Robot Kinematic of Three-Link Manipulator and Computation of Joint Torque by Phase Variable Method

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Abstract In this paper, the analytical solution of the dynamic model of the three-link robotic manipulator has been presented where the mathematical formulation of direct kinematics is presented using Newton–Euler approach. The robotic manipulator can be assumed as a three-link robotic arm where all the joints are revolute and planar. This study establishes the relationship between the angular position of each link member, angular velocity and angular acceleration using forward equation of motion, and the actuation torque at each joint is calculated by backward equations. The three simultaneous differential equations of second order corelating torque and angular position of each joint-link pair can be reformed into six ordinary differential equations of first order. The analytical solution represents the dynamic response of each link rotation with respect to time for a constant torque applied to each joint.

Keywords Direct kinematic · Three-link · MATLAB · Newton–Euler · Joint torque phase variable

1 Introduction

The study of robot and robot dynamics has been carried out over a decades using different types of approaches. One of the methods is to reduce the order of the multi-robot system with some of the properties of the model [1]. Deshpande et al. [2] compare computational complexities of various forms of the equations of motion for open chain systems for dynamics of robot manipulator. Yamane and Nakamura [3] describe a simulation model of human body for its motion analysis, and the algorithms

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were developed in robotics which are combined with physiological models to solve the difficulties arise of handling real human structure.

Dynamic modeling and simulation of the industrial robot, Stiiubli TX40, are presented by Cheraghpour et al. where a precise simulation for experimental analysis of kinematics, dynamics, and control is described [4]. In this study, inertial and geometric parameters are accurately measured and recorded in the software database. Dynamic modeling of a nonholonomic wheeled manipulator is carried out which consists of elastic joints and a self-directed wheeled movable platform [5, 6]. Gibbs–Appell (G–A) recursive algorithm is adopted to avoid computation of Lagrange multipliers. The proposed algorithm recursively and methodically derives the dynamic equation, moreover all mathematical processes are done by 3×1 and 3×3 matrices to improve the computational complexity. Also, local coordinate system is assigned to all dynamic equations of a link by which the algorithm was generalized where the implementation and simulation of a greater degrees of freedom wheeled movable robotic manipulator become easier. However, the damping and friction were not considered to improve joint modeling and flexibility. Lagrange–Euler dynamics are presented in [7–9] where the aim is to derive simple and well-structured dynamic equations of motion. Again, the Newton–Euler model was used to form the dynamic equations and a comparative study is given experiencing major difficulties in using both methods. For development of a robotic system, complete modeling of a well-known robotic manipulator UR5 is presented [10] where the dynamic properties, inertia matrix, coriolis and centrifugal matrix, and gravity vector are derived based on the Lagrange method, and the derived mathematical model has been implemented in MATLAB. In many research article, forward kinematics of a manipulator are discussed [11, 12] with different joint structures such as triangular prism structured links where the varying positions of the end effector were calculated and plotted against joint angles in MATLAB.

In this paper, the dynamic equations of motion have been derived using Newton–Euler formulation. The direct kinematic solution is obtained analytically by expressing the three nonhomogeneous nonlinear second order differential equations into six differential equations of first order and rearranging them in phase variable form. The angular displacement and angular velocities of each link are presented graphically with respect to time for constant torques applied for a specific period of time using MATLAB. Then, the optimum torque ratio is obtained which gives the feasible angular positions of three-links for practical implementations.

2 Three-Link Manipulator

In this work, a robotic manipulator of three-links of lengths l_1, l_2, l_3 and three joints J_1, J_2, J_3 is shown in Fig. 1. The links are having mass m_1, m_2, m_3 and link parameters are $\alpha_1 = \alpha_2 = \alpha_3 = 0$. All the revolving rigid joints can rotate in a two-dimensional space about the z-axis. The link parameters to be considered as $\alpha_1 = \alpha_2 = \alpha_3 = 0$,

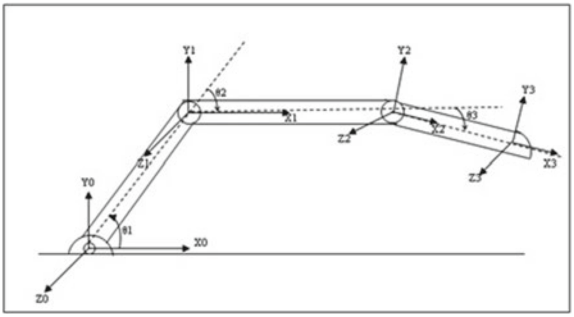

Fig. 1 Three-link manipulator with coordinate assignment

and θ_1, θ_2, and θ_3 are considered as angular displacement of each joint-link pair of the system.

