac{\partial}{\partial x}$$

where I is the identity operator. The reduced problem corresponding to (17), (18) is defined by

$$\frac{\partial \hat{\vec{v}}_0}{\partial t} + \hat{A}\hat{\vec{v}}_0 = \hat{\vec{f}}, \text{ on } \Omega, \; \hat{\vec{v}}_0 = \hat{\vec{v}} \text{ on } \Gamma_B.$$

The operator $\hat{L}$ satisfies the following maximum principle:

Lemma 4 *Let the assumptions (4)–(6) hold. Let $\vec{\psi} = (\psi_1, ..., \psi_n)^T$ be any vector-valued function in the domain of $\hat{L}$ such that $\mu_0\psi_n(0,t) \geq 0$, $\mu_1\psi_n(1,t) \geq 0$, $\vec{\psi}(x,0) \geq \vec{0}$. Then $\hat{L}\vec{\psi}(x,t) \geq \vec{0}$ on Ω implies that $\vec{\psi}(x,t) \geq \vec{0}$ on $\bar{\Omega}$.*

Lemma 5 *Let the assumptions (4)–(6) hold. If $\vec{\psi} = (\psi_1, ..., \psi_n)^T$ is any vector-valued function in the domain of $\hat{L}$, then, for each $i = 1, ..., n$ and $(x,t) \in \bar{\Omega}$,*

$$|\psi_i(x,t)| \leq \max\left\{\| \mu_0\psi(0,t) \|, \| \mu_1\psi(1,t) \|, \| \vec{\psi}(x,0) \|, \frac{1}{\alpha} \| \hat{L}\vec{\psi} \|\right\}.$$

A standard estimate of the solution $\hat{\vec{v}}$ to the problem (17), (18) and its derivatives is contained in the following lemma.

Lemma 6 *Let $\hat{\vec{v}}$ be the solution of (17), (18). Then, for all $(x,t) \in \bar{\Omega}$ and each $i = 1, ..., n$,*

$$|\hat{v}_i(x,t)| \quad \leq C(\| \alpha(t) \| + \| \beta(t) \| + \| \vec{\delta}(x) \| + \| \hat{\vec{f}} \|),$$
$$|\frac{\partial^l \hat{v}_i}{\partial t^l}(x,t)| \leq C(\| \hat{\vec{v}} \| + \textstyle\sum_{q=0}^{l} \| \frac{\partial^q \hat{\vec{f}}}{\partial t^q} \|),\ l = 1, 2,$$
$$|\frac{\partial \hat{v}_i}{\partial x}(x,t)| \leq C\varepsilon_n^{-1/2}(\| \hat{\vec{v}} \| + \| \hat{\vec{f}} \| + \| \frac{\partial \hat{\vec{f}}}{\partial t} \| + \varepsilon_n^{1/2} \| \frac{\partial \hat{\vec{f}}}{\partial x} \|),$$
$$|\frac{\partial^2 \hat{v}_i}{\partial x^2}(x,t)| \leq C\varepsilon_n^{-1}(\| \hat{\vec{v}} \| + \| \hat{\vec{f}} \| + \| \frac{\partial \hat{\vec{f}}}{\partial t} \| + \varepsilon_n \| \frac{\partial \hat{\vec{f}}}{\partial x} \| + \varepsilon_n \| \frac{\partial^2 \hat{\vec{f}}}{\partial x^2} \|),$$
$$|\frac{\partial^3 \hat{v}_i}{\partial x^3}(x,t)| \leq C\varepsilon_n^{-3/2}(\| \hat{\vec{v}} \| + \| \hat{\vec{f}} \| + \| \frac{\partial \hat{\vec{f}}}{\partial t} \| + \| \frac{\partial^2 \hat{\vec{f}}}{\partial t^2} \| + \varepsilon_n^{3/2} \| \frac{\partial \hat{\vec{f}}}{\partial x} \| + \varepsilon_n^{3/2} \| \frac{\partial^2 \hat{\vec{f}}}{\partial x^2} \|)$$
$$+\varepsilon_n^{3/2} \| \frac{\partial^3 \hat{\vec{f}}}{\partial x^3} \|),$$
$$|\frac{\partial^4 \hat{v}_i}{\partial x^4}(x,t)| \leq C\varepsilon_n^{-2}(\| \hat{\vec{v}} \| + \| \hat{\vec{f}} \| + \| \frac{\partial \hat{\vec{f}}}{\partial t} \| + \| \frac{\partial^2 \hat{\vec{f}}}{\partial t^2} \| + \varepsilon_n^2 \| \frac{\partial \hat{\vec{f}}}{\partial x} \| + \varepsilon_n^2 \| \frac{\partial^2 \hat{\vec{f}}}{\partial x^2} \|)$$
$$+\varepsilon_n^2 \| \frac{\partial^3 \hat{\vec{f}}}{\partial x^3} \|) + \varepsilon_n^2 \| \frac{\partial^4 \hat{\vec{f}}}{\partial x^4} \|),$$
$$\|\frac{\partial^2 \hat{v}_i}{\partial x \partial t}(x,t)| \leq C\varepsilon_n^{-1/2}(\| \hat{\vec{v}} \| + \| \hat{\vec{f}} \| + \| \frac{\partial \hat{\vec{f}}}{\partial t} \| + \| \frac{\partial^2 \hat{\vec{f}}}{\partial t^2} \|) + \varepsilon_n^{1/2} \| \frac{\partial \hat{\vec{f}}}{\partial x} \|),$$
$$|\frac{\partial^3 \hat{v}_i}{\partial x^2 \partial t}(x,t)| \leq C\varepsilon_n^{-1}(\| \hat{\vec{v}} \| + \| \hat{\vec{f}} \| + \| \frac{\partial \hat{\vec{f}}}{\partial t} \| + \| \frac{\partial^2 \hat{\vec{f}}}{\partial t^2} \|) + \varepsilon_n \| \frac{\partial \hat{\vec{f}}}{\partial x} \| + \varepsilon_n \| \frac{\partial^2 \hat{\vec{f}}}{\partial x^2} \|).$$

Bounds on the smooth component $\vec{\vartheta}$ of $\vec{v}$ and its derivatives are contained in

Lemma 7 *Let the assumptions (4)–(6) hold. Then there exists a constant C, such that, for each $(x,t) \in \bar{\Omega}$ and $i = 1, ..., n$,*

$$|\frac{\partial^l \vartheta_i}{\partial t^l}(x,t)| \le C,\ l=0,1,2, \qquad |\frac{\partial^l \vartheta_i}{\partial x^l}(x,t)| \le C,\ l=1,2,$$
$$|\frac{\partial^l \vartheta_i}{\partial x^l}(x,t)| \le C\varepsilon_i^{-(l-2)/2},\ l=3,4, \quad |\frac{\partial^{l+1} \vartheta_i}{\partial x^l \partial t}(x,t)| \le C,\ l=1,2.$$

Proof From (8)–(10), it is observed that the components $\vartheta_{i,j},\ i=1,...,n,\ j=i,i+1,...,n$ satisfy the following systems of equations:

$$\begin{aligned}
\frac{\partial \vartheta_{1,n}}{\partial t} + a_{11}\vartheta_{1,n} + a_{12}\vartheta_{2,n} + \cdots + a_{1n}\vartheta_{n,n} &= \frac{\varepsilon_1}{\varepsilon_n}\frac{\partial^2 \vartheta_{0,1}}{\partial x^2}\\
\frac{\partial \vartheta_{2,n}}{\partial t} + a_{21}\vartheta_{1,n} + a_{22}\vartheta_{2,n} + \cdots + a_{2n}\vartheta_{n,n} &= \frac{\varepsilon_2}{\varepsilon_n}\frac{\partial^2 \vartheta_{0,2}}{\partial x^2}\\
&\vdots\\
\frac{\partial \vartheta_{n-1,n}}{\partial t} + a_{n-11}\vartheta_{1,n} + a_{n-12}\vartheta_{2,n} + \cdots + a_{n-1n}\vartheta_{n,n} &= \frac{\varepsilon_{n-1}}{\varepsilon_n}\frac{\partial^2 \vartheta_{0,n-1}}{\partial x^2}\\
\frac{\partial \vartheta_{n,n}}{\partial t} - \varepsilon_n\frac{\partial^2 \vartheta_{n,n}}{\partial x^2} + a_{n1}\vartheta_{1,n} + a_{n2}\vartheta_{2,n} + ... + a_{nn}\vartheta_{n,n} &= \frac{\partial^2 \vartheta_{0,n}}{\partial x^2}
\end{aligned} \tag{19}$$

with

$$(\vartheta_{n,n} - \sqrt{\varepsilon_n}\frac{\partial \vartheta_{n,n}}{\partial x})(0,t) = 0,\ (\vartheta_{n,n} + \sqrt{\varepsilon_n}\frac{\partial \vartheta_{n,n}}{\partial x})(1,t) = 0,\ \vartheta_{i,n}(x,0) = 0, \tag{20}$$

where $\vartheta_{0,i},\ i=1,...,n$ is the solution of the reduced problem (3).

$$\begin{aligned}
\frac{\partial \vartheta_{1,n-1}}{\partial t} + a_{11}\vartheta_{1,n-1} + ... + a_{1n-1}\vartheta_{n-1,n-1} &= \frac{\varepsilon_1}{\varepsilon_{n-1}}\frac{\partial^2 \vartheta_{1,n}}{\partial x^2}\\
\frac{\partial \vartheta_{2,n-1}}{\partial t} + a_{21}\vartheta_{1,n-1} + ... + a_{2n-1}\vartheta_{n-1,n-1} &= \frac{\varepsilon_2}{\varepsilon_{n-1}}\frac{\partial^2 \vartheta_{2,n}}{\partial x^2}\\
&\vdots\\
\frac{\partial \vartheta_{n-2,n-1}}{\partial t} + a_{n-21}\vartheta_{1,n-1} + ... + a_{n-2n-1}\vartheta_{n-1,n-1} &= \frac{\varepsilon_{n-2}}{\varepsilon_{n-1}}\frac{\partial^2 \vartheta_{n-2,n}}{\partial x^2}\\
\frac{\partial \vartheta_{n-1,n-1}}{\partial t} - \varepsilon_{n-1}\frac{\partial^2 \vartheta_{n-1,n-1}}{\partial x^2} + a_{n-11}\vartheta_{1,n-1} + ... + a_{n-1n-1}\vartheta_{n-1,n-1}&\\
&= \frac{\partial^2 \vartheta_{n-1,n}}{\partial x^2}
\end{aligned} \tag{21}$$

with

$$\begin{aligned}
(\vartheta_{n-1,n-1} - \sqrt{\varepsilon_{n-1}}\frac{\partial \vartheta_{n-1,n-1}}{\partial x})(0,t) = 0,\ \ (\vartheta_{n-1,n-1} + \sqrt{\varepsilon_{n-1}}\frac{\partial \vartheta_{n-1,n-1}}{\partial x})(1,t) = 0,\\
\vartheta_{i,n-1}(x,0) = 0,
\end{aligned} \tag{22}$$

and so on.

Lastly,

$$\begin{aligned} \frac{\partial \vartheta_{1,2}}{\partial t} + a_{11}\vartheta_{1,2} + a_{12}\vartheta_{2,2} &= \frac{\varepsilon_1}{\varepsilon_2}\frac{\partial^2 \vartheta_{1,3}}{\partial x^2} \\ \frac{\partial \vartheta_{2,2}}{\partial t} - \varepsilon_2 \frac{\partial^2 \vartheta_{2,2}}{\partial x^2} + a_{21}\vartheta_{1,2} + a_{22}\vartheta_{2,2} &= \frac{\partial^2 \vartheta_{2,3}}{\partial x^2} \end{aligned} \tag{23}$$

with

$$(\vartheta_{2,2} - \sqrt{\varepsilon_2}\frac{\partial \vartheta_{2,2}}{\partial x})(0,t) = 0,\;\; (\vartheta_{2,2} + \sqrt{\varepsilon_2}\frac{\partial \vartheta_{2,2}}{\partial x})(1,t) = 0,\;\; \vartheta_{i,2}(x,0) = 0, \tag{24}$$

and

$$\frac{\partial \vartheta_{1,1}}{\partial t} - \varepsilon_1 \frac{\partial^2 \vartheta_{1,1}}{\partial x^2} + a_{11}\vartheta_{1,1} = \frac{\partial^2 \vartheta_{1,2}}{\partial x^2} \tag{25}$$

with

$$(\vartheta_{1,1} - \sqrt{\varepsilon_1}\frac{\partial \vartheta_{1,1}}{\partial x})(0,t) = 0,\;\; (\vartheta_{1,1} + \sqrt{\varepsilon_1}\frac{\partial \vartheta_{1,1}}{\partial x})(1,t) = 0,\;\; \vartheta_{1,1}(x,0) = 0. \tag{26}$$

From the expressions (19)–(26) and using Lemma (6) for $\vec{\vartheta}$, it is found that for $i = 1, ..., n,\;\; j = i, i+1, ..., n,\;\; i \leq j,\;\; k = 1, 2, 3, 4,\;\; l = 0, 1, 2,\;\; m = 1, 2,$

$$\begin{gathered} |\frac{\partial^l \vartheta_{i,j}}{\partial t^l}(x,t)| \leq C(1 + \prod_{r=j+1}^{n} \varepsilon_r^{-1}),\quad |\frac{\partial^k \vartheta_{i,j}}{\partial x^k}(x,t)| \leq C(1 + \varepsilon_j^{-k/2} \prod_{r=j+1}^{n} \varepsilon_r^{-1}), \\ |\frac{\partial^{m+1} \vartheta_{i,j}}{\partial x^m \partial t}(x,t)| \leq C(1 + \varepsilon_j^{-m/2} \prod_{r=j+1}^{n} \varepsilon_r^{-1}). \end{gathered} \tag{27}$$

From (11), (12) and (27), the following bounds for $\vartheta_i,\;\; i = 1, 2, ..., n$ hold:

$$\begin{aligned} |\frac{\partial^l \vartheta_i}{\partial t^l}(x,t)| \leq C,\;\; l = 0, 1, 2, \quad & |\frac{\partial^l \vartheta_i}{\partial x^l}(x,t)| \leq C,\;\; l = 1, 2, \\ |\frac{\partial^l \vartheta_i}{\partial x^l}(x,t)| \leq C\varepsilon_i^{-(l-2)/2},\;\; l = 3, 4, \quad & |\frac{\partial^{l+1} \vartheta_i}{\partial x^l \partial t}(x,t)| \leq C,\;\; l = 1, 2. \end{aligned}$$

The layer functions $B_i^L,\;\; B_i^R,\;\; B_i,\;\; i = 1,\; \ldots,\; n,$ associated with the solution $\vec{v},$ are defined on $\bar{\Omega}$ by

$$B_i^L(x) = e^{-x\sqrt{\alpha/\varepsilon_i}},\;\; B_i^R(x) = B_i^L(1-x),\;\; B_i(x) = B_i^L(x) + B_i^R(x).$$

The following elementary properties of these layer functions, for all $1 \leq i < j \leq n$ and $0 \leq x < y \leq 1$, should be noted:

$$B_i(x) = B_i(1-x). \tag{28}$$

$$B_i^L(x) < B_j^L(x), \ B_i^L(x) > B_i^L(y), \ 0 < B_i^L(x) \le 1. \tag{29}$$

$$B_i^R(x) < B_j^R(x), \ B_i^R(x) < B_i^R(y), \ 0 < B_i^R(x) \le 1. \tag{30}$$

$$B_i(x) \text{ is monotone decreasing for increasing } x \in [0, \frac{1}{2}]. \tag{31}$$

$$B_i(x) \text{ is monotone increasing for increasing } x \in [\frac{1}{2}, 1]. \tag{32}$$

$$B_i(x) \le 2B_i^L(x) \text{ for } x \in [0, \frac{1}{2}], \ B_i(x) \le 2B_i^R(x) \text{ for } x \in [\frac{1}{2}, 1]. \tag{33}$$

$$B_i^H(2\frac{\sqrt{\varepsilon_i}}{\sqrt{\alpha}} \ln N) = N^{-2}. \tag{34}$$

The interesting points $x_{i,j}^{(s)}$ are now defined.

Definition 1 For B_i^L, B_j^L, each $i, j, 1 \le i \ne j \le n$ and each $s, s > 0$, the point $x_{i,j}^{(s)}$ is defined by

$$\frac{B_i^H(x_{i,j}^{(s)})}{\varepsilon_i^s} = \frac{B_j^H(x_{i,j}^{(s)})}{\varepsilon_j^s}. \tag{35}$$

It is remarked that

$$\frac{B_i^R(1 - x_{i,j}^{(s)})}{\varepsilon_i^s} = \frac{B_j^R(1 - x_{i,j}^{(s)})}{\varepsilon_j^s}. \tag{36}$$

In the next lemma, the existence, uniqueness and ordering of the points $x_{i,j}^{(s)}$ are established. Sufficient conditions for them to lie in the domain $\bar{\Omega}$ are also provided.

Lemma 8 *For all i, j such that $1 \le i < j \le n$ and $0 < s \le 3/2$, the points $x_{i,j}^{(s)}$ exist, are uniquely defined and satisfy the following inequalities:*

$$\frac{B_i^H(x)}{\varepsilon_i^s} > \frac{B_j^H(x)}{\varepsilon_j^s}, \ x \in [0, x_{i,j}^{(s)}), \ \frac{B_i^H(x)}{\varepsilon_i^s} < \frac{B_j^H(x)}{\varepsilon_j^s}, \ x \in (x_{i,j}^{(s)}, 1]. \tag{37}$$

In addition, the following ordering holds:

$$x_{i,j}^{(s)} < x_{i+1,j}^{(s)}, \text{ if } i + 1 < j \text{ and } x_{i,j}^{(s)} < x_{i,j+1}^{(s)}, \text{ if } i < j. \tag{38}$$

Also

$$x_{i,j}^{(s)} < 2s\frac{\sqrt{\varepsilon_j}}{\sqrt{\alpha}} \text{ and } x_{i,j}^{(s)} \in (0, \frac{1}{2}) \text{ if } i < j. \tag{39}$$

Analogous results hold for B_i^R, B_j^R *and the points* $1 - x_{i,j}^{(s)}$.

Proof The proof is as given in [8].

Bounds on the singular component $\vec{\omega}$ of $\vec{v}$ and its derivatives are contained in

Lemma 9 *Then there exists a constant* C, *such that, for each* $(x, t) \in \bar{\Omega}$ *and* $i = 1, ..., n$,

$$\begin{aligned}
&|\frac{\partial^l \omega_i}{\partial t^l}(x, t)| \le C B_n(x), \text{ for } l = 0, 1, 2,\\
&|\frac{\partial^l \omega_i}{\partial x^l}(x, t)| \le C \sum_{r=i}^{n} \frac{B_r(x)}{\varepsilon_r^{\frac{l}{2}}}, \text{ for } l = 1, 2,\\
&|\frac{\partial^3 \omega_i}{\partial x^3}(x, t)| \le C \sum_{r=1}^{n} \frac{B_r(x)}{\varepsilon_r^{\frac{3}{2}}},\\
&|\varepsilon_i \frac{\partial^4 \omega_i}{\partial x^4}(x, t)| \le C \sum_{r=1}^{n} \frac{B_r(x)}{\varepsilon_r}.
\end{aligned}$$

Proof To derive the bound of $\vec{\omega}$, define $\vec{\psi}^{\pm}(x, t) = (\psi_1, ..., \psi_n)^T$, where

$$\psi_i^{\pm}(x, t) = Ce^{\alpha t} B_n(x) \pm \omega_i(x, t), \text{ for each } i = 1, \dots, n.$$

For a proper choice of C, $\vec{b}_0\vec{\psi}^{\pm}(0, t) \ge \vec{0}$, $\vec{b}_1\vec{\psi}^{\pm}(1, t) \ge \vec{0}$ and $\vec{\psi}^{\pm}(x, 0) \ge \vec{0}$. Also, for $(x, t) \in \Omega$, $\vec{H}\vec{\psi}^{\pm}(x, t) \ge \vec{0}$. By Lemma 1, $\vec{\psi}^{\pm} \ge \vec{0}$ on $\bar{\Omega}$ and it follows that

$$|\omega_i(x, t)| \le Ce^{\alpha t} B_n(x) \text{ or } |\omega_i(x, t)| \le C B_n(x).$$

Differentiating the homogeneous equation satisfied by ω_i, partially with respect to 't', and using Lemma 1, it is not hard to see that

$$|\frac{\partial \omega_i}{\partial t}(x, t)| \le C B_n(x).$$

Note that
$$|\frac{\partial^2 \omega_i}{\partial x \partial t}(x, t)| \le |\frac{\partial^2 v_i}{\partial x \partial t}(x, t)| + |\frac{\partial^2 v_i}{\partial x \partial t}(x, t)|.$$

Thus,
$$|\frac{\partial^2 \omega_i}{\partial x \partial t}(x, t)| \le C\varepsilon_i^{\frac{-1}{2}}(\| \vec{v} \| + \| \vec{f} \| + \| \frac{\partial \vec{f}}{\partial t} \| + \| \frac{\partial^2 \vec{f}}{\partial t^2} \|).$$

Similarly,
$$|\frac{\partial^3 \omega_i}{\partial x^2 \partial t}(x, t)| \le C\varepsilon_i^{-1}(\| \vec{v} \| + \| \vec{f} \| + \| \frac{\partial \vec{f}}{\partial t} \| + \| \frac{\partial^2 \vec{f}}{\partial t^2} \|).$$

As before, using suitable barrier functions, it is not hard to verify that

$$|\frac{\partial^{l+1}\omega_i}{\partial x^l \partial t}(x,t)| \le C\varepsilon_i^{\frac{-l}{2}} B_n(x),\ \ l=1,2.$$

Differentiating the equation satisfied by ω_i partially with respect to 't' once and rearranging yields

$$|\frac{\partial^2\omega_i}{\partial t^2}(x,t)| \le C B_n(x).$$

The bounds on $\dfrac{\partial^l\omega_i}{\partial x^l}$, $l=1,2,3,4$ and $i=1,\dots,n$ are now derived by induction on n. It is then assumed that the required bounds on $\dfrac{\partial\omega_i}{\partial x}$, $\dfrac{\partial^2\omega_i}{\partial x^2}$, $\dfrac{\partial^3\omega_i}{\partial x^3}$ and $\dfrac{\partial^4\omega_i}{\partial x^4}$ hold for all systems up to order $n-1$. Define $\tilde{\vec{\omega}}=(\omega_1,\dots,\omega_{n-1})$, then $\tilde{\vec{\omega}}$ satisfies the system

$$\frac{\partial\tilde{\vec{\omega}}}{\partial t} - \tilde{E}\frac{\partial^2\tilde{\vec{\omega}}}{\partial x^2} + \tilde{A}\tilde{\vec{\omega}} = \vec{g}, \tag{40}$$

with $\vec{b}_0\tilde{\vec{\omega}}(0,t)=\vec{b}_0(\tilde{\vec{v}}-\tilde{\vec{\vartheta}})(0,t),\ \vec{b}_1\tilde{\vec{\omega}}(1,t)=\vec{b}_1(\tilde{\vec{v}}-\tilde{\vec{\vartheta}})(1,t),\ \tilde{\vec{\omega}}(x,0)=\tilde{\vec{0}}.$

Here, $\tilde{E}$ and $\tilde{A}$ are the matrices obtained by deleting the last row and last column from E, A, respectively, the components of $\vec{g}$ are $g_i=-a_{in}\omega_n$ for $1\le i\le n-1$ and $\tilde{\vec{\vartheta}}=\tilde{\vec{v}}_0+\tilde{\vec{\gamma}}$ is the corresponding component decomposition of $\tilde{\vec{v}}$ similar to (11) of $\vec{\vartheta}$. Now decompose $\tilde{\vec{\omega}}$ into smooth and singular components to get $\tilde{\vec{\omega}}=\vec{p}+\vec{q}$, where $\vec{H}\vec{p}=\vec{g},\ \vec{b}_0\vec{p}(0,t)=\vec{b}_0(\tilde{\vec{v}}_0+\tilde{\vec{\gamma}})(0,t),\ \vec{b}_1\vec{p}(1,t)=\vec{b}_1(\tilde{\vec{v}}_0+\tilde{\vec{\gamma}})(1,t),\ \vec{p}(x,0)=(\tilde{\vec{v}}_0+\tilde{\vec{\gamma}})(x,0)$ and $L\vec{q}=\vec{0},\ \vec{b}_0\vec{q}(0,t)=\vec{b}_0\tilde{\vec{\omega}}(0,t)-\vec{b}_0\vec{p}(0,t),\ \vec{b}_1\vec{q}(1,t)=\vec{b}_1\tilde{\vec{\omega}}(1,t)-\vec{b}_1\vec{p}(1,t),\ \vec{q}(x,0)=\tilde{\vec{\omega}}(x,0)-\vec{p}(x,0).$

Consider the equation of the system satisfied by ω_i,

$$\frac{\partial\omega_i}{\partial t} - \varepsilon_i\frac{\partial^2\omega_i}{\partial x^2} + \sum_{j=1}^{n} a_{ij}\omega_j = 0.$$

By using mean-value theorem, the bound on $\dfrac{\partial\omega_i}{\partial x}$, for each (x,t), is determined as follows:

$$|\frac{\partial\omega_i}{\partial x}(x,t)| \le C\varepsilon_i^{\frac{-1}{2}} B_n(x).$$

Rearranging the equation of the system satisfied by ω_i, yields

$$|\frac{\partial^2\omega_i}{\partial x^2}(x,t)| \le C\varepsilon_i^{-1} B_i(x).$$

Differentiating the equation satisfied by ω_i with respect to 'x' once and twice and rearranging, the following bounds are derived:

$$|\frac{\partial^3\omega_i}{\partial x^3}(x,t)| \le C\sum_{r=1}^{n}\varepsilon_r^{-\frac{3}{2}}B_r(x),\ \ |\varepsilon_i\frac{\partial^4\omega_i}{\partial x^4}(x,t)| \le C\sum_{r=1}^{n}\varepsilon_r^{-1}B_r(x).$$

Using the bounds on ω_n, $\frac{\partial\omega_n}{\partial x}$, $\frac{\partial^2\omega_n}{\partial x^2}$, $\frac{\partial^3\omega_n}{\partial x^3}$ and $\frac{\partial^4\omega_n}{\partial x^4}$, it is seen that the function $\vec{g}$ in (40) and its derivatives $\frac{\partial\vec{g}}{\partial x}$, $\frac{\partial^2\vec{g}}{\partial x^2}$, $\frac{\partial^3\vec{g}}{\partial x^3}$, $\frac{\partial^4\vec{g}}{\partial x^4}$ are bounded by $CB_n(x)$, $C\frac{B_n(x)}{\sqrt{\varepsilon_n}}$, $C\frac{B_n(x)}{\varepsilon_n}$, $C\sum_{r=1}^{n}\frac{B_r(x)}{\varepsilon_r^{\frac{3}{2}}}$ and $C\varepsilon_n^{-1}\sum_{r=1}^{n}\frac{B_r(x)}{\varepsilon_r}$, respectively. Introducing the functions $\vec{\psi}^{\pm}(x,t) = Ce^{\alpha t}B_n(x)\vec{e} \pm \vec{p}(x,t)$, it is easy to see that $\vec{b}_0\vec{\psi}^{\pm}(0,t) = Ce^{\alpha t}B_n(0)\vec{e} \pm \vec{b}_0\vec{p}(0,t) \ge \vec{0}$, $\vec{b}_1\vec{\psi}^{\pm}(1,t) = Ce^{\alpha t}B_n(1)\vec{e} \pm \vec{b}_1\vec{p}(1,t) \ge \vec{0}$, $\vec{\psi}^{\pm}(x,0) = CB_n(x)\vec{e} \pm \vec{p}(x,0) \ge \vec{0}$ and

$$(L\vec{\psi}^{\pm})_i(x,t) = C(-\varepsilon_i\frac{\alpha}{\varepsilon_n} + \alpha e^{\alpha t} + \sum_{j=1}^{n}a_{ij})B_n(x) \pm (L\vec{p})_i \ge 0,\ \text{as} -\frac{\varepsilon_i}{\varepsilon_n} \ge -1.$$

Applying Lemma 1, it follows that $\| \vec{p}(x,t) \| \le CB_n(x)$.

Defining the barrier functions through $\vec{\theta}^{\pm}(x,t) = C\varepsilon_n^{-\frac{l}{2}}e^{\alpha t}B_n(x)\vec{e} \pm \frac{\partial^l\vec{p}}{\partial x^l}$, $l = 1,2$ and using Lemma 1 for the problem satisfied by $\vec{p}$ and the bounds of the derivatives of $\vec{g}$, the bounds of $\frac{\partial\vec{p}}{\partial x}$ and $\frac{\partial^2\vec{p}}{\partial x^2}$ are derived.

The bounds for $\frac{\partial^l\vec{p}}{\partial x^l}$, $l = 3,4$ follow from the defining equation of $\vec{p}$.
By induction, the following bounds for $\vec{q}$ hold for $i = 1,\dots,n-1$:

$$\begin{aligned}
|\frac{\partial q_i}{\partial x}(x,t)| &\le C\left[\frac{B_i(x)}{\sqrt{\varepsilon_i}} + \dots + \frac{B_{n-1}(x)}{\sqrt{\varepsilon_{n-1}}}\right], & |\frac{\partial^2 q_i}{\partial x^2}(x,t)| &\le C\left[\frac{B_i(x)}{\varepsilon_i} + \dots + \frac{B_{n-1}(x)}{\varepsilon_{n-1}}\right],\\
|\frac{\partial^3 q_i}{\partial x^3}(x,t)| &\le C\left[\frac{B_1(x)}{\varepsilon_1^{3/2}} + \dots + \frac{B_{n-1}(x)}{\varepsilon_{n-1}^{3/2}}\right], & |\varepsilon_i\frac{\partial^4 q_i}{\partial x^4}(x,t)| &\le C\left[\frac{B_1}{\varepsilon_1} + \dots + \frac{B_{n-1}(x)}{\varepsilon_{n-1}}\right].
\end{aligned}$$

Combining the bounds for the derivatives of p_i and q_i, it follows that, for $i = 1,2,\dots,n-1$,

$$\begin{aligned}
&|\frac{\partial^l\tilde{\omega}_i}{\partial x^l}(x,t)| \le C\sum_{r=i}^{n-1}\frac{B_r(x)}{\varepsilon_r^{\frac{l}{2}}}\ \text{for } l = 1,2,\\
&|\frac{\partial^3\tilde{\omega}_i}{\partial x^3}(x,t)| \le C\sum_{r=1}^{n-1}\frac{B_r(x)}{\varepsilon_r^{\frac{3}{2}}},\ \ |\varepsilon_i\frac{\partial^4\tilde{\omega}_i}{\partial x^4}(x,t)| \le C\sum_{r=1}^{n-1}\frac{B_r(x)}{\varepsilon_r}.
\end{aligned}$$

4 The Shishkin Mesh

A piecewise-uniform Shishkin mesh with $N \times M$ mesh-elements is now constructed. Let $\Omega_t^M = \{t_k\}_{k=1}^M$, $\Omega_x^N = \{x_j\}_{j=1}^{N-1}$, $\bar{\Omega}_t^M = \{t_k\}_{k=0}^M$, $\bar{\Omega}_x^N = \{x_j\}_{j=0}^N$, $\Omega^{N,M} = \Omega_x^N \times \Omega_t^M$, $\bar{\Omega}^{N,M} = \bar{\Omega}_x^N \times \bar{\Omega}_t^M$ and $\Gamma^{N,M} = \Gamma \cap \bar{\Omega}^{N,M}$. The mesh $\bar{\Omega}_t^M$ is chosen to be a uniform mesh with M mesh-intervals on $[0, T]$.

The mesh $\bar{\Omega}_x^N$ is a piecewise-uniform mesh on $[0, 1]$ constructed by dividing $[0, 1]$ into $2n + 1$ mesh-intervals given by

$$[0, \tau_1] \cup \cdots \cup (\tau_{n-1}, \tau_n] \cup (\tau_n, 1 - \tau_n] \cup (1 - \tau_n, 1 - \tau_{n-1}] \cup \cdots \cup (1 - \tau_1, 1].$$

The n parameters τ_r, $r = 1, ..., n$, which determine the points separating the uniform meshes, are defined by

$$\tau_n = \min\left\{\frac{1}{4}, 2\frac{\sqrt{\varepsilon_n}}{\sqrt{\alpha}} \ln N\right\} \tag{41}$$

and for $r = n - 1, \ldots 1$,

$$\tau_r = \min\left\{\frac{r\tau_{r+1}}{r+1}, 2\frac{\sqrt{\varepsilon_r}}{\sqrt{\alpha}} \ln N\right\}. \tag{42}$$

Also, $\tau_0 = 0$, $\tau_{n+1} = \frac{1}{2}$. Clearly,

$$0 < \tau_1 < \ldots < \tau_n \le \frac{1}{4}, \qquad \frac{3}{4} \le 1 - \tau_n < \ldots < 1 - \tau_1 < 1.$$

Then, on the subinterval $(\tau_n, 1 - \tau_n]$ a uniform mesh with $\frac{N}{2}$ mesh-points is placed and on each of the subintervals $(\tau_r, \tau_{r+1}]$ and $(1 - \tau_{r+1}, 1 - \tau_r]$, $r = 0, 1, \ldots, n - 1$, a uniform mesh of $\dfrac{N}{4n}$ mesh-points is placed. In practice, it is convenient to take

$$N = 4nq, \ q \ge 3, \tag{43}$$

where n is the number of distinct singular perturbation parameters involved in (1). This construction leads to a class of 2^n piecewise-uniform Shishkin meshes $\bar{\Omega}^{N,M}$.

In particular, when all the parameters τ_r, $r = 1, \ldots, n$, are with the left choice, the Shishkin mesh $\bar{\Omega}^{N,M}$ becomes the classical uniform mesh with the transition parameters $\tau_r = \dfrac{r}{4n}$, $r = 1, \ldots, n$, and with the step size N^{-1} throughout from $\bar{\Omega}_x^N$. The Shishkin mesh suggested here has the following features: (i) when all the transition parameters have the left choice, it is the classical uniform mesh and (ii) it is coarse in the outer region and becomes finer and finer toward the left and right boundaries. From the above construction, it is clear that the transition points $\{\tau_r, 1 - \tau_r\}_{r=1}^n$ on $[0, 1]$ are the only points at which the mesh-size can change and that it does not necessarily change at each of these points. The following notations are introduced: if $x_j = \tau_r$, then $h_r^- = x_j - x_{j-1}$, $h_r^+ = x_{j+1} - x_j$, $J = \{\tau_r, 1 - \tau_r : h_r^+ \neq h_r^-\}$. In general, for each point x_j in the mesh-interval $(\tau_{r-1}, \tau_r]$,

$$x_j - x_{j-1} = 4nN^{-1}(\tau_r - \tau_{r-1}). \tag{44}$$

Also, for $x_j \in (\tau_n, 1 - \tau_n]$ $x_j - x_{j-1} = 2N^{-1}(1 - 2\tau_n)$ and for $x_j \in (0, \tau_1]$ and $x_j \in (1 - \tau_1, 1)$, $x_j - x_{j-1} = 4nN^{-1}\tau_1$. Thus, for $1 \leq r \leq n - 1$, the change in the mesh-size at the point $x_j = \tau_r$ is

$$h_r^+ - h_r^- = 4nN^{-1}(d_r - d_{r-1}), \tag{45}$$

where

$$d_r = \frac{r\tau_{r+1}}{r+1} - \tau_r \tag{46}$$

with the convention $d_0 = 0$. Notice that $d_r \geq 0$, $\Omega^{N,M}$ is a classical uniform mesh when $d_r = 0$ for all $r = 1 \ldots n$ and, from (42) and (43), that

$$\tau_r \leq C\sqrt{\varepsilon_r} \ln N, \quad 1 \leq r \leq n. \tag{47}$$

It follows from (44) and (47) that for $r = 1, \ \ldots \ ,n - 1$,

$$h_r^- + h_r^+ \leq C\sqrt{\varepsilon_{r+1}} N^{-1} \ln N. \tag{48}$$

Also

$$\tau_r = \frac{r}{s}\tau_s, \text{ when } d_r = \cdots = d_s = 0, \ 1 \leq r \leq s \leq n. \tag{49}$$

Lemma 10 *Assume that $d_r > 0$ for some r, $1 \leq r \leq n$. Then the following inequalities hold:*

$$B_r^H(1 - \tau_r) \leq B_r^H(\tau_r) = N^{-2}. \tag{50}$$

$$x_{r-1,r}^{(s)} \ \leq \ \tau_r - h_r^- \text{ for } 0 < s \leq 3/2. \tag{51}$$

$$B_q^L(\tau_r - h_r^-) \leq C B_q^L(\tau_r) \text{ for } 1 \leq r \leq q \leq n. \tag{52}$$

$$\frac{B_q^L(\tau_r)}{\sqrt{\varepsilon_q}} \leq C\frac{1}{\sqrt{\varepsilon_r} \ln N} \text{ for } 1 \leq q \leq n. \tag{53}$$

Analogous results hold for B_r^R.

