Structural Damage Quantification Using Flexibility Matrix-Based Approach

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1 Introduction

Vibration-based damage detection methods are widely used in structural health monitoring for the last few decades. Any damage in a structure leads to the change in the eigen properties of the structure. However, it is very difficult to measure these modal parameters of the higher modes in actual scenario. The damage can be localized in case of shear buildings by observing the change in first mode shape slopes [1]. Discontinuity in the mode shape slope and curvature [2] can also identify damage location of any structure. The damage quantification technique is the emerging field of research over damage identification in the recent years. The flexibility matrix of a structure can be estimated approximately by a few lower modes and is very much sensitive to damage. Usually, a generalized flexibility matrix-based method [3] detects damage with good accuracy while reducing the effect of higher-order modes. The damage severity can be also determined by Flexibility Proportional Coordinate Modal Assurance Criterion [4] (FPCoMAC) to identify how error due to random variation propagates in the flexibility matrix of the structure by observing the relative change in flexibility matrix with respect to the flexibility matrix of an undamaged structure at any location.

A new approach to detect damage based on the change in flexibility matrix has been proposed in this study. The approach involved derivation of the change in flexibility matrix with respect to damaged story stiffness in terms of modal parameters using weightage factor matrix correlations with mode shapes and its partial derivative for both the first and second-order sensitivity of the flexibility matrix. The variation of the change in the flexibility matrix of a structure due to damage has been identified by

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D. Maity et al. (eds.), *Recent Advances in Computational and Experimental Mechanics, Vol–I*, Lecture Notes in Mechanical Engineering,
https://doi.org/10.1007/978-981-16-6738-1_49

considering higher-order terms. Simulation has been performed on damaged shear building to illustrate the effectiveness of the proposed method.

2 Methodology

The change in the flexibility matrix of a structure due to stiffness reduction at any member has been derived for undamped free vibration of an n degrees-of-freedom (DoFs) spring-mass system. The intact structure has been considered as the initial model. The undamaged structure modal parameters are the only input variables required to identify the change in the structure due to damage. Damage due to stiffness degradation has been considered in this article.

The basic equation required for the formulation of the derivative of the flexibility matrix F with respect to the change in stiffness in any member t is the eigenvalue problem for the total system. Consider K, M, and F as the stiffness matrix, mass matrix, and flexibility matrix of an undamped freely vibrating n DoF spring-mass system of respectively as shown in Fig. 1. $\lambda^{(i)}$ represents a square of circular frequency, and $\phi_p^{(i)}$ is the i^{th} mode shape at the p^{th} degree of freedom. The eigenvalue problem of any system can be formulated as,

$$K\Phi = M\Phi\Lambda \tag{1}$$

where Φ is the mass-normalized mode shape matrix and Λ is the eigenvalue matrix with square of the circular frequencies as the diagonal. The mass-normalized mode shape matrix implies,

$$\Phi^{T}M\Phi = I \tag{2}$$

The definition of flexibility matrix states that it is simply the inverse of the stiffness matrix of the structure, such that

$$FK = I \tag{3}$$

Pre-multiplying both sides of Eq. (1) with F and substituting Eqs. (2) and (3), a simplified expression of the flexibility matrix in terms of mode shapes and circular frequencies can be obtained as,

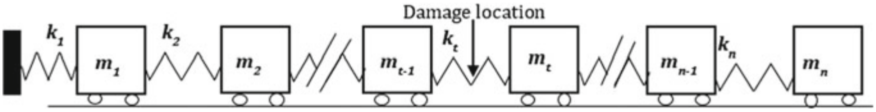

Fig. 1 Schematic diagram of an undamped spring-mass system [2] with damage between $(t-1)^{th}$ and t^{th} oscillator

$$F = \Phi\Lambda^{-1}\Phi^{\mathrm{T}} \tag{4}$$

Damage has been introduced between the $(t-1)^{\mathrm{th}}$ and t^{th} masses by degrading the stiffness by a small amount ΔK. The first and second derivatives of the flexibility matrix with respect to damaged stiffness k_t can be expressed as follows:

