Recent Trends

AS THIS IS THE FINAL CHAPTER OF THIS OVERVIEW we should have a look at the 'recent' developments in graph algorithmics.[1] To keep this brief short let me select one topic — namely treewidth — and use that as a chassis to explain various 'recent' concepts.

[1] Better late than never ...

4.1 Triangulations

A triangulation of a graph is an embedding of it in a chordal graph.

4.1.1 Chordal Graphs

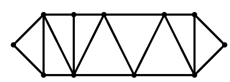


Figure 4.1: A chordal graph

Definition 4.1. A graph is chordal if it has no chordless cycle[2] of length more than 3.

[2] A chordless cycle is a cycle without chords, that is, it is an induced cycle.

We could say that chordal graphs are $\{C_n\}$–free graphs, (for all $n \geqslant 4$), except that this notation looks a bit weird (since it is not finite). Trees are well–known examples of chordal graphs and in general — one could say — that chordal graphs have a 'tree-like'–structure.

The way in which the structure of chordal graphs resembles that of trees is best conveyed via their minimal separators.

Exercise 4.1

Prove that every connected chordal graph, with at least two vertices, has a simplicial vertex, that is, a vertex whose neighborhood is a clique.
Hint: Consider a feasible partition, see Definition 2.69 on page 76.

T. Kloks, M. Xiao, *A Guide to Graph Algorithms*, https://doi.org/10.1007/978-981-16-6350-5_4

Lemma 4.2. *A connected graph is chordal if and only if all its minimal separators are cliques.* [3]

Proof. First, assume that G has a minimal separator S with two nonadjacent vertices — say a and b. Let S be a minimal $x|y$-separator — and let C_x and C_y be the components of $G - S$ that contain x and y. Then every vertex of S has a neighbor in C_x and in C_y.

Two chordless $a \sim b$ – paths — one with its internal vertices in C_x and the other with its internal vertices in C_y — form a chordless cycle in G of length at least 4. — So — when G has a minimal separator that is not a clique then G is not chordal.

Now assume that every minimal separator in G is a clique. When G is a clique itself we are done, since cliques are chordal graphs. We proceed by induction on the number of vertices in the graph. [4]

Let S be a minimal separator and let $C_1, \cdots, C_t$ be the components of $G - S$. We claim that every minimal separator in each graph $C_i \cup S$ is a clique.

— To see that — let S' be a minimal $x|y$-separator in $C_1 \cup S$. Since S is a clique it can be contained in at most one component of

$$(C_1 \cup S) \setminus S'.$$

It follows that S' is a minimal separator in G — and so — S' is a clique.

By induction — on the number of vertices — each graph $C_i \cup S$ is chordal — and since any chordless cycle of length at least 4 would be contained in one $C_i \cup S$ — such a chordless cycle cannot exist in G.

This proves the lemma. □

[3] We exclude disconnected graphs, because, those have $\varnothing$ as a minimal separator, and $\varnothing$ is not a clique.

[4] Alternatively, we could proceed by induction on the number of minimal separators in the graph.

Usually trees have leaves. In the language of chordal graphs leaves are called simplicials.

Definition 4.3. A vertex is simplicial if its neighborhood is either $\varnothing$ or a clique.

Lemma 4.4. *Every chordal graph has a simplicial.*

Proof. If G is a clique we are done since then every vertex is a simplicial.

Consider a feasible partition $\{X, S, C\}$.[5] Then S is a clique since it is a minimal separator in a chordal graph. By induction on the number of vertices in the graph, $G[X]$ has a simplicial vertex. Since S is a clique this is also a simplicial in G.

[5] See Section 2.9.1.

This proves the lemma. □

Since cycles have no simplicials we have the following characterization.

> A graph is chordal if and only if every induced subgraph of G has a simplicial.

Equivalently — we have —

Corollary 4.5. *A graph is chordal if and only if it has a perfect elimination order — that is — an ordering of its vertices — say*

$$x_1 \quad \cdots \quad x_n \qquad\qquad —$$

such that

Abuse coming up ! !

$$\forall_i \quad x_i \quad \textit{is simplicial in} \quad G[x_i \cdots x_n].$$

Exercise 4.2

Prove that every chordal graph that is not a clique has at least two simplicial vertices.

4.1.2 Clique – Trees

From a computational point of view the clique – tree of a chordal graph says it all.

Definition 4.6. A clique tree of a graph G is a pair $(T, \mathcal{C})$ where T is a tree and $\mathcal{C}$ is the set of all maximal cliques in G. Furthermore, there is a bijection [6] $V(T) \to \mathcal{C}$ which satisfies the property [7]

> for each vertex $x \in V(G)$ the maximal cliques that contain x form a subtree of T under the bijection.

[6] In future we'll simply identify each vertex of T with one maximal clique of G.
[7] Let's call this the subtree property.

Theorem 4.7. *A graph is chordal if and only if it has a clique – tree.*

Proof. Assume G has a clique – tree $(T, \mathcal{C})$. Consider a clique $C \in \mathcal{C}$ that is a leaf of T. We contradict the maximality of C when we assume that every vertex of C is also an element of the only neighbor of C in T. Therefore — by the subtree – property — there is a vertex in C that appears in no other element of $\mathcal{C}$. — That vertex — is a simplicial of G.

Exercise 4.3

Show that if a graph has a clique-tree then so does every induced subgraph.

Notice that having a clique – tree is a hereditary property — that is — if a graph G has a clique – tree then so does every induced subgraph of G. This shows that G is chordal, since every induced subgraph has a simplicial.

Assume G is chordal. When G is a clique it has a clique – tree and then we are done. Otherwise — let x be a simplicial of G. By induction on $|V(G)|$ the chordal graph $G - x$ has a clique - tree, say $(T', \mathcal{C}')$.

Since x is simplicial its neighborhood $N(x)$ is a clique. Let $P \in \mathcal{C}'$ contain $N(x)$. Create a new node for $N[x]$ and attach it in T to P. It is readily checked that this creates a clique – tree for G.

This proves the theorem. $\square$

Exercise 4.4

Construct a clique – tree for the graph in Figure 4.1 on Page 129. Which chordal graphs have a clique – tree that is a path?
HINT: Chordal graphs that have a clique – tree which is a path are called interval graphs. Consider the intersection graph of a set of intervals on the real line.

Exercise 4.5

Show that a graph is chordal if and only if it is the intersection graph of a set of subtrees in a tree. By that we mean that there exists a tree T and a collection of n subtrees of T

$$\{ T_x \mid x \in V(G) \}$$

such that

$$\{ x, y \} \in E(G) \quad \Leftrightarrow \quad V(T_x) \cap V(T_y) \neq \varnothing .$$

HINT: Assume G is chordal. For a vertex $x \in V(G)$ consider the subtree T_x of all the maximal cliques in the clique – tree T that contain the vertex x.

Exercise 4.6

Let G be a connected chordal graph and let $(T, \mathcal{C})$ be a clique-tree for G. Let

$$\mathcal{S} = \{ C_i \cap C_j \mid \{ C_i, C_j \} \in E(T) \}. \tag{4.1}$$

Show that $\mathcal{S}$ is the set of minimal separators of G.

4.2 Treewidth

Definition 4.8. A chordal embedding or triangulation of a graph G is a chordal graph — say H [8] — with

$$V(H) = V(G) \quad \text{and} \quad E(G) \subseteq E(H). \qquad (4.2)$$

[8] H i !

Notice that every graph has a chordal embedding — just add all edges to G to make it a clique. The triangulation is minimal if the removal of an added edge creates a chordless cycle. [9]

[9] which has to be a 4-cycle

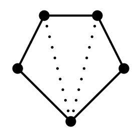


Figure 4.2: A chordal embedding of C_5: The dotted lines are added in the embedding.

The objective of the treewidth problem is to find a chordal embedding with smallest clique number.

Definition 4.9. The treewidth of a graph $G = (V, E)$ is defined as

$$\mathsf{treewidth}(G) = \min\{\, \omega(H) - 1 \mid H \text{ is a chordal embedding of } G \,\}. \qquad (4.4)$$

We use $\mathsf{tw}(G)$ to denote the treewidth of G.

Exercise 4.7

Show that the number of minimal triangulations of the cycle C_n is the Catalan number Cat_{n-2}. That is, it satsifies the recurrence

$$\mathrm{Cat}_{n-2} = \frac{4 \cdot n - 10}{n - 1} \cdot \mathrm{Cat}_{n-3} \qquad (4.3)$$

with $\mathrm{Cat}_0 = 1$. So, the number of minimal triangluations of C_5 is 5.

Treewidth of Claw – Free Graphs

Computing the treewidth of a graph is NP – complete. — However — it is solvable in polynomial time for many special classes of graphs.

As an example — since we are already a bit familiar with the structure of claw – free graphs — let's have a quick look at the computational complexity of treewidth for claw – free graphs.

Arnborg et al. showed that the treewidth problem remains NP – complete for bipartite graphs — and also for — cobipartite graphs. [10]

[10] A graph is cobipartite if its complement is bipartite.

Exercise 4.8

Show that the treewidth problem remains NP – complete when restricted to claw – free graphs.

4.2.1 Treewidth and brambles

LET'S START WITH THE DEFINITION OF A BRAMBLE.

Definition 4.10. Let G be a graph. Two subsets $A, B \subseteq V$ touch if $A \cap B \neq \varnothing$ or there exist $a \in A$ and $b \in B$ with $\{a, b\} \in E$.

Definition 4.11. Let G be a graph. A bramble $B = \{B_i\}$ is a set of subsets $B_i \subseteq V$ whis satisfies the following properties.

1. each subset B_i induces a connected subgraph of G
2. Each pair B_i and B_j touch.

NOW LET'S DEFINE THE ORDER OF A BRAMBLE.

A set Z is a hitting set for $\{B_i\}$ if $Z \cap B_i \neq \varnothing$ for all $B_i \in B$.

Definition 4.12. Let $B = \{B_i\}$ be a bramble. The order of B is the minimal number of elements in a hitting set for B.

AND NOW WE DEFINE THE BRAMBLE NUMBER OF A GRAPH.

Exercise: What is the bramble number of an independent set? What is the bramble number of a clique? How to compute the bramble number of a graph from the bramble numbers of its components? Design an algorithm to compute the bramble number in cographs.

We write $\mathsf{tw}(G)$ for the treewidth of G.

Definition 4.13. The bramble number of a graph G is the maximal order of a bramble in G.

We denote the bramble number of G by $b(G)$.

The reason we did all that is the following theorem proved by Seymour and Thomas in 1993.

Theorem 4.14 (Seymour and Thomas)**.** *For any graph G the following equality holds.*

$$\mathsf{tw}(G) + 1 \quad = \quad b(G)$$

There is an elegant proof of Theorem 4.14 by Bellenbaum and Diestel.

P. Bellenbaum and R. Diestel, Two short proofs concerning treedecompositions, *Combinatorics, Probability, and Computing* **11** (2002), pp. 541–547.

The following lemma gives a short proof of the theorem for chordal graphs. (Perhaps it helps the reader to get a feel for brambles and for chordal graph.)

Lemma 4.15. *For any chordal graph* G

$$b(G) \quad = \quad \omega(G).$$

Proof. The graph is chordal so there is a tree T and a collection of subtrees of T — say $\{T_x \mid x \in V(G)\}$ — with the property that for any two subtrees T_x and T_y

$$V(T_x) \quad \cap \quad V(T_y) \quad \neq \quad \varnothing \qquad \Leftrightarrow \qquad \{x, y\} \quad \in \quad E(G)$$

Let $\{B_i\}$ be a bramble. By definition each $G[B_i]$ is connected. Let

$$T_i \quad = \quad \bigcup_{x \in B_i} T_x$$

Then T_i is a subtree of T.

Every pair B_i and B_j touch. This implies

$$V(T_i) \quad \cap \quad V(T_j) \quad \neq \quad \varnothing.$$

Since every pair of subtrees T_i and T_j share a point of T there exists a point c in T that is in every tree T_i. [11]

Let $C = \{ x \in V \mid c \in V(T_x) \}$. Then C is a clique in G — and so — $|C| \leqslant \omega$. — Furthermore — the set C is a hitting set for the bramble.

We conclude that $b(G) \leqslant \omega(G)$ since any bramble has order at most $\omega(G)$.

To see that $\omega \leqslant b$ let M be a clique in G with ω vertices. Define a bramble $\{\{x\} \mid x \in M\}$. — Clearly — the order of this bramble is ω. So $\omega \leqslant b(G)$.

This proves the lemma. □

[11] The Helly property.

Exercise 4.9

Show that any graph G satisfies

$$\mathsf{b}(\mathsf{G}) \quad \leqslant \quad \mathsf{tw}(\mathsf{G}) + 1.$$

Exercise 4.10

(a) Let G be a graph and let H be a minimal triangulation of G. Show that any bramble in G is a bramble in H of the same order.

(b) Show that for any graph

$$\mathsf{b}(\mathsf{G}) \quad = \quad \mathsf{tw}(\mathsf{G}) + 1.$$

Further reading

The paper below introduces brambles — although in this paper brambles are called 'screens.'

P. Seymour and R. Thomas, Graph searching and a minimax theorem for treewidth, *Journal of Combinatorial Theory, Series B* **58** (1993), pp. 239–257.

The paper that christens the concept as brambles is this.

B. Reed, Treewidth and tangles, a new measure of connectivity and some applications. In: Vol. 241 of *LMS Lecture Note Series*, Cambridge University Press, (1997), pp. 87–162.

4.2.2 Tree - decompositions

Graphs of treewidth k are exactly the graphs that have tree - decompositions of width k.

Definition 4.16. Let G be a graph. A tree - decomposition for G is a pair $(T, \{X_i\})$ where

1. T is a tree with a root
2. each node $i \in V(T)$ corresponds with a bag $X_i \subseteq V(G)$
3. every vertex of G is in a bag
4. every edge of G is contained in a bag
5. for each vertex $x \in V(G)$ the nodes i with $x \in X_i$ form a subtree of T.

The width of a tree - decomposition is the maximal size of a bag minus one.

A graph has treewidth k if and only if it has a tree - decomposition with width $\leqslant k$.

Make a clique of every bag; this embeds the graph in a chordal graph with the tree - decomposition as a clique - tree.

In 1996 H. Bodlaender designed a linear time algorithm to compute a tree - decomposition of minimal width for graphs of bounded treewidth.

H. Bodlaender, A linear time algorithm for finding tree - decompositions of small treewidth, *SIAM J. Comput.* **25** (1996), pp. 1305–1317.

Exercise 4.11

Show that a graph G has a nice tree - decomposition — that is — a tree - decomposition $(T, \{X_i\})$ which satisfies the following.

1. the width of T is $\mathrm{tw}(G)$
2. every node of T has at most two children
3. if a node i has one child j then $|X_i| = |X_j| + 1$ and $X_j \subset X_i$ or $|X_i| = |X_j| - 1$ and $X_i \subset X_j$
4. if a node i has two children p and q then $X_i = X_p = X_q$
5. T has at most $4n$ nodes (where $n = |V(G)|$).

4.2.3 Example: Steiner tree

As an example let us take a look at an algorithm that solves the Steiner tree problem on graphs of bounded treewidth. To be precise we take a close look at the following paper.

M. Chimani, P. Mutzl and B. Zey, Improved Steiner tree algorithms for bounded treewidth, *Journal of Discrete Algorithms* **16** (2012), pp. 67–78.

Partitions of a set

As usual we start with something else.

Let S be a set. Say that S has k elements. We wish to enumerate all the partitions of S.

One way to do this is recursive and goes as follows. Choose an arbitrary element, say $a \in S$. Choose a part with j elements in $S \setminus a$ and put a in that part. Partition the remaining vertices in all possible ways.

A (mock) closed-form for B_k is

$$B_k = (1 + B)^{k-1}.$$

This algorithm gives the following formula for the number of partitions of a set with k elements.

$$B_k \quad = \quad \sum_{j=0}^{k-1} \binom{k-1}{j} \cdot B_{k-1-j}.$$

To make it all work out nicely we choose $B_0 = 1$.

These are the Bell numbers

1 1 2 5 15 52
203 877 4140 ⋯

There is a simple algorithm to make a table of these numbers similar to Pascal's triangle. De Bruijn (in his book "Asymptotic Methods in Analysis") gives a nice asymptotic expression for $\ln(B_n)/n$ with a marvelous Big-Oh - term $O(\ln\ln n/(\ln n)^2)$. A more recent bound for the Bell numbers is

$$B_n < \left(0.792n/\ln(n+1)\right)^n.$$

Exercise 4.12

Design an algorithm that runs in $O(k \cdot B_k)$ time to make a list of all the partitions of a set with k elements.

Assume that we wish to enumerate all partitions of S where possibly one part is special.

This is done by adding a 'ghost - element' g to the set. Now enumerate all partitions of $S \cup \{g\}$. The part that contains the ghost element is the special part.

The algorithm gives the following formula for the number of partitions with a special part.

$$B_k^* = B_{k+1}.$$

The numbers $B_k^* - B_k$ appear in the second diagonal of the Bell triangle. (These are the number of partitions that have a special part.)

Steiner trees

Let G be a graph and let $\Omega \subseteq V$ be a subset of the vertices. The elements of Ω are called terminals. The Steiner tree problem is to connect the terminals by a tree in G with a smallest number of edges.

Equivalently: the Steiner tree problem asks for a connected subgraph of G with a smallest number of edges which contains all terminals. Clearly, this can only exist if the terminals are in a component of the graph.

Processing the tree - decomposition

To solve the Steiner tree problem we have a nice tree - decomposition $(T, \{X_i\})$ at our disposal. Let $i \in V(T)$. Let T_i be the subtree of T which is rooted at node i.

Chimani et al consider the problem where the edges have weights. We don't do that.

- $V_i \subseteq V(G)$ is the set of vertices that appear in bags of T_i
- $\Omega_i \subseteq \Omega$ is the set of terminals that appear in bags of T_i.

Let S be a Steiner tree in G. This induces a forest in $G[V_i]$. The only vertices of V_i that have neighbors in $V \setminus V_i$ are vertices of X_i. Assume that there is at least one terminal in $V \setminus V_i$. Then the terminals of $\Omega_i \setminus X_i$ are connected (in a forest) to some vertices in X_i. The algorithm stores at the node i the sub - forest of S on the vertices of X_i.

A forest on X_i is represented as a partition of X_i. One part of this partition may be special; a set of vertices that is not in the forest. The other parts represent the components of the forest.

A partition of X_i has a cost. The cost of a partition is the smallest number of edges in a Steiner forest in V_i which induces the partition.

The cost of a partition is the sum of the costs of the parts.

We may assume that every node in T is of one of the following types.

See Exercise 4.11.

1. a start node satisfies $|X_i| = 1$
2. an introduce node i has exactly one child j and the bags satisfy $X_j \subset X_i$ and X_i has exactly one vertex that is not in X_j
3. a forget node i has exactly one child j and the bags satisfy $X_i \subset X_j$ and X_j has exactly one vertex that is not in X_i
4. a join node i has two children p and q and the bags at these nodes satisfy $X_i = X_p = X_q$.

A node in the tree T is processed at a time after the completion of processing its children. Below we describe — for each type of node — the computational process.

Bottoms up!

Process at a start node

Let i be a start node — that is — X_i contains one vertex $x \in V(G)$. A table at i has the partitions of X_i with a possible special part. So at i we have a table with two entries $\{x\}$; in one of the entries $\{x\}$ is marked 'special.'

The special part contains the vertices of X_i that are not in the Steiner tree.

We need to supply a cost to each part of a table entry. When the vertex x is a terminal and a part that contains x is marked as a 'special' then the partition has cost ∞. In all other cases the partition has cost 0.

Process at an introduce node

Let i be an introduce node with a child j. Let x be the vertex that is in $X_i \setminus X_j$. Then

$$N[x] \quad \cap \quad V_i \qquad \subseteq \qquad X_i.$$

THE FIRST STEP of the process is to generate a table with all partitions of X_i with one part possibly marked as special. That is equivalent to the generation of all partitions of X_j with two ghost elements — of which one is x.

The algorithm needs to compute a cost of each part of a partition P at the node i. This is obtained from the costs of partitions Q at node j that are compatible with P — as follows.

some parts of Q connect with x into one part of P.

A partition Q of X_j is compatible with P if each part of Q either

- is equal to a part in P or
- is a subset of the part in P that contains x.

When a part of P is equal to a part of Q then its cost is copied. Let P_x be the part of the partition that contains x. Let Q be compatible with P and let

$$P_x \setminus x \quad = \quad \bigcup Q_i,$$

where $\{Q_i\}$ is the partition of $P_x \setminus x$ into parts of Q.

The cost of the part P_x is ∞ when $x \in \Omega$ and P_x is marked as special. — Otherwise — it is the smallest value — over all partitions Q that are compatible with P — of $\sum \mathsf{cost}(Q_i)$ **plus**

- ∞ when x is not connected to some Q_i
- the number of parts Q_i when they all connect to x.

The only neighbors of x in V_i are vertices in X_i.

Exercise 4.13

Show that a table for the node i can be computed in $O(B_{k+2}^2 \cdot k)$ time where $k = \mathsf{tw}(G)$.

Exercise 4.14

Let i be an introduce node with child j and let $x \in X_i \setminus X_j$. Consider partitions of X_j with a 2-coloring on its parts. Let the parts of one color union with x to make up one part in a partition of X_i. Show that the number of 2-colored partitions (of a set with k elements) satisfies $B_0^{(2)} = 1$, $B_1^{(2)} = 2$ and

$$B_k^{(2)} = \sum_{j=0}^{k} \binom{k}{j} \cdot B_j \cdot B_{k-j}.$$

Show that $B_k^{(2)} \leqslant B_k \cdot B_{k+1}$.

Process at a forget node

Not much happens in this case. We have $X_i = X_j \setminus \nu$ for some $\nu \in X_j$. Call two partitions of X_j equivalent if they are the same when the vertex x is removed. Compute the cost of an equivalence class (which is a partition of X_i) as the smallest cost of the elements in the class.

We leave it as an exercise to show that forget nodes are processed within $O(B_{k+2}^2 \cdot k)$ time where $k = \mathrm{tw}(G)$.

Process at a join node

Let i be a join - node with two children p and q. We have for the sets of vertices

$$V_p \cup V_q = V_i \quad \text{and} \quad V_p \cap V_q = X_i$$

and for the sets of edges in $G[V_p]$ and in $G[V_q]$:

$$E_p \cup E_q = E_i \quad \text{and} \quad E_p \cap E_q = E_i.$$

A Steiner tree has edges in E_p or in E_q or in both — that is — we can split it up in two forests; one with edges in E_p and the other with edges in E_q. The monochromatic parts are a partition of X_i (with some part that is not used).

The algorithm needs to check if the two forests at p and q add up as a tree (to make one part of a partition at node i).

Consider all $2^{k-1}k^{k-2}$ trees in K_k where $k = |X_i|$ with a 2-coloring on the edges. Let the partition at p have the components of the tree with edges in color 1 and let the partition at q have the components of the tree with edges in color 2. (Both partitions may have a special part.) The cost of the tree is the sum of the costs of the partitions of p and q.

This yields an algorithm to compute a cost for all partitions at join - node i. To compute a table at node i it uses $O((2k)^{k-1}B_{k+1})$ time. The paper by Chimani et al improves on this. [12]

[12] We have $B_k < (k/\ln k)^k$ (when $k \to \infty$).

Consider partitions P and Q at nodes p and q. Construct a graph on parts $\{P_i\}$ and $\{Q_i\}$ that make up a part of a partition at node i. The following algorithm checks if the union of $\{P_i\}$ and $\{Q_i\}$ is a proper part at node i.

Exercise: $k!/B_k \to \infty$ and $k!/B_k^2 \to 0$ (as $k \to \infty$).

Construct a graph whose vertices are the parts in $\{P_i\}$ and the parts in $\{Q_i\}$. Two parts are connected by an edge if they share a vertex in X_i. This graph has $O(\mathsf{tw}(G))$ vertices and edges. A depth - first - search on this graph detects whether it is connected and if there is any cycle in linear time.

In their paper Chimani et al show that this gives an algorithm that computes a table for a join - node in $O(B^2_{\mathsf{tw}+2} \cdot \mathsf{tw})$ time.

Conclusion

The cases described above add up to prove the theorem of Chimani, Mutzel and Zey.

We are not aware of any (nontrivial) lowerbounds. — Perhaps — there is room for improvement.

Theorem 4.17. *There is an algorithm that takes $O(k \cdot B^2_{k+1} \cdot n)$ time to solve the Steiner tree problem for graphs with treewidth k.*

Must - reads on Steiner trees

S. Dreyfus and R. Wagner, The Steiner problem in graphs, *Networks* **1** (1972), pp. 195–207.

A. Aitken, A problem in combinations, *Mathematical Notes* **28** (1933), pp. 18–23.

A. Marcus and G. Tardos proved the Stanley - Wilf conjecture in 2004. [13]

[13] ACTUALLY they proved the Füredi - Hajnal conjecture about $\{0,1\}$ - matrices.

Theorem 4.18. *For every permutation* π *there is a constant* C *such that the number of permutations in* S_n *that avoid* π *as a pattern is at most* C^n.

(The constant $C(\pi)$ is an exponential function of π for almost all permutations.)

Any permutation has a cycle decomposition. (That shows that $k! > B_k$ when $k \geqslant 3$; B_k is the number of permutation in which each cycle is ordered. The average number of cycles is the harmonic number $H_n \approx \ln(n)$) The number of permutations drops dramatically when we forbid a pattern.

A. Marcus and G. Tador, Excluded permutation matrices and the Stanley - Wilf conjecture, *Journal of Combinatorial Theory, Series A* **107** (2004), pp. 153–160.

Exercise 4.15

How many pairs (α, β) of permutations in S_n are there that have cycles C_α and C_β such that the union of $\alpha \setminus C_\alpha$ and $\beta \setminus C_\beta$ is a tree of cycles — that is — a cactus?

A connected graph is a cactus if any two cycles share at most one vertex. SO every block is an edge or a cycle.

4.2.4 Treewidth of Circle Graphs

Consider a circle in the Euclidean plane. A chord of the circle is a line segment that connects two points of the circle.

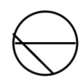


Figure 4.3: A circle and two chords in it. The circle graph corresponding to this model has two, adjacent vertices.

Definition 4.19. A circle graph is an intersection graph of a set of chords of a circle in the Euclidean plane. — That is — the vertex set of the circle graph is the set of chords of the circle and two vertices are adjacent whenever their chords intersect.

— As an example — we show that there is a nice algorithm that computes the treewidth of circle graphs.

To compute the treewidth of a graph we need to find a triangulation of it that minimizes the clique number. In a chordal embedding of a graph all minimal separators are cliques. — It follows that they are non-crossing.

Crossing Separators

Definition 4.20. One minimal separator S_1 crosses another one S_2 if there exist two components of $G - S_2$ — each containing a vertex of S_1.

We say that two minimal separators are parallel if they are noncrossing — that is — if neither crosses the other.

Exercise 4.16

Show that the crossing relation, on the set of minimal separators of a graph, is symmetric.

Exercise 4.17

Show that there is a $1-1$ correspondence between the set of minimal triangulations of a graph and the maximal sets of pairwise parallel minimal separators. — For example — in a chordal graph all minimal separators are pairwise parallel — and so — the graph has only one chordal embedding — namely the graph itself.

Minimal Separators in Circle Graphs

Consider an intersection model of a circle graph.[14] — That is — let C be a circle in the Euclidean plane and let G be a set of chords of C. We may assume that no two chords share an endpoint.

[14] Circle graphs can be recognized in 'almost' linear time.

Definition 4.21. A scanline is a chord of C that shares no endpoint with any chord of G.

Lemma 4.22. *Assume the graph G is connected. Let S be the set of chords of a minimal separator in G. There is a scanline t such that the elements of S are exactly the chords that cross t.*

Proof. Remove the chords from the intersection model that correspond with vertices of S. Then each part that remains connected corresponds to a component of $G - S$.

Let S be a minimal $a|b$-separator and let C_a and C_b be the components of $G - S$ that contain a and b. Choose a scanline t that separates the component C_a from C_b.[15] The chords

[15] The chords are straight line segments, therefore, there is a straight scanline that separates the convex hull of the chords of a component.

that cross t are exactly the chords of S.

This proves the lemma. □

Consider a polygon P with $2n$ corners — one between every two consecutive endpoints of G.

[16] Let's call chords of P, diagonals.

Definition 4.23. A plane triangulation of P is a maximal set of noncrossing chords in P. [16]

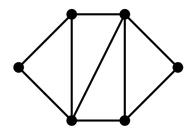


Figure 4.4: A plane triangulation of a 6-sided polygon.

Definition 4.24. Let T be a plane triangulation of P. The weight of a triangle in T is the number of chords in G that cross some sides of the triangle. The weight of the triangulation T is the maximal weight over the triangles contained in T.

Notice that, if a chord of G cross some side of a triangle, then it crosses exactly two sides of the triangle.

An algorithm to compute the treewidth of circle graphs

To compute the minimal weight of all the triangulations of the polygon P we use dynamic programming.

Let ℓ be the number of corners of P [17] and let them be numbered

$$s_0, s_1, \cdots, s_{\ell-1}.$$

[17] If $n = |V(G)|$ then the circle graph model has n chords, with $2n$ endpoints. between any two consecutive endpoints, we have a corner of P. Thus, the polygon P has $\ell = 2n$ corners.

Denote the number of chords in G — that cross the line (s_i, s_j) — by $c(i, j)$.

Let $P(i, t)$ denote the sub−polygon with corners

$$s_i, s_{i+1}, \cdots, s_{i+t-1}$$

where we take indices modulo ℓ. Let $w(i, t)$ denote the weight of a minimal triangulation of $P(i, t)$.

Organize the computation by increasing length of the subpolygons. For $t = 2$ let $w(i, 2) = 0$ for all $i \in \{0, \dots, \ell-1\}$. Then we have — for $t \geqslant 3$ —

$$w(i, t) = \min_{2 \leqslant j < t} \max \{w(i, j), w(i+j-1, t-j+1), F(i, j)\} \tag{4.5}$$

where

$$F(i, j) = \frac{1}{2} \cdot (c(i, i+j-1) + c(i+j-1, i+t-1) + c(i, i+t-1)). \tag{4.6}$$

That is so because every chord of G crosses — zero — or exactly two sides of every triangle.

Theorem 4.25. *The minimal weight of a triangulation of* P *can be computed in* $O(n^3)$ *time. The treewidth of* G *is*

$$\mathrm{tw}(G) = w(0, \ell) - 1.$$

Proof. There are $O(n^2)$ diagonals in P. A check whether a diagonal of P and a chord of G cross each other can be achieved in $O(1)$ time. Thus — to compute the numbers $c(i, j)$ of chords that cross a diagonal (s_i, s_j) takes $O(n^3)$ time.

By Equation (4.5) the minimal weight of a triangulation is obtained in $O(n^3)$ time.

To compute the treewidth of G we need to select a maximal set of parallel minimal separators. This problem is equivalent to finding a triangulation of P. The clique number of the minimal triangulation of G is the maximal number of chords that cross a triangle in the minimal triangulation of P.

This proves the theorem. □

Exercise 4.18

Describe all minimal triangulations of the circle graph in the figure on Page 149. What is the treewidth of this graph?

I have not given you the definition of rankwidth [18] yet but — for the record — I put down the following research problem here.

[18] Rankwidth is a parametrization of the class of distance-hereditary graphs.

Exercise 4.19

Research problem:
Design a polynomial-time algorithm to compute the rankwidth of circle graphs.

David Chandler once showed me an algorithm, but, as far as I know, the details were not written down fully.

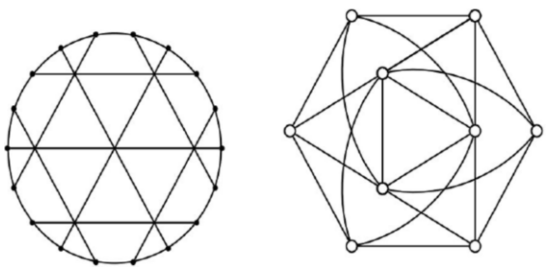


Figure 4.5: A circle graph and its model.

4.3 On the treewidth of planar graphs

> BEFORE WE SAY ANYTHING ELSE let us mention that at present ie anno Domini 2021 the complexity of computing treewidth of planar graphs is OPEN.

In a remarkable paper Seymour and Thomas showed (in 1994) that treewidth of planar graphs can be approximated within a factor 3/2. Their algorithm computes a decomposition in $O(n^4)$ time. In this chapter we describe their method.

LET G BE A GRAPH. In this chapter the definition of a graph is relaxed so that a graph may have multiple edges and loops. [19]

[19] The dual of a plane graph may have loops and multiple edges. Without this relaxation the dual of a plane graph would not be a graph.

CONSIDER A TERNARY TREE T and a 1-1 map from the vertices of G to the leaves of T. [20] Identify an edge $e \in E(T)$ with a bipartition of $V(G)$ where two elements occupy a similar part if they appear in the same component of $T \setminus e$. [21]

[20] A tree is ternary if every vertex has degree 1 or 3.

[21] The tree is called a 'routing tree' of the graph. Routing trees with small 'congestion' (ie carving width) are of importance for the design of telephone networks.

The collection of (all) the classes of bipartitions of V over all edges of T defines a 'cross-free set-system' on $V(G)$. [22]

[22] A set-system $\mathcal{C}$ is laminar if for any two elements A and B of $\mathcal{C}$ at least one of the three sets $A \cap B$, $A \setminus B$ and $B \setminus A$ is empty. Notice that 'being laminar' is a property that is independent of the ground set V. ('Being cross-free' is not.)

Definition 4.26. Let V be a finite set with at least two elements. Two subsets A and B (subsets of V) cross if

$$A \cap B \qquad A \setminus B \qquad B \setminus A \qquad V \setminus (A \cup B)$$

are all non-empty.

A family $\mathcal{C}$ of subsets of V is a carving if

C1 $\varnothing, V \notin \mathcal{C}$

C2 no two elements of $\mathcal{C}$ cross

C3 $\mathcal{C}$ is maximal subject to the above.

Exercise 4.20

Let $\mathcal{C}$ be a carving of a finite set V, $|V| \geqslant 2$. Show that there exists a ternary tree T and a 1-1 map $V(G) \to \mathsf{leaves}(T)$ such that $C \in \mathcal{C}$ if and only if there exists an edge e in T such that C is identified with the set of leaves in one of the two components of $T \setminus e$.

