Chapter 2
Symmetries of Shapes

In this chapter, we are interested in the symmetric properties of plane figures. By a symmetry of a plane figure we mean a motion of the plane that moves the figure so that it falls back on itself.

2.1 Symmetry

One of the most important and beautiful themes unifying many areas of modern mathematics is the study of symmetry. Many of us have an intuitive idea of symmetry, and we often think about certain shapes or patterns as being more or less symmetric than others. In this chapter we sharpen the concept of "shape" into a precise definition of "symmetry".

Definition 2.1 A *transformation of the plane* is a function $f : \mathbb{R}^2 \to \mathbb{R}^2$.

Transformation involves moving an object from its original position to a new position. The object in the new position is called the image. Each point in the object is mapped to another point in the image.

A geometric shape or object is symmetric if it can be divided into two or more identical pieces that are arranged in an organized fashion. This means that an object is symmetric if there is a transformation that moves individual pieces of the object, but doesn't change the overall shape. The type of symmetry is determined by the way the pieces are organized.

Example 2.2 Symmetry occurs in nature in many ways; for example, the human form is symmetric, see Fig. 2.1.

Example 2.3 The heart carved out is an example of symmetry, see Fig. 2.2.

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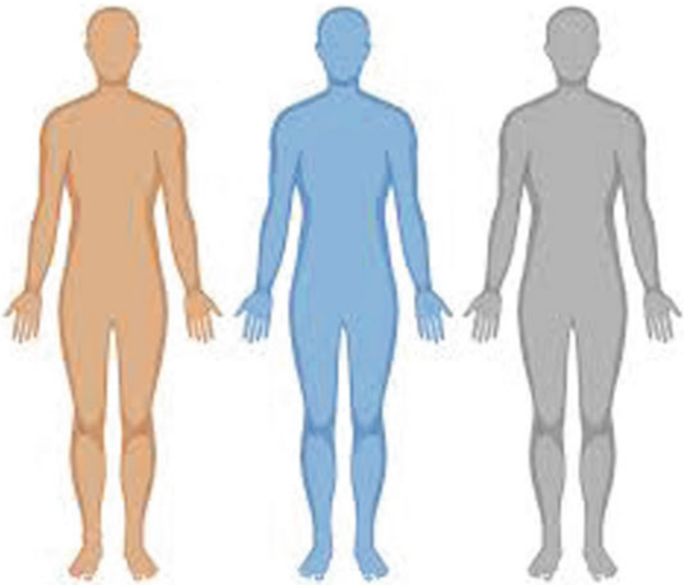
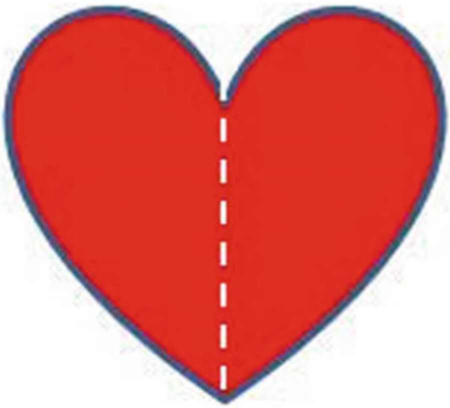


Fig. 2.1 Symmetry of human form

Fig. 2.2 The heart carved out



Definition 2.4 The symmetry through a line $\mathcal{L}$ is a transformation of the plane which sends point P into point Q such that $\mathcal{L}$ is the midperpendicular to segment PQ. Such a transformation is also called the *axial symmetry* and $\mathcal{L}$ is called the *axis of the symmetry*. If a figure turns into itself under the symmetry through line $\mathcal{L}$, then $\mathcal{L}$ is called the axis of symmetry of this figure.

The line of symmetry can be vertical, horizontal, or diagonal. There may be one or more lines of symmetry, see Fig. 2.3.