The initial conditions can be considered as

$$\omega_0 = 0, \dot{\omega}_0 = 0, v_0 = 0$$

where ω_0 is the initial angular velocity, $\dot{\omega}_0$ is the initial angular acceleration, v_0 is the initial linear velocity, and $\dot{v}_0$ is the initial linear acceleration.

Therefore,

$$\dot{v}_0 = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix}$$

$$\dot{v}_0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}$$

$$g = 9.8\ \text{m/s}^2$$

Joint variables: angular position, angular speed, and angular acceleration

$$q_i = \theta_i = [\theta_1, \theta_2, \theta_3];$$

$$\dot{q}_i = \dot{\theta}_i = \left[\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3\right];$$

$$\ddot{q}_i = \ddot{\theta}_i = \left[\ddot{\theta}_1, \ddot{\theta}_2, \ddot{\theta}_3\right];$$

Link variable:

$$F_i, f_i, n_i, \tau_i$$

where

F_i: Total external force exerted on link i at the center of mass

f_i: Force exerted on link i by link $i-1$

n_i: Moment exerted on link i by link $i-1$

τ_i: Torque applied to joint i.

3 Computed Joint Torque by Newton–Euler Equations of Motion

The Newton–Euler approach has been adopted to compute the kinematic model relating joint torque and angular displacement as given in (1) to (3).

Joint torque computed to each joint actuator for link $i = 3, 2, 1$ [for link3, link2, link1 are τ_3, τ_2, τ_1, respectively]

$$\begin{aligned}\tau_3 = &\frac{1}{2}m_3 l_1 l_3\left[\cos(\theta_2+\theta_3)\ddot{\theta}_1 + \sin(\theta_2+\theta_3)\dot{\theta}_1^2\right] \\ &+ \frac{1}{2}m_3 l_2 l_3\left[\sin\theta_3\left(\dot{\theta}_1+\dot{\theta}_2\right)^2 + \cos\theta_3\left(\ddot{\theta}_1+\ddot{\theta}_2\right)\right] \\ &+ \frac{1}{3}m_3 l_3^2\left(\ddot{\theta}_1+\ddot{\theta}_2+\ddot{\theta}_3\right) + \frac{1}{2}m_3 l_3 g\cos(\theta_1+\theta_2+\theta_3)\end{aligned} \tag{1}$$

$$\begin{aligned}\tau_2 = &\frac{1}{2}m_3 l_1 l_3\left[\cos(\theta_2+\theta_3)\ddot{\theta}_1 + \sin(\theta_2+\theta_3)\dot{\theta}_1^2\right] \\ &+ \frac{1}{2}m_3 l_2 l_3\left[\sin\theta_3\left(\dot{\theta}_1+\dot{\theta}_2\right)^2 + \cos\theta_3\left(\ddot{\theta}_1+\ddot{\theta}_2\right)\right] \\ &+ \frac{1}{3}m_3 l_3^2\left(\ddot{\theta}_1+\ddot{\theta}_2+\ddot{\theta}_3\right) + \frac{1}{2}m_3 l_3 g\cos(\theta_1+\theta_2+\theta_3) \\ &+ \left(m_3+\frac{1}{2}m_2\right)l_1 l_2\left[\cos\theta_2\ddot{\theta}_1 + \sin\theta_2\dot{\theta}_1^2\right] \\ &+ \left(m_3+\frac{1}{3}m_2\right)l_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right) \\ &+ \frac{1}{2}m_3 l_2 l_3\left[\cos\theta_3\left(\ddot{\theta}_1+\ddot{\theta}_2+\ddot{\theta}_3\right) - \sin\theta_3\left(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3\right)^2\right] \\ &+ \left(m_3+\frac{1}{2}m_2\right)l_2 g\cos(\theta_1+\theta_2)\end{aligned} \tag{2}$$