$$\frac{\partial F}{\partial k_t} = \frac{\partial \Phi}{\partial k_t}\Lambda^{-1}\Phi^{\mathrm{T}} + \Phi\frac{\partial \Lambda^{-1}}{\partial k_t}\Phi^{\mathrm{T}} + \Phi\Lambda^{-1}\frac{\partial \Phi^{\mathrm{T}}}{\partial k_t} \tag{5}$$

$$\begin{aligned}\frac{\partial^2 F}{\partial k_t^2} &= \left(\frac{\partial^2 \Phi}{\partial k_t^2}\Lambda^{-1}\Phi^{\mathrm{T}} + \Phi\frac{\partial^2 \Lambda^{-1}}{\partial k_t^2}\Phi^{\mathrm{T}} + \Phi\Lambda^{-1}\frac{\partial^2 \Phi^{\mathrm{T}}}{\partial k_t^2}\right)\\ &\quad + 2\left(\frac{\partial \Phi}{\partial k_t}\frac{\partial \Lambda^{-1}}{\partial k_t}\Phi^{\mathrm{T}} + \frac{\partial \Phi}{\partial k_t}\Lambda^{-1}\frac{\partial \Phi^{\mathrm{T}}}{\partial k_t} + \Phi\frac{\partial \Lambda^{-1}}{\partial k_t}\frac{\partial \Phi^{\mathrm{T}}}{\partial k_t}\right)\end{aligned} \tag{6}$$

Both the first and second derivatives of the mode shape matrix can be expressed as a correlation with the mode shape matrix by a weightage factor Ψ and Γ, respectively, as given in Eqs. (7) and (8). The double derivative of the square of the natural frequency with respect to the damaged story stiffness can be obtained in terms of Ψ, Γ, and Ω as given in Eq. (9). The expressions of Ψ, Γ, and Ω and the detailed calculation of the derivation $\partial^2\Lambda/\partial k_t^2$ has been provided in the Appendix.

$$\frac{\partial \Phi}{\partial k_t} = \Phi\Psi \tag{7}$$

$$\frac{\partial^2 \Phi}{\partial k_t^2} = \Phi\Gamma \tag{8}$$

$$\frac{\partial^2 \Lambda}{\partial k_t^2} = \left(\Gamma^{\mathrm{T}}\Lambda + \Lambda\Gamma\right) - 2(\Psi\Omega - \Omega\Psi + \Psi\Lambda\Psi) \tag{9}$$

Substituting Eqs. (7), (8), and (9) in Eqs. (5) and (6), the first and second derivatives of the flexibility matrix can be expressed in the simplified form as,

$$\frac{\partial F}{\partial k_t} = \Phi\left[\left(\Psi\Lambda^{-1} - \Lambda^{-1}\Psi\right) - \Lambda^{-1}\frac{\partial \Lambda}{\partial k_t}\Lambda^{-1}\right]\Phi^{\mathrm{T}} \tag{10}$$

$$\begin{aligned}&\frac{\partial^2 F}{\partial k_t^2}\\ &= \Phi\left[\begin{array}{l}\left(\Gamma\Lambda^{-1} + \Lambda^{-1}\Gamma^{T}\right) + 2\left(\Lambda^{-1}\frac{\partial \Lambda}{\partial k_t}\Lambda^{-1}\Psi - \Psi\Lambda^{-1}\frac{\partial \Lambda}{\partial k_t}\Lambda^{-1} - \Psi\Lambda^{-1}\Psi\right)\\ +\Lambda^{-1}\left\{2\left(\frac{\partial \Lambda}{\partial k_t}\Lambda^{-1}\frac{\partial \Lambda}{\partial k_t} + (\Psi\Omega - \Omega\Psi) + \Psi\Lambda\Psi\right) - \left(\Gamma^{T}\Lambda + \Lambda\Gamma\right)\right\}\Lambda^{-1}\end{array}\right]\Phi^{\mathrm{T}}\end{aligned} \tag{11}$$