Proof The proof is as given in [8].

5 The Discrete Problem

In this section, a classical finite difference operator with an appropriate Shishkin mesh is used to construct a numerical method for the problem (1), (2) which is shown later to be first-order parameter-uniform convergent in time and essentially first-order parameter-uniform convergent in the space variable.

The discrete initial-boundary value problem is now defined on any mesh by the finite difference method

$$D_t^- \vec{V}(x_j, t_k) - E\delta_x^2 \vec{V}(x_j, t_k) + A\vec{V}(x_j, t_k) = \vec{f}(x_j, t_k) \text{ on } \Omega^{N,M}, \tag{54}$$

$$\begin{gathered} \vec{V}(0, t_k) - E_* D_x^+ \vec{V}(0, t_k) = \vec{\phi}_L(t_k), \;\; \vec{V}(1, t_k) + E_* D_x^- \vec{V}(1, t_k) = \vec{\phi}_R(t_k) \\ \vec{V}(x_j, 0) = \vec{\phi}_B(x_j). \end{gathered} \tag{55}$$

This is used to compute numerical approximations to the solution of (1), (2). It is assumed henceforth that the mesh is a Shishkin mesh, as defined in the previous section. The problem (54) and (55) can also be written in the operator form

$$\vec{H}^{N,M} \vec{V} \; = \; \vec{f} \text{ on } \Omega^{N,M},$$

$$\vec{b}_0^{N,M} \vec{V}(0, t_k) = \vec{\phi}_L(t_k) \;\; \vec{b}_1^{N,M} \vec{V}(1, t_k) = \vec{\phi}_R(t_k) \;\; \vec{V}(x_j, 0) = \vec{\phi}_B(x_j), \tag{56}$$

where

$$\vec{H}^{N,M} \; = ID_t^- - E\delta_x^2 + AI, \; \vec{b}_0^{N,M} = I - E_* D_x^+, \; \vec{b}_1^{N,M} = I + E_* D_x^-$$

and D_t^-, δ_x^2, D_x^+ and D_x^- are the difference operators

$$D_t^- \vec{V}(x_j, t_k) \; = \; \frac{\vec{V}(x_j, t_k) - \vec{V}(x_j, t_{k-1})}{t_k - t_{k-1}},$$

$$\delta_x^2 \vec{V}(x_j, t_k) \; = \; \frac{D_x^+ \vec{V}(x_j, t_k) - D_x^- \vec{V}(x_j, t_k)}{(x_{j+1} - x_{j-1})/2},$$

$$D_x^+ \vec{V}(x_j, t_k) \; = \; \frac{\vec{V}(x_{j+1}, t_k) - \vec{V}(x_j, t_k)}{x_{j+1} - x_j},$$

$$D_x^- \vec{V}(x_j, t_k) \; = \; \frac{\vec{V}(x_j, t_k) - \vec{V}(x_{j-1}, t_k)}{x_j - x_{j-1}}.$$

For any function $\vec{Z} = (Z_1, ..., Z_n)^T$ defined on the Shishkin mesh $\bar{\Omega}^{N,M}$, the following norm $||\vec{Z}|| = \max\limits_{1 \le i \le n} \max\limits_{\substack{0 \le j \le N, \\ 0 \le k \le T}} |Z_i(x_j, t_k)|$ is defined.

The following discrete results are analogous to those for the continuous case.

Lemma 11 *Then, for any vector-valued mesh-function $\vec{\psi}$, the inequalities $\vec{b}_0^{N,M}\vec{\psi}(0, t_k) \geq \vec{0}$, $\vec{b}_1^{N,M}\vec{\psi}(1, t_k) \geq \vec{0}$, $\vec{\psi}(x_j, 0) \geq \vec{0}$ and $\vec{H}^{N,M}\vec{\psi} \geq \vec{0}$ on $\Omega^{N,M}$ imply that $\vec{\psi} \geq \vec{0}$ on $\bar{\Omega}^{N,M}$.*

Proof Let i^*, j^*, k^* be such that $\psi_{i^*}(x_{j^*}, t_{k^*}) = \min\limits_i \min\limits_{j,k} \psi_i(x_j, t_k)$ and assume that the lemma is false. Then $\psi_{i^*}(x_{j^*}, t_{k^*}) < 0$. Also, $\psi_{i^*}(x_{j^*}, t_{k^*}) - \psi_{i^*}(x_{j^*}, t_{k^*-1}) \leq 0$, $\psi_{i^*}(x_{j^*}, t_{k^*}) - \psi_{i^*}(x_{j^*-1}, t_{k^*}) \leq 0$, $\psi_{i^*}(x_{j^*+1}, t_{k^*}) - \psi_{i^*}(x_{j^*}, t_{k^*}) \geq 0$. Thus, $D_t^- \psi_{i^*}(x_{j^*}, t_{k^*}) \leq 0$, $\delta_x^2 \psi_{i^*}(x_{j^*}, t_{k^*}) > 0$.

If $x_{j^*} = 0$, then $(\vec{b}_0^{N,M}\vec{\psi})_{i^*}(0, t_{k^*}) = \psi_{i^*}(0, t_{k^*}) - \sqrt{\varepsilon_{i^*}} D_x^+ \psi_{i^*}(0, t_{k^*}) < 0$, a contradiction. Therefore, $x_{j^*} \neq 0$, for the same reason $x_{j^*} \neq 1$. Further,

$$\begin{aligned}\left(\vec{H}^{N,M}\vec{\psi}\right)_{i^*}(x_{j^*}, t_{k^*}) &= D_t^- \psi_{i^*}(x_{j^*}, t_{k^*}) - \varepsilon_{i^*}\delta_x^2 \psi_{i^*}(x_{j^*}, t_{k^*}) \\ &\quad + \sum_{q=1}^{n} a_{i^*q}(x_{j^*}, t_{k^*})\psi_q(x_{j^*}, t_{k^*}) \\ &< 0,\end{aligned}$$

which is a contradiction. Hence the result.

An immediate consequence of this is the following discrete stability result.

Lemma 12 *Then, for any vector-valued mesh-function $\vec{\psi}$ on $\bar{\Omega}^{N,M}$ and $i = 1, \ldots, n$,*

$$|\psi_i(x_j, t_k)| \leq \max\left\{||\vec{b}_0^{N,M}\vec{\psi}(0, t_k)||,\ ||\vec{b}_1^{N,M}\vec{\psi}(1, t_k)||,\ ||\vec{\psi}(x_j, 0)||,\ \frac{1}{\alpha}||\vec{H}^{N,M}\vec{\psi}||\right\}.$$

Proof Define the mesh-functions

$$\begin{aligned}\vec{\theta}^{\pm}(x_j, t_k) = \max\{||\vec{b}_0^{N,M}\vec{\psi}(0, t_k)||,\ ||\vec{b}_1^{N,M}\vec{\psi}(1, t_k)||,\ ||\vec{\psi}(x_j, 0)||,\ \tfrac{1}{\alpha}||\vec{H}^{N,M}\vec{\psi}||\}\vec{e} \\ \pm\vec{\psi}(x_j, t_k),\ \ (x_j, t_k) \in \bar{\Omega}^{N,M}.\end{aligned}$$

Using the properties of A, it is not hard to verify that $\vec{b}_0^{N,M}\vec{\theta}^{\pm}(0, t_k) \geq \vec{0}$, $\vec{b}_1^{N,M}\vec{\theta}^{\pm}(1, t_k) \geq \vec{0}$, $\vec{\theta}^{\pm}(x_j, 0) \geq \vec{0}$ and $\vec{H}^{N,M}\vec{\theta}^{\pm} \geq \vec{0}$ on $\Omega^{N,M}$. It follows from Lemma 11 that $\vec{\theta}^{\pm} \geq \vec{0}$ on $\bar{\Omega}^{N,M}$.

The following comparison principle will be used in the proof of the error estimate.

Lemma 13 *Assume that for each $i = 1, \ldots, n$, the vector-valued mesh-functions $\vec{\xi}$ and $\vec{Z}$ satisfy $|(\vec{b}_0^{N,M}\vec{Z})_i(0, t_k)| \leq (\vec{b}_0^{N,M}\vec{\phi})_i(0, t_k)$, $|(\vec{b}_1^{N,M}\vec{Z})_i(1, t_k)| \leq (\vec{b}_1^{N,M}\vec{\phi})_i(1, t_k)$, $|Z_i(x_j, 0)| \leq \phi_i(x_j, 0)$, and $|(\vec{H}^{N,M}\vec{Z})_i| \leq (\vec{H}^{N,M}\vec{\xi})_i$ on $\Omega^{N,M}$. Then, for each $i = 1, \ldots, n$,*

$$|Z_i| \leq \phi_i \text{ on } \overline{\Omega}^{N,M}.$$

Proof Define the mesh-functions $\vec{\psi}^{\pm}$ by

$$\vec{\psi}^{\pm} = \vec{\phi} \pm \vec{Z}.$$

Then, for each $i = 1, \ldots, n$, $\psi_i^{\pm}$ satisfy $(\vec{b}_0^{N,M}\vec{\psi}^{\pm})_i(0, t_k) \geq 0$, $(\vec{b}_1^{N,M}\vec{\psi}^{\pm})_i(1, t_k) \geq 0$, $\psi_i^{\pm}(x_j, 0) \geq 0$, and $(\vec{H}^{N,M}\vec{\psi}^{\pm})_i(x_j, t_k) \geq 0$ on $\Omega^{N,M}$.

The required result follows from the Lemma 11.

6 The Local Truncation Error

From Lemma 12, it is seen that in order to bound the error $\vec{V} - \vec{v}$, it suffices to bound $\vec{H}^{N,M}(\vec{V} - \vec{v})$. Note that, for $(x_j, t_k) \in \Omega^{M,N}$,

$$\vec{H}^{N,M}(\vec{V} - \vec{v}) = \vec{f} - \vec{H}^{N,M}\vec{v} = \vec{H}\vec{v} - \vec{H}^{N,M}\vec{v} = (\vec{H} - \vec{H}^{N,M})\vec{v}.$$

It follows that

$$\vec{H}^{N,M}(\vec{V} - \vec{v}) = (\frac{\partial}{\partial t} - D_t^-)\vec{v} - E(\frac{\partial^2}{\partial x^2} - \delta_x^2)\vec{v}.$$

Let $\vec{E}$, $\vec{W}$ be the discrete analogous to $\vec{\vartheta}$, $\vec{\omega}$, respectively.

$$\vec{H}^{N,M}\vec{E} = \vec{f} \text{ on } \Omega^{N,M},$$

$$\vec{b}_0^{N,M}\vec{E}(0, t_k) = \vec{b}_0\vec{\vartheta}(0, t_k),\ \vec{b}_1^{N,M}\vec{E}(1, t_k) = \vec{b}_1\vec{\vartheta}(1, t_k),\ \vec{E}(x_j, 0) = \vec{\vartheta}(x_j, 0),$$

$$\vec{H}^{N,M}\vec{W} = \vec{0} \text{ on } \Omega^{N,M},$$

$$\vec{b}_0^{N,M}\vec{W}(0, t_k) = \vec{b}_0\vec{\omega}(0, t_k),\ \vec{b}_1^{N,M}\vec{W}(1, t_k) = \vec{b}_1\vec{\omega}(1, t_k),\ \vec{W}(x_j, 0) = \vec{\omega}(x_j, 0),$$

where $\vec{\vartheta}$ and $\vec{\omega}$ are the solutions of (13), (14) and (15), (16), respectively. Further,

$$\vec{b}_0^{N,M}(\vec{E} - \vec{\vartheta})(0, t_k) = (\frac{\partial}{\partial x} - D_x^+)\vec{\vartheta}(0, t_k),$$

$$\vec{b}_1^{N,M}(\vec{E} - \vec{\vartheta})(1, t_k) = (D_x^- - \frac{\partial}{\partial x})\vec{\vartheta}(1, t_k),$$

$$\vec{b}_0^{N,M}(\vec{W} - \vec{\omega})(0, t_k) = (\frac{\partial}{\partial x} - D_x^+)\vec{\omega}(0, t_k),$$

$$\vec{b}_1^{N,M}(\vec{W} - \vec{\omega})(1, t_k) = (D_x^- - \frac{\partial}{\partial x})\vec{\omega}(1, t_k),$$

$$\vec{H}^{N,M}(\vec{E}-\vec{\vartheta})(x_j,t_k) = ((\frac{\partial}{\partial t} - D_t^-)\vec{\vartheta} - E(\frac{\partial^2}{\partial x^2} - \delta_x^2)\vec{\vartheta})(x_j,t_k),$$

$$\vec{H}^{N,M}(\vec{W}-\vec{\omega})(x_j,t_k) = ((\frac{\partial}{\partial t} - D_t^-)\vec{\omega} - E(\frac{\partial^2}{\partial x^2} - \delta_x^2)\vec{\omega})(x_j,t_k),$$

and so, for each $i = 1, 2$,

$$|(\vec{b}_0^{N,M}(\vec{E}-\vec{\vartheta}))_i(0,t_k)| = |(\frac{\partial}{\partial x} - D_x^+)v_i(0,t_k)|,$$

$$|(\vec{b}_1^{N,M}(\vec{E}-\vec{\vartheta}))_i(1,t_k)| = |(D_x^- - \frac{\partial}{\partial x})v_i(1,t_k)|,$$

$$|(\vec{b}_0^{N,M}(\vec{W}-\vec{\omega}))_i(0,t_k)| = |(\frac{\partial}{\partial x} - D_x^+)\omega_i(0,t_k)|,$$

$$|(\vec{b}_1^{N,M}(\vec{W}-\vec{\omega}))_i(1,t_k)| = |(D_x^- - \frac{\partial}{\partial x})\omega_i(1,t_k)|,$$

$$|(\vec{H}^{N,M}(\vec{E}-\vec{\vartheta}))_i(x_j,t_k)| \le |((\frac{\partial}{\partial t} - D_t^-)v_i)(x_j,t_k)| + |(\varepsilon_i(\frac{\partial^2}{\partial x^2} - \delta_x^2)v_i)(x_j,t_k)|, \tag{57}$$

$$|(\vec{H}^{N,M}(\vec{W}-\vec{\omega}))_i(x_j,t_k)| \le |((\frac{\partial}{\partial t} - D_t^-)\omega_i)(x_j,t_k)| + |(\varepsilon_i(\frac{\partial^2}{\partial x^2} - \delta_x^2)\omega_i)(x_j,t_k)|. \tag{58}$$

Therefore, the local truncation error of the smooth and singular components can be treated separately. Note that, for any smooth function ψ and for each $(x_j, t_k) \in \Omega^{N,M}$, the following distinct estimates of the local truncation error hold:

$$|(\frac{\partial}{\partial t} - D_t^-)\psi(x_j,t_k)| \le C(t_k - t_{k-1}) \max_{s \in [t_{k-1},t_k]} |\frac{\partial^2 \psi}{\partial t^2}(x_j,s)|, \tag{59}$$

$$|(\frac{\partial}{\partial x} - D_x^-)\psi(x_j,t_k)| \le C(x_j - x_{j-1}) \max_{s \in [x_{j-1},x_j]} |\frac{\partial^2 \psi}{\partial x^2}(s,t_k)|, \tag{60}$$

$$|(\frac{\partial}{\partial x} - D_x^+)\psi(x_j,t_k)| \le C(x_{j+1} - x_j) \max_{s \in [x_j,x_{j+1}]} |\frac{\partial^2 \psi}{\partial x^2}(s,t_k)|, \tag{61}$$

$$|(\frac{\partial^2}{\partial x^2} - \delta_x^2)\psi(x_j,t_k)| \le C \max_{s \in I_j} |\frac{\partial^2 \psi}{\partial x^2}(s,t_k)|, \tag{62}$$

$$|(\frac{\partial^2}{\partial x^2} - \delta_x^2)\psi(x_j,t_k)| \le C(x_{j+1} - x_{j-1}) \max_{s \in I_j} |\frac{\partial^3 \psi}{\partial x^3}(s,t_k)|. \tag{63}$$

Here $I_j = [x_{j-1}, x_{j+1}]$.

7 Error Estimate

The proof of the theorem on the error estimate is broken into two parts. First, a theorem concerning the error in the smooth component is established. Then the error in the singular component is estimated.

Define the barrier function through

$$\vec{\xi}(x_j, t_k) = C[M^{-1} + (r+1)N^{-1}\ln N + (N^{-1}\ln N)\sum_{\{r:\,\tau_r\in J\}} \frac{\tau_r}{\sqrt{\varepsilon_i}}\theta_r(x_j, t_k)]\vec{e},$$

where C is sufficiently large and θ_r is a piecewise linear polynomial for each $x_j = \tau_r \in J$ defined by

$$\theta_r(x,t) = \begin{cases} \dfrac{x}{\tau_r}, & 0 \le x \le \tau_r, \\ 1, & \tau_r < x < 1-\tau_r, \\ \dfrac{1-x}{\tau_r}, & 1-\tau_r \le x \le 1. \end{cases}$$

Also note that

$$(\vec{H}^{N,M}\theta_r\vec{e})_i(x_j, t_k) \ge \begin{cases} \alpha\theta_r(x_j, t_k), & \text{if } x_j \notin J \\ \alpha + \dfrac{2\varepsilon_i}{\tau_r(h_r^- + h_r^+)}, & \text{if } x_j \in J,\ x_j \in \{\tau_r, 1-\tau_r\}. \end{cases} \tag{64}$$

Then, on $\Omega^{M,N}$, the components ϕ_i of $\vec{\xi}$ satisfy

$$0 \le \phi_i(x_j, t_k) \le C(M^{-1} + N^{-1}\ln N),\ \ 1 \le i \le n. \tag{65}$$

Also,

$$(\vec{b}_0\vec{\phi})_i(0,t)) \ge C(M^{-1} + N^{-1}\ln N),\ \ (\vec{b}_1\vec{\phi})_i(1,t)) \ge C(M^{-1} + N^{-1}\ln N). \tag{66}$$

For $x_j \notin J$,

$$(\vec{H}^{N,M}\vec{\xi})_i(x_j, t_k) \ge C(M^{-1} + N^{-1}\ln N) \tag{67}$$

and, for $x_j \in J$, using (47), (48) and (64),

$$(\vec{H}^{N,M}\vec{\xi})_i(x_j, t_k) \ge C(M^{-1} + N^{-1}\ln N). \tag{68}$$

The following theorem gives the estimate for the error in the smooth component.

Theorem 1 *Let $\vec{\vartheta}$ denote the smooth component of the solution to the problem (1), (2) and $\vec{E}$ denote the smooth component of the solution to the problem (54), (55). Then*

$$||\vec{E}-\vec{\vartheta}|| \le C(M^{-1}+N^{-1}\ln N). \tag{69}$$

Proof From the expression (61),

$$\begin{aligned}|(\vec{b}_0^{N,M}(\vec{E}-\vec{\vartheta}))_i(0,t_k)| &\le C\sqrt{\varepsilon_i}(x_1-x_0)\max_{s\in[x_0,x_1]}|\frac{\partial^2 v_i}{\partial x^2}(s,t_k)| \\ &\le CN^{-1}.\end{aligned} \tag{70}$$

From the expression (60),

$$\begin{aligned}|(\vec{b}_1^{N,M}(\vec{E}-\vec{\vartheta}))_i(1,t_k)| &\le C\sqrt{\varepsilon_i}(x_N-x_{N-1})\max_{s\in[x_{N-1},x_N]}|\frac{\partial^2 v_i}{\partial x^2}(s,t_k)| \\ &\le CN^{-1}.\end{aligned} \tag{71}$$

Thus from (70), (71) and (66),

$$\begin{aligned}|(\vec{b}_0^{N,M}(\vec{E}-\vec{\vartheta}))_i(0,t_k)| &\le (\vec{b}_0^{N,M}\vec{\phi})_i(0,t_k),\\ |(\vec{b}_1^{N,M}(\vec{E}-\vec{\vartheta}))_i(1,t_k)| &\le (\vec{b}_1^{N,M}\vec{\phi})_i(1,t_k),\\ |(\vec{E}-\vec{\vartheta})_i(x_j,0)| &\le \phi_i(x_j,0).\end{aligned} \tag{72}$$

For each mesh-point x_j, there are two possibilities: either $x_j \notin J$ or $x_j \in J$.
If $x_j \notin J$, using the bounds of the derivatives of $\vec{\vartheta}$ and the expressions (59) and (63),

$$|(\vec{H}^{N,M}(\vec{E}-\vec{\vartheta}))_i(x_j,t_k)| \le C[M^{-1}+N^{-1}]. \tag{73}$$

On the other hand, if $x_j \in J$, then $x_j \in \{\tau_r, 1-\tau_r\}$, for some r, $1 \le r \le n$.
Consider the case $x_j = \tau_r$ and for $x_j = 1-\tau_r$ the proof is analogous.
Using the bounds of the derivatives of $\vec{\vartheta}$ and the expressions (59) and (63),

$$|(\vec{H}^{M,N}(\vec{E}-\vec{\vartheta}))_i(x_j,t_k)| \le C[M^{-1}+N^{-1}\ln N]. \tag{74}$$

From (72), (73) and (74) and the comparison principle, the required result is obtained.
In order to estimate the error of the singular component, the following lemmas are required.

Lemma 14 *Assume that $x_j \notin J$. Then, on $\Omega^{N,M}$, for each $1 \le i \le n$,*

$$|(\vec{H}^{N,M}(\vec{W}-\vec{w}))_i(x_j,t_k)| \le C(M^{-1}+\frac{(x_{j+1}-x_{j-1})}{\sqrt{\varepsilon_1}}). \tag{75}$$

The following decomposition in the singular components ω_i is used in the next lemma:

$$\omega_i = \sum_{m=1}^{r+1} \omega_{i,m}, \tag{76}$$

where the components $\omega_{i,m}$ are defined by

$$\omega_{i,r+1} = \begin{cases} p_i^{(s)} & \text{on } [0, x_{r,r+1}^{(s)}) \\ \omega_i & \text{on } [x_{r,r+1}^{(s)}, 1 - x_{r,r+1}^{(s)}] \\ q_i^{(s)} & \text{on } (1 - x_{r,r+1}^{(s)}, 1] \end{cases}$$

where

$$p_i^{(s)}(x,t) = \begin{cases} \sum_{k=0}^{3} \dfrac{\partial^k \omega_i}{\partial x^k}(x_{r,r+1}^{(s)}, t) \dfrac{(x - x_{r,r+1}^{(s)})^k}{k!}, & s = \frac{3}{2}, \\ \\ \sum_{k=0}^{4} \dfrac{\partial^k \omega_i}{\partial x^k}(x_{r,r+1}^{(s)}, t) \dfrac{(x - x_{r,r+1}^{(s)})^k}{k!}, & s = 1, \end{cases}$$

$$q_i^{(s)}(x,t) = \begin{cases} \sum_{k=0}^{3} \dfrac{\partial^k \omega_i}{\partial x^k}(1 - x_{r,r+1}^{(s)}, t) \dfrac{(x - (1 - x_{r,r+1}^{(s)}))^k}{k!}, & s = \frac{3}{2}, \\ \\ \sum_{k=0}^{4} \dfrac{\partial^k \omega_i}{\partial x^k}(1 - x_{r,r+1}^{(s)}, t) \dfrac{(x - (1 - x_{r,r+1}^{(s)}))^k}{k!}, & s = 1, \end{cases}$$

and, for each m, $r \geq m \geq 2$,

$$\omega_{i,m} = \begin{cases} p_i^{(s)} & \text{on } [0, x_{m-1,m}^{(s)}) \\ \omega_i - \sum_{k=m+1}^{r+1} \omega_{i,k} & \text{on } [x_{m-1,m}^{(s)}, 1 - x_{m-1,m}^{(s)}] \\ q_i^{(s)} & \text{on } (1 - x_{m-1,m}^{(s)}, 1] \end{cases}$$

where

$$p_i^{(s)}(x,t) = \begin{cases} \sum_{k=0}^{3} \dfrac{\partial^k \omega_i}{\partial x^k}(x_{m,m+1}^{(s)}, t) \dfrac{(x - x_{m,m+1}^{(s)})^k}{k!}, & s = \frac{3}{2}, \\ \\ \sum_{k=0}^{4} \dfrac{\partial^k \omega_i}{\partial x^k}(x_{m,m+1}^{(s)}, t) \dfrac{(x - x_{m,m+1}^{(s)})^k}{k!}, & s = 1, \end{cases}$$

$$q_i^{(s)}(x,t) = \begin{cases} \sum_{k=0}^{3} \dfrac{\partial^k \omega_i}{\partial x^k}(1 - x_{m,m+1}^{(s)}, t) \dfrac{(x - (1 - x_{m,m+1}^{(s)}))^k}{k!}, & s = \frac{3}{2}, \\ \\ \sum_{k=0}^{4} \dfrac{\partial^k \omega_i}{\partial x^k}(1 - x_{m,m+1}^{(s)}, t) \dfrac{(x - (1 - x_{m,m+1}^{(s)}))^k}{k!}, & s = 1, \end{cases}$$

and

$$\omega_{i,1} = \omega_i - \sum_{k=2}^{r+1} \omega_{i,k} \text{ on } [0, 1].$$

Notice that the decomposition (76) depends on the choice of the polynomials $p_i^{(s)}$, $q_i^{(s)}$ and the definition of $x_{i,j}^{(s)}$ given by (35).

The following lemma provides estimates of the derivatives of the components

$\omega_{i,m}$, $1 \leq m \leq r+1$ of ω_i, $1 \leq i \leq n$.

Lemma 15 *Assume that $d_r > 0$ for some r, $1 \leq r \leq n$. Then, for each $1 \leq i \leq n$, the components in the decomposition (76) satisfy the following estimates for each q and r, $1 \leq q \leq r$, and all $(x_j, t_k) \in \Omega^{N,M}$:*

$$|\frac{\partial^2 \omega_{i,q}}{\partial x^2}(x_j, t_k)| \leq C \min\{\frac{1}{\varepsilon_q}, \frac{1}{\varepsilon_i}\} B_q(x_j),$$

$$|\frac{\partial^3 \omega_{i,q}}{\partial x^3}(x_j, t_k)| \leq C \min\{\frac{1}{\varepsilon_i \sqrt{\varepsilon_q}}, \frac{1}{\varepsilon_q^{3/2}}\} B_q(x_j),$$

$$|\frac{\partial^3 \omega_{i,r+1}}{\partial x^3}(x_j, t_k)| \leq C \min\{\textstyle\sum_{q=r+1}^n \dfrac{B_q(x_j)}{\varepsilon_i \sqrt{\varepsilon_q}}, \textstyle\sum_{q=r+1}^n \dfrac{B_q(x_j)}{\varepsilon_q^{3/2}}\},$$

$$|\frac{\partial^4 \omega_{i,q}}{\partial x^4}(x_j, t_k)| \leq C \frac{B_q(x_j)}{\varepsilon_i \varepsilon_q},$$

$$|\frac{\partial^4 \omega_{i,r+1}}{\partial x^4}(x_j, t_k)| \leq C \textstyle\sum_{q=r+1}^n \dfrac{B_q(x_j)}{\varepsilon_i \varepsilon_q}.$$

Lemma 16 *Assume that $d_r > 0$ for some r, $1 \leq r \leq n$. Then, if $x_j \notin J$,*

$$|(\vec{H}^{N,M}(\vec{W} - \vec{\omega}))_i(x_j, t_k)| \leq C[M^{-1} + B_r(x_{j-1}) + \frac{(x_{j+1} - x_{j-1})}{\sqrt{\varepsilon_{r+1}}}], \tag{77}$$

and if $x_j \in J$,

$$|(\vec{H}^{N,M}(\vec{W} - \vec{\omega}))_i(x_j, t_k)| \leq C[M^{-1} + N^{-1} \ln N]. \tag{78}$$

Lemma 17 *Then, on $\Omega^{N,M}$, for each $1 \leq i \leq n$, the following estimates hold:*

$$|(\vec{H}^{N,M}(\vec{W} - \vec{w}))_i(x_j, t_k)| \leq C(M^{-1} + B_n(x_{j-1})). \tag{79}$$

The following theorem gives the estimate for the error in the singular component.

Theorem 2 *Let $\vec{\omega}$ denote the singular component of the solution to the problem (1), (2) and $\vec{W}$ be the singular component of the solution to the problem (54), (55). Then*

$$||\vec{W} - \vec{\omega}|| \leq C(M^{-1} + N^{-1} \ln N). \tag{80}$$

Proof From the expression (61),

$$\begin{aligned} |(\vec{b}_0^{N,M}(\vec{W} - \vec{\omega}))_i(0, t_k)| &\leq C\sqrt{\varepsilon_i}(x_1 - x_0) \max_{s \in [x_0, x_1]} |\frac{\partial^2 \omega_i}{\partial x^2}(s, t_k)| \\ &\leq CN^{-1} \ln N. \end{aligned} \tag{81}$$

From the expression (60),

$$\begin{aligned} |(\vec{b}_1^{N,M}(\vec{W} - \vec{\omega}))_i(1, t_k)| &\leq C\sqrt{\varepsilon_i}(x_N - x_{N-1}) \max_{s \in [x_{N-1}, x_N]} |\frac{\partial^2 \omega_i}{\partial x^2}(s, t_k)| \\ &\leq CN^{-1} \ln N. \end{aligned} \tag{82}$$

Thus from (81), (82) and (66),

$$\begin{aligned} |(\vec{b}_0^{N,M}(\vec{W} - \vec{\omega}))_i(0, t_k)| &\leq (\vec{b}_0^{N,M}\vec{\phi})_i(0, t_k), \\ |(\vec{b}_1^{N,M}(\vec{W} - \vec{\omega}))_i(1, t_k)| &\leq (\vec{b}_1^{N,M}\vec{\phi})_i(1, t_k), \\ |(\vec{W} - \vec{\omega})_i(x_j, 0)| &\leq \phi_i(x_j, 0). \end{aligned} \tag{83}$$

In the remaining portion, it is shown that for all i, j, k, and some constant C,

$$|(\vec{H}^{N,M}(\vec{W} - \vec{\omega}))_i(x_j, t_k)| \leq (\vec{H}^{N,M}\vec{\xi})_i(x_j, t_k). \tag{84}$$

This is proved for each mesh-point $x_j \in \Omega_x^N$ by considering separately the 8 kinds of subintervals
(a) $(0, \tau_1)$,
(b) $[\tau_1, \tau_2)$,
(c) $[\tau_m, \tau_{m+1})$ for some m, $2 \leq m \leq n-1$,
(d) $[\tau_n, 1/2)$,
(e) $[1/2, 1-\tau_n]$,
(f) $(1-\tau_{m+1}, 1-\tau_m]$, for some m, $2 \leq m \leq n-1$,
(g) $(1-\tau_2, 1-\tau_1]$ and
(h) $(1-\tau_1, 1)$.

(a) Clearly, $x_j \notin J$ and $x_{j+1} - x_{j-1} \leq C\sqrt{\varepsilon_1}N^{-1} \ln N$.
Then, Lemma 14 and expression (67) give (84). Similar arguments hold for the case (e).

(b) There are 2 possibilities: (b1) $d_1 = 0$ and (b2) $d_1 > 0$.

(b1) Since $\tau_1 = \dfrac{\tau_2}{2}$ and the mesh is uniform in $(0, \tau_2)$, it follows that $x_j \notin J$, and $x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_1}N^{-1}\ln N$. Then Lemma 14 and expression (67) give (84).
(b2) Either $x_j \notin J$ or $x_j \in J$.
If $x_j \notin J$ then $x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_2}N^{-1}\ln N$ and by Lemma 10, $B_1(x_{j-1}) \le B_1^H(x_{j-1}) \le B_1^H(\tau_1 - h_1^-) \le CN^{-2}$, so Lemma 16 (77) with $r = 1$ and expression (67) give (84).
On the other hand, if $x_j \in J$, the expression (78) of Lemma 16 with $r = 1$ and expression (68) give (84). Similar arguments hold for the case (f).

(c) There are 3 possibilities:
(c1) $d_1 = d_2 = \cdots = d_m = 0$,
(c2) $d_r > 0$ and $d_{r+1} = \ldots = d_m = 0$ for some r, $1 \le r \le m-1$ and
(c3) $d_m > 0$.
(c1) Since the mesh is uniform in $(0, \tau_{m+1})$, it follows that $x_j \notin J$ and $x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_1}N^{-1}\ln N$. Then Lemma 14 and expression (67) give (84).
(c2) Either $x_j \notin J$ or $x_j \in J$.
If $x_j \notin J$ then $\tau_{r+1} = C\tau_{m+1}$, $x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_{m+1}}N^{-1}\ln N$ and by Lemma 10 $B_r(x_{j-1}) \le B_r^H(x_{j-1}) \le B_r^H(\tau_m - h_m^-) \le B_r^H(\tau_r - h_r^-) \le CN^{-2}$. Thus, expression (77) of Lemma 16 and expression (67) give (84).
On the other hand, if $x_j \in J$, then $x_j = \tau_m$. The expression (78) of Lemma 16 with $r = m$ and expression (68) give (84).
(c3) Either $x_j \notin J$ or $x_j \in J$.
If $x_j \notin J$ then $x_{j+1} - x_{j-1} \le C\sqrt{\varepsilon_{m+1}}N^{-1}\ln N$. From 10 $B_m(x_{j-1}) \le B_m^H(x_{j-1}) \le B_m^H(\tau_m - h_m^-) \le CN^{-2}$. Expression (77) of Lemma 16 with $r = m$ and expression (67) give (84).
On the other hand, if $x_j = \tau_m$, expression (78) of Lemma 16 with $r = m$ and expression (68) give (84). Similar arguments hold for the case (g).

(d) There are 3 possibilities:
(d1) $d_1 = \ldots = d_n = 0$,
(d2) $d_r > 0$ and $d_{r+1} = \ldots = d_n = 0$ for some r, $1 \le r \le n-1$ and
(d3) $d_n > 0$.
(d1) Since the mesh is uniform in Ω_x^N, it follows that $x_j \notin J$, $\frac{1}{\sqrt{\varepsilon_1}} \le C\ln N$ and $x_{j+1} - x_{j-1} \le CN^{-1}$. Then Lemma 14 and expression (67) give (84).
(d2) Either $x_j \notin J$ or $x_j \in J$.
If $x_j \notin J$ then $\frac{1}{\sqrt{\varepsilon_{r+1}}} \le C\ln N$, $x_{j+1} - x_{j-1} \le CN^{-1}$ and by Lemma 10, $B_r(x_{j-1}) \le B_r^H(x_{j-1}) \le B_r^H(\tau_n - h_n^-) \le B_r^H(\tau_r - h_r^-) \le CN^{-2}$. The expression (77) of Lemma 16 and expression (67) give (84).
On the other hand, if $x_j \in J$, then $x_j = \tau_n$.
The expression (78) of Lemma 16 and expression (68) give (84).
(d3) By Lemma 10 with $r = n$, $B_n(x_{j-1}) \le B_n^H(x_{j-1}) \le B_n^H(\tau_n - h_n^-) \le CN^{-2}$. Then Lemma 17 and expression (67) give (84). Similar arguments hold for the case (h).

By using comparison principle, the required result is established from (83) and (84).

The following theorem gives a parameter-uniform bound which is first order in time and essentially first order in space for the error incurred in the computed solution.

Theorem 3 *Let $\vec{v}$ denote the solution to the problem (1), (2) and $\vec{V}$ denote the solution to the problem (54), (55). Then*

$$||\vec{V} - \vec{v}|| \leq C(M^{-1} + N^{-1} \ln N).$$

Proof An application of the triangular inequality and the results of Theorems 1 and 2 lead to the required result.

8 Numerical Illustration

The following example illustrates the above-proposed numerical method in this section. The parameter-uniform order of convergence and the parameter-uniform error constants are computed. To get the order of convergence in the variable t exclusively, a fine Shishkin mesh is considered for x and the resulting problem is solved for various uniform meshes with respect to t. A variant of the two-mesh algorithm for a vector problem which is found in [2] for a scalar problem is applied to get parameter-uniform t-order of convergence and the error constant. A uniform mesh is considered for t and the resulting problem is solved for various piecewise-uniform fine Shishkin meshes with respect to x to get the order of convergence in the variable x exclusively. The numerical results are presented in Tables 1 and 2.