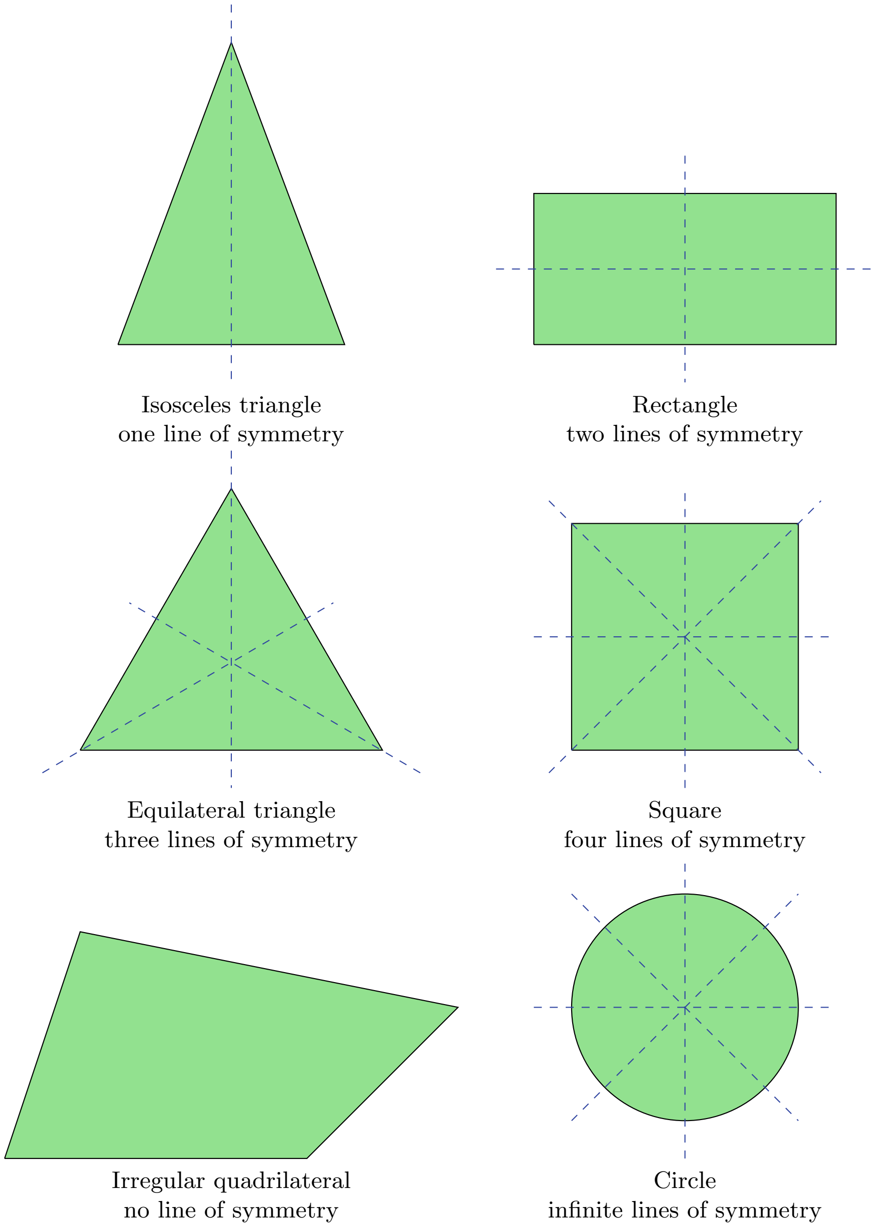


Fig. 2.3 Line of symmetry

Fig. 2.4 Font Geneva

Exercises

1. Prove that the inverse of a bijective transformation is a bijective transformation.
2. The letters in Fig. 2.4 are in the font Geneva. Some of them have one line of symmetry, some have two, some have none, and some have point symmetry. The latter are invariant under a half-turn. Which ones are in the first set? The second? The third? The fourth?

2.2 Translations

Translation is a term used in geometry to describe a function that moves an object a certain distance. The object is not altered in any other way. It is not rotated, reflected, or resized. In a translation, every point of the object must be moved in the same direction and for the same distance.

Definition 2.5 A *translation* is an object from one location to another, without any change in size or orientation.