Let G be a graph. For $X \subseteq V$ let $\delta(X)$ denote the set of edges in G that have exactly one endpoint in X. [23]

[23] In this chapter we show that the carving width is computable in polynomial time for planar graphs.

Definition 4.27. Let G be a graph with at least two vertices. The width of a carving $\mathcal{C}$ of V is

$$\max\{\,|\delta(X)| \mid X \in \mathcal{C}\,\}.$$

The carving width of G is the smallest width of a carving of V.

Let $p : E \to \mathbb{Z}^{\geqslant 0}$. For $A \subseteq E$ let $p(A) = \sum_{e \in A} p(e)$. [24] The p-carving width of a graph with at least 2 vertices is the minimum over all carvings $\mathcal{C}$ of V of the maximum $p(\delta(A))$ for $A \in \mathcal{C}$.

[24] $\mathbb{Z}^{\geqslant 0} = \mathbb{N} \cup \{0\}$. We could replace each edge of G by $p(e)$ parallel lines and compute the carving width of this auxiliary 'graph.' However it is our aim to find a 'strongly polynomial' algorithm for p-carving width — the timebound of our algorithm is a polynomial in $n + m$ (which gives a much better bound than a polynomial of the same degree in $n + \sum p(e)$).

We show below that the p-carving width of a connected planar graph is at least k if and only if either

1. there exists a vertex v satisfying $p(\delta(v)) \geqslant k$
2. it has an 'antipodality' of p-range at least k.

In other words the minimal $k \in \mathbb{N} \cup \{0\}$ such that there is a carving $\mathcal{C}$ of $V(G)$ satisfying

$$\forall_{X \in \mathcal{C}} \quad p(\delta(X)) \quad \leqslant \quad k$$

equals the maximal $k \in \mathbb{N} \cup \{0\}$ such that G has an antipodality of p-range at least k or a vertex satisfying $p(\delta(x)) \geqslant k$. [25]

[25] Slogan:

$$\min \textsf{carvingwidth} = \max \textsf{antipodality}$$

Next, we will show that there is a very easy algorithm (using the concept of 'round set') to find an antipodality of maximal p-range.

4.3.1 Antipodalities

A walk in a graph is a sequence

$$v_0 \quad e_1 \quad v_1 \quad e_2 \quad v_2 \quad e_3 \quad \cdots \quad e_k \quad v_k$$

with all $v_i \in V$ and all $e_i \in E$ and all $e_i = \{v_{i-1}, v_i\}$. The walk is closed if $v_0 = v_k$.

Let Σ be a sphere [26] and let a graph G be embedded on Σ. Let $R(G)$ denote the regions of the embedding — that is — the maximal open sets (faces) in Σ that do not contain any vertex nor hit any edge.

[26] A sphere is a balloon. The rubber edition was invented by Michael Faraday in 1824. An embedding of a graph is a drawing of it on a balloon such that edges only meet at endpoints. The regions of the drawing are the maximal connected areas of the balloon's surface that are not touched by the edges or vertices of the drawing. The regions are 'open sets:' if you take any point in a region, then a small enough disc with positive radius around the point will be contained in that region.

Assume that G is not empty and let G^* denote the dual of G. Each vertex $v \in V$ is contained in exactly one region $r \in R(G^*)$; we denote $v^* = r$. Similarly, each region $r \in R(G)$ contains exactly one vertex $r^* \in V(G^*)$ (and that vertex is denoted r^*). For an edge $e \in E$ we let e^* be the unique edge of G^* that crosses e in Σ.

Let $p : E(G) \to \mathbb{Z}^{\geqslant 0}$. For a walk

$$v_0 \quad e_1 \quad \cdots \quad e_k \quad v_k$$

in the dual G^* define the p-length as

$$p(f_1) \quad + \quad \cdots \quad + \quad p(f_k)$$

where $f_i = e_i^*$.

Definition 4.28. An antipodality of p-range $\geqslant k$ is a function α with domain $E(G) \cup R(G)$ such that for all $e \in E$ $\alpha(e)$ is a subgraph and for all regions $r \in R$ $\alpha(r)$ is a nonempty subset of V satisfying

(A1) for each edge $e \in E(G)$ the subgraph $\alpha(e)$ does not contain an endpoint of e

(A2) if an edge e and a region r are incident then $\alpha(r) \subseteq V(\alpha(e))$ and every component of $\alpha(e)$ has a vertex in $\alpha(r)$

(A3) if $e \in E$ and $f \in E(\alpha(e))$ then every closed walk in G^* that contains e^* and f^* has p-length at least k.

Intermezzo: Round Sets

Let N be a graph with vertex set $V(N) = I$ and let M be a graph with a partition of its vertices

$$\{X_i \mid i \in I\}.$$

Assume

$$x \in X_i \quad \text{and} \quad y \in X_j \quad \text{and} \quad \{x, y\} \in E(M) \quad \Rightarrow \quad i \neq j \quad \text{and} \quad \{i, j\} \in E(N).$$

(There is a homomorphism $M \to N$.)

Definition 4.29. A set $Z \subseteq V(M)$ is round if

$$\forall_{\{i,j\} \in E(N)} \quad \forall_{x \in Z \cap X_i} \quad \exists_{y \in Z \cap X_j} \quad \{x, y\} \in E(M).$$

We show that there is a greedy algorithm to decide whether M has a nonempty round set.

Exercise 4.21

Prove the following. Let Z be round and let $Z \subseteq S$. Assume that there exists a vertex $x \in S$ that has no neighbors in $S \cap X_j$ for some $j \in N_N(i)$. Then $Z \subseteq S \setminus \{x\}$.

HINT: If $x \in Z \cap X_i$ then it must have a neighbor in $Z \cap X_j \subseteq S \cap X_j$ since Z is round.

Exercise 4.22

Define the bipartite graph H as follows. The two color classes of H are I and $V(M)$. A vertex $x \in V(M)$ is adjacent to $j \in I$ if $x \in X_i$ and i adjacent to j in N.

Design an algorithm that find a maximal round set in M. Your algorithm should run in

$$O(\quad |V(M)| \quad + \quad |E(M)| \quad + \quad |E(H)| \quad).$$

HINT: For $\{x, j\} \in E(H)$ let

$$d(x, j) \quad = \quad |\, N_M(x) \cap X_j \,|.$$

Construct a stack that contains the vertices $v \in V(M)$ that satisfy $d(v, j) = 0$ for some $j \in I$. Initialize $Z = V(M)$. At the end of the recursion described below Z will be a maximal round set of M.

During the recursion vertices of L are deleted from L and from Z. This needs an update of values $d(v, j)$ (since a vertex in M is deleted). To estimate the timebound describe in detail how the stack L and the values $d(v, j)$ are maintained — in other words — present the algorithm and prove its correctness. [27]

[27] A stack is a linear data structure in which elements are added or deleted only from the top-end.

Notice that the property of being round is maintained under unions — so — there is a unique maximal round set.

We show that there exists an $O(m^2)$ - algorithm to decide if a graph has an antipodality of p-range $\geqslant k$.

Let G be a connected planar graph which is not empty. Let $p : E(G) \to \mathbb{Z}^{\geqslant 0}$ and let $k \in \mathbb{Z}^{\geqslant 0}$. For $e \in E(G)$ let $\varphi(e)$ be the following subgraph. The vertices of $\varphi(e)$ are $V \setminus e$ (that is; all vertices except the endpoints of e) and the edges of $\varphi(e)$ are $f \in E$ for which $f \cap e = \varnothing$ and for which every closed walk in the dual G^* that contains e^* and f^* has p-length $\geqslant k$.

Lemma 4.30. *If there exists an antipodality of* p*-range* $\geqslant k$ *then there exists one — say* α *— such that* $\alpha(e)$ *is a union of components of* $\varphi(e)$ *for all* $e \in E$.

Proof. Let β be an antipodality of p-range $\geqslant k$. By definition $\beta(e)$ is a subgraph of $\varphi(e)$.

For $e \in E$ define $\alpha(e)$ as the union of those components of $\varphi(e)$ that intersect $\beta(e)$. For all regions r define $\alpha(r) = \beta(r)$.

The map α satisfies the first and third antipodality-condition (Definition 4.28). To check the second condition let $e \in E$ and $r \in R$ and assume that e and r are incident. Then since β is an antipodality

$$\alpha(r) \quad = \quad \beta(r) \quad \subseteq \quad V(\beta(e)) \quad \subseteq \quad V(\alpha(e)).$$

Every component of $\alpha(e)$ contains a component of $\beta(e)$. This implies that every component of $\alpha(e)$ intersects $\alpha(r)$.

This proves that α is an antipodality of p-range at least k. $\square$

We show that the set of components of $\varphi(e)$ for all $e \in E$ can be computed in $O(m^2)$ time.

Exercise 4.23

Let G be a connected plane graph with a dual G^*. Let $p : E \to \mathbb{Z}^{\geqslant 0}$. For two vertices $x, y \in V(G^*)$ denote their p-distance in G^* as $d^*(x, y)$. [28]

[28] The p-distance is the shortest p-length of a path (that runs between two vertices in the dual graph).

Let $e^* = \{x, y\}$ and $f^* = \{p, q\}$ be edges of G^*; $e^* \cap f^* = \varnothing$. There is a closed walk in G^* of p-length $< k$ if and only if either

$$\begin{aligned} d^*(x,p) + d^*(y,q) &< k - p(e) - p(f) \quad \text{or} \\ d^*(x,q) + d^*(y,p) &< k - p(e) - p(f). \end{aligned}$$

Exercise 4.24

Design an algorithm to compute the following.
INPUT: A connected plane graph G with a dual G^*. A function $p : E \to \mathbb{Z}^{\geqslant 0}$ and integer $k \in \mathbb{Z}^{\geqslant 0}$.
Output: The graph $\phi(e)$ and a list of the components of $\phi(e)$ for every edge $e \in E(G)$.

Your algorithm should run in $O(m^2)$ time.
HINT: Frederickson shows that the all pairs shortest paths problem can be solved on planar graphs in $O(n^2)$ time.

We describe an algorithm to decide whether G has an antipodality of p-range $\geqslant k$. Below we prove the correctness.

The input is a connected plane graph G with a dual G^*, a function $p : E \to \mathbb{Z}^{\geqslant 0}$ and an integer $k \geqslant 0$.

1. For each edge $e \in E(G)$ compute C_e ie the set of components of $\phi(e)$.

2. Construct a graph M with vertices

$$V(M) = \bigcup_{i \in I} X_i,$$

where $I = E \cup R$ and a partition $\{X_i \mid i \in I\}$ of $V(M)$ is defined as follows

(i) For each $r \in R$ let

$$X_r = \{(r, x) \mid x \in V\}$$

(ii) For each $e \in E$ let

$$X_e = \{(e, C) \mid C \in C_e\}.$$

A vertex (e, C) is adjacent to a vertex (r, x) in M if [29]

$$e \subseteq \bar{r} \quad \text{and} \quad x \in V(C).$$

3. Use Exercise 4.22 to check if M has a nonempty round set. The graph N has vertex set $I = E \cup R$. The adjacencies in N running between edges and regions is defined by their incidence in G.

4. The graph G has an antipodality of p-range $\geqslant k$ precisely when M has a nonempty round set.

[29] We write $\bar{r}$ for the geometric closure of the region. Each region is an open set. An edge e and region r are incident if $e \subseteq \bar{r}$.

Remark 4.31. To compute a round set in M construct a bipartite graph H with color classes $V(M)$ and $V(N)$ as follows. When an edge e and a region r are incident then e is adjacent to every vertex in X_r and r is adjacent to every vertex in X_e.

For $v \in V(M)$ and $j \in V(N)$ such that $\{v, j\} \in E(H)$ define $d(v, j)$ as the number of vertices in X_j that is adjacent to v in M ie

$$d(v, j) = \begin{cases} |C| & \text{if } j \in R,\ v = (e, C),\ (e \in E,\ C \in C_e) \\ 1 & \text{if } j \in E,\ v = (r, x),\ (r \in R,\ x \notin j) \\ 0 & \text{if } j \in E,\ v = (r, x),\ (r \in R,\ x \in j). \end{cases}$$

Create a stack with elements $v \in M$ for which there is a $j \in N$ satisfying $d(v, j) = 0$. Repeatedly delete those elements from M and update the stack.

Theorem 4.32. *Let G be a connected planar graph and let $p : E \to \mathbb{Z}^{\geqslant 0}$ and $k \in \mathbb{N}$. There exists an $O(m^2)$ - algorithm that decides if G has an antipodality of p-range $\geqslant k$. Here $m = |V(G)| + |E(G)|$.*

Proof. To prove the correctness let $Z \subseteq V(M)$. Define, for $e \in E$ and for $r \in R$, a subgraph $\alpha(e)$ and a subset of vertices $\alpha(r)$:

$$\begin{aligned} \alpha(e) &= \cup \{C \in C_e \mid (e, C) \in Z\} \\ \alpha(r) &= \{v \in V \mid (r, v) \in Z\}. \end{aligned}$$

Then α satisfies the first and third condition. It satisfies the second condition of Definition 4.28 on Page 152 exactly when Z is round.

The selected $\alpha(e)$ and $\alpha(r)$ are not empty exactly when $Z \neq \varnothing$. [30] — That is — the graph G has an antipodality of p-range $\geqslant k$ exactly when Z is round in M and $Z \neq \varnothing$.

This proves the theorem. □

[30] Notice that when one of the $\alpha(e)$ and $\alpha(r)$ is empty then they are all empty. This follows from the second antipodality condition.

4.3.2 Tilts and slopes

It is our aim to prove the following theorem.

Theorem 4.33. *Let G be a connected planar graph with $|V| \geqslant 2$. Let $p : E \to \mathbb{N}$ and let $k \in \mathbb{Z}^{\geqslant 0}$. Then G has p-carvingwidth at least k if and only if it has a vertex x with $p(\delta(x)) \geqslant k$ or an antipodality of p-range at least k.*

In this section we show 'only if'.

Let V be a set and let $\kappa : 2^V \to \mathbb{Z}$. [31] The function κ is submodular if it satisfies for all $X, Y \in 2^V$

$$\kappa(X \cup Y) + \kappa(X \cap Y) \quad \leqslant \quad \kappa(X) + \kappa(Y).$$

For example, the function $\kappa(X) = p(\delta(X)) + c$ (for any constant c) is a submodular function $2^V \to \mathbb{Z}^{\geqslant 0}$. [32]

[31] In this section we slip and slide all the way down the rabbit hole;

bias → tilt →
slope → antipodality

[32] The edges that are not counted on the left are those with one end in $X \setminus Y$ and the other in $Y \setminus X$.

Let $\kappa : 2^V \to \mathbb{Z}$. Call a set X efficient if $\kappa(X) \leqslant 0$. Assume that κ satisfies

(a) $\kappa(X) = \kappa(V \setminus X)$ for all $X \subseteq V$

(b) κ is submodular

(c) all $X \subseteq V$ of cardinality one are efficient.

A bias is a collection $\mathcal{B}$ of efficient subsets of V such that

(B1) for each efficient set X exactly one of X and $V \setminus X$ is in $\mathcal{B}$

(B2) $X, Y, Z \in \mathcal{B} \Rightarrow X \cup Y \cup Z \neq V$

(B3) $|X| = 1 \Rightarrow X \in \mathcal{B}$.

Robertson and Seymour proved the following lemma in their 'Graph Minors X.' [33]

[33] That is 'Graph Minors. X. Obstructions to tree-decomposition.' It appeared in: Journal of Combinatorial Theory, Series B, **52** (1991), pp. 153–190.

Lemma 4.34. *Exactly one of the following statements is true*

1. *there is a carving* $\mathcal{C}$ *such that* $\kappa(X) \leqslant 0$ *for all* $X \in \mathcal{C}$
2. *there is a bias* $\mathcal{B}$.

Proof. We only prove that not both statements are true.

Assume there is a carving $\mathcal{C}$ (as stated in the lemma) and a bias $\mathcal{B}$. We derive a contradiction as follows.

The carving $\mathcal{C}$ corresponds with a routing tree T. Let e be an edge of the routing tree. Let X and $V \setminus X$ be the leaves in the components of $T - e$. Since all sets in $\mathcal{C}$ are efficient both X and $V \setminus X$ are efficient. Since $\mathcal{B}$ is a bias exactly one of X and $V \setminus X$ is in $\mathcal{B}$.

An incident pair (v, e) is a pair with $v \in V(T)$ and $e \in E(T)$ and $v \in e$. Let $X(v, e)$ be the set of leaves in the component of $T - e$ that contains x. Call a pair (v, e) passive if $X(v, e) \notin \mathcal{B}$. By the above there are exactly $|E(T)|$ incident pairs that are passive.

There are $|E(T)| + 1$ vertices in T. It follows that there is a vertex $v \in V(T)$ such that for that all edges e that have v as an endpoint $X(v, e) \notin \mathcal{B}$.

Now assume that such a vertex v is a leaf. Then $|X(v, e)| = 1$ and since $\mathcal{B}$ is a bias $X(v, e) \in \mathcal{B}$ which contradicts the assumption.

Assume that v is not a leaf. Then there are three edges incident with v. The sets of leaves $X(v, e_i)$ satisfy

$$\bar{X}(v, e_1) \cup \bar{X}(v, e_2) \cup \bar{X}(v, e_3) = V$$
$$\text{where} \quad \bar{X}(v, e_i) = V \setminus X(v, e_i) \in \mathcal{B}.$$

— So — assuming that all three $X(v, e_i) \notin \mathcal{B}$ contradicts the assumption that $\mathcal{B}$ is a bias.

This shows a small part of the proof. FOR THE SAKE OF BREVITY we direct the reader to 'Graph Minors X' for the remainder of the proof. $\square$

Definition 4.35. A <u>tilt</u> of order k is a collection $\mathcal{B}$ of subsets of V that satisfy $p(\delta(X)) < k$ and [34]

(T1) Let $X \subseteq V$. Then $\mathcal{B}$ contains exactly one of X and $V \setminus X$ if and only if $p(\delta(X)) < k$.

(T2) When $X, Y, Z \in \mathcal{B}$ then $X \cup Y \cup Z \neq V$.

(T3) For all $x \in V$ $\{x\} \in \mathcal{B}$.

[34] A 'tilt' is a sloping surface (one that makes you glide down). It's of course just a linear transformation of a bias.

Corollary 4.36. *Let G be a graph with at least two vertices. Let $p : E \to \mathbb{N}$ and let $k \in \mathbb{N}$ be so that $p(\delta(v)) < k$ for all $v \in V$. Then G has p-carvingwidth $\geqslant k$ if and only if G has a tilt of order k.*

Proof. Let

$$\kappa(X) \quad = \quad p(\delta(X)) - k + 1.$$

Then $\kappa(X) \leqslant 0$ if and only if $p(\delta(X)) < k$. The statement follows from Lemma 4.34. $\square$

LET US GET BACK TO THAT SURFACE Σ. Let G be embedded in Σ and let $k \in \mathbb{N}$. We show how the circuits in the graph define a slope. [35]

[35] We call a 'circuit' the embedding of a cycle in Σ. Every circuit cuts the sphere in two discs. So, topologically speaking, every circuit is an 'equator.'

Definition 4.37. A <u>slope</u> of order $k/2$ is a function ins that assigns to every circuit C of length $< k$ exactly one of the two closed discs in Σ that have C as boundary. Furthermore, the function ins satisfies the following conditions.

(S1) Let C and C' be circuits of length $< k$ and assume that C is drawn within $\mathsf{ins}(C')$. Then

$$\mathsf{ins}(C) \quad \subseteq \quad \mathsf{ins}(C').$$

(S2) Let P_1, P_2 and P_3 be three paths of length $< k$ that run between two vertices u and v but that are otherwise vertex disjoint. Then

$$\mathsf{ins}(P_1 \cup P_2) \quad \cup \quad \mathsf{ins}(P_1 \cup P_3) \quad \cup \quad \mathsf{ins}(P_2 \cup P_3) \quad \neq \quad \Sigma$$

A slope is uniform if for every region $r \in R(G)$ there is a circuit C of length $< k$ such that

$$r \quad \subseteq \quad \mathsf{ins}(C).$$

Definition 4.38. Let $X \subseteq V$ be nonempty such that $G[X]$ is connected and $V \setminus X \neq \varnothing$ and $G[V \setminus X]$ connected. Then $\delta(X)$ is a bond.

Exercise 4.25

Show that the dual of a bond is the set of edges of a circuit in G^*.

Lemma 4.39. *Let* G *be connected and drawn on a sphere* Σ*. Let* $p : E(G) \to \mathbb{N}$*. Let* $k \in \mathbb{N}$ *be such that* $p(\delta(x)) < k$ *for all* $x \in V(G)$*. Define the graph* G' *as the graph obtained from the dual* G^* *by subdividing each edge* $e^* \in E(G^*)$ $p(e) - 1$ *times.* [36] *If* G *has a tilt of order* k *then* G' *has a uniform slope of order* $k/2$*.*

[36] A subdivision replaces an edge (in this case an edge of G^*) by a path of length 2. Subdividing ℓ times introduces ℓ new vertices in Σ on e which replace the edge by a path of length $\ell + 1$.

Proof. Let $\mathcal{B}$ be a tilt of order k in G. We define a slope in G' of order $k/2$ and show that it is uniform.

Let C be a circuit in G' of length $< k$ and let Δ_1 and Δ_2 be the two discs in Σ with boundary C. Notice that

$$p(\delta(V(G) \cap \Delta_i)) \quad = \quad |E(C)| \quad < \quad k.$$

— So — exactly one of the sets $V(G) \cap \Delta_i$ is an element of $\mathcal{B}$ — say — $V(G) \cap \Delta_1 \in \mathcal{B}$. Define $\mathsf{ins}(C) = \Delta_1$. Then (by the tilt conditions T1 and T2) ins is a slope of order $k/2$ in G'. We show that ins is uniform.

Let $r \in R(G')$. Then $r \in R(G^*)$; say $r = v^*$ for some $v \in V(G)$. By T3: $\{v\} \in \mathcal{B}$. Choose $X \in \mathcal{B}$ maximal such that $v \in X$ and such that $G[X]$ is connected.

Let $Y = V(G) \setminus X$. By T2: $Y \neq \varnothing$. We show that $G[Y]$ is connected (and so $\delta(X)$ is a bond).

Let $Y_1, Y_2, \cdots$ be the components of $G[Y]$. Each $G[X \cup Y_i]$ is connected since G and $G[X]$ are connected. Also $\delta(Y_i) \subseteq \delta(X)$. Since X is maximal $X \cup Y_i \notin \mathcal{B}$. Notice that

$$\begin{aligned} \delta(X \cup Y_i) \quad \subseteq \quad \delta(X) \quad \Rightarrow \quad p(\delta(X \cup Y_i)) \quad < \quad k \quad \Rightarrow \\ Y \setminus Y_i \quad = \quad V(G) \setminus (X \cup Y_i) \quad \in \quad \mathcal{B}. \quad \text{((By T1))} \end{aligned}$$

When $t \geqslant 2$ then X, $Y \setminus Y_1$ and $Y \setminus Y_2$ are all in $\mathcal{B}$ and their union is $V(G)$. This contradicts T2.

— So — $t \leqslant 1$ and $t \neq 0$ (by T2). It follows that $\delta(X)$ is a bond. Let $C = \{e^* \mid e \in \delta(X)\}$. Then C is the set of edges of a circuit in G^*. The subdivisions transform C into a circuit C' of G'. Then

$$|E(C')| = p(\delta(X)) < k$$

and $r \subseteq \mathsf{ins}(C') = \mathsf{ins}(C)$ since $X \in \mathcal{B}$.

This proves that ins is a uniform slope of order $k/2$ in G'. □

Robertson and Seymour proved the following lemma in their 'Graph Minors XI.' We omit the proof. [37]

[37] Graph Minors. XI. Circuits on a surface. Journal of Combinatorial Theory, Series B **60** (1994), pp. 72–106.

A closed walk W 'captures' a point $x \in \Sigma$ if it passes through x or there is a circuit C of length $< k$ that satisfies $E(C) \subseteq E(W)$ and $x \in \mathsf{ins}(C)$.

Lemma 4.40. *Let G be drawn on a sphere Σ and let $k \in \mathbb{N}$. Let ins be a slope of order $k/2$ and let $x \in \Sigma$.*
Let $N_x \subseteq \Sigma$ be the set of $y \in \Sigma$ for which there is a closed walk W in G of length $< k$ that captures x and y. Then either $\Sigma - N_x$ is an open disc or $N_x = \varnothing$. Furthermore if ins is uniform then $N_x \neq \varnothing$.

WE COME TO THE FINAL STEP in showing that carvingswidth at least k and for all vertices $p(\delta(x)) < k$ implies an antipodality of p-range at least k. Actually, the following theorem shows that there is an antipodality which is connected. An antipodality α is connected if $\alpha(e)$ is connected for all $e \in E(G)$.

In the following theorem let G, G^*, G', p and k be as in Lemma 4.39.

Theorem 4.41. *If G' has a uniform slope of order $k/2$ then G has a connected antipodality of p-range $\geqslant k$.*

Proof. For $x \in \Sigma$ let N_x be as in Lemma 4.40 but for the graph G' instead of G.

Define α as follows. We show below that α is a connected antipodality of p-range at least k.

For $r \in R(G)$ let

$$\alpha(r) = \{v \in V \mid v^* \subseteq \Sigma - N_{r^*}\}$$

and for $e \in E(G)$ let

$$\begin{aligned} V(\alpha(e)) &= \{v \in V \mid v^* \subseteq \Sigma - N_{x(e)}\} \\ E(\alpha(e)) &= \{f \in E \mid f^* \subseteq \Sigma - N_{x(e)}\}. \end{aligned}$$

Notice that this is a subgraph. [38]

By Lemma 4.40 $N_{x(e)}$ is an open disc — so — $\alpha(e)$ is a connected subgraph of G.

We show that α is an antipodality of p-range $\geqslant k$. To prove A1 let $e \in E$ and let v be an endpoint of e. Since ins is uniform there exists a circuit C in G' of length $< k$ such that $v^* \subseteq \mathsf{ins}(C)$. Then $e^* \subseteq \mathsf{ins}(C)$. So there is a circuit of length $< k$ that captures each point of v^* and $x(e)$ and this implies $v^* \subseteq N_{x(e)}$ — so — $v \notin V(\alpha(e))$.

To see A2 let $e \in E(G)$ and $r \in R(G)$ and let e and r be incident. Then e^* and r^* are incident in G^* and $N_{x(e)} \subseteq N_{r^*}$ (since any walk in G' that captures $x(e)$ also captures r^*) and so

$$\Sigma - N_{r^*} \subseteq \Sigma - N_{x(e)}.$$

[38] To see that let f be an edge with endpoint v. Notice that

$$f^* \subseteq \Sigma - N_{x(e)} \Rightarrow v^* \subseteq \Sigma - N_{x(e)}$$

This show that $\alpha(r) \subseteq V(\alpha(e))$.

To see A3 let $e \in E$ and $f \in E(\alpha(e))$. No closed walk of length $< k$ captures both $x(e)$ an $x(f)$. This implies that no closed walk in G^* of length $< k$ contains e^* and f^*.

This proves the theorem. $\square$

This shows the following halfway result.

> Let G be a connected graph with at least two vertices drawn on a sphere Σ with a dual G^*. Let $p : E(G) \to \mathbb{N}$ and $k \in \mathbb{N}$. Assume that $p(\delta(x)) < k$ for each vertex $x \in V(G)$ and that the p-carving width of G is $\geqslant k$. Then G has an antipodality of p-range $\geqslant k$.

4.3.3 Bond carvings

Take a deep breath: it's time to start the proof of the 'if' - part of Theorem 4.33 on Page 157.

Let's get in the mood with a groovy exercise. [39]

[39] The word 'groovy' is a bit outdated (it means 'cool' — if that's not already outdated also); it comes from the 'groove' in a vinyl record; 'the music is in the groove' ie in the carving of the record.

Exercise 4.26

Let $\mathcal{C}$ be a carving of a set V.

(i) $X \in \mathcal{C} \quad \Rightarrow \quad V \setminus X \in \mathcal{C}$

(ii) if $X \in \mathcal{C}$ and $|X| \geqslant 2$ then X has a unique partition $\{Y, Z\}$ with $Y, Z \in \mathcal{C}$. [40]

[40] So —

(a) $Y, Z \neq \varnothing$

(b) $Y \cap Z = \varnothing$

(c) $Y \cup Z = X$.

The fact that $Y, Z \in \mathcal{C}$ implies (also) that $Y, Z \neq \varnothing$.

Hint: Use Exercise 4.20 on Page 150. A routing tree is ternary. The set X is the set of leaves of a branch and X has at least two leaves. (A branch of a tree is a component of the forest obtained by removing one edge of the tree.)

Recall Definition 4.38 on Page 160: a set $\delta(X)$ is a bond if X and $V \setminus X$ induce connected subgraphs in G.

Definition 4.42. A carving $\mathcal{C}$ of a graph G is a bond - carving if $\delta(X)$ is a bond for all $X \in \mathcal{C}$.

Here's one more easy exercise.

Exercise 4.27

A carving $\mathcal{C}$ is a bond - carving if and only if X is connected for all $X \in \mathcal{C}$. [41]

For disjoint sets X and Y denote the set of edges that have one endpoint in X and the other in Y as $\delta(X, Y)$. Let $\mathcal{C}$ be a carving of G. A triad is a partition $\{X, Y, Z\}$ of V with $X, Y, Z \in \mathcal{C}$. — Clearly — each $X \in \mathcal{C}$ is in at most one triad. It is in a (unique) triad when

$$|X| \quad \leqslant \quad n - 2.$$

Define a 'measure' μ on the carvings of G as follows. Assume that G is connected and let $T = \{X, Y, Z\}$ be a triad. Then at most one of $\delta(X, Y)$, $\delta(X, Z)$, and $\delta(Y, Z)$ is empty. [42] If one is empty — say $\delta(X, Y) = \varnothing$ — then let $\mu(T) = |Z| - 1$. If none of $\delta(X, Y)$, $\delta(X, Z)$ and $\delta(Y, Z)$ is empty then let $\mu(T) = 0$. Define

$$\mu(\mathcal{C}) \quad = \quad \sum_{T \text{ a triad of } \mathcal{C}} \mu(T).$$

(The summation is over all triads $T = \{X, Y, Z\}$ in $\mathcal{C}$. [43])

Lemma 4.43. *Let $\mathcal{C}$ be a carving of G. Assume that G is connected and that $|V| \geqslant 2$. Let $\{A_1, A_2, B_1, B_2\} \subseteq \mathcal{C}$ be a partition of V such that* [44]

(a) $A_1 \cup A_2 \in \mathcal{C}$

(b) $\delta(A_1, A_2) = \varnothing$

(c) $\delta(A_1, B_1) \neq \varnothing$ *and* $\delta(A_2, B_2) \neq \varnothing$.

[41] Abuse! We mean of course that $G[X]$ is connected when $\mathcal{C}$ is a bond carving and $X \in \mathcal{C}$.

[42] Otherwise, one of X, Y and Z is disconnected from the rest.

[43] That is — over all points of degree 3 in a routing tree.

[44] Take an edge e of a routing tree T such that both components of $T \setminus e$ have at least two leaves. Let $\{A_1, A_2\}$ and $\{B_1, B_2\}$ be a partition of the two sets of leaves of $T \setminus e$. The lemma rebuilds the tree (grouping $A_1 \cup B_1$ and $A_2 \cup B_2$ into sets of leaves) into one that has a smaller μ-value.

Define a carving $\mathcal{C}'$ *of* G *as follows*

$$\mathcal{C}' \;=\; (\,\mathcal{C} \;\setminus\; \{A_1 \cup A_2,\, B_1 \cup B_2\,\}) \;\cup\; \{A_1 \cup B_1,\, A_2 \cup B_2\,\}.$$

Then $\mu(\mathcal{C}') < \mu(\mathcal{C})$.

Proof.

$$\begin{aligned}\mu(\mathcal{C}) \;-\; \mu(\mathcal{C}') \;=&\\ &\mu(A_1 \cup A_2, B_1, B_2) \;+\; \underset{\sim\sim\sim\sim\sim\sim\sim\sim}{\mu(A_1, A_2, B_1 \cup B_2)}\\ &\quad - \mu(A_1 \cup B_1, A_2, B_2) \;-\; \mu(A_1, B_1, A_2 \cup B_2).\end{aligned}$$

Since $\{A_1, A_2, B_1 \cup B_2\}$ is a triad and $\delta(A_1, A_2) = \varnothing$ we have (by definition of μ)

$$\mu(A_1, A_2, B_1 \cup B_2) \;=\; |B_1 \cup B_2| - 1 \;=\; |B_1| + |B_2| - 1.$$

We prove the claim via the principle of contradiction; assume

$$\begin{aligned}\mu(A_1 \cup B_1, A_2, B_2) \;+\; \mu(A_1, B_1, A_2 \cup B_2) \;\geqslant&\\ &\mu(A_1 \cup A_2, B_1, B_2) \;+\; |B_1 \cup B_2| \;-\; 1.\end{aligned} \tag{4.7}$$

Since B_1 and B_2 are disjoint we conclude

$$\begin{aligned}(\,\mu(A_1 \cup B_1, A_2, B_2) \;-\; (\,|B_2| - 1\,)\,) \;+&\\ &(\,\mu(A_1, B_1, A_2 \cup B_2) \;-\; (\,|B_1| - 1\,)\,) \;>\; 0.\end{aligned}$$

— So — without loss of generality we may assume that

$$\mu(A_1, B_1, A_2 \cup B_2) \;>\; |B_1| \;-\; 1. \tag{4.8}$$

Notice that (4.8) implies that $\delta(A_1, A_2 \cup B_2) \neq \varnothing$ since otherwise the equation would be an equality. Since we are also given that $\delta(A_1, B_1) \neq \varnothing$ (4.8) implies that $\delta(B_1, A_2 \cup B_2) = \varnothing$ (since otherwise $\mu(A_1, B_1, A_2 \cup B_2) = 0$). — So — we obtain that

$$\mu(A_1, B_1, A_2 \cup B_2) \;=\; |A_1| - 1 \quad \text{and} \quad \delta(B_1, A_2 \cup B_2) = \varnothing.$$

— Clearly — this implies

$$\begin{aligned}\delta(B_1, B_2) \;\subseteq\; \delta(B_1, A_2 \cup B_2) \;=\; \varnothing \;\Rightarrow&\\ &\mu(A_1 \cup A_2, B_1, B_2) \;=\; |A_1 \cup A_2| - 1.\end{aligned}$$

Rewriting our assumption (4.7) we find that

$$\begin{aligned} \mu(A_1 \cup B_1, A_2, B_2) + |A_1| - 1 \quad &\geqslant \quad |A_1 \cup A_2| - 1 + |B_1 \cup B_2| - 1 \quad \Rightarrow \\ \mu(A_1 \cup B_1, A_2, B_2) \quad &\geqslant \quad |A_2 \cup B_1 \cup B_2| - 1 \quad > \\ &\max\{\,|A_2| - 1,\ |B_2| - 1\,\}. \end{aligned} \tag{4.9}$$

By assumption $\delta(A_2, B_2) \neq \varnothing$ — so — $\mu(A_1 \cup B_1, A_2, B_2) \neq |A_1 \cup B_1| - 1$. Then $\mu(A_1 \cup B_1, A_2, B_2)$ must be one of $|A_2| - 1$, $|B_2| - 1$, or 0. This contradicts (4.9).