Example 1 Consider the problem

$$\frac{\partial \vec{v}}{\partial t} - E\frac{\partial^2 \vec{v}}{\partial x^2} + A\vec{v} = \vec{f} \text{ on } (0, 1) \times (0, 1],$$

$$\vec{b}_0 \vec{v}(0, t) = \vec{\xi}_L, \;\; \vec{b}_1 \vec{v}(1, t) = \vec{\xi}_R, \;\; \vec{v}(x, 0) = \vec{\xi}_B$$

where $E = diag(\varepsilon_1, \varepsilon_2)$, $A = \begin{pmatrix} 4+3t & -1 \\ -1 & 4+3t \end{pmatrix}$, $\vec{f} = \begin{pmatrix} 2+e^{3t} \\ 2+e^{3t} \end{pmatrix}$,
$\vec{\xi}_L = \begin{pmatrix} 1+t^8 \\ 1+t^8 \end{pmatrix}$, $\vec{\xi}_R = \begin{pmatrix} 1+t^8 \\ 1+t^8 \end{pmatrix}$, $\vec{\xi}_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

For various values of ε_1 and ε_2, the maximum errors, the $\vec{\varepsilon}$-uniform order of convergence and the $\vec{\varepsilon}$-uniform error constant are computed. The value of η is varied as shown in the tables. α is taken to be 0.9. Fixing a fine Shishkin mesh with 128 points horizontally, the problem is solved by the method suggested above. The order of convergence and the error constant for $\vec{v}$ are calculated for t, and the results are presented in Table 1. A fine uniform mesh on t with 32 points is considered, and the

Table 1 Values of $D_{\vec{\varepsilon}}^N$, D^N, p^N, p^* and $C_{p^*}^N$ for $\varepsilon_1 = \dfrac{\eta}{16}$, $\varepsilon_2 = \dfrac{\eta}{8}$, $\alpha = 2.9$ and $N = 128$

η	Number of mesh-points M			
	32	64	128	256
2^{-7}	0.153E-01	0.783E-02	0.397E-02	0.199E-02
2^{-8}	0.155E-01	0.788E-02	0.397E-02	0.200E-02
2^{-9}	0.156E-01	0.790E-02	0.398E-02	0.200E-02
2^{-10}	0.156E-01	0.790E-02	0.398E-02	0.200E-02
2^{-11}	0.156E-01	0.790E-02	0.398E-02	0.200E-02
D^N	0.156E-01	0.790E-02	0.398E-02	0.200E-02
p^N	0.980E+00	0.990E+00	0.995E+00	
$C_{p^*}^N$	0.945E+00	0.945E+00	0.939E+00	0.929E+00
Computed t-order of $\vec{\varepsilon}$—uniform convergence, $p^* = 0.9803767$				
Computed $\vec{\varepsilon}$—uniform error constant, $C_{p^*}^* = 0.9451866$				

Table 2 Values of $D_{\vec{\varepsilon}}^N$, D^N, p^N, p^* and $C_{p^*}^N$ for $\varepsilon_1 = \dfrac{\eta}{16}$, $\varepsilon_2 = \dfrac{\eta}{8}$, $\alpha = 2.9$ and $M = 32$

η	Number of mesh-points N			
	32	64	128	256
2^{-7}	0.530E-01	0.339E-01	0.172E-01	0.689E-02
2^{-8}	0.530E-01	0.339E-01	0.172E-01	0.689E-02
2^{-9}	0.530E-01	0.339E-01	0.172E-01	0.689E-02
2^{-10}	0.530E-01	0.339E-01	0.172E-01	0.689E-02
2^{-11}	0.530E-01	0.339E-01	0.172E-01	0.689E-02
D^N	0.530E-01	0.339E-01	0.172E-01	0.689E-02
p^N	0.644E+00	0.977E+00	0.132E+01	
$C_{p^*}^N$	0.137E+01	0.137E+01	0.109E+01	0.679E+00
Computed x-order of $\vec{\varepsilon}$—uniform convergence, $p^* = 0.6436486$				
Computed $\vec{\varepsilon}$—uniform error constant, $C_{p^*}^* = 1.371360$				

order of convergence for $\vec{v}$ in the variable x is calculated and the results are presented in Table 2.

It is evident from Figs. 1 and 2 that the solution $\vec{v}$ exhibits parabolic twin boundary layers at $(0, t)$ and $(1, t)$, $0 \le t \le 1$. Further, the t-order of convergence and the x-order of convergence of the numerical method presented in Tables 1 and 2 agree with the theoretical result.

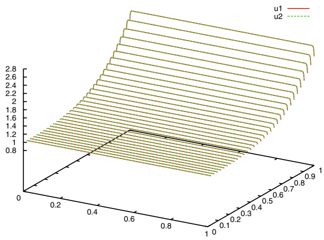


Fig. 1 The numerical approximation of $\vec{v}$ for $\varepsilon_1 = 2^{-15}$, $\varepsilon_2 = 2^{-14}$ and $M = 32$

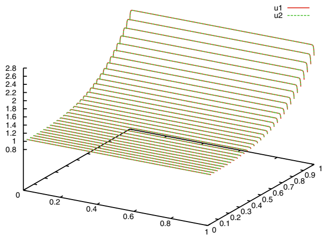


Fig. 2 The numerical approximation of $\vec{v}$ for $\varepsilon_1 = 2^{-15}$, $\varepsilon_2 = 2^{-14}$ and $N = 128$

References

1. Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Fitted numerical methods for singular perturbation problems. World Scientific Revised Edition (2012)
2. Farrell, P.A., Hegarty, A., Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: In: Knops, R.J., Morton, K.W. (eds.), Robust computational techniques for boundary layers. Applied Mathematics & Mathematical Computation. Champman & Hall/CRC Press (2000)
3. Franklin, V., Paramasivam, M., Miller, J.J.H., Valarmathi, S.: Second order parameter-uniform convergence for a finite difference method for a singularly perturbed linear parabolic system. Int. J. Numer. Anal. Model. **10**(1), 178–202 (2013)
4. Friedman, A.: Partial Differential Equations of Parabolic Type. Prentice Hall (1964)
5. Ladyzhenskaya, O.A., Ural'tseva, N.N.: Linear and Quasilinear Elliptic Equtions. Academic Press, New York, London (1968)
6. Miller, J.J.H., O'Riordan, E., Shishkin, G.I., Shishkina, L.P.: Fitted mesh methods for problems with parabolic boundary layers. Math. Proc. R. Irish Acad. **98**(A), 173–190 (1998)
7. Gracia, J.L., Lisbona, F.J., O'Riordan, E.: A coupled system of singularly perturbed parabolic reaction-diffusion equations. Adv. Comput. Math. **32**, 43–61 (2010)
8. Paramasivam, M., Valarmathi, S., Miller, J.J.H.: Second order parameter-uniform convergence for a finite difference method for a singularly perturbed linear reaction diffusion system. Math. Commun. **15**(2) (2010)
9. Ishwariya, R., Miller, J.J.H., Valarmathi, S.: A parameter uniform essentially second order convergence of a fitted mesh method for a class parabolic singularly perturbed Robin problem of reaction-diffusion equation (communicated)
10. Roos, H.-G., Stynes, M., Tobiska, L.: Numerical Methods for Singularly Perturbed Differential Equations. Springer (1996)
11. Doolan, E.P., Miller, J.J.H., Schilders, W.H.A.: Uniform Numerical Methods for Problems with Initial and Boundary Layers. Boole Press, Dublin, Ireland (1980)

Finite Difference Methods with Interpolation for First-Order Hyperbolic Delay Differential Equations

S. Karthick and V. Subburayan

Abstract First-order hyperbolic delay differential equations are considered in this article. Finite difference methods with piecewise linear interpolation are suggested to solve the problems. The proposed methods are consistent, conditionally stable and hence they converge by Lax-Richtmyer equivalence theorem. It is observed that the solution of the first-order hyperbolic delay differential equation exhibits wave nature and due to the presence of delay argument one or more additional waves appear in the solution. Further, the wave propagation occurs in forward and backward direction, respectively, if $a > 0$ and $a < 0$. Numerical results are presented to validate the theoretical results.

Keywords Hyperbolic delay differential equations · Piecewise linear interpolation · Forward time backward space scheme · Forward time forward space scheme · Matrix method

2010 MSC Subject Classification 35L02; 35L03; 35L40; 65M06; 65M12

1 Introduction

Hyperbolic delay differential equations appear in many branches of applied science. For example, the study of distributed networks containing lossless transmission lines [1], the theoretical analysis of neuronal variability [2] have been modeled as hyperbolic delay differential equations. The numerical methods for ordinary delay differential equations and hyperbolic partial differential equations have been well studied in the literature, to list a few: [3–13] and the references therein. In the recent past years, there has been growing interest in developing numerical methods for hyperbolic delay differential equation. To cite a few, the authors Kapil K Sharma and Paramjeet Singh have applied Forward Time Backward Space (FTBS) and Backward Time Backward

S. Karthick · V. Subburayan (✉)
Department of Mathematics, Faculty of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur 603 203, TN, India

Valarmathi S et al. (eds.), *Differential Equations and Applications*, Springer Proceedings in Mathematics & Statistics 368,
https://doi.org/10.1007/978-981-16-7546-1_8

Space (BTBS) numerical methods suggested in [4] for hyperbolic delay differential equations [14–16]. The existence results pertaining to the hyperbolic system of equations have been well addressed in [17]. Protter and Weinberger [18] have studied extensively the maximum principle for hyperbolic, parabolic, and elliptical differential equations. For first-order impulsive hyperbolic equation, comparison principles have been studied in [19]. The maximum principle for a modified triangle-based adaptive difference scheme for hyperbolic conservation laws is discussed in [20]. In this present paper, as mentioned in the abstract, we consider first-order hyperbolic delay differential equations and suggested the numerical methods with interpolation. They have been proved that the methods are convergence of order one in time and space.

The article is organized as follows: in Sect. 2 the problems under study are stated. The finite difference schemes with linear interpolation are presented in Sect. 3. Further, we proved that the methods are consistent and stable. The error analysis of the methods is carried out in the Sect. 4. The paper is concluded with the Concluding Remark as the Sect. 6.

2 Problem Statements

In the present section, motivated by the works of [1, 2, 14, 15], we consider the following two problems. The delay argument δ is a fixed positive constant throughout the article.

2.1 Problem I

Consider the first-order hyperbolic delay differential equation

$$L_1 u(x,t) := \frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} - b(x,t)u(x-\delta,t) = 0,\ (x,t) \in (0, x_f] \times (0,T], \tag{1}$$

$$u(x,t) = \phi_l(x,t),\ (x,t) \in [-\delta, 0] \times [0,T], \tag{2}$$

$$u(x,0) = u_0(x),\ x \in [0, x_f]. \tag{3}$$

The coefficient functions a, b and the data functions u_0, and ϕ_l are sufficiently differentiable on their domains. In this problem $a(x,t) > 0$, $\forall (x,t)$ and $\phi_l(0,0) = u_0(0)$.

2.2 Problem II

Further we consider the first-order hyperbolic delay differential equation

$$L_2u(x,t) := \frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} - b(x,t)u(x-\delta,t) = 0,\ (x,t) \in [0,x_f) \times (0,T], \tag{4}$$

$$u(x,t) = \phi_l(x,t),\ (x,t) \in [-\delta,0) \times [0,T], \tag{5}$$

$$u(x,0) = u_0(x),\ x \in [0,x_f], \tag{6}$$

$$u(x_f,t) = \phi_r(t),\ t \in [0,T]. \tag{7}$$

Here also the functions a, b, u_0, ϕ_l and ϕ_r are sufficiently differentiable on their domains. Here $a(x,t) < 0,\ \forall(x,t)$ and $u_0(x_f) = \phi_r(0)$.

3 Finite Difference Methods

This section presents the mesh selection procedure and the finite difference methods for the above stated problems (1)–(3) and (4)–(7). In the subsequent sections we use the following: U_i^j denotes the numerical solution at the mesh point (x_i, t_j), and $a(x_i,t_j) = a_i^j$, $b(x_i,t_j) = b_i^j$.

3.1 Mesh Points

Let N and M be the number of mesh points in $[0,x_f]$ and $[0,T]$, respectively. Define $\Delta x = x_f/N$ and $\Delta t = T/M$. Then the mesh $\bar{\Omega}^{N,M}$ is defined as $\bar{\Omega}^{N,M} = \{(x_i,t_j)|i = 0,1,\ldots,N,\ \ j = 0,1,\ldots,M\}$, where $x_k = k\Delta x$ and $t_m = m\Delta t$.

3.2 Finite Difference Scheme with Piecewise Linear Interpolation for the Problem I

The Forward Time Backward Space (FTBS) finite difference scheme with piecewise linear interpolation for the above problem (1)–(3) is as follows:

$$L_1^{N,M}U_i^j := D_t^+U_i^j + a_i^j D_x^- U_i^j - b_i^j \phi_l(x_i-\delta, t_j) = 0,\ x_i - \delta \le 0, \tag{8}$$

$$L_1^{N,M}U_i^j := D_t^+U_i^j + a_i^j D_x^- U_i^j - b_i^j \left[U_k^j l_k(x_i-\delta) + U_{k+1}^j l_{k+1}(x_i-\delta)\right] = 0,\ x_i - \delta > 0, \tag{9}$$

$$U_i^0 = u_0(x_i),\ i = 0, 1, \cdots, N,\ U_0^j = \phi_l(x_0, t_j),\ j = 0, 1, \ldots, M \tag{10}$$

where $D_t^+U_i^j = \dfrac{U_i^{j+1} - U_i^j}{\Delta t}$, $D_x^-U_i^j = \dfrac{U_i^j - U_{i-1}^j}{\Delta x}$, $l_k(x) = \dfrac{x_{k+1} - x}{\Delta x}$ and $l_{k+1}(x) = \dfrac{x - x_k}{\Delta x}$.

3.3 Finite Difference Scheme with Piecewise Linear Interpolation for the Problem II

The Forward Time Forward Space (FTFS) finite difference scheme with piecewise linear interpolation for the above problem (4)–(7) is as follows:

$$L_2^{N,M}U_i^j := D_t^+U_i^j + a_i^j D_x^+ U_i^j - b_i^j \phi_l(x_i-\delta, t_j) = 0,\ x_i - \delta \le 0, \tag{11}$$

$$L_2^{N,M}U_i^j := D_t^+U_i^j + a_i^j D_x^+ U_i^j - b_i^j \left[U_k^j l_k(x_i-\delta) + U_{k+1}^j l_{k+1}(x_i-\delta)\right] = 0,\ x_i - \delta > 0, \tag{12}$$

$$U_i^0 = u_0(x_i),\ i = 0, 1, \cdots, N,\ U_N^j = \phi_r(t_j),\ j = 0, 1, \ldots, M \tag{13}$$

where $D_x^+U_i^j = \dfrac{U_{i+1}^j - U_i^j}{\Delta x}$.

3.4 Consistency of the Schemes

Following the arguments of [14, 15] we prove the consistency of the proposed schemes. Consider the scheme (8)–(10). Let $e_i^j = u(x_i, t_j) - U_i^j$, then

$$\begin{aligned} L_1^{N,M}e_i^j &= D_t^+e_i^j + a_i^j D_x^- e_i^j - b_i^j \begin{cases} 0, & x_i - \delta \le 0, \\ e_k^j l_k(x_i-\delta) + e_{k+1}^j l_{k+1}(x_i-\delta), & x_i - \delta > 0, \end{cases} \\ &= L_1^{N,M}u(x_i, t_j) - L_1 u(x_i, t_j) \\ &= O(\Delta x) + O(\Delta t) + O((\Delta x)^2). \end{aligned}$$

Therefore the scheme is consistent. Similarly one can prove that

$$L_2^{N,M} e_i^j = D_t^+ e_i^j + a_i^j D_x^+ e_i^j - b_i^j \begin{cases} 0, & x_i - \delta \le 0, \\ e_k^j l_k(x_i - \delta) + e_{k+1}^j l_{k+1}(x_i - \delta), & x_i - \delta > 0, \end{cases}$$
$$= O(\Delta x) + O(\Delta t) + O((\Delta x)^2).$$

3.5 Matrix Method for Stability of the Schemes

This section presents the stability of the proposed schemes. First we consider the scheme (8)–(10).

Lemma 1 *If* $|a_j^n \lambda| + 2|b_j^n|\Delta t \le 1$ *then the scheme is (8)–(10) stable, where* $\lambda = \frac{\Delta t}{\Delta x}$.

Proof Rewrite the scheme (8)–(10) as

$$U_j^{n+1} = (1 - a_j^n \lambda) U_j^n + a_j^n \lambda U_{j-1}^n + \Delta t b_j^n \begin{cases} \phi_l(x_j - \delta, t_n), & j = 1, 2, \dots, r, \text{ and } x_j - \delta \le 0 \\ [U_k^n l_k(x_j - \delta) + U_{k+1}^n l_{k+1}(x_j - \delta)], & j = r+1, \dots, N. \end{cases}$$

For simplicity let us assume that the coefficient functions a and b are constants. Then,

$$U_j^{n+1} = (1 - a\lambda) U_j^n + a\lambda U_{j-1}^n + b\Delta t [U_k^n l_k(x_j - \delta) + U_{k+1}^n l_{k+1}(x_j - \delta)], \; j = 1, \dots, N$$

Further, the above scheme can be written in the following matrix form

$$U^{n+1} = AU^n + B^n + C^n, \; \forall\, n.$$

Using the above recursion formula, we get

$$U^n = A^n U^0 + \sum_{l=0}^{n} \left(A^{n-l} B^l + A^{n-l} C^l \right)$$

where $A = \begin{pmatrix} V_{r\times r} & \mathbf{0}_{r\times(N-r)} \\ W_{(N-r)\times r} & V_{(N-r)\times(N-r)} \end{pmatrix}$,

$$V_{r\times r} = (v_{l,k}), \; v_{l,k} = \begin{cases} 1 - a\lambda, & \text{if } l = k = i, \; i = 1, \dots, r, \\ a\lambda, & \text{if } l = i, \; k = i-1, \; i = 2, \dots, r, \\ 0, & \text{otherwise} \end{cases}$$

$$W_{(N-r)\times r} = (w_{l,k}), \; w_{l,k} = \begin{cases} b\Delta t l_i(x_{r+i} - \delta), & \text{if } l = r+i, \; k = i, \; i = 1, \dots, N-r, \\ b\Delta t l_{i+1}(x_{r+i} - \delta), & \text{if } l = r+i, \; k = i+1, \; i = 1, \dots, N-r, \\ a\lambda, \; l = k = r+1, & \\ 0, & \text{otherwise} \end{cases}$$

$$V_{(N-r)\times(N-r)} = (v_{l,k}),\ v_{l,k} = \begin{cases} 1 - a\lambda, & \text{if } l = k = i,\ i = r+1, \ldots, N, \\ a\lambda, & \text{if } l = i,\ k = i-1,\ i = r+2, \ldots, N, \\ 0, & \text{otherwise} \end{cases}$$

$$B^n = (a\lambda U_0^n, \underbrace{0, \ldots, 0)^t}_{k \text{ times}},\ C^n = (b\Delta t\phi_l(x_1 - \delta, t_n), \ldots, b\Delta t\phi_l(x_r - \delta, t_n), \underbrace{0, \ldots, 0)^t}_{k \text{ times}}.$$

Here, the superscript on B and C is the index for the time level t_n, whereas the superscript on A indicates the multiplicative power. It is observed that

$$A^n = \begin{pmatrix} V^n_{r\times r} & \mathbf{0}_{r\times(N-r)} \\ W^*_{(N-r)\times r} & V^n_{(N-r)\ \times(N-r)} \end{pmatrix},\ W^*_{(N-r)\times r} = \sum_{j=0}^{n} V^j_{(N-r)\ \times(N-r)} W_{(N-r)\times r} V^{n-j}_{r\times r}.$$

We use the matrix norm $\| A \| = \max\limits_{1\le i\le N} \sum_{j=1}^{N} |a_{i,j}|$.

$$\begin{aligned} \| A \| &= \max_{1\le i\le N} \sum_{l=1}^{N} |a_{i,l}| \\ &= |1 - a\lambda| + |a\lambda| + |b|\Delta t|l_k(x_j - \delta)| + |b|\Delta t|l_{k+1}(x_j - \delta)| \\ &\le |1 - a\lambda| + |a\lambda| + 2|b|\Delta t \end{aligned}$$

and $\| A^n \| \le \| A \|^n \le (|1 - a\lambda| + |a\lambda| + 2|b|\Delta t)^n$. If $|a\lambda| + 2|b|\Delta t \le 1$, then $|a\lambda| \le 1$. The stable region is plotted in Fig. 1.

It is easy to see that $\| A^n \|$ are bounded uniformly for $0 \le n\Delta t \le T$ (See [4]), we get

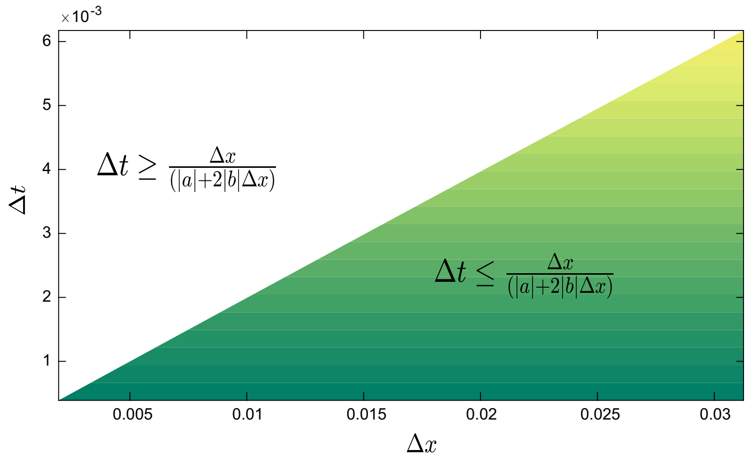


Fig. 1 Stability Region

$$\Delta x \parallel U^n \parallel \leq C_T \left(\Delta x \parallel U^0 \parallel + \Delta t \sum_{j=0}^{n} \parallel U_0^j \parallel + \Delta t \Delta x \parallel \phi_l(x, t_n) \parallel \right) \tag{14}$$

where C_T is a constant.

Similarly, one can prove that the scheme (11)–(13) is also stable.

Remark 1 From the Lax-Richtmyer equivalence theorem [4], the proposed scheme (8)–(10) is convergent.

Remark 2 For variable coefficients problems, one can use the appropriate bounds of $A,\ B,\ C$ to derive the above result.

4 Error Analysis

In this section, we present the convergence results of the proposed schemes. Consider the scheme (8)–(10).

Definition 1 [4] A finite difference scheme $P_{\Delta x,\Delta t}u = R_{\Delta x,\Delta t} f$ is consistent with the differential equation $Pu = f$ is accurate of order p in time and order q in space if for any smooth function u

$$P_{\Delta x,\Delta t}u - R_{\Delta x,\Delta t} Pu = O(\Delta x^q) + O(\Delta t^p).$$

We say that the scheme accurate of order (q, p).

Using the arguments given in [14, 15] one can prove the following theorems.

Theorem 1 *Let $u(x, t)$ be the solution of the problem (1)–(3) and U_i^j be its numerical solution defined by (8)–(10). Then $\mid u(x_i, t_j) - U_i^j \mid \leq C(N^{-1} + M^{-1}),\ \forall i,\ \forall j.$*

Theorem 2 *Let $u(x, t)$ be the solution of the problem (4)–(7) and U_i^j be its numerical solution defined by (11)–(13). Then $\mid u(x_i, t_j) - U_i^j \mid \leq C(N^{-1} + M^{-1}),\ \forall i,\ \forall j.$*

5 Numerical Examples

Two examples are taken in this section to illustrate the numerical methods presented in this paper. We use the half mesh principle to estimate the maximum error. For this we put

$$E^{N,M} = \max_{0\leq i\leq N,\ 0\leq j\leq M} \mid U_i^j(\Delta x, \Delta t) - U_i^j(\Delta x/2, \Delta t/2) \mid,$$

where $U_i^j(\Delta x, \Delta t)$ and $U_i^j(\Delta x/2, \Delta t/2)$ are the numerical solution at the node (x_i, t_j) with mesh sizes $(\Delta x, \Delta t)$ and $(\Delta x/2, \Delta t/2)$, respectively.

Example 1 Consider the following first-order hyperbolic delay differential equation

$$\begin{aligned}
&\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} = b(x,t)u(x-\delta,t),\ (x,t) \in (0,2] \times (0,5],\\
&u(x,t) = \phi_l(x,t),\ (x,t) \in [-\delta,0] \times [0,5],\\
&u(x,0) = u_0(x),\ x \in [0,2].
\end{aligned}$$

Numerical solution of the above problem for different cases are analyzed in the following six cases. In all the cases, we assumed that

$$\begin{aligned}
&a(x,t) = \frac{1+x^2}{1+2tx+2x^2+x^4},\ (x,t) \in [0,2] \times [0,5],\\
&\phi_l(x,t) = 0,\ (x,t) \in [0,2] \times [0,5],\\
&u_0(x) = x(2-x)\exp(-(4x-1)^2),\ x \in [0,2].
\end{aligned}$$

Case 1: In this case $b(x,t) = 0.5$ and the delay argument is $\delta = 1$. Due to the presence of the delay term, an additional wave propagation occurs in the forward direction of x at δ unit distance. It is depicted in the Fig. 2 and for different time levels the solution curves are plotted in Fig. 3.

Case 2: In this case it is assumed that $b(x,t) = 2$ and $\delta = 1$. If the magnitude of the function $b(x,t)$ increases then the amplitude of the additional waves increases. It is depicted in Fig. 4 and the numerical solution at different time level is presented in Fig. 5.

Case 3: Here it is assumed that $b(x,t) = 2$ but $\delta = 0.5$. It is observed that in every δ unit distance in the x-direction the waves are created. Numerical solution and numerical solution at different time levels are presented in the Figs. 6 and 7, respectively.

Case 4: If $b(x,t) = -1$ and $\delta = 1$ then the wave propagation occurs in positive x-direction and in δ unit distance additional waves are created. The numerical solution and its different time levels are plotted in the Figs. 8 and 9, respectively.

Case 5: If $b(x,t) = 0$ (that is, the given differential equation is not a delay differential equation), then there is no additional wave propagation in the x-direction, see Figs. 10 and 11.

Example 2 Consider the following first-order hyperbolic delay differential equation

$$\begin{aligned}
&\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} = b(x,t)u(x-\delta,t),\ (x,t) \in [0,2) \times (0,5],\\
&u(x,t) = \phi_l(x,t),\ (x,t) \in [-\delta,0) \times [0,5],\\
&u(2,t) = \phi_r(2,t),\ t \in [0,5],\\
&u(x,0) = u_0(x),\ x \in [0,2].
\end{aligned}$$

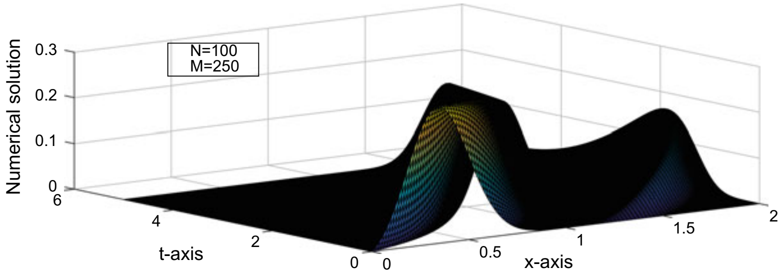


Fig. 2 The surface plot of the numerical solution of Example 1 for the Case 1

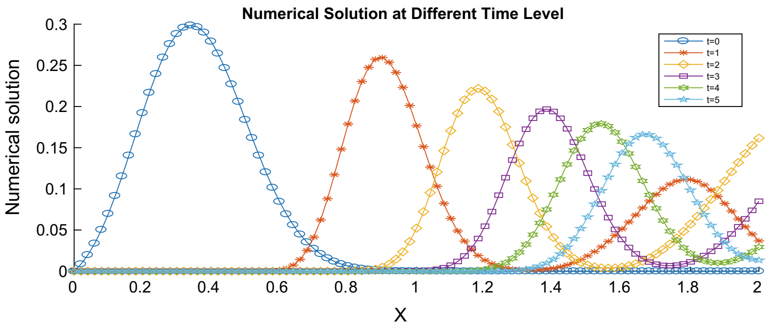


Fig. 3 Numerical solution of Example 1 at different time level for Case 1

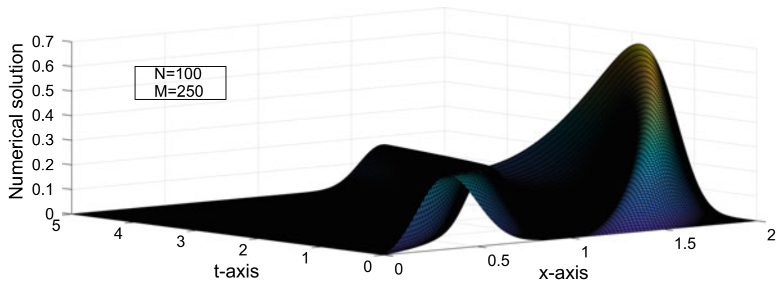


Fig. 4 The surface plot of the numerical solution of Example 1 for the Case 2

Numerical solution of the above problem for two different cases are analyzed. Here, we assumed that

$$a(x,t) = -1,\ (x,t) \in [0,2] \times [0,5],$$
$$\phi_l(x,t) = 0,\ (x,t) \in [0,2] \times [0,5],\ \delta = 1.$$

Case 1 Here it is assumed that $b(x,t) = 0.5$, $u_0(x) = 0$ and $\phi_r(2,t) = \exp(-(10t-1)^2)t$. The numerical solution of the above problem is plotted

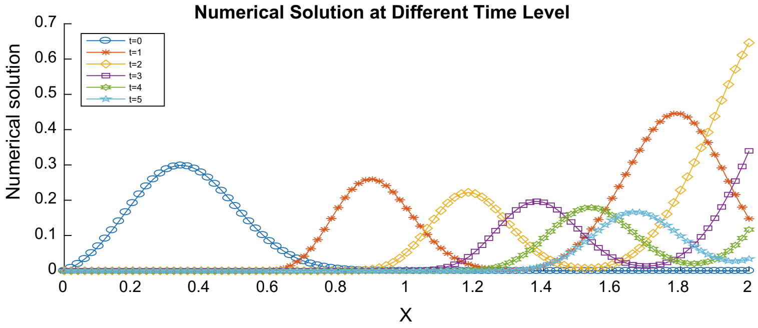


Fig. 5 Numerical solution of Example 1 at different time level for Case 2

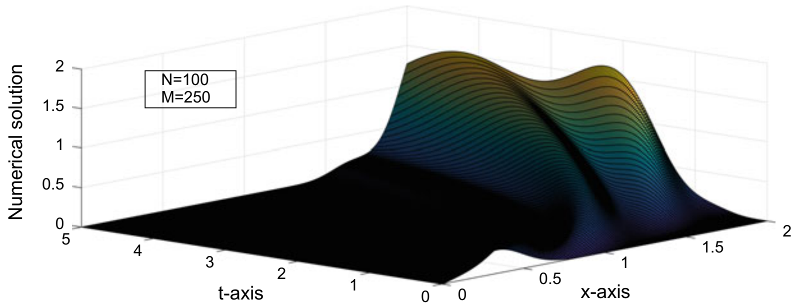


Fig. 6 The surface plot of the numerical solution of Example 1 for the Case 3

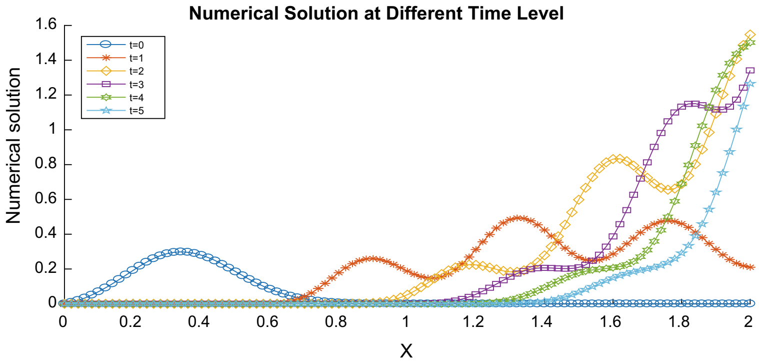


Fig. 7 Numerical solution of Example 1 at different time level for Case 3

in the following Figs. 12 and 13. Here also the additional waves are created and they move in the negative x-direction (Tables 1 and 2).