$$\begin{aligned}\tau_1 = &\frac{1}{2}m_3 l_1 l_3\left[\cos(\theta_2+\theta_3)\ddot{\theta}_1 + \sin(\theta_2+\theta_3)\dot{\theta}_1^2\right] \\ &+ \frac{1}{2}m_3 l_2 l_3\left[\sin\theta_3\left(\dot{\theta}_1+\dot{\theta}_2\right)^2 + \cos\theta_3\left(\ddot{\theta}_1+\ddot{\theta}_2\right)\right] \\ &+ \frac{1}{3}m_3 l_3^2\left(\ddot{\theta}_1+\ddot{\theta}_2+\ddot{\theta}_3\right) + \frac{1}{2}m_3 l_3 g\cos(\theta_1+\theta_2+\theta_3)\end{aligned}$$

$$
\begin{aligned}
&+\left(m_3+\frac{1}{2}m_2\right)l_1l_2\left[\cos\theta_2\ddot{\theta}_1+\sin\theta_2\dot{\theta}_1^2\right]\\
&+\left(m_3+\frac{1}{3}m_2\right)l_2^2\left(\ddot{\theta}_1+\ddot{\theta}_2\right)\\
&+\frac{1}{2}m_3l_2l_3\left[\cos\theta_3\left(\ddot{\theta}_1+\ddot{\theta}_2+\ddot{\theta}_3\right)-\sin\theta_3\left(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3\right)^2\right]\\
&+\left(m_3+\frac{1}{2}m_2\right)l_2g\cos(\theta_1+\theta_2)+\left(m_3+m_2+\frac{1}{3}m_1\right)l_1^2\ddot{\theta}_1\\
&+\left(m_3+\frac{1}{2}m_2\right)l_1l_2\left[\cos\theta_2\left(\ddot{\theta}_1+\ddot{\theta}_2\right)-\sin\theta_2\left(\dot{\theta}_1+\dot{\theta}_2\right)^2\right]\\
&-\frac{1}{2}m_3l_1l_3\left[\sin(\theta_2+\theta_3)\left(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3\right)^2-\cos(\theta_2+\theta_3)\left(\ddot{\theta}_1+\ddot{\theta}_2+\ddot{\theta}_3\right)\right]\\
&+\left(m_3+m_2+\frac{1}{2}m_1\right)l_1g\cos\theta_1
\end{aligned}
\tag{3}
$$

4 Kinematic Solution of Mathematical Modeling of N–E Model of Three-Link Manipulator

As the dynamics relating joint torque and angular displacement of three coupled joint-link are highly nonlinear in nature, a phase variable form has been adopted. In order to derive a systematic procedure, the three nonlinear differential equation of order two is transformed into a phase variable form representing a system of six first order differential equations.

Assuming x_1, x_2, x_3, x_4, $x_{5,}$ and x_6 are six phase variables given by

$$x_1=\dot{\theta}_1 \tag{4}$$

$$x_2=\dot{\theta}_2 \tag{5}$$

$$x_3=\dot{\theta}_3 \tag{6}$$

$$x_4=\theta_1 \tag{7}$$

$$x_5=\theta_2 \tag{8}$$

$$x_6=\theta_3 \tag{9}$$

Therefore,

$$\dot{x}_1 = \ddot{\theta}_1 \tag{10}$$

$$\dot{x}_2 = \ddot{\theta}_2 \tag{11}$$

$$\dot{x}_3 = \ddot{\theta}_3 \tag{12}$$

$$\dot{x}_4 = \dot{\theta}_1 = x_1 \tag{13}$$

$$\dot{x}_5 = \dot{\theta}_2 = x_2 \tag{14}$$

$$\dot{x}_6 = \dot{\theta}_3 = x_3 \tag{15}$$

Nondimensionalizing (1), (2), and (3), we have

$$\begin{aligned}\tau_3 = {} & \frac{1}{2}\cos(\theta_2+\theta_3)\ddot{\theta}_1 + \frac{1}{2}\sin(\theta_2+\theta_3)\dot{\theta}_1^2 \\ & + \frac{1}{2}\sin\theta_3\left(\dot{\theta}_1+\dot{\theta}_2\right)^2 + \frac{1}{2}\cos\theta_3\left(\ddot{\theta}_1+\ddot{\theta}_2\right) \\ & + \frac{1}{3}\left(\ddot{\theta}_1+\ddot{\theta}_2+\ddot{\theta}_3\right) + \frac{1}{2}\cos(\theta_1+\theta_2+\theta_3)\end{aligned} \tag{16}$$