This proves the lemma. □

We come to the main result of this section. When G is a 2-connected graph and $p : E \to \mathbb{N}$ then G has a bond-carving of minimal p-width. [45]

[45] A graph is 2-connected if every minimal separator in it has at least two vertices.

Theorem 4.44. *Assume that a graph G is 2-connected and that it has at least two vertices. Let $p : E \to \mathbb{N}$ and assume that G has p-carving width $< k$. Then G has a bond - carving $\mathcal{C}$ such that $p(\delta(X)) < k$ for all $X \in \mathcal{C}$.*

Proof. Since we introduced that measure μ we might as well use it. Let $\mathcal{C}$ be a carving of G such that $p(\delta(X)) < k$ for all $X \in \mathcal{C}$ and assume that $\mu(\mathcal{C})$ is minimal (subject to the above). We claim that $\mathcal{C}$ is a bond-carving.

Assume not. Then there exists $X \in \mathcal{C}$ which is not connected. Choose X such that it has minimal size. (Clearly $|X| > 1$ since it induces a disconnected graph).

Since $\mathcal{C}$ is a carving there exists a partition $\{X_1, X_2\}$ of X with both sets $X_i \in \mathcal{C}$. By the minimality of X both X_1 and X_2 are connected. Since X is not connected $\delta(X_1, X_2) = 0$.

This shows that $\mathcal{C}$ has a triad $\{A_1, A_2, B\}$ such that

i. $\delta(A_1, A_2) = \varnothing$

ii. $|B|$ minimal subject to i.

The set B is a separator for A_1 and A_2. Since we assume that G is 2-connected $|B| \geqslant 2$. It follows that B has a partition $\{B_1, B_2\}$ with both $B_i \in \mathcal{C}$.

Claim **A**: $\delta(A_1, B_1) \neq \varnothing$ or $\delta(A_2, B_1) \neq \varnothing$. That is so because the triad $\{B_1, A_1 \cup A_2, B_2\}$ has $|B_2| < |B|$ and so $\delta(A_1 \cup A_2, B_1) \neq \varnothing$ (since we chose B minimal). — For the same reason — we have $\delta(A_1 \cup A_2, B_2) \neq \varnothing$.

First assume that $\delta(A_1, B_1) \neq \varnothing$ and $\delta(A_2, B_2) \neq \varnothing$. Claim: $p(\delta(A_1 \cup B_1)) \geqslant k$. To see that by Lemma 4.43 $\mu(\mathcal{C}') < \mu(\mathcal{C})$ and so (since $\mu(\mathcal{C}$ is minimal) there exists $X \in \mathcal{C}'$ such that $p(\delta(X)) \geqslant k$. By definition of $\mathcal{C}'$ either $X = A_1 \cup B_1$ or $X = A_2 \cup B_2$. In either case

$$p(\delta(A_1 \cup B_1)) = p(\delta(A_2 \cup B_2)) = p(\delta(X)) \geqslant k. \tag{4.10}$$

By similarity we also have that $\delta(A_1, B_2) \neq \varnothing$ and $\delta(A_2, B_1) \neq \varnothing$ implies

$$p(\delta(A_1 \cup B_2)) \geqslant k. \tag{4.11}$$

Since G is connected at least one of $\delta(A_1, B_1)$ and $\delta(A_1, B_2)$ is not empty and the same holds when A_2 replaces A_1. By Claim **A** we may assume that $\delta(A_1, B_1) \neq \varnothing$ and $\delta(A_2, B_2) \neq \varnothing$. By Equation (4.10) $p(\delta(A_1 \cup B_1)) \geqslant k$. Since $p(\delta(B_1)) < k$ and $\delta(A_1, A_2) = \varnothing$,

$$\delta(A_1 \cup B_1) \not\subseteq \delta(B_1) \quad \Rightarrow \quad \delta(A_1, B_2) \neq \varnothing,$$

and — similarly — $\delta(A_2, B_1) \neq \varnothing$. By (4.11) $p(\delta(A_1 \cup B_2)) \geqslant k$.

We derive a contradiction as follows

$$\begin{aligned} 2 \cdot k \leqslant \; & p(\delta(A_1 \cup B_1)) + p(\delta(A_1 \cup B_2)) = \\ & p(\delta(A_1, B_2)) + p(\delta(A_2, B_1)) + p(\delta(B_1, B_2)) + \\ & p(\delta(A_1, B_1)) + p(\delta(A_2, B_2)) + p(\delta(B_1, B_2)) = \\ & p(\delta(B_1)) + p(\delta(B_2)) < 2 \cdot k. \end{aligned}$$

(where the last line follows from $B_1, B_2 \in \mathcal{C}$)

This proves the theorem. □

4.3.4 Carvings and antipodalities

In this section we complete the proof of Theorem 4.33 on Page 157. We present the proof in seven easy steps.

Let G be a connected planar graph with at least two vertices. Let G be drawn on a sphere Σ and let G^* be the dual. Let $p : E \to \mathbb{N}$ and let $k \in \mathbb{Z}^{\geqslant 0}$.

Let α be an antipodality of p-range $\geqslant k$. In this section we show that G has p-carving width $\geqslant k$. [46]

A pair (P, ν) with $P \subseteq V$ and $\nu \in P$ is a <u>limb</u> if

$$\delta(P) \subseteq \delta(\nu) \quad \text{and} \quad \exists_{e \text{ with endpoint } \nu} \quad V(\alpha(e)) \cap P \neq \varnothing.$$

[46] Recall the definition of an antipodality. For each region $\varnothing \neq \alpha(r) \subseteq V$ and for each edge $\alpha(e)$ is a subgraph (so it has at least one vertex) that does not contain an endpoint of e. The function α satisfies the following conditions.

(A1) for each edge $e \in E(G)$ the subgraph $\alpha(e)$ does <u>not</u> contain an endpoint of e

(A2) if an edge e and a region r are incident then $\alpha(r) \subseteq V(\alpha(e))$ and every component of $\alpha(e)$ has a vertex in $\alpha(r)$

(A3) if $e \in E$ and $f \in E(\alpha(e))$ then every closed walk in G^* that contains e^* and f^* has p-length at least k.

Exercise 4.28

When (P, ν) is a limb and $P \neq V$ then ν is a cutvertex and P is the union of a collection of components of $G - \nu$ and $\{\nu\}$.
Hint: Use that $\delta(P) \subseteq \delta(\nu)$.

Lemma 4.45. *If* (P, ν) *is a limb then*

$$\forall_{e \text{ with endpoint } \nu} \quad V(\alpha(e)) \cap P \setminus \{\nu\} \quad \neq \quad \varnothing.$$

Proof. Let $e_1, \cdots, e_t$ be the edges that are incident with ν in the cyclic order in which they are drawn in Σ. Assume $V(\alpha(e_1)) \cap P \neq \varnothing$. We prove the lemma via contradiction: let $i > 1$ be minimal $V(\alpha(e_i)) \cap P = \varnothing$.

Then $V(\alpha(e_{i-1})) \cap P \neq \varnothing$. Let H be a component of $\alpha(e_{i-1})$ with $V(H) \cap P \neq \varnothing$. Then $\nu \notin V(H)$ since $\alpha(e_{i-1})$ does not contain an endpoint of e_{i-1} (since α is an antipodality).

Since $\delta(P) \subseteq \delta(\nu)$ and $\nu \notin V(H)$ and H is connected

$$V(H) \quad \subseteq \quad P.$$

Let $r \in R(G)$ be the region that is incident with e_{i-1} and e_i. By the second antipodality condition

$$V(H) \cap \alpha(r) \neq \varnothing \quad \text{and} \quad \alpha(r) \subseteq V(\alpha(e_i)).$$

This implies

$$\varnothing \neq V(H) \cap \alpha(r) \subseteq P \cap \alpha(r) \subseteq P \cap V(\alpha(e_i)).$$

This contradicts the choice of i.

This proves the lemma. $\square$

Notice that all vertices have degree at least 1 since G is connected and $|V| \geqslant 2$. It follows that for all vertices (V, ν) is a limb. Choose a limb (P, ν) such that $|P|$ is as small as possible.

Lemma 4.46. $G[P \setminus \{\nu\}]$ *is connected.*

Proof. Notice that $P \setminus \{\nu\} \neq \varnothing$ since [47]

$$V(\alpha(e)) \cap (P \setminus \{\nu\}) = V(\alpha(e)) \cap P \neq \varnothing.$$

[47] When (P, ν) is a limb then $P \neq \{\nu\}$ since $P \cap V(\alpha(e)) \neq \varnothing$ for some e incident with ν and $\alpha(e)$ does not contain ν.

Suppose $P \setminus \nu$ is not connected. Then there exist subsets P_1 and P_2 satisfying

1. $P_1 \cup P_2 = P$
2. $P_1 \cap P_2 = \{\nu\}$
3. $\delta(P_1 \setminus \nu, P_2 \setminus \nu) = \varnothing$
4. $P_1 \setminus \nu \neq \varnothing$ and $P_2 \setminus \nu \neq \varnothing$.

Choose e incident with ν such that $P \cap V(\alpha(e)) \neq \varnothing$. Then one of $P_1 \cap V(\alpha(e))$ and $P_2 \cap V(\alpha(e))$ is $\neq \varnothing$. — That is — (P_1, ν) or (P_2, ν) is a limb. This contradicts the choice of (P, ν) as a limb with P minimal.

This proves the lemma. $\square$

Exercise 4.29

Show that $P \setminus \{\nu\} \neq \varnothing$ and that $\delta(P \setminus \{\nu\}, \nu) \neq \varnothing$.

Let B be a maximal 2-connected subgraph of G that contains ν and a neighbor of ν in P. [48] Since $\delta(P) \subseteq \delta(\nu)$

$$V(B) \subseteq P.$$

[48] An edge forms a 2-connected subgraph. So B contains at least ν and a neighbor of ν in P. Since B is 2-connected and $\delta(P)) \subseteq \delta(\nu)$ B does not contain any vertex outside P; otherwise B would contain a cutvertex; namely ν.

Lemma 4.47. *Every neighbor of* v *in* P *is in* $V(B)$*:*

$$N[v] \cap P \quad \subseteq \quad V(B).$$

Proof. By definition of B v has a neighbor $u_1 \in V(B)$. Assume v has another neighbor u_2 in P. Since $G[P \setminus \{v\}]$ is connected there is a circuit C in $G[P]$ that contains v, u_1, and u_2.

Then

$$|\,V(B) \cap C\,| \quad \geqslant \quad 2 \quad \Rightarrow \quad C \subseteq V(B).$$

This implies $u_2 \in V(B)$ and this proves the lemma. □

Exercise 4.30

For $X \subseteq B$ show that there exists a unique set $\tilde{X}$ that satisfies

$$\tilde{X} \cap B = X \quad \text{and} \quad \delta(\tilde{X}) = \delta(X, B \setminus X).$$

HINT: Consider the union of components of $V \setminus B$ that have a neighbor in X. No component can have a neighbor in X and in $B \setminus X$ since B is a biconnected component.

Exercise 4.31

When $v \notin X$ then $\tilde{X} \subseteq P \setminus \{x\}$.

Exercise 4.32

Let a graph H be a subgraph of a graph G. Let $p : E \to \mathbb{N}$. Show that the p-carving width of G is at least the p-carving width of H.
HINT: Let T be a routing tree for G. Remove leaves that are not in $V(H)$. Let T' be the result of this. How can we change T' into a routing tree for H?

ASSUME THAT B HAS p-CARVING WIDTH $< k$. We show that this assumption leads to a contradiction.

By Theorem 4.44 there exists a bond - carving $\mathcal{C}$ such that for all $X \subseteq B$

$$p(\delta(X, B \setminus X)) \quad = \quad p(\delta(\tilde{X})) \quad < \quad k.$$

Define

$$\mathcal{C}' \quad = \quad \{X \in \mathcal{C} \mid v \notin X \quad \text{and} \quad \exists_{e \in \delta(\tilde{X})} \quad V(\alpha(e)) \cap \tilde{X} \neq \varnothing\}.$$

Lemma 4.48. $\mathcal{C}' \neq \varnothing$.

Proof. We show that $X = B \setminus \{v\} \in \mathcal{C}'$.

Clearly $X \in \mathcal{C}$. Notice that $\tilde{X} = P \setminus \{v\}$ (since every neighbor of v in P is in B). Choose $e \in E(B)$ with endpoint v. Since (P, v) is a limb (by Lemma 4.45)

$$V(\alpha(e)) \cap \tilde{X} \neq \varnothing.$$

Since $e \in \delta(\tilde{X})$ this proves that $X \in \mathcal{C}'$. □

Choose $X \in \mathcal{C}'$ such that $|X|$ is minimal.

Lemma 4.49. $|X| \neq 1$.

Proof. Assume $X = \{u\}$. Then $\delta(\tilde{X}) \subseteq \delta(u)$ and $V(\alpha(e)) \cap \tilde{X} \neq \varnothing$ for some $e \in \delta(\tilde{X})$ (since $X \in \mathcal{C}'$). This shows that $(\tilde{X}, u)$ is a limb and $\tilde{X} \subseteq P \setminus v$. This contradicts the choice of (P, v) (as a limb with $|P|$ is as small as possible). □

Since $\mathcal{C}$ is a carving there exist $X_1, X_2 \in \mathcal{C}$ such that $\{X_1, X_2\}$ is a partition of X. By the minimality of $|X|$ neither X_1 nor X_2 is in $\mathcal{C}'$.

Lemma 4.50. *For all* $e \in \delta(\tilde{X}_1) \cup \delta(\tilde{X}_2)$

$$E(\alpha(e)) \cap \Big(\delta(\tilde{X}_1) \cup \delta(\tilde{X}_2)\Big) \quad = \quad \varnothing.$$

Proof. Let $e, f \in \delta(\tilde{X}_1) \cup \delta(\tilde{X}_2)$. One of $\delta(X)$, $\delta(X_1)$ and $\delta(X_2)$ contains both e and f — say D.

Since $\mathcal{C}$ is a bond-carving D is a bond and this implies that there is a circuit C in G^*

$$E(C) = \{ f^* \mid f \in D \}$$

and $e^*, f^* \in E(C)$.

The circuit C has p-length $p(D) < k$. By the antipodality condition A_3: $f \notin E(\alpha(e))$.

This proves the lemma. □

Lemma 4.51. *For all* $e \in \delta(\tilde{X})$

$$V(\alpha(e)) \cap \tilde{X}_1 = \varnothing.$$

Proof. Since $\delta(X)$ is a bond we may choose an ordering of the edges in $\delta(\tilde{X})$

$$e_1 \quad \cdots \quad e_t$$

such that there is a region r_i in G incident with e_{i-1} and e_i.

Notice that $B \setminus X_2$ is connected. That is so because $B \setminus X_2 \in \mathcal{C}$ and $\mathcal{C}$ is a bond carving. It follows that

$$\delta(\tilde{X}_1) \cap \tilde{X} \neq \varnothing.$$

— So — we may choose $e_1 \in \delta(\tilde{X}_1)$. Since $X_1 \notin \mathcal{C}'$ we have

$$V(\alpha(e_1)) \cap \tilde{X}_1 = \varnothing.$$

Let i be minimal such that $V(\alpha(e_i)) \cap \tilde{X}_1 \neq \varnothing$. [49] Let H be a component of $\alpha(e_i)$ that intersects $\tilde{X}_1$. By Lemma 4.50 $E(H) \cap \delta(\tilde{X}_1) = \varnothing$. Since H is connected

$$V(H) \subseteq \tilde{X}_1.$$

By the second antipodality condition

$$V(H) \cap \alpha(r_i) \neq \varnothing$$

[49] We go via contradiction.

and $\alpha(r_i) \subseteq V(\alpha(e_{i-1}))$. This implies $V(\alpha(e_{i-1})) \cap \tilde{X}_1 \neq \varnothing$ and this is a contradiction. — So — $V(\alpha(e)) \cap \tilde{X}_1 = \varnothing$ for all $e \in \delta(\tilde{X})$.

The same argument shows that $V(\alpha(e)) \cap \tilde{X}_2 = \varnothing$ for all $e \in \delta(\tilde{X})$. Since $\{X_1, X_2\}$ is a partition of X $V(\alpha(e)) \cap \tilde{X} = \varnothing$ for all $e \in \delta(\tilde{X})$. This contradicts the assumption that $X \in \mathcal{C}'$.

This proves the lemma. □

In other words the assumption on Page 170 is wrong: B has p-carving width $\geqslant k$. This shows that G has p-carving width $\geqslant k$ since B is a subgraph of G.

The following theorem summarizes the results.

Theorem 4.52. *Let G be a connected planar graph with at least two vertices. Let G be drawn on a sphere with a dual G^*. Let $p : E \to \mathbb{N}$ and let $k \in \mathbb{Z}^{\geqslant 0}$. The following statements are equivalent.*

1. *the graph G has p-carving width at least k*
2. *the graph G has a tilt of p-order k*
3. *the graph G' has a uniform slope of order $k/2$* [50]
4. *the graph G has a connected antipodality of p-range at least k*
5. *the graph G has an antipodality of p-range at least k.*

[50] G' is obtained from G^* by subdividing edges $e^* \in E(G^*)$ $p(e)-1$ times.

Remark 4.53. By Theorem 4.32 there exists an $O(m^2)$ - algorithm to decide if the p-carving width $\geqslant k$ $(m = |V| + |E|)$. In their paper Robertson and Seymour show that the result above can be used to compute the 'branchwidth' of planar graphs in $O(m^2)$ time. The branchwidth parameter approximates the treewidth of graphs within a factor 3/2.

4.4 Tree - degrees of graphs

Chordal graphs are the intersection graphs of subtrees of a tree (see Exercise 4.5 on Page 133). In this section we look at graphs that have a — slightly — different intersection model.

Definition 4.54. A graph G is the edge intersection - graph of a tree T if there is a collection of subtrees

$$\{ T_x \mid x \in V \}$$

such that

$$\{x, y\} \in E(G) \quad \Leftrightarrow \quad E(T_x) \cap E(T_y) \neq \varnothing.$$

The definition above is — a kind of — a joke. Namely any graph is the edge intersection - graph of a family of subtrees of a tree. To see that consider a star $K_{1,z}$ and a bijection from its leaves to the maximal cliques of G. [51] For $x \in V(G)$ let T_x be the subtree of the star that connects all leaves that contain x.

[51] So z is the number of maximal cliques in G. Two subtrees of a star share a line if and only if they share a leaf ie if and only if the vertices share a maximal clique.

Exercise 4.33

Show that the above constructs an edge intersection - model for any graph.

Hint: Two vertices are adjacent only if they are in a maximal clique — that is — only if their subtrees share a leaf.

Exercise 4.34

Let G be a graph without isolated vertices [52] and let $\mathcal{C}$ be a collection of maximal cliques that cover the edges of G — that is — every edge of G is contained in one of the cliques of $\mathcal{C}$. The

[52] If x is an isolated vertex let T_x be a subtree that contains only one (arbitrary) vertex.

minimal number of cliques in a cover of E is the edge clique - cover of G. Denote the edge - clique cover - number of a graph G as $cc(G)$. Its computation is NP-complete.

Say that G has an edge clique - cover with k cliques. Show that G is the edge intersection graph of a tree with maximal degree k.

Definition 4.55. The tree - degree $\tau(G)$ of a graph G is the minimal $k \in \mathbb{N} \cup \{0\}$ such that G is the edge intersection - graph of subtrees of a tree that has maximal degree $\leqslant k$.

Exercise 4.35

Let G be connected. Show that $\tau(G) = 1$ if and only if G is a clique with at least two vertices.

Assume that G has $k \geqslant 2$ components G_i that have at least one edge. Show that

$$\tau(G) \quad = \quad \max\{\tau(G_i), 2 \mid 1 \leqslant i \leqslant k\}$$

4.4.1 Intermezzo: Interval graphs

Consider a set of n intervals on the real line — say — $I_1, \cdots, I_n$. Construct a graph with vertex set $[n]$. Let two vertices be adjacent if the intervals have a nonempty intersection. The graph is called an interval graph.

Definition 4.56. A graph is an interval graph is it is the intersection graph of a collection of intervals on a line.

Exercise 4.36

Show that interval graphs are chordal.

A COLLECTION OF INTERVALS ON THE REAL LINE SATISFIES THE HELLY PROPERTY: if any pair of a collection of intervals intersects then they they all contain a common point on the real line. [53] Suppose we scan the line from left to right and for each point we record all the intervals that contain that point. That will give us a list of the cliques of G.

Exercise 4.37

Show that a graph G is an interval graph if and only if its maximal cliques can be put in a linear order [54]

$$C_1 \quad \cdots \quad C_t$$

such that for any vertex the cliques that contain it form an interval.

Exercise 4.38

Let G be an interval graph and let $C_1, \cdots, C_t$ be a consecutive clique arrangement. Show that the collection

$$\{\, C_i \cap C_{i+1} \mid 1 \leqslant i < t \,\}$$

is the collection of minimal separators in G.

THE REASON FOR INTRODUCING INTERVAL GRAPHS is the following presentation of graphs that have tree - degree at most two and three.

Exercise 4.39

A graph satisfies $\tau \leqslant 2$ if and only if it is an interval graph.

A graph satisfies $\tau \leqslant 3$ if and only if it is a chordal graph.

[53] The Helly property also holds for subtrees of a tree: when $\mathcal{C}$ is a collection of subtrees of a tree T of which every pair intersects in some point of T. Then T contains a point that in in all the subtrees of $\mathcal{C}$. The same is not true when we consider edge - intersections: for example, consider three paths in a claw each connecting one pair of leaves.

[54] Since G is chordal it has at most n maximal cliques. The linear order of maximal cliques is called a consecutive clique arrangement.

Remark 4.57. It can be shown that for all graphs [55]

$$\tau(G) \quad \leqslant \quad \mathsf{cc}(G).$$

Equality holds when G has no clique separator.

[55] M. Chang, T. Kloks and H. Müller, *On the tree-degree of graphs*, Springer, Lecture Notes in Computer Science **2204** (2001), pp. 44–54.

Let $k \in \mathbb{N}$ and let $\mathcal{G}$ be a class of graphs that satisfy $\tau \leqslant k$. There exists a polynomial - time algorithm to compute the treewidth of a graph in $\mathcal{G}$. The reason is that the number of minimal separators in a graph of $\mathcal{G}$ is at most $3m \cdot 2^{\tau-2}$. There exists a polynomial - time algorithm to compute the treewidth of graphs that have only a polynomial number of minimal separators. [56] [57]
An upper-bound for the treewidth is

$$\mathsf{tw} \quad \leqslant \quad \tau \cdot \omega.$$

The computation of $\tau(G)$ remains NP-complete even when restricted to plane triangulations (G is a plane graph and every face is a triangle).

[56] V. Bouchitté and I. Todinca, *Minimal triangulations of graphs with "few" minimal separators*, Springer, Lecture Notes in Computer Science **1461** (1998), pp. 344–355.
[57] T. Kloks and D. Kratsch, Finding all minimal separators of a graph. Technical report 9327, Eindhoven University of Technology, 1993.

4.5 Modular decomposition

A module is a spacelab.

Let's start with a definition; then we know what we are talking about.

Definition 4.58. Let G be a graph. A module in G is a set of vertices $X \subseteq V$ with the property that every vertex of $V \setminus X$ is either adjacent to every vertex of X or not adjacent to any vertex in X.

The components of a graph as well as the components of the complement are modules.

Clearly every graph has modules. The trivial modules are $\varnothing$, V and the subsets with one vertex. A graph is prime if it only has trivial modules.

Exercise 4.40

(a) Let G be a graph and let X be a module in G. A set $\mathsf{Y} \subseteq \mathsf{X}$ is a module in G if and only if Y is a module in $\mathsf{G}[\mathsf{X}]$.

(b) Let M be a module in a connected graph G and assume that M contains two vertices that are at distance > 2 in G. Then $\mathsf{M} = \mathsf{V}$.

HINT: If $z \notin \mathsf{M}$ then z is not adjacent to anything in M (otherwise every pair of vertices in M is at distance $\leqslant 2$). But G is connected — so — $\mathsf{M} = \mathsf{V}$.

(c) Let X be a module in G and let $z \in \mathsf{V} \setminus \mathsf{X}$. Show that X is a module in $\mathsf{G}[\,\mathsf{N}(z)\,]$ or in $\mathsf{G}[\,\mathsf{V} \setminus \mathsf{N}[z]\,]$.

(d) The set of modules $\mathcal{M}$ of a graph is a partitive family — that is —

- all trivial modules are in $\mathcal{M}$
- when X and Y are modules that overlap then

$$\mathsf{X} \cap \mathsf{Y} \quad \mathsf{X} \cup \mathsf{Y} \quad \mathsf{X} \setminus \mathsf{Y} \quad \text{and} \quad (\mathsf{X} \setminus \mathsf{Y}) \cup (\mathsf{Y} \setminus \mathsf{X})$$

are all modules.

Two sets A and B overlap if $\mathsf{A} \setminus \mathsf{B} \neq \varnothing$ and $\mathsf{B} \setminus \mathsf{A} \neq \varnothing$ and $\mathsf{A} \cap \mathsf{B} \neq \varnothing$.

Definition 4.59. A module is strong if it does not overlap other modules.

Every graph has a partition of its vertices with parts that are strong modules.

Exercise 4.41

(a) Let A be a strong module. The smallest strong module that properly contains A is unique.

(b) Let S be a minimal separator in G and let M be a module in G that has a vertex of S but that is not contained in S. Then M contains all vertices of components that are close to S.

LET US SEE if a BFS - tree can be of any use in spotting a module.

A BFS - tree is a spanning tree with the property that every path from a vertex in the tree to the root is a shortest path in the graph.

In this chapter a BFS - tree T has a left - to - right order of the vertices in every level with the following property. If a vertex x has a parent y in T then x is not adjacent to any vertex in the level of y that is to the left of y.

Exercise 4.42

1. Let M be a module and let x be a vertex not in M. Let T be a BFS - tree with root x. Then all vertices of M have the same parent in T.
2. Let C be a component of the graph induced by the vertices in a level > 1 in a BFS - tree with root x. Let M be a strong module that contains x. Then $C \subset M$ or $C \cap M = \varnothing$.
3. Let x be a vertex and let C be a cocomponent in $G[N(x)]$. Let M be a strong module that contains x. Then $C \subset M$ or $C \cap M = \varnothing$.

4.5.1 Modular decomposition tree

The nodes of a modular decomposition tree are the nonempty modules that overlap with no other modules. The ancestor relation is containment.

LET G BE A GRAPH. A modular decomposition - tree for the graph G is a rooted tree with the vertices of G as leaves defined as follows.

Exercise 4.43

Show that a decomposition tree (as defined below) has the following property. The leaves of a branch are a module in G.

1. if G is disconnected then the root is labeled as a <u>parallel node</u>. Its children are the roots of the decompositions of the components.

2. if $\bar{G}$ is disconnected then the root is labeled as a <u>series node</u>. Its children are the roots of the decomposition trees of the cocomponents.

3. if G and $\bar{G}$ are both connected then there is a set $U \subseteq V$ and a partition $\mathcal{P}$ of V with the following properties.

   - $|U| > 3$
   - $G[U]$ is a maximal prime subgraph of G
   - every part $P \in \mathcal{P}$ satisfies $|U \cap P| = 1$.

   The node is a <u>prime node</u> and labeled as $G[U]$. Its children are the roots of a decomposition tree for the parts in $\mathcal{P}$.

Every graph with less than 4 vertices is a cograph. The decomposition tree for a cograph has only series nodes and parallel nodes.

Can two adjacent nodes in a modular decomposition tree be of the same type?

Exercise 4.44

Let G be a cograph. What are the strong modules in G?

HINT: The cotree is a modular decomposition (without any prime nodes). Every branch in a cotree is a module. Every induced subgraph has a module with two vertices.

4.5.2 A linear - time modular decomposition

In this section we take a look at the following paper.

M. Tedder, D. Corneil, M. Habib and C. Paul, Simple, linear - time modular decomposition. Manuscript on arXiv: 07010.3901, 2008.

The algorithm that is presented in this paper computes a modular decomposition tree as follows.

We assume that G is connected. Choose an arbitrary vertex x and let $\{L_i\}$ be a partition of V into the levels of a BFS-tree with $L_0 = \{x\}$ and $L_1 = N(x)$ and so on. The algorithm recursively computes a modular decomposition tree $T(L_i)$ for each layer L_1 and orders them as

$$T(L_1) \quad \{x\} \quad T(L_2) \quad T(L_3) \quad \cdots$$

Recall: The strong modules that do not contain x are modules in $N(x)$ or in $V \setminus N[x]$. The difficulties are

1. find the strong modules that contain x
2. construct the modular decomposition tree.

The aim is to compute a linear order of the vertices that is factorizing — that is — all the vertices of any strong module are consecutive.

We describe three procedures to achieve this. As an invariant we have an ordered partition of the vertices (each part is a set of leaves in a tree) which satisfies the condition that all the strong modules that do not contain x appear in consecutive parts.

The invariant is a relaxation of the desired result.

Below we add a second invariant that describes the positions of the strong modules that contain x.

Refinement

Let the current list of trees be

$$\mathsf{T}(\mathsf{N}_0) \quad \{x\} \quad \mathsf{T}(\mathsf{N}_1) \quad \cdots \quad \mathsf{T}(\mathsf{N}_k)$$

The paper of Tedder et al uses

$$\mathsf{N}_0 = \mathsf{N}(x) = \mathsf{L}_1.$$

Let M be a strong module that does not contain x. Then $M \subseteq \mathsf{N}_i$ for some i — furthermore — it is either a node in $\mathsf{T}(\mathsf{N}_i)$ or it is a set of nodes that all have the same parent.

The refinement - operation rebuilds $\mathsf{T}(\mathsf{N}_i)$ in case M is not a node. Let y be the parent of elements of M and assume that y has some children that are not in M. The procedure creates a new child y' of y. The children of y that are in M become children of y'.

Exercise 4.45

After a refinement the strong modules that do not contain x appear in consecutive parts.

When the refinement procedure terminates the nodes that do not have marked children are the strong modules containing x. The next procedure creates a new list of trees. This needs to be in order so that the invariant remains true. A “marker” helps us decide upon the new order.

Definition 4.60. An edge $\{x, y\} \in \mathsf{E}(\mathsf{G})$ is active if its endpoints are in different N_i's.

Exercise 4.46

Upon termination of the refinement - procedure the strong modules that contain x are exactly the nodes that do not have marked children.

Algorithm 6: Part 1 of **refine**: decide on left-split, right-split, and marker

input an ordered list of trees:

$$T(N_0) \quad \{x\} \quad T(N_1) \quad \cdots \quad T(N_k)$$

$\alpha(\nu) = \{\, y \in N(\nu) \mid \{\nu, y\} \text{ is active} \,\}$;
Refine with $X = \alpha(\nu)$ and
if ν is left of x OR refine operates on a tree left of x **then**
  Refine with **left - splits** and marker=**left**
else
  Refine with **right - splits** and marker=**right**
end if

The presentation of the refinement - code is unorthodox. The first part shows what markers and splits are used. The details of the refinement procedure are in the second part.

A SECOND INVARIANT limits the trees in the sequence that contain vertices of strong modules that contain x. This invariant is formulated as follows.

Let the ordered set of trees — after a refinement — be

$$T_k \quad \cdots \quad T_1 \quad \{x\} \quad T'_1 \quad \cdots T'_\ell.$$

Let M be a strong module with $x \in M$. There exist trees T_i and T'_j such that

$$\begin{array}{lll} M & \supset & T_{i-1} \cup \cdots \cup T_1 \cup \{x\} \cup T'_1 \cup \cdots \cup T'_{j-1} \\ \text{and} \quad M & \subseteq & T_i \cup \cdots \cup T_1 \cup \{x\} \cup T'_1 \cup \cdots \cup T'_j. \end{array}$$

Remark 4.61. If $M \neq V$ is a module and $x \in M$ then M has only vertices in levels 0, 1 and 2. If some vertex in level 2 is not in M then it is not adjacent to any vertex of M. So M intersects the second level in some union of components. (The same must hold

Algorithm 7: Part 2 of **refine**: Refinements by a set X

Let $T_1 \cdots T_k$ be the maximal subtrees with all leaves in X;
Let $P_1 \cdots P_\ell$ be the parents of $T_1, \cdots, T_k$;

for all P_i NOT prime **do**
  Partition the children of P_i ;
  A are the children of P_i that are T_j's and B are the other children;
  Create trees T_a and T_b with roots that have children A and B;
  Assign the label (series, parallel or prime) of P_i to A and B;
  if P_i is a root **then**
    if left - split **then**
      replace P_i with T_a T_b
    else
      replace P_i with T_b T_a
    end if;
  else
    replace children of P_i with T_a and T_b
  end if;
  Use the marker (left or right) to mark roots of T_a and T_b and all their ancestors ▷ (When a node is not a module then neither is any of its ancestors.)
end for;
for all P_i prime **do**
  Mark P_i; Mark all its children; Mark all its ancestors
end for

for $\bar{G}$ — so — M intersects $N(x)$ in a union of cocomponents.) When the second level is disconnected then the root of $T(L_2)$ is a parallel node and M is a set of children of the root.