Case 2 In this case it assumed that $b(x,t) = 1,\ u_0(x) = x(2-x)\exp(-4(4x-1)^2)$ and $\phi_r(2,t) = 0$. The numerical solution of the above problem is plotted

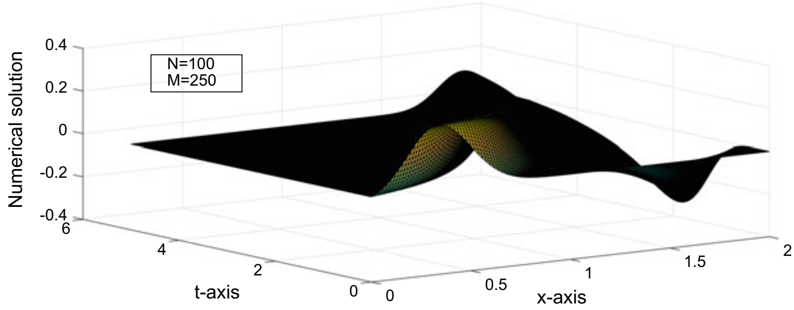


Fig. 8 The surface plot of the numerical solution of Example 1 for the Case 4

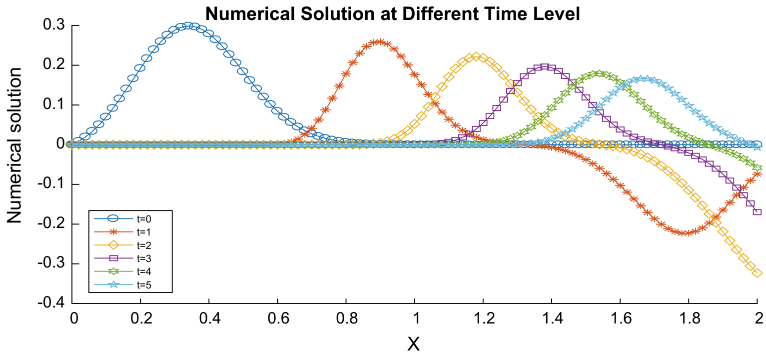


Fig. 9 Numerical solution of Example 1 at different time level for Case 4

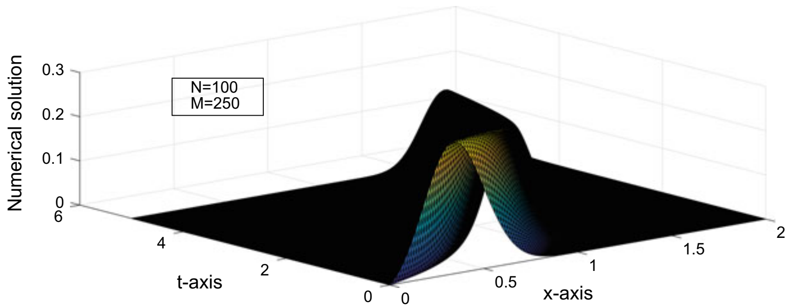


Fig. 10 The surface plot of the numerical solution of Example 1 for the Case 5

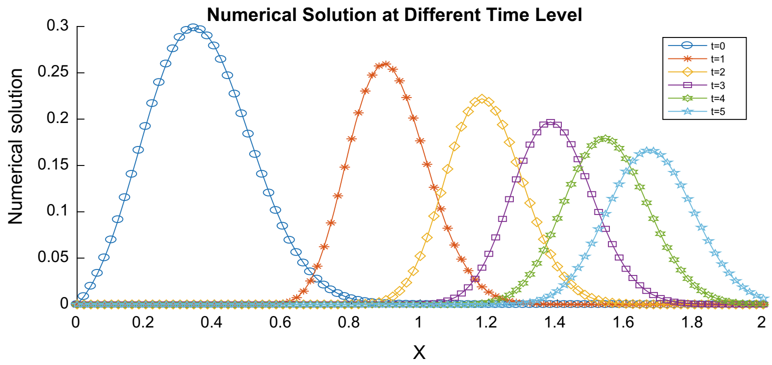


Fig. 11 Numerical solution of Example 1 at different time level for Case 5

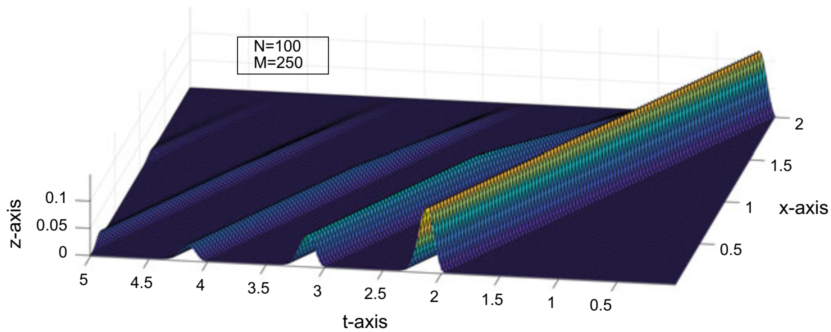


Fig. 12 The surface plot of the numerical solution of Example 2 for the Case 1

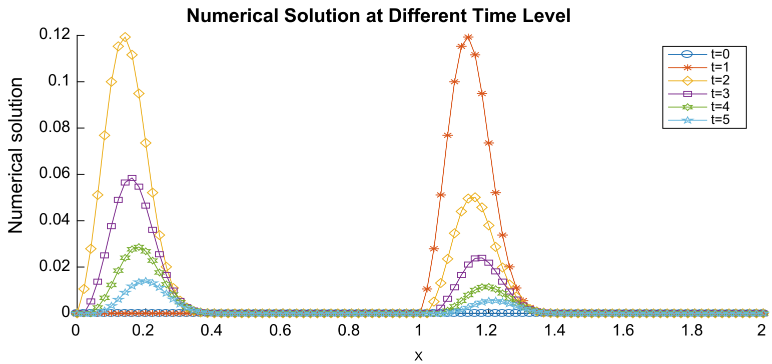


Fig. 13 Numerical solution of Example 2 at different time level for Case 1

Table 1 Case 1: Maximum error for the Example 1

N					
M ↓	64	128	256	512	1024
2048	4.5243e-03	7.2475e-03	1.4026e-02	3.7054e-02	1.3188e+16
4096	2.2289e-03	3.5481e-03	6.2369-03	1.3608e-02	3.6488e-02
8192	1.1063e-03	1.7556e-03	2.9537e-03	6.0507e-03	1.3400e-02
16384	5.5112e-04	8.7325e-04	1.4404e-03	2.8655e-03	5.9582e-03
32768	2.7506e-04	4.3550e-04	7.1127e-04	1.3973e-03	2.8217e-03

Table 2 Case 2: Maximum error for the Example 1

N					
M ↓	64	128	256	512	1024
2048	7.1756e-03	1.1203e-02	1.6021e-02	3.7054e-02	1.3188e+16
4096	3.5525e-03	5.5075e-03	7.7946e-03	1.3608e-02	3.6488e-02
8192	1.7676e-03	2.7310e-03	3.8461e-03	6.0507e-03	1.3400e-02
16384	8.8165e-04	1.3599e-03	1.9106e-03	2.8655e-03	5.9582e-03
32768	4.4029e-04	6.7853e-04	9.5222e-04	1.3973e-03	2.8217e-03

Table 3 Case 2: Maximum error for the Example 2

N					
M ↓	64	128	256	512	1024
2048	7.0881e-03	1.1139e-02	1.5024e-02	2.9687e-02	1.3100e+16
4096	3.4629e-03	5.3935e-03	7.2177e-03	1.0318e-02	2.5154e-02
8192	1.7117e-03	2.6545e-03	3.5390e-03	4.4702e-03	9.6547e-03
16384	8.5099e-04	1.3169e-03	1.7525e-03	2.0955e-03	4.3030e-03
32768	4.2429e-04	6.5588e-04	8.7206e-04	1.0326e-03	2.0383e-03

in the following Figs. 12 and 13. The Table 3 presents the maximum error for this case.

6 Concluding Remarks

In this article, hyperbolic partial delay differential equations are considered. Numerical methods FTBS scheme and FTFS schemes combined with piecewise linear interpolations are suggested, respectively, for the cases $a(x,t) > 0$ and $a(x,t) < 0$. The methods are proved that they are of almost first-order convergence in space and time. It is observed that space delay argument produces some non-trivial lineament

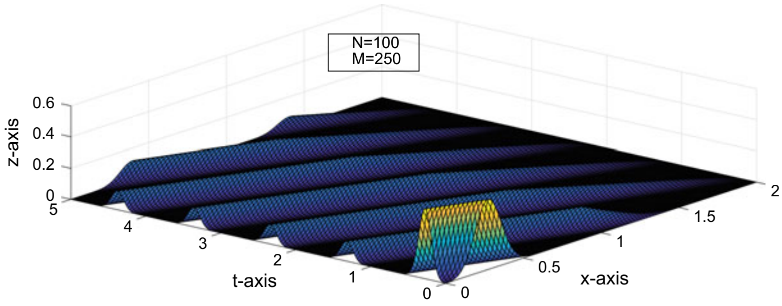


Fig. 14 The surface plot of the numerical solution of Example 2 for the Case 2

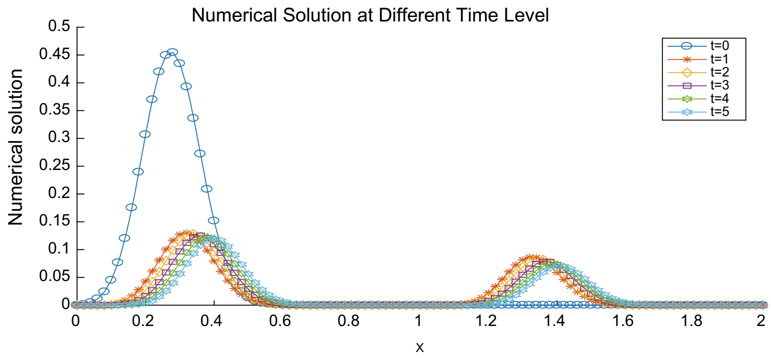


Fig. 15 Numerical solution of Example 2 at different time level for Case 1

in the solution, see Figs. 2 and 3. In the Example 1, we analyzed various cases. Due to presence of the delay argument δ, there are some additional propagated waves that occurred in the solution, see Figs. 2, 4, 6, and 8. Figures 3, 5, 7, 9, and 11 present the numerical solution of the problem for different time levels for the above five cases. If the size of the delay argument δ reduces, then the solution exhibits more number of propagated waves (See Fig. 6). Further if the magnitude of the coefficient function $b(x, t)$ increases, then the amplitude of the propagated waves increases (See Fig. 4). Tables 1 and 2 present the maximum errors of the Example 1 for the case (1) and case (2), respectively and the Table 3 presents the maximum errors of the Example 2 for the case (2). It is observed that when $N = 1024$ and $M = 2048$ the magnitude of the maximum error is very high. If N and M satisfies the relation $|a_j^n \lambda| + 2|b_j^n \Delta t| \leq 1$, then this unacceptable situation will not arise. Further, for fixed N, if the number of mesh points in t direction increases then the maximum error decreases, whereas for fixed M, if N increases then the maximum error increases. This is due to the violation of the relation among N and M. If $a(x, t) > 0$ then the waves move in positive x-direction, whereas if $a(x, t) < 0$ then the waves move in negative x-direction, see the Figs. 12, 13, 14, and 15.

References

1. Kuang, Y.: Delay Differential Equations with Applications in Population Dynamics, vol. 191. Academic Press (1993)
2. Richard, B.S.: A theoretical analysis of neuronal variability. Biophys. J. **5**, 173–194 (1965)
3. Alfredo, B., Zennaro, M.: Numerical Methods for Delay Differential Equations. Oxford University Press (2013)
4. Strikwerda, J.C.: Finite Difference Schemes and Partial Differential Equations. SIAM, Philadelphia (2004)
5. Godlewski, E., Pierre-Arnaud, R.: Hyperbolic Systems of Conservation Laws. Ellipses (1991)
6. Leveque, R.J.: Numerical Methods for Conservation Laws, vol. 3. Birkhäuser, Basel (1992)
7. Smoller, J.: Shock Waves and Reaction-Diffusion Equations, vol. 258. Springer Science and Business Media (2012)
8. Thomas, J.W.: Numerical Partial Differential Equations: Conservation Laws and Elliptic Equations, vol. 33. Springer Science and Business Media (2013)
9. Morton, K.W.: Arthur Kevin Parrott: Generalised Galerkin methods for first-order hyperbolic equations. J. Comput. Phys. **36**(2), 249–270 (1980)
10. Kreiss, H.-O., Scherer, G.: Finite element and finite difference methods for hyperbolic partial differential equations. Mathematical Aspects of Finite Elements in Partial Differential Equations, pp. 195–212. Academic Press (1974)
11. Nordström, J., Gong, J.: A stable hybrid method for hyperbolic problems. J. Comput. Phys. **212**(2), 436–453 (2006)
12. Gustafsson, B., Kreiss, H.-O., Sundström, A.: Stability theory of difference approximations for mixed initial boundary value problems II. Math. Comput. **26**(119), 649–686 (1972)
13. Jan, N.: Error bounded schemes for time-dependent hyperbolic problems. SIAM J. Sci. Comput. **30**(1), 46–59 (2008)
14. Kapil, K., PS, S.: Hyperbolic partial differential-difference equation in the mathematical modelling of neuronal firing and its numerical solution. Appl. Math. Comput. **201**, 229–238 (2008)
15. Singh, P., Sharma, K.K.: Numerical solution of first-order hyperbolic partial differential–difference equation with shift. Numer. Methods Partial Differ. Equ.: Int. J. **26**(1), 107–116 (2010)
16. Chauhan, V., Singh, N.K.: Finite difference approximation for first-order hyperbolic partial differential equation arising in neuronal variability with shifts. Int. J. Sci. Eng. Res. **3** (2015)
17. Mizohata, S., Venkatesha Murthy, M.K., Singbal, B.V.: Lectures on Cauchy Problem, vol. 35. Tata Institute of Fundamental Research, Bombay (1965)
18. Protter, M.H., Hans, F.W.: Maximum Principles in Differential Equations. Springer Science and Business Media (2012)
19. Bainov, D.D., Minchev, E: Comparison principles for impulsive hyperbolic equations of first order. J. Comput. Appl. Math. **60**(3), 379–388 (1995)
20. Liu, X.-D.: A maximum principle satisfying modification of triangle based adaptive stencils for the solution of scalar hyperbolic conservation laws. SIAM J. Numer. Anal. **30**(3), 701–716 (1993)

Fitted Mesh Methods for a Class of Weakly Coupled System of Singularly Perturbed Convection–Diffusion Equations

Saravanasankar Kalaiselvan, John J. H. Miller, and Valarmathi Sigamani

Abstract In this paper, a class of singularly perturbed coupled linear systems of second-order ordinary differential equations of convection–diffusion type is considered on the interval [0, 1]. Due to the presence of different perturbation parameters multiplying the diffusion terms of the coupled system, each of the solution components exhibits multiple layers in the neighbourhood of the origin. This fact is proved in the estimates of the derivatives of the solution. A numerical method composed of an upwind finite difference scheme applied on a piecewise uniform Shishkin mesh that resolves all the layers is suggested to solve the problem. The method is proved to be almost first-order convergent in the maximum norm uniformly in all the perturbation parameters. Numerical examples are provided to support the theory.

Keywords Singular perturbation problems · System of convection-diffusion equations · Finite difference method · Shishkin Mesh · Parameter uniform method

1 Introduction

Singularly perturbed differential equations of convection–diffusion type appear in several branches of applied mathematics. Roos et al. [1] describes linear convection–diffusion equations and related non-linear flow problems. Modelling real-life problems such as fluid flow problems, control problems, heat transport problems, river networks results in singularly perturbed convection–diffusion equations. Some of those models were discussed in [2]. A form of linearized Navier Stokes equations called Oseen system of equations, which models many of the physical problems, is a system of singularly perturbed convection–diffusion equations. Also systems

S. Kalaiselvan (✉) · V. Sigamani
Department of Mathematics, Bishop Heber College, Tiruchirappalli 620017, TN, India

J. J. H. Miller
Trinity College, Dublin, Ireland
e-mail: jmiller@tcd.ie

Valarmathi S et al. (eds.), *Differential Equations and Applications*,
Springer Proceedings in Mathematics & Statistics 368,
https://doi.org/10.1007/978-981-16-7546-1_9

of singularly perturbed convection–diffusion equations have applications in control problems [3].

For a broad introduction to singularly perturbed boundary value problems of convection–diffusion type and robust computational techniques to solve them, one can refer to [4–6]. In [7], a coupled system of two singularly perturbed convection–diffusion equations is analysed and a parameter uniform numerical method is suggested to solve the same. Here, in this paper, the following weakly coupled system of n-singularly perturbed convection–diffusion equations is considered.

$$L\mathbf{u}(x) \equiv E\mathbf{u}''(x) + A(x)\mathbf{u}'(x) - B(x)\mathbf{u}(x) = \mathbf{f}(x), \; x \in \Omega = (0, 1) \tag{1}$$

$$\mathbf{u}(0) = \mathbf{l}, \quad \mathbf{u}(1) = \mathbf{r}, \tag{2}$$

where $\mathbf{u}(x) = \big(u_1(x), u_2(x), \ldots, u_n(x)\big)^T$, $\mathbf{f}(x) = \big(f_1(x), f_2(x), \ldots, f_n(x)\big)^T$,

$$E = \begin{bmatrix} \varepsilon_1 & 0 & \ldots & 0 \\ 0 & \varepsilon_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \varepsilon_n \end{bmatrix}, A = \begin{bmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & a_n \end{bmatrix}, \; B = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{bmatrix}.$$

Here, $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ are distinct small positive parameters and for convenience, it is assumed that $\varepsilon_i < \varepsilon_j$, for $i < j$. The functions a_i, b_{ij} and f_i, for all i and j, are taken to be sufficiently smooth on $\overline{\Omega}$. It is further assumed that, $a_i(x) \geq \alpha > 0$, $b_{ij}(x) < 0$, $i \neq j$ and $\sum_{j=1}^{n} b_{ij}(x) \geq \beta > 0$, for all $i = 1, 2, \ldots, n$. The case $a_i(x) < 0$ can be treated in a similar way with a transformation of x to $1 - x$.

In [9], Linss has analysed a broader class of weakly coupled system of singularly perturbed convection–diffusion equations and presented an estimate of the derivatives of u_i depending only on ε_i, for $i = 1, 2, \ldots, n$. He has claimed first order and almost first-order convergence if solved on Bakhvalov and Shishkin meshes, respectively, with the classical finite difference scheme.

The reduced problem corresponding to (1)–(2) is

$$\begin{aligned} L_0\mathbf{u}_0(x) &\equiv A(x)\mathbf{u}_0'(x) - B(x)\mathbf{u}_0(x) = \mathbf{f}(x), \; x \in \Omega \\ \mathbf{u}_0(1) &= \mathbf{r}, \end{aligned} \tag{3}$$

where $\mathbf{u}_0(x) = (u_{01}(x), u_{02}(x), ..., u_{0n}(x))^T$.

If $u_k(0) \neq u_{0k}(0)$ for any k such that $0 \leq k \leq n$, then a boundary layer of width $O(\varepsilon_k)$ is expected near $x = 0$ in each of the solution component u_i, $1 \leq i \leq k$.

Notations. For any real valued function y on D, the norm of y is defined as $\|y\|_D = \sup_{x \in D}|y(x)|$. For any vector valued function $\mathbf{z}(x) = (z_1(x), z_2(x), \ldots,$

$z_n(x))^T$, $\|\mathbf{z}\|_D = \max\left\{\|z_1\|_D, \|z_2\|_D, \ldots, \|z_n\|_D\right\}$. For any mesh function Y on a mesh $D^N = \left\{x_j\right\}_{j=0}^{N}$, $\|Y\|_{D^N} = \max_{0\le j\le N}|Y(x_j)|$ and for any vector valued mesh function $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)^T$, $\|\mathbf{Z}\|_{D^N} = \max\left\{\|Z_1\|_{D^N}, \|Z_2\|_{D^N}, \ldots, \|Z_n\|_{D^N}\right\}$.

Throughout this paper, C denotes a generic positive constant which is independent of the singular perturbation and discretization parameters.

2 Analytical Results

In this section, a maximum principle, a stability result and estimates of the derivatives of the solution of the system of Eqs. (1)–(2) are presented.

Lemma 1 (Maximum Principle) *Let $\boldsymbol{\psi} = (\psi_1, \psi_2, ..., \psi_n)^T$ be in the domain of L with $\boldsymbol{\psi}(0) \ge \mathbf{0}$ and $\boldsymbol{\psi}(1) \ge \mathbf{0}$. Then $L\boldsymbol{\psi} \le \mathbf{0}$ on Ω implies that $\boldsymbol{\psi} \ge \mathbf{0}$ on $\overline{\Omega}$.*

Lemma 2 (Stability Result) *Let $\boldsymbol{\psi}$ be in the domain of L, then for $x \in \overline{\Omega}$ and $1 \le i \le n$*

$$|\psi_i(x)| \le \max\left\{\|\boldsymbol{\psi}(0)\|,\ \|\boldsymbol{\psi}(1)\|,\ \frac{1}{\beta}\|L\boldsymbol{\psi}\|\right\}.$$

Theorem 1 *Let $\mathbf{u}$ be the solution of (1)–(2), then for $x \in \overline{\Omega}$ and $1 \le i \le n$, the following estimates hold.*

$$|u_i(x)| \le C\max\left\{\|\mathbf{l}\|,\ \|\mathbf{r}\|,\ \frac{1}{\beta}\|\mathbf{f}\|\right\}, \tag{4}$$

$$|u_i^{(k)}(x)| \le C\varepsilon_i^{-k}\Big(\|\mathbf{u}\| + \varepsilon_i\|\mathbf{f}\|\Big)\ \ for\ \ k = 1, 2, \tag{5}$$

$$|u_i^{(3)}(x)| \le C\varepsilon_i^{-2}\varepsilon_1^{-1}\Big(\|\mathbf{u}\| + \varepsilon_i\|\mathbf{f}\|\Big) + \varepsilon_i^{-1}|f_i'(x)|. \tag{6}$$

Proof The estimate (4) follows immediately from Lemma 2 and Eq. (1). Let $x \in [0, 1]$, then for each $i,\ 1 \le i \le n$, there exists $a \in [0, 1 - \varepsilon_i]$ such that $x \in N_a = [a, a + \varepsilon_i]$. By the mean value theorem, there exists $y_i \in (a, a + \varepsilon_i)$ such that

$$u_i'(y_i) = \frac{u_i(a+\varepsilon_i) - u_i(a)}{\varepsilon_i}$$

and hence

$$|u_i'(y_i)| \le C\varepsilon_i^{-1}\|\mathbf{u}\|.$$

Also,

$$u_i'(x) = u_i'(y_i) + \int_{y_i}^{x} u_i''(s)ds.$$

Substituting for $u_i''(s)$ from (1), $|u_i'(x)| \le C\varepsilon_i^{-1}\Big(\|\mathbf{u}\| + \varepsilon_i\|\mathbf{f}\|\Big)$. Again from (1), $|u_i''(x)| \le C\varepsilon_i^{-2}\Big(\|\mathbf{u}\| + \varepsilon_i\|\mathbf{f}\|\Big)$. Differentiating (1) once and substituting the above bounds lead to

$$|u_i^{(3)}(x)| \le C\varepsilon_i^{-2}\varepsilon_1^{-1}\Big(\|\mathbf{u}\| + \varepsilon_i\|\mathbf{f}\|\Big) + \varepsilon_i^{-1}|f_i'(x)|.$$

2.1 Shishkin Decomposition of the Solution

The solution $\mathbf{u}$ of the problem (1)–(2) can be decomposed into smooth $\mathbf{v} = (v_1, ..., v_n)^T$ and singular $\mathbf{w} = (w_1, ..., w_n)^T$ components given by $\mathbf{u} = \mathbf{v} + \mathbf{w}$, where

$$L\mathbf{v} = \mathbf{f}, \quad \mathbf{v}(0) = \boldsymbol{\gamma}, \quad \mathbf{v}(1) = \mathbf{r}, \tag{7}$$

$$L\mathbf{w} = \mathbf{0}, \quad \mathbf{w}(0) = \mathbf{l} - \mathbf{v}(0), \quad \mathbf{w}(1) = \mathbf{0}, \tag{8}$$

where $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)^T$ is to be chosen.

2.1.1 Estimates for the Bounds on the Smooth Components and Their Derivatives

Theorem 2 *For a proper choice of* $\boldsymbol{\gamma}$, *the solution of the problem (7) satisfies for* $1 \le i \le n$ *and* $0 \le k \le 3$,

$$|v_i^{(k)}(x)| \le C(1 + \varepsilon_i^{2-k}), \quad x \in \overline{\Omega}.$$

Proof Considering the layer pattern of the solution, first, the decomposition is done with ε_n, for all the components of $\mathbf{v}$. The second level decomposition with ε_{n-1} is for the first $n-1$ components of $\mathbf{v}$. Then, the decomposition continues with ε_{n-2} for the first $n-2$ components of $\mathbf{v}$ and so on. It is carried out in the following way. First, the smooth component $\mathbf{v}$ is decomposed into

$$\mathbf{v} = \mathbf{y}_n + \varepsilon_n\mathbf{z}_n + \varepsilon_n^2\mathbf{q}_n \tag{9}$$

where $\mathbf{y}_n = (y_{n1}, y_{n2}, \ldots, y_{nn})^T$ is the solution of

$$A(x)\mathbf{y}_n'(x) - B(x)\mathbf{y}_n(x) = \mathbf{f}(x), \quad \mathbf{y}_n(1) = \mathbf{r}, \tag{10}$$

$\mathbf{z}_n = (z_{n1}, z_{n2}, \ldots, z_{nn})^T$ is the solution of

$$A(x)\mathbf{z}_n'(x) - B(x)\mathbf{z}_n(x) = -\varepsilon_n^{-1}E\mathbf{y}_n''(x), \ \ \mathbf{z}_n(1) = \mathbf{0} \tag{11}$$

and $\mathbf{q}_n = (q_{n1}, q_{n2}, \ldots, q_{nn})^T$ is the solution of

$$L\mathbf{q}_n(x) = -\varepsilon_n^{-1} E\mathbf{z}_n''(x),\ \ \mathbf{q}_n(1) = \mathbf{0} \text{ and } \mathbf{q}_n(0) \text{ remains to be chosen.} \tag{12}$$

Using the fact that $\varepsilon_n^{-1}E$ is a matrix of bounded entries, and from the results in [10] for (10) and (11), it is not hard to see that

$$\|\mathbf{y}_n^{(k)}\| \leq C \ \text{ and } \ \|\mathbf{z}_n^{(k)}\| \leq C,\ \ 0 \leq k \leq 3. \tag{13}$$

Now, using Theorem 1 and (13), with the choice that $q_{nn}(0) = 0$,

$$|q_{nn}^{(k)}(x)| \leq C\varepsilon_n^{-k},\ \ 0 \leq k \leq 3. \tag{14}$$

Then from (9), it is clear that $v_n(0) = \gamma_n = y_{nn}(0) + \varepsilon_n z_{nn}(0)$. Also from (13) and (14),

$$|v_n^{(k)}(x)| \leq C(1 + \varepsilon_n^{2-k}),\ \ 0 \leq k \leq 3. \tag{15}$$

Now, having found the estimates of $v_n^{(k)}$, to estimate the bounds $v_i^{(k)}$, for $1 \leq i \leq n-1$, the following notations are introduced, for $1 \leq l \leq n$,

$$E_l = \begin{bmatrix} \varepsilon_1 & 0 & \ldots & 0 \\ 0 & \varepsilon_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \varepsilon_l \end{bmatrix}, A_l = \begin{bmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & a_l \end{bmatrix},\ B_l = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1l} \\ b_{21} & b_{22} & \ldots & b_{2l} \\ \vdots & \vdots & & \vdots \\ b_{l1} & b_{l2} & \ldots & b_{ll} \end{bmatrix},$$

$\tilde{\mathbf{q}}_l = \left(q_{l1}, q_{l2}, \ldots, q_{l(l-1)}\right)^T, \mathbf{g}_{(l-1)} = \left(g_{(l-1)1}, g_{(l-1)2}, \ldots, g_{(l-1)(l-1)}\right)^T$, with $g_{(l-1)j} = -\dfrac{\varepsilon_j}{\varepsilon_l} z_{lj}'' + b_{jl}q_{ll}$.

Now, considering the first $(n-1)$ equations of the system (12), it follows that

$$\tilde{L}_n\tilde{\mathbf{q}}_n \equiv E_{n-1}\tilde{\mathbf{q}}_n''(x) + A_{n-1}(x)\tilde{\mathbf{q}}_n'(x) - B_{n-1}(x)\tilde{\mathbf{q}}_n(x) = \mathbf{g}_{n-1}(x), \tag{16}$$

where $\tilde{\mathbf{q}}_n(1) = \mathbf{0}$ and $\tilde{\mathbf{q}}_n(0)$ remains to be chosen.

Furthermore, decomposing $\tilde{\mathbf{q}}_n$ in a similar way to (9), we obtain

$$\tilde{\mathbf{q}}_n = \mathbf{y}_{n-1} + \varepsilon_{n-1}\mathbf{z}_{n-1} + \varepsilon_{n-1}^2\mathbf{q}_{n-1} \tag{17}$$

where $\mathbf{y}_{n-1} = \left(y_{(n-1)1}, y_{(n-1)2}, \ldots, y_{(n-1)(n-1)}\right)^T$ is the solution of the problem

$$A_{n-1}(x)\mathbf{y}_{n-1}'(x) - B_{n-1}(x)\mathbf{y}_{n-1}(x) = \mathbf{g}_{n-1}(x),\ \ \mathbf{y}_{n-1}(1) = \mathbf{0}, \tag{18}$$

$\mathbf{z}_{n-1} = \left(z_{(n-1)1}, z_{(n-1)2}, \ldots, z_{(n-1)(n-1)}\right)^T$ is the solution of the problem

$$A_{n-1}(x)\mathbf{z}'_{n-1}(x) - B_{n-1}(x)\mathbf{z}_{n-1}(x) = -\varepsilon_{n-1}^{-1}E_{n-1}\mathbf{y}''_{n-1}(x),\ \ \mathbf{z}_{n-1}(1) = \mathbf{0} \quad (19)$$

and $\mathbf{q}_{n-1} = \left(q_{(n-1)1}, q_{(n-1)2}, \ldots, q_{(n-1)(n-1)}\right)^T$ is the solution of the problem

$$\tilde{L}_n\mathbf{q}_{n-1}(x) = -\varepsilon_{n-1}^{-1}E_{n-1}\mathbf{z}''_{n-1}(x),\ \ \mathbf{q}_{n-1}(1) = \mathbf{0} \text{ and } \mathbf{q}_{n-1}(0) \text{ remains to be chosen.} \quad (20)$$

Now choose $\mathbf{q}_{n-1}(0)$ so that its $(n-1)$th component is zero (i.e. $q_{(n-1)(n-1)}(0) = 0$). Problem (18) is similar to the problem (11). Using the estimates (13)–(14), the solution of the problem (18) satisfies the following bound for $0 \le k \le 3$.

$$\|\mathbf{y}_{n-1}^{(k)}\| \le C\left(1 + \varepsilon_n^{1-k}\right). \quad (21)$$

Using (21) and Lemma 2.2 in [10], the solution of the problem (19) satisfies

$$\|\mathbf{z}_{n-1}\| \le C\varepsilon_n^{-1}. \quad (22)$$

and from (19), for $1 \le k \le 3$,

$$\|\mathbf{z}_{n-1}^{(k)}\| \le C\varepsilon_n^{-k}. \quad (23)$$

Now, using Theorem 1 and (23), the following estimate holds:

$$|q_{(n-1)(n-1)}^{(k)}(x)| \le C\varepsilon_n^{-2}\varepsilon_{n-1}^{-k}, \quad 0 \le k \le 3. \quad (24)$$

By the choice of $q_{(n-1)(n-1)}(0)$, from (9) and (17), it is clear that $v_{n-1}(0) = \gamma_{n-1} = y_{n(n-1)}(0) + \varepsilon_n z_{n(n-1)}(0) + \varepsilon_n^2 y_{(n-1)(n-1)}(0) + \varepsilon_n^2\varepsilon_{n-1}z_{(n-1)(n-1)}(0)$. Also, the estimates (21)–(24) imply that

$$|v_{n-1}^{(k)}(x)| \le C(1 + \varepsilon_{n-1}^{2-k}). \quad (25)$$

Proceeding in a similar way, one can derive singularly perturbed systems of l equations, $l = n-2,\ n-3, \ldots,\ 2,\ 1$,

$$\tilde{L}_{l+1}\tilde{\mathbf{q}}_{l+1} \equiv E_l\tilde{\mathbf{q}}''_{l+1}(x) + A_l(x)\tilde{\mathbf{q}}'_{l+1}(x) - B_l(x)\tilde{\mathbf{q}}_{l+1}(x) = \mathbf{g}_l(x), \quad (26)$$

with $\tilde{\mathbf{q}}_{l+1}(1) = \mathbf{0}$ and $\tilde{\mathbf{q}}_{l+1}(0)$, to be chosen.
Now, decomposing $\tilde{\mathbf{q}}_{l+1}$ in a similar way to (9), we obtain

$$\tilde{\mathbf{q}}_{l+1} = \mathbf{y}_l + \varepsilon_l\mathbf{z}_l + \varepsilon_l^2\mathbf{q}_l \quad (27)$$

where $\mathbf{y}_l = (y_{l1}, y_{l2}, \ldots, y_{ll})^T$ and $\mathbf{z}_l = (z_{l1}, z_{l2}, \ldots, z_{ll})^T$ satisfy

$$A_l(x)\mathbf{y}_l{}'(x) - B_l(x)\mathbf{y}_l(x) = \mathbf{g}_l(x), \quad \mathbf{y}_l(1) = \mathbf{0}, \quad (28)$$

$$A_l(x)\mathbf{z}_l{}'(x) - B_l(x)\mathbf{z}_l(x) = -\varepsilon_l^{-1}E_l\mathbf{y}_l{}''(x),\ \ \mathbf{z}_l(1) = \mathbf{0} \quad (29)$$

respectively and $\mathbf{q}_l = (q_{l1}, q_{l2}, \ldots, q_{ll})^T$ is the solution of the problem

$$\tilde{L}_{l+1}\mathbf{q}_l(x) = -\varepsilon_l^{-1} E_l \mathbf{z}_l''(x),\ \ \mathbf{q}_l(1) = \mathbf{0} \text{ where } \mathbf{q}_l(0) \text{ remains to be chosen.} \tag{30}$$

We choose $\mathbf{q}_l(0)$ so that its lth component is zero (i.e. $q_{ll}(0) = 0$).
From (28) it is clear that, for $0 \le k \le 3$,

$$\|\mathbf{y}_l^{(k)}\| \le C\left(1 + \varepsilon_{l+1}^{1-k}\right) \prod_{i=l+2}^{n} \varepsilon_i^{-2}. \tag{31}$$

Using (31) in (29), $\|\mathbf{z}_l\| \le C\left(1 + \varepsilon_{l+1}^{-1}\right) \prod_{i=l+2}^{n} \varepsilon_i^{-2}$ and for $1 \le k \le 3$,

$$\|\mathbf{z}_l^{(k)}\| \le C\left(1 + \varepsilon_{l+1}^{-k}\right) \prod_{i=l+2}^{n} \varepsilon_i^{-2}. \tag{32}$$

Now, using Theorem 1 for $\mathbf{q}_l$, we obtain

$$|q_{ll}^{(k)}(x)| \le C\varepsilon_l^{-k} \prod_{i=l+1}^{n} \varepsilon_i^{-2},\quad 0 \le k \le 3. \tag{33}$$

Since $q_{ll}(0) = 0$, it is clear that

$$v_l(0) = \gamma_l = y_{nl}(0) + \varepsilon_n z_{nl}(0) + \varepsilon_n^2 y_{(n-1)l}(0) + \ldots + \left(\prod_{j=l+1}^{n} \varepsilon_j^2\right) \varepsilon_l z_{ll}(0).$$

Also, the estimates (31)–(33) imply that

$$|v_l^{(k)}(x)| \le C(1 + \varepsilon_l^{2-k}),\quad 0 \le k \le 3. \tag{34}$$

Thus, by the choice made for $\gamma_n,\ \gamma_{n-1}, \ldots, \gamma_2,\ \gamma_1$, the solution $\mathbf{v}$ of the problem (7) satisfies the following bound for $1 \le i \le n$ and $0 \le k \le 3$

$$|v_i^{(k)}(x)| \le C(1 + \varepsilon_i^{2-k}),\quad x \in \overline{\Omega}. \tag{35}$$

2.1.2 Estimates for the Bounds on the Singular Components and Their Derivatives

Let $\mathscr{B}_i(x),\ 1 \le i \le n$, be the layer functions defined on [0, 1] as

$$\mathscr{B}_i(x) = \exp(-\alpha x/\varepsilon_i). \tag{36}$$

Theorem 3 *Let* $\mathbf{w}(x)$ *be the solution of (8), then for* $x \in \overline{\Omega}$ *and* $1 \leq i \leq n$ *the following estimates hold.*

$$|w_i(x)| \leq C\mathscr{B}_n(x), \tag{37}$$
$$|w_i'(x)| \leq C\Big(\varepsilon_i^{-1}\mathscr{B}_i(x) + \varepsilon_n^{-1}\mathscr{B}_n(x)\Big), \tag{38}$$
$$|w_i^{(2)}(x)| \leq C\sum_{k=i}^{n}\varepsilon_k^{-2}\mathscr{B}_k(x), \tag{39}$$
$$|w_i^{(3)}(x)| \leq C\varepsilon_i^{-1}\Big(\sum_{k=1}^{i-1}\varepsilon_k^{-1}\mathscr{B}_k(x) + \sum_{k=i}^{n}\varepsilon_k^{-2}\mathscr{B}_k(x)\Big). \tag{40}$$

Proof Consider the barrier function $\boldsymbol{\phi} = (\phi_1, \phi_2, \dots, \phi_n)^T$ defined by $\phi_i(x) = C\mathscr{B}_n(x)$, $1 \leq i \leq n$. Put $\boldsymbol{\psi}^{\pm}(x) = \boldsymbol{\phi}(x) \pm \mathbf{w}(x)$, then for sufficiently large C, $\boldsymbol{\psi}^{\pm}(0) \geq \mathbf{0}$, $\boldsymbol{\psi}^{\pm}(1) \geq \mathbf{0}$ and $L\boldsymbol{\psi}^{\pm}(x) \leq \mathbf{0}$. Using Lemma 1, it follows that, $\boldsymbol{\psi}^{\pm}(x) \geq \mathbf{0}$. Hence, estimate (37) holds. From (8), for $1 \leq i \leq n$

$$\varepsilon_i(w_i')'(x) + a_i(x)(w_i')(x) = g_i(x) \tag{41}$$

where $g_i(x) = \sum_{j=1}^{n} b_{ij}(x)w_j(x)$. Let $\mathscr{A}_i(x) = \int_0^x a_i(s)ds$, then solving (41) leads to

$$w_i'(x) = w_i'(0)\exp\big(-\mathscr{A}_i(x)/\varepsilon_i\big) + \varepsilon_i^{-1}\int_0^x g_i(t)\exp\big(-(\mathscr{A}_i(x) - \mathscr{A}_i(t))/\varepsilon_i\big)dt.$$

Using Theorem 1 for $\mathbf{w}$, $|w_i'(0)| \leq C\varepsilon_i^{-1}$. Further from the inequalities, $\exp\big(-(\mathscr{A}_i(x) - \mathscr{A}_i(t))/\varepsilon_i\big) \leq \exp\big(-\alpha(x-t)/\varepsilon_i\big)$ for $t \leq x$ and $|g_i(t)| \leq C\mathscr{B}_n(t)$, it is clear that

$$|w_i'(x)| \leq C\varepsilon_i^{-1}\exp\big(-\alpha x/\varepsilon_i\big) + C\varepsilon_i^{-1}\int_0^x \exp\big(-\alpha t/\varepsilon_n\big)\exp\big(-\alpha(x-t)/\varepsilon_i\big)dt.$$

Using integration by parts, it is not hard to see that

$$|w_i'(x)| \leq C\varepsilon_i^{-1}\exp\big(-\alpha x/\varepsilon_i\big) + C\varepsilon_n^{-1}\exp\big(-\alpha x/\varepsilon_n\big). \tag{42}$$

Differentiating (41) once leads to

$$\varepsilon_i(w_i'')'(x) + a_i(x)(w_i'')(x) = h_i(x) \equiv g_i'(x) - a_i'(x)w_i'(x). \tag{43}$$

Then,

$$w_i''(x) = w_i''(0)\exp\big(-\mathscr{A}_i(x)/\varepsilon_i\big) + \varepsilon_i^{-1}\int_0^x h_i(t)\exp\big(-(\mathscr{A}_i(x)-\mathscr{A}_i(t))/\varepsilon_i\big)dt.$$

Using $|w_i''(0)| \leq C\varepsilon_i^{-2}$, $|h_i(t)| \leq C\sum_{k=1}^{n}\varepsilon_k^{-1}\mathscr{B}_k(t)$ and hence

$$|w_i''(x)| \leq C\sum_{k=i}^{n}\varepsilon_k^{-2}\mathscr{B}_k(x). \tag{44}$$

Using the bounds given in (42) and (44) in (43), (40) can be derived.