A horizontal translation refers to an object from left to right or vice versa along the x-axis (the horizontal access). A vertical translation refers to an object up or down along the y-axis (the vertical access). In many cases, a translation will be both horizontal and vertical, resulting in a diagonal object across the coordinate plane, for example, see Fig. 2.5. The trivial translation is the translation through zero distance; all other translations are non-trivial.

Definition 2.6 Let P and Q be two points in a plane. The *translation* from P to Q is transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that

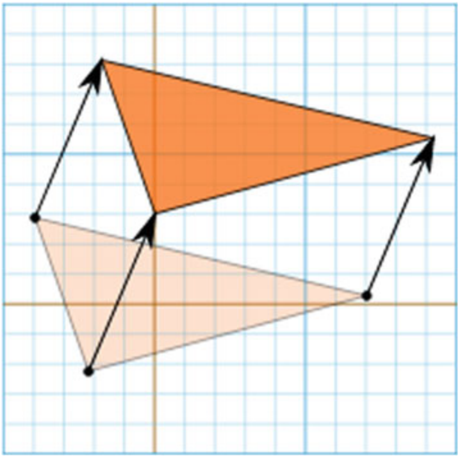


Fig. 2.5 A translation

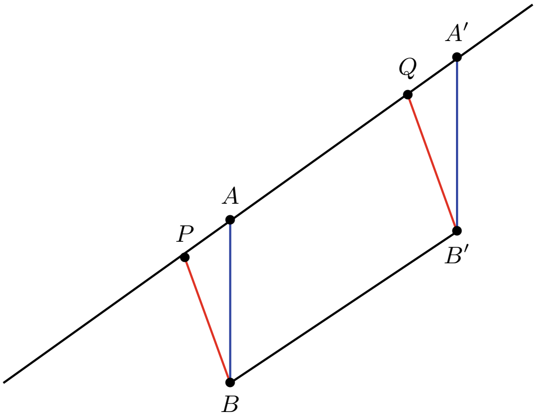


Fig. 2.6 The quadrilateral $PQB'B$ and $AA'B'B$ are parallelograms

(1) $Q = f(P)$;
(2) If $P = Q$, then f is the identity;
(3) If $P \neq Q$, let A be any point on PQ and let B be any point of PQ; let $A' = f(A)$ and $B' = f(B)$. Then quadrilaterals $PQB'B$ and $AA'B'B$ are parallelograms, see Fig. 2.6.

When $P \neq Q$ one can think of a translation as a slide in the direction of vector PQ. If A is any point and $\mathcal{L}$ is the line through A parallel to PQ, then $f(A)$ is the point on $\mathcal{L}$ whose distance from A in the direction of vector PQ is PQ.

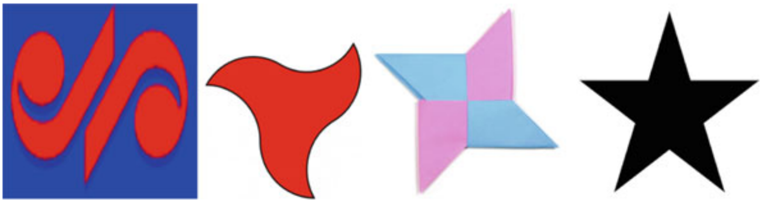


Fig. 2.7 Some shapes with rotational symmetries

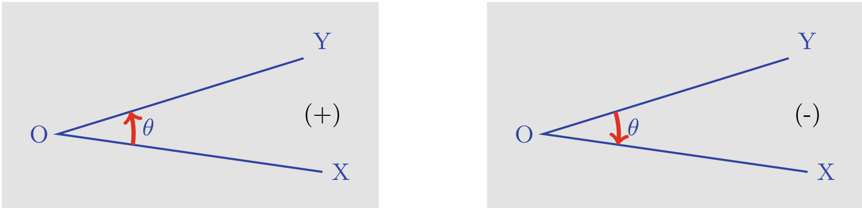


Fig. 2.8 Positively (+) and negatively (−) oriented angle XOY

Exercises

1. Prove that the composition of two translations is a translation.
2. Prove that
   (a) A composition of translations commutes;
   (b) The inverse of a translation is a translation.