Exercise 4.47

Let M be a strong module that contains x and a vertex at distance > 2 from x. Then $M = V$. This shows that the second invariant is true for the initial set of trees

$$T(N_0) \quad \{x\} \quad T(N_1) \quad \cdots \quad T(N_k)$$

with $T_i = T(N_0)$ and T_j either $T(N_1)$ or $T(N_k)$. We leave it as an exercise to check that this invariant remains true after refinement.

Promotion

Algorithm 8: Promotion

while there is a root r and child c both marked left **do**
  Move the branch with root c to the left of r
end while;
while there is a root r and child c both marked right **do**
  Move the branch with root c to the right of r
end while;
If a marked root has only one child then replace it with that child;
Delete all marked roots that have no children;
Remove all marks

Exercise 4.48

Upon termination of the promotion - procedure the ordered list of trees is a factorizing permutation.

Assembly

At this point we have an ordered list of trees. The nodes of these trees (except x) are the strong modules that do not contain x and each of these is properly decomposed. What's left to do is to identify the strong modules that contain x.

The strong modules that contain x are nested in intervals around x. The (co-)components of $G[N_i]$ appear consecutive. The list of trees is a list of (co-)components

$$C_\kappa \quad \cdots \quad C_1 \quad \{x\} \quad C'_1 \quad \cdots \quad C_\lambda$$

where $\{C_i\}$ are the cocomponents of $G[N_0]$ and $\{C'_i\}$ are the components of $G[N_i]$ for $i > 0$. The strong components that contain x form a nested family of intervals.

By Exercise 4.42 we have the following lemma.

Lemma 4.62. *Let M be the smallest strong module that contains x. Then M satisfies one of the following.*

1. *M is a series - module: M is a maximal consecutive collection that contains x and no C'_i*
2. *M is parallel: M is a consecutive collection that contains x and no C_i. Furthermore all the C'_j that are in M are in N_1 and they have no edge to their right*
3. *M is prime: M is a consecutive collection which includes $\{x\}$, C_1 and C'_1.*

The lemma above is used to compute the strong modules that contain x as follows.

1. For C'_i in N_1 determine if it has a vertex with a neighbor in N_j for $j > 1$.
2. Determine a μ - value for each C_i and C'_i as follows.

- to find $\mu(C_i)$: find C'_j with smallest j such that C_i has no neighbors in C'_ℓ for $\ell \geqslant j$. then

$$\mu(C_i) \quad = \quad \begin{cases} x & \text{if } j = 1 \\ C'_{j-1} & \text{otherwise} \end{cases}$$

- $\mu(C'_i)$ is defined "symmetrically."

3. The cases in the lemma above are now easily recognized. If there is no series or prime module then M contains C_1 and C'_1. When C_i is added to M — recursively — add $C'_1 \cup \cdots \cup \mu(C_i)$ (and a symmetric rule applies when C'_j is added).

Conclusion

Theorem 4.63. *There exists a linear time algorithm to compute a modular decomposition tree of a graph.*

Exercise 4.49

Let's do it again: Design an algorithm to compute a modular decomposition via a depth - first search tree.

Further reading

T. Uno and M. Yagiura, Fast algorithms to enumerate all common intervals of two permutations, *Algorithmica* **26** (2000), pp. 290 – 309.

4.5.3 Exercise

Let $\mathcal{G}$ be a class of graph. A graph G is a probe graph of $\mathcal{G}$ if its set of vertices partitions into probes and nonprobes such that G embeds in a graph of $\mathcal{G}$ by adding edges between nonprobes.

Exercise 4.50

Design an algorithm to check if a graph is a probe permutation graph. You may assume that a partition of the vertices into probes and nonprobes is a part of the input.

Hint: A graph G is a permutation graphs if and only if G and $\bar{\mathsf{G}}$ are both comparability graphs. There is a linear - time algorithm to construct a transitive orientation of a comparability graph via modular decomposition. This does not imply a linear - time recognition algorithm for comparability graphs.

The modules in a permutation graphs can be represented by boxes in the permutation graph diagram. The following paper addresses the problem in detail.

D. Chandler, Maw-Shang Chang, T. Kloks, J. Liu, Sheng-Lung Peng, On probe permutation graphs, *Discrete Applied Mathematics* **157** (2009), pp. 2611–2619.

4.6 Rankwidth

Let G be a graph and let $\mathcal{C}$ be a carving of G. The cut matrix of a set $\mathsf{X} \in \mathcal{C}$ is the submatrix of the adjacency matrix with X as rows and $\mathsf{V} \setminus \mathsf{X}$ as columns. Let $\mathsf{rank}(\mathsf{X})$ be the rank over $\mathsf{GF}[2]$ of the cutmatrix associated with X. [58] [59]

Definition 4.64. A graph has rankwidth $\leqslant \mathsf{k}$ if it has a carving $\mathcal{C}$ such that for every $\mathsf{X} \in \mathcal{C}$ $\mathsf{rank}(\mathsf{X}) \leqslant \mathsf{k}$.

[58] The rows of a cutmatrix form elements of a vector space over $\mathsf{GF}[2]$; the Galois field with two elements 0 and 1. (BTW Évariste Galois was a French mathematician.) The rank of a cutmatrix is the maximal number of rows of which no nonempty subset adds up to 0. (Addition of vectors is element-wise and obeys $0+0=0$, $1+0=1$, and $1+1=0$).

[59] The routing tree is restricted so that internal nodes have degree 3. This is of importance; a star would allow a decomposition of rankwidth 1 of any graph.

Exercise 4.51

Let $\mathcal{C}$ be a carving of a graph G of rankwidth $\leqslant k$. Show that every $X \in \mathcal{C}$ has a partition into at most 2^k classes such that the vertices of each class all have the same neighbors in $V \setminus X$.

Hliněný and Oum showed (in 2008) that there is a fixed-parameter $O(n^3)$ algorithm that recognizes graphs of rankwidth at most k (for $k \in \mathbb{N}$). (Computing the rankwidth of a graph is NP-complete.) Hliněný and Oum's algorithm finds a carving of rankwidth $\leqslant k$ if there exists one. [60]

Geelen, Kwon, McCarthy and Wollan show that any circle graph H is a vertex - minor of a graph with sufficiently large rankwidth.
J. Geelen, O. Kwon, R. McCarthy and P. Wollan, The grid theorem for vertex - minors. Manuscript on arXiv: 1909.08113, 2019.

4.6.1 Distance hereditary - graphs

It makes sense to have a close look at graphs that have rankwidth at most one. In this section we characterize those graphs.

Definition 4.65. A graph G is distance - hereditary if for any two nonadjacent vertices x and y all chordless $x{\sim}y$-paths have the same length. [61]

Pioneering work on distance hereditary - graphs was done by Howorka. We summarize some results below. [62]

Exercise 4.52

1. Show that a distance hereditary - graph G does not have an induces subgraph which is isomorphic to a house, hole, domino or gem. [63]

[60] P. Hliněný and S. Oum, *Finding branch-decompositions and rank-decompositions*, SIAM Journal on Computing **38** (2008), pp. 1012–1032.

[61] A path is chordless if only consecutive pairs of vertices are adjacent in the graph.

[62] Are you getting confused? If all else fails you can always have a look at: Ton Kloks and Yue-Li Wang's Advances in graph algorithms.

[63] A domino is a tile in a game and the game is called 'dominoes.' It seems that the game originated in China. (A domino is not a pizza!) The domino is a rectangle that is partitioned into two squares. Each square has a number of dots on it. For plural we use 'dominoes'.

2. Show that a graph is distance - hereditary if and only if it does not contain a house, hole, domino or gem as an induced subgraph. [64]

HINT: The class of graphs that are distance - hereditary is hereditary — that is — the class is closed under taking induced subgraphs. The house, holes, domino and gem are exactly the smallest graphs that have two nonadjacent vertices that are connected by two chordless paths of different length (so the two nonadjacent vertices are in a cycle).

[64]

(i) a <u>hole</u> in a graph is a chordless cycle of length at least 5

(ii) a <u>house</u> is C_5 with one chord

(iii) a <u>gem</u> is C_5 with two chords that share an endpoint

(iv) a <u>domino</u> is C_6 with one chord that connects two vertices that are at distance 3 in C_6.

Exercise 4.53

Show that the class of graphs that are distance - hereditary is closed under the following operations.

1. add a pendant vertex to the graph — that is — add one new vertex and give it exactly one neighbor

2. add a (true or false) twin — that is — add one new vertex and give it exactly the same (closed or open) neighborhood as one other vertex.

Exercise 4.54

Let T be a routing tree for a graph G of rankwidth $k \geqslant 1$. Let the graph G' be obtained from G by adding one new vertex that gets the same (open or closed) neighborhood as a vertex of G. (That is; G' is obtained from G 'by creating a twin.') Construct a routing tree for G' (of width k).

One other (similar) question: how do you construct a routing tree for the graph obtained from G by adding a pendant vertex (adjacent to exactly one other vertex in the graph)?

Exercise 4.55

When G is distance - hereditary then every neighborhood induces a cograph. [65]

[65] Recall Chapter 2.11: a graph is a cograph if it does not contain a path with 4 vertices as an induced subgraph.

HINT: Assume G has an induced P_4 with all its points in a neighborhood — say — $N(x)$ contains an induced P_4. Then G has a gem.

Theorem 4.66. *A graph has rankwidth $\leqslant 1$ if and only if it is distance - hereditary.*

Proof. Assume a graph G is distance - hereditary. By Exercise 4.53 G has a routing tree with the following property. The set of leaves — say X — of a branch has a partition $\{X_1, X_2\}$ (where we allow parts to be empty) such that the vertices of each part have the same neighbors in $V \setminus X$. Furthermore when both parts are nonempty then the vertices of one of the two parts have no neighbors in $V \setminus X$. [66] — In other words — all nonzero rows of the cutmatrix of X are the same. This shows that G has rankwidth $\leqslant 1$.

A carving $\mathcal{C}$ of rankwidth $\leqslant 1$ has the property that $\mathsf{rank}(X) \leqslant 1$ for every $X \in \mathcal{C}$. This implies that all vertices of X that have neighbors in $V \setminus X$ have the same neighbors in $V \setminus X$.

Let $X \in \mathcal{C}$ with $|X| = 2$. If X has two vertices that have neighbors in $V \setminus X$ then those two are twins. If X has only one vertex with neighbors in $V \setminus X$ then the other one is a pendant or an isolated vertex. [67] By induction it follows that G is distance - hereditary.

This proves the theorem. □

[66] A suitable routing tree is easily constructed via an elimination by pendant vertices and elements of twins.

[67] Clearly the class of distance hereditary - graphs is (also) closed under adding isolated vertices.

4.6.2 Intermezzo: Perfect graphs

The classes of chordal graphs, bipartite graphs and distance hereditary - graphs share an interesting property — namely — they are perfect.

Definition 4.67. A graph G is perfect if every induced subgraph H satisfies
$$\chi(H) \quad = \quad \omega(H).$$

There are two important things to say about perfect graphs. They are called 'the perfect graph theorem' and 'the strong perfect graph theorem.' We present them without proof.

Theorem 4.68. *A graph is perfect if and only if its complement is perfect.*

Theorem 4.69. *A graph is perfect if and only if it does not contain an odd hole or an odd antihole as an induced subgraph.* [68]

[68] An odd hole in a graph is an induced cycle of odd length at least 5. An odd antihole in a graph is an odd hole in the complement.

Remark 4.70. A few other things: Claude Berge introduced perfect graphs after listening to a lecture by Claude Shannon. [69] In 1963 he proposed two conjectures that are now the two theorems above. [70]

[69] A lecture about the Shannon capacity of a graph. See: C. Shannon, *The zero-error capacity of a noisy channel*, IRE Trans. Inform. Theory (1956), 8–19.

[70] A graph is Berge if it does not contain an odd hole or odd antihole.

Important classes of perfect graphs are bipartite graphs, line graphs of bipartite graphs, chordal graphs and comparability graphs (and of course the classes of complements of these graphs).

One other characterization states that a graph is perfect if every induced subgraph H satisfies [71]

$$\alpha(H) \cdot \omega(H) \quad \geqslant \quad |V(H)|.$$

[71] Notice that this is not true for odd cycles of length more than 3.

Lovász proved the perfect graph theorem in 1972. Chudnovsky, Cornuéjols, Liu, Seymour and Vušković showed (in 2008) that there is a polynomial - time algorithm to check if a graph is perfect.

Grötschel, Lovász, and Schrijver showed (in 1988) that $\alpha(G)$, $\omega(G)$, and $\chi(G)$ are computable in polynomial time on perfect graphs.

4.6.3 χ - Boundedness

As we already mentioned problems that can be formulated in monadic second-order logic (without using quantification over subsets of edges) can be solved in $O(n^3)$ time on graphs of bounded rankwidth. When a routing tree is a part of the input then this reduces to $O(n)$ time.

Exercise 4.56

Let $k \in \mathbb{N}$. Design a monadic second-order formula that checks if a graph G can be (properly) colored with at most k colors — that is — the formula expresses $\chi(G) \leqslant k$.

Hint: A graph is k-colorable if there exists a partition of $V(G)$ into at most k classes that are all independent sets in G.

— The idea is — to prove that χ is computable in $O(n^3)$ time for graphs of bounded rankwidth and bounded clique number by providing an upper bound

$$\chi \quad \leqslant \quad f(\omega) \tag{4.12}$$

for some function $f : \mathbb{N} \to \mathbb{N}$ such that (4.12) holds true for all graphs of rankwidth $\leqslant k$.

Definition 4.71. A class of graphs $\mathcal{G}$ is χ-bounded if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\chi(G) \quad \leqslant \quad f(\omega(G))$$

for all $G \in \mathcal{G}$.

Many classes of graphs are χ - bounded; for example intersection graphs of axis - parallel boxes in d - space; graphs without odd holes; graphs without long holes; graphs that do no contain a subdivision of a tree. *Circle graphs* are polynomially χ - bounded.

Z. Dvořák and D. Král', *Classes of graphs with small rank decompositions are χ-bounded*, Manuscript on arXiv: 1107.2161, 2011.

In this section we show that for $k \in \mathbb{N}$ the class of graphs of rankwidth $\leqslant k$ is χ-bounded.

Exercise 4.57

Show that the class of distance hereditary - graphs is χ-bounded.
Hint: Design an algorithm that colors a distance hereditary - graph with ω colors.

We show that there is a partition of V that splits up all maximum cliques.

Lemma 4.72. *Let G be a connected graph with at least two vertices and of rankwidth $\leqslant k$. There exists a partition of $V(G)$ into at most $3 \cdot 2^k$ classes such that each class induces a subgraph with clique number less than $\omega(G)$.*

Proof. We may assume that the graph G has no false twins; ie the graph has no two vertices x and y with $N(x) = N(y)$. [72]

Let T be a routing tree. To facilitate the description of the partitioning - procedure we give T a root r which is a leaf and not a vertex of G. [73]

The tree T is ternary so we can label the lines of T with labels from $\{1, 2, 3\}$ such that each internal point of T is incident with an edge of each label. [74]

For $\nu \in V(T)$ let V_ν denote the set of vertices of G that are leaves of the subtree T_ν rooted at ν. Since T has rankwidth k we can fix a partition

$$V_\nu \quad = \quad \{\, V_\nu^0, \quad V_\nu^1, \quad \cdots, \quad V_\nu^d \,\}$$

such that

(a) the vertices of V_ν^0 have no neighbors outside V_ν

(b) all vertices of V_ν^j for $j > 0$ have the same neighbors outside V_ν

(c) $d < 2^k$.

Define a coloring ϕ of the vertices of G as follows. [75] To color $x \in V(G)$ find the point ℓ on the path from x to r in T that is furthest from r such that $N[x]$ is contained in the set of leaves of T_ℓ. — In other words V_ℓ is the minimal element of the carving that contains $N[x]$. By definition $x \in V_\ell^0$. Assume that x is a leaf of T_p for a child p of ℓ. Let $\alpha \in \{1, 2, 3\}$ be the label of the edge $\{\ell, p\} \in E(T)$.

In the partition of V_p find β such that $x \in V_p^\beta$. Notice that by the choice of ℓ [76]

[72] Otherwise we can remove an element x of a twin $\{x, y\}$; compute the partition for the graph $G - x$; and put x in the same class as y. In a similar manner we could reduce the graph so that it contains no pendant vertices either.

[73] That is, we let T be a routing tree for a graph $G + r$ in which $r \notin V(G)$ is added as an isolated vertex (a 'root') to the graph G.

[74] That is so because a tree is bipartite and so its line-graphs is perfect.

[75] The coloring is not necessarily proper. The color classes of ϕ form the partition that is claimed in the lemma.

[76] The node p is further from r than ℓ; so V_p does not contain $N(x)$ — that is — x has a neighbor in $V \setminus V_p$.

$$0 \quad < \quad \beta \quad < \quad 2^k.$$

Color the vertex x with a pair:

$$\phi'(x) \quad = \quad (\alpha, \beta).$$

Map the colors that are used by ϕ' to a set of colors that is not used by ϕ to color $V_\ell^1, V_\ell^2, \cdots$. (Each of these classes is monochromatic.) This defines ϕ.

This procedure describes a coloring of $V(G)$ that uses less than $3 \cdot 2^k$ colors. This coloring is not necessarily proper — however — it colors no maximum clique monochromatic.

TO SEE THAT let K be a clique in G of size $\omega(G)$ and assume that ϕ colors K monochromatic. Then there exists a vertex $z \in V(T)$ such that

$$K \quad \subseteq \quad V_z^j$$

for some $j > 0$. [77] This implies that the vertices of K have a common neighbor in $V \setminus V_z$. This contradicts that K has size $\omega(G)$.

[77] EXERCISE !

This proves the lemma. □

THE REST IS A WALK IN THE PARK.

Theorem 4.73. *Let* $k \in \mathbb{N}$. *The class of graphs of rankwidth* $\leqslant k$ *is* χ*-bounded.*

Proof. Define a function $f : \mathbb{N} \to \mathbb{N}$

$$\text{for } s \in \mathbb{N} : \quad f(s) \quad = \quad 2^{k \cdot s} \cdot 3^{s-1}$$

Let G be a graph of rankwidth $\leqslant k$. We show that $\chi \leqslant f(\omega)$ by induction on ω.

When $\omega = 1$ then $\chi = 1 \leqslant f(1)$.

When $\omega > 1$ then by Lemma 4.72 there is a coloring ϕ of G that uses at most $3 \cdot 2^k$ colors and has the property no maximum clique is monochromatic.

The subgraphs induced by the color classes of ϕ have rankwidth $\leqslant k$ — so — each color class of ϕ has a proper coloring with at most $f(\omega - 1)$ colors.

Color a vertex x by a pair:

(1) the index of the color class of ϕ that contains it

(2) the color assigned by the proper coloring.

The number of colors used by this proper coloring of G is at most

$$3 \cdot 2^k \cdot f(\omega - 1) = f(\omega)$$

This proves the theorem. □

H. Guo, T. Kloks, H. Wang and M. Xiao, *On conflict - free colorings of some classes of graphs*. Manuscript 2019.

Exercise 4.58

A conflict - free coloring of a graph is a function $c : V \to \mathbb{N}$ which satisfies the following condition

$$\forall_{x \in V} \quad \exists_{y \in N[x]} \quad \forall_{z \in N[x] \setminus \{y\}} \quad c(y) \quad \neq \quad c(z). \tag{4.13}$$

Show that there is an $O(n^3)$ algorithm to compute the conflict-free chromatic number $\chi_{CF}(G)$ for graphs of bounded rankwidth and bounded clique number.

Hint: Clearly $\chi_{CF}(G) \leqslant \chi(G)$. A class of graphs of rankwidth $\leqslant k$ is χ-bounded. This solves the problem since (4.13) is a formula in monadic second-order logic.
The real question is to find a good upper bound for

$$\max\{\chi_{CF}(G) \mid \mathsf{rankwidth}(G) \leqslant k\}.$$

Remark 4.74. Gyárfás did a lot of pioneering work on χ-bounded classes of graphs. [78] One conjecture of his is that for any tree T the class of graphs that do not have T as an induced subgraph is χ-bounded. A weaker statement is true — namely — for every tree T the class of graphs that do not contain any subdivision of T as an induced subgraph is χ-bounded.

[78] A. Gyárfás, *Problems from the world surrounding perfect graphs*, Proceedings of the international conference on combinatorial analysis and its applications (Pokrzywna, 1985), Zastos. Mat. **19** (1987), pp. 413–441.
To pronounce Gyárfás, try to say something like "garfas."

By Erdős' result the class of triangle-free graphs is *not* χ-bounded. Scott and Seymour proved (in 2016) that, for all $\kappa \geqslant 0$, if G is a a graph with

$$\omega \leqslant \kappa \quad \text{and} \quad \chi > 2^{2^{\kappa+2}}$$

then G has an odd hole.

For more general information on χ-boundedness we direct the reader to the survey by Scott and Seymour. [79]

[79] A. Scott and P. Seymour, *A survey of χ-boundedness*. Manuscript on arXiv: 1812.07500, 2018.

4.6.4 Governed decompositions

Bonamy and Pilipczuk prove in 2020 that graphs of bounded diversity are polynomially χ - bounded. For this purpose they derive the lemma below.

To present this lemma we need some definitions.

Definition 4.75. A generalized decomposition of a graph G is a pair (T, η) where T is a rooted tree and η a map $\eta : V(G) \to V(T)$.

Let (T, η) be a generalized decomposition. For an edge $e = \{u, v\} \in E(G)$ let $\eta(e)$ be the least common ancestor of u and v in T. For nodes $x, y \in V(T)$ write $x \preceq y$ when x is an ancestor of y. For a node $x \in V(T)$ define the graph $G<x>$ as the graph with the following sets of vertices and edges.

$$\begin{aligned} V<x> &= \{\, y \in V(G) \mid x \preceq \eta(y) \,\} \\ E<x> &= \{\, e \in E(G) \mid \eta(e) = x \,\} \end{aligned}$$

Definition 4.76. Let $\mathcal{C}$ be a class of graphs. A generalized decomposition (T, η) of a graph is $\mathcal{C}$ - governed if the graph $G<x> \in \mathcal{C}$ for all $x \in V(T)$.

Let (T, η) be a generalized decomposition of a graph. The (generalized) diversity of a branch rooted at a node $x \in V(T)$ is the number of neighborhood - classes of $G<x>$ — that is — the number of equivalence classes when two vertices of $G<x>$ are equivalent if they have the same neighbors in $V(G) \setminus V<x>$.

Let (T, η) be a generalized decomposition of diversity k. A tagging is a set of functions $\lambda_x : V<x> \to [k]$ $(x \in V(T))$ such that u and v in $V<x>$ have the same neighbors in $V(G) \setminus V<x>$ when $\lambda_x(u) = \lambda_x(v)$.

Let $e = \{x, y\} \in E(T)$ and let x be the parent of y. Let $\rho(e) : [k] \to [k]$ be a function which assigns $\rho(e)(i) = j$ if

$$u \in V<y> \quad \text{and} \quad \lambda_y(u) = i \quad \Rightarrow \quad \lambda_x(u) = j.$$

The functions $\rho(e)$ $(e \in E(T))$ form a labeling of $E(T)$ with elements of $\mathcal{F} = [k]^{[k]}$.

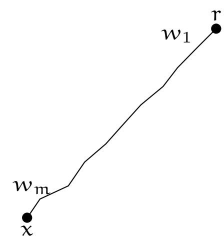


Figure 4.6: Edge - labels are elements of $\mathcal{F} = [k]^{[k]}$. A word w_x is the sequence of edge labels from the root to x. It relates to the function $w_1 \circ \cdots \circ w_m$.

Bonamy and Pilipczuk call the Kruskalian decompositions: splendid.

Let $\mathcal{F} = [k]^{[k]}$ — that is — $\mathcal{F}$ is the collection of functions $[k] \to [k]$. Denote the composition of two functions f and g in $\mathcal{F}$ as $f \circ g$.

Definition 4.77. A subset $A \subseteq \mathcal{F}$ is forward Ramsey if for all $e, f \in A$

$$e = e \circ f.$$

Definition 4.78. A decomposition (T, η) of a graph is Kruskalian if the set of edge - labels is forward Ramsey.

A decomposition (T, η) is shallow if every path to the root has at most two nodes.

Bonamy and Pilipczuk prove the following lemma. We take a look at the proof.

Lemma 4.79. *Let $\mathcal{C}$ be a hereditary class of graphs. There exists $p \in 2^{O(k \log k)}$ and a sequence of hereditary classes*

$$\mathcal{D}_0 \quad \subseteq \quad \mathcal{D}_1 \quad \cdots \quad \subseteq \quad \mathcal{D}_p$$

such that

1. $\mathcal{D}_0 = \mathcal{C}$
2. $\mathcal{D}_p$ *is the class of graphs that have a* $\mathcal{C}$ *- governed decomposition of generalized diversity* k
3. *for* $i \in [p]$ *all graphs in* $\mathcal{D}_i$ *have a* $\mathcal{D}_{i-1}$ *- governed decomposition of generalized diversity* k *which is Kruskalian or shallow.*

4.6.5 Forward Ramsey splits

T. Colcombet, A combinatorial theorem for trees. In Proceedings ICALP'07, Springer, LNCS 4596, pp. 901–912, 2007.

To prove Lemma 4.79 the authors make use of a lemma that they attribute to Thomas Colcombet. We present the lemma without proof.

Consider a tree T with a root and an edge - labeling $\rho : E(T) \to \mathcal{F}$ where $\mathcal{F} = [k]^{[k]}$. A 'word' is a sequence of edge labels on a path from the root to a node.

Let $\mathcal{F}^*$ the set of words of finite length with letters in $\mathcal{F}$. For a nonempty word $w \in \mathcal{F}^*$ let $\phi(w) \in \mathcal{F}$ denote the function that is the composition of the letters in w.

Let w be a nonempty word say $w = w_1 \cdots w_n$. Let $w[x, y]$ denote the word $w_{x+1} \cdots w_y$ (for $0 \leqslant x < y \leqslant n$). A split of height h of w is a map $\{0, \cdots, n\} \to [h]$. Two positions $0 \leqslant x < y \leqslant n$ are s - equivalent if

$$s(x) \quad = \quad s(y) \qquad \text{and} \qquad s(z) \quad \leqslant \quad s(x) \qquad \text{for all } x \leqslant z \leqslant y.$$

Definition 4.80. A split is forward Ramsey if for every two s - equivalent pairs $x < y$ and $x' < y'$

$$\phi(w[x,y]) \quad = \quad \phi(w[x,y]) \quad \circ \quad \phi(w[x',y']).$$

Lemma 4.81 (Colcombet). *There exists a map* $\mu : \mathcal{F}^* \to \left[\, k^k \,\right]$ *with the following property. Let* w *be a nonempty word and let* s_w *be the split* $s_w(x) = \mu(\, w[0,x]\,)$. *Then* s_w *is forward Ramsey.*

Notice that we define the 'forward Ramsey - concept' for

1. Sets $A \subseteq \mathcal{F}$
2. Splits of words in $\mathcal{F}^*$
3. Splits of $\mathcal{F}$ - labeled trees.

For the tree T define a split of height h as a function $V(T) \to [h]$. For a node x let w_x be sequence of edge - labels on the path from the root to x. The split is forward Ramsey if it is a forward Ramsey split of w_x for every $x \in V(T)$.

4.6.6 Factorization of trees

Let (T, r) be a rooted tree. A factorization is a partition of $V(T)$ in parts that induce subtrees. The parts are called factors. The root of a factor is the vertex closest to r.

Let $\mathcal{P}$ be a factorization. The quotient tree $T/\mathcal{P}$ is obtained from T by shrinking each factor to its root.

Assume that T has an edge - labeling $\rho : E(T) \to \mathcal{F}$. The quotient tree has an edge - labeling $\rho/\mathcal{P}$ defined as follows. Let $e = \{x, y\}$ be an edge of $T/\mathcal{P}$ and assume that x is the parent of y. Let $e_1 \dots e_m$ be the sequence of edges on the path in T from the root of x to the root of y. Define the edge label of the quotient tree as follows

$$\rho/\mathcal{P}\,(e) \quad = \quad \rho(e_1) \circ \cdots \circ \rho(e_m).$$

Bonamy and Pilipczuk prove the following lemma.

Lemma 4.82. *There exists a sequence* $(\mathcal{T}_i)_{i=0}^{3|\mathcal{F}|}$ *which satisfies the following.* [80]

- *each* $\mathcal{T}_i$ *is a class of edge - labeled trees with labels from* $\mathcal{F}$
- *for all* i $\mathcal{T}_i \subseteq \mathcal{T}_{i+1}$

[80] The sequence has $3 \cdot k^k + 1$ elements.

- *the class* $\mathcal{T}_0$ *has only one element which is a tree with one node. The class* $\mathcal{T}_{3|\mathcal{F}|}$ *is the class of all edge - labeled trees with labels from* $\mathcal{F}$
- *for all* $i > 0$ *every tree in* $\mathcal{T}_i$ *has a factorization with factors in* $\mathcal{T}_{i-1}$ *and with a quotient tree that is either Kruskalian or shallow.*

Proof. Let (T, ρ) be a rooted tree with an $\mathcal{F}$ - labeling on its edges. Define the level of (T, ρ) as follows.

$$\text{level}\,(T, \rho) \quad = \quad \min \quad \{\, h \quad | \quad (T, \rho) \quad \text{has a split of height } h\,\}$$

The level of (T, ρ) is at most k^k by Colcombet's lemma.

For $x \in V(T)$ define $t(x) = \mu(w[0, x])$.

Also — define a complexity of (T, ρ) as follows.

- if T has only one node then then the complexity of (T, ρ) is 0
- otherwise — when T has at least two nodes — the complexity is the smallest number k such that (T, ρ) has a factorization of which every factor has complexity $< k$ and for which the quotient tree is Kruskalian or shallow.

We claim that

$$\text{complexity}\,(T, \rho) \qquad \leqslant \qquad 3 \cdot \text{level}\,(T, \rho). \qquad (\dagger)$$

This claim implies the lemma: let $\mathcal{T}_i$ be the set of trees with edge - labels in $\mathcal{F}$ that have complexity i.

We prove $(\dagger)$ by induction on the level.

The following observation is our primary tool.

Exercise 4.59

Let $A, B \subseteq \mathcal{F}$ and assume that $A \cap B \neq \varnothing$. If A and B are forward Ramsey then so is $A \cup B$.

Recall the definition 4.77 of forward Ramsey - subsets of $\mathcal{F}$.

Assume that the level of (T, ρ) is 1 — that is — T has a split of height one: $t(x) = 1$ for all $x \in V(T)$.

Let's start with the base case.

Exercise 4.60

Let $y \in V(T)$ be a child of the root and let $A_y \subseteq \mathcal{F}$ be those elements of $\mathcal{F}$ that are assigned by ρ to edges of the subtree rooted at y. The set A_y is forward Ramsey.

HINT: Let z be a child of y. The elements of w_z are forward Ramsey. All words w_z contain the element $\rho(r, y)$. The previous exercise shows that $A_y \cup \{\rho(r, y)\}$ is forward Ramsey. This implies that A_y is forward Ramsey.

Exercise 4.61

For every child y of the root the complexity of the tree rooted at y is at most one.

HINT: Take a factorization into single nodes. All factors have complexity zero and the quotient tree is just the tree T_y. By the previous exercise it is Kruskalian. This shows that the complexity of (T, ρ) is at most two because we can take a factorization with the root as one factor and each subtree rooted at a child y of r as a factor. Then the quotient is shallow.

This proves (†) for the base case; when the level of T is one.

That base case was easy enough; let's get on with the induction step; I bet it goes 'in the same way.' (After all; we only have a short list of ingredients to cook the proof.)

Induction step: Assume that the level of (T, ρ) is $\ell > 1$ and let t be a split of T of height ℓ. Define $X \subseteq V(T)$ as follows

$$X \quad = \quad \{\, x \in V(T) \mid t(x) \quad = \quad \ell \,\}.$$

Define a factorization $\mathcal{P}$ of T as follows. Two nodes x and y are in the same part if either

1. neither x nor y has an ancestor in X (which implies $x, y \notin X$)
2. x and y have the same least ancestor in X.

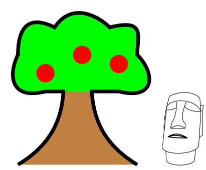


Figure 4.7: An artist's impression of a factorization.

For $x \in X$ let $\mathcal{P}_x$ be the factorization of the subtree rooted at x.

Let $y \in X$ have an ancestor in $X \setminus y$. Then $(T_y/\mathcal{P}_y, \rho_y/\mathcal{P}_y)$ is Kruskalian. The argument to prove this is similar to the one asked for in Exercise 4.60: Let $x \in X$ be the least ancestor of y not equal to y and let $z \in X$ be a descendant of y in X. Let Q_{xz} be the path in $T/\mathcal{P}$ from x to z; then $\{x, y\}$ is the first edge of Q_{xz}. The set B_z

of elements assigned by $\rho/\mathcal{P}$ to edges of Q_{xz} is forward Ramsey. It follows that the union of $\cup B_z$ for $z \in X$ that are descendants of y is forward Ramsey.

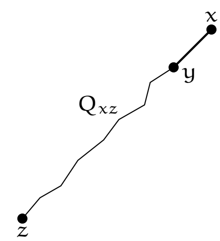


Figure 4.8: Illustration to show that $(T_y/\mathcal{P}_y, \rho_y/\mathcal{P}_y)$ is Kruskalian.

CLAIM: Let $y \in X$ and assume that y has an ancestor in $X \setminus y$. Then the complexity of (T_y, ρ_y) is at most $3\ell - 1$. By the previous observation $(T_y/\mathcal{P}_y, \rho_y/\mathcal{P}_y)$ is Kruskalian. Let F be a factor of $\mathcal{P}_y$. We show that F has complexity at most $3\ell - 2$. The only node of F that is at height ℓ is the root of F. Take a factorization with the root as one factor and with every branch of the root as one factor. Each branch has level $\leqslant \ell - 1$ — and so — (by the induction hypothesis) its complexity is at most $3\ell - 3$. The root is a factor with one node and so it has complexity zero. The factorization is shallow — and so — F has complexity at most $3\ell - 1$.

Let R be the subtree of T induced by the nodes that have at most one ancestor in X.

R Is the maximal subtree of T in which no path to the root has two elements of X.

CLAIM: The complexity of R is at most $3 \cdot \ell - 1$.