Substituting Eqs. (23–32) in Eqs. (10) and (11), the simplified expression of the derivatives of the flexibility matrix in terms of only modal parameters can be formulated considering h as the intermediate distance between t and $(t-1)$ mass locations. If $()^{(i)}$ and $()^{(j)}$ represent two different modes and $()_p$ and $()_q$ represent index of each element of the whole sensitivity matrix,

$$\left(\frac{\partial F}{\partial k_t}\right)_{p,q} = -h^2\left(\sum_{i=1}^{n}\phi_p^{(i)}\phi_q^{(i)}\left(\frac{\phi'^{(i)}_t}{\lambda^{(i)}}\right)^2 + \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\left(\phi_q^{(i)}\phi_p^{(j)}+\phi_p^{(i)}\phi_q^{(j)}\right)\phi'^{(i)}_t\phi'^{(j)}_t}{\lambda^{(i)}\lambda^{(j)}}\right) \quad (12)$$

$$\left(\frac{\partial^2 F}{\partial k_t^2}\right)_{p,q} = 2h^4\left(\sum_{i=1}^{n}\phi_q^{(i)}\frac{\phi'^{(i)}_t}{\lambda^{(i)}}\right)\left(\sum_{i=1}^{n}\phi_p^{(i)}\frac{\phi'^{(i)}_t}{\lambda^{(i)}}\right)\left(\sum_{i=1}^{n}\frac{\left(\phi'^{(i)}_t\right)^2}{\lambda^{(i)}}\right) \quad (13)$$

An infinite DoF spring-mass system can be considered as an axially vibrating bar element of length L and shear wave velocity c with one end fixed [5].

$$\lambda^{(i)} = \left(\frac{(2i-1)\pi c}{2L}\right)^2 \quad (14)$$

$$\phi^{(i)}(x) = b\sin\left(\frac{(2i-1)\pi x}{2L}\right) \quad (15)$$

where $b=\sqrt{2/m}$ and m is the total mass of the bar to be discretized into N number of element. Equations (12) and (13) can be simplified into cosine form for a continuous system by substituting Eqs. (14) and (15) depending on the stiffness k of each element and the damage location of the system is x_t. The expressions are obtained as shown in Eqs. (16) and (17).

$$\left(\frac{\partial F}{\partial k_t}\right)_{p,q} = -\left(\frac{4}{kN\pi}\right)^2\left(\begin{array}{l}\displaystyle\sum_{i=1}^{n}\frac{\left(\cos\frac{(2i-1)\pi x_t}{2L}\right)^2\sin\frac{(2i-1)\pi x_p}{2L}\sin\frac{(2i-1)\pi x_q}{2L}}{(2i-1)^2}\\ \displaystyle+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{\cos\frac{(2i-1)\pi x_t}{2L}\cos\frac{(2j-1)\pi x_t}{2L}\left(\sin\frac{(2i-1)\pi x_q}{2L}\sin\frac{(2j-1)\pi x_p}{2L}+\sin\frac{(2i-1)\pi x_p}{2L}\sin\frac{(2j-1)\pi x_q}{2L}\right)}{(2i-1)(2j-1)}\end{array}\right) \quad (16)$$

$$\left(\frac{\partial^2 F}{\partial k_t^2}\right)_{p,q} = \frac{64}{N^4\pi^2k^3}\left(\sum_{i=1}^{n}\frac{\sin\frac{(2i-1)\pi x_q}{2L}\cos\frac{(2i-1)\pi x_t}{2L}}{(2i-1)}\right)$$

$$\times\left(\sum_{i=1}^{n}\frac{\sin\frac{(2i-1)\pi x_p}{2L}\cos\frac{(2i-1)\pi x_t}{2L}}{(2i-1)}\right)\left(\sum_{i=1}^{n}\left(\cos\frac{(2i-1)\pi x_t}{2L}\right)^2\right) \quad (17)$$