$$\begin{aligned}\tau_2 = {} & \frac{1}{2}\cos(\theta_2+\theta_3)\ddot{\theta}_1 + \frac{1}{2}\sin(\theta_2+\theta_3)\dot{\theta}_1^2 \\ & + \frac{1}{2}\sin\theta_3\left(\dot{\theta}_1+\dot{\theta}_2\right)^2 + \frac{1}{2}\cos\theta_3\left(\ddot{\theta}_1+\ddot{\theta}_2\right) \\ & + \frac{1}{3}\left(\ddot{\theta}_1+\ddot{\theta}_2+\ddot{\theta}_3\right) + \frac{1}{2}\cos(\theta_1+\theta_2+\theta_3) \\ & + \frac{3}{2}\cos\theta_2\ddot{\theta}_1 + \frac{3}{2}\sin\theta_2\dot{\theta}_1^2 + \frac{4}{3}\left(\ddot{\theta}_1+\ddot{\theta}_2\right) \\ & + \frac{1}{2}\cos\theta_3\left(\ddot{\theta}_1+\ddot{\theta}_2+\ddot{\theta}_3\right) - \frac{1}{2}\sin\theta_3\left(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3\right)^2 \\ & + \frac{3}{2}\cos(\theta_1+\theta_2)\end{aligned} \tag{17}$$

$$\begin{aligned}\tau_1 = {} & \frac{1}{2}\cos(\theta_2+\theta_3)\ddot{\theta}_1 + \frac{1}{2}\sin(\theta_2+\theta_3)\dot{\theta}_1^2 + \frac{1}{2}\sin\theta_3\left(\dot{\theta}_1+\dot{\theta}_2\right)^2 \\ & + \frac{1}{2}\cos\theta_3\left(\ddot{\theta}_1+\ddot{\theta}_2\right) + \frac{1}{3}\left(\ddot{\theta}_1+\ddot{\theta}_2+\ddot{\theta}_3\right) \\ & + \frac{1}{2}\cos(\theta_1+\theta_2+\theta_3) + \frac{3}{2}\cos\theta_2\ddot{\theta}_1 + \frac{3}{2}\sin\theta_2\dot{\theta}_1^2 \\ & + \frac{4}{3}\left(\ddot{\theta}_1+\ddot{\theta}_2\right) + \frac{1}{2}\cos\theta_3\left(\ddot{\theta}_1+\ddot{\theta}_2+\ddot{\theta}_3\right)\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\sin\theta_3\left(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3\right)^2+\frac{3}{2}\cos(\theta_1+\theta_2)+\frac{7}{3}\ddot{\theta}_1\\
&+\frac{3}{2}\cos\theta_2\left(\ddot{\theta}_1+\ddot{\theta}_2\right)-\frac{3}{2}\sin\theta_2\left(\dot{\theta}_1+\dot{\theta}_2\right)^2\\
&-\frac{1}{2}\sin(\theta_2+\theta_3)\left(\dot{\theta}_1+\dot{\theta}_2+\dot{\theta}_3\right)^2+\frac{1}{2}\cos(\theta_2+\theta_3)\left(\ddot{\theta}_1+\ddot{\theta}_2+\ddot{\theta}_3\right)+\frac{5}{2}\cos\theta_1
\end{aligned}
\tag{18}
$$

Using (16), (17), and (18), $\ddot{\theta}_1, \ddot{\theta}_2,$ and $\ddot{\theta}_3$ can be formed as algebraic sum of $\theta_1, \theta_2, \theta_3$ and their first derivatives which give all first order nonlinear equations. Therefore, (10) to (15) are expressed as six equations of first order which can be solved graphically using MATLAB.

4.1 Transient Behavior of Link and Joint Movement

Figures 2, 3, and 4 describe the dynamic behavior of ω and θ of each link corresponding to each joint actuation torque, τ_1, τ_2, and τ_3, for different torque ratio applied for a finite length of time (0.5 s). The angular position versus torque characteristics are shown in Figs. 5, 6, and 7 for different torque ratio.