As the estimates of the derivatives are to be used in the different segments of the piecewise uniform Shishkin meshes, the estimates are improved using the layer interaction points as given below.

2.1.3 Improved Estimates for the Bounds on the Singular Components and Their Derivatives

For $\mathscr{B}_i, \mathscr{B}_j$, each $i, j,\ 1 \leq i < j \leq n$ and each $s = 1, 2$ the point $x_{i,j}^{(s)}$ is defined by

$$\frac{\mathscr{B}_i(x_{i,j}^{(s)})}{\varepsilon_i^s} = \frac{\mathscr{B}_j(x_{i,j}^{(s)})}{\varepsilon_j^s}. \tag{45}$$

Lemma 3 *For all i, j such that $1 \leq i < j \leq n$ and $s = 1, 2$ the points $x_{i,j}^{(s)}$ exist, are uniquely defined and satisfy the following inequalities*

$$\frac{\mathscr{B}_i(x)}{\varepsilon_i^s} > \frac{\mathscr{B}_j(x)}{\varepsilon_j^s},\quad x \in [0, x_{i,j}^{(s)}),\quad \frac{\mathscr{B}_i(x)}{\varepsilon_i^s} < \frac{\mathscr{B}_j(x)}{\varepsilon_j^s},\ x \in (x_{i,j}^{(s)}, 1]. \tag{46}$$

In addition, the following ordering holds

$$x_{i,j}^{(s)} < x_{i+1,j}^{(s)},\ \textit{if}\ i+1 < j\ \textit{ and }\ x_{i,j}^{(s)} < x_{i,j+1}^{(s)},\ \textit{if}\ i < j. \tag{47}$$

Proof Proof is similar to the Lemma 2.3.1 of [8].

Consider the following decomposition of $w_i(x)$

$$w_i = \sum_{q=1}^{n} w_{i,q}, \tag{48}$$

where the components $w_{i,q}$ are defined as follows.

$$w_{i,n} = \begin{cases} \displaystyle\sum_{k=0}^{3} \frac{(x - x^{(2)}_{n-1,n})^k}{k!} w_i^{(k)}(x^{(2)}_{n-1,n}) \text{ on } [0, x^{(2)}_{n-1,n}) \\ w_i \qquad\qquad \text{otherwise} \end{cases} \tag{49}$$

and, for each $q, \quad n-1 \ge q \ge i,$

$$w_{i,q} = \begin{cases} \displaystyle\sum_{k=0}^{3} \frac{(x - x^{(2)}_{q-1,q})^k}{k!} \, p_{i,q}^{(k)}(x^{(2)}_{q-1,q}) \text{ on } [0, x^{(2)}_{q-1,q}) \\ p_{i,q} \qquad\qquad \text{otherwise} \end{cases} \tag{50}$$

and, for each $q, \quad i-1 \ge q \ge 2,$

$$w_{i,q} = \begin{cases} \displaystyle\sum_{k=0}^{3} \frac{(x - x^{(1)}_{q-1,q})^k}{k!} \, p_{i,q}^{(k)}(x^{(1)}_{q-1,q}) \text{ on } [0, x^{(1)}_{q-1,q}) \\ p_{i,q} \qquad\qquad \text{otherwise} \end{cases} \tag{51}$$

with $p_{i,q} = w_i - \displaystyle\sum_{k=q+1}^{n} w_{i,k}$

and

$$w_{i,1} = w_i - \sum_{k=2}^{n} w_{i,k} \text{ on } [0, 1]. \tag{52}$$

Theorem 4 *For each q and i, $1 \le q \le n$, $1 \le i \le n$ and all $x \in \overline{\Omega}$, the components in the decomposition (48) satisfy the following estimates.*

$$|w'''_{i,q}(x)| \le C\,\varepsilon_i^{-1}\varepsilon_q^{-2}\,\mathscr{B}_q(x), \quad \textit{if } i \le q, \quad |w'''_{i,q}(x)| \le C\,\varepsilon_i^{-2}\varepsilon_q^{-1}\,\mathscr{B}_q(x), \quad \textit{if } i > q,$$

$$|w''_{i,q}(x)| \le C\,\varepsilon_i^{-1}\varepsilon_q^{-1}\,\mathscr{B}_q(x), \quad \textit{if } i \le q < n, \quad |w''_{i,q}(x)| \le C\,\varepsilon_i^{-2}\,\mathscr{B}_q(x), \quad \textit{if } i > q,$$

$$|w'_{i,q}(x)| \le C\,\varepsilon_i^{-1}\,\mathscr{B}_q(x), \quad \textit{if } q < n.$$

Proof Differentiating (49) thrice,

$$|w'''_{i,n}(x)| = \begin{cases} |w'''_i(x^{(2)}_{n-1,n})| \text{ on } [0, x^{(2)}_{n-1,n}) \\ |w'''_i(x)| \qquad \text{otherwise} \end{cases}.$$

Then for $x \in [0, x^{(2)}_{n-1,n})$, using Theorem 3,

$$|w'''_{i,n}(x)| \leq C\varepsilon_i^{-1}\Big(\sum_{k=1}^{i-1}\varepsilon_k^{-1}\mathscr{B}_k(x^{(2)}_{n-1,n}) + \sum_{k=i}^{n}\varepsilon_k^{-2}\mathscr{B}_k(x^{(2)}_{n-1,n})\Big).$$

Since $x^{(2)}_{k,n} \leq x^{(2)}_{n-1,n}$ for $k < n$, using (46) $\varepsilon_k^{-2}\mathscr{B}_k(x^{(2)}_{n-1,n}) \leq \varepsilon_n^{-2}\mathscr{B}_n(x^{(2)}_{n-1,n})$ and hence

$$|w'''_{i,n}(x)| \leq C\varepsilon_i^{-1}\varepsilon_n^{-2}\mathscr{B}_n(x^{(2)}_{n-1,n}) \leq C\varepsilon_i^{-1}\varepsilon_n^{-2}\mathscr{B}_n(x). \tag{53}$$

For $x \in [x^{(2)}_{n-1,n}, 1]$,

$$|w'''_{i,n}(x)| = |w'''_i(x)| \leq C\varepsilon_i^{-1}\Big(\sum_{k=1}^{i-1}\varepsilon_k^{-1}\mathscr{B}_k(x) + \sum_{k=i}^{n}\varepsilon_k^{-2}\mathscr{B}_k(x)\Big).$$

As $x \geq x^{(2)}_{n-1,n}$, using (46) $\varepsilon_k^{-2}\mathscr{B}_k(x) \leq \varepsilon_n^{-2}\mathscr{B}_n(x)$ and hence for $x \in [x^{(2)}_{n-1,n}, 1]$

$$|w'''_{i,n}(x)| \leq C\varepsilon_i^{-1}\varepsilon_n^{-2}\mathscr{B}_n(x). \tag{54}$$

From (49) and (50), it is not hard to see that for each q, $n-1 \geq q \geq i$ and $x \in [x^{(2)}_{q,q+1}, 1]$, $w_{i,q}(x) = p_{i,q}(x) = w_i(x) - \sum_{k=q+1}^{n} w_{i,k}(x) = w_i(x) - w_i(x) = 0$. Differentiating (50) thrice, on $x \in [0, x^{(2)}_{q-1,q})$

$$|w'''_{i,q}(x)| = |p'''_{i,q}(x^{(2)}_{q-1,q})| \leq C\varepsilon_i^{-1}\varepsilon_q^{-2}\mathscr{B}_q(x).$$

For $x \in [x^{(2)}_{q-1,q}, x^{(2)}_{q,q+1})$, using Lemma 3,

$$|w'''_{i,q}(x)| \leq C\,\varepsilon_i^{-1}\varepsilon_q^{-2}\,\mathscr{B}_q(x). \tag{55}$$

From (50) and (51), it is not hard to see that for each q, $i-1 \geq q \geq 2$ and $x \in [x^{(1)}_{q,q+1}, 1]$, $w_{i,q}(x) = 0$. Differentiating (51) thrice on $x \in [0, x^{(1)}_{q-1,q})$

$$|w'''_{i,q}(x)| = |p'''_{i,q}(x^{(1)}_{q-1,q})| \leq C\varepsilon_i^{-2}\varepsilon_q^{-1}\mathscr{B}_q(x).$$

For $x \in [x^{(1)}_{q-1,q}, x^{(1)}_{q,q+1})$, using Lemma 3,

$$|w'''_{i,q}(x)| \leq C\varepsilon_i^{-2}\varepsilon_q^{-1}\mathscr{B}_q(x). \tag{56}$$

From (51) and (52), it is not hard to see that $w_{i,1}(x) = 0$ for $x \in [x^{(1)}_{1,2}, 1]$ and for $x \in [0, x^{(1)}_{1,2})$, $|w'''_{i,1}(x)| \leq |w'''_i(x)| \leq C\varepsilon_i^{-2}\varepsilon_1^{-1}\mathscr{B}_1(x)$. Since $w''_{i,q}(1) = 0$, for $q < n$, it follows that for any $x \in [0, 1]$ and $i > q$,

$$|w''_{i,q}(x)| = \Big|\int_x^1 w^{(3)}_{i,q}(t)dt\Big| \le C\int_x^1 \varepsilon_i^{-2}\varepsilon_q^{-1}\mathscr{B}_q(t)dt \le C\varepsilon_i^{-2}\mathscr{B}_q(x).$$

Hence,

$$|w''_{i,q}(x)| \le C\,\varepsilon_i^{-2}\,\mathscr{B}_q(x),\quad \text{for}\ \ i > q. \tag{57}$$

Similar arguments lead to

$$|w''_{i,q}(x)| \le C\,\varepsilon_i^{-1}\varepsilon_q^{-1}\,\mathscr{B}_q(x),\quad \text{for}\ \ i \le q, \tag{58}$$

and

$$|w'_{i,q}(x)| \le C\,\varepsilon_i^{-1}\,\mathscr{B}_q(x),\ \ 1 \le i \le n, 1 \le q \le n. \tag{59}$$

3 Numerical Method

To solve the BVP (1)–(2), a numerical method comprising of a Classical Finite Difference(CFD) Scheme and a piecewise uniform Shishkin mesh fitted on the domain [0, 1] is suggested.

3.1 Shishkin Mesh

A piecewise uniform Shishkin mesh with N mesh-intervals is now constructed. The mesh $\overline{\Omega}^N$ is a piecewise uniform mesh on [0, 1] obtained by dividing [0, 1] into $n+1$ mesh-intervals as $[0,\tau_1]\cup[\tau_1,\tau_2]\cup\cdots\cup[\tau_{n-1},\tau_n]\cup[\tau_n,1]$. Transition parameters $\tau_r,\ 1\le r\le n$, are defined as $\tau_n = \min\left\{\dfrac{1}{2},\ 2\dfrac{\varepsilon_n}{\alpha}\ln N\right\}$ and, for $r = n-1, \ldots 1,\ \tau_r = \min\left\{\dfrac{r\tau_{r+1}}{r+1},\ 2\dfrac{\varepsilon_r}{\alpha}\ln N\right\}$. On the sub-interval $[\tau_n, 1]$, $\frac{N}{2}+1$ mesh-points are placed uniformly and on each of the subintervals $[\tau_r, \tau_{r+1})$, $r = n-1, \ldots 1$, a uniform mesh of $\frac{N}{2n}$ mesh-points is placed. A uniform mesh of $\frac{N}{2n}$ mesh-points is placed on the sub-interval $[0, \tau_1)$.

The Shishkin mesh is coarse in the outer region and becomes finer and finer in the inner (layer) regions. From the above construction, it is clear that the transition points $\tau_r,\ r = 1,\ldots,n$, are the only points at which the mesh-size can change and that it does not necessarily change at each of these points.

If each of the transition parameters $\tau_r,\ r = 1,\ldots,n$, are with the left choice, the Shishkin mesh $\overline{\Omega}^N$ becomes the classical uniform mesh with $\tau_r = \frac{r}{2n},\ r = 1,\ldots,n$, and hence the step size is N^{-1}.

The following notations are introduced: $h_j = x_j - x_{j-1}$ and if $x_j = \tau_r$, then $h_r^- = x_j - x_{j-1}$, $h_r^+ = x_{j+1} - x_j$, $J = \{\tau_r : h_r^+ \neq h_r^-\}$. Let $H_r = 2n\, N^{-1}(\tau_r - \tau_{r-1})$, $2 \leq r \leq n$ denote the step size in the mesh interval $(\tau_{r-1}, \tau_r]$. Also, $H_1 = 2\,nN^{-1}\tau_1$ and $H_{n+1} = 2\,N^{-1}(1-\tau_n)$. Thus, for $1 \leq r \leq n-1$, the change in the step size at the point $x_j = \tau_r$ is

$$h_r^+ - h_r^- = 2\,nN^{-1}\Big(\frac{(r+1)}{r}d_r - d_{r-1}\Big), \tag{60}$$

where $d_r = \frac{r\tau_{r+1}}{r+1} - \tau_r$ with the convention $d_n = 0$, when $\tau_n = 1/2$. The mesh $\overline{\Omega}^N$ becomes a classical uniform mesh when $d_r = 0$ for all $r = 1, \ldots, n$ and $\tau_r \leq C\,\varepsilon_r \ln N$, $1 \leq r \leq n$. Also $\tau_r = \frac{r}{s}\tau_s$ when $d_r = \cdots = d_s = 0$, $1 \leq r \leq s \leq n$.

3.2 Discrete Problem

To solve the BVP (1)–(2) numerically the following upwind classical finite difference scheme is applied on the mesh $\overline{\Omega}^N$.

$$L^N\mathbf{U}(x_j) \equiv E\delta^2\mathbf{U}(x_j) + A(x_j)D^+\mathbf{U}(x_j) - B(x_j)\mathbf{U}(x_j) = \mathbf{f}(x_j), \tag{61}$$

$$\mathbf{U}(x_0) = \mathbf{l},\ \mathbf{U}(x_N) = \mathbf{r}, \tag{62}$$

where $\mathbf{U}(x_j) = (U_1(x_j), U_2(x_j), \ldots, U_n(x_j))^T$ and for $1 \leq j \leq N-1$,

$$D^+U(x_j) = \frac{U(x_{j+1}) - U(x_j)}{h_{j+1}},\quad D^-U(x_j) = \frac{U(x_j) - U(x_{j-1})}{h_j},$$

$$\delta^2 U(x_j) = \frac{1}{\overline{h_j}}\Big(D^+U(x_j) - D^-U(x_j)\Big),$$

with

$$\overline{h_j} = \frac{(h_j + h_{j+1})}{2}.$$

4 Numerical Results

In this section a discrete maximum principle, a discrete stability result and the first-order convergence of the proposed numerical method are established.

Lemma 4 (Discrete Maximum Principle) *Assume that the vector valued mesh function* $\boldsymbol{\psi}(x_j) = (\psi_1(x_j), \psi_2(x_j), \ldots, \psi_n(x_j))^T$ *satisfies* $\boldsymbol{\psi}(x_0) \geq \mathbf{0}$ *and* $\boldsymbol{\psi}(x_N) \geq \mathbf{0}$. *Then* $L^N\boldsymbol{\psi}(x_j) \leq \mathbf{0}$ *for* $1 \leq j \leq N-1$ *implies that* $\boldsymbol{\psi}(x_j) \geq \mathbf{0}$ *for* $0 \leq j \leq N$.

Lemma 5 (Discrete Stability Result) *If* $\boldsymbol{\psi}(x_j) = (\psi_1(x_j), \psi_2(x_j), \ldots, \psi_n(x_j))^T$ *is any vector valued mesh function defined on* $\overline{\Omega}^N$, *then for* $1 \le i \le n$ *and* $0 \le j \le N$,

$$|\psi_i(x_j)| \le \max\left\{\|\boldsymbol{\psi}(x_0)\|,\ \|\boldsymbol{\psi}(x_N)\|,\ \frac{1}{\beta}\ \|L^N\boldsymbol{\psi}\|_{\Omega^N}\right\}.$$

4.1 Error Estimate

Analogous to the continuous case, the discrete solution $\mathbf{U}$ can be decomposed into $\mathbf{V}$ and $\mathbf{W}$ as defined below.

$$L^N\mathbf{V}(x_j) = \mathbf{f}(x_j),\ \text{for } 0 < j < N,\ \mathbf{V}(x_0) = \mathbf{v}(x_0),\ \mathbf{V}(x_N) = \mathbf{v}(x_N) \quad (63)$$

$$L^N\mathbf{W}(x_j) = \mathbf{0},\ \text{for } 0 < j < N,\ \mathbf{W}(x_0) = \mathbf{w}(x_0),\ \mathbf{W}(x_N) = \mathbf{w}(x_N) \quad (64)$$

Lemma 6 *Let* $\mathbf{v}$ *be the solution of (7) and* $\mathbf{V}$ *be the solution of (63), then*

$$\|\mathbf{V} - \mathbf{v}\|_{\overline{\Omega}^N} \le CN^{-1}.$$

Proof For $1 \le j \le N-1$,

$$L^N(\mathbf{V}-\mathbf{v})(x_j) = \begin{pmatrix} \varepsilon_1(\frac{d^2}{dx^2} - \delta^2)v_1(x_j) + a_1(x_j)(\frac{d}{dx} - D^+)v_1(x_j) \\ \varepsilon_2(\frac{d^2}{dx^2} - \delta^2)v_2(x_j) + a_2(x_j)(\frac{d}{dx} - D^+)v_2(x_j) \\ \vdots \\ \varepsilon_n(\frac{d^2}{dx^2} - \delta^2)v_n(x_j) + a_n(x_j)(\frac{d}{dx} - D^+)v_n(x_j) \end{pmatrix}.$$

By the standard local truncation used in the Taylor expansions,

$$|\varepsilon_i\left(\frac{d^2}{dx^2} - \delta^2\right)v_i(x_j) + a_i(x_j)\left(\frac{d}{dx} - D^+\right)v_i(x_j)| \le C(x_{j+1} - x_{j-1})(\varepsilon_i\|v_i^{(3)}\| + \|v_i^{(2)}\|).$$

Since $(x_{j+1} - x_{j-1}) \le CN^{-1}$, by using (35),

$$\|L^N(\mathbf{V}-\mathbf{v})\|_{\Omega^N} \le CN^{-1}.$$

As v and V agree at the boundary points, using Lemma 5,

$$\|\mathbf{V} - \mathbf{v}\|_{\overline{\Omega}^N} \le CN^{-1}. \quad (65)$$

To estimate the error in the singular component $(\mathbf{W} - \mathbf{w})$, the mesh functions $B_i^N(x_j)$ for $1 \le i \le n$ on $\overline{\Omega}^N$ are defined by

$$B_i^N(x_j) = \prod_{k=1}^{j} \left(1 + \frac{\alpha h_k}{2\varepsilon_i}\right)^{-1}$$

with $B_i^N(x_0) = 1$. It is to be observed that B_i^N are monotonically decreasing.

Lemma 7 *The singular components W_i, $1 \le i \le n$ satisfy the following bound on $\overline{\Omega}^N$;*

$$|W_i(x_j)| \le C B_n^N(x_j).$$

Proof Consider the following vector valued mesh functions on $\overline{\Omega}^N$,

$$\boldsymbol{\psi}^{\pm}(x_j) = C B_n^N(x_j)\mathbf{e} \pm \mathbf{W}(x_j)$$

where $\mathbf{e}$ is the n- vector $\mathbf{e} = (1, 1, \dots, 1)^T$.
Then for sufficiently large C, $\boldsymbol{\psi}^{\pm}(x_0) \ge \mathbf{0}$, $\boldsymbol{\psi}^{\pm}(x_N) \ge \mathbf{0}$ and $L^N\boldsymbol{\psi}^{\pm}(x_j) \le \mathbf{0}$, for $1 \le j \le N-1$. Using Lemma 4, $\boldsymbol{\psi}^{\pm}(x_j) \ge \mathbf{0}$ on $\overline{\Omega}^N$, which implies that

$$|W_i(x_j)| \le C B_n^N(x_j).$$

Lemma 8 *Assume that $d_r = 0$, for $r = 1, 2, \dots, n$. Let $\mathbf{w}$ be the solution of (8) and $\mathbf{W}$ be the solution of (64). Then*

$$\|\mathbf{W} - \mathbf{w}\|_{\overline{\Omega}^N} \le CN^{-1}\ln N.$$

Proof By the standard local truncation used in the Taylor expansions,

$$\left|\varepsilon_i(\frac{d^2}{dx^2} - \delta^2)w_i(x_j) + a_i(x_j)(\frac{d}{dx} - D^+)w_i(x_j)\right| \le C(x_{j+1} - x_{j-1})(\varepsilon_i\|w_i^{(3)}\| + \|w_i^{(2)}\|)$$

where the norm is taken over the interval $[x_{j-1}, x_{j+1}]$.
Since $d_r = 0$, the mesh is uniform, $h = N^{-1}$ and $\varepsilon_k^{-1} \le C\ln N$. Then,

$$|(\mathbf{L}^N(\mathbf{W} - \mathbf{w}))_i(x_j)| \le C\,N^{-1}\Big(\sum_{k=1}^{i-1}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + \sum_{k=i}^{n}\varepsilon_k^{-2}\mathscr{B}_k(x_{j-1})\Big) \quad (66)$$

$$\le CN^{-1}\ln N + CN^{-1}\ln N\sum_{k=i}^{n}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}). \quad (67)$$

Consider the barrier function $\boldsymbol{\phi} = (\phi_1(x_j), \phi_2(x_j), \dots, \phi_n(x_j))^T$ given by

$$\phi_i(x_j) = CN^{-1}\ln N + \frac{CN^{-1}\ln N}{\gamma(\alpha - \gamma)}\Big(\sum_{k=i}^{n}\exp(2\gamma h/\varepsilon_k)Y_k(x_j)\Big), \text{ on } \overline{\Omega}^N,$$

where γ is a constant such that $0 < \gamma < \alpha$,

$$Y_k(x_j) = \frac{\lambda_k^{N-j} - 1}{\lambda_k^N - 1} \text{ with } \lambda_k = 1 + \frac{\gamma h}{\varepsilon_k}.$$

It is not hard to see that, $0 \le Y_k(x_j) \le 1, \quad D^+ Y_k(x_j) \le -\frac{\gamma}{\varepsilon_k} \exp(-\gamma x_{j+1}/\varepsilon_k)$ and $(\varepsilon_k \delta^2 + \gamma D^+) Y_k(x_j) = 0$. Hence,

$$(L^N \boldsymbol{\phi})_i(x_j) \le -CN^{-1} \ln N - CN^{-1} \ln N \sum_{k=i}^{n} \varepsilon_k^{-1} \mathscr{B}_k(x_{j-1}).$$

Consider the discrete functions

$$\boldsymbol{\psi}^{\pm}(x_j) = \boldsymbol{\phi}(x_j) \pm (\mathbf{W} - \mathbf{w})(x_j), x_j \in \overline{\Omega}^N.$$

Then for sufficiently large C, $\boldsymbol{\psi}^{\pm}(x_0) > \mathbf{0}$, $\boldsymbol{\psi}^{\pm}(x_N) \ge \mathbf{0}$ and $L^N \boldsymbol{\psi}^{\pm}(x_j) \le \mathbf{0}$ on Ω^N. Using Lemma 4, $\boldsymbol{\psi}^{\pm}(x_j) \ge \mathbf{0}$ on $\overline{\Omega}^N$. Hence, $|(\mathbf{W} - \mathbf{w})_i(x_j)| \le CN^{-1} \ln N$ for $1 \le i \le n$, implies that

$$\|(\mathbf{W} - \mathbf{w})\|_{\overline{\Omega}^N} \le CN^{-1} \ln N. \tag{68}$$

Lemma 9 *Let* $\mathbf{w}$ *be the solution of (8) and* $\mathbf{W}$ *be the solution of (64); then*

$$\|\mathbf{W} - \mathbf{w}\|_{\overline{\Omega}^N} \le CN^{-1} \ln N.$$

Proof This is proved for each mesh point $x_j \in (0, 1)$ by dividing $(0, 1)$ into $n + 1$ subintervals (a) $(0, \tau_1)$, (b) $[\tau_1, \tau_2)$, (c) $[\tau_m, \tau_{m+1})$ for some m, $2 \le m \le n - 1$ and (d) $[\tau_n, 1)$.

For each of these cases, an estimate for the local truncation error is derived and a barrier function is defined. Lastly, using these barrier functions, the required estimate is established.

Case (a): $x_j \in (0, \tau_1)$.
Clearly $x_{j+1} - x_{j-1} \le C\varepsilon_1 N^{-1} \ln N$. Then, by standard local truncation used in Taylor expansions, the following estimates hold for $x_j \in (0, \tau_1)$ and $1 \le i \le n$.

$$\begin{aligned} |(\mathbf{L}^N(\mathbf{W} - \mathbf{w}))_i(x_j)| &\le C\,(x_{j+1} - x_{j-1})(\varepsilon_i \|w_i^{(3)}\| + \|w_i^{(2)}\|) \\ &\le C\,N^{-1} \ln N \sum_{k=i}^{n} \varepsilon_k^{-1} \mathscr{B}_k(x_{j-1}). \end{aligned}$$

Consider the following barrier functions for $x_j \in (0, \tau_1)$ and $1 \le i \le n$.

$$\phi_i(x_j) = CN^{-1}\ln N\sum_{k=i}^{n}\exp(2\alpha H_1/\varepsilon_k)B_k^N(x_j) + \sum_{k=1}^{n}B_k^N(\tau_k). \tag{69}$$

Case (b): $x_j \in [\tau_1, \tau_2)$.
There are 2 possibilities: **Case (b1):** $\mathbf{d_1 = 0}$ and **Case (b2):** $\mathbf{d_1 > 0}$.
Case (b1): $\mathbf{d_1 = 0}$
Since the mesh is uniform in $(0, \tau_2)$, it follows that $x_{j+1} - x_{j-1} \le C\,\varepsilon_1 N^{-1}\ln N$, for $x_j \in [\tau_1, \tau_2)$. Then,

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,N^{-1}\ln N\sum_{k=i}^{n}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}). \tag{70}$$

Now for $x_j \in [\tau_1, \tau_2)$ and $1 \le i \le n$, define,

$$\phi_i(x_j) = CN^{-1}\ln N\sum_{k=i}^{n}\exp(2\alpha H_2/\varepsilon_k)B_k^N(x_j) + \sum_{k=2}^{n}B_k^N(\tau_k). \tag{71}$$

Case (b2): $\mathbf{d_1 > 0}$.
For this case, $x_{j+1} - x_{j-1} \le C\,\varepsilon_2 N^{-1}\ln N$, and hence for $x_j \in [\tau_1, \tau_2)$

$$\begin{aligned}\left|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)\right| &\le \left|\varepsilon_i(\frac{d^2}{dx^2}-\delta^2)w_i(x_j)\right| + C\left|(\frac{d}{dx}-D^+)w_i(x_j)\right| \\ &\le \left|\varepsilon_i(\frac{d^2}{dx^2}-\delta^2)\sum_{k=1}^{n}w_{i,k}\right| + C\left|(\frac{d}{dx}-D^+)\sum_{k=1}^{n}w_{i,k}\right|.\end{aligned}$$

By the standard local truncation used in Taylor expansions

$$\begin{aligned}|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le{}& C\,\varepsilon_i|w_{i,1}^{(2)}(x_{j-1})| + C\,(x_{j+1}-x_{j-1})\varepsilon_i\sum_{k=2}^{n}|w_{i,k}^{(3)}(x_{j-1})| \\ &+C\,|w_{i,1}^{(1)}(x_{j-1})| + C\,(x_{j+1}-x_{j-1})\sum_{k=2}^{n}|w_{i,k}^{(2)}(x_{j-1})|.\end{aligned} \tag{72}$$

Now using Theorem 4, it is not hard to derive that

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_1(x_j)| \le C\,N^{-1}\ln N\sum_{k=2}^{n}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + C\,\varepsilon_1^{-1}\mathscr{B}_1(x_{j-1}) \tag{73}$$

and for $2 \le i \le n$,

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,N^{-1}\ln N\sum_{k=i}^{n}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + C\,\varepsilon_i^{-1}\mathscr{B}_1(x_{j-1}). \quad (74)$$

Define

$$\phi_1(x_j) = CN^{-1}\ln N\sum_{k=2}^{n}\exp(2\alpha H_2/\varepsilon_k)B_k^N(x_j) + C\,B_1^N(x_j) + C\,\sum_{k=2}^{n}B_k^N(\tau_k)$$

and for $2 \le i \le n$,

$$\phi_i(x_j) = CN^{-1}\ln N\sum_{k=i}^{n}\exp(2\alpha H_2/\varepsilon_k)B_k^N(x_j) + C\,B_1^N(x_j) + C\,\sum_{k=2}^{n}B_k^N(\tau_k).$$

Case (c): $x_j \in (\tau_m, \tau_{m+1}]$**.** There are 3 possibilities:

Case (c1): $d_1 = d_2 = \cdots = d_m = 0$,
Case (c2): $d_r > 0$ and $d_{r+1} = \ \ldots\ = d_m = 0$ for some r, $1 \le r \le m-1$ and
Case (c3): $d_m > 0$.

Case (c1): $d_1 = d_2 = \cdots = d_m = 0$,
Since $\tau_1 = C\tau_{m+1}$ and the mesh is uniform in $(0, \tau_{m+1})$, it follows that, for $x_j \in (\tau_m, \tau_{m+1}]$, $x_{j+1} - x_{j-1} \le C\,\varepsilon_1 N^{-1}\ln N$ and hence

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \ \le\ C\,N^{-1}\ln N\sum_{k=i}^{n}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}). \quad (75)$$

For $1 \le i \le n$,

$$\phi_i(x_j) = CN^{-1}\ln N\sum_{k=i}^{n}\exp(2\alpha H_{m+1}/\varepsilon_k)B_k^N(x_j) + C\sum_{k=m+1}^{n}B_k^N(\tau_k). \quad (76)$$

Case (c2): $d_r > 0$ and $d_{r+1} = \ \ldots\ = d_m = 0$ for some r, $1 \le r \le m-1$
Since, $\tau_{r+1} = C\tau_{m+1}$, the mesh is uniform in (τ_r, τ_{m+1}), it follows that $x_{j+1} - x_{j-1} \le C\,\varepsilon_{r+1}N^{-1}\ln N$, for $x_j \in (\tau_m, \tau_{m+1}]$.
By the standard local truncation used in Taylor expansions

$$\begin{aligned}|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,\varepsilon_i\sum_{k=1}^{r}|w_{i,k}^{(2)}(x_{j-1})| + C\,(x_{j+1}-x_{j-1})\varepsilon_i\sum_{k=r+1}^{n}|w_{i,k}^{(3)}(x_{j-1})|\\ +C\sum_{k=1}^{r}|w_{i,k}^{(1)}(x_{j-1})| + C\,(x_{j+1}-x_{j-1})\sum_{k=r+1}^{n}|w_{i,k}^{(2)}(x_{j-1})|.\end{aligned} \quad (77)$$

Now using Theorem 4, it is not hard to derive that for $i \le r$

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,N^{-1}\ln N \sum_{k=r+1}^{n} \varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + C\sum_{k=i}^{r}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1})$$

and for $i > r$

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,N^{-1}\ln N \sum_{k=i}^{n} \varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + C\varepsilon_i^{-1}\mathscr{B}_r(x_{j-1}).$$

Now define, for $i \le r$

$$\phi_i(x_j) = CN^{-1}\ln N \sum_{k=r+1}^{n} \exp\left(\frac{2\alpha H_{m+1}}{\varepsilon_k}\right) B_k^N(x_j) + C\sum_{k=i}^{r} B_k^N(x_j) + C\sum_{k=m+1}^{n} B_k^N(\tau_k)$$

and for $i > r$

$$\phi_i(x_j) = CN^{-1}\ln N \sum_{k=i}^{n} \exp\left(\frac{2\alpha H_{m+1}}{\varepsilon_k}\right) B_k^N(x_j) + CB_r^N(x_j) + C\sum_{k=m+1}^{n} B_k^N(\tau_k).$$

Case (c3): $d_m > 0$
Replacing r by m in the arguments of the previous case **Case(c2)** and using $x_{j+1} - x_{j-1} \le C\varepsilon_{m+1}N^{-1}\ln N$, the following estimates hold for $x_j \in (\tau_m, \tau_{m+1}]$.
For $i \le m$,

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,N^{-1}\ln N \sum_{k=m+1}^{n} \varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + C\sum_{k=i}^{m}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) \tag{78}$$

and for $i > m$

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,N^{-1}\ln N \sum_{k=i}^{n} \varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + C\,\varepsilon_i^{-1}\mathscr{B}_m(x_{j-1}). \tag{79}$$

For $i \le m$, define,

$$\phi_i(x_j) = CN^{-1}\ln N \sum_{k=m+1}^{n} \exp\left(\frac{2\alpha H_{m+1}}{\varepsilon_k}\right) B_k^N(x_j) + C\sum_{k=i}^{m} B_k(x_j) + C\sum_{k=m+1}^{n} B_k^N(\tau_k)$$

and for $i > m$

$$\phi_i(x_j) = CN^{-1}\ln N \sum_{k=i}^{n} \exp\left(\frac{2\alpha H_{m+1}}{\varepsilon_k}\right) B_k^N(x_j) + CB_m(x_j) + C\sum_{k=m+1}^{n} B_k^N(\tau_k).$$

Case (d): There are 3 possibilities.

Case (d1): $d_1 = \ldots = d_n = 0$,
Case (d2): $d_r > 0$ and $d_{r+1} = \ldots = d_n = 0$ for some r, $1 \le r \le n-1$ and
Case (d3): $d_n > 0$.