The change in the flexibility matrix can be identified by only the diagonal terms of the derived matrix. Thus, to reduce the complicacy of formulations, Eqs. (12) and (13) can be simplified as,

$$\left(\frac{\partial F}{\partial k_t}\right)_{p,p}=-h^2\left(\sum_{i=1}^{n}\left(\frac{\phi_p^{(i)}\phi{\prime}_t^{(i)}}{\lambda^{(i)}}\right)^2+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{2\phi_p^{(i)}\phi_p^{(j)}\phi{\prime}_t^{(i)}\phi{\prime}_t^{(j)}}{\lambda^{(i)}\lambda^{(j)}}\right) \quad (18)$$

$$\left(\frac{\partial^2 F}{\partial k_t^2}\right)_{p,p}=2h^4\left(\sum_{i=1}^{n}\phi_p^{(i)}\frac{\phi{\prime}_t^{(i)}}{\lambda^{(i)}}\right)^2\left(\sum_{i=1}^{n}\frac{\left(\phi{\prime}_t^{(i)}\right)^2}{\lambda^{(i)}}\right) \quad (19)$$

Similarly, Eqs. (16) and (17) for the diagonal terms of a continuous system can be simplified into cosine form as,

$$\left(\frac{\partial F}{\partial k_t}\right)_p = -\left(\frac{4}{kN\pi}\right)^2\left(\begin{array}{l}\sum_{i=1}^{n}\left(\frac{\cos\frac{(2i-1)\pi x_t}{2L}\sin\frac{(2i-1)\pi x_p}{2L}}{2i-1}\right)^2\\+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\frac{2\cos\frac{(2i-1)\pi x_t}{2L}\cos\frac{(2j-1)\pi x_t}{2L}\left(\sin\frac{(2i-1)\pi x_p}{2L}\sin\frac{(2j-1)\pi x_p}{2L}\right)}{(2i-1)(2j-1)}\end{array}\right) \quad (20)$$

$$\left(\frac{\partial^2 F}{\partial k_t^2}\right)_p=\frac{64}{N^4\pi^2k^3}\left(\sum_{i=1}^{n}\frac{\sin\frac{(2i-1)\pi x_p}{2L}\cos\frac{(2i-1)\pi x_t}{2L}}{(2i-1)}\right)^2\left(\sum_{i=1}^{n}\left(\cos\frac{(2i-1)\pi x_t}{2L}\right)^2\right) \quad (21)$$

The flexibility matrix of the damaged structure F_d can be expressed in terms of the flexibility matrix of the undamaged structure F_u by expanding in form of Taylor series up to second order as shown in Eq. (22).

$$F_d = F_u - \Delta K\frac{\partial F}{\partial k_t}+\frac{1}{2}\frac{\partial^2 F}{\partial k_t^2}(\Delta K)^2 \quad (22)$$

The proposed methodology in the form of a flowchart has been described in Fig. 2. The change in the flexibility matrix due to damage can be regarded as a severity-sensitive parameter. It can provide a clear knowledge about the damage location and its quantification for a specified problem by only considering the diagonal terms of the global flexibility matrix of the structure.