To see that notice that for every $x \in R \cap X$ the subtree R_x has complexity at most $3\ell - 2$ (because the only node of R_x of height ℓ is the root). [81]

[81] So — if the root of T is in X then we are done.

Let R' be the subtree of nodes that have no ancestor in X. Then R' has a forward Ramsey split of height $\ell - 1$ and so its complexity is at most $3\ell - 3$.

This proves the claim because $\{R', R_x \,|\, x \in R \cap X\}$ is a shallow factorization of R with all factors having complexity at most $3\ell - 2$.

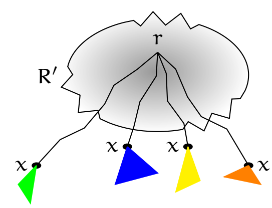


Figure 4.9: The figure illustrates the factorization of R into R' and subtrees R_x.

IT'S TIME TO FINISH THE PROOF OF (†). Let $\mathcal{Q}$ be the partition of $V(T)$ with factors R and T_y for the nodes $y \in X$ that have exactly one ancestor $\neq y$ in X. Then all factors have complexity $\leqslant 3 \cdot \ell - 1$. — Furthermore— the quotient is shallow — and so — the complexity of T is at most $3 \cdot \ell$. □

The lemma of Bonamy and Pilipczuk follows from Lemma 4.82.

Lemma 4.83. *Let $\mathcal{C}$ be a hereditary class of graphs and let $p = 3 \cdot k^k$. There exists a sequence of hereditary classes*

$$\mathcal{D}_0 \quad \subseteq \quad \mathcal{D}_1 \quad \cdots \quad \subseteq \quad \mathcal{D}_p$$

such that

1. $\mathcal{D}_0 = \mathcal{C}$
2. $\mathcal{D}_p$ *is the class of graphs that have a* $\mathcal{C}$ *- governed decomposition of generalized diversity* k
3. *for* $i \in [p]$ *all graphs in* $\mathcal{D}_i$ *have a* $\mathcal{D}_{i-1}$ *- governed decomposition of generalized diversity* k *which is Kruskalian or shallow.*

Proof. By Lemma 4.82 there exists a sequence $(\mathcal{T}_i)_1^{3|\mathcal{F}|}$ of trees with edge - labels in $\mathcal{F}$ such that trees in $\mathcal{T}_i$ have a factorization that has all factors in $\mathcal{T}_{i-1}$ and a quotient which is Kruskalian or shallow.

Let $\mathcal{D}_i$ be the class of graphs that have a $\mathcal{C}$ - governed decomposition of diversity k which is a tree in $\mathcal{T}_i$.

WE LEAVE IT TO THE READER to check that the classes $\mathcal{D}_i$ satisfy the required properties. □

EXERCISE: Show that the class of circle graphs is closed under vertex - minors.

Remark 4.84. Geelen et al shows that for any circle graph H the graphs that do not have H as a vertex - minor have bounded rankwidth. — So — these graphs are polynomially χ - bounded.

4.6.7 Kruskalian decompositions

WE END THIS CHAPTER with some remarks about the Kruskalian decompositions. These observations show that a class of graphs of bounded rankwidth is polynomially χ - bounded.

Let $\mathcal{C}$ be a hereditary class of graphs and denote the closure of $\mathcal{C}$ under modular substitution by $\mathcal{C}^*$. — For example — when $\mathcal{C}$

be the class of graphs that have at most two vertices then $\mathcal{C}^*$ is the class of cographs.

Notice that $\mathcal{C}^*$ is the class of graphs that have a $\mathcal{C}$ - governed decomposition of diversity one.

When $\mathcal{C}$ is polynomially χ - bounded then so is $\mathcal{C}^*$.

The fact that T is Kruskalian implies that there is a partition of $V(G)$ into at most k parts such that for all parts the decomposition has an edge - labeling which a constant function.

For each part G^i partition the levels of T in odd and even. This defines two spanning subgraphs of G^i — say G^i_0 and G^i_1 where G^i_0 has the edges of G^i defined by the nodes in even levels of T and G^i_1 has the edges of G^i defined by nodes in odd levels of T.

Since the labeling is a constant function G^i_0 and G^i_1 are both in $\mathcal{C}^*$.

4.6.8 Exercise

Exercise 4.62

Show that the class of AT - free graphs is linearly χ - bounded.

Permutation graphs and interval graphs are AT - free. These classes are perfect; so $\chi = \omega$. These classes are complements of comparability graphs. That is ; they are intersection graphs of continuous functions $f : [0, 1] \to \mathbb{R}$. (This implies that cocomparability graphs are AT - free.) AT - free graphs are not perfect as C_5 is AT - free.

HINT: Every connected AT - free graph has a dominating pair — that is a pair $\{a, b\}$ of vertices with the property that every path that runs between a and b is a dominating set in the graph.

Remark 4.85. Graphs of bounded linear rankwidth are linearly χ - bounded.

J. Nešetřil, P. Ossona de Mendez, R. Rabinovich and S. Siebertz, Linear rankwidth meets stability. Manuscript on arXiv: 1911.07748, 2019.

4.7 Clustered coloring

Hadwiger conjectures that graphs without K_t - minor can be colored with $t-1$ colors. This has been proved for $t \leqslant 6$ (and it is open for $t \geqslant 7$).

The conjecture dates back to 1943.

EXERCISE: What are the graphs that do not have K_3 as a minor? Are they colorable with two colors? K_4 poses similar questions that you should be able to answer. The case $t = 5$ is equivalent to the 4-color theorem. (That is so because by Wagner's theorem every 4 - connected graph is planar if and only if it has no K_5 - minor.)

Definition 4.86. Let G be a graph. A cluster coloring of G with k colors and cluster number c is a map $f : (V) \to [k]$ such that for each $i \in [k]$ the graph induced by the vertices of color i has no component with more than c vertices.

Definition 4.87. Let $\mathcal{G}$ be a class of graphs. The cluster chromatic number of $\mathcal{G}$ is the least number k for which there exists $c \in \mathbb{N}$ such that all graphs in $\mathcal{G}$ can be cluster colored with k colors and cluster number c.

In 2018 Jan van den Heuvel and David Wood proved the theorem below. In this chapter we take a look at the proof.

Theorem 4.88. *Let* $t \geqslant 4$. *Every* K_t *minor free - graph has a cluster coloring that uses* $(2t-2)$ *colors and has cluster number* $\lceil (t-2)/2 \rceil$.

4.7.1 Bandwidth and BFS - trees with few leaves

To prove Theorem 4.88 we start with some easy exercises. (The rest turns out to be easy as well.)

Definition 4.89. A graph has bandwidth k if its vertices can be ordered $v_1 \cdots v_n$ such that

$$\{v_i, v_j\} \in E \quad \Rightarrow \quad |i - j| \leqslant k.$$

Clearly when a graph has bandwidth k then it is a subgraph of an interval graph with clique number $k+1$ — that is — the pathwidth of a graph is at most its bandwidth.

Let G be a graph and let T be a BFS - tree of G. The tree partitions the vertices of G in layers [82] — say $V_0 \cdots V_\ell$ — where V_i is the set of vertices that are in G at distance i from the root. For a vertex $x \in V_i$ its <u>parent</u> is its neighbor in T in layer V_{i-1}.

[82] the BFS - levels

The BFS - tree <u>orders each layer</u> V_i such that

1. the parent of a vertex $x \in V_i$ is the first neighbor of x in V_{i-1}
2. if $\{x, y\} \in E(G)$ and $x \in V_i$ and $y \in V_{i-1}$ then there does not exist $\{a, b\} \in E$ with a before x in V_i and b after y in V_{i-1}.

Exercise 4.63

Let $k \in \mathbb{N}$ and let G be a connected graph that has a BFS - tree with at most k leaves. Show that the bandwidth of G is at most k.

Hint: Notice that each layer has at most k vertices. Take a linear order

$$V_0 \qquad V_1 \qquad \cdots \qquad V_\ell.$$

Let G be a graph and let T be a BFS - tree of G. A <u>BFS - subtree</u> of T is a subtree of T that contains the root of T. (A BFS - subtree is a BFS - tree of the graph induced in G by its vertices.)

Exercise 4.64

Let G be a connected graph and let T be a BFS - tree of G. Let S be a BFS - subtree of T with k leaves. Every vertex of G has at most $2k$ neighbors in $V(S)$.

Hint: Let P be a path in S that runs from the root to a leaf. It is sufficient to show that every vertex x in G has at most 2 neighbors in P. That is clearly so when $x \in P$ because P is a shortest path. Let $x \notin P$ and assume that x is adjacent to three vertices in P — say — a, b and c. Let $a \in V_{i-1}$, $b \in V_i$ and $c \in V_{i+1}$. Let y be the parent of x. Then — by the first rule — y appears before a and (by the second rule) x appears before b in V_i. But then (by the first rule) c is adjacent to x instead of b in T.

4.7.2 Connected partitions

In their paper Van den Heuvel and Wood prove the following theorem which implies Theorem 4.88.

Let G be a graph. A connected partition $\{H_1, \cdots, H_\ell\}$ is an ordered partition of $V(G)$ such that each part H_i induces a connected subgraph of G. Two parts are connected if there is an edge in G with an endpoint in both.

Theorem 4.90. *Let* $t \geqslant 4$ *and let* G *be* K_t *- minor free. Then* G *has a connected partition* $\{H_1, \cdots, H_\ell\}$ *which satisfies for all* i*:*

- H_i *is adjacent to at most* $t-2$ *parts of* $H_1 \cdots H_{i-1}$
- *every vertex of the induced subgraph* H_i *has degree at most* $t-2$
- *the induced subgraph* H_i *is 2 - colorable with cluster* $\lceil (t-2)/2 \rceil$.

In an early paper Van den Heuvel et al. show that each part H_i has a BFS - tree with at most $t-3$ leaves. — So — a 2 - coloring of the BFS - layers yields a 2 - coloring of the part with cluster number $t-3$.

Exercise 4.65

Show that Theorem 4.90 implies Theorem 4.88.

HINT: Color the parts H_i such that adjacent parts receive different colors. — Obviously — $t-1$ colors are sufficient. Color the vertices of the graph G with a pair of colors: one element of the pair is the color of the part that contains the vertex and the other element of the pair is the color that the vertex receives by a 2 - coloring of the part with cluster number $\lceil (t-2)/2 \rceil$.

The Steiner tree problem asks for a tree in G (of minimum weight) which spans all vertices of a set of 'terminals.' This problem is fixed - parameter tractable (with parameter the number of terminals).

A 'minimal' induced connected subgraph H as specified in Lemma 4.91 is computable by a greedy algorithm.

THEOREM 4.90 IS PROVED *via* the following two lemmas.

Lemma 4.91. *Let* G *be a connected graph and let* $A \subseteq V(G)$ *with* $|A| = k \geqslant 2$*. Let* H *be a connected induced subgraph of* G *with* $A \subseteq V(H)$ *and*

- *every vertex of* $H - A$ *is a cutvertex of* H
- *for every vertex* $x \in H - A$ *every component of* $H - x$ *has a vertex of* A.

Then the graph H *has the following properties.*

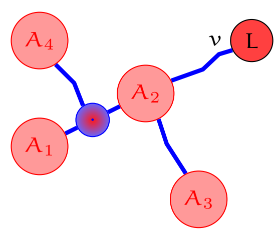
1. *every tree contained in* H *has at most* k *leaves*
2. *every vertex of* H *has degree at most* k
3. H *has bandwidth at most* $k-1$
4. H *has a 2 - coloring with cluster* $\lceil k/2 \rceil$
5. H *can be colored with colors red and blue such that*
   - *the red subgraph of* H *has at most* $k-2$ *vertices*
   - *the blue subgraph of* H *is a union of at most* $k-1$ *paths.*

Proof. All the leaves of a spanning tree T of H are in A (by the minimality of H). It follows that any tree in H has at most k leaves — and so — every vertex of H has degree at most k. By Exercise 4.63 H has bandwidth $k-1$ (since H has a BFS - tree with a root in A).

We show that H has a 2 - coloring with cluster $\lceil k/2 \rceil$. When $|V(H)| = k$ then this is obvious. We proceed by induction on $|V(H)|$.

When $|V(H)| > k$ then H has a cutvertex. Let $\nu \notin A$ be a cutvertex such that some component L of $H - \nu$ has the least number of elements in A. — Clearly — $V(L) \subseteq A$ and $|V(L)| \leqslant \frac{k}{2}$.

Every vertex of $H - A$ is a cutvertex of H.



The figure illustrates the induced subgraph H.

Let $H' = H - L$ and let

$$A' = (A \setminus V(L)) \cup \{\nu\}.$$

Then H' is a minimal connected subgraph of G that spans A' and $|V(H')| < |V(H)|$ and $|A'| \leqslant k$. By induction H' has a 2 - coloring with cluster number $\lceil k/2 \rceil$. To color H color the vertices of L by the color that is not used by ν.

We show that H has the required red / blue - coloring.

We use induction on k. When $k = 2$ then H is a path between two vertices of A. In that case color all vertices of H blue.

Assume $k \geqslant 3$ and let $x \in A$ be such that $H - x$ is connected. Let H' be a minimal connected subgraph of $H - x$ that contains $A \setminus x$. By induction H' can be colored with $\leqslant k-3$ red vertices and $\leqslant k-2$ blue paths. When $x \in V(H')$ then we are done.

Otherwise when $x \notin V(H')$ let $P = [x \cdots u, v, w]$ be a shortest path $x \rightsquigarrow A \setminus x$ in H — that is — v is the only vertex of $P \setminus w$ that has neighbors in $V(H')$. Color v red and the path $P \setminus \{v, w\}$ blue. This colors all vertices of H with a color red or blue. There are at most $k-2$ red vertices and the blue vertices form a union of at most $k-1$ paths. — Furthermore — $V(H) \subseteq V(P) \cup V(H')$ (by the minimality of H).

This proves the lemma. □

To compute H as mentioned in Lemma 4.92 construct a BFS - tree rooted at a vertex of A. Extract a minimal subtree that spans all vertices of A. Repeatedly remove vertices that are not in A and that are not cutvertices.

Lemma 4.92. *Let G be a connected graph, let $A \subseteq V(G)$ and let $|A| = k \geqslant 2$. There exists an induced subgraph H which satisfies all the items mentioned in Lemma 4.91 and which — furthermore — has the property that every vertex in G has at most $2k-2$ neighbors in $V(H)$.*

Proof. Let T be a BFS - tree with a root in A. Let S be the minimal subtree of T that contains all elements of A. Let H be a minimal induced subgraph of $G[V(S)]$ — as mentioned in Lemma 4.91 — which spans A. Then $V(H) \subseteq V(S)$.

The tree S has at most $k-1$ leaves. By Exercise 4.64 every vertex of G has at most $2(k-1)$ neighbors in $V(S)$.

The claims follow from Lemma 4.91.

This proves the lemma. □

4.7.3 A decomposition of K_t minor free graphs

In this section we prove Theorem 4.90 (which implies Theorem 4.88).

Definition 4.93. Let $\{H_1, \cdots, H_\ell\}$ be a connected partition of a graph G. It has width k if for all t every component of $G - \bigcup_{t=1}^{i} H_t$ is adjacent to at most k of the graphs $H_1, \cdots, H_i$.

The following theorem implies Theorem 4.90.

Theorem 4.94. *Let $t \geqslant 4$ and let G be a graph without K_t as a minor. There is a connected partition $\{H_1, \cdots, H_\ell\}$ of width $t-2$ such that each H_i satisfies the following conditions.*

(a) H_i *is a graph of which every vertex has degree at most* $t-2$

(b) H_i *has bandwidth* $t-3$

(c) H_i *has a 2 - coloring with cluster* $\lceil (t-2)/2 \rceil$

(d) H_i *can be colored with colors red and blue such that*

- *there are at most* $t-4$ *red vertices*
- *the blue vertices induce a union of at most* $t-3$ *paths.*

— *Furthermore* — *let* C *be a component of* $G - \bigcup_{t=1}^{i} H_t$. *Then* C *satisfies the following.*

(i) *at most* $t-2$ *of the graphs* $H_1, \cdots, H_i$ *are adjacent to* C *and those are pairwise adjacent*

(ii) *every vertex of* C *has at most* $2t-6$ *neighbors in each of the sets* $V(H_1), \cdots, V(H_i)$.

Proof. We may assume that G is connected. Construct the H_i one by one as follows. Choose an arbitrary vertex x of G and let $V(H_1) = \{x\}$. Then H_1 and every component of $G - H_1$ satisfy all the items.

Assume there is a component C in $G - \bigcup_{t=1}^{i} H_t$. Let $Q_1, \cdots, Q_k$ be the elements of $\{H_1, \cdots, H_i\}$ that are adjacent to C. We may assume that $k \leqslant t-2$ and that the Q_i's are pairwise adjacent.

Define H_{i+1} as follows. For $j \in [k]$ let $v_j \in C$ be a vertex with a neighbor in Q_j. When $k = 1$ then let $V(H_{i+1}) = \{v_1\}$. When $k \geqslant 2$ let H_{i+1} be the graph as produced in Lemma 4.92 which contains $\{v_1, \cdots, v_k\}$.

Let C' be a component of $G - \bigcup_{t=1}^{i+1} H_t$. Notice that either $C' \subset C$ or $C \cap C' = \varnothing$.

Assume $C' \cap C = \varnothing$. Then C' is a component of $G - \bigcup_{t=1}^{i} H_t$ and C' is not adjacent to H_{i+1}.

Assume $C' \subset C$. By induction and Lemma 4.92 every vertex of C' has at most $2t - 6$ neighbors in each of $H_1, \cdots, H_{i+1}$. The neighbors of C' are a subset of $Q_1, \cdots, Q_k, H_{i+1}$ and these are pairwise adjacent.

Suppose $k = t - 2$. Contract each of $Q_1, \cdots, Q_{t-2}, H_{i+1}$ and C' to a single vertex. This produces K_t which contradicts that G does not have K_t as a minor.

This proves the theorem. □

4.7.4 Further reading

In case you haven't read this; it's a "golden oldie."

S. Dreyfus and R. Wagner, The Steiner problem in graphs, *Networks* **1** (1972), pp. 195–207.

In 2018 Chung - Hung Liu and Sang-il Oum found that the cluster chromatic number of K_t - minor graph graphs is at most $3(t-1)$. [83]

[83] The cluster numbers of these colorings are very large.

C.-H. Liu and S.-i. Oum, Partitioning H - minor free graphs into three subgraphs with no large components, *Journal of Combinatorial Theory* **128** (2018), pp. 114–133.

In 2020 Chun - Hung Liu determines the cluster chromatic number up to a small additive constant for graphs without H - immersion.

Chun - Hung Liu, Immersions and clustered coloring. Manuscript on arXiv: 2007.00259, 2020.

Definition 4.95. A coloring of a graph with k colors and defect d is a coloring $f : V(G) \to [k]$ such that in every monochromatic component every vertex has degree at most d.

In their paper Van den Heuvel and Wood show that K_t - minor free graphs can be colored with $t-1$ colors and defect $t-2$. Edwards et al. show that the class of K_t - minor free graphs has defective chromatic number equal to $t-1$.

A class of graphs has defective chromatic number k if there exists $d \in \mathbb{N}$ such that every graph in the class has a coloring with k colors and defect d.

The paper of Edwards et al. shows — also — that the class of graphs without topological K_t minor is colorable with $t-1$ colors and defect $O(t^4)$.

K. Edwards, D. Y. Kang, J. Kim, S.-i. Oum and P. Seymour, A relative of Hadwiger's conjecture, *SIAM Journal on Discrete Mathematics* **29** (2015), pp. 2385 – 2388.

Let T be a tree with n edges. When n is large enough K_{2n+1} can be packed with $2n+1$ copies of T.

The edges of K_{2n+1} can be colored with $2n+1$ colors such that each color induces a copy of T. This is called Ringel's conjecture.

R. Montgomery, A. Prokovskiy and B. Sudakov, A proof of Ringel's conjecture. Manuscript on arXiv:2001.02665, 2020.

Let T be an oriented tree on n vertices. When n is large enough any tournament with $2n-1$ vertices contains a copy of T.

D. Kühn, R. Mycroft and D. Osthus, A proof of Sumner's universal tournament conjecture for large tournaments. Manuscript on arXiv: 1010.4430, 2010.

4.8 Well - Quasi Orders

We might as well put the definitions of well - quasi orders here.

Definition 4.96. A quasi - order is a set with binary relation which is reflexive and transitive.

Notice that a quasi - order is similar to a partial order — except that — quasi - orders are not necessarily anti - symmetric.

Definition 4.97. A quasi - order is a well - quasi order if — for any infinite sequence of elements $x_1, x_2, \cdots$ — there exist indices $i < j$ such that $x_i \preceq x_j$.

Exercise 4.66

Let T be a tree not necessarily finite with a root. Define the run - out of a point x in T as the supremum of the lengths of paths in T that have x as endpoint.

Prove Kőnig's infinity lemma:
If every point of T has finite out - degree and some point P has infinite run - out then there is an infinite path that starts in P.
Notice that the condition that the out - degree is finite is essential.

Hint: At any point P with infinite run - out there must be a successor Q which has also infinite run - out.

This proof is not constructive; only a proof by contradiction shows the existence of a successor. So it is not a proof in the sense of L.E.J. Brouwer.

4.8.1 Higman's Lemma

Let A be a finite alphabet of letters and consider an infinite sequence

$$w_1 \quad w_2 \quad w_3 \quad \cdots$$

of 'words' — that is — finite nonempty sequences of letters

$$w_i = w_i[1] \quad w_i[2] \quad \ldots \quad w_i[k]$$

where k is the length of the word w_i and $w_i(\ell) \in A$ for $\ell \in [k]$. [84] Higman's lemma asserts that there exist $i < j$ such that w_i is a subsequence of w_j. By that we mean that there is an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that

$$w_i[k] = w_j[f(k)]$$

for $k = 1$ up to the length of w_i.

[84] The notation A^* is used for the set of finite sequences over the alphabet A. So we have $w_i \in A^*$ for $i \in \mathbb{N}$. Perhaps we should have used 'k_i' instead of k for the length of the word w_i. Just note that words may have different lengths. We assume that the words are finite and nonempty so their lengths are in $\mathbb{N}$.

Below we present Nash-Williams' proof of the lemma.

Lemma 4.98. *The set A^* of finite nonempty sequences over a finite alphabet A is well-quasi ordered by the subsequence relation.*

Proof. Nash-Williams introduces the notion of a 'bad sequence.' A sequence (w_i) $(i \in \mathbb{N})$ is bad if for no pair $i < j$ w_i is a subsequence of w_j. Assume that there exists a bad sequence.

Construct a minimal bad sequence as follows. Let $x_1 \in A^*$ be a word of minimal length that starts some bad sequence. Let $x_2 \in A^*$ be a word of minimal length such that there exists a bad sequence that starts with x_1, x_2. For $i \in \mathbb{N}$ let $x_{i+1} \in A^*$ be of minimal length such that there is a bad sequence starting with $x_1 \cdots x_{i+1}$.

Choose an infinite subsequence of (x_i) — say (y_i) — such that all words y_i start with the same letter. From each y_i remove the first letter and call the new sequence (y'_i). Then (y'_i) is a bad sequence.

Let $y_1 = x_n$. The sequence

$$x_1 \quad \cdots \quad x_{n-1} \quad y'_1 \quad y'_2 \quad \cdots$$

is bad also. But this contradicts the choice of x_n since y'_1 is a shorter word and $x_1 \cdots x_{n-1}\, y'_1$ starts a bad sequence.

This proves the lemma. $\square$

4.8.2 Kruskal's Theorem

Kruskal's theorem extends Higman's lemma to sequences of labeled trees.

Consider an infinite sequence of rooted trees [85] — say

$$T_1 \quad T_2 \quad \cdots$$

Assume that the vertices of each tree have been labeled with elements from some finite set — say $[k]$ — for $k \in \mathbb{N}$. [86]

[85] A root in a tree is one vertex that is labeled as 'root.'

[86] To spell: 'labeling' and 'labelling' are both OK.

Kruskal's theorem says that there exist $i < j$ such that T_i can be 'embedded' into T_j. This is defined as follows.

Definition 4.99. Define for two labeled trees T_i and T_j

$$T_i \quad \preceq \quad T_j$$

if there is an injective map

$$f : V(T_i) \quad \rightarrow \quad V(T_j)$$

that satisfies

A function $A \rightarrow B$ is injective if every $b \in B$ is the image of at most one element in A.

1. f maps the root of T_i to the root of T_j
2. f preserves labels — that is —
$$\mathsf{label}(f(x)) = \mathsf{label}(x),$$
3. for any two vertices x and y of T_i their common ancestor is mapped to the common ancestor of $f(x)$ and $f(y)$.

We present Kruskal's theorem without proof.

Theorem 4.100. *The set of rooted trees with vertices labeled from a finite set is well - quasi ordered. — That is — let* $k \in \mathbb{N}$ *and let* $T_1, T_2, \cdots$ *be an infinite sequence of rooted trees with vertices labeled from* $[k]$*. Then there exist indices* $i < j$ *that satisfy*

$$T_i \preceq T_j.$$

The theorem remains true when the trees are labeled with elements of a well - quasi order. Nash–Williams gives an elegant proof of Kruskal's theorem.

4.8.3 Gap embeddings

Let $(Q, \ll)$ be a quasi - order and let $k \in \mathbb{N}$. Let $\mathcal{T}$ be the set of triples (T, f, a) where

- T is a rooted tree
- $f : V(T) \to Q$
- $a : E(T) \to [k]$.

Define a quasi - order $\preceq$ on $\mathcal{T}$ as follows. Let $t = (T, f, a)$ and $r = (R, g, b)$ be two elements of $\mathcal{T}$. Let $t \preceq r$ if there exists an injective map $\eta : V(T) \to V(R)$ which satisfies the following conditions.

1. η maps the root of T to the root of R and η maps the common ancestor of any two points in T to the common ancestor of their images in R
2. for an $x \in V(T)$: $g(\eta(x)) \ll f(x)$
3. for any edge $e = \{x, y\} \in E(T)$: $a(e) \leqslant b(e')$ for all $e' \in E(R)$ that lie on the path from $\eta(x)$ to $\eta(y)$.

The theorem remains valid when $\mathbb{N}$ is replaced with — say — $\mathbb{N} \cup \{i, 0, \infty\}$.

Theorem 4.101. *Let $(Q, \leqslant)$ be a well - quasi order and let $k \in \mathbb{N}$. The collection $\mathcal{T}$ of labeled trees — as defined above — is well - quasi -ordered by the relation $\preceq$.*

I. Tzameret, *Kruskal - Friedman gap embedding theorems over well - quasi - orderings*. Thesis, Tel Aviv University 2002.

Remark 4.102. Above we assume that the edge - labels are from a totally ordered set $[k]$. Tzameret shows that this can be relaxed — but — <u>not</u> 'all the way.'

4.9 Threshold graphs and threshold - width

THRESHOLD GRAPHS WERE GIVEN THEIR NAME BY CHVÁTAL AND HAMMER. Below we present one way to define this class of graphs.

Definition 4.103. A graph is a threshold graph if every induced subgraph has an isolated vertex or a universal vertex. [87]

[87] A vertex is isolated if its neighborhood is empty. A vertex is universal if it is adjacent to all other vertices. When a graph has only one vertex it is both; isolated and universal.

Exercise 4.67

A graph is a threshold graph if and only if its complement is that. Show that a graph is a threshold graph if and only if it has no induced P_4, C_4, or $2K_2$.

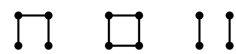


Figure 4.10: The figure shows P_4, C_4 and $2K_2$.

Exercise 4.68

A graph is a split graph if its vertices partition into a clique and an independent set. Show that a graph is a split graph if and only if it does not contain $2K_2$, C_4 or C_5 as an induced subgraph. Show that every threshold graph is a split graph.

The threshold dimension $\theta(G)$ of a graph $G = (V, E)$ is the minimum $k \in \mathbb{N}$ for which there are k threshold graphs $G_i = (V, E_i)$ with $\bigcup E_i = E$. There exists an $O(n^3)$ - algorithm to check if $\theta(G) \leqslant 2$. To check if $\theta(G) \leqslant 3$ is NP-complete.

Exercise 4.69

What is the threshold dimension of a trivially perfect graph? [88]

[88] Recall from Section 2.9.3 on page 80 that a graph is trivially perfect if it has no induced P_4 and no induced C_4. Since $\bar{C}_4 = 2K_2$ a graph is a threshold graph if and only if it and its complement are trivially perfect.

4.9.1 Threshold - width

A — SOMEWHAT — SIMILAR CONCEPT is that of the threshold - width of a graph.

Definition 4.104. A graph G has threshold - width $\leqslant k$ if it has k independent sets $N_1, \cdots, N_k$ such that there is a threshold graph H that has G as a spanning subgraph and every edge of H — which is not an edge of G — has both endpoints in N_i for some $i \in [k]$.

Denote the smallest k such that a graph G has threshold - width $\leqslant k$ as $\tau(G)$. In this section we show that $\tau \leqslant k$ is fixed parameter - tractable. [89]

[89] EVEN BETTER!

Theorem 4.105. *There exists a characterization of the graphs that satisfy $\tau \leqslant k$ by a finite collection of forbidden induced subgraphs.*

Proof. Observe that the class of graphs that satisfy $\tau \leqslant k$ is hereditary. — So — there exists a collection $\mathcal{F}$ of graphs F that satisfy $\tau(F) > k$ and for every vertex $x \in V(F)$ $\tau(F-x) \leqslant k$.

IT IS OUR JOB TO PROVE THAT $|\mathcal{F}|$ IS FINITE.

A graph is a threshold - graph if its vertices can be put in a linear order such that each vertex is either adjacent to all vertices that come after it or not adjacent to any vertex that comes after it.

Let G be a graph that has threshold - width $\leqslant k$. Let $N_1, \cdots, N_k$ be k independent sets in G that witness this. We identify the graph with a word with letters from a finite alphabet as follows.

Label each vertex x with a $(0, 1)$-vector $\ell(x)$ of length k with the i^{th} entry 1 if $x \in N_i$. Then G has a linear order of its vertices such that each vertex x is — either

(i) not adjacent to any vertex that comes after it

(ii) adjacent to exactly all vertices y that come after x for which $\ell(x) \cdot \ell(y) = 0$.

(Here $\ell(x) \cdot \ell(y)$ denotes the inner product of the two labels; it is not zero when x and y occupy a similar independent set N_i.)

Identify a graph with a sequence of labels in an elimination order as above. Notice that if a graph H has a sequence which is a subsequence of the sequence of the graph G then H is isomorphic to an induced subgraph of G. [90]

[90] Exercise !

We proceed by contradiction. Let

$$F_1 \quad F_2 \quad \cdots \tag{4.14}$$

be an infinite sequence of different elements of $\mathcal{F}$. Each graph F in this sequence has $\tau(F) > k$ — but — we can choose an arbitrary vertex r in it such that $\tau(F - r) \leqslant k$.

Denote a choice for a vertex in the graph F_i as r_i. Each $F_i - r_i$ identifies with a word via the labeling procedure described above with letters from a finite alphabet.

Extend the labels of the vertices of $F_i - r_i$ with one <u>additional</u> $(0, 1)$-label: 1 if the vertex is adjacent to r_i and 0 otherwise. Notice that if a graph $F_i - r_i$ has a sequence which is a subsequence of a graph $F_j - r_j$ then F_i is isomorphic to an induced subgraph of F_j. [91]

[91] Exercise !

Replace the sequence (4.14) by a sequence of words of finite length with letters from a finite alphabet. We can make use of Higman's Lemma (Lemma 4.98 on Page 214): the sequence must contain elements $i < j$ such that the word F_i is a subsequence of the word F_j — that is — F_i is isomorphic to an induced subgraph of F_j. This is a contradiction — so — $|\mathcal{F}| \in \mathbb{N}$.

This proves the theorem. $\square$

Lemma 4.106. *For any graph its rankwidth is at most* 2^{τ}.

Proof. Let G be a graph, let $N_1, \cdots, N_k$ be k independent sets in G, and let H be an embedding of G in a threshold graph with every $e \in E(H) \setminus E(G)$ contained in some N_i.

The graph H has rankwidth 1 (it is distance - hereditary). Let $\mathcal{C}$ be a carving of H with $\mathsf{rank}(X) \leqslant 1$ for every $X \in \mathcal{C}$. The rank of the cut matrix $(X, V \setminus X)$ of the adjacency matrix of G is at most 2^k. To see that observe that each independent set N_i is a 0-submatrix in the adjacency matrix of G. It follows that for $X \in \mathcal{C}$ there are at most 2^k different neighborhoods in $V \setminus X$. [92]

[92] This bound is sharp.

This proves the lemma. □

Corollary 4.107. *Problems that can be formulated in monadic second-order logic can be solved in* $O(n^3)$ *time for graphs of bounded threshold - width.*

Theorem 4.108. *Threshold -width is fixed - parameter tractable.*

Proof. The class of graphs with threshold - width $\leqslant k$ is characterized by a finite collection of forbidden induced subgraphs. This shows that the recognition can be formulated in monadic second-order logic.

The graphs of threshold - width $\leqslant k$ have rankwidth at most 2^k. By Courcelle's theorem there exists an $O(n^3)$ - algorithm to recognize graphs of threshold - width $\leqslant k$.

This proves the theorem. □

Remark 4.109. Theorem 4.108 can also be proved via the formulation of an elimination order — that is — a graph has threshold - width $\leqslant k$ if and only if every induced subgraph has an isolated vertex or a vertex x which is adjacent to all other vertices except those that are in one of the k independent sets that contain x. This property be formulated in monadic second order logic.

4.9.2 On the complexity of threshold - width

In this section we show that computing the threshold - width of a graph is NP-complete.

LET US LOOK AT AN 'EASIER' PROBLEM — FIRST. Let $\mathcal{K}$ be the class of all cliques ie all complete graphs. The <u>$\mathcal{K}$ - width</u> of a graph is the minimum number of independent sets $N_1, \cdots, N_k$ in the graph such that every nonedge of G has its endpoints in one of the N_i. [93]

[93] We could call this the 'clique - width' of the graph. Unfortunately, there is another concept closely related to rankwidth that carries that name. So we just call it $\mathcal{K}$ - width.