2.3 Rotation Symmetries

An equilateral triangle can be rotated by 120°, 240°, or 360° angles without really changing it. If you were to close your eyes, and a friend rotated the triangle by one of those angles, then after opening your eyes you would not notice that anything had changed. In contrast, if that friend rotated the triangle by 33° or 85°, you would notice that the bottom edge of the triangle is no longer perfectly horizontal. Many other shapes that are not regular polygons also have rotational symmetries. Each shape illustrated in Fig. 2.7, for example, has rotational symmetries.

In order to define the rotation we need the notions of "*oriented angle*" and of its "*signed measure*".

The oriented angle XOY is an angle in which we distinguish the order of its sides OX, OY (see Fig. 2.8). If the transition from OX to OY is opposite to the direction of the clock's hands, then we consider the angle as being "positively oriented", or

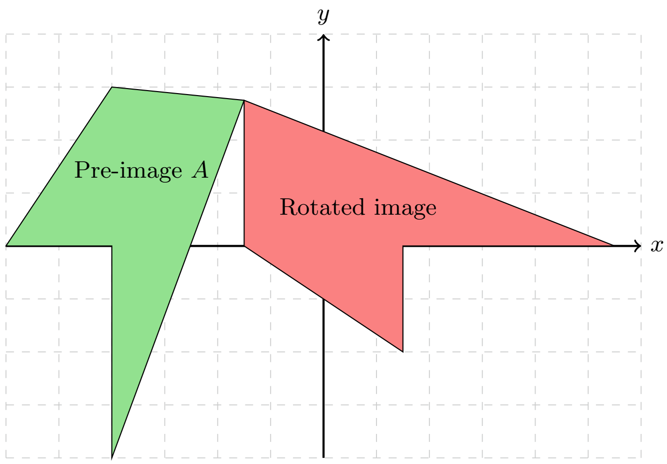


Fig. 2.9 Example of a rotation

simply a positive angle. If the transition is in the same direction as the clock's, we consider the angle as being "negatively oriented", or simply a negative angle.

Definition 2.7 A *rotation* in $\mathbb{R}^2$ is a circular movement of an object around a center of rotation.

Example 2.8 An equilateral triangle can be rotated by 120°, 240°, or 360° angles without really changing it. If you were to close your eyes, and a friend rotated the triangle by one of those angles, then after opening your eyes you would not notice that anything had changed.

Example 2.9 Figure 2.9 shows that the pre-image A is rotated 90° counterclockwise about the center point A to form the rotated image.

Exercises

1. Show that the composition of two rotations is a rotation.
2. Find the image of the ellipse $x^2/4 + y^2/9 = 1$ under the 60° rotation about $(0, 0)$.
3. A rotation of about $(-1, 0)$ is followed by a rotation of about $(1, 0)$. The first rotation is applied again after that. Analyze the composite of these three rotations.

Fig. 2.10 Beautiful reflections in nature

2.4 Mirror Reflection Symmetries

Another type of symmetry that we can find in two-dimensional geometric shapes is mirror reflection symmetry. More specifically, we can draw a line through some shapes and reflect the shape through this line without changing its appearance.

Definition 2.10 A *reflection* is defined by its axis or line of symmetry, i.e., the *mirror line*. Each point $P(x, y)$ is mapped onto the point $P'(x', y')$ which is the mirror image of (x, y) in the mirror line. This yields that PP' is perpendicular to the mirror. A reflection preserves distances.

Example 2.11 Many objects in nature appear the same on the left and right; for instance, see Figs. 2.10 and 2.11. The left half of a butterfly appears the same as the right half, and if we were to place a mirror down the center to reflect the left half, the resulting butterfly would look the same as the original, see Fig. 2.11.

Example 2.12 In Fig. 2.12, we can observe some reflections in nature.