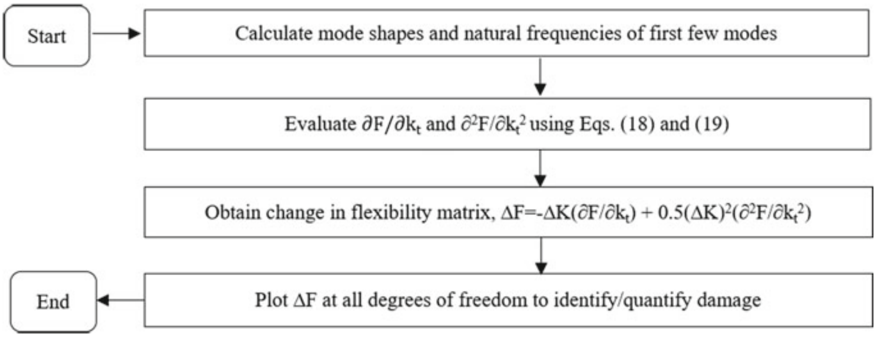

Fig. 2 Flowchart of the proposed methodology

3 Numerical Study

A simulation study has been performed on a 6-story shear building of uniform mass and stiffness, i.e., $m_1 = m_2 = \ldots = m_6 = 200$ kg and $k_1 = k_2 = \ldots = k_6 = 250$ kN/m, respectively, as shown in Fig. 3(a). Eigenvalue analysis has been performed to obtain the natural frequencies and mode shapes of the structure. A damage scenario

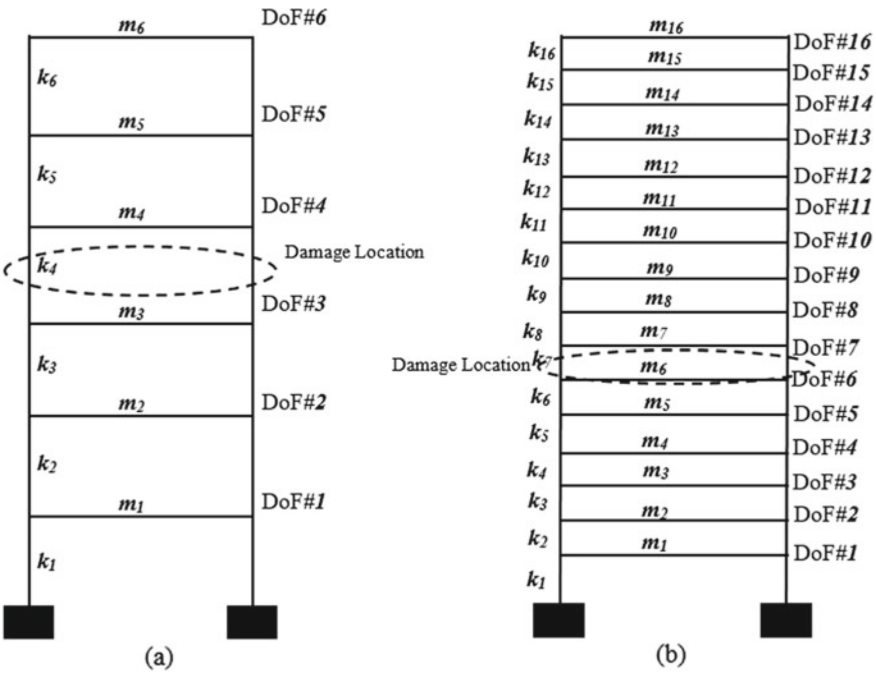

Fig. 3 Schematic diagram of the numerical model **a** 6-DoF and **b** 16-DoF

has been introduced by reducing the stiffness between DoF#3 and #4 by 20%. The flexibility matrix of both the intact and damaged structure has been evaluated using Eq. (4). Substituting the results of the eigenvalue problem of the intact structure in Eqs. (18) and (19), the first and second derivatives of the flexibility matrix with respect to damaged story stiffness have been calculated.

A similar study has been performed on a 16-story shear building of same mass and stiffness as the 6-story shear building. Damage has been introduced between DoF#6 and #7 by 20% reduction in the initial stiffness as shown in Fig. 3b by using Eqs. (20) and (21). The objective is to observe the effectiveness of the cosine form of the derived formulation for systems with higher number of DoFs such that it can be considered as a continuous system.

4 Results and Discussion

The change in flexibility matrix due to damage in a structure has been evaluated from the proposed formulation, i.e., Eqs. (10) and (11) for a shear building as shown in Fig. 4. It has been observed that the consideration up to the second derivative of the flexibility matrix of the intact structure with respect to the damaged story stiffness gave close to exact results, when compared to the conventional values.