Lemma 4.110. *$\mathcal{K}$ - Width is NP-complete.*

Proof. The problem is equivalent to finding a cover of the edges of $\bar{G}$ with a minimal number of cliques. Kou, Stockmeyer, and Wong proved that this is NP-complete. [94] □

[94] L. Kou, L. Stockmeyer and C. Wong, *Covering edges by cliques with regard to keyword conflicts and intersection graphs*. Communications of the ACM **21** (1978), pp. 135–139.

Theorem 4.111. *Threshold - width is NP-complete.*

Proof. We reduce $\mathcal{K}$-width to threshold - width.

Let G be a graph for which we would like to compute the $\mathcal{K}$-width. Starting with G construct the graph G':

(a) add a clique C with n^2 vertices and make every vertex of C adjacent to every vertex of G

(b) add one more vertex ω and make it adjacent to all vertices of G.

CLEARLY if we add an edge between every nonadjacent pair in G then G' becomes a threshold graph. We claim that this is the best way to embed G' into a threshold graph.

Let x and y be a nonadjacent pair in G. When x and y are not adjacent in a threshold embedding of G' then ω is adjacent to all vertices of C in that embedding. That is so because a threshold graph has no C_4. HOWEVER to make ω adjacent to all vertices of C needs n^2 independent sets (since C is a clique).

This proves that the threshold - width of G' equals the $\mathcal{K}$ - width of G.

By Lemma 4.110 this proves the theorem. □

4.9.3 A fixed - parameter algorithm for threshold - width

We have shown that threshold - width is fixed - parameter tractable.

In this section we present an algorithm.

To get in the mood we start with some — easy — exercises.

Exercise 4.70

Show that a graph is a threshold graph if and only if for any pair of its vertices x and y

$$N(x) \subseteq N[y] \quad \text{or} \quad N(y) \subseteq N[x].$$

In other words a graph is a threshold graph if and only if its vertices can be put into a linear order

$$x_1 \quad \cdots \quad x_n$$

such that

$$1 \leqslant i < j \leqslant n \quad \Rightarrow \quad N(x_i) \subseteq N[x_j].$$

Exercise 4.71

Let G be a graph and let $\{N_1, \cdots, N_k\}$ be a 'witness' — ie — a collection of k independent sets in G. Design an algorithm to check if G can be embedded into a threshold graph H such that every edge of H which is not an edge of G has both endpoints in some N_i. Hint: Define a k-universal vertex as a vertex for which the sets N_i that contain it cover all its nonneighbors.

Exercise 4.72

Let G be a graph of threshold - width $\leqslant k$. Let $N_1, \cdots, N_k$ be k independent sets in G that witness this. The witness $\{N_1, \cdots, N_k\}$ is well-linked if every N_i is a maximal independent set in G. Prove that every graph of threshold - width $\leqslant k$ has a well-linked witness.

Exercise 4.73

Assume G has a well-linked witness $\{N_1, \cdots, N_k\}$ and a threshold embedding H. Label each vertex as in the proof of Theorem 4.105 with a vector $\ell(x)$ of length k. Assume two vertices x and y satisfy $N_H(x) \subseteq N_H[y]$. Then

$$N_G(x) \subseteq N_G[y] \quad \Leftrightarrow \quad \ell(x) \geqslant \ell(y).$$

Definition 4.112. Let $k \in \mathbb{N}$. A set M of vertices in a graph is called a k-probe module if either

1. $|M| \geqslant 3$ and every pair of vertices in M is a false twin (in the graph)
2. $|M| \geqslant k+3$ and every pair of vertices in M is a true twin.

Lemma 4.113. *Let G be a graph; let $k \in \mathbb{N}$; and let M be a k-probe module in G. Then for any $x \in M$*

$$\tau(G) \leqslant k \quad \Leftrightarrow \quad \tau(G-x) \leqslant k.$$

Proof. — Clearly — $\tau(G-x) \leqslant \tau(G)$ for any $x \in V(G)$. Assume $\tau(G-x) \leqslant k$ for some x in a k-probe module M. Let H be a threshold embedding of $G-x$. Since H is a split graph its vertices partition into a clique — say C — and an independent set.

First assume that M is an independent set - module. Then

$$|M \setminus x| \quad \geqslant \quad 2.$$

Let $y \in M \backslash x$. If y is in the independent set (that is, if $y \in V(H) \backslash C$) then we can let x be a false twin of y. This produces a threshold embedding of G of width $\leqslant k$. When $(M \setminus x) \subseteq C$ then let x be a true twin of an arbitrary element of $M \setminus x$. — Again — this produces a threshold embedding of G with width $\leqslant k$.

Assume that M is a clique module. Then it has at least $k+3$ elements. At least $k+1$ of those are in C. Choose $z \in M \cap C$ such that $N_H[z]$ is minimal. Assume that z has a neighbor u in H which is not a neighbor of z in G. Then u is a neighbor in H but not in G of every vertex of $M \cap C$. Since M is a clique in G the vertex u is contained in at least $k+1$ independent sets. This is a contradiction.

— So —

$$N_H[z] \quad = \quad N_G[z]$$

and we can let x be a twin of z.

This proves the lemma. $\square$

Definition 4.114. A vertex x is <u>maximal</u> if for all $y \in V$

$$N(y) \subseteq N(x) \quad \Rightarrow \quad N(y) = N(x) \quad \text{and}$$
$$N[y] \subseteq N[x] \quad \Rightarrow \quad N[y] = N[x].$$

Lemma 4.115. *Let G be a graph with threshold - width $\leqslant k$ and assume that G has no k-probe module. Then the number of maximal vertices in G is at most*

$$2^{k+1} + k.$$

Proof. Let H be a threshold embedding of G with a well-linked witness $\{N_1, \cdots, N_k\}$.

Partition the vertices of H into equivalence classes $M_0, M_1, \cdots$ of vertices with the same open - or closed neighborhood. (Each M_i is a clique or an independent set in H.) Order the classes such that for each $x_i \in M_i$ and $x_{i+1} \in M_{i+1}$:

$$N_H(x_{i+1}) \subseteq N_H[x_i].$$

(So when H is connected M_0 is its set of universal vertices.)

Partition each M_i into sets of vertices that have the same label. [95] These 'label-sets' are modules in G. Since there is no k-probe module each label-set has at most 2 vertices when it is independent and at most $k+2$ when it is a clique. It follows that for each i

[95] Two vertices have the same label if every N_i contains both of them or neither of them.

$$|M_i| \quad \leqslant \quad 2(2^k-1)+(k+2) \quad = \quad 2^{k+1}+k.$$

Notice that there are at most 2^k label-sets of maximal vertices. At most 2^k-1 are in independent sets N_i and they have at most 2 vertices. At most one is a clique and it has at most $k+2$ vertices.

This proves the lemma. □

Lemma 4.116. *Let G be a graph of threshold - width $\leqslant k$. Assume that G has no isolated vertices and no k-probe modules. There exists a set $Y \subseteq V$, $|Y| \leqslant 2^{2(k+1)}$, such that every threshold embedding that is a witness has its set of universal vertices $M_0 \subseteq Y$. The set Y can be computed in linear time.*

Proof. Since G has no isolated vertices H is connected. Let M_0 be its set of universal vertices.

To compute a set Y that contains M_0 start with $Y = \varnothing$. Repeatedly add the set of vertices to Y that are maximal in G and remove those from the graph.

After at most 2^k repetitions each label-set of M_0 is contained in Y. Each set of maximal elements has size at most $2^{k+1}+k$ which shows

$$|Y| \quad \leqslant \quad 2^k \cdot (2^{k+1}+k) \quad \leqslant \quad 2^{2(k+1)}$$

This proves the lemma. □

Exercise 4.74

Let U be a set of labeled vertices. As usual; each label is a $(0,1)$-vector of length k and for each entry i the vertices — say N_i — that have a 1 in that entry form an independent set.

The set U is a probe clique if the inner product of any two labels of elements in U is zero — that is — if the nonedges of U are exactly the pairs that share some N_i.

Call $U \subseteq V(G)$ probe universal if each $x \notin U$ can be given a label such that $U \cup x$ is a probe clique.

Let U be a probe universal set and let $x \notin U$ be such that the set

$$U' = U \cup N(x)$$

can be labeled as probe universal with the same number of nonempty label-sets as U. When there is such a vertex then choose x such that $N(x)$ is minimal. When G has an embedding with U as a universal set then G has an embedding with U' as a universal set.

Theorem 4.117. *Let $k \in \mathbb{N}$. There exists an $O(n^2)$ algorithm that recognizes graphs of threshold width $\leqslant k$.*

Proof. We may assume that G has no isolated vertices or k-probe modules.

By Lemma 4.116 there is a constant number of feasible universal sets.

Assume there exists a vertex x that can be labeled such that $N(x)$ extends the universal set in such a way that it does not increase the number of label-sets. By Exercise 4.74 the algorithm can safely extend the probe universal set with $N(x)$. Next the algorithm removes the vertex x and tries to find another greedy extension.

When there are no more greedy extension the algorithm computes the set Y as in Lemma 4.116. It then tries all subsets of Y as possible extensions of the probe universal set.

There can be at most 2^k steps in this algorithm that increase the number of label-sets in the probe universal set. The set Y of maximal elements can be computed in $O(n^2)$ time.

This proves the theorem. □

Exercise 4.75

Define a width - parameter for the class of distance-hereditary graphs as follows. A graph G has DH-width $\leqslant k$ if it has k independent sets $N_1, \cdots, N_k$ and an embedding H which is distance-hereditary such that every edge of H which is not an edge of G has both endpoints in some N_i.

Is DH-width fixed-parameter tractable?

HINT: Is there a monadic second-order formulation of $\mathsf{DH}(G) \leqslant k$?

4.10 Black and white - coloring

CLAUDE BERGE POSED THE PROBLEM to put b black queens and w white queens on a chess board so that no two queens of opposite colors hit each other. [96]

[96] We talk about Western chess played on a 8×8 board. A black and white queen hit each other if they are placed in the same row, column, or diagonal provided no other piece is placed between them.

Definition 4.118. Let G be a graph and let $b, w \in \mathbb{N}$. A black and white - coloring of G chooses b black vertices and w white vertices such that no black vertex is adjacent to any white vertex.

Exercise 4.76

Show that the black and white - coloring problem can be solved in linear time on graphs of bounded treewidth.

Exercise 4.77

Show that there is a polynomial-time algorithm to solve the black and white - coloring problem on cographs.

Hint: Define a boolean variable $\phi(G, b, w)$ with value true if the graph G has a black and white coloring with b black vertices and w white vertices.

Let G_1 and G_2 are two cographs and let G be the join or union of the two. Express $\phi(G, w, b)$ as a function of $\phi(G_1, w_1, b_1)$ and $\phi(G_2, w_2, b_2)$.

This leads to an (n^5) algorithm to check if a cograph has a black and white coloring (for any values b and w). Improve your algorithm to solve the black and white - coloring problem on cographs so that it runs in $O(n^3)$.

4.10.1 The complexity of black and white - coloring

In this section we show that the black and white - coloring problem is NP-complete even when restricted to the class of splitgraphs (see Exercise 4.68 for the definition of a splitgraph.) [97]

[97] A graph is a splitgraph if it is a clique or an independent set or else V partitions in $\{C, S\}$ where C induces a clique and S a stable set. A graph is a splitgraph if it has no induced C_4, C_5 or $2K_2$.

Theorem 4.119. *The black and white coloring - problem is* NP-*complete on splitgraphs.*

Proof. Splitgraphs are closed under complementations. The 'inverse black and white coloring - problem' is similar except that it is required that every black vertex is adjacent to every white vertex.

Let G be a graph. Construct a splitgraph $H = (S \cup C, E')$ as follows. The clique C of H is $V(G)$. The independent set S of H is $E(G)$. A vertex of C is adjacent to a vertex of S in H if the vertex is not an endpoint of the edge in G.

Let Ω be a maximum clique in G of size k and let

$$V' = V(G) \setminus \Omega.$$

The computation of the clique number in G remains NP-complete when n is even, $k = \frac{n}{2}$ and $n > 6$. [98] — Henceforth — we

[98] D. Johnson, *The* NP-*completeness column — an ongoing guide*, Journal of Algorithms **8** (1987), pp. 438–448.

assume that.

Notice that H has an inverse black and white coloring with

$$b = k \quad \text{and} \quad w = k + \binom{k}{2} = \binom{k+1}{2} \tag{4.15}$$

To see that — color all vertices of V' black. They are in H adjacent to all edges in S that have both ends in Ω and to all vertices of $C \setminus V'$.

For the converse assume that the splitgraph H has an inverse black and white coloring with b and w as in Equation (4.15). Since S is an independent set all colored vertices of S are the same color.

First assume that S contains no white vertices. Then C contains a set W of white vertices and vertices in $C \setminus W$ are black. However $w > 2k = n$ (since $n > 6$) — so — this is not possible. So the inverse black and white - coloring has white vertices in S.

Let S_w be the set of white vertices in S and let $V' \subseteq C$ be the set of k black vertices. If an edge of G is in S_w then it is not incident with a vertex of V'. All those edges are incident with vertices in $V(G) \setminus V'$. Since $|V \setminus V'| = k$ and $|S_w| = \binom{k}{2}$ the only possibility is that S_w is the set of edges of a k-clique $V \setminus V'$.

This proves the theorem. $\square$

4.11 k – Cographs

In this section we illustrate the importance of Kruskal's theorem.

Recall the definition of a cograph; Definition 2.78.[99]
By Theorem 2.79 (on Page 84) cographs can be encoded into cotrees.

Definition 4.120. Let G be a graph. A <u>cotree</u> for G is a pair (T, f) — where

1. T is a rooted binary tree [100]
2. $f : V(G) \to \mathsf{leaves}(T)$ is a bijection that identifies each vertex of G with one leaf of T

[99] A graph is a cograph if it has no induced P_4.

Figure 4.11: P_4

[100] A rooted tree is <u>binary</u> if either it has only one point (which is then both root and leaf) or the root has degree 2, all leaves have degree 1, and all other vertices have degree 3. Notice that there is no binary tree with two points.

3. each internal node of T is labeled as $\otimes$ or $\oplus$

4.

$$\{x, y\} \in E(G) \quad \Leftrightarrow$$
$$\text{the least common ancestor of } f(x) \text{ and } f(y) \text{ in } T \text{ is labeled } \otimes. \tag{4.16}$$

By Theorem 2.79 a graph is a cograph if and only if it has a cotree.

We parametrize the class of cographs as follows.[101]

[101] To spell, both 'parametrize' and 'parameterize' are OK.

Definition 4.121. Let $k \in \mathbb{N}$. A graph G is a k-cograph if it has a decomposition (T, f) such that

I. T is a rooted binary tree

II. each leaf of T is labeled with an element from $[k]$

III. f is a bijection $V(G) \rightarrow \mathsf{leaves}(T)$

IV. all internal nodes are labeled by a a symmetric binary relation on $[k]$ (eg represented by a symmetric Boolean $k \times k$-matrix)

V.

$$\{x, y\} \in E(G) \quad \Leftrightarrow$$
$$\text{the lowest common ancestor of } f(x) \text{ and } f(y) \text{ is labeled } \sigma,$$
$$\text{with } \sigma \text{ such that}$$
$$\sigma(\mathsf{label}(f(x)), \mathsf{label}(f(y))) = \text{true}. \tag{4.17}$$

— Notice that — ordinary cographs are 1-cographs.

Exercise 4.78

Show that — for each $k \in \mathbb{N}$ — the class of k–cographs is hereditary.
Hint: Let G be a k–cograph and let (T, f) be a decomposition tree for G. For an induced subgraph H consider the subtree of T that contains all vertices of H.

Exercise 4.79

Show that a class of k–cographs is closed under creating twins. A twin is a pair of vertices — say x and y — such that [102]

$$N[x] = N[y] \quad \text{if } x \text{ and } y \text{ are adjacent}$$
$$N(x) = N(y) \quad \text{otherwise.}$$

[102] When the pair is adjacent, the twin is called a true twin. When the pair is nonadjacent, it is called a false twin.

When G is a k–cograph — and H is obtained from G by creating a twin of some vertex in G — then H is also a k–cograph.

4.11.1 Recognition of k – Cographs

In this section we show that there exists a characterization of k–cographs by a finite set of forbidden induced subgraphs.

For $k \in \mathbb{N}$ denote the class of k-cographs as $\mathcal{C}(k)$.

Theorem 4.122. *There exists a finite set of graphs* S_k — *such that — a graph* $G \in \mathcal{C}(k)$ *if and only if* G *contains no element of* S_k *as an induced subgraph.*

Proof. The set S_k is the set of inclusion–minimal graphs that are not k–cographs. We show that S_k is finite.

Assume that S_k is not finite. Then we can choose an infinite sequence of pairwise different graphs in S_k — say

$$G_1, G_2, \cdots.$$

Since each graph G_i is inclusion–minimal — we have that $G_i - r_i$ is a k–cograph — for each vertex $r_i \in V(G_i)$. Pick one arbitrary vertex r_i in each G_i.

Consider a sequence of rooted binary trees

$$T_1, T_2, \cdots$$

where T_i is a k–cotree for the graph $G_i - r_i$.

Extend the labels of the vertices in each T_i as follows. In T_i, give a vertex $z \in V(G_i - r_i)$ an extra label

$$\mathsf{label}(z) = \begin{cases} 0 & \text{if } z \text{ is not adjacent to } r_i, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

— By Kruskal's theorem — the newly labeled sequence of trees $T_1, T_2, \cdots$ satisfies

$$\exists_i \quad \exists_j \quad i, j \in \mathbb{N} \quad \text{and} \quad i < j \quad \text{and} \quad T_i \preceq T_j .$$

But —owing to the new labeling — this implies that G_i is an induced subgraph of G_j — which is a contradiction.

This proves the theorem. □

Corollary 4.123. *For $k \in \mathbb{N}$, there exists a polynomial–time algorithm that recognizes k–cographs.* [103]

[103] But ...nobody knows what it is, as long as S_k is unknown.

This follows from the fact that S_k is finite. Namely we can test whether one element of S_k —say with t vertices — is an induced subgraph of a graph in $O\left(n^{t+2}\right)$ time. So the recognition takes time

$$O\left(|S_k| \cdot n^{t+2}\right) = O\left(n^{t+2}\right), \quad \text{where } t = \max\{|V(S)| \mid S \in S_k\}. \tag{4.18}$$

4.11.2 Recognition of k–Cographs — revisited

Actually we can do much better. If I gave you the definition, it would be easy for you to check that k–cographs have <u>rankwidth</u> at most k. Courcelle showed that

> every problem that can be formulated in MS_1 can be solved in $O\left(n^3\right)$ time for graphs of rankwidth at most k.

It is an easy exercise to show that — for any graph S — there is a monadic second–order formula expressing that S is an induced subgraph of a graph G.

Theorem 4.124. *The recognition of* k *– cographs is fixed parameter – tractable. There exists an* $O\left(n^3\right)$ *algorithm to tests if a graph is a* k *– cograph.*

Exercise 4.80

Prove that any graph is a k-cograph, for some $k \in \mathbb{N}$. Define the

$$\text{cograph-width}(G) = \min\{k \mid G \text{ is a } k\text{-cograph}\}.$$

Show that cograph – width is fixed parameter – tractable — that is — show that there exists an $O\left(f(k) \cdot n^3\right)$ algorithm that checks if the cograph – width is at most k (for some function $f : \mathbb{N} \to \mathbb{N}$.

4.11.3 Treewidth of Cographs

Exercise 4.81

Prove the following lemma.

Lemma 4.125. *Let a graph* G *be the join of two graphs — say*

$$G = G_1 \otimes G_2.$$

Then

$$\text{tw}(G) = \min\{\text{tw}(G_1) + |V(G_2)|,\ \text{tw}(G_2) + |V(G_1)|\}.$$

Hint: Let H be a chordal embedding of G. If both H_1 and H_2 have nonadjacent vertices H has a C_4.

Exercise 4.82

Prove the following lemma.

Lemma 4.126. *Let a graph* G *be the union of two graphs — say that* $G = G_1 \oplus G_2$. *Then*

$$\text{tw}(G) = \max\{\text{tw}(G_1),\ \text{tw}(G_2)\}.$$

Exercise 4.83

Design a linear – time algorithm that computes the treewidth of cographs. You may assume that a cotree is part of the input.

4.12 Minors

IT IS A BIT UNFORTUNATE that graphs are not well-quasi-ordered by the induced subgraph relation. — For example — Figure 4.12 shows an infinite sequence of graphs, $T_1, \cdots$ that are pairwise incomparable [104] with respect to the induced subgraph relation — that is —

$$\forall_{i \neq j} \quad \neg(T_i \preceq_{\text{ind}} T_j), \qquad \text{where } \preceq_{\text{ind}} \text{ is the induced subgraph relation.} \quad (4.19)$$

[104] they are pairwise 'dissimilar.'

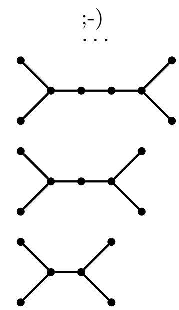


Figure 4.12: A sequence of graphs that is not well-quasi-ordered by the induced subgraph relation.

The way to avoid any infinite sequence of incomparable graphs, is to define a minor – ordering.

Definition 4.127. A graph H is a minor of a graph G if there is a sequence of

1. vertex deletions
2. edge deletions and
3. edge contractions

— performed on the graph G — that turns it into H.

Exercise 4.84

Prove that a graph H is a minor of a graph G if and only if $V(G)$ can be partitioned into sets

$$V_1, V_2, \cdots, V_h \quad \text{where } V(H) = [h]$$

— such that —

1. each $G[V_i]$ is connected

2.

$$\{i, j\} \in E(H) \quad \Rightarrow \quad \exists_{v_i \in V_i} \ \exists_{v_j \in V_j} \ \{v_i, v_j\} \in E(G).$$

Hint: — Since each $G[V_i]$ is connected — it can be contracted to one vertex. The second condition guarantees that H is a subgraph of the remainder. Notice that this proves the following corollary.

Corollary 4.128. *For any graph* H *the property that a graph* G *contains* H *as a minor can be formulated in* MS_1*.*

4.12.1 The Graph Minor Theorem

Robertson and Seymour proved the following theorem — which extends Kruskal's.

Theorem 4.129. *The class of all graphs is well-quasi-ordered by the minor relation.*

— Equivalently — we have the following result.

Theorem 4.130. *Every class of graphs that is closed under minors has a finite obstruction set.*

Let $\mathcal{G}$ be a class of graphs that is closed under taking minors. — So —

$$G \in \mathcal{G} \quad \text{and} \quad H \preceq_{\text{minor}} G \quad \Rightarrow \quad H \in \mathcal{G}.$$

Then there is a finite set $\mathcal{O}$ of graphs — called the obstruction set — which characterizes $\mathcal{G}$ in the following sense:

$$G \in \mathcal{G} \quad \Leftrightarrow \quad \forall_{O \in \mathcal{O}} \quad \neg\,(O \preceq_{\text{minor}} G).$$

To see that Theorem 4.130 follows from Theorem 4.129, let $\mathcal{G}$ be a class of graphs that is closed under minors. Let $\mathcal{O}$ [105]

[105] $\mathcal{O}$ is the set of minimal elements, under the minor relation, that are not in $\mathcal{G}$.

be the set of graphs O that satisfy

$$O \notin \mathcal{G} \quad \text{and} \quad \forall_{O'} \quad (O' \preceq_{\mathsf{minor}} O \quad \text{and} \quad O' \neq O) \quad \Rightarrow \quad O' \in \mathcal{G}.$$

If $|\mathcal{O}|$ is infinite [106] we can choose an infinite sequence $O_1, \cdots$ of graphs in $\mathcal{O}$ that are all different. — However — by Theorem 4.129 there must exist

$$i < j \quad \text{such that} \quad O_i \preceq_{\mathsf{minor}} O_j$$

which is a contradiction.

This proves Theorem 4.130.

[106] ∞

Exercise 4.85

Show that the following classes of graphs are closed under taking minors;

1. the class of planar graphs [107]
2. the class of graphs with treewidth at most k, for $k \in \mathbb{N}$.

What is the obstruction set for the class of planar graphs?
Hint: Recall Kuratowski's theorem — A graph is planar if and only if it has no element of

$$\{K_5, K_{3,3}\}$$

as a minor. [108]

[107] A graph is planar if it can be drawn in the plane without crossing edges. A plane graph is a planar graph together with an embedding of it in the plane.

[108] Harary dedicated his book to Kuratowski, "who gave K_5 and $K_{3,3}$ to those who thought planarity was nothing but topology."

4.13 General Partition Graphs

Definition 4.131. A graph G is a general partition graph if there exists a set S and a map which assigns a subset $S_x \subseteq S$ to every vertex $x \in V$ such that [109]

1. for all pairs of vertices x and y

$$\{x, y\} \in E \quad \Leftrightarrow \quad S_x \cap S_y \neq \varnothing$$

[109] The graph is a partition graph if it satisfies the three conditions and furthermore no two S_x and S_y $(x \neq y)$ are the same.

2. $S = \cup_x S_x$

3. when M is a maximal independent set then

$$\{ S_m \mid m \in M \} \qquad \text{is a partition of } S.$$

In this section we show that for every class of graphs which is not the class of all graphs and which is closed under taking minors there is a polynomial - time algorithm to check if a graph in the class is a general partition graph.

So — for example — there exists an efficient algorithm to check if a planar graph is a general partition graph.

Exercise 4.86

Show that a graph is a general partition graph if and only if it has a set $\mathcal{C}$ of cliques such that

(a) every edge $\{x, y\} \in E$ has both endpoints in some clique $C \in \mathcal{C}$ — that is — $\mathcal{C}$ covers the edges of G

(b) every maximal independent set hits every $C \in \mathcal{C}$.

Exercise 4.87

Show that every cograph is a general partition graph.

Exercise 4.88

Let G be a graph. Let the graph H be obtained from G by adding one vertex to every edge in G and making that adjacent to the two endpoints of the edge. Then H has only one clique - cover with $|E(G)|$ maximal cliques. Furthermore every maximal independent set hits every clique in this cover. So H is a general partition graph.

At some point in history it was discovered that general partition graphs satisfy the triangle condition: [110]

[110] It can be shown that an AT-free graph satisfies the triangle condition if and only if it is a general partition graph. This can be checked on AT-free graphs in polynomial time. It is conjectured that the triangle condition is co-NP-complete.

A graph satisfies the triangle condition if for every maximal independent set M and every edge $\{x, y\}$ in $G - M$ there is a vertex $m \in M$ such that $\{x, y, m\}$ is a triangle in G.

Exercise 4.89

Design an algorithm to check if a planar graph satisfies the triangle condition. [111]

[111] T. Kloks, C. Lee, J. Liu and H. Müller, *On the recognition of general partition graphs*. Proceedings WG 2003, Springer-Verlag, Lecture Notes in Computer Science 2880 (2003), pp. 273–283.

Exercise 4.90

Not all graphs that satisfy the triangle condition are general partition graphs. For example the figure on Page 149 shows a circle graph that satisfies the triangle condition but is not a general partition graph.
Show that every general partition graph satisfies the triangle condition.

Our claim that we can test if a graph is general partition for minor - closed classes follows easily from the following lemma.

Lemma 4.132. *Let* $k \in \mathbb{N}$ *and let* $\mathcal{G}$ *be a class of graphs that satisfy* $\omega \leqslant k$. *There exists a polynomial - time algorithm to check if a graph in* $\mathcal{G}$ *is a general partition graph.*

Proof. We use Exercise 4.86.

Let $\mathcal{G}$ be a class of graphs with clique number $\leqslant k$. — Clearly — graphs in $\mathcal{G}$ have only $O(n^k)$ maximal cliques and we can compute a list of maximal cliques in polynomial time — eg — via the algorithm of Bron and Kerbosch.

Let C be a maximal clique for which there is a maximal independent set M such that

$$C \cap M = \varnothing.$$

Then C is not in a clique cover of which every element is hit by every maximal independent set. Call C intolerable.

We can recognize whether C is intolerable in polynomial time as follows. Let $C = \{x_1, \cdots, x_\ell\}$. For every choice of $y_i \in N(x_i)$ check if $\{y_1, \cdots, y_\ell\}$ is an independent set. If there exists a choice $Y = \{y_1, \cdots, y_\ell\}$ which is independent then Y is contained in a maximal independent set M with $C \cap M = \varnothing$.

Let $\mathcal{C}$ be the set of tolerable cliques (ie those that are not intolerable). We have that G is a general partition graph if and only if $\mathcal{C}$ covers all edges of G.

This proves the lemma. □

Corollary 4.133. *Let $\mathcal{G}$ be a class of graphs which is not the class of all graphs and which is closed under taking minors. There exists a polynomial - time algorithm to check if a graph in $\mathcal{G}$ is a general partition graph.*

Proof. By the graph minor theorem the class $\mathcal{G}$ has a finite obstruction set $\mathcal{F}$. This set is not empty since $\mathcal{G}$ does not contain all graphs. Let

$$k = \min\{\,|V(F)| \mid F \in \mathcal{F}\,\}.$$

No graph in $\mathcal{G}$ can have a clique of size $> k$ (since it would have $F \in \mathcal{F}$ as a subgraph). □

Exercise 4.91

The red maximal independent set problem is the following. GIVEN a graph G and a coloring of its vertices with colors red and blue. QUESTION: does G have a maximal independent set with only red vertices?

This problem is NP-complete even when restricted to planar graphs. Show that there is a polynomial - time algorithm to solve RED MAXIMAL INDEPENDENT SET for graphs with $\omega \leqslant k$. [112]

[112] T. Kloks, D. Kratsch, C. Lee and J. Liu, *Improved bottleneck domination algorithms*. Discrete Applied Mathematics **154** (2006), pp. 1578 – 1592.

4.14 Tournaments

An orientation of a graph G assigns to every edge $\{x, y\} \in E(G)$ an orientation — say — either xy or yx. [113]

[113] We may also write $x \to y$ instead of xy.

Definition 4.134. A tournament is an orientation of a complete graph.

4.14.1 Tournament games

Two players play a game. The board they use is a tournament. They both choose a point of the tournament. When they chose the same point the outcome of the game is a draw. Otherwise the player who chose the head of the arc formed by the two chosen points is the winner. [114]

[114] It's like the "paper, scissors and stone game."

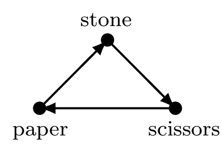


Figure 4.13: The paper, scissors and stone game

Let T be a tournament. A probability distribution is a function $w : V(T) \to [0, 1]$ which satisfies

$$\sum_{x \in V(T)} w(x) = 1.$$

For a subset $S \subseteq V(T)$ we write $w(S) = \sum_{x \in S} w(x)$.

A player wishes to find a winning probability distribution — that is — he wishes to find a probability distribution which satisfies

$$\forall_{x \in V(T)} \quad w(I(x)) \geqslant w(O(x))$$

where $I(x)$ and $O(x)$ are the sets of vertices that beat x (so the arrows point from $I(x)$ towards x) and are beaten by x (the arrows point away from x towards $O(x)$). [115]

[115] A winning distribution has the property that for every $x \in V(T)$ it is at least as likely to beat x as it is to lose to x.

Fisher and Ryan show that every tournament has a winning probability distribution. In this section we show their proof.

In their proof they make use of Farkas' lemma.

Lemma 4.135. *Given a matrix* M *and a vector* b *over the reals. Exactly one of the two following systems of linear inequalities has a solution.*

1. $Mx = b$ *and* $x \geqslant 0$
2. $M^T y \geqslant 0$ *and* $b^T y < 0$.

Let T be a tournament. The <u>payoff matrix</u> K has entries

$$k_{ij} \quad = \quad \begin{cases} 0 & \text{if } i = j \\ -1 & \text{if } i \rightarrow j \\ 1 & \text{if } i \leftarrow j. \end{cases}$$

Since T is a tournament $K^T = -K$ — that is — K is *skew - symmetric.*

Theorem 4.136. *Every tournament has a winning probability distribution. A winning distribution* w *satisfies*

$$w(x) > 0 \quad \Rightarrow \quad w(I(x)) \quad = \quad w(O(x)).$$

Proof. By definition; a distribution w is winning if

$$w \quad \geqslant \quad 0 \qquad \text{and} \qquad 1^T w \quad = \quad 1 \qquad \text{and} \qquad Kw \quad \leqslant \quad 0.$$

Assume that there is no winning distribution. Then the following system has no solution

$$\begin{pmatrix} K & I \\ 1^T & 0^T \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} w \\ z \end{pmatrix} \geqslant \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By Farkas' lemma the following system has a solution

$$\begin{pmatrix} -K & 1 \\ I & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \geqslant \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0^T & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} < 0.$$

This implies $u \geqslant 0$ and $Ku < 0$. But then $w = u/(1^T u)$ is a winning probability distribution — a contradiction.

When w is winning $w_i(Kw)_i \leqslant 0$ for all i. Since K is skew-symmetric $w^T K w = -w^T K w = 0$ and this implies

$$w_i \quad > \quad 0 \qquad \Rightarrow \qquad w(I(i)) \quad = \quad w(O(i)).$$

This proves the theorem. ∎

4.14.2 Trees in tournaments

Sumner conjectures that every tournament with $2(n-1)$ vertices contains any oriented tree with n vertices. This is true when n is large enough.

A weaker result was obtained by A. El Sahili in 2004.

Theorem 4.137 (El Sahili)**.** *Every tournament with* $3(n-1)$ *vertices contains every oriented tree with* n *vertices.*

In this section we take a look at the proof.

MEDIAN ORDERS of digraphs were introduced by Havet and Thomassé. A median order of a digraph is defined as follows.

Exercise: Design an algorithm to compute a median order.

HINT: Notice that the problem is feedback arc set.

Definition 4.138. A median order of a digraph is an ordering of the vertices $v_1 \cdots v_n$ which maximizes

$$|\{\,(v_i, v_j) \in E \quad | \quad i < j\,\}|.$$

Exercise 4.92

Let $v_1 \cdots v_n$ be a median order of the vertices of a digraph.

1. any interval $v_{i+1} \cdots v_j$ is a median order of the digraph induced by $\{v_{i+1}, \cdots, v_j\}$
2. let $I = \{v_{i+1}, \cdots, v_j\}$. Then $|N^+(v_i) \cap I| \geqslant |N^-(v_i) \cap I|$.

Exercise: What are the median orders in graphs with a transitive orientation?