A *glide reflection* is a composition of transformations. In a glide reflection, a translation is first performed on the figure, then it is reflected over a line, which is parallel to the direction of the previous translation. Reversing the order of combining gives the same result. Glide reflections with non-trivial translation have no fixed points. The composition of a reflection in a line and a translation in a perpendicular direction is a reflection in a parallel line. However, a glide reflection cannot be reduced like that. Thus the effect of a reflection on a line combined with a translation in one of

Fig. 2.11 Butterfly and reflection

Fig. 2.12 Some reflections in nature

Fig. 2.13 Footprints fixed by a glide reflection

the directions of that line is a glide reflection, with a special case as just a reflection. Therefore, the only required information is the translation rule and a line to reflect over, the resulting orientation of the two figures is opposite.

Example 2.13 In your mind, picture the footprints you leave when walking in the sand. Imagine a line $\mathcal{L}$ positioned midway between your left and right footprints. In your mind, slide the entire pattern one-half step in a direction parallel to $\mathcal{L}$ then reflect in line $\mathcal{L}$. The image pattern exactly superimposes on the original pattern. This transformation is an example of a glide reflection with axis $\mathcal{L}$ (see Fig. 2.13).

Alternatively, we can think of a glide reflection with axis $\mathcal{L}$ as a reflection in line $\mathcal{L}$ followed by a translation parallel to $\mathcal{L}$.

Definition 2.14 A non-identity transformation f is an *involution* if and only if $f^2 = id$.

Note that an involution f has the property that $f = f^{-1}$.

Theorem 2.15 *A reflection is an involution.*

Proof Left as an exercise for the reader. ■

Exercises

1. What conjectures can you make about a figure reflected in two lines?
2. Prove that a non-identity translation is not a reflection.
3. Show that the composition of translations and non-trivial rotation is a rotation.
4. Point P is reflected in two parallel lines, $\mathcal{L}$ and $\mathcal{L}'$, to form P' and P''. The distance from $\mathcal{L}$ to $\mathcal{L}'$ is 10 cm. What is the distance PP''.
5. Words such as **MOM** and **RADAR** that spell the same forward and backward, are called *palindromes*.
   (a) When reflected in their vertical midlines, **MOM** remains **MOM** but the **R**s and **D** in **RADAR** appear backward. Find at least five other words like **MOM** that are preserved under reflection in their vertical midlines.
   (b) When reflected in their horizontal midlines, **MOM** becomes **WOW**, but **BOB** remains **BOB**. Find at least five other words like **BOB** that are preserved under reflection in their horizontal midlines.

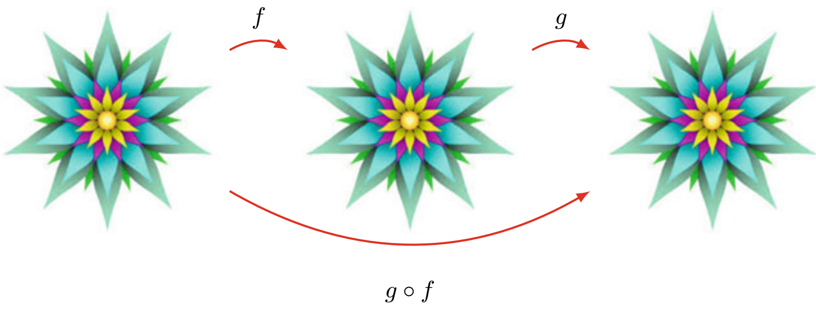


Fig. 2.14 $S(F)$ is closed under composition of functions

6. Show that any glide reflection can be written as the composition $R \circ S$ of a reflection R and a rotation S. Comment on the uniqueness of this decomposition.
7. Show that any glide reflection can be written as the product of reflections in the sides of an equilateral triangle.
8. Find all values for a and b such that $f(x, y) = (ay, x/b)$ is an involution.

2.5 Congruence Transformations

We say that two plane figures are *congruent* if they have the same shape and size. In other words, two plane figures are congruent if one figure can be moved so that it fits exactly on top of the other figure. This movement can always be affected by a sequence of translations, rotations, and reflections. Each part of one figure can be matched with a part of the other figure, and matching angles have the same size, matching intervals have the same length, and matching regions have the same area.