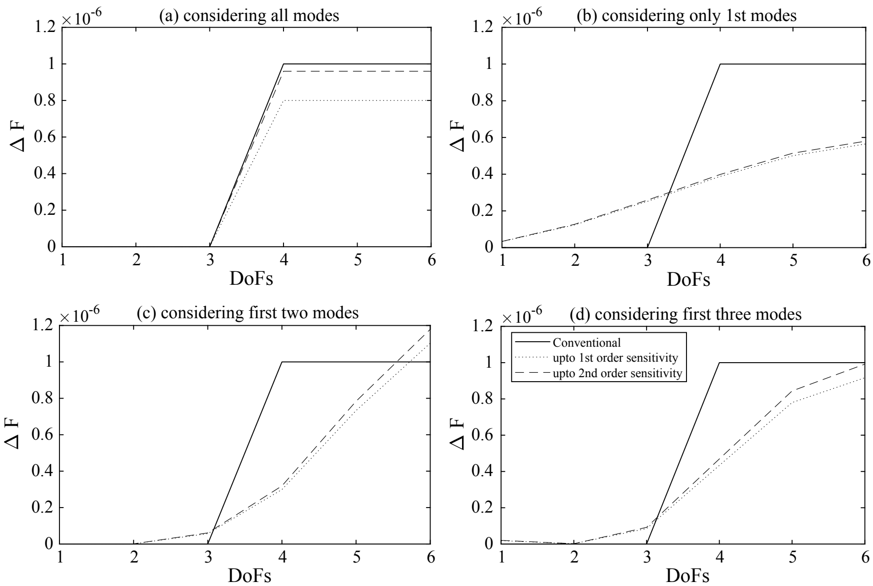

Fig. 4 Comparison of the change in flexibility matrix of the 6-story shear building due to stiffness reduction at 4^{th} story as obtained considering **a** all modes, **b** only first mode, **c** first 2 modes, and **d** first 3 modes

The percentage difference of results considering only up to the first derivative is 20%, whereas it reduces to 4% when the second derivative has been considered. The damage location can be easily identified from the plot between DoF#3 and #4 as obtained by only considering the first derivative of the intact flexibility matrix. However, the effect of damage to the flexibility of the structure can be accurately determined using the proposed approach. These are observed when all modes are considered numerically.

However, only the first few modes are available in the practical scenario. So the proposed approach has been studied for the first few modes. It has been observed that modal data of minimum first three modes are required to predict the damage severity. The magnitude of the results is approximately equal to the results obtained by considering the effect of all modes. However, damage location cannot be identified accurately. Instead, a range of location where the damage may occur can be predicted from the change in the slope of the plot.

A similar study performed for 16-story shear building with damage in between #6 and #7 DoFs using Eqs. (20) and (21) as shown in Fig. 5. A similar trend observed that the expression for the second derivative of the flexibility matrix increases the accuracy of the results. Modal data of minimum first three modes are required to predict the damage severity as it provides result close to exact result.

Though the damage location cannot be identified accurately, the formulation for the continuous system can quantify damage as can be obtained by considering all the modes. Thus, damage quantification in continuous systems or higher DoFs systems is possible by the proposed method.