Case (d1): $d_1 = \ldots = d_n = 0$,
The mesh is uniform in [0, 1] and the result is established in the Lemma 8.
Case (d2): $d_r > 0$ and $d_{r+1} = \ldots = d_n = 0$ for some r, $1 \le r \le n-1$
In this case from the definition of τ_n it follows that $x_{j+1} - x_{j-1} \le C\,\varepsilon_{r+1} N^{-1} \ln N$ and arguments similar to the **Case(c2)** lead to the following estimates for $x_j \in (\tau_n, 1]$.
For $i \le r$,

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,N^{-1}\ln N \sum_{k=r+1}^{n} \varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + C\sum_{k=i}^{r}\varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) \tag{80}$$

and for $i > r$

$$|(\mathbf{L}^N(\mathbf{W}-\mathbf{w}))_i(x_j)| \le C\,N^{-1}\ln N \sum_{k=i}^{n} \varepsilon_k^{-1}\mathscr{B}_k(x_{j-1}) + C\,\varepsilon_i^{-1}\mathscr{B}_r(x_{j-1}). \tag{81}$$

Define the barrier functions ϕ_i for $i \le r$ by

$$\phi_i(x_j) = CN^{-1}\ln N \sum_{k=r+1}^{n} \exp(2\alpha H_{n+1}/\varepsilon_k)B_k^N(x_j) + C\sum_{k=i}^{r} B_k^N(x_j) \tag{82}$$

and for $i > r$

$$\phi_i(x_j) = CN^{-1}\ln N \sum_{k=i}^{n} \exp(2\alpha H_{n+1}/\varepsilon_k)B_k^N(x_j) + CB_r^N(x_j). \tag{83}$$

Case (d3): $d_n > 0$
Now $\tau_n = 2\dfrac{\varepsilon_n}{\alpha}\ln N$. Then on $(\tau_n, 1]$,

$$\begin{aligned}|(W_i - w_i)(x_j)| &\le |W_i(x_j)| + |w_i(x_j)|\\ &\le CB_n^N(x_j) + C\mathscr{B}_n(x_j), \text{ using Lemma 7 and Theorem 3}\end{aligned}$$

Hence,

$$|(W_i - w_i)(x_j)| \le CN^{-1}, \quad \text{on } [\tau_n, 1]. \tag{84}$$

Now using the estimates derived and the barrier functions ϕ_i, $1 \le i \le n$, defined for all the four cases, the main proof is split into two cases

Case 1: $d_n > 0$. Consider the following discrete functions for $0 \le j \le N/2$,

$$\boldsymbol{\psi}^{\pm}(x_j) = \boldsymbol{\phi}(x_j) \pm (\mathbf{W} - \mathbf{w})(x_j) \tag{85}$$

where $\boldsymbol{\phi}(x_j) = (\phi_1(x_j), \phi_2(x_j), \ldots, \phi_n(x_j))^T$.
For sufficiently large C, it is not hard to see that

$$\boldsymbol{\psi}^{\pm}(x_0) \geq \mathbf{0},\ \ \boldsymbol{\psi}^{\pm}(x_{\frac{N}{2}}) \geq \mathbf{0} \text{ and } L^N \boldsymbol{\psi}^{\pm}(x_j) \leq \mathbf{0}, \text{ for } 0 < j < N/2.$$

Then by Lemma 4, $\boldsymbol{\psi}^{\pm}(x_j) \geq \mathbf{0}$ for $0 \leq j \leq N/2$. Consequently,

$$|(W_i - w_i)(x_j)| \leq CN^{-1}, \quad \text{on } [0, \tau_n]. \tag{86}$$

Hence, (84) and (86) imply that, for $d_n > 0$

$$\|(\mathbf{W} - \mathbf{w})\|_{\overline{\Omega}^N} \leq CN^{-1} \ln N. \tag{87}$$

Case 2: $d_n = 0$. Consider the following discrete functions for $0 \leq j \leq N$,

$$\boldsymbol{\psi}^{\pm}(x_j) = \boldsymbol{\phi}(x_j) \pm (\mathbf{W} - \mathbf{w})(x_j). \tag{88}$$

For sufficiently large C, it is not hard to see that

$$\boldsymbol{\psi}^{\pm}(x_0) \geq \mathbf{0},\ \ \boldsymbol{\psi}^{\pm}(x_N) \geq \mathbf{0} \text{ and } L^N \boldsymbol{\psi}^{\pm}(x_j) \leq \mathbf{0}, \text{ for } 0 < j < N.$$

Then by Lemma 4, $\boldsymbol{\psi}^{\pm}(x_j) \geq \mathbf{0}$ for $0 \leq j \leq N$. Hence, for $d_n = 0$,

$$\|(\mathbf{W} - \mathbf{w})\|_{\overline{\Omega}^N} \leq CN^{-1} \ln N.$$

Theorem 5 *Let u be the solution of the problem (1)–(2) and U be the solution of the problem (61)–(62), then,*

$$\|(\mathbf{u} - \mathbf{U})\|_{\overline{\Omega}^N} \leq CN^{-1} \ln N.$$

Proof From the Eqs. (7), (8), (63) and (64), we have

$$\begin{aligned} \|(\mathbf{u} - \mathbf{U})\|_{\overline{\Omega}^N} &= \|(\mathbf{v} + \mathbf{w} - \mathbf{V} + \mathbf{W})\|_{\overline{\Omega}^N} \\ &\leq \|(\mathbf{v} - \mathbf{V}\|_{\overline{\Omega}^N} + \|(\mathbf{w} - \mathbf{W})\|_{\overline{\Omega}^N} \end{aligned}$$

Then the result follows from Lemmas 6 and 9.

5 Numerical Illustrations

Example 1 Consider the following boundary value problem for the system of convection–diffusion equations on (0, 1)

$$
\begin{aligned}
&\varepsilon_1 u_1''(x) + (1+x)u_1'(x) - 4u_1(x) + 2u_2(x) + u_3(x) = -e^x,\\
&\varepsilon_2 u_2''(x) + (2+x^2)u_2'(x) + u_1(x) - 6u_2(x) + 2u_3(x) = -\sin x,\\
&\varepsilon_3 u_3''(x) + (e^x)u_3'(x) + 3u_1(x) + 2u_2(x) - 8u_3(x) = -\cos x,\\
\text{with}\quad & u_1(0)=1,\ u_2(0)=1,\ u_3(0)=1\ u_1(1)=0,\ u_2(1)=0\ u_3(1)=0.
\end{aligned}
$$

The above problem is solved using the suggested numerical method and plot of the approximate solution for $N = 1536$, $\varepsilon_1 = 5^{-4}$, $\varepsilon_2 = 3^{-4}$, $\varepsilon_3 = 2^{-5}$ is shown in Fig. 1.

Parameter uniform error constant and the order of convergence of the numerical method for $\varepsilon_1 = \eta/625$, $\varepsilon_2 = \eta/81$ and $\varepsilon_3 = \eta/32$ are computed using a variant of the two mesh algorithm suggested in [6] and are shown in Table 1.

It is found that the parameter ε_i for any i, influences the components $u_1, u_2, \ldots, u_i$ and causes multiple layers for these components, in the neighbourhood of the origin and has no significant influence on $u_{i+1}, u_{i+2}, \ldots, u_n$. The following examples illustrate this.

Example 2 Consider the following boundary value problem for the system of convection–diffusion equations on (0, 1)

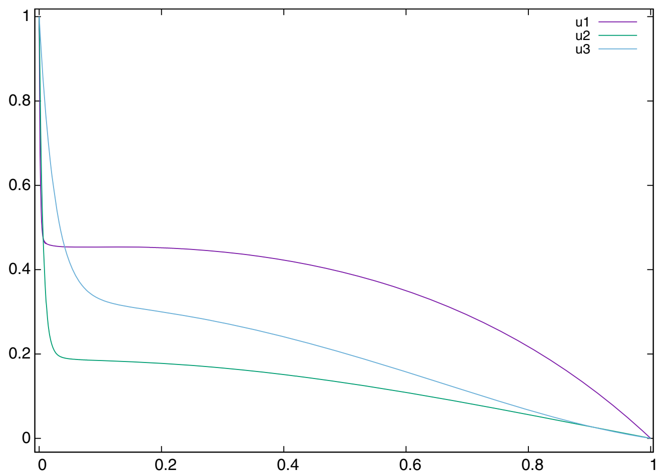


Fig. 1 Approximate solution of Example 1

Table 1 Maximum errors and order of convergence

η	Number of mesh elements N				
	96	192	384	768	1536
2^{0}	$0.1604E-01$	$0.9767E-02$	$0.5495E-02$	$0.2860E-02$	$0.1430E-02$
2^{-1}	$0.1626E-01$	$0.9895E-02$	$0.5560E-02$	$0.2893E-02$	$0.1446E-02$
2^{-2}	$0.1637E-01$	$0.9955E-02$	$0.5587E-02$	$0.2905E-02$	$0.1451E-02$
2^{-3}	$0.1643E-01$	$0.9983E-02$	$0.5598E-02$	$0.2910E-02$	$0.1452E-02$
2^{-4}	$0.1645E-01$	$0.9995E-02$	$0.5603E-02$	$0.2911E-02$	$0.1453E-02$
2^{-5}	$0.1647E-01$	$0.1000E-01$	$0.5604E-02$	$0.2911E-02$	$0.1453E-02$
2^{-6}	$0.1647E-01$	$0.1000E-01$	$0.5604E-02$	$0.2911E-02$	$0.1453E-02$
2^{-7}	$0.1648E-01$	$0.1000E-01$	$0.5604E-02$	$0.2911E-02$	$0.1453E-02$
2^{-8}	$0.1648E-01$	$0.1000E-01$	$0.5604E-02$	$0.2911E-02$	$0.1453E-02$
D^N	$0.1648E-01$	$0.1000E-01$	$0.5604E-02$	$0.2911E-02$	$0.1453E-02$
P^N	$0.7203E+00$	$0.8358E+00$	$0.9447E+00$	$0.1002E+01$	
C_p^N	$0.1123E+01$	$0.1123E+01$	$0.1037E+01$	$0.8877E+00$	$0.7300E+00$

The computed order of ε_i-uniform convergence, $p^* = 0.7203$.
The computed ε_i-uniform error constant, $C_{p^*}^N = 1.1235$.
From Table 1, it is to be noted that the error decreases as the number of mesh elements N increases. Also for each N, the error stabilises as η tends to zero

$$\begin{aligned}
&\varepsilon_1 u_1''(x) + (1+x)u_1'(x) - 4u_1(x) + 2u_2(x) + u_3(x) = 1 - x,\\
&\varepsilon_2 u_2''(x) + (2+x^2)u_2'(x) + 2u_1(x) - 6u_2(x) + 3u_3(x) = 3 - 3x,\\
&\varepsilon_3 u_3''(x) + u_3'(x) + 3u_1(x) + 3u_2(x) - 7u_3(x) = 7x - 8,\\
\text{with}\quad &u_1(0) = 0,\ u_2(0) = 1,\ u_3(0) = 1\ u_1(1) = 0,\ u_2(1) = 0\ u_3(1) = 0
\end{aligned}$$

The above problem is solved using the suggested numerical method. As $u_2(0) \neq u_{02}(0)$ and $u_i(0) = u_{0i}(0),\ i = 1, 3$ for this problem, a layer of width $O(\varepsilon_2)$ is expected to occur in the neighbourhood of the origin for u_1 and u_2 but not for u_3. Further u_1 cannot have ε_1 layer or ε_3 layer. The plot of an approximate solution of this problem for $N = 384$, $\varepsilon_1 = 5^{-4}$, $\varepsilon_2 = 3^{-4}$, $\varepsilon_3 = 2^{-5}$ is shown in Fig. 2a–d.

Example 3 Consider the following boundary value problem for the system of convection–diffusion equations on (0, 1)

$$\begin{aligned}
&\varepsilon_1 u_1''(x) + (1+x)u_1'(x) - 4u_1(x) + 2u_2(x) + u_3(x) = x,\\
&\varepsilon_2 u_2''(x) + (2+x^2)u_2'(x) + 2u_1(x) - 6u_2(x) + 3u_3(x) = 3x,\\
&\varepsilon_3 u_3''(x) + u_3'(x) + 3u_1(x) + 3u_2(x) - 7u_3(x) = 1 - 7x,\\
\text{with}\quad &u_1(0) = 0,\ u_2(0) = 0,\ u_3(0) = 1\ u_1(1) = 0,\ u_2(1) = 0\ u_3(1) = 1.
\end{aligned}$$

The above problem is solved using the suggested numerical method. As $u_3(0) \neq u_{03}(0)$ and $u_i(0) = u_{0i}(0),\ i = 1, 2$ for this problem, a layer of width $O(\varepsilon_3)$ is expected to occur in the neighbourhood of the origin for u_1, u_2 and u_3. Further

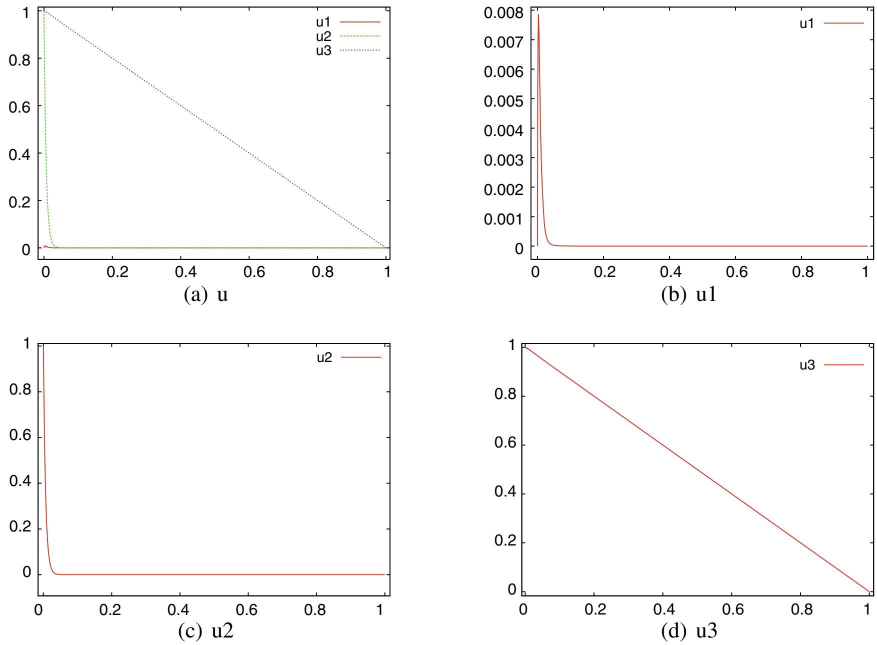


Fig. 2 Approximation of solution components of Example 2

u_1 will not have ε_1 layer or ε_2 layer. Similarly u_2 will not have ε_2 layer. The plot of an approximate solution of this problem for $N = 384$, $\varepsilon_1 = 5^{-4}$, $\varepsilon_2 = 3^{-4}$, $\varepsilon_3 = 2^{-5}$ is shown in Fig. 3a–d.

6 Conclusions

The method presented in this paper is the extension of the work done for the scalar problem in [4]. The novel estimates of derivatives of the solution help us to establish the desired error bound for the Classical Finite Difference Scheme when applied on any of the 2^n Shishkin meshes.

The examples given are to facilitate the reader to note the effect of coupling with the assumed order of the perturbation parameters.

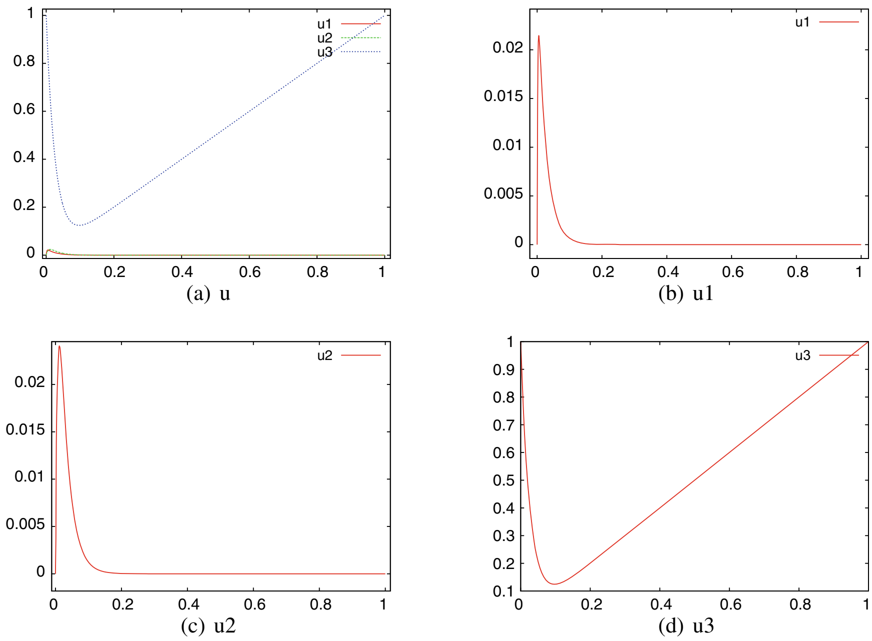


Fig. 3 Approximation of solution components of Example 3

Acknowledgements The first author wishes to acknowledge the financial support extended through Junior Research Fellowship by the University Grants Commission, India, to carry out this research work. Also the first and the third authors thank the Department of Science & Technology, Government of India for the support to the Department through the DST-FIST Scheme to set up the Computer Lab where the computations have been carried out.

References

1. Roos, H.-G., Stynes, M., Tobiska, L.: Robust Numerical Methods for Singularly Perturbed Differential Equations. Springer Series in Computational Mathematics, Berlin (1996)
2. Morton, K.W.: Numerical solution of convection diffusion problems. Appl. Math. Math. Comput. (1995)
3. Kokotovic, P.V.: Applications of singularly perturbation techniques to control problems. SIAM Rev. **26**(4), 501–550 (1984)
4. Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Fitted Numerical Methods for Singular Perturbation Problems. World Scientific Publishing Co., Singapore (1996)
5. Doolan, E.P., Miller, J.J.H., Schilders, W.H.A.: Uniform Numerical Methods for Problems with Initial and Boundary Layers. Boole press, Dublin, Ireland (1980)
6. Farrell, P.A., Hegarty, A., Miller, J.J.H., O'Riordan, E., Shishkin, G.I.: Robust Computational Techniques for Boundary Layers. Chapman and Hall/CRC Press, Boca Raton (2000)
7. Saravana Sankar, K., Miller, J.J.H., Valarmathi, S.: A parameter uniform fitted mesh method for a weakly coupled system of two singularly perturbed convection-diffusion equations. Math. Commun. **24**, 193–210 (2019)

8. Paramasivam, M., Valarmathi, S., Miller, J.J.H.: Second order parameter-uniform convergence for a finite difference method for a singularly perturbed linear reaction-diffusion system. Math. Commun. **15**(2), 587–612 (2010)
9. Linss, T.: Analysis of an upwind finite difference scheme for a system of coupled singularly perturbed convection-diffusion equations. Computing **79**(1), 23–32 (2007)
10. Valarmathi, S., Miller, J.J.H.: A parameter-uniform finite difference method for singularly perturbed linear dynamical systems. Int. J. Numer. Anal. Model. **7**(3), 535–548 (2010)

Numerical Analysis of a Finite Difference Method for a Linear System of Singularly Perturbed Parabolic Delay Differential Reaction-Diffusion Equations with Discontinuous Source Terms

Parthiban Saminathan and Franklin Victor

Abstract On the rectangular domain $\Delta = \{(x, t) : 0 < x < 2, 0 < t \leq T\}$, a singularly perturbed linear system of parabolic second-order delay differential equations of reaction-diffusion type with discontinuous source terms is considered. In Δ, there is a discontinuity in the source terms at (d, t). The components of the solution of this system exhibit boundary layers near $x = 0$ and $x = 2$, and interior layers at $(1, t)$ and/or (d, t) and $(1 + d, t)$ as the highest order space derivative is multiplied by a singular perturbation parameter. To approximate the solution, a numerical method based on the classical finite difference scheme on a Shishkin piecewise-uniform mesh is proposed. For all values of singular perturbation parameters, the method is shown to be first-order convergent. The theoretical results are supported by numerical illustrations.

1 Introduction

Singularly perturbed delay differential equations are used to model a variety of phenomena such as population dynamics, physiology, and control systems. However, in some circumstances, the condition that the source function be continuous may be impractical. The paper considers a singularly perturbed system of reaction-diffusion type linear parabolic delay differential equations with discontinuous source terms. Because of the perturbation parameters, delay term, and discontinuous source terms, the components of the solution of these systems include boundary and interior layers.

A singularly perturbed linear system of second-order partial differential equations of the parabolic reaction-diffusion type has been considered in [3] with provided initial and boundary conditions. Each equation diffusion term is multiplied by a small positive parameter. It is assumed that these singular perturbation parameters are distinct. Overlapping layers appear in the solution components. Shishkin piecewise-uniform meshes are introduced, and they are used in conjunction with a classical finite

P. Saminathan (✉) · F. Victor
Department of Mathematics, Bishop Heber College, Tiruchirappalli, TN, India

Valarmathi S et al. (eds.), *Differential Equations and Applications*,
Springer Proceedings in Mathematics & Statistics 368,
https://doi.org/10.1007/978-981-16-7546-1_10

difference discretization to construct a numerical method for solving this problem. The numerical approximations obtained with this method are first-order convergent in time and essentially second-order convergent in the space variable in the maximum norm, uniformly with respect to all of the parameters. The work in the previous study is extended to a linear system of singularly perturbed parabolic delay differential reaction-diffusion equations with discontinuous source terms in this paper.

The following is the outline of the paper. The problem is defined in Sect. 2, and the existence and regularity of the solution are discussed. Section 3 establishes the maximum principle for the differential operator and, as a result, the stability result. Standard estimates of the solution derivatives are also presented. Improved estimates for the derivatives of solution components are presented in Sect. 4. Section 5 provides piecewise-uniform Shishkin meshes, whereas Sect. 6 defines the discrete problem and establishes the discrete maximum principle and discrete stability properties. The numerical analysis is presented in Sect. 7 along with the error bounds. Section 8 provides numerical illustrations, and Sect. 9 includes the conclusion.

2 The Continuous Problem

A singularly perturbed boundary value problem for a linear system of n parabolic second-order delay differential equations of reaction-diffusion type with discontinuous source terms is considered in the rectangular domain Δ. Discontinuity occurs in the source terms at $(d, t) \in \Delta$. Introduce the notations $\Delta^- = (0, d) \times (0, T]$, $\overline{\Delta^-} = [0, d-] \times [0, T]$, $\Delta^+ = (d, 2) \times (0, T]$, $\overline{\Delta^+} = [d+, 2] \times [0, T]$. The corresponding boundary value problem is

$$\begin{aligned}
&\mathbf{L}\mathbf{u}(x,t) = u_t(x,t) - E\mathbf{u}_{xx}(x,t) + A(x,t)\mathbf{u}(x,t) + B(x,t)\mathbf{u}(x-1,t) = \mathbf{f}(x,t)\\
&\text{on } \Delta^- \cup \Delta^+, \mathbf{u} \text{ given on } \Gamma, \mathbf{f}(d-,t) \neq \mathbf{f}(d+,t) \text{ and}\\
&\mathbf{u}(x,t) = \boldsymbol{\chi}(x,t), (x,t) \in [-1,0] \times [0,T],
\end{aligned} \tag{1}$$

where $\Gamma = \Gamma_L \cup \Gamma_B \cup \Gamma_R$ with $\mathbf{u}(0, t) = \boldsymbol{\chi}(0, t)$ on $\Gamma_L = \{(0, t) : 0 \le t \le T\}$, $\mathbf{u}(x, 0) = \boldsymbol{\phi}_B(x)$ on $\Gamma_B = \{(x, 0) : 0 \le x \le 2\}$, and $\mathbf{u}(2, t) = \boldsymbol{\phi}_R(t)$ on $\Gamma_R = \{(2, t) : 0 \le t \le T\}$. For all $(x, t) \in \overline{\Delta}$, $\mathbf{u}(x, t) = (u_1(x), u_2(x), \dots, u_n(x))^T$ and $\mathbf{f}(x, t) = (f_1(x), f_2(x), \dots, f_n(x))^T$.$E$, $A(x, t)$ and $B(x, t)$ are $n \times n$ matrices. $E = diag(\bar{\varepsilon})$, $\bar{\varepsilon} = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ and $B(x, t) = diag(\mathbf{b}(x, t))$, $\mathbf{b}(x, t) = (b_1(x, t), b_2(x, t), \dots, b_n(x, t))$. The function $\boldsymbol{\chi}$ is sufficiently smooth on $[-1, 0] \times [0, T]$.

The cases (i) $(d, t) \in (0, 1) \times (0, T]$, (ii) $(d, t) \in (1, 2) \times (0, T]$, and (iii) $(d, t) = (1, t) \times (0, T]$ are treated separately. The problem is analogous to the problem in [2] for case (iii). This paper looks at the cases (i) and (ii).

The singular perturbation parameters satisfy

$$0 < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_n << 1$$

For all $(x,t) \in [0,2] \times [0,T]$, it is also assumed that the entries $a_{ij}(x,t)$ of $A(x,t)$ and the components $b_i(x,t)$ of $\mathbf{b}(x,t)$ satisfy

$$b_i(x,t), a_{ij}(x,t) \le 0 \ \text{ for } \ 1 \le i \ne j \le n, a_{ii}(x,t) > \sum_{i \ne j} |a_{ij}(x,t) + b_i(x,t)| \tag{2}$$

$$\text{and } \ 0 < \alpha < \min_{\substack{(x,t)\in[0,2]\times[0,T] \\ i=1,2\ldots,n}} (\sum_{j=1}^{n} a_{ij}(x) + b_i(x)), \ \text{ for some } \alpha \tag{3}$$

The functions $a_{ij}(x,t)$ and $b_i(x,t)$, $1 \le i, j \le n$ are in $C^2([0,2] \times [0,T])$.
The following notations, which will be used subsequently, are introduced. $\tilde{\Delta} = \{(0,1-) \times (0,T]\} \cup \{(1+,2) \times (0,T]\}$, $\overline{\tilde{\Delta}} = \{[0,1-] \times [0,T]\} \cup \{[1+,2] \times [0,T]\}$, $\tilde{\Delta}_1 = \{(0,d) \times (0,T]\} \cup \{(d,1-) \times (0,T]\}$, $\tilde{\Delta}_2 = \{(1+,1+d) \times (0,T]\} \cup \{(1+d,2) \times (0,T]\}$, $\tilde{\Delta}_3 = \{(1+,d) \times (0,T]\} \cup \{(d,2) \times (0,T]\}$, $\Delta_1 = \{[0,d) \times [0,T]\} \cup \{(d,1-) \times [0,T]\} \cup \{(1+,1+d) \times [0,T]\} \cup \{(1+d,2] \times [0,T]\}$, $\Delta_2 = \{[0,1-) \times [0,T]\} \cup \{(1+,d) \times [0,T]\} \cup \{(d,2] \times [0,T]\}$, $\overline{\Delta} = \Delta \cup \Gamma$. The problem (1) can be rewritten for the Case (i) as follows:

$$\mathbf{L}_1\mathbf{u}(x,t) = \mathbf{u}_t(x,t) - E\mathbf{u}_{xx}(x,t) + A(x,t)\mathbf{u}(x,t) = \mathbf{g}(x,t), \ \text{on } \tilde{\Delta}_1 \tag{4}$$

where $\mathbf{g}(x,t) = \mathbf{f}(x,t) - B(x,t)\boldsymbol{\chi}(x-1,t)$

$$\mathbf{L}_2\mathbf{u}(x,t) = \mathbf{u}_t(x,t) - E\mathbf{u}_{xx}(x,t) + A(x,t)\mathbf{u}(x,t) + B(x,t)\mathbf{u}(x-1,t) = \mathbf{f}(x,t), \text{on } \ \tilde{\Delta}_2 \tag{5}$$

and for Case (ii) as follows:

$$\mathbf{L}_1\mathbf{u}(x,t) = \mathbf{u}_t(x,t) - E\mathbf{u}_{xx}(x,t) + A(x,t)\mathbf{u}(x,t) = \mathbf{g}(x,t), \ \text{on } (0,1) \times (0,T] \tag{6}$$

where $\mathbf{g}(x,t) = \mathbf{f}(x,t) - B(x,t)\boldsymbol{\chi}(x-1,t)$

$$\mathbf{L}_2\mathbf{u}(x,t) = \mathbf{u}_t(x,t) - E\mathbf{u}_{xx}(x,t) + A(x,t)\mathbf{u}(x,t)B(x,t)\mathbf{u}(x-1,t) = \mathbf{f}(x,t), \text{on } \ \tilde{\Delta}_3 \tag{7}$$

$\mathbf{u}(x,t) = \boldsymbol{\chi}(x,t)$ on $[-1,0] \times [0,T]$, $\mathbf{f}(d-,t) \ne \mathbf{f}(d+,t)$, $\mathbf{u}(0,t) = \boldsymbol{\chi}(0,t)$,
$\mathbf{u}(d-,t) = \mathbf{u}(d+,t)$,
$\mathbf{u}_x(d-,t) = \mathbf{u}_x(d+,t)$, $\mathbf{u}(x,0) = \boldsymbol{\phi}_B(x)$ on $\Gamma_{B_1} = \{(x,0) : 0 \le x \le 1-\}$,
$\mathbf{u}(1-,t) = \mathbf{u}(1+,t)$,
$\mathbf{u}_x(1-,t) = \mathbf{u}_x(1+,t)$, $\mathbf{u}(x,0) = \boldsymbol{\phi}_B(x)$ on $\Gamma_{B_2} = \{(x,0) : 1+ \le x \le 2\}$,
$\mathbf{u}(2,t) = \boldsymbol{\phi}_R(t)$ on Γ_R.

The reduced problem corresponding to (4)–(5) is defined by

$$\mathbf{u}_{t_0}(x,t) + A(x,t)\mathbf{u}_0(x,t) = \mathbf{g}(x,t), \text{ on } \tilde{\Delta}_1 \tag{8}$$

$$\mathbf{u}_{t_0}(x,t) + A(x,t)\mathbf{u}_0(x,t) + B(x,t)\mathbf{u}_0(x-1,t) = \mathbf{f}(x,t), \text{ on } \tilde{\Delta}_2. \tag{9}$$

and the reduced problem corresponding to (6)–(7) is defined by

$$\mathbf{u}_{t_0}(x,t) + A(x,t)\mathbf{u}_0(x,t) = \mathbf{g}(x,t), \text{ on } (0,1)\times(0,T] \tag{10}$$

$$\mathbf{u}_{t_0}(x,t) + A(x,t)\mathbf{u}_0(x,t) + B(x,t)\mathbf{u}_0(x-1,t) = \mathbf{f}(x,t), \text{ on } \tilde{\Delta}_3. \tag{11}$$

The solution $\mathbf{u}$ has the following layer pattern. Each component u_i for $i = 1, 2, \ldots, n$ exhibits twin layers at $x = 0$ and $x = 2$ and twin interior layers at $x = 1$ of width $O(\sqrt{\varepsilon_n})$, while the components u_i for $i = 1, 2, \ldots, n-1$ have additional twin layers at $x = 0$ and $x = 2$ and twin interior layers at $x = 1$ of width $O(\sqrt{\varepsilon_{n-1}})$, the components u_i for $i = 1, 2, \ldots, n-2$ have additional twin boundary layers at $x = 0$ and $x = 2$ and twin interior layers at $x = 1$ of width $O(\sqrt{\varepsilon_{n-2}})$ and so on.

In addition to this, at $x = d$ and $x = 1 + d$ in Case (i) and at $x = d$ in Case (ii), the components u_i for $i = 1, 2, \ldots, n$ exhibit twin interior layers of width $O(\sqrt{\varepsilon_n})$, while the components u_i for $i = 1, 2, \ldots, n-1$ have additional twin interior layers of width $O(\sqrt{\varepsilon_{n-1}})$, the components u_i for $i = 1, 2, \ldots, n-2$ have additional twin interior layers of width $O(\sqrt{\varepsilon_{n-2}})$ and so on.

The compatibility conditions for Γ corners $(0, 0)$ and $(2, 0)$ are derived using similar arguments as in [2] Theorem 2.1.

3 Analytical Results

The proofs of most of the results in this section could be derived by using similar arguments as in paper [2] and hence proofs are omitted.

Theorem 1 *The given problem* (1) *has a solution*

$$\mathbf{u} \in \mathscr{C} = C^0_\lambda([0,2]\times[0,T]) \cap C^1_\lambda((0,2)\times(0,T]) \cap C^4_\lambda(\overline{\tilde{\Delta}}\backslash\{d\}).$$

Lemma 1 *Let conditions* (2) *and* (3) *hold. Let* $\boldsymbol{\psi}(x,t)$ *be any function in the domain of* $\mathbf{L}$ *such that* $\boldsymbol{\psi}(0,t) \geq 0$, $\boldsymbol{\psi}(2,t) \geq 0$, $\mathbf{L}_1\boldsymbol{\psi}(x,t) \geq 0$ *on* $\tilde{\Delta}_1$, $\mathbf{L}_2\boldsymbol{\psi}(x,t) \geq 0$ *on* $\tilde{\Delta}_2$ *in Case (i) and* $\mathbf{L}_1\boldsymbol{\psi}(x,t) \geq 0$ *on* $(0,1)\times(0,T]$, $\mathbf{L}_2\boldsymbol{\psi}(x,t) \geq 0$ *on* $\tilde{\Delta}_3$ *in Case (ii) and* $[\boldsymbol{\psi}](d,t) = 0$, $[\boldsymbol{\psi}](1,t) = 0$, $\left[\boldsymbol{\psi}_x\right](d,t) \leq 0$ *and* $\left[\boldsymbol{\psi}_x\right](1,t) \leq 0$ *then* $\boldsymbol{\psi}(x,t) \geq 0$ *on* $[0,2]\times[0,T]$.

The uniqueness of the analytical solution follows from the maximum principle for all $(x,t) \neq (d,t)$ and for (d,t) it follows from the continuity of the solution.