For instance, our reflections in a mirror have the same shape and size as we do, so we would say that we are congruent to our reflection in a mirror.

Definition 2.16 A *congruence transformation* is a transformation under which the image and pre-image are congruent.

We denote the set of all symmetries of a plane figure F by $S(F)$. The elements of $S(F)$ are distance-preserving functions $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(F) = F$. So we can form the composite of any two elements f and g in $S(F)$ to obtain the function $g \circ f : \mathbb{R}^2 \to \mathbb{R}^2$. Let $f, g \in S(F)$. Since f and g both map F to itself, so must $g \circ f$; and since f and g both preserve distance, so must $g \circ f$. Hence $g \circ f \in S(F)$. We describe this situation by saying that the set $S(F)$ is closed under composition of functions, see Fig. 2.14.

Example 2.17 We consider some examples of composition in $S(\square)$, the set of symmetries of the square. Any non-trivial translation alters the location of the square in

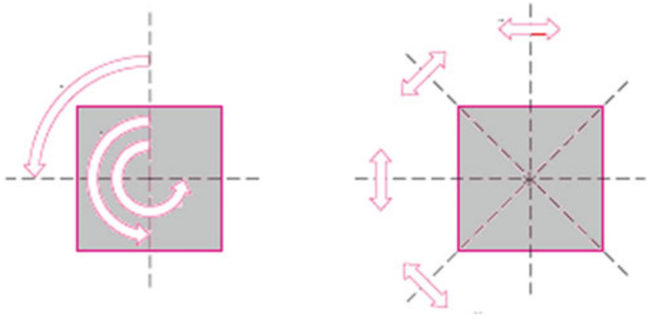
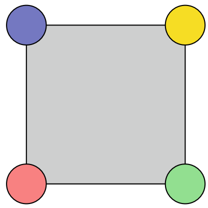


Fig. 2.15 Rotations and reflections of a square

Fig. 2.16 Initial position



the plane and so cannot be a symmetry of the square. Therefore, we consider only rotations and reflections as potential symmetries of the square. Figure 2.15 shows our labeling for the following elements:

R_0 = Rotation of 0°	S_0 = Reflection about a horizontal axis
R_1 = Rotation of 90°	S_1 = Reflection about a vertical axis
R_2 = Rotation of 180°	S_2 = Reflection about the main diagonal
R_3 = Rotation of 270°	S_3 = Reflection about the other diagonal

We want to find $R_1 \circ S_0$ and $S_2 \circ R_1$. We consider the initial position as Fig. 2.16 to keep track of the composition of the symmetries.

Figure 2.17 shows the effect of $R_1 \circ S_0$, i.e., first S_0 and then R_1. Comparing the initial and final positions, we observe that the effect of $R_1 \circ S_0$ is to reflect the square in the diagonal from bottom left to top right. This is the symmetry that we have called S_2, and hence $R_1 \circ S_0 = S_2$.

Exercises

1. With the notation given in Example 2.17, find the following composites of symmetries of the square: $R_2 \circ S_3$, $R_2 \circ S_2$, and $R_3 \circ_3$.