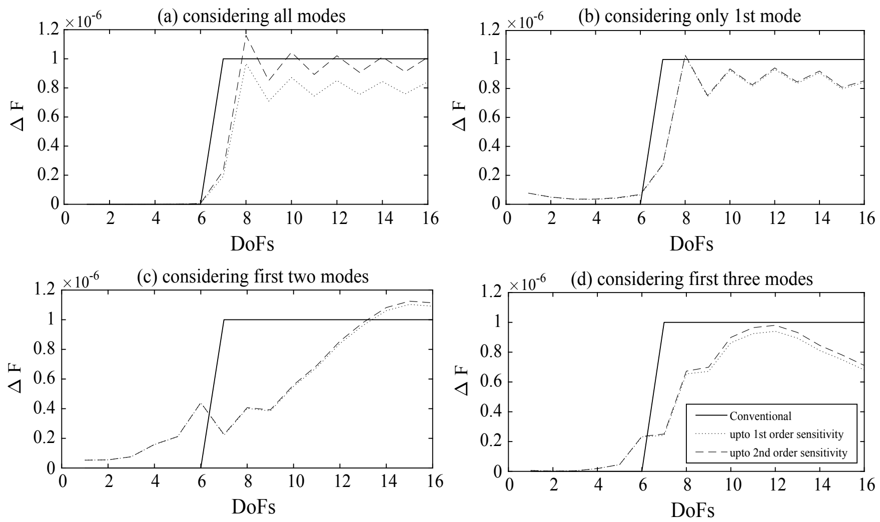

Fig. 5 Comparison of the change in flexibility matrix of 16-story shear building due to stiffness reduction at 7th story as obtained considering **a** all modes, **b** only first mode, **c** first 2 modes, and **d** first 3 modes

5 Conclusion

Flexibility-based method is becoming a popular method for structural damage detection due to its easy measurement in a real structure. In this study, a simplified formulation of the change in flexibility matrix between intact and damaged structure has been derived using the first and second derivative of flexibility matrix of the intact structure with respect to damaged story stiffness. The double derivative enhances the accuracy of the results. Precise results can be obtained using the proposed method when all the modes are considered. The damage severity can be predicted even by considering modal data of minimum first three modes of the structure. Thus, the proposed method can quantify damage in terms of the change in flexibility matrix as observed from the simulation. However, its applicability needs to be validated through experimental investigation in future.

6 Appendix

- Some expressions required for the formulation of the derivatives of the flexibility matrix are summarized in the following equations:

$$\Lambda = \begin{bmatrix} \lambda^{(1)} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda^{(n)} \end{bmatrix} \tag{23}$$

$$\Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda^{(i)}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{\lambda^{(i)}} \end{bmatrix} \tag{24}$$

Sensitivity matrix [1] of the square of the circular frequency with respect to the damaged story stiffness is given as,

$$\frac{\partial \Lambda}{\partial k_t} = \begin{bmatrix} \left(\phi_t^{(1)} - \phi_{t-1}^{(1)}\right)^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \left(\phi_t^{(n)} - \phi_{t-1}^{(n)}\right)^2 \end{bmatrix} \tag{25}$$

$$\frac{\partial \Lambda^{-1}}{\partial k_t} = -\Lambda^{-1} \frac{\partial \Lambda}{\partial k_t} \Lambda^{-1} \tag{26}$$

The weightage factor matrix [2] Ψ correlating mode shape matrix with the first partial derivative is given as,

$$\Psi = \begin{bmatrix} 0 & \cdots & \frac{\left(\phi_t^{(1)}-\phi_{t-1}^{(1)}\right)\left(\phi_t^{(n)}-\phi_{t-1}^{(n)}\right)}{\lambda^{(n)}-\lambda^{(1)}} \\ \vdots & \ddots & \vdots \\ -\frac{\left(\phi_t^{(1)}-\phi_{t-1}^{(1)}\right)\left(\phi_t^{(n)}-\phi_{t-1}^{(n)}\right)}{\lambda^{(n)}-\lambda^{(1)}} & \cdots & 0 \end{bmatrix} \tag{27}$$

The weightage factor matrix [1] Γ correlating mode shape matrix with the second partial derivative is given as,