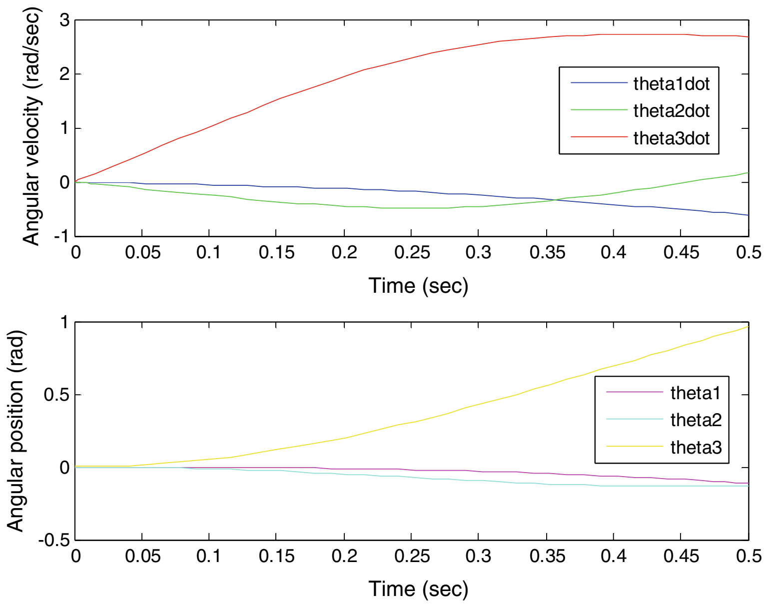

Fig. 2 Transient of angular velocity and angular position for τ_1, τ_2, τ_3:1, 1, 1 (in Nm), respectively

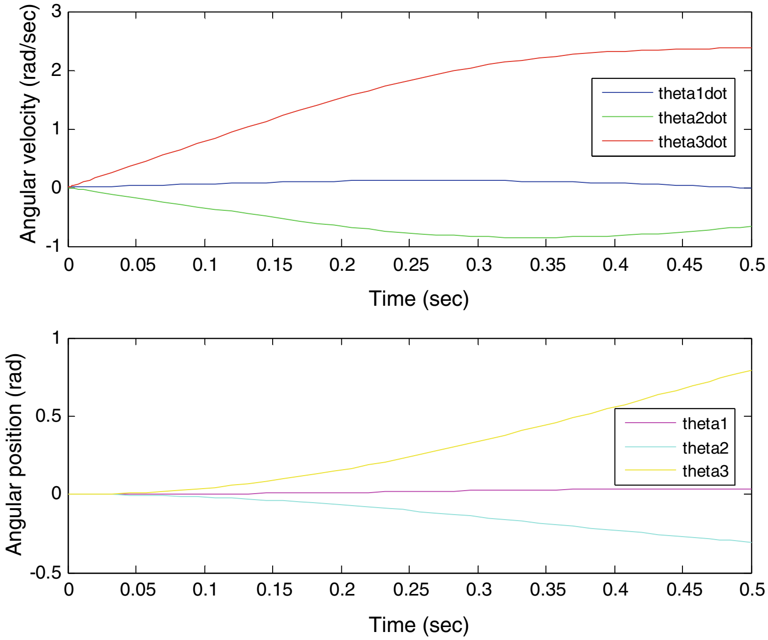

Fig. 3 Transient of angular velocity and angular position for τ_1, τ_2, τ_3:4, 2, 1 (in Nm), respectively

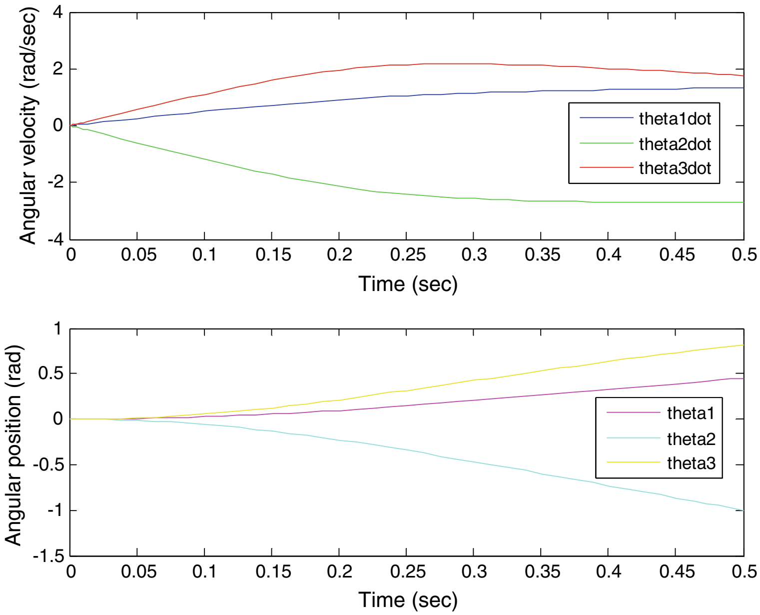

Fig. 4 Transient response of angular velocity and angular position for τ_1, τ_2, τ_3:9, 3, 1 (in Nm), respectively