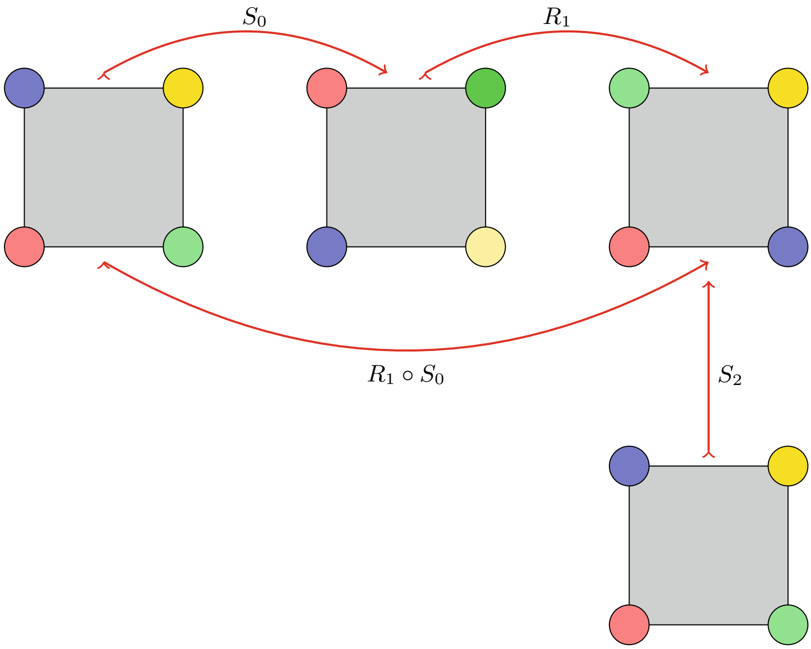


Fig. 2.17 $R_1 \circ S_0 = S_2$

2.6 Worked-Out Problems

Problem 2.18 Let ABC be a triangle with the vertices labeled clockwise such that $AC = BC$ and $\angle ACB = \pi/2$. Let S_{AB} be the reflection in the line AB, S_{AC} be the reflection in the line AC, and R be the rotation by $\pi/2$ counterclockwise around B. Identify the composition $R \circ S_{AB} \circ S_{AC}$.

Solution We can solve problems like this one using the following simple strategy. Find three points which form a triangle and see where the composition of isometries takes them. Next, it's time to guess what the isometry is. If your guess is correct for the three vertices of the triangle, then it must be correct. And this is because the theorem above guarantees that if you know what an isometry does to three corners of a triangle, then you know what the isometry does to every point in the plane. Figure 2.18 shows triangle ABC drawn on a grid of squares. Since we want to choose three points which form a triangle, we may as well choose the points A, B, and C.

- It is easy to check that $S_{AC}(A) = A$, $S_{AB}(A) = A$ and $R(A) = P$. In other words, $R \circ S_{AB} \circ s_{AC}(A) = P$;
- It is easy to check that $S_{AC}(B) = Q$, $S_{AB}(Q) = N$ and $R(N) = Q$. In other words, $R \circ S_{AB} \circ S_{AC}(B) = Q$;
- It is easy to check that $S_{AC}(C) = C$, $S_{AB}(C) = M$ and $R(M) = C$. In other words, $R \circ S_{AB} \circ S_{AC}(C) = C$.

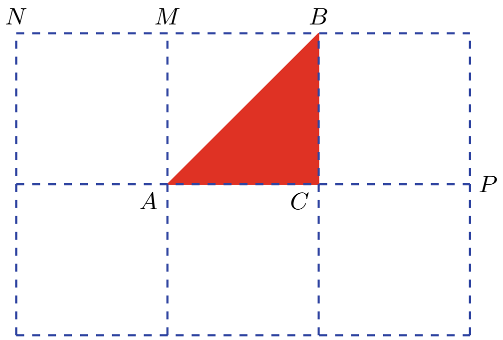


Fig. 2.18 Triangle ABC drawn on a grid of squares

Hence, can you think of an isometry which takes A to P, B to Q, and C to C? If you think hard enough, you should realize that it's just a rotation by π around C. So, we have managed to deduce that the composition $R \circ S_{AB} \circ S_{AC}$ is a rotation by π around C. ■

Problem 2.19 Let $ABCD$ be a rectangle with the vertices labeled counterclockwise such that $BC = 2AB$. Suppose that

- S_{AB} is the reflection in the line AB;
- R_B is the counterclockwise rotation by $\pi/2$ about B;
- T_{DB} is the translation which takes D to B;
- G_{CD} is the glide reflection in the line CD which takes C to D.

Identify the composition $S_{AB} \circ R_B \circ T_{DB} \circ G_{CD}$.

Solution Figure 2.19 shows rectangle $ABCD$ drawn on a grid of squares. Since we want to choose three points which form a triangle, we may as well choose the points A, B, and C.