$$\Gamma = \begin{bmatrix} -\sum_{\substack{m=1\\m\neq 1}}^{n}\left(\frac{\left(\phi_t^{(1)}-\phi_{t-1}^{(1)}\right)\left(\phi_t^{(m)}-\phi_{t-1}^{(m)}\right)}{\lambda^{(1)}-\lambda^{(m)}}\right)^2 & \cdots & 2\frac{\begin{pmatrix}\left(\phi_t^{(n)}-\phi_{t-1}^{(n)}\right)\\ \times\sum_{\substack{m=1\\m\neq n}}^{n}\frac{\left(\phi_t^{(m)}-\phi_{t-1}^{(m)}\right)^2}{\left(\lambda^{(n)}-\lambda^{(m)}\right)}\\ -\frac{\left(\phi_t^{(n)}-\phi_{t-1}^{(n)}\right)^3}{\left(\lambda^{(n)}-\lambda^{(1)}\right)}\end{pmatrix}\left(\phi_t^{(1)}-\phi_{t-1}^{(1)}\right)}{\left(\lambda^{(n)}-\lambda^{(1)}\right)} \\ \vdots & \ddots & \vdots \\ 2\frac{\begin{pmatrix}\left(\phi_t^{(1)}-\phi_{t-1}^{(1)}\right)\\ \times\sum_{\substack{m=1\\m\neq 1}}^{n}\frac{\left(\phi_t^{(m)}-\phi_{t-1}^{(m)}\right)^2}{\left(\lambda^{(1)}-\lambda^{(m)}\right)}\\ -\frac{\left(\phi_t^{(1)}-\phi_{t-1}^{(1)}\right)^3}{\left(\lambda^{(1)}-\lambda^{(n)}\right)}\end{pmatrix}\left(\phi_t^{(n)}-\phi_{t-1}^{(n)}\right)}{\left(\lambda^{(1)}-\lambda^{(n)}\right)} & \cdots & -\sum_{\substack{m=1\\m\neq n}}^{n}\left(\frac{\left(\phi_t^{(n)}-\phi_{t-1}^{(n)}\right)\left(\phi_t^{(m)}-\phi_{t-1}^{(m)}\right)}{\lambda^{(n)}-\lambda^{(m)}}\right)^2 \end{bmatrix} \tag{28}$$

$$\Omega = \Phi^{\mathrm{T}}\frac{\partial K}{\partial k_t}\Phi = \begin{bmatrix} \left(\phi_t^{(1)}-\phi_{t-1}^{(1)}\right)^2 & \cdots & \left(\phi_t^{(1)}-\phi_{t-1}^{(1)}\right)\left(\phi_t^{(n)}-\phi_{t-1}^{(n)}\right) \\ \vdots & \ddots & \vdots \\ \left(\phi_t^{(1)}-\phi_{t-1}^{(1)}\right)\left(\phi_t^{(n)}-\phi_{t-1}^{(n)}\right) & \cdots & \left(\phi_t^{(n)}-\phi_{t-1}^{(n)}\right)^2 \end{bmatrix} \tag{29}$$

- Derivation of $\partial^2\Lambda/\partial k_t^2$ from the basic eigenvalue formulation:

Pre-multiplying both sides of Eq. (1) and substituting the property of mass-normalized mode shapes, Λ will be obtained as

$$\Lambda = \Phi^{\mathrm{T}} K \Phi \tag{30}$$

Differentiating Eq. (30) twice with respect to damaged story stiffness and substituting Eqs. (7), (8), and (25), the expression of $\partial^2\Lambda/\partial k_t^2$ is obtained as

$$\frac{\partial^2 \Lambda}{\partial k_t^2} = \left(\Gamma^{\mathrm{T}} \Lambda + \Lambda \Gamma\right) - 2(\Psi \Omega - \Omega \Psi + \Psi \Lambda \Psi) \tag{31}$$

Differentiating Eq. (26) with respect to damaged story stiffness and substituting Eq. (9), the expression for $\partial^2 \Lambda^{-1} / \partial k_t^2$ can be obtained as,

$$\frac{\partial^2 \Lambda^{-1}}{\partial k_t^2} = \Lambda^{-1}\left[-2\frac{\partial \Lambda}{\partial k_t}\Lambda^{-1}\frac{\partial \Lambda}{\partial k_t} - \left(\Gamma^{\mathrm{T}} \Lambda + \Lambda \Gamma\right) + 2(\Psi \Omega - \Omega \Psi + \Psi \Lambda \Psi)\right]\Lambda^{-1} \tag{32}$$

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