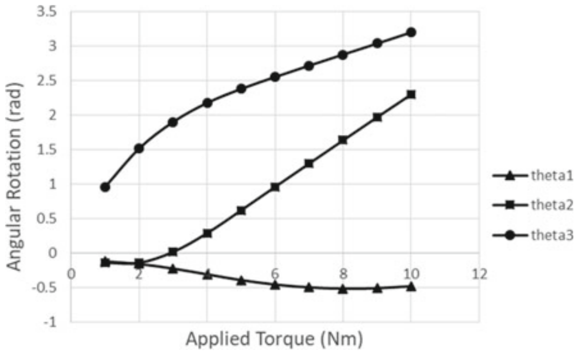

Fig. 5 Angular position versus applied torque (τ_1:τ_2:τ_3::1:1:1 (in Nm); time: 0.5 s)

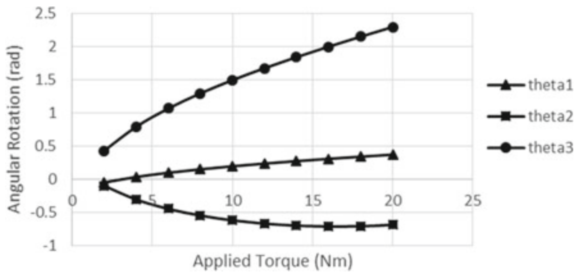

Fig. 6 Angular position versus applied torque (τ_1:τ_2:τ_3::4:2:1 (in Nm); time: 0.5 s)

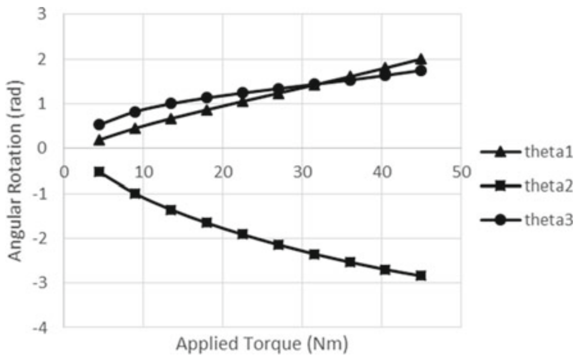

Fig. 7 Angular position versus applied torque (τ_1:τ_2:τ_3::9:3:1 (in Nm); time: 0.5 s)

4.2 Characteristics of Link and Joint Movement with Applied Torque

See Figs. 5, 6, and 7.

Table 1 Angular position due to application of constant torques applied for 0.5 s

τ_1 (in Nm)	τ_2 (in Nm)	τ_3 (in Nm)	θ_1 (rad)	θ_2 (rad)	θ_3 (rad)
1	1	1	−0.1146	−0.1338	0.9641
4	2	1	0.0359	−0.3048	0.7929
9	3	1	0.4517	−1.0001	0.8221

5 Results and Discussion

Table 1 represents the results obtained from Figs. 2 to 4 which describe the angular position of each link after 0.5 s due to application of constant torques.

From Table 1, it is evident that in nondimentionalize condition of the manipulator, applying joint torques in equal proportion are not able to lift up the link1 and link2 as both θ_1 and θ_2 are negative. For practical realization, it can be assumed that $0 \leq \theta_1 \leq \frac{\pi}{2}$, $0 \leq \theta_2 \leq \frac{\pi}{2}$, and $0 \leq \theta_3 \leq \frac{\pi}{2}$. And if θ_2 is negative, it must follow the condition $|\theta_2| \leq \theta_1$. Therefore, to satisfy this condition the joint torques must be applied in the proportion of τ_1:τ_2:τ_3::9:3:1 or higher (refer Table 1).

6 Conclusion

The dynamic equations, as stated in (1) to (3), describe the nonlinear nature of the system. Here, the open loop response of the system has been studied and result is as expected. Figures 5–7 show the angular rotation versus torque characteristics and the most linear characteristics and hence optimum torque can be obtained for the torque ratios of $\tau 1$:$\tau 2$:$\tau 3$::9:3:1 as shown in Fig. 7 using MATLAB. Moreover, the system can be assumed to have a combination of three inverted pendulum will also be a chaotic system. As a very consequences, the system can be controlled with sliding mode controller as well as with proportional derivative controller which will give the system more stability.

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