- It is easy to check that $G_{CD}(A) = E, T_{DB}(E) = D, R_B(D) = F$ and $S_{AB}(F) = J$. In other words, $S_{AB} \circ R_B \circ T_{DB} \circ G_{CD}(A) = J$;
- It is easy to check that $G_{CD}(B) = H, T_{DB}(H) = C, R_B(C) = I$, and $S_{AB}(I) = I$. In other words, $S_{AB} \circ R_B \circ T_{DB} \circ G_{CD}(B) = I$;
- It is easy to check that $G_{CD}(C) = D$, $T_{DB}(D) = B$, $R_B(B) = B$, and $S_{AB}(B) = B$. In other words, $S_{AB} \circ R_B \circ T_{DB} \circ G_{CD}(C) = B$.

So can you think of an isometry which takes A to J, B to I, and C to B? If you think hard enough, you should realize that it is a rotation, although you might not be sure of where the center lies. However, we can use the fact that if a rotation takes X to Y, then the center of rotation must lie on the perpendicular bisector of xy. In particular, the center of the rotation that we are interested in must lie on the perpendicular bisector of AJ as well as the perpendicular bisector of BI. And there is only one point which does that namely, the point O labeled in Fig. 2.19. It is now easy to deduce that the composition must be a rotation about O by $\angle AOJ = \pi/2$ in the clockwise direction. ■

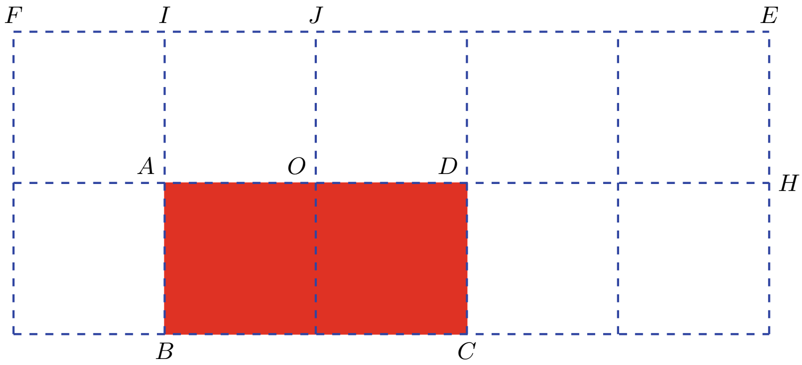
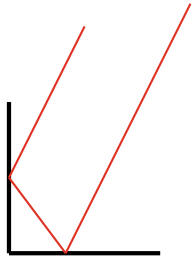


Fig. 2.19 Rectangle $ABCD$ drawn on a grid of squares

Fig. 2.20 What letter has been folded once to make this shape?

Fig. 2.21 A ray of light is reflected by two perpendicular flat mirrors



2.7 Supplementary Exercises

1. Prove that the composition of two reflections with parallel axes is a translation perpendicular to these axes by a distance twice that from the first axis to the second.
2. Prove that the composition of two reflections with axes meeting at a point is a rotation about that point through an angle twice that from the first axis to the second.

3. Prove that the composition of three reflections is a reflection if three axes are parallel or concurrent, and otherwise is a glide reflection.
4. Five reflections are composed, with axes in order the lines $x = 0$, $x + y = 6$, $y = 6$, $y = x + 2$, $y = x + 8$. Is the composition a reflection or a glide reflection? Give details.
5. What capital letters could be cut out of paper and given a single fold to produce Fig. 2.20?
6. Give an example of a bijection $f : \mathbb{R}^2 \to \mathbb{R}^2$ that preserves angles but not distances. Describe in general terms the effect of f on lines, circles, and triangles.
7. Prove that the composition of a non-trivial rotation and a reflection is glide reflection except when the axis of the reflection passes through the center of the rotation, in which case it is a reflection.
8. A ray of light is reflected by two perpendicular flat mirrors. Prove that the emerging ray is parallel to the initial incoming ray, as indicated in Fig. 2.21.