Lemma 2 *Let conditions* (2) *and* (3) *hold. Let* $\boldsymbol{\psi}$ *be any function in the domain of* $\mathbf{L}$ *such that* $[\boldsymbol{\psi}](1,t)=0$, $[\boldsymbol{\psi}](d,t)=0$, $\left[\boldsymbol{\psi}_x\right](1,t)=0$ *and* $\left[\boldsymbol{\psi}_x\right](d,t)=0$ *in cases (i) and (ii), then for all* $(x,t)\in[0,2]\times[0,T]$,

$$|\boldsymbol{\psi}(x,t)|\le\max\left\{||\boldsymbol{\psi}||_{\Gamma},\ \frac{1}{\alpha}||\mathbf{f}||_{\Delta^-\cup\Delta^+}\right\}+|[\mathbf{f}](d,t)|$$

Standard estimates of the solution of (1) and its derivatives are contained in the following lemma:

Lemma 3 *Let conditions* (2) *and* (3) *hold. Let* $\mathbf{u}$ *be the solution of* (1)*. Then, in Case (i), for all* $(x,t)\in\overline{\tilde{\Delta}}\backslash\{(d,t),(1+d,t)\}$,

$$\begin{aligned}
|\mathbf{u}_t^k(x,t)| &\le C(||\mathbf{u}||_{\Gamma}+\textstyle\sum_{q=0}^{k}||\mathbf{f}_t^q||),\ k=0,1,2\\
|\mathbf{u}_x^k(x,t)| &\le CE^{\frac{-k}{2}}(||\mathbf{u}||_{\Gamma}+||\mathbf{f}||_{\Delta_1}+||\mathbf{f}_t||_{\Delta_1}),\ k=1,2\\
|\mathbf{u}_x^k(x,t)| &\le CE^{\frac{-k}{2}}(||\mathbf{u}||_{\Gamma}+||\mathbf{f}||_{\Delta_1}+||\mathbf{f}_t||_{\Delta_1}+||\mathbf{f}_t^2||_{\Delta_1}+E^{\frac{k-2}{2}}||\mathbf{f}_x^{k-2}||_{\Delta_1}),\\
&\qquad k=3,4\\
|\mathbf{u}_x^{k-1}(x,t)\mathbf{u}_t(x,t)| &\le CE^{\frac{1-k}{2}}(||\mathbf{u}||_{\Gamma}+||\mathbf{f}||_{\Delta_1}+||\mathbf{f}_t||_{\Delta_1}+||\mathbf{f}_t^2||_{\Delta_1}),\ k=2,3.
\end{aligned}$$

and in Case (ii), for all $(x,t)\in\overline{\tilde{\Delta}}\backslash\{(d,t)\}$,

$$\begin{aligned}
|\mathbf{u}_t^k(x,t)| &\le C(||\mathbf{u}||_{\Gamma}+\textstyle\sum_{q=0}^{k}||\mathbf{f}_t^q||),\ k=0,1,2\\
|\mathbf{u}_x^k(x,t)| &\le CE^{\frac{-k}{2}}(||\mathbf{u}||_{\Gamma}+||\mathbf{f}||_{\Delta_2}+||\mathbf{f}_t||_{\Delta_2}),\ k=1,2\\
|\mathbf{u}_x^k(x,t)| &\le CE^{\frac{-k}{2}}(||\mathbf{u}||_{\Gamma}+||\mathbf{f}||_{\Delta_2}+||\mathbf{f}_t||_{\Delta_2}+||\mathbf{f}_{tt}||_{\Delta_2}+E^{\frac{k-2}{2}}||\mathbf{f}_x^{k-2}||_{\Delta_2}),\\
&\qquad k=3,4\\
|\mathbf{u}_x^{k-1}(x,t)\mathbf{u}_t(x,t)| &\le CE^{\frac{1-k}{2}}(||\mathbf{u}||_{\Gamma}+||\mathbf{f}||_{\Delta_2}+||\mathbf{f}_t||_{\Delta_2}+||\mathbf{f}_{tt}||_{\Delta_2}),\ k=2,3.
\end{aligned}$$

The Shishkin decomposition of the solution $\mathbf{u}$ of (1) is $\mathbf{u}=\mathbf{v}+\mathbf{w}$, where the smooth component $\mathbf{v}$ is the solution of

$$\begin{aligned}
&\mathbf{L}_1\mathbf{v}=\mathbf{g}\text{ on }(0,d-)\times(0,T],\ \mathbf{v}(0,t)=\mathbf{u}_0(0,t),\ \mathbf{v}(x,0)=\boldsymbol{\phi}_B(x),\ \mathbf{v}(d-,t)=\mathbf{u}_0(d-,t)\\
&\mathbf{L}_1\mathbf{v}=\mathbf{g}\text{ on }(d+,1-)\times(0,T],\ \mathbf{v}(d+,t)=\mathbf{u}_0(d+,t),\ \mathbf{v}(x,0)=\boldsymbol{\phi}_B(x),\\
&\qquad\mathbf{v}(1-,t)=\mathbf{u}_0(1-,t)\\
&\mathbf{L}_2\mathbf{v}=\mathbf{f}\text{ on }(1+,2)\times(0,T],\ \mathbf{v}(1+,t)=\mathbf{u}_0(1+,t),\ \mathbf{v}(x,0)=\boldsymbol{\phi}_B(x),\ \mathbf{v}(2,t)=\mathbf{u}_0(2,t)
\end{aligned}$$

and the singular component $\mathbf{w}$ is the solution of

$$\begin{aligned}
&\mathbf{L}_1\mathbf{w}=0\text{ on }\tilde{\Delta}_1,\quad \mathbf{L}_2\mathbf{w}=0\text{ on }(1,2)\times(0,T]\\
&\text{with }\mathbf{w}(0,t)=\mathbf{u}(0,t)-\mathbf{v}(0,t),\ [\mathbf{w}](1,t)=-[\mathbf{v}](1,t),\ [\mathbf{w}](d,t)=-[\mathbf{v}](d,t),\\
&\qquad\mathbf{w}(x,0)=0,\\
&[\mathbf{w}_x](1,t)=-[\mathbf{v}_x](1,t),\ [\mathbf{w}_x](d,t)=-[\mathbf{v}_x](d,t),\ \mathbf{w}(2,t)=\mathbf{u}(2,t)-\mathbf{v}(2,t).
\end{aligned}$$

for Case (i) and the smooth component $\mathbf{v}$ is the solution of

$$\begin{aligned}
&\mathbf{L_1 v} = \mathbf{g} \text{ on } (0, 1-) \times (0, T], \mathbf{v}(0, t) = \mathbf{u}_0(0, t), \mathbf{v}(x, 0) = \boldsymbol{\phi}_B(x), \mathbf{v}(1-, t) = \mathbf{u}_0(1-, t)\\
&\mathbf{L_2 v} = \mathbf{f} \text{ on } (1+, d-) \times (0, T], \mathbf{v}(1+, t) = \mathbf{u}_0(1+, t), \mathbf{v}(x, 0) = \boldsymbol{\phi}_B(x),\\
&\qquad\qquad \mathbf{v}(d-, t) = \mathbf{u}_0(d-, t)\\
&\mathbf{L_2 v} = \mathbf{f} \text{ on } (d+, 2) \times (0, T], \mathbf{v}(d+, t) = \mathbf{u}_0(d+, t), \mathbf{v}(x, 0) = \boldsymbol{\phi}_B(x), \mathbf{v}(2, t) = \mathbf{u}_0(2, t)
\end{aligned}$$

and the singular component $\mathbf{w}$ is the solution of

$$\begin{aligned}
&\mathbf{L_1 w} = 0 \text{ on } (0, 1) \times (0, T], \quad \mathbf{L_2 w} = 0 \text{ on } \tilde{\Delta}_3\\
&\text{with } \mathbf{w}(0, t) = \mathbf{u}(0, t) - \mathbf{v}(0, t), [\mathbf{w}](1, t) = -[\mathbf{v}](1, t), [\mathbf{w}](d, t) = -[\mathbf{v}](d, t),\\
&\qquad\qquad \mathbf{w}(x, 0) = 0,\\
&[\mathbf{w}_x](1, t) = -[\mathbf{v}_x](1, t), [\mathbf{w}_x](d, t) = -[\mathbf{v}_x](d, t), \mathbf{w}(2, t) = \mathbf{u}(2, t) - \mathbf{v}(2, t).
\end{aligned}$$

for Case (ii).
The singular component is given a further decomposition, for Case (i),

$$\mathbf{w}(x, t) = \tilde{\mathbf{w}}_1(x, t) + \tilde{\mathbf{w}}_2(x, t) + \hat{\mathbf{w}}(x, t)$$

where $\tilde{\mathbf{w}}_1$ is the solution of

$$\begin{aligned}
&\tilde{\mathbf{w}}_{1_t}(x, t) - E\tilde{\mathbf{w}}_{1_{xx}}(x, t) + A(x, t)\tilde{\mathbf{w}}_1(x, t) = 0 \text{ on } (0, d) \times (0, T],\\
&\tilde{\mathbf{w}}_1(x, 0) = 0, \tilde{\mathbf{w}}_1(0, t) = \mathbf{w}(0, t), \tilde{\mathbf{w}}_1(d, t) = K_1,\\
&\tilde{\mathbf{w}}_1(x, t) = 0 \text{ on } (d, 2) \times (0, T]
\end{aligned}$$

$\tilde{\mathbf{w}}_2$ is the solution of

$$\begin{aligned}
&\tilde{\mathbf{w}}_{2_t}(x, t) - E\tilde{\mathbf{w}}_{2_{xx}}(x, t) + A(x, t)\tilde{\mathbf{w}}_2(x, t) = 0 \text{ on } (d, 1) \times (0, T],\\
&\tilde{\mathbf{w}}_2(x, 0) = 0, \tilde{\mathbf{w}}_2(d, t) = K_2, \tilde{\mathbf{w}}_2(1, t) = K_3,\\
&\tilde{\mathbf{w}}_2(x, t) = 0 \text{ on } ((0, d) \cup (1, 2)) \times (0, T]
\end{aligned}$$

and $\hat{\mathbf{w}}$ is the solution of

$$\begin{aligned}
&\hat{\mathbf{w}}_t(x, t) - E\hat{\mathbf{w}}_{xx}(x, t) + A(x, t)\hat{\mathbf{w}}(x, t) + B(x, t)\hat{\mathbf{w}}(x - 1, t) = 0 \text{ on } (1, 2) \times (0, T],\\
&\hat{\mathbf{w}}(x, 0) = 0, \hat{\mathbf{w}}(1, t) = K_4, \hat{\mathbf{w}}(2, t) = \mathbf{w}(2, t),\\
&\hat{\mathbf{w}}(x, t) = 0 \text{ on } (0, 1) \times (0, T]
\end{aligned}$$

Here, K_1, K_2, K_3, and K_4 are constants to be chosen in such a way that the jump conditions at $x = d$ and $x = 1$ are satisfied.
For Case (ii)

$$\mathbf{w}(x, t) = \tilde{\mathbf{w}}(x, t) + \hat{\mathbf{w}}_1(x, t) + \hat{\mathbf{w}}_2(x, t)$$

where $\tilde{\mathbf{w}}$ is the solution of

$$\tilde{\mathbf{w}}_t(x,t) - E\tilde{\mathbf{w}}_{xx}(x,t) + A(x,t)\tilde{\mathbf{w}}(x,t) = 0 \text{ on } (0,1)\times(0,T],$$
$$\tilde{\mathbf{w}}(x,0) = 0,\ \tilde{\mathbf{w}}(0,t) = \mathbf{w}(0,t),\ \tilde{\mathbf{w}}(1,t) = K_5,$$
$$\tilde{\mathbf{w}}(x,t) = 0 \text{ on } (1,2)\times(0,T]$$

$\hat{\mathbf{w}}_1$ is the solution of

$$\hat{\mathbf{w}}_{1_t}(x,t) - E\hat{\mathbf{w}}_{1_{xx}}(x,t) + A(x,t)\hat{\mathbf{w}}_1(x,t) + B(x,t)\hat{\mathbf{w}}_1(x-1,t) = 0 \quad \text{on } (1,d)\times(0,T],$$
$$\hat{\mathbf{w}}_1(x,0) = 0,\ \hat{\mathbf{w}}_1(1,t) = K_6,\ \hat{\mathbf{w}}_1(d,t) = K_7,$$
$$\hat{\mathbf{w}}_1(x,t) = 0 \text{ on } ((0,1)\cup(d,2))\times(0,T]$$

and $\hat{\mathbf{w}}_2$ is the solution of

$$\hat{\mathbf{w}}_{2_t}(x,t) - E\hat{\mathbf{w}}_{2_{xx}}(x,t) + A(x,t)\hat{\mathbf{w}}_2(x,t) + B(x,t)\hat{\mathbf{w}}_2(x-1,t) = 0 \quad \text{on } (d,2)\times(0,T],$$
$$\hat{\mathbf{w}}_2(x,0) = 0,\ \hat{\mathbf{w}}_2(d,t) = K_8,\ \hat{\mathbf{w}}_2(2,t) = \mathbf{w}(2,t),$$
$$\hat{\mathbf{w}}_2(x,t) = 0 \text{ on } (0,d)\times(0,T]$$

Here, K_5, K_6, K_7 and K_8 are constants to be chosen in such a way that the jump conditions at $x = 1$ and $x = d$ are satisfied.
In Case (i) and (ii), the bounds on the smooth component $\mathbf{v}$ of $\mathbf{u}$ and its derivatives are contained in the following.

Lemma 4 *Let conditions* (2) *and* (3) *hold. Then the smooth component* $\mathbf{v}$ *and its derivatives satisfy, for all* $(x,t) \in \overline{\tilde{\Delta}}\backslash\{(d,t),(1+d,t)\}$*, for Case (i) and* $(x,t) \in \overline{\tilde{\Delta}}\backslash\{(d,t)\}$ *for Case (ii),*
$|\mathbf{v}_t^k(x,t)| \le C,$ *for* $k = 0,1,2$
$|\mathbf{v}_x^k(x,t)| \le C(1 + E^{1-\frac{k}{2}}),$ *for* $k = 0,1,2,3$
$|\mathbf{v}_x^{k-1}(x,t)\mathbf{v}_t^k(x,t)| \le C,$ *for* $k = 2,3.$

The layer functions $B^l_{1,i}, B^r_{1,i}, B^l_{2,i}, B^r_{2,i}, B^l_{3,i}, B^r_{3,i}, B^l_{4,i}, B^r_{4,i}, B_{1,i}, B_{2,i}, B_{3,i}, B_{4,i}$, $i = 1,2,\ldots,n$, associated with the solution $\mathbf{u}$ of Case (i), are defined by

$$B^l_{1,i}(x) = e^{-x\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},\ B^r_{1,i}(x) = e^{-(d-x)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},\ B_{1,i}(x) = B^l_{1,i}(x) + B^r_{1,i}(x), \text{ on } [0,d]\times[0,T],$$
$$B^l_{2,i}(x) = e^{-(x-d)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},\ B^r_{2,i}(x) = e^{-(1-x)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},\ B_{2,i}(x) = B^l_{2,i}(x) + B^r_{2,i}(x), \quad \text{on } [d,2]\times[0,T],$$
$$B^l_{3,i}(x) = e^{-(x-1)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},\ B^r_{3,i}(x) = e^{-(1+d-x)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},\ B_{3,i}(x) = B^l_{3,i}(x) + B^r_{3,i}(x), \quad \text{on } [1,1+d]\times[0,T],$$
$$B^l_{4,i}(x) = e^{-(x-(1+d))\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},\ B^r_{4,i}(x) = e^{-(2-x)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},\ B_{4,i}(x) = B^l_{4,i}(x) + B^r_{4,i}(x), \quad \text{on } [1+d,2]\times[0,T].$$

The layer functions $B^l_{1,i}$, $B^r_{1,i}$, $B^l_{2,i}$, $B^r_{2,i}$, $B^l_{3,i}$, $B^r_{3,i}$, $B_{1,i}$, $B_{2,i}$, $B_{3,i}$, associated with the solution $\mathbf{u}$ of Case (ii), are defined by

$$\begin{aligned}
&B^l_{1,i}(x) = e^{-x\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},\ B^r_{1,i}(x) = e^{-(1-x)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},\ B_{1,i}(x) = B^l_{1,i}(x) + B^r_{1,i}(x),\ \text{on } [0,1]\times[0,T],\\
&B^l_{2,i}(x) = e^{-(x-1)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},\ B^r_{2,i}(x) = e^{-(d-x)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},\ B_{2,i}(x) = B^l_{2,i}(x) + B^r_{2,i}(x),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } [1,d]\times[0,T],\\
&B^l_{3,i}(x) = e^{-(x-d)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon_i}}},\ B^r_{3,i}(x) = e^{-(2-x)\frac{\sqrt{\alpha}}{\sqrt{\varepsilon}}},\ B_{3,i}(x) = B^l_{3,i}(x) + B^r_{3,i}(x),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } [d,2]\times[0,T].
\end{aligned}$$

The existence, uniqueness and the properties of $x^{(s)}_{i,j}$ can be verified as in [3]. In cases (i) and (ii), the bounds on the singular component $\mathbf{w}$ of $\mathbf{u}$ and its derivatives are contained in the following.

Lemma 5 *Let conditions* (2) *and* (3) *hold. Then there exists a constant* C, *such that, for* $i = 1, 2, \ldots, n$ *and for* $k = 0, 1, 2, 3$,

$$\left.\begin{aligned}\left|\mathbf{w}^k_t(x,t)\right| &\le CB_{1,i}(x),\\ \left|\mathbf{w}^k_x(x,t)\right| &\le CE^{\frac{-k}{2}}B_{1,i}(x)\end{aligned}\right\}\ \textit{for } (x,t)\in(0,d)\times(0,T]$$

$$\left.\begin{aligned}\left|\mathbf{w}^k_t(x,t)\right| &\le CB_{2,i}(x),\\ \left|\mathbf{w}^k_x(x,t)\right| &\le CE^{\frac{-k}{2}}B_{2,i}(x)\end{aligned}\right\}\ \textit{for } (x,t)\in(d,1)\times(0,T]$$

$$\left.\begin{aligned}\left|\mathbf{w}^k_t(x,t)\right| &\le CB_{3,i}(x),\\ \left|\mathbf{w}^k_x(x,t)\right| &\le CE^{\frac{-k}{2}}B_{3,i}(x)\end{aligned}\right\}\ \textit{for } (x,t)\in(1,1+d)\times(0,T]$$

$$\left.\begin{aligned}\left|\mathbf{w}^k_t(x,t)\right| &\le CB_{4,i}(x),\\ \left|\mathbf{w}^k_x(x,t)\right| &\le CE^{\frac{-k}{2}}B_{4,i}(x)\end{aligned}\right\}\ \textit{for } (x,t)\in(1+d,2)\times(0,T]$$

in Case (i) and

$$\left.\begin{aligned}\left|\mathbf{w}^k_t(x,t)\right| &\le CB_{1,i}(x),\\ \left|\mathbf{w}^k_x(x,t)\right| &\le CE^{\frac{-k}{2}}B_{1,i}(x)\end{aligned}\right\}\ \textit{for } (x,t)\in(0,1)\times(0,T]$$

$$\left.\begin{aligned}\left|\mathbf{w}^k_t(x,t)\right| &\le CB_{2,i}(x),\\ \left|\mathbf{w}^k_x(x,t)\right| &\le CE^{\frac{-k}{2}}B_{2,i}(x)\end{aligned}\right\}\ \textit{for } (x,t)\in(1,d)\times(0,T]$$

$$\left.\begin{aligned}\left|\mathbf{w}^k_t(x,t)\right| &\le CB_{3,i}(x),\\ \left|\mathbf{w}^k_x(x,t)\right| &\le CE^{\frac{-k}{2}}B_{3,i}(x)\end{aligned}\right\}\ \textit{for } (x,t)\in(d,2)\times(0,T]$$

in Case (ii).

4 Improved Estimates

In the following lemma, sharper estimates of the smooth component are presented.

Lemma 6 *Let conditions* (2) *and* (3) *hold. Then the smooth component* $\mathbf{v}$ *of the solution* $\mathbf{u}$ *of* (1) *satisfies,*

$$|\mathbf{v}^k_x(x,t)| \le C\,(1 + B_{1,2}(x)),\ \textit{for } k = 0,1,2\ \ \textit{and}\ \ |\mathbf{v}^3_x(x,t)| \le C\left(1 + \sum_{q=i}^{n}\frac{B_{1,q}(x)}{\sqrt{\varepsilon_q}}\right)$$

on $(0,d)\times(0,T]$,

$|\mathbf{v}_x^k(x,t)| \leq C\,(1+B_{2,2}(x)),\ for\ k=0,1,2\ \ and\ \ |\mathbf{v}_x^3(x,t)| \leq C\left(1+\sum_{q=i}^{n}\frac{B_{2,q}(x)}{\sqrt{\varepsilon_q}}\right)$
on $(d,1)\times(0,T]$,

$|\mathbf{v}_x^k(x,t)| \leq C\,(1+B_{3,2}(x)),\ for\ k=0,1,2\ \ and\ \ |\mathbf{v}_x^3(x,t)| \leq C\left(1+\sum_{q=i}^{n}\frac{B_{3,q}(x)}{\sqrt{\varepsilon_q}}\right)$
on $(1,1+d)\times(0,T]$,

$|\mathbf{v}_x^k(x,t)| \leq C\,(1+B_{4,2}(x)),\ for\ k=0,1,2\ \ and\ \ |\mathbf{v}_x^3(x,t)| \leq C\left(1+\sum_{q=i}^{n}\frac{B_{4,q}(x)}{\sqrt{\varepsilon_q}}\right)$
on $(1+d,2)\times(0,T]$,
in Case (i) and

$|\mathbf{v}_x^k(x,t)| \leq C\,(1+B_{1,2}(x)),\ for\ k=0,1,2\ \ and\ \ |\mathbf{v}_x^3(x,t)| \leq C\left(1+\sum_{q=i}^{n}\frac{B_{1,q}(x)}{\sqrt{\varepsilon_q}}\right)$
on $(0,1)\times(0,T]$,

$|\mathbf{v}_x^k(x,t)| \leq C\,(1+B_{2,2}(x)),\ for\ k=0,1,2\ \ and\ \ |\mathbf{v}_x^3(x,t)| \leq C\left(1+\sum_{q=i}^{n}\frac{B_{2,q}(x)}{\sqrt{\varepsilon_q}}\right)$
on $(1,d)\times(0,T]$,

$|\mathbf{v}_x^k(x,t)| \leq C\,(1+B_{3,2}(x)),\ for\ k=0,1,2\ \ and\ \ |\mathbf{v}_x^3(x,t)| \leq C\left(1+\sum_{q=i}^{n}\frac{B_{3,q}(x)}{\sqrt{\varepsilon_q}}\right)$
on $(d,2)\times(0,T]$,
in Case (ii).

5 The Shishkin Mesh

Case (i):
A piecewise-uniform Shishkin mesh with $M \times N$ mesh-intervals is now constructed. Let $\varDelta_t^M = \{t_k\}_{k=1}^M$, $\overline{\varDelta}_t^M = \{t_k\}_{k=0}^M$, $\varDelta_x^N = \varDelta_{x_1}^N \cup \varDelta_{x_2}^N \cup \varDelta_{x_3}^N$ where $\varDelta_{x_1}^N = \{x_j\}_{j=1}^{\frac{N}{4}-1}$, $\varDelta_{x_2}^N = \{x_j\}_{j=\frac{N}{4}+1}^{\frac{N}{2}-1}$, $\varDelta_{x_3}^N = \{x_j\}_{j=\frac{N}{2}+1}^{N-1}$, $x_{\frac{N}{4}} = d$ and $x_{\frac{N}{2}} = 1$. Then $\overline{\varDelta}_{x_1}^N = \{x_j\}_{j=0}^{\frac{N}{4}}$, $\overline{\varDelta}_{x_2}^N = \{x_j\}_{j=\frac{N}{4}}^{\frac{N}{2}}$, $\overline{\varDelta}_{x_3}^N = \{x_j\}_{j=\frac{N}{2}}^{N}$, $\overline{\varDelta}_{x_1}^N \cup \overline{\varDelta}_{x_2}^N \cup \overline{\varDelta}_{x_3}^N = \overline{\varDelta}_x^N = \{x_j\}_{x=0}^N$, $\varDelta^{M,N} = \varDelta_t^M \times \varDelta_x^N$, $\overline{\varDelta}^{M,N} = \overline{\varDelta}_t^M \times \overline{\varDelta}_x^N$, $\varDelta_1^{M,N} = \varDelta_t^M \times \varDelta_{x_1}^N$, $\overline{\varDelta}_1^{M,N} = \overline{\varDelta}_t^M \times \overline{\varDelta}_{x_1}^N$, $\varDelta_2^{M,N} = \varDelta_t^M \times \varDelta_{x_2}^N$, $\overline{\varDelta}_2^{M,N} = \overline{\varDelta}_t^M \times \overline{\varDelta}_{x_2}^N$, $\varDelta_3^{M,N} = \varDelta_t^M \times \varDelta_{x_3}^N$, $\overline{\varDelta}_3^{M,N} = \overline{\varDelta}_t^M \times \overline{\varDelta}_{x_3}^N$, $\overline{\varDelta}^{M,N} = \overline{\varDelta}_1^{M,N} \cup \overline{\varDelta}_2^{M,N} \cup \overline{\varDelta}_3^{M,N}$ and $\varGamma^{M,N} = \varGamma \cap \overline{\varDelta}^{M,N}$.
The mesh $\overline{\varDelta}_t^M$ is chosen to be a uniform mesh with M mesh-intervals on $[0,T]$. The mesh $\overline{\varDelta}_x^N$ is chosen to be a piecewise-uniform mesh with N mesh-intervals on $[0,2]$. The interval $[0,d]$ is divided into $2n+1$ sub-intervals as follows $[0,\tau_1] \cup (\tau_1,\tau_2] \cup \cdots \cup (\tau_{n-1},\tau_n] \cup (\tau_n, d-\tau_n] \cup (d-\tau_n, d-\tau_{n-1}] \cup \cdots \cup (d-\tau_2, d-\tau_1] \cup (d-$

$\tau_1, d]$. The parameters $\tau_r, r = 1, 2 \ldots, n$, which determine the points separating the uniform meshes, are defined by $\tau_0, \tau_{n+1} = \frac{d}{2}$,

$$\tau_n = \min\left\{\frac{d}{4}, \frac{2\sqrt{\varepsilon_n}}{\sqrt{\alpha}} \ln N\right\} \text{and for } k = 1, 2, \ldots, n-1, \tau_k = \min\left\{\frac{k\tau_{k+1}}{k+1}, \frac{2\sqrt{\varepsilon_k}}{\sqrt{\alpha}} \ln N\right\}. \tag{12}$$

Then, on the sub-interval $(\tau_n, d - \tau_n]$ a uniform mesh with $\frac{N}{4}$ mesh points is placed and on each of the sub-intervals $[0, \tau_1]$, $(d - \tau_1, d]$, $(\tau_k, \tau_{k+1}]$, and$(d - \tau_{k+1}, d - \tau_k]$, $k = 1, \ldots, n-1$, a uniform mesh of $\frac{N}{16n}$ mesh points is placed. Similarly, the interval $(d, 1]$ is divided into $2n + 1$ sub-intervals $(d, d + \eta_1]$, $(d + \eta_1, d + \eta_2]$, ..., $(d + \eta_{n-1}, d + \eta_n]$, $(d + \eta_n, 1 - \eta_n]$, ..., $(d + \eta_2, 1 - \eta_2]$, $(1 - \eta_2, 1 - \eta_1]$ and $(1 - \eta_1, 1]$.

The parameters $\eta_r, r = 1, ..., n$, which determine the points separating the uniform meshes, are defined by $\eta_0 = d$, $\eta_{n+1} = \frac{1-d}{2}$,

$$\eta_n = \min\left\{\frac{1-d}{4}, \frac{2\sqrt{\varepsilon_n}}{\sqrt{\alpha}} \ln N\right\} \text{and for } k = 1, \ldots, n-1, \eta_k = \min\left\{\frac{k\,\eta_{k+1}}{k+1}, \frac{2\sqrt{\varepsilon_k}}{\sqrt{\alpha}} \ln N\right\}. \tag{13}$$

The interval $(1, 1 + d]$ is divided into $2n + 1$ sub-intervals $(1, 1 + \tau_1]$, $(1 + \tau_1, 1 + \tau_2]$, ..., $(1 + \tau_{n-1}, 1 + \tau_n]$, $(1 + \tau_n, 1 + d - \tau_n]$, $(1 + d - \tau_2, 1 + d - \tau_1]$ and $(1 + d - \tau_1, 1 + d]$ and the interval $(1 + d, 2]$ is divided into $2n + 1$ sub-intervals $(1 + d, 1 + d + \eta_1]$, $(1 + d + \eta_1, 1 + d + \eta_2]$, ..., $(1 + d + \eta_{n-1}, 1 + d + \eta_n]$, $(1 + d + \eta_n, 2 - \eta_n]$, ..., $(2 - \eta_2, 2 - \eta_1]$ and $(2 - \eta_1, 2]$ having the same mesh pattern as in [0, 1].

In practice, it is convenient to take $N = 32k,\ \ k \geq 2$.

From the above construction of $\overline{\Delta}^N$, it is clear that the transition points $\{\tau_r, d - \tau_r, d + \eta_r, 1 - \eta_r,\ 1 + \tau_r, 1 + d - \tau_r,\ 1 + d + \eta_r, 2 - \eta_r\}$, $r = 1, ..., n$, are the only points at which the mesh-size can change and that it does not necessarily change at each of these points. The following notations are introduced: $h_j = x_j - x_{j-1}$, $h_{j+1} = x_{j+1} - x_j$ and if x_j is a transition point, then $h_j^- = x_j - x_{j-1}$, $h_j^+ = x_{j+1} - x_j$, $J = \{x_j : h_j^+ \neq h_j^-\}$.

Case (ii):

In this case, a piecewise-uniform Shishkin mesh with $M \times N$ mesh-intervals is now constructed.

Let $\Delta_t^M = \{t_k\}_{k=1}^M$, $\overline{\Delta}_t^M = \{t_k\}_{k=0}^M$, $\Delta_x^N = \Delta_{x_1}^N \cup \Delta_{x_2}^N \cup \Delta_{x_3}^N$ where $\Delta_{x_1}^N = \{x_j\}_{j=1}^{\frac{N}{3}-1}$, $\Delta_{x_2}^N = \{x_j\}_{j=\frac{N}{3}+1}^{\frac{2N}{3}-1}$, $\Delta_{x_3}^N = \{x_j\}_{j=\frac{2N}{3}+1}^{N-1}$, $x_{\frac{N}{3}} = 1$ and $x_{\frac{2N}{3}} = d$. Then $\overline{\Delta}_{x_1}^N = \{x_j\}_{j=0}^{\frac{N}{3}}$, $\overline{\Delta}_{x_2}^N = \{x_j\}_{j=\frac{N}{3}}^{\frac{2N}{3}}$, $\overline{\Delta}_{x_3}^N = \{x_j\}_{j=\frac{2N}{3}}^N$, $\overline{\Delta}_{x_1}^N \cup \overline{\Delta}_{x_2}^N \cup \overline{\Delta}_{x_3}^N = \overline{\Delta}_x^N = \{x_j\}_{x=0}^N$,
$\Delta^{M,N} = \Delta_t^M \times \Delta_x^N$, $\overline{\Delta}^{M,N} = \overline{\Delta}_t^M \times \overline{\Delta}_x^N$, $\Delta_1^{M,N} = \Delta_t^M \times \Delta_{x_1}^N$, $\overline{\Delta}_1^{M,N} = \overline{\Delta}_t^M \times \overline{\Delta}_{x_1}^N$, $\Delta_2^{M,N} = \Delta_t^M \times \Delta_{x_2}^N$, $\overline{\Delta}_2^{M,N} = \overline{\Delta}_t^M \times \overline{\Delta}_{x_2}^N$, $\Delta_3^{M,N} = \Delta_t^M \times \Delta_{x_3}^N$, $\overline{\Delta}_3^{M,N} = \overline{\Delta}_t^M \times \overline{\Delta}_{x_3}^N$, $\overline{\Delta}^{M,N} = \overline{\Delta}_1^{M,N} \cup \overline{\Delta}_2^{M,N} \cup \overline{\Delta}_3^{M,N}$ and $\Gamma^{M,N} = \Gamma \cap \overline{\Delta}^{M,N}$.

The interval [0, 1] is subdivided into $2n + 1$ sub-intervals as follows

$$[0, \tau_1] \cup (\tau_1, \tau_2] \cup ... \cup (\tau_{n-1}, \tau_n] \cup (\tau_n, 1 - \tau_n] \cup ... \cup (1 - \tau_2, 1 - \tau_1] \cup (1 - \tau_1, 1].$$

The parameters $\tau_r, r = 1, ..., n$, which determine the points separating the uniform meshes, are defined by $\tau_0 = 0,\ \tau_{n+1} = \frac{1}{2}$,

$$\tau_n = \min\left\{\frac{1}{4}, \frac{2\sqrt{\varepsilon_n}}{\sqrt{\alpha}} \ln N\right\} \text{ and for} k = 1, .., n-1, \tau_k = \min\left\{\frac{k\,\tau_{k+1}}{k+1}, \frac{2\sqrt{\varepsilon_k}}{\sqrt{\alpha}} \ln N\right\}. \quad (14)$$

On the sub-interval $(\tau_n, 1-\tau_n]$ a uniform mesh with $\dfrac{N}{6}$ mesh points is placed and on each of the sub-intervals $[0, \tau_1],\ (1-\tau_1, 1],\ (\tau_k, \tau_{k+1}]$ and $(1-\tau_{k+1}, 1-\tau_k],\ k = 1, ..., n-1$, a uniform mesh of $\dfrac{N}{12n}$ mesh points is placed.
The interval $(1, d]$ is divided into $2n+1$ sub-intervals $(1, 1+\sigma_1],\ (1+\sigma_1, 1+\sigma_2], ...,\ (1+\sigma_{n-1}, 1+\sigma_n],\ (1+\sigma_n, d-\sigma_n],\ (d-\sigma_2, d-\sigma_1]$ and $(d-\sigma_1, d]$.
The parameters $\sigma_r, r = 1, ..., n$, which determine the points separating the uniform meshes, are defined by $\sigma_0 = 1,\ \sigma_{n+1} = \frac{1+d}{2}$,

$$\sigma_n = \min\left\{\frac{d-1}{4}, \frac{2\sqrt{\varepsilon_n}}{\sqrt{\alpha}} \ln N\right\} \text{ and for} k = 1, .., n-1, \sigma_k = \min\left\{\frac{k\,\sigma_{k+1}}{k+1}, \frac{2\sqrt{\varepsilon_k}}{\sqrt{\alpha}} \ln N\right\}. \quad (15)$$

And the interval $(d, 2]$ is divided into $2n+1$ sub-intervals $(d, d+\gamma_1], (d+\gamma_1, d+\gamma_2], ..., (d+\gamma_{n-1}, d+\gamma_n], (d+\gamma_n, 2-\gamma_n], ..., (2-\gamma_2, 2-\gamma_1]$ and $(2-\gamma_1, 2]$.
The parameters $\gamma_r, r = 1, ..., n$, which determine the points separating the uniform meshes, are defined by $\gamma_0 = d,\ \gamma_{n+1} = \frac{2+d}{2}$,

$$\gamma_n = \min\left\{\frac{2-d}{4}, \frac{2\sqrt{\varepsilon_n}}{\sqrt{\alpha}} \ln N\right\} \text{ and for } k = 1, ..., n-1, \gamma_k = \min\left\{\frac{k\,\gamma_{k+1}}{k+1}, \frac{2\sqrt{\varepsilon_k}}{\sqrt{\alpha}} \ln N\right\}. \quad (16)$$

In practice, it is convenient to take $N = 24k,\ \ k \geq 2$.
From the above construction of $\overline{\Delta}^N$, it is clear that the transition points $\{\tau_r, 1-\tau_r, 1+\sigma_r, d-\sigma_r, d+\gamma_r, 2-\gamma_r\}, r = 1, ..., n$, are the only points at which the mesh-size can change and that it does not necessarily change at each of these points. The following notations are introduced: $h_j = x_j - x_{j-1}, h_{j+1} = x_{j+1} - x_j$ and if x_j is a transition point, then $h_j^- = x_j - x_{j-1},\ h_j^+ = x_{j+1} - x_j, J = \{x_j : h_j^+ \neq h_j^-\}$.

6 The Discrete Problem

In this section, a numerical method for (1) is constructed using a classical finite difference operator and an appropriate Shishkin mesh, which is later proven to be first-order parameter-uniform in time and essentially first-order parameter-uniform in the space variable.

The finite difference method can now solve the discrete initial boundary value problem on any mesh.

$$\begin{aligned}\mathbf{L}^{M,N}\mathbf{U}(x_j,t_k) &= D_t^-\mathbf{U}(x_j,t_k) - E\delta_x^2\mathbf{U}(x_j,t_k) + A(x_j,t_k)\mathbf{U}(x_j,t_k)\\ +B(x_j,t_k)\mathbf{U}(x_j-1,t_k) &= \mathbf{f}(x_j,t_k) \ \text{ on } \ \Delta^{M,N}, 0\le j\le N, 0\le k\le M\end{aligned} \tag{17}$$

$\mathbf{U}=\mathbf{u}$ on $\Gamma^{M,N}$ and $\mathbf{U}(x_j-1,t_k)=\boldsymbol{\phi}(x_j-1,t_k)$ for $(x_j,t_k)\in\Delta_1^{M,N}\cup\Delta_2^{M,N}$ in Case (i) and for $(x_j,t_k)\in\Delta_1^{M,N}$ in Case (ii).
The problem (17) can be rewritten as,

$$\begin{aligned}\mathbf{L}_1^{M,N}\mathbf{U}(x_j,t_k) = D_t^-\mathbf{U}(x_j,t_k) - E\delta_x^2\mathbf{U}(x_j,t_k) + A(x_j,t_k)\mathbf{U}(x_j,t_k)\\ =\mathbf{g}(x_j,t_k) \ \text{ on } \ \Delta_1^{M,N}\cup\Delta_2^{M,N}\end{aligned} \tag{18}$$

where $\mathbf{g}(x_j,t_k)=\mathbf{f}(x_j,t_k)-B(x_j,t_k)\boldsymbol{\chi}(x_j-1,t_k)$

$$\begin{aligned}\mathbf{L}_2^{M,N}\mathbf{U}(x_j,t_k) = D_t^-\mathbf{U}(x_j,t_k) - E\delta_x^2\mathbf{U}(x_j,t_k) + A(x_j,t_k)\mathbf{U}(x_j,t_k)\\ +B(x_j,t_k)\mathbf{U}(x_j-1,t_k)=\mathbf{f}(x_j,t_k) \ \text{ on } \ \Delta_3^{M,N}\end{aligned} \tag{19}$$

$$\mathbf{U}=\mathbf{u} \ \text{ on } \ \Gamma^{M,N}$$

$$D_x^-\mathbf{U}(x_{\frac{N}{4}},t_k)=D_x^+\mathbf{U}(x_{\frac{N}{4}},t_k),\ D_x^-\mathbf{U}(x_{\frac{N}{2}},t_k)=D_x^+\mathbf{U}(x_{\frac{N}{2}},t_k),$$

in Case (i) and

$$L_1^{M,N}\mathbf{U}(x_j,t_k) = D_t^-\mathbf{U}(x_j,t_k) - E\delta_x^2\mathbf{U}(x_j,t_k) + A(x_j,t_k)\mathbf{U}(x_j,t_k) = \mathbf{g}(x_j,t_k) \ \text{ on } \ \Delta_1^{M,N} \tag{20}$$
$$\text{where } \mathbf{g}(x_j,t_k)=\mathbf{f}(x_j,t_k)-B(x_j,t_k)\boldsymbol{\chi}(x_j-1,t_k)$$

$$\begin{aligned}L_2^{M,N}\mathbf{U}(x_j,t_k) = D_t^-\mathbf{U}(x_j,t_k) - E\delta_x^2\mathbf{U}(x_j,t_k) + A(x_j,t_k)\mathbf{U}(x_j,t_k)\\ +B(x_j,t_k)\mathbf{U}(x_j-1,t_k)=\mathbf{f}(x_j,t_k) \ \text{ on } \ \Delta_2^{M,N}\cup\Delta_3^{M,N}\end{aligned} \tag{21}$$

$$\mathbf{U}=\mathbf{u} \ \text{ on } \ \Gamma^{M,N}$$

$$D_x^-\mathbf{U}(x_{\frac{N}{3}},t_k)=D_x^+\mathbf{U}(x_{\frac{N}{3}},t_k),\ D_x^-\mathbf{U}(x_{\frac{2N}{3}},t_k)=D_x^+\mathbf{U}(x_{\frac{2N}{3}},t_k),$$

in Case (ii).
The results for the discrete case are similar to those for the continuous case.