Definition 4.139. Let A and D be digraphs. An embedding of A in D is an injection $f : V(A) \to V(D)$ which satisfies

$$(a, b) \in E(A) \quad \Rightarrow \quad (f(a), f(b)) \in E(D).$$

A embeds in D if A is isomorphic to a subgraph of D.

Definition 4.140. Let A and D be digraphs and let $M = v_1 \cdots v_n$ be a median order of D. An embedding f of A in D is an M-embedding if for every final segment $I = \{v_{i+1} \cdots v_n\}$

$$\boxed{\quad |f(A) \cap I| \quad < \quad \frac{1}{2} \cdot |I| + 1. \quad}$$

A. El Sahili remarks that this lemma is suggested in the paper by Havet and Thomassé on median orders.

Lemma 4.141. *Let T be a tournament with at least three vertices and let $M = v_1 \cdots v_n$ be a median order of D. Let T' be the tournament induced by $\{v_1, \cdots, v_{n-2}\}$ and let $M' = v_1 \cdots v_{n-2}$. Let A be a digraph with a leaf (x, y).* [116] *Assume that $A - y$ has an M' - embedding f' in T'. Then A has an M - embedding in T which extends f'.*

[116] In general a leaf is a vertex with exactly one neighbor. Here we assume that the vertex y has exactly one in - neighbor x and no out-neighbor.

Proof. Let $A' = A - y$. Let f' be an M' - embedding of A' in T'. Let $f'(x) = v_i$ and let $I' = \{v_{i+1} \cdots v_{n-2}\}$. Then

$$|f'(A') \cap I'| \quad < \quad \frac{1}{2} \cdot |I'| + 1.$$

Let $I = \{v_{i+1} \cdots v_n\}$. Since T is a tournament and M is a median order

$$|N_T^+(v_i) \cap I| \quad \geqslant \quad \frac{1}{2} \cdot |I| = \frac{1}{2} \cdot |I'| + 1.$$

So

$$|N_T^+(v_i) \cap I| \quad > \quad |f'(A') \cap I'| \quad = \quad |f'(A') \cap I|.$$

We conclude that v_i has an outneighbor $v_j \in I \setminus f'(A')$ — say v.

Define $f(y) = v$ and $f = f'$ everywhere else. Then f is an M - embedding of A in T.

This proves the lemma. □

When a digraph has a leaf y then it has a median order with y at the end.

A branching is a rooted tree with an orientation that is directed away from the root.

Exercise 4.93

Let A be a branching on n vertices and let T be a tournament with $2(n-1)$ vertices. Then A has an M - embedding in T for every median order M of T.

Hint: Use Lemma 4.141.

Well - rooted trees

Definition 4.142. A digraph A is t - embeddable if A has an M - embedding in every tournament that has t vertices and median order M.

Definition 4.143. An oriented tree with a root is well - rooted if its root is a source.

Let A be a well - rooted tree. An edge of A is backward if its head is closer to the root. Let b be the number of backward edges.

If there are no backward edges the tree A is a branching.

The set of backward edges — if any — span a digraph — say B. Let c be the number of components of B.

B is a forest and it has has b edges. If c is the number of components, then what can you say about the number of vertices in B? (It's $b + c$.) B has no isolated vertices so $b > c$ unless both are zero.

FINALLY; let $d = b - c$.

Lemma 4.144. *Let A be a well - rooted tree with n vertices. Then A is $(2n + 2d)$ - embeddable with d defined as above.*

Proof. By induction on c. First assume $c = 0$. Then A is a branching and the claim follows from Exercise 4.93.

Assume $c > 0$. If A has a leaf y with parent x such that (x, y) is a forward arc then the claim follows from Lemma 4.141. So we may assume that every leaf of A is a source.

Let T be a tournament with $2(n + d)$ vertices and median order M — say $M = v_1 \cdots v_{2n+2d}$. We show that A has an M - embedding f in T.

Let B' be a component of B which contains a leaf of A. Let y be its root and let x be the parent of y. Then (x, y) is a forward arc.

Recall that A is a well - rooted tree — so — y is not the root.

Let $n' = |V(B')|$. Then $A - B'$ has $n - n'$ vertices and $b - (n' - 1)$ backward edges and $c - 1$ backward components.

Let T' be the tournament induced by the initial segment

$$M' = v_1 \cdots v_{2n+2d-4(n'-1)}.$$

BY INDUCTION A' has an M' - embedding f' in T'.

Let A'' be obtained from A' by adding $2(n'_1)$ forward edges with tail x. Let S be the set of heads of these edges. By Lemma 4.141 A'' has an M - embedding f'' in T which extends f'.

Let U be the subtournament of T induced by $f''(S)$. Then U has $2(n'-1)$ vertices. By Exercise 4.93 U contains B'. Let g be an isomorphism from B' to a subtournament of U which is isomorphic to B'.

Define the map $f : V(A) \to V(T)$ as follows.

$$f(x) \quad = \quad \begin{cases} f'(x) & \text{if } x \in V(A') \\ g(x) & \text{if } x \in B'. \end{cases}$$

Then f is an M - embedding of A in T.

This proves the lemma. □

We now prove Theorem 4.137.

Theorem 4.145. *Every oriented tree with* $n \geqslant 2$ *vertices is* $3(n-1)$ *- embeddable.*

Proof. For any root that we choose in A we may assume that the number of forward arcs is at least the number of backward arcs — otherwise — we consider the problem of embedding the 'inverse' of A. [117]

[117] The inverse is obtained by replacing every arc (x, y) by its inverse (y, x).

We may assume that A is not a branching. Choose a root r in A which minimizes d. We may assume that r is a source. — To see that — assume there is a vertex v incident with an arc (v, r). If we choose v as the root then d decreases or else (if $\{v, r\}$ is one backward component) the tree is well rooted with v as a root and the same value d.

Since A is well - rooted we can apply Lemma 4.144: A is $2(n+d)$ - embeddable. We have

$$b \quad \leqslant \quad \frac{n-1}{2} \qquad \text{and} \qquad c \quad \geqslant \quad 1 \qquad \Rightarrow \qquad d \quad \leqslant \quad \frac{n-3}{2}.$$

This proves the theorem. □

Remark 4.146. In their paper Havet and Thomassé conjecture the following: Let A be an oriented tree with at most k leaves. Every tournament on $n+k-1$ vertices contains A.

Further reading

F. Havet and S. Thomassé, Median orders of tournaments: a tool for the second neighborhood problem and Sumner's conjecture, *Journal of Graph Theory* **35** (2000), pp. 244–256.

D. Kühn, R. Mycroft and D. Osthus, An approximate version of Sumner's universal tournament conjecture. Manuscript on arXiv: 1010.4429, 2010.

4.14.3 Immersions in tournaments

Let G and H be digraphs. [118] The digraph H immerses in G if there is a map $\eta : H \to G$ which satisfies the following criteria.

[118] A digraph is an oriented graph. Each edge has an orientation; either xy or yx.

1. $\eta(x) \in V(G)$ for every $x \in V(H)$
2. when $x, y \in V(H)$ and $x \neq y$ then $\eta(x) \neq \eta(y)$
3. $\eta(xy)$ is a directed path in G from $\eta(x)$ to $\eta(y)$ for every edge $xy \in E(H)$
4. when e and f are edges of H and $e \neq f$ then $\eta(e)$ and $\eta(f)$ are edge - disjoint.

The digraph H strongly immerses in G when — additionally — the following condition is satisfied.

5. When $x \in V(H)$ and $e \in EH)$ and x is not an endpoint of e then $\eta(x)$ is not on the path $\eta(e)$.

To define immersions for graphs replace 'arc' with 'edge' in the definition.

Exercise 4.94

Let H and G be graphs. Show that H immerses in G if and only if H is an induced subgraph of a graph G' obtained from G *via* a sequence of edge lifts.

An edge - lift takes two edges that share an endpoint — say $\{x, a\}$ and $\{x, b\}$ — and replaces it with one edge $\{a, b\}$.

Chudnovsky and Seymour prove that tournaments are well - quasi ordered by strong immersion. In this section we review their proof.

The same is *not* true for digraphs. To see that consider the set of even length cycles and orient the edges so that there is no directed path with more than two vertices. No element of this set immerses in another one.

What happened earlier ..

In their paper Chudnovsky and Seymour use the following result (which they published in a separate paper). Let G be a digraph. A layout is a linear order of its vertices. Let

$$v_1 \quad \cdots \quad v_n$$

be a layout of G. The layout has cutwidth k if for each i there are at most k arcs that have their tail in $\{v_1, \cdots, v_i\}$ and their head in $\{v_{i+1}, \cdots, v_n\}$. The digraph G has cutwidth k if it has a layout of cutwidth k.

Theorem 4.147. *Let $\mathcal{S}$ be a set of tournaments. The following two statements are equivalent.*

1. *there exists $k \in \mathbb{N}$ such that all tournaments in $\mathcal{S}$ have cutwidth at most k*
2. *there exists a digraph H such that H does not strongly immerse in any tournament of $\mathcal{S}$.*

Remark 4.148. The two statements are also equivalent with this: there exists $k \in \mathbb{N}$ such that every vertex of a tournaments in $\mathcal{S}$ is in at most k edge - disjoint directed cycles.

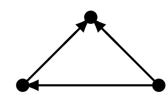


Figure 4.14: A transitive tournament

An orientation of a graph is transitive if xy and yz imply xz. A comparability graph is a graph that allows a transitive orientation of its edges.

Exercise 4.95

Let H be a cyclic triangle. The tournaments in which H does not immerse are the transitive tournaments. Transitive tournaments have cutwidth 0.

We first show that it is sufficient to prove that tournaments of cutwidth at most k are well - quasi ordered.

Lemma 4.149. *Assume that for every $k \in \mathbb{N}$ the class of tournaments of cutwidth k is well - quasi -ordered by strong immersions. This implies that the class of all tournaments is well - quasi ordered by strong immersions.*

Proof. By means of contradiction; let (T_i) be a sequence of tournaments such that no T_i strongly immerses in T_j whenever $i < j$. Let $\mathcal{T} = \{T_i\}$ and let

$$\mathcal{S} \quad = \quad \mathcal{T} \setminus \{T_1\}.$$

Then there is a digraph that does not strongly immerse in any tournament of $\mathcal{S}$ — namely — T_1.

By Theorem 4.147 all tournaments of $\mathcal{S}$ have cutwidth at most k (for some $k \in \mathbb{N}$). This contradicts the assumption. $\square$

Linked layouts

Let G be a digraph and let

$$\mu \quad = \quad x_1 \quad \cdots \quad x_n$$

be a layout of G. Write $B_i = \{x_1, \cdots, x_i\}$, $A_i = \{x_{i+1}, \cdots, x_n\}$ and let F_i be the set of edges with tail in B_i and head in A_i.

The layout μ is linked if for all $h < j$ such that $|F_h| = F_j| = t$ and for all $h \leqslant i \leqslant j$ $|F_i| \geqslant t$ there are t edge - disjoint directed paths from B_h to A_j.

Lemma 4.150. *Let* G *be a digraph of cutwidth* k*. There is a linked layout of* G *of cutwidth* k*.*

Proof. For a layout μ of cutwidth at most k let

$$n_s = |\{ i \mid |F_i| = s \}|.$$

Choose a layout μ of cutwidth at most k such that the sequence $(n_0, n_1, \cdots)$ is lexicographically as large as possible.

Assume that this layout μ is not linked. Then there exist $h < j$ with $F_h| = |F_j| = t$ and for all $h \leqslant i \leqslant j$ $|F_i| \geqslant t$ and there are not t edge-disjoint paths from B_h to A_j. By Menger's theorem [119] there exists a partition $\{P, Q\}$ of $V(G)$ with $B_h \subseteq P$ and $A_j \subseteq Q$ and there are less than t arcs from P to Q. Let F be the set of arcs with tail in P and head in Q. Choose the partition $\{P, Q\}$ such that $|F|$ is as small as possible.

[119] Max flow = min cut

Let $p = |P|$ and let

$$\mu' = x'_1 \quad \cdots \quad x'_p \quad x'_{p+1} \quad \cdots \quad x'_n$$

be the layout that puts all elements of P before the elements of Q and that keeps the ordering within the parts P and Q the same as in μ.

We claim that μ' has cutwidth k. We first show that $|F'_i| \leqslant k$ for all $i \neq p$ (where F'_i is the set of edges with tail in $B'_i = \{x'_1, \cdots, x'_i\}$ and head in $A'_i = \{x'_{i+1}, \cdots, x'_n\}$).

To see that let $i < p$ and choose r such that

$$B'_i = B_r \cap P \quad \text{and} \quad A'_i = A_r \cup Q.$$

Then $r < j$ and $A_j \cap (B_r \cup P) = \varnothing$.

Since we chose $|F|$ minimal and since $B_h \subseteq P \subseteq B_r \cup P$ we have that $|N^+(B_r \cup P)| \geqslant |F|$. [120]

We have [121]

[120] $N^+(S)$ is the set of edges with tail in S and head in $V \setminus S$.

[121] We use the fact that $|N^+(S)|$ $(S \subseteq V)$ is a submodular function — that is — for any sets S and T: $|N^+(S)| + |N^+(T)| \geqslant |N^+(S \cup T)| + |N^+(S \cap T)|$. This property is also known as the principle of diminishing returns.

$$|N^+(B_r)| + |N^+(P)| \quad \geqslant \quad |N^+(B_r \cap P)| + |N^+(B_r \cup P)| \quad \text{that is}$$
$$|F_r| + |F| \quad \geqslant \quad |F_i'| + |N^+(B_r \cup P)| \quad \geqslant \quad |F_i'| + |F|$$
and this implies
$$|F_i'| \quad \leqslant \quad |F_r|.$$

It follows that $|F_i'| \leqslant k$ for $i < p$ and similarly $|F_i'| \leqslant k$ for $i > p$. Since $F_p' = F$ and $|F| \leqslant k$ this proves that μ' has cutwidth k.

We claim that $n_s' \geqslant n_s$ for $s < t$. Let $0 \leqslant s \leqslant t-1$ and let r be such that $|F_r| = s$. We actually have that $F_r' = F_r$. To see that observe that we must have $r < h$ or $r > j$. Assume $r < h$. Then $B_r \subseteq P$ and so $B_r' = B_r$ and $F_r' = F_r$.

This shows that $n_s' = n_s$ for all $s < t$ and

$$|F_r'| \quad < \quad t \qquad \Rightarrow \qquad r \quad < \quad h \qquad \text{or} \qquad r \quad > \quad j$$

We arrived at a contradiction since $|F_p'| < t$ and $h \leqslant p \leqslant j$. Therefore μ is linked.

This proves the lemma. $\square$

Gap sequences

Let $(Q, \ll)$ be a quasi - order and let $k \in \mathbb{N}$. A (Q, k) - gap sequence is a triple (P, f, a) where

- P is a path
- f is a map $V(P) \to Q$
- a is a map $E(P) \to \{0, \cdots, k\}$.

Define a quasi - order on (Q, k) - gap sequences as follows. For two (Q, k) - gap sequences (P, f, a) and (R, g, b) let $(P, f, a) \preceq (R, g, b)$ if

$$P \quad = \quad p_1 \quad \cdots \quad p_m \qquad \text{and} \qquad R \quad = \quad r_1 \quad \cdots \quad r_n$$

and there exists a map

$$1 \quad \leqslant \quad s(1) \quad < \quad s(2) \quad < \quad \cdots \quad < s(m) \quad \leqslant \quad n$$

such that

- for all i: $f(p_i) \ll g(r_{s(i)})$
- for all i: if $e = p_i p_{i+1}$ then $a(e) \leqslant b(e')$ for all $e' \in E(R)$ that are on the path from $r_{s(i)}$ to $r_{s(i+1)}$.

By Theorem 4.101 when $(Q, \ll)$ is a well - quasi order then $\preceq$ is a well - quasi order on (Q, k) - gap sequences.

Marches

A march μ is a sequence

$$e_1 \quad \cdots \quad e_k$$

of elements. The set $\{e_1, \cdots, e_k\}$ is the support of μ and $k = |\mu|$ is the length of μ. We write $e_i = \mu(i)$.

Define an equivalence on pairs of marches as follows. Two pairs pairs of marches (μ_1, ν_1) and (μ_2, ν_2) are equivalent if

- $|\mu_1| = |\mu_2|$
- $|\nu_1| = |\nu_2|$
- for all i and j $\mu_1(i) = \nu_1(j) \Leftrightarrow \mu_2(i) = \nu_2(j)$.

Codewords

A codeword of type k is a pair (P, f) where

- P is a path say $P = p_1 \cdots p_n$
- f is a function with domain $V(P)$ which maps a vertex p_i of P to a pair of marches (μ_i, ν_i) both of length at most k such that
  - $|\nu_i| = |\mu_{i+1}|$
  - $|\mu_1| = |\nu_n| = 0$.

The cutsize function $a : E(P) \to \{0, \cdots, k\}$ maps each edge (p_i, p_{i+1}) to $|\nu_i| = |\mu_{i+1}|$.

Let $\mathcal{C}_k$ be the set of all codewords of type k. Define a quasi - order on $\mathcal{C}_k$ as follows. Let (P, f) and (R, g) be two codewords of type k and let a and b be their cutsize functions. Then (P, f, a)

and (R, g, b) are (Q, k) - gap sequences where Q is the set of pairs of marches ordered by equivalence. Let $(P, f) \preceq (R, g)$ if $(P, f, a) \preceq (R, g, b)$.

Lemma 4.151. *For each* k $(\mathcal{C}_k, \preceq)$ *is well - quasi ordered.*

Proof. The set Q of pairs of marches of length at most k is well - quasi ordered by equivalence because there are only a finite number of equivalence classes. □

Let G be a tournament of cutwidth k and let

$$x_1 \quad \cdots \quad x_n$$

be a linked layout of G of cutwidth k. Define B_i, A_i and F_i as before. We have that for all $h < j$ that satisfy

$$|F_h| = |F_j| = t \quad \text{and} \quad \forall_{h \leqslant i \leqslant j} \quad |F_i| \geqslant t \tag{4.20}$$

there are t edge disjoint paths $P_1, \cdots, P_t$ from B_h to A_j.

The following lemma makes sure that we can find marches with support F_i such that the s^{th} elements of them are edges of P_s.

Lemma 4.152. *There exist marches* μ_i *with support* F_i *such that all* $h < j$ *that satisfy* (4.20) *there are edge - disjoint paths* $P_1, \cdots, P_t$ *such that for* $s \in [t]$ *the* s^{th} *term of* μ_h *and the* s^{th} *term of* μ_j *are edges of* P_s.

Exercise 4.96

Prove Lemma 4.152.

HINT: Fix t. Let $i(1) < \cdots < i(\ell)$ be the indices i with $|F_i| = t$. For $j = 1, \cdots, \ell$ choose the march $\mu_{i(j)}$ (ie choose a linear order of the elements of $F_{i(j)}$) such that the s^{th} element extends the path P_s.

Encoding

Let G be a tournament and let $g_1 \cdots g_n$ be a layout of G with cutwidth k. Map G to a codeword of type k as follows.

Let G denote the path with vertices $g_1, \cdots, g_n$. Define $\mu_0 = \mu_n = \varnothing$ and let μ_i be a march as in Lemma 4.152. Let g be the map

$$g(g_i) \quad = \quad (\mu_{i-1}, \mu_i)$$

for $i \in [n]$. Then (G, g) is a codeword of type k.

We use the same symbol G for the tournament and its layout.

Lemma 4.153. *Let G and H be tournaments of cutwidth k and let (G, g) and (H, h) be codewords of G and H. Assume $(H, h) \preceq (G, g)$ in $(\mathcal{C}_k, \preceq)$. Then H immerses strongly in G.*

table of notations:

H	G
$(h_j)_1^m$	$(g_i)_1^n$
D_j, C_j, F_j	B_i, A_i, E_i
marches: ν_j	marches: μ_i

Proof. There are linked layouts of the tournaments G and H that give rise to the codewords (G, g) and (H, h) — say —

$$G \quad = \quad g_1 \quad \cdots \quad g_n \qquad \text{and} \qquad H \quad = \quad h_1 \quad \cdots \quad h_m.$$

For the layout (g_i) define B_i and A_i as above and let E_i be the set of edges with tail in B_i and head in A_i. Similarly define D_j, C_j and F_j for the layout (h_j) of H. Denote the cutsize functions of (g_i) and (h_j) as b and a.

Let the μ_i be marches with support E_i as in Lemma 4.152 and — similarly — let the ν_i be marches with support F_i.

We have that $(H, h) \preceq (G, g)$ which implies there are

$$1 \quad \leqslant \quad r(1) \quad < \quad r(2) \quad < \quad \cdots \quad < \quad r(m) \quad \leqslant \quad n$$

such that

- $h(h_i)$ and $g(g_{r(i)})$ are equivalent pairs of marches
- if $e = h_i h_{i+1}$ then $a(e) \leqslant b(e')$ for every edge e' on the path $g_{r(i)} \rightsquigarrow g_{r(i+1)}$.

Since $h(h_i) = (\nu_{i-1}, \nu_i)$ and $g(g_{r(i)}) = (\mu_{r(i)-1}, \mu_{r(i)})$ are equivalent pairs of marches we have that

$$|F_i| = |E_{r(i)}|, \quad |F_{i-1}| = |E_{r(i)-1}| \quad \text{and} \quad |F_{i-1} \cap F_i| = |E_{r(i)-1} \cap E_{r(i)}|$$

The second property implies

$$|E_{r(i)}| = |E_{r(i+1)-1}| = |F_i| \quad \text{and}$$
$$|E_j| \geqslant |F_i| \quad \text{for all } r(i) \leqslant j \leqslant r(i+1) - 1.$$

Let $1 \leqslant i \leqslant m$. For $e \in F_i$ there are directed paths $P_i(e)$ in G with the following properties.

(a) the paths in $\{P_i(e) \mid e \in F_i\}$ are pairwise edge - disjoint

(b) the first edge of $P_i(e)$ is in $E_{r(i)}$ and it has tail $g_{r(i)}$ if and only if e has tail h_i

(c) the last edge of $P_i(e)$ is in $E_{r(i+1)}$ and it has head $g_{r(i+1)}$ if and only if e has head h_{i+1}

(d) all internal vertices of $P_i(e)$ are in $\{g_{r(i)+1}, \cdots, g_{r(i+1)-1}\}$

(e) let e be the s^{th} term of march ν_i. The first edge of $P_i(e)$ is the s^{th} term of $\mu_{r(i)}$ and the last edge of $P_i(e)$ is the s^{th} term of $\mu_{r(i+1)-1}$.

Let $e = h_h h_j$ be an edge of H with $h < j$. Then $e \in F_i$ for $h \leqslant i < j$. The reader is invited to check that the appropriate paths $P_i(e)$ glue together — to be precise — let $e \in E(H)$ and let $e = h_h h_j$ for $h < j$. There is a directed path $\eta(e) = g_{r(h)} \rightsquigarrow g_{r(j)}$ in G such that

(f) no vertex of $\{g_{r(1)}, \cdots, g_{r(m)}\}$ is an internal vertex of $\eta(e)$

(g) all the paths in $\{\eta(e) \mid e = h_h h_j \text{ where } h < j\}$ are pairwise edge - disjoint

(h) if e is the s^{th} term of ν_h then the first edge of $\eta(e)$ is the s^{th} term of $\mu_{r(h)}$

(i) if e is the t^{th} term of ν_{j-1} then the last edge of $\eta(e)$ is the t^{th} term of $\mu_{r(j)-1}$.

Let $h < j$ and let $e = h_j h_h$ be an edge of H. Then $g_{r(j)} g_{r(h)}$ is an edge of G. For these edges define $\eta(e)$ as the edge $g_{r(j)} g_{r(h)}$; this is a directed path in G of length one and it is edge - disjoint from the paths $\eta(e)$ for edges in H that point forward in its layout.

Define $\eta(h_i) = g_{r(i)}$. This completes the definition of η which is a strong immersion of H in G.

This proves the lemma. □

Theorem 4.154. *The class of tournaments is well - quasi ordered by the strong immersion - relation.*

Proof. By Lemma 4.149 it is sufficient to show that a class of tournaments of cutwidth at most k is well - quasi ordered by strong immersions. Let (T_i) be a sequence of tournaments of cutwidth at most k. Their codewords are elements of $\mathcal{C}_k$ and these are well - quasi ordered. — So — there exist $i < j$ such that the codeword of T_i is dominated by the codeword of T_j. By Lemma 4.153 this implies that T_i strongly immerses in T_j. □

Remark 4.155. Orientations of complete bipartite graphs are well - quasi ordered under strong immersions — moreover — the immersion relation respects the parts of the bipartition.

4.14.4 Domination in tournaments

Exercise 4.97

Every acyclic digraph has a unique independent dominating set.

HINT: This result has been attributed to Von Neumann and Morgenstern.

J. von Neumann and O. Morgenstern, *Theory of games and economic behavior*, Princeton University Press, 1944.

In 2017 Bousquet, Lochet and Thomassé proved the Erdős - Sands - Sauer - Woodrow conjecture. In this section we review their proof.

> There exists a function $g : \mathbb{N} \to \mathbb{N}$ so that if the arcs of a tournament are colored with k colors there is a set S with at most $g(k)$ vertices such that for every vertex x there is a monochromatic path from S to x.

The transive closure of a binary relation $(Q, \preceq)$ is the smallest transitive relation $(Q, \leqslant)$ which contains $(Q, \preceq)$.

A multiset is a set together with a multiplicity function which maps the elements of the set to $\mathbb{N}$. It is sometimes called a bag.

Let T be a tournament and let the arcs be colored with k colors. In order to formulate the conjecture above as a domination problem we would want each color to induce a quasi - order — so — we take the transitive closure of the set of arcs of each color. — Clearly — this may introduce multiple arcs between pairs of vertices.

A digraph is an orientation of a graph. So it has no loops, no multiple arcs and no directed cycles of length two.

Definition 4.156. A complete multi digraph is a set D of vertices and a multiset of arcs A such that

1. every arc is an ordered pair of vertices
2. every two vertices form at least one arc

A complete multi digraph can have cycles of length two (but no loops).

A multi digraph is a set of vertices together with a multiset of ordered pairs of distinct vertices. Multi digraphs can have oriented cycles of length two but not of length one.

For a (multi-) digraph (D, A) and $x \in D$ define the closed in-neighborhood as

$$N^{-}[x] \quad = \quad \{x\} \quad \cup \quad \{\, y \mid (y, x) \in A\,\}.$$

For a set S we write $N^{-}[S] = \cup_{x \in S} N^{-}[x]$. Similarly define N^{+}.

A set S is domination if $N^{+}[S] = V$. The domination number of the digraph $\gamma(D)$ is the smallest cardinality of a dominating set.

We prove the following theorem. (Clearly this implies the ESSW - conjecture, above.)

Theorem 4.157. *There exists a function* $f : \mathbb{N} \to \mathbb{N}$ *with the following property. Let* T *be a complete multi digraph whose arcs are the union of* k *quasi - orders then* $\gamma(T) \leqslant f(k)$.

The proof of the theorem makes use of two lemmas.

Let T be a complete multi digraph and let the arcs of T be covered with k quasi - orders — say — $(T, \leqslant_i)$, $(i \in [k])$. For $x \in V(T)$ we write $N_i^{-}[x]$ for the closed in-neighborhood of x in $(T, \leqslant_i)$.

Lemma 4.158. *Let* T *be a complete multi digraph whose set of arcs is the union of* k *quasi - orders. There exists a probability distribution* $w : V(T) \to [0, 1]$ *and a partition* $\{T_1, \cdots, T_k\}$ *of* $V(T)$ *such that for each* $x \in T_i$

$$w(N^{-}[x]) \quad \geqslant \quad \frac{1}{2k}.$$

Proof. By Theorem 4.136 (on Page 241) there is a probability distribution $w : V(T) \to [0, 1]$ such that $w(N^{-}[x]) \geqslant 1/2$ for all $x \in V(T)$.

Define for $i \in [k]$

$$T_i \quad = \quad \{\, x \;\mid\; w(N_i^-[x]) \;\geqslant\; 1/2k \,\}.$$

Then $\cup T_i = V(T)$.

— Clearly — the sets T_i can be reduced so that the result forms a partition of $V(T)$.

This proves the lemma. □

Let $(P, \preceq)$ be a quasi - order. We identify $(P, \preceq)$ with a digraph (P, A) where the set of arcs is the set of ordered pairs xy with $x \preceq y$ and $x \neq y$. [122]

[122] So $N^-[x]$ is well-defined on the elements of a quasi - order.

Definition 4.159. Let $(P, \preceq)$ be a quasi - order. A set $A \subseteq P$ is ε-dense in P if there is a probability distribution w on P which satisfies

$$\forall_{x \in A} \quad w(N^-[x]) \quad \geqslant \quad \varepsilon.$$

Lemma 4.160. *There exists a function $g : [0,1] \to \mathbb{N}$ with the following property. In every quasi - order $(P, \preceq)$ if $C \subseteq B$ are subsets of P such that B is ε-dense in P and C is ε-dense in B then there exists a set of $g(\varepsilon)$ elements of P that dominate C.*

Proof. Let $w : P \to [0,1]$ and $w_B : B \to [0,1]$ be probability distributions that show that B is ε-dense in P and C is ε-dense in B — that is —

$$\forall_{x \in B} \quad w(N^-[x]) \quad \geqslant \quad \varepsilon \quad \text{and} \quad \forall_{x \in C} \quad w_B(N^-([x]) \quad \geqslant \quad \varepsilon.$$

Define the function $g(\varepsilon) = \left\lfloor \frac{\ln(\varepsilon)}{\ln(1-\varepsilon)} \right\rfloor + 1$.

Select —at random and according to probability distribution w — a multiset S of $g(\varepsilon)$ elements of P. Then

$$\forall_{x \in B} \quad \mathbb{P}(x \in N^+[S]) \quad \geqslant \quad 1 - (1-\varepsilon)^{g(\varepsilon)} \quad > \quad 1 - \varepsilon.$$

By linearity of expectation of w_B there exists a set S such that $w_B(N^+[S]) > 1-\varepsilon$.

Since $w_B(N^-[x]) \geqslant \varepsilon$ for all $x \in C$ we have that $N^-[x]$ intersects $N^+[S]$ for all $x \in C$. By transitivity this implies that S dominates C.

This proves the lemma. □

We now present the proof of Theorem 4.157.

Theorem. *Let $k \in \mathbb{N}$ and let T be a complete multi digraph whose arcs are the union of k quasi - orders then $\gamma(T) = O(k^{k+2} \cdot \ln(2k))$.*

Proof. Let $P_1 = \{T_1, \cdots, T_k\}$ be a partition of $V(T)$ as mentioned in Lemma 4.158. Repeat this partitioning process $k+1$ times to obtain a sequence of partitions $P_1, \cdots, P_{k+1}$ which we specify as

$$P_i \quad = \quad \{\, T_{j_1 \dots j_i} \,|\, j_1, \cdots, j_i \in [k]\,\}$$

so that for each $\ell \leqslant k+1$ $T_{j_1 \cdots j_\ell}$ is a subset of $T_{j_1 \cdots j_{\ell-1}}$.

Let $w_{j_1 \cdots j_\ell - 1}$ be a probability distribution (as in Lemma 4.158) such that

$$w_{j_1 \cdots j_{\ell-1}}(N^-_{j_\ell}[x]) \quad \geqslant \quad \frac{1}{2k}$$

for all $x \in T_{j_1 \cdots j_\ell}$.

In this formula $N^-_{j_\ell}([x])$ denotes the closed in-neighborhood of x in the j_ℓ^{th} quasi order.

By the pigeonhole principle every sequence $j_1 \cdots j_{k+1}$ in $[k]^{k+1}$ contains $i < \ell$ such that $j_i = j_\ell$. Apply Lemma 4.160 with

$$P = T_{j_1 \cdots j_{\ell-1}} \quad B = T_{j_1 \cdots j_i} \quad \text{and} \quad C = T_{j_1 \cdots j_\ell}.$$

It follows that there exists a set of at most $g(1/2k)$ elements that dominates $T_{j_1 \cdots j_\ell}$ and so it dominates $T_{j_1 \cdots j_{k+1}}$.

We can conclude that $\gamma(T) \leqslant k^{k+1} \cdot g(1/2k)$. Notice that $g(1/2k) \leqslant \ln(2k) \cdot (2k - 1/2 + o(1))$ — that is — $\gamma(T) = O(k^{k+2} \cdot \ln(2k))$.

This proves the theorem. □

As far as we know the following conjecture is open. (It was posed by Sanders, Sauer and Woodrow in 1982.)

Conjecture 4.161. *There exists a function* $f \in \mathbb{N}^{\mathbb{N}}$ *with the following property. Let* D *be a multi digraph whose set of arcs is a union of* k *quasi - orders. Then* D *has a dominating set which is the union of* $f(k)$ *independent sets.*

4.15 Immersions

In this chapter graphs and digraphs are allowed to have multiple edges but no loops unless stated otherwise.

In this chapter we show that digraphs without k - alternating paths are well quasi - ordered by strong immersions.

Robertson and Seymour proved that the class of all graphs is well quasi - ordered by weak immersions. Whether the same holds true for strong immersions is an open problem. [123]

[123] See Page 246 for the definitions of immersions.

Exercise 4.98

A graph is 'subcubic' if every vertex has degree at most 3. Show that the class of subcubic graphs is well quasi - ordered by strong immersions. — Also — for subcubic graphs H is a topological minor of G if and only if it is a minor.
Hint: Let G and H be subcubic. Show that H immerses in G if and only if H is a minor of G.

Chun-Hung Liu and Irene Muzi show that digraphs without k-alternating paths are well quasi - ordered by strong immersions. Before we take a closer look at their proof let us take some time off to meditate on an important result on topological minors.

Chun-Hung Liu and Irene Muzi, Well - quasi - ordering digraphs with no long alternating paths by the strong immersion relation. Manuscript on arXiv: 2007.15822, 2020.

4.15.1 Intermezzo: Topological minors

Relax: we present only facts; no proofs; just try to understand what's going on...

Definition 4.162. Let G and H be graphs. The graph H is a topological minor of G if some subgraph of G is isomorphic to a subdivision of H.

A graph H is a topological minor of G if there is a homeomorphic embedding of H in G — that is — a map $\eta : H \to G$ such that

1. the map $\eta : V(H) \to V(G)$ is injective
2. η maps each edge $\{x, y\} \in E(H)$ to a path $\eta(x) \rightsquigarrow \eta(y)$ in G such that distinct edges of H map to paths in G that have no vertices in common other than endpoints.