Lemma 7 *Let conditions (2) and (3) hold. Then, for any mesh function* $\overrightarrow{\Psi}(x_j,t_k)$, $0\le j\le N, 0\le k\le M$, *the inequalities* $\overrightarrow{\Psi}\ge 0$ *on* $\Gamma^{M,N}$, $\mathbf{L}_1^{M,N}\overrightarrow{\Psi}(x_j,t_k)\ge 0$, *on* $\Delta_1^{M,N}\cup\Delta_2^{M,N}$, $\mathbf{L}_2^{M,N}\overrightarrow{\Psi}(x_j,t_k)\ge 0$ *on* $\Delta_3^{M,N}$ *and* $D_x^+\overrightarrow{\Psi}(x_j,t_k)-D_x^-\overrightarrow{\Psi}(x_j,t_k)\le 0$, $j=\frac{N}{4},\frac{N}{2}$, *in Case (i) and*

$\mathbf{L}_1^{M,N}\overrightarrow{\Psi}(x_j,t_k) \geq 0$, *on* $\Delta_1^{M,N}$, $\mathbf{L}_2^{M,N}\overrightarrow{\Psi}(x_j,t_k) \geq 0$ *on* $\Delta_2^{M,N} \cup \Delta_3^{M,N}$ *and* $D_x^+\overrightarrow{\Psi}(x_j,t_k) - D_x^-\overrightarrow{\Psi}(x_j,t_k) \leq 0$, $j = \frac{N}{3}, \frac{2N}{3}$, *in Case (ii) imply that* $\overrightarrow{\Psi}(x_j,t_k) \geq 0$ *on* $\overline{\Delta}^{M,N}$.

Proof The result is obtained by applying the same arguments as in [2] Lemma 6.1. The following discrete stability result is an immediate result of this.

Lemma 8 *Let conditions* (2) *and* (3) *hold. Then, for any mesh function* $\overrightarrow{\Psi}$ *satisfying, for* $i = 1,\ldots,n$, $D_x^+\overrightarrow{\Psi}(x_j,t_k) - D_x^-\overrightarrow{\Psi}(x_j,t_k) = 0$, $j = \frac{N}{4}, \frac{N}{2}$, *in Case (i), then for* $0 \leq j \leq N, 0 \leq k \leq M$

$$|\overrightarrow{\Psi}(x_j,t_k)| \leq \max\left\{||\overrightarrow{\Psi}||_{\Gamma^{M,N}},\ \frac{1}{\alpha}||\mathbf{L}_1^{M,N}\overrightarrow{\Psi}||_{\Delta_1^{M,N}\cup\Delta_2^{M,N}},\ \frac{1}{\alpha}||\mathbf{L}_2^{M,N}\overrightarrow{\Psi}||_{\Delta_3^{M,N}}\right\},$$

and $D_x^+\overrightarrow{\Psi}(x_j,t_k) - D_x^-\overrightarrow{\Psi}(x_j,t_k) = 0$, $j = \frac{N}{4}, \frac{N}{2}$, *in Case (ii), then for* $0 \leq j \leq N, 0 \leq k \leq M$.

$$|\overrightarrow{\Psi}(x_j,t_k)| \leq \max\left\{||\overrightarrow{\Psi}||_{\Gamma^{M,N}},\ \frac{1}{\alpha}||\mathbf{L}_1^{M,N}\overrightarrow{\Psi}||_{\Delta_1^{M,N}},\ \frac{1}{\alpha}||\mathbf{L}_2^{M,N}\overrightarrow{\Psi}||_{\Delta_2^{M,N}\cup\Delta_3^{M,N}}\right\},$$

Proof It is not difficult to derive the results using similar reasons as in Lemma 7.2 of [2].

7 Error Estimate

The discrete solution **U** can be decomposed into **V** and **W**, which can then be further decomposed as follows:

$$\mathbf{V} = \begin{cases} \tilde{\mathbf{V}} & \text{on } \bar{\Delta}_1^{M,N} \\ \hat{\mathbf{V}} & \text{on } \bar{\Delta}_2^{M,N} \\ \check{\mathbf{V}} & \text{on } \bar{\Delta}_3^{M,N} \end{cases},\ \mathbf{W} = \begin{cases} \tilde{\mathbf{W}} & \text{on } \bar{\Delta}_1^{M,N} \\ \hat{\mathbf{W}} & \text{on } \bar{\Delta}_2^{M,N} \\ \check{\mathbf{W}} & \text{on } \bar{\Delta}_3^{M,N} \end{cases}$$

where

$$\mathbf{L}_1^{M,N}\tilde{\mathbf{V}}(x_j,t_k) = \mathbf{g}(x_j,t_k),\ (x_j,t_k) \in \Delta_1^{M,N} \tag{22}$$

$$\mathbf{L}_1^{M,N}\hat{\mathbf{V}}(x_j,t_k) = \mathbf{g}(x_j,t_k),\ (x_j,t_k) \in \Delta_2^{M,N} \tag{23}$$

$$\tilde{\mathbf{V}}(0,t_k) = \mathbf{v}(0,t_k),\ \tilde{\mathbf{V}}(x_{N/4-1},t_k) = \mathbf{v}(d-,t_k),\ \hat{\mathbf{V}}(x_{N/4+1},t_k) = \mathbf{v}(d+,t_k),$$

$$\hat{\mathbf{V}}(x_{N/2-1},t_k) = \mathbf{v}(1-,t_k),\ \tilde{\mathbf{V}}(x_j,0) = \boldsymbol{\phi}_B(x_j),\ j = 1,2,\ldots,\tfrac{N}{4},$$

$$\hat{\mathbf{V}}(x_j, 0) = \boldsymbol{\phi}_B(x_j),\ j = \tfrac{N}{4}+1, \ldots, \tfrac{N}{2}$$

$$\mathbf{L}_2^{M,N}\check{\mathbf{V}}(x_j, t_k) = \mathbf{f}(x_j, t_k),\ (x_j, t_k) \in \Delta_3^{M,N}, \tag{24}$$

$$\check{\mathbf{V}}(x_{N/2+1}, t_k) = \mathbf{v}(1+, t_k), \check{\mathbf{V}}(x_N, t_k) = \mathbf{v}(2, t_k), \check{\mathbf{V}}(x_j, 0) = \boldsymbol{\phi}_B(x_j),\ j = \tfrac{N}{2}+1, \ldots N$$

and

$$\mathbf{L}_1^{M,N}\tilde{\mathbf{W}}(x_j, t_k) = 0,\ (x_j, t_k) \in \Delta_1^{M,N},$$

$$\mathbf{L}_1^{M,N}\hat{\mathbf{W}}(x_j, t_k) = 0,\ (x_j, t_k) \in \Delta_2^{M,N},$$

$$\mathbf{L}_2^{M,N}\check{\mathbf{W}}(x_j, t_k) = 0,\ (x_j, t_k) \in \Delta_3^{M,N},$$

$$\tilde{\mathbf{W}}(0, t_k) = \mathbf{w}(0, t_k), \hat{\mathbf{W}}(x_N, t_k) = \mathbf{w}(2, t_k),$$

$$\tilde{\mathbf{V}}(x_{N/4}, t_k) + \tilde{\mathbf{W}}(x_{N/4}, t_k) = \hat{\mathbf{V}}(x_{N/4}, t_k) + \hat{\mathbf{W}}(x_{N/4}, t_k), \tag{25}$$

$$\hat{\mathbf{V}}(x_{N/2}, t_k) + \hat{\mathbf{W}}(x_{N/2}, t_k) = \check{\mathbf{V}}(x_{N/2}, t_k) + \check{\mathbf{W}}(x_{N/2}, t_k);$$

$$D_x^-\tilde{\mathbf{W}}(x_{N/4}, t_k) + D_x^-\tilde{\mathbf{V}}(x_{N/4}, t_k) = D_x^+\hat{\mathbf{W}}(x_{N/4}, t_k) + D_x^+\hat{\mathbf{V}}(x_{N/4}, t_k),$$

$$D_x^-\hat{\mathbf{W}}(x_{N/2}, t_k) + D_x^-\hat{\mathbf{V}}(x_{N/2}, t_k) = D_x^+\check{\mathbf{W}}(x_{N/2}, t_k) + D_x^+\check{\mathbf{V}}(x_{N/2}, t_k),$$

$$\mathbf{W}(x_j, 0) = 0,\ j = 1, \ldots, N.$$

in Case (i) and

$$\mathbf{L}_1^{M,N}\tilde{\mathbf{V}}(x_j, t_k) = \mathbf{g}(x_j, t_k),\ (x_j, t_k) \in \Delta_1^{M,N}, \tag{26}$$

$$\tilde{\mathbf{V}}(0, t_k) = \mathbf{v}(0, t_k), \tilde{\mathbf{V}}(x_{N/3-1}, t_k) = \mathbf{v}(1-, t_k), \tilde{\mathbf{V}}(x_j, 0) = \boldsymbol{\phi}_B(x_j),\ j = 1, 2 \ldots \tfrac{N}{3}$$

$$\mathbf{L}_2^{M,N}\hat{\mathbf{V}}(x_j, t_k) = \mathbf{f}(x_j, t_k),\ (x_j, t_k) \in \Delta_2^{M,N}, \tag{27}$$

$\hat{\mathbf{V}}(x_{N/3+1}, t_k) = \mathbf{v}(1+, t_k), \hat{\mathbf{V}}(x_{2N/3-1}, t_k) = \mathbf{v}(d-, t_k), \hat{\mathbf{V}}(x_j, 0) = \boldsymbol{\phi}_B(x_j),$
$j = \tfrac{N}{3}+1, \ldots, \tfrac{2N}{3},$

$$\mathbf{L}_2^{M,N}\check{\mathbf{V}}(x_j, t_k) = \mathbf{f}(x_j, t_k),\ (x_j, t_k) \in \Delta_3^{M,N}, \tag{28}$$

$$\check{\mathbf{V}}(x_{2N/3+1}, t_k) = \mathbf{v}(d+, t_k), \check{\mathbf{V}}(x_N, t_k) = \mathbf{v}(2, t_k), \check{\mathbf{V}}(x_j, 0) = \boldsymbol{\phi}_B(x_j),\ j = \tfrac{2N}{3}+1, \ldots, N$$

and

$$\mathbf{L}_1^{M,N}\tilde{\mathbf{W}}(x_j,t_k)=0,\ (x_j,t_k)\in\Delta_1^{M,N},$$

$$\mathbf{L}_2^{M,N}\hat{\mathbf{W}}(x_j,t_k)=0,\ (x_j,t_k)\in\Delta_2^{M,N},$$

$$\mathbf{L}_2^{M,N}\check{\mathbf{W}}(x_j,t_k)=0,\ (x_j,t_k)\in\Delta_3^{M,N},$$

$$\tilde{\mathbf{W}}(0,t_k)=\mathbf{w}(0,t_k),\ \hat{\mathbf{W}}(x_N,t_k)=\mathbf{w}(2,t_k),$$

$$\tilde{\mathbf{V}}(x_{N/3},t_k)+\tilde{\mathbf{W}}(x_{N/3},t_k)=\hat{\mathbf{V}}(x_{N/3},t_k)+\hat{\mathbf{W}}(x_{N/3},t_k), \tag{29}$$

$$\hat{\mathbf{V}}(x_{2N/3},t_k)+\hat{\mathbf{W}}(x_{2N/3},t_k)=\check{\mathbf{V}}(x_{2N/3},t_k)+\check{\mathbf{W}}(x_{2N/3},t_k);$$

$$D_x^-\tilde{\mathbf{W}}(x_{N/3},t_k)+D_x^-\tilde{\mathbf{V}}(x_{N/3},t_k)=D_x^+\hat{\mathbf{W}}(x_{N/3},t_k)+D_x^+\hat{\mathbf{V}}(x_{N/3},t_k),$$

$$D_x^-\hat{\mathbf{W}}(x_{2N/3},t_k)+D_x^-\hat{\mathbf{V}}(x_{2N/3},t_k)=D_x^+\check{\mathbf{W}}(x_{2N/3},t_k)+D_x^+\check{\mathbf{V}}(x_{2N/3},t_k),$$

$$\mathbf{W}(x_j,0)=0,\ j=1,\ldots,N.$$

in Case (ii).

The error at each point $(x_j,t_k)\in\overline{\Delta}^{M,N}$ is denoted by $\mathbf{e}(x_j,t_k)=\mathbf{U}(x_j,t_k)-\mathbf{u}(x_j,t_k)$. Then the local truncation error $\mathbf{L}^{M,N}\mathbf{e}(x_j,t_k)$, for $j\neq N/4, N/2$ in Case (i) and $j\neq N/3, 2N/3$ in Case (ii), has the decomposition

$$\mathbf{L}^{M,N}\mathbf{e}(x_j,t_k)=\mathbf{L}^{M,N}(\mathbf{V}-\mathbf{v})(x_j,t_k)+\mathbf{L}^{M,N}(\mathbf{W}-\mathbf{w})(x_j,t_k).$$

In the following theorem, the error in the smooth and singular components is bounded.

Lemma 9 *Let conditions* (2) *and* (3) *hold. If* $\mathbf{v}(x_j,t_k)$ *denotes the smooth component of the solution of* (1) *and* $\mathbf{V}(x_j,t_k)$ *the smooth component of the solution of the problem* (17)*, then, for* $i=1,2,\ldots,n$,

$$||\mathbf{L}_1^{M,N}(\mathbf{V}-\mathbf{v})_i(x_j,t_k)||\leq C(M^{-1}+(N^{-1}\ln N)^2),\ \textit{for } l=1,2 \textit{ and } j\neq\frac{N}{4},\frac{N}{2}$$

in Case (i) and

$$||\mathbf{L}_1^{M,N}(\mathbf{V}-\mathbf{v})_i(x_j,t_k)|| \le C(M^{-1}+(N^{-1}\ln N)^2),\ \textit{for } l=1,2 \textit{ and } j \neq \frac{N}{3}, \frac{2N}{3}$$

in Case (ii).
If $\mathbf{w}(x_j, t_k)$ *denotes the singular component of the solution of* (1) *and* $\mathbf{W}(x_j, t_k)$ *the singular component of the solution of the problem* (17)*, then, for* $i = 1, 2, \ldots, n$,

$$||\mathbf{L}_1^{M,N}(\mathbf{W}-\mathbf{w})_i(x_j,t_k)|| \le C(M^{-1}+(N^{-1}\ln N)^2),\ \textit{for } l=1,2 \textit{ and } j \neq \frac{N}{4}, \frac{N}{2}$$

in Case (i) and

$$||\mathbf{L}_1^{M,N}(\mathbf{W}-\mathbf{w})_i(x_j,t_k)|| \le C(M^{-1}+(N^{-1}\ln N)^2),\ \textit{for } l=1,2 \textit{ and } j \neq \frac{N}{3}, \frac{2N}{3}$$

in Case (ii).

Proof: The needed bounds hold because the expressions for the local truncation error in **V** and **W**, as well as estimates for the derivatives of the smooth and singular components, are exactly in the form found in [3].
Case (i) :
At the points x_j, $j = \frac{N}{4}, \frac{N}{2}$,

$$(D_x^+ - D_x^-)\mathbf{e}(x_j,t_k) = (D_x^+ - D_x^-)(\mathbf{U}-\mathbf{u})(x_j,t_k), 0 \le k \le M$$

Recall that $(D_x^+ - D_x^-)\mathbf{U}(x_j,t_k) = 0$ for $j = \frac{N}{4}, \frac{N}{2}$. Let $h^* = \max\{h_{N/4}, h_{N/2}\}$, where $h_j = h_j^- = h_j^+$, $h_j^- = x_j - x_{j-1}$ and $h_j^+ = x_{j+1} - x_j$ for $j = \frac{N}{4}, \frac{N}{2}$.
Then

$$|(D_x^+ - D_x^-)\mathbf{e}(x_j,t_k)| \le C\frac{h^*}{\varepsilon}, \quad \text{for } j = \frac{N}{4}, \frac{N}{2}. \tag{30}$$

Define, for $i = 1, 2, \ldots, n$, a set of discrete barrier functions on $\overline{\Delta}^{M,N}$ by

$$\omega_i(x_j,t_k)=\begin{cases}\dfrac{\Pi_{l=1}^{j}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_l\right)}{\Pi_{l=1}^{N/4}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_l\right)}, & 0\le j\le \frac{N}{4}\\ \dfrac{\Pi_{l=j}^{(3N/8)-1}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_{l+1}\right)}{\Pi_{l=N/4}^{(3N/8)-1}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_{l+1}\right)}, & \frac{N}{4}\le j\le \frac{3N}{8}\\ \dfrac{\Pi_{l=3N/8}^{j-1}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_l\right)}{\Pi_{l=3N/8}^{(N/2)-1}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_l\right)}, & \frac{3N}{8}\le j\le \frac{N}{2}\\ \dfrac{\Pi_{l=j}^{(5N/8)-1}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_{l+1}\right)}{\Pi_{l=N/2}^{(5N/8)-1}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_{l+1}\right)}, & \frac{N}{2}\le j\le \frac{5N}{8}\\ \dfrac{\Pi_{l=5N/8}^{j-1}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_l\right)}{\Pi_{l=5N/8}^{(3N/4)-1}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_l\right)}, & \frac{5N}{8}\le j\le \frac{3N}{4}\\ \dfrac{\Pi_{l=j}^{N-1}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_{l+1}\right)}{\Pi_{l=3N/4}^{N-1}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_{l+1}\right)}, & \frac{3N}{4}\le j\le N\end{cases} \tag{31}$$

It is not hard to see that,

$$\begin{aligned}(\mathbf{L_1}^{M,N}\boldsymbol{\omega})_i(x_j,t_k) &= D_t^-\omega_i(x_j,t_k)-\varepsilon_i\,\delta_x^2\omega_i(x_j,t_k)+\sum_{l=1}^{n}a_{il}(x_j,t_k)\omega_l(x_j,t_k)\\ &> -\alpha\omega_i(x_j,t_k)+\sum_{l=1}^{i}a_{il}(x_j,t_k)\omega_i(x_j,t_k)+\sum_{l=i+1}^{2}a_{il}(x_j,t_k).\end{aligned}$$

And

$$
\begin{aligned}
(\mathbf{L_2}^{M,N}\boldsymbol{\omega})_i(x_j,t_k) &= D_t^-\omega_i(x_j,t_k) - \varepsilon_i\,\delta_x^2\omega_i(x_j,t_k)\\
&\quad + \sum_{l=1}^{2} a_{il}(x_j,t_k)\omega_l(x_j,t_k) + b_i(x_j,t_k)\omega_i(x_j-1,t_k)\\
&\geq -\alpha\omega_i(x_j,t_k) + \sum_{l=1}^{i} a_{il}(x_j,t_k)\omega_i(x_j,t_k)\\
&\quad + \sum_{l=i+1}^{2} a_{il}(x_j,t_k) + b_i(x_j,t_k).
\end{aligned}
$$

Case (ii) :
At the points x_j, $j = \frac{N}{3}, \frac{2N}{3}$,

$$
(D_x^+ - D_x^-)\mathbf{e}(x_j,t_k) = (D_x^+ - D_x^-)(\mathbf{U}-\mathbf{u})(x_j,t_k),\, 0 \leq k \leq M
$$

Recall that $(D_x^+ - D_x^-)\mathbf{U}(x_j,t_k) = 0$ for $j = \frac{N}{3}, \frac{2N}{3}$. Let $h^* = \max\{h_{N/3}, h_{2N/3}\}$, where $h_j = h_j^- = h_j^+$, $h_j^- = x_j - x_{j-1}$ and $h_j^+ = x_{j+1} - x_j$ for $j = \frac{N}{3}, \frac{2N}{3}$. Then

$$
|(D_x^+ - D_x^-)\mathbf{e}(x_j,t_k)| \leq C\frac{h^*}{\varepsilon},\ \text{for } j = \frac{N}{3}, \frac{2N}{3}.
$$

Define, for $i = 1, 2, \ldots, n$, a set of discrete barrier functions on $\overline{\Delta}^{M,N}$ by

$$
\omega_i(x_j,t_k) = \begin{cases}
\dfrac{\Pi_{l=1}^{j}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_l\right)}{\Pi_{l=1}^{N/3}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_l\right)}, & 0 \leq j \leq \frac{N}{3}\\[3ex]
\dfrac{\Pi_{l=j}^{(N/2)-1}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_{l+1}\right)}{\Pi_{l=N/3}^{(N/2)-1}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_{l+1}\right)}, & \frac{N}{3} \leq j \leq \frac{N}{2}\\[3ex]
\dfrac{\Pi_{l=N/2}^{j-1}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_l\right)}{\Pi_{l=N/2}^{(2N/3)-1}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_l\right)}, & \frac{N}{2} \leq j \leq \frac{2N}{3}\\[3ex]
\dfrac{\Pi_{l=j}^{N-1}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_{l+1}\right)}{\Pi_{l=2N/3}^{N-1}\left(1+\sqrt{\dfrac{\alpha}{\varepsilon_i}}h_{l+1}\right)}, & \frac{2N}{3} \leq j \leq N
\end{cases} \tag{32}
$$

It is not hard to see that,

$$
\begin{aligned}
(\mathbf{L_1}^{M,N}\boldsymbol{\omega})_i(x_j,t_k) &= D_t^-\omega_i(x_j,t_k) - \varepsilon_i\,\delta_x^2\omega_i(x_j,t_k) + \sum_{l=1}^{n} a_{il}(x_j,t_k)\omega_l(x_j,t_k)\\
&> -\alpha\omega_i(x_j,t_k) + \sum_{l=1}^{i} a_{il}(x_j,t_k)\omega_i(x_j,t_k) + \sum_{l=i+1}^{2} a_{il}(x_j,t_k).
\end{aligned}
$$

And

$$
\begin{aligned}
(\mathbf{L_2}^{M,N}\boldsymbol{\omega})_i(x_j,t_k) &= D_t^-\omega_i(x_j,t_k) - \varepsilon_i\,\delta_x^2\omega_i(x_j,t_k)\\
&\quad + \sum_{l=1}^{2} a_{il}(x_j,t_k)\omega_l(x_j,t_k) + b_i(x_j,t_k)\omega_i(x_j-1,t_k)\\
&\ge -\alpha\omega_i(x_j,t_k) + \sum_{l=1}^{i} a_{il}(x_j,t_k)\omega_i(x_j,t_k)\\
&\quad + \sum_{l=i+1}^{2} a_{il}(x_j,t_k) + b_i(x_j,t_k).
\end{aligned}
$$

We now state and prove the main theoretical result of this section.

Theorem 2 *Let* $\mathbf{u}(x_j,t_k)$ *denote the exact solution of* (1) *and* $\mathbf{U}(x_j,t_k)$ *the solution of* (17). *Then, for* $0 \le j \le N,\ 0 \le k \le M$,

$$||\mathbf{U}(x_j,t_k) - \mathbf{u}(x_j,t_k)|| \le C(M^{-1} + N^{-1}\ln N). \tag{33}$$

Proof Consider the mesh function $\overrightarrow{\Psi}^{\pm}$ given by $\Psi_i^{\pm}(x_j,t_k) = C_1(M^{-1} + N^{-1}\ln N) + C_2\frac{h^*}{\sqrt{\varepsilon_i}}\omega_i(x_j,t_k) \pm e_i(x_j,t_k), i = 1,2,\ldots,$ $n, 0 \le j \le N, 0 \le k \le M$, where C_1 and C_2 are constants. Then the result follows by using the mesh function $\overrightarrow{\Psi}^{\pm}$, Lemma 9, Lemma 7 and the procedure adopted in Lemma 7.2 in [2].

8 Numerical Illustration

The numerical method provided in this section ε-uniform convergence is illustrated in the examples below. For numerical example, a singularly perturbed boundary value problem is considered for a linear system of n parabolic second-order delay differential equations of reaction-diffusion type with discontinuous source terms.

The problem is solved using the method provided in Sect. 6 by fixing a piecewise uniform Shishkin mesh with 96 points in space. For t, the order of convergence and the error constant are calculated. A uniform mesh in time is considered, with 16 points. For x, the order of convergence and the error constant are calculated. Tables 1,

Table 1 Values of $D^{M,N}$, $p^{M,N}$, p^*, $C_{p^*}^{M,N}$ and $C_{p^*}^*$ for $\varepsilon_1 = \eta/64$, $\varepsilon_2 = \eta/32$, $\varepsilon_3 = \eta/16$, $N = 96$ and $\alpha = 0.9$

η	Number of mesh points M				
	32	64	128	256	512
2^{-3}	1.013E-02	5.366E-03	2.774E-03	1.411E-03	7.116E-04
2^{-6}	9.726E-03	5.095E-03	2.610E-03	1.321E-03	6.650E-04
2^{-9}	9.592E-03	5.021E-03	2.573E-03	1.303E-03	6.556E-04
2^{-12}	9.591E-03	5.020E-03	2.573E-03	1.303E-03	6.556E-04
2^{-15}	9.591E-03	5.020E-03	2.573E-03	1.303E-03	6.556E-04
$D^{M,N}$	1.013E-02	5.366E-03	2.774E-03	1.411E-03	7.116E-04
$p^{M,N}$	0.9169	0.9521	0.9752	0.9874	
$C_{p^*}^{M,N}$	0.5167	0.5167	0.5043	0.4843	0.4612
t-order of convergence, p^*= 0.9169					
The error constant, $C_{p^*}^*$= 0.5167					

Where $D^{M,N}$—the ε-uniform maximum point-wise errors, $p^{M,N}$—the ε- uniform order of local convergence, p^*—the ε- uniform order of convergence, $C_{p^*}^{M,N} = \frac{D^{M,N} N^{p^*}}{1-2^{-p^*}}$ and $C_{p^*}^*$—error constant

2, 3, and 4 give the parameter-uniform order of convergence and the error constant, respectively, using a variant of the two mesh algorithm found in [7] (Figs. 1 and 2).

Example: Consider the following problem

$$\mathbf{u}_t(x,t) - E\mathbf{u}_{xx}(x,t) + A(x,t)\mathbf{u}(x,t) + B(x,t)\mathbf{u}(x-1,t) = \mathbf{f}(x,t),$$
$$(x,t) \in (0,2) \times [0,T], \quad (34)$$

$\mathbf{u}(x,t) = (1,1,1)^T$, for $(x,t) \in [-1,0] \times [0,T]$, $\mathbf{u}(0,t) = (1,1,1)^T$, $\mathbf{u}(x,0) = (1,1,1)^T$ and $\mathbf{u}(2,t) = (1,1,1)^T$.

where

$$E{=}diag(\varepsilon_1,\varepsilon_2,\varepsilon_3), A(x,t){=}\begin{pmatrix} 5+t & -2 & -1 \\ -2 & 5 & -1 \\ -1 & -1 & 5+x \end{pmatrix}, B(x,t) = diag(-1,-1,-1),$$

$$\mathbf{f} = \begin{cases} (1+t, 5, 1+x)^T, & (x,t) \in (0,d) \times [0,T] \\ (0, 1.5, 1)^T, & (x,t) \in (d,2) \times [0,T]. \end{cases}$$

Case (i) $(d,t) = (0.4,t)$
Case (ii) $(d,t) = (1.4,t)$

Table 2 Values of $D^{M,N}$, $p^{M,N}$, p^*, $C_{p^*}^{M,N}$ and $C_{p^*}^*$ for $\varepsilon_1 = \eta/64$, $\varepsilon_2 = \eta/32$, $\varepsilon_3 = \eta/16$, $M = 16$ and $\alpha = 0.9$

η	Number of mesh points N			
	96	192	384	768
2^{-3}	1.440E-02	7.659E-03	3.890E-03	1.953E-03
2^{-6}	2.562E-02	1.893E-02	1.066E-02	5.499E-03
2^{-9}	1.056E-02	1.079E-02	7.925E-03	4.905E-03
2^{-12}	1.056E-02	1.079E-02	7.925E-03	4.905E-03
2^{-15}	1.056E-02	1.079E-02	7.925E-03	4.905E-03
$D^{M,N}$	2.562E-02	1.893E-02	1.066E-02	5.499E-03
$p^{M,N}$	0.4365	0.8281	0.9554	
$C_{p^*}^{M,N}$	0.7195	0.7195	0.5484	0.3827
x-order of convergence, p^*= 0.4365				
The error constant, $C_{p^*}^*$= 0.7195				

Table 3 Values of $D^{M,N}$, $p^{M,N}$, p^*, $C_{p^*}^{M,N}$ and $C_{p^*}^*$ for $\varepsilon_1 = \eta/64$, $\varepsilon_2 = \eta/32$, $\varepsilon_3 = \eta/16$, $N = 96$ and $\alpha = 0.9$

η	Number of mesh points M				
	32	64	128	256	512
2^{-3}	1.013E-02	5.366E-03	2.774E-03	1.411E-03	7.116E-04
2^{-6}	9.726E-03	5.095E-03	2.610E-03	1.321E-03	6.650E-04
2^{-9}	9.544E-03	4.993E-03	2.559E-03	1.297E-03	6.530E-04
2^{-12}	9.544E-03	4.993E-03	2.559E-03	1.297E-03	6.530E-04
2^{-15}	9.544E-03	4.993E-03	2.559E-03	1.297E-03	6.530E-04
$D^{M,N}$	1.013E-02	5.366E-03	2.774E-03	1.411E-03	7.116E-04
$p^{M,N}$	0.9169	0.9521	0.9752	0.9874	
$C_{p^*}^{M,N}$	0.5167	0.5167	0.5043	0.4843	0.4612
t-order of convergence, p^*= 0.9169					
The error constant, $C_{p^*}^*$= 0.5167					

9 Conclusion

In this study, a first-order convergent numerical technique for a parabolic system of singularly perturbed reaction-diffusion delay differential equations with discontinuous source terms is proposed and studied. The location of the point of discontinuity influences the solution profile. The occurrence of interior layers is also influenced by the delay term. Due to the presence of the delay term, an additional interior layer occurs at the point $1 + d$ if the point of discontinuity d is located in the interval $(0, 1)$. The numerical results right close the established convergence analysis.

Table 4 Values of $D^{M,N}$, $p^{M,N}$, p^*, $C_{p^*}^{M,N}$ and $C_{p^*}^*$ for $\varepsilon_1 = \eta/64$, $\varepsilon_2 = \eta/32$, $\varepsilon_3 = \eta/16$, $M = 16$ and $\alpha = 0.9$

η	Number of mesh points N			
	96	192	384	768
2^{-3}	1.48E-02	7.83E-03	3.97E-03	1.99E-03
2^{-6}	2.66E-02	1.94E-02	1.09E-02	5.61E-03
2^{-9}	1.06E-02	1.08E-02	7.92E-03	4.91E-03
2^{-12}	1.06E-02	1.08E-02	7.92E-03	4.91E-03
2^{-15}	1.06E-02	1.08E-02	7.92E-03	4.91E-03
$D^{M,N}$	2.66E-02	1.94E-02	1.09E-02	5.61E-03
$p^{M,N}$	0.452	0.835	0.957	
$C_{p^*}^{M,N}$	0.778	0.778	0.597	0.421
x-order of convergence, p^*= 0.452				
The error constant, $C_{p^*}^*$= 0.778				

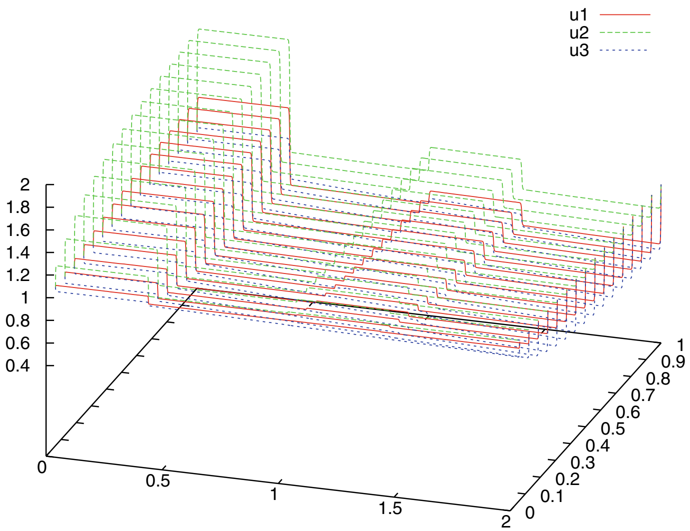


Fig. 1 The Figure displays the numerical solution for the problem (34), computed for $M = 16$, $N = 96$, and $\varepsilon = 2^{-3}$. The solution components $u_1(x, t)$, $u_2(x, t)$, and $u_3(x, t)$ have boundary layers at $(0, t)$ and $(2, t)$ and interior layers at $(0.4, t)$, $(1, t)$, and $(1.4, t)$

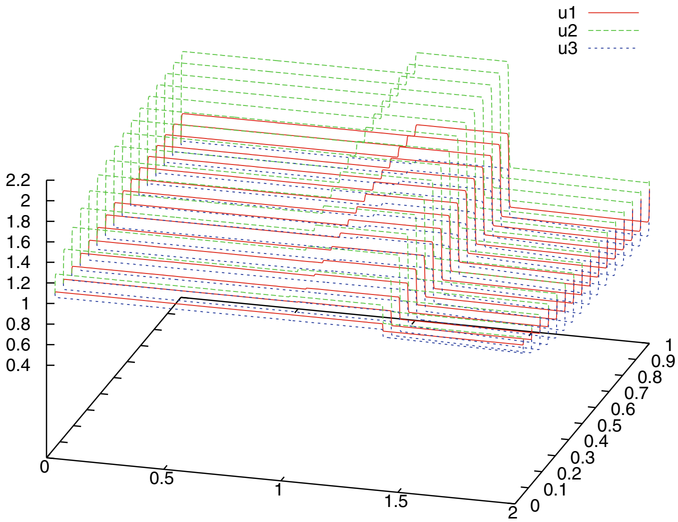


Fig. 2 The Figure displays the numerical solution for the problem (34), computed for $M = 16$, $N = 96$ and $\varepsilon = 2^{-6}$. The solution components $u_1(x, t)$, $u_2(x, t)$ and $u_3(x, t)$ have boundary layers at $(0, t)$ and $(2, t)$ and interior layers at $(1, t)$ and $(1.4, t)$

References

1. Saminathan, P., Sigamani, V., Victor, F.: Numerical method for a singularly perturbed boundary value problem for a linear parabolic second order delay differential equation, differential equations and numerical analysis. Springer Proceedings in Mathematics and Statistics, vol. 172 (2016)
2. Saminathan, P., Sigamani, V., Victor, F.: Parameter uniform numerical method for a singularly perturbed boundary value problem for a linear system of parabolic second order delay differential equations. Malaya J. Matematik textbf7(2), 147–160 (2019)
3. Franklin, V., Paramasivam, M., Miller, J.J.H., Valarmathi, S.: Second order parameter-uniform convergence for a finite difference method for a singularly perturbed linear parabolic system. Int. J. Numer. Anal. Model. **10**(1), 178–202
4. Manikandan, M., Shivaranjani, N., Miller, J.J.H., Valarmathi, S.: A parameter uniform first order convergent numerical method for a boundary value problem for a singularly perturbed delay differential equation. Advances in Applied Mathematics, vol. 87, pp. 71–88. Springer Proceedings in Mathematics and Statistics
5. Mariappan, M., Miller, J.J.H., Sigamani, V.: A parameter-uniform first order convergent numerical method for a system of singularly perturbed second order delay differential equations. In: Knobloch, P. (ed.), Boundary and Interior Layers, Computational and Asymptotic Methods—BAIL 2014, Lecture Notes in Computational Science and Engineering, vol. 108. Springer International Publishing Switzerland (2015)
6. Franklin, V., Miller, J.J.H., Valarmathi, S.: Second order parameter-uniform convergence for a finite difference method for a partially singularly perturbed linear parabolic system. Math. Commun. 1**9**, 469–495 (2014)
7. Farrell, A., Hegarty, A.F., Miller, J.J.H., Riordan, E.O', Shishkin, G.I.: Robust Computational Techniques for Boundary Layers. Chapman and hall, CRC, Boca Raton, FL, USA (2000)