Exercise 4.99

Show that K_5 is a minor of the Peterson graph but that it is not a topological minor.

Remark 4.163. Grohe, Marx, Wollan, and Kawarabashi show that finding a topological minor is fixed parameter - tractable: there exists a cubic algorithm to test if a graph H of 'constant size' is a topological minor of a graph G.

Graphs are not well quasi - ordered by topological minors. — To see that — let P_i be a path with i vertices and construct a graph G_i as follows.

- duplicate every edge of P_i
- attach two new vertices to each end of P_i.

The sequence (G_i) is an infinite antichain in the topological minor order.

Chun-Hung Liu and Robin Thomas prove that this is the only obstruction.

For subcubic graphs the topological minor relation is equivalent with the minor relation. Let G be an arbitrary graph. We can map it to a subcubic graph G' as follows. Replace a vertex x by a cycle with $d(x)$ vertices. Each vertex in the cycle receives one neighbor of x as a neighbor outside the cycle. For what classes of graphs holds $G' \leqslant_{\text{top minor}} H' \Rightarrow G \leqslant_{\text{top minor}} H$?

Definition 4.164. A Robertson chain of length k is a graph obtained from a path of length k by duplicating each edge.

Theorem 4.165. *Let $k \in \mathbb{N}$ and let $(Q, \leqslant_Q)$ be a well quasi - order. Let (G_i) be a sequence of graphs without Robertson chain of length k and let $\phi_i : V(G) \to Q$ be a labeling of the vertices of G_i with elements of Q. There exist $j < j'$ and a homeomorphism $\eta : G_j \to G_{j'}$ which satisfies*

$$\forall_{x \in V(G_j)} \qquad \phi_j(x) \quad \leqslant_Q \quad \phi_{j'}(\eta(x)).$$

Further reading on topological minors:

C.-H. Liu, *Graph structures and well-quasi-ordering*, PhD dissertation, Georgia Institute of Technology, 2014.

C.-H. Liu and R. Thomas, Robertson's conjecture I. Well - quasi - ordering bounded treewidth graphs by the topological minor relation. Manuscript on arXiv: 2006.00192, 2020.

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4.15.2 Strong immersions in series - parallel digraphs

A thread is a digraph whose underlying graph is a path. A thread P in a digraph D is k-alternating if it changes direction k times — that is — if it has k vertices that have in-degree in P equal to 0 or out-degree in P equal to 0.

Chun-Hung Liu and Irene Muzi prove the following theorem.

Theorem 4.166. *Let* $k \in \mathbb{N}$ *and let* (D_i) *be a sequence of digraphs without* k*-alternating thread. Let* $(Q, \leqslant)$ *be a well quasi - order and for all* i *let* $\phi_i : V(D_i) \to Q$. *There exist* $j < j'$ *and a strong immersion* η *of* D_j *into* $D_{j'}$ *such that for all* $x \in V(D_j)$

$$\phi_j(x) \quad \leqslant \quad \phi_{j'}(\eta(x)).$$

In this chapter we take a close look at the proof of this theorem. The proof is by induction on k. We present the base case $k = 1$ as an exercise.

Exercise 4.100

Let D be a digraph in which every thread is a directed path. Then D is obtained from a directed path or cycle (possibly of length two) by multiplying edges.

Exercise 4.101

Let (D_i) be a sequence of digraphs without 1-alternating thread. Let $(Q, \leqslant)$ be a well quasi - order and let $\phi_i : V(D_i) \to Q$. There exist $j < j'$ such that there is a strong immersion η of D_j in $D_{j'}$ which satisfies

$$\forall_{x \in V(D_j)} \qquad \phi_j(x) \quad \leqslant \quad \phi_{j'}(\eta(x)).$$

Hint: Use the gap theorem.

Perhaps we should do this in class... Work this out in detail!

Step number two is a proof of the fact that one - way series - parallel triples are well quasi - ordered by strong immersions.

A biconnected multigraph is a two - terminal graph if and only if it is confluent. That is, for any two edges every cycle that contain them, meets the endpoints in the same relative order. A multigraph is confluent if and only if it contains no subgraph which is a subdivision of K_4 — so — its underlying simple graph has treewidth two.

Definition 4.167. A triple (D, s, t) is a two - terminal graph if D is a multigraph and $s, t \in V(D)$ and either

- $V(D) = \{s, t\}$ and $E(D) = \{\{s, t\}\}$
- D is a series composition: there exist two - terminal graphs (D_1, s_1, t_1) and (D_2, s_2, t_2) and $s = s_1$ and $t = t_2$ and D is obtained from the union of D_1 and D_2 by identifying t_1 and s_2
- D is a parallel composition: (D, s, t) is obtained from a union of two - terminal graphs (D_1, s_1, t_1) and (D_2, s_2, t_2) by identifying $s = s_1 = s_2$ and $t = t_1 = t_2$.

The two vertices s and t are the 'terminals' of the graph.

4.15.3 Intermezzo on 2 - trees

The underlying simple graph of a 2 - terminal graph is a partial 2 - tree — that is — it is a subgraph of a 2 - tree.

To define a 2 - tree: any graph that is an edge is a 2-tree. When T is a 2-tree and t a triangle then a new 2-tree is obtained from the disjoint union by identifying the endpoints of an edge in T with the endpoints of an edge in t.

The partial 2 - trees are the graphs of treewidth 2. They are the graphs that do not have K_4 as a minor.

'To understand what the elements of a combinatorial structure look like you should try to enumerate them.' (De Bruijn.)

Labeled and unlabeled biconnected partial two-trees can be enumerated (like trees). The enumeration of the 'rooted' graphs (where the root is a pair $\{s, t\}$) serves as a first step. See Chapter 4 in: Ton Kloks, "Treewidth." PhD Thesis, 1993.

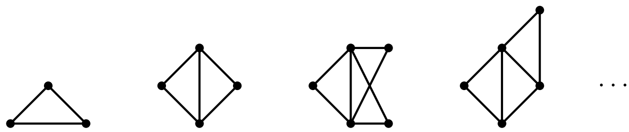


Figure 4.15: Enumeration of 2-trees

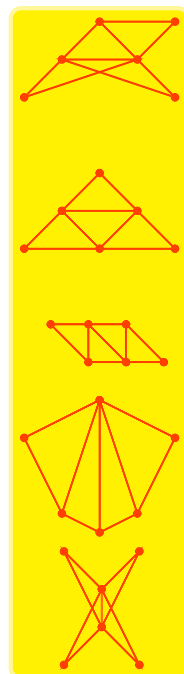


Figure 4.16: A clip from the cover

Exercise 4.102

Any 2-tree has an orientation which is acyclic.

Let G be a biconnected graph of treewidth two. A cell - completion of G is obtained from G as follows. Let s and t be nonadjacent vertices in G. If $G - s - t$ has at least three components then add an edge $\{s, t\}$ in the cell - completion.

> When G is biconnected and has treewidth two then its cell - completion is unique and it is a tree of cycles.

A tree of cycles is a graph defined recursively, as follows.

(i) any graph that is a cycle is a tree of cycles

(ii) Let C be a cycle and let T be a tree of cycles. Then another tree of cycles is obtained from the union by identifying the endpoints of an edge in C with the endpoints of an edge in T.

4.15.4 Series parallel - triples

When a two-terminal graph is not biconnected then its cutvertices and blocks form a path: every cutvertex is in two blocks, every block is incident with at most two cutvertices, and there are two blocks that are incident with exactly one cutvertex.

The underlying simple graph of a two - terminal graph has treewidth two — that is — a graph without a subgraph homeomorphic to K_4. However, notice that the claw can not be generated as a two - terminal graph: every cutvertex in a two - terminal graph separates the graph in two components; one contains s and the other contains t.
A graph is the underlying graph of a two - terminal graph if and only if it is a graph of treewidth two of which the cutvertices and blocks form a path. Every block has a minimal triangulation (into a 2-tree) in which the two cutvertices (including s and t) form an edge.

The Figure 4.17 shows a minimal triangulation of a 2-terminal graph. To specify the 2-terminal graph each edge of this minimal triangulation is labeled with a multiplicity; ie an element of $\in \mathbb{N} \cup \{0\}$. (The multiplicity - labels are not shown.)

The only edges in a minimal triangulation of a block that can have multiplicity zero are edges that are minimal separators.

Since a 2-terminal graph has treewidth 2 each block in a minimal triangulation is a 2-tree. It has a coloring with three colors such that every pair of colors induces a tree. Similarly, a 2-tree has a 3-partition of its edges such that each part is a tree.

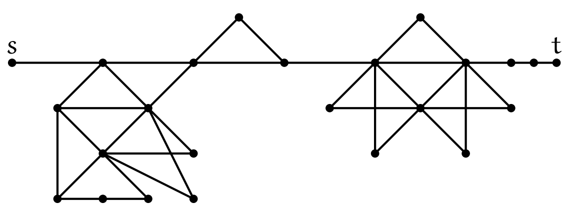


Figure 4.17: The figure shows a minimal triangulations of a 2-terminal graph.

A one-way series-parallel digraph is an orientation of a 2-terminal graph such that all threads that run from s to t are directed paths.

Definition 4.168. A series - parallel triple (D, s, t) is a directed graph D whose underlying graph is connected and s and t are distinct vertices of D such that every thread with ends s and t is a directed path and every cutvertex separates s and t.

A series - parallel triple is one - way if every s, t - thread is a directed path from s to t or if every s, t - thread is a directed path from t to s.

The proof of the following lemma is an easy exercise.

Lemma 4.169. *A series - parallel triple is an orientation of a two - terminal graph* (D, s, t) *such that every thread with ends* s *and* t *is a directed path.*

Definition 4.170. Let $(Q, \leqslant)$ be a well quasi - order. Let (D_i, s_i, t_i) (for $i \in \{1, 2\}$) be two series - parallel triples and let $\varphi_i : V(D_i) \to Q$. The pair (D_2, φ_2) simulates (D_1, φ_1) if there exists a strong immersion $\eta : D_1 \to D_2$ which satisfies

$$\eta(s_1) = s_2 \quad \text{and} \quad \eta(t_1) = t_2 \quad \text{and}$$
$$\forall_{x \in V(D_1)} \quad \varphi_1(x) \leqslant \varphi_2(\eta(x))$$

Definition 4.171. A collection $\mathcal{F}$ of series parallel triples is well - simulated if for every well quasi - order $(Q, \leqslant)$ in any sequence $((D_i, \varphi_i))$ of Q-labeled elements of $\mathcal{F}$ there exist $j < j'$ such that $(D_{j'}, \varphi_{j'})$ simulates (D_j, φ_j).

The minimal triangulations of the underlying simple graphs in a parallel composition are obtained by gluing 2-trees together along their root - edges $\{s, t\}$. (See Figure 4.17.) A parallel composition encodes as a 'Higman - word' over an alphabet formed by the constituents of the composition.

Parallel compositions

Lemma 4.172. *Let* $\mathcal{F}$ *be a set of well - simulated one way series parallel triples. Let* $\mathcal{F}^p$ *be the set of parallel compositions of elements of* $\mathcal{F}$*. Then* $\mathcal{F}^p$ *is well - simulated.*

Proof. Let $(Q, \leqslant)$ be a well quasi - order and let (D_i, φ_i) be a sequence of Q-labeled series - parallel triples in $\mathcal{F}^p$. By assumption each D_i is a parallel composition of a collection of — say ℓ_i ($\ell_i \in \mathbb{N}$) — series - parallel triples that are in $\mathcal{F}$:

$$D_i \quad \text{is a parallel composition of} \quad \{\, D_{i,j} \mid j \in [\ell_i] \,\}$$

Define a word a_i — which encode (D_i, ϕ_i) — as the sequence

$$a_i \quad = \quad (D_{i,1}, \psi_1) \quad \cdots \quad (D_{i,\ell_i}, \psi_{\ell_i})$$

where the ψ_j in this formula are simply the restrictions of ϕ_i to $V(D_{i,j})$ (for $j \in [\ell_i]$).

It now follows from Higman's Lemma that there exist $j < j'$ such that $(D_{j'}, \phi_{j'})$ simulates (D_j, ϕ_j).

This proves the lemma. □

$\mathcal{F}$ - Series parallel trees

Series compositions are not easy to deal with since immersions may 'stretch out' the domain. To handle this we introduce series - parallel trees.

Figure 4.17 suggests an encoding of one way series parallel triple as a 'word' over an alphabet which is the set of blocks and to use Higman's lemma - with - a - gap. The alphabet (set of blocks) is well quasi - ordered by homeomorphic embedding.

Rooted digraphs (D, r) are digraphs with a root. We let strong immersions of rooted digraphs preserve the root.

Definition 4.173. A set of rooted digraphs is well - behaved if for any well quasi - order $(Q, \leqslant)$ and any sequence in the set of rooted digraphs (D_i, r_i) with a labeling $\phi_i : V(D_i) \to Q$ there exist $j < j'$ such that there is a strong immersion η of (D_j, r_j) in $(D_{j'}, r_{j'})$ which satisfies $\eta(r_j) = r_{j'}$ and

$$\forall_{x \in V(D_j)} \qquad \phi_j(x) \quad \leqslant \quad \phi_{j'}(\eta(x))$$

Let (D, r) be a rooted digraph. Associate with (D, r) a rooted tree T of which the nodes are the cutvertices (including r) and the blocks of D. A block and a cutvertex are adjacent in T when the block contains the cutvertex. The root of T maps to the root of the digraph.

Definition 4.174. Let $\mathcal{F}$ be a set of rooted digraphs. A rooted digraph (D, r) is an $\mathcal{F}$ series parallel tree if

1. the block that contains r is in $\mathcal{F}$

2. If B is a block and c is the cutvertex that separates it from its parent then $(B, c) \in \mathcal{F}$
3. every thread from r to a cutvertex is a directed path
4. every block contains at most two cutvertices of D; so every block B which has a child - block is a series parallel triple. [124]

[124] Blocks that have a child are called middle blocks.

Truncations and portraits

Let (B, x, y) be a middle block of a series parallel tree. Let $\{X, Y\}$ be a partition of $V(B)$ such that $x \in X$, $y \in Y$ and the number of edges with one end in X and the other end in Y is equal to the maximal number of edge - disjoint threads between x and y. A truncation is the series parallel triple obtained by shrinking one of the two parts X or Y to one vertex.

A block is a middle block if it has two cutvertices.

$\{X, Y\}$ is a minimum cut.

In an $\mathcal{F}$ - series parallel tree add the two truncations of every middle block to the tree; by subdividing the two edges incident with the middle block. The new trees are called portraits.

The gap - theorem — applied to these portraits — proves the following lemma. (We omit the proof.)

Lemma 4.175.

Let $\mathcal{F}$ be a set of rooted digraphs which behaves well

- *$\mathcal{F}'$ is the set of series parallel triples (D, s, t) with $(D, s) \in \mathcal{F}$ and $t \in V(D - s)$*
- *$\mathcal{F}''$ is the set of truncations of elements of $\mathcal{F}'$.*

If $\mathcal{F}'$ and $\mathcal{F}''$ are well - simulated then the set of $\mathcal{F}$ - series parallel trees behaves well.

Let $\mathcal{F}$ be a set of one way series parallel - triples and assume that $\mathcal{F}$ is well - simulated. Let $\mathcal{F}^s$ denote the set of all series extensions of elements of $\mathcal{F}$. The following exercise initiates a proof to show that $\mathcal{F}^s$ is well - simulated.

The series extensions of $\mathcal{F}$ is the set $\mathcal{F}^*$ of which the elements in a word are chained by identifying t_i and s_{i+1}.

Exercise 4.103

Let $\mathcal{F}$ be a set of one way series parallel triples which is well - simulated;

- $\mathcal{F}^s$ is the set of one way series parallel triples that are series extensions of elements of $\mathcal{F}$
- $\mathcal{F}^t$ is the set of all truncations of elements of $\mathcal{F}$.

If $\mathcal{F}^t$ is well - simulated then $\mathcal{F}^s$ is well - simulated.

Hint: Use Lemma 4.175.

4.15.5 A well quasi - order for one way series parallel - triples

In this section let $\mathcal{F}$ be a set of one way series parallel triples. For $k \in \mathbb{N}$ let $\mathcal{F}^k \subseteq \mathcal{F}$ be the set of one way series parallel triples that do not contain a k-alternating path.

Exercise 4.104

Let (D, s, t) be a one way series parallel triple. Design an algorithm to calculate the maximal number $k \in \mathbb{N}$ for which (D, s, t) has a k-alternating path.

Chun-Hung Liu and Irene Muzi prove the following lemma.

Lemma 4.176. *Let $(Q, \leqslant)$ be a well quasi - order. Let $((D_i, s_i, t_i))$ be a sequence in $\mathcal{F}^k$ and let $\phi_i : V(D_i) \to Q$. There exist $j < j'$ and a strong immersion $\eta : D_j \to D_{j'}$ such that $\eta(s_j) = s_{j'}$ and $\eta(t_j) = t_{j'}$ and*

$$\forall_{x \in V(D_j)} \qquad \phi_j(x) \quad \leqslant \quad \phi_{j'}(\eta(x))$$

Proof. Cover the set $\mathcal{F}$ with the following collections of one way series parallel triples.

1. A_0 is the set of series parallel triples that consist of one edge
2. $A_{0,0} = A_0$

For k and i in $\mathbb{N} \cup \{0\}$ define

3. $A_{k,2i+1}$ is the set of all parallel extensions of elements in $A_{k,2i}$

4. $A_{k,2i+2}$ is the set of all series extensions of elements in $A_{k,2i+1}$
5. $A_{k+1} \subseteq \mathcal{F}$ is the set of one way series parallel triples that have no $(k+1)$ - alternating path that starts in s or t
6. $A_{k+1,0}$ is the set of elements in $\mathcal{F}$ such that either
   - every $(k+1)$ - alternating path with s on one end contains t and there is no $(k+1)$ - alternating path with t on one end or
   - every $(k+1)$ - alternating path with t on one end contains s and there is no $(k+1)$ - alternating path with s on one end.

The lemma is proved in the following steps.

(a) every series - irreducible triple in A_{k+1} is in $A_{k,3}$

(b) $A_{k+1} \subseteq A_{k,4}$

(c) for $(D, s, t) \in \mathcal{F}$:

- if $(D, s, t) \in A_k$ then every truncation[125] is in A_k
- if $(D, s, t) \in A_{k,0}$ then every truncation is in $A_{k,0}$.

[125] with respect to a partition $\{S, T\}$ with $s \in S$ and $t \in T$ and a minimal number of crossing edges.

The next claim is proved via Lemma 4.175.

(d) if A_k is well - simulated then $A_{k,0}$ is well - simulated

(e) let $k > 0$ and $\ell \geqslant 0$. All truncations of elements of $A_{k,\ell}$ are elements of $A_{k,\ell}$.

(f) for $k, \ell \geqslant 0$ if $A_{k,0}$ is well - simulated then $A_{k,\ell}$ is well - simulated.

It now easily follows by induction on k that A_k is well - simulated: This is clearly true for $k = 0$. When A_{k-1} is well - simulated then by (d) $A_{k-1,0}$ is well - simlulated. By (f) $A_{k-1,4}$ is well - simulated and since $A_k \subseteq A_{k-1,4}$ (by (b)) A_k is well - simulated.

This proves the lemma since — obviously — $\mathcal{F}^k \subseteq A_k$. $\square$

4.15.6 Series parallel separations

Definition 4.177. Let D be a digraph. A separation of D is a pair of edge disjoint subgraphs (A, B) such that $A \cup B = D$. The order of the separation is $|V(A \cap B)|$.

The intersection $A \cap B$ of two digraphs is of course what you think it is.

Definition 4.178. A series parallel separation of a digraph D is a separation (A, B) of D with $V(A \cap B) = \{s, t\}$ and such that (A, s, t) is a one way series parallel triple.

Exercise 4.105

Let G be a graph of treewidth w. Let $\mathcal{S}$ be a collection of subsets of V and assume that $G[S]$ is connected for each $S \in \mathcal{S}$. Let $k \in \mathbb{N}$. One of the two following statements holds true.

- there exist k pairwise disjoint elements of $\mathcal{S}$
- there exists a subset $Z \subseteq V(G)$ $|Z| \leqslant (k-1)(w+1)$ and $Z \cap S \neq \varnothing$ for each $S \in \mathcal{S}$.

Hint: We may as well assume that G is a w - tree. First consider the case $w = 1$ — that is — G is a tree. The Erdős - Pósa property says the following. Let $\mathcal{A}$ be a collection of subtrees of G. For every k either $\mathcal{A}$ has k elements that are vertex - disjoint or G has a subset of less than k vertices which hits every element of $\mathcal{A}$.

The subtrees are vertices in a chordal graph. When every pair of subtrees intersects then they have a point in common.

Let's get to the point.

A graph is biconnected if it has no separator with less than two vertices — that is — the graph is connected and has no cutvertex.

Lemma 4.179. *There exists a function $f : \mathbb{N} \to \mathbb{N}$ with the following property. Let D be a digraph whose underlying graph is biconnected and assume that D has no $(t+1)$ - alternating path. There exists a set $Z \subseteq V(D)$ $|Z| \leqslant f(t)$ such that every t - alternating path P satisfies one of the following two statements.*

- *there is a series parallel separation (A, B) with $P \subseteq A$*
- $V(P) \cap Z \neq \varnothing$.

Proof. If a digraph has two vertex disjoint threads and $2t+3$ vertex disjoint threads that run between them then D has a $(t+1)$ - alternating path. This implies that the underlying graph of D has no subdivision of a $2 \times k$ -wall (for sufficiently large k). By the grid minor - theorem there exists $w \in \mathbb{N}$ such that D has treewidth w.

Exercise: Let D be an orientation of a ladder, say with $2t+3$ steps. Show that D has an $(t+1)$ - alternating path.

Hint: There are $t+2$ steps whose orientation from one stringer to the other is the same.

Let $k \in \mathbb{N}$ and let G be a graph. Either the treewidth of G is at most k or G has an $f(k) \times f(k)$ - grid as a minor, for some function $f : \mathbb{N} \to \mathbb{N}$. This is the grid minor - theorem.

Let $f(t) = 4(w+1)$. (This is a function of t; we show below that this works.)

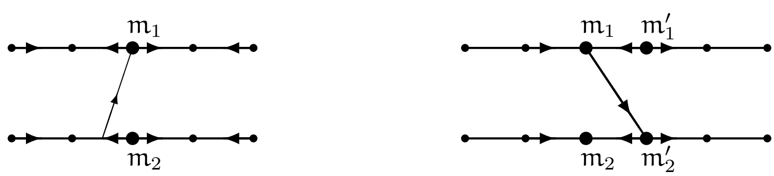
Let P be a t - alternating path in D.

- when t is odd then let m denote the pivot in the middle
- when t is even then let m and m' denote the two middle - pivots. By symmetry we may assume that m is a sink and m' is a source.

Let P_1 and P_2 be two vertex - disjoint t - alternating paths in D. Denote the pivots in the middle of P_i as m_i and m'_i $(i \in \{1,2\})$.

Let P be a thread that that runs between P_1 and P_2.

1. if t is odd then $V(P_1) \cap V(P) = m_1$ and $V(P_2) \cap V(P)$ is between the $(\lceil \frac{t}{2} \rceil - 1)^{\text{th}}$ pivot and the $(\lceil \frac{t}{2} \rceil + 1)^{\text{th}}$ pivot of P_2 or vice versa. Furthermore, if $V(P) \cap P_2 \neq \{m_2\}$ then P is a directed path
2. if t is even then P is a directed path between m_1 and m'_2 or vice versa.



t is odd $\qquad$ t is even

Figure 4.18: The figure illustrates two disjoint t - alternating paths — P_1 and P_2 — interacting with a thread P that runs between them.

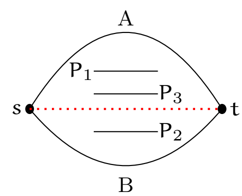
By assumption the underlying graph of D is biconnected and since there are no three disjoint threads between P_1 and P_2 it follows that there exists a separation (A, B) of order two with $P_1 \subseteq A$ and $P_2 \subseteq B$. — Furthermore — there exist two disjoint directed paths Q_1 and Q_2 that run between P_1 and P_2 and which satisfy the following.

- if t is even then Q_1 is a directed path from m_1 to m_2' and Q_2 is a directed path from m_2 to m_1'
- if t is odd then Q_1 has endpoint m_1 and Q_2 has endpoint m_2. Furthermore, the other end of Q_1 is not m_2 and the other end of Q_2 is not m_1.

We show the following.

> For any FIVE vertex - disjoint t - alternating paths $P_1, \cdots, P_5$ there exists a series parallel separation (A, B) with $P_i \subseteq A$ for some $i \in [5]$.

Assume the paths exist. Let (A, B) be a separation of order two —say $V(A) \cap V(B) = \{s, t\}$ — such that $P_1 \subseteq A$ and $P_2 \subseteq B$. By assumption (A, B) and (B, A) are not series parallel separations. At most two of the three other paths can intersect $\{s, t\}$. Assume $V(P_3) \cap \{s, t\} = \varnothing$ and $P_3 \subseteq A - s - t$.



Let $Q_{3,1}$ be a directed path from P_3 to P_1 as mentioned above. Then $Q_{3,1} \subseteq A$.

To see that first assume that $Q_{3,1}$ contains $\{s, t\}$. Then we can replace the part that passes through s and t by a thread in B. The result should be a directed path from m_3 to $P_1 \setminus \{m_1\}$. However, (B, A) is not a series parallel separation and so B contains a thread between s and t that is not a directed path. This proves $Q_{3,1} \subseteq A$.

We claim that there is a separation (A', B') of order two with $V(A' \cap B') = \{m_1, m_2\}$ and

$$P_1 \cup P_3 \quad \subseteq \quad A' \qquad \text{and} \qquad P_2 \quad \subseteq \quad B'$$

To prove that we show that there is no thread in $D - \{m_1, m_2\}$ between m_3 and $P_2 \setminus \{m_2\}$. That is so because a merge of such a thread with $Q_{3,1}$ would be a thread between $P_1 \setminus \{m_1\}$ and $P_2 \setminus \{m_2\}$ which is a contradiction. So no component of $D - \{s, t\}$ intersects

P_3 and $P_2 - m_2$ and no component of $D - \{s, t\}$ intersects $P_1 - m_1$ and $P_2 - m_2$. This proves the claim.

If $Q_{2,3} \cap P_1 = \varnothing$ then $Q_{2,3} \subseteq A'$. Merge $Q_{2,3}$ with a thread in B' to obtain thread $m_2 \rightsquigarrow m_1$. Since (B', m_1, m_2) is not a one way series parallel triple there is a thread $m_2 \rightsquigarrow m_1$ which is not a directed path. So $Q_{2,3} \cap P_1 \neq \varnothing$.

If $Q_{2,3} \cap P_1 \neq \varnothing$ then let P'' be the subthread of $Q_{2,3}$ from $P_3 - m_3$ to P_1. Then P'' has end m_1 and $P'' \subseteq A'$ (since $m_2 \notin V(P'')$). The concatenation $P'' \cup Q_{1,2}$ is a thread from $P_3 - m_3$ to $P_2 - m_2$ and this is a contradiction.

Let $\mathcal{S}$ be the collection of vertex - sets of t - alternating paths P for which there is no series parallel separation (A, B) with $P \subseteq A$. By Exercise 4.105 there exists a set Z of vertices in D with $|Z| \leqslant 4(w+1)$ which hits every set $S \in \mathcal{S}$.

This proves the lemma. □

AHEAD LIES A CLEAR ROAD TO GLORY; we should examine the extreme series parallel separations.

Way to go! There are less than 5 extremes! A good name for maximal separations is 'asteroidal.'

Definition 4.180. A series parallel separation (A, B) of a digraph is maximal if there exists no series parallel separation (A', B') in the digraph with $A \subset A'$.

Lemma 4.181. *Let D be a digraph whose underlying graph is biconnected. Assume that $D \neq X \cup Y$ for one way series parallel triples (X, s, t) and (Y, t, s). If (A_i, B_i) are two distinct maximal series parallel separations then*

$$A_1 \subseteq B_2 \quad \text{and} \quad A_2 \subseteq B_1.$$

Proof. For $i \in [2]$ let $A_i \cap B_i = \{s_i, t_i\}$ such that every thread $s_i \rightsquigarrow t_i$ in A_i is a directed path from s_i to t_i.

Assume that $t_2 \in V(A_1)$ and that $s_2 \in V(B_1)$ (see Figure 4.19). Let P_2 and P_2' be threads that run between s_2 and t_2 in A_2 and

B_2. Every thread in A_1 from s_1 to t_1 contains t_2 (since it connects $A_2 \setminus B_2$ with $B_2 \setminus A_2$) and $s_1 \in V(P_2)$.

It follows that $(A_1 \cup A_2, s_2, t_1)$ is a one way series parallel triple. By assumption $D \neq A_1 \cup A_2$ so $E(B_1 \cap B_2) \neq \varnothing$. This shows that $(A_1 \cup A_2, B_1 \cap B_2)$ is a series parallel separation and $A_1 \subset A_1 \cup A_2$ (since $s_2 \notin V(A_1)$). This contradicts the assumption that (A_1, B_1) is maximal.

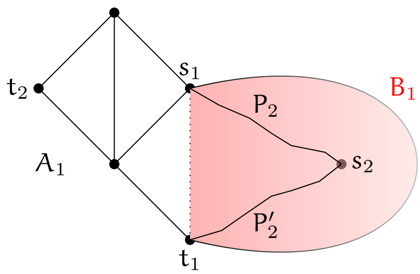


Figure 4.19: Illustration of Case 1

Case 2: Assume that $\{s_2, t_2\} \subseteq V(A_1)$. Then $B_1 \subseteq A_2$ or $B_1 \subseteq B_2$. When $B_1 \subseteq B_2$ then $A_2 \subseteq A_1$ and this contradicts that (A_2, B_2) is maximal.

So we have $B_1 \subseteq A_2$. Then (B_1, t_1, s_1) is a one way series parallel triple and so D is the union of one way series parallel triples A_1 and B_1. This is a contradiction.

Since (A_1, B_1) is maximal $A_1 \not\subseteq A_2$ and so $A_1 \subseteq B_2$.

This proves the lemma. $\square$

To summarize: Let D be a digraph whose underlying graph is biconnected and assume that D is not a union of one way series parallel triples (X, s, t) and (Y, t, s). The collection $\mathcal{S}$ of maximal series parallel separations of D satisfies the following.

- for every series parallel separation (A, B) of D there exists $(A', B') \in \mathcal{S}$ with $A \subseteq A'$
- when $(A_1, B_1) \in \mathcal{S}$ and $(A_2, B_2) \in \mathcal{S}$ then $A_1 \subseteq B_2$ and $A_2 \subseteq B_1$.

4.15.7 Coda

In this section we prove Theorem 4.166.

IN CASE YOU LOST TRACK; it's the theorem below.

Theorem 4.182 (Liu and Muzi's theorem)**.** *Let* $k \in \mathbb{N}$ *and let* (D_i) *be a sequence of digraphs without* k *- alternating path. Let* $(Q, \leqslant)$ *be a well quasi - order and for* $i \in \mathbb{N}$ *let* $\phi_i : V(D_i) \to Q$*. Then there exist* $j < j'$ *and a strong immersion* $\eta : D_j \to D_{j'}$ *such that for all* $x \in V(D_j)$

$$\phi_j(x) \quad \leqslant \quad \phi_{j'}(\eta(x)).$$

LET'S GET IN THE MOOD and start with an easy exercise.

"In the mood" is a tune by Glen Miller. In a future edition of this book we will let you listen to it!

Exercise 4.106

Let D be a digraph whose underlying graph is biconnected. Let r, x and y be three vertices of D and assume that every thread from r to $\{x, y\}$ is a directed path. Then for one of x and y there are directed paths to and from r.

HINT: See the figure.

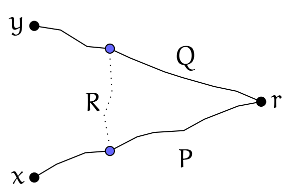


Figure 4.20: Let P and Q be threads from r to x and y with $V(P) \cap V(Q) = \{r\}$. (Exercise: show that P and Q exist.) Let R be a thread that connects $P \setminus r$ with $Q \setminus r$. (R exists.) When all threads from r to $\{x, y\}$ are directed paths then one endpoint of R must be one of x or y.

To prove Theorem 4.166 we order the set of rooted digraphs.

For $t, k \in \mathbb{N} \cup \{0\}$ let $\mathcal{F}_{t,k}$ be the set of those rooted digraphs (D, r) that satisfy the following properties.

here we go again ... see Lemma 4.176 on Page 268.

- the underlying graph of D is connected
- r is not a cutvertex
- D has no $(\mathsf{t}+1)$ - alternating path
- no block of D has a t - alternating path
- no k - alternating path in D has r as an endpoint.

Define the classes $\mathcal{F}_t$, $\mathcal{F}_t^b$ and $\mathcal{F}_t^*$ as follows.

- $\mathcal{F}_t$ is the set of rooted digraphs of which the underlying graph is connected and which has no t - alternating path
- $\mathcal{F}_t^b$ is the set of rooted digraphs with no t - alternating path and of which the underlying graph is biconnected.[126]
- $\mathcal{F}_t^*$ is the set of rooted digraphs without t - alternating path.

[126] This includes the case where the underlying graph is one vertex or two vertices that are adjacent: a graph is biconnected if it is connected and has no cutvertex.

Exercise 4.107

1. $\mathcal{F}_t^b \subseteq \mathcal{F}_t \subseteq \mathcal{F}^*$
2. $\varnothing \subseteq \mathcal{F}_{t,0} \subseteq \cdots \subseteq \mathcal{F}_{t,t+1} = \sum_{k \geqslant 0} \mathcal{F}_{t,k}$.

Exercise 4.108

If $\mathcal{F}_{t,t+1}$ behaves well then $\mathcal{F}_t \cup \mathcal{F}_t^*$ behaves well.

HINT: By Higman's lemma if $\mathcal{F}_t$ behaves well then so does $\mathcal{F}_t^*$. So it is suffient to prove that $\mathcal{F}_t$ behaves well. To show this apply Higman's lemma (on words that are composed of letters in the well - behaved set $\mathcal{F}_{t,t+1}$).

Lemma 4.183. *If $\mathcal{F}_t^b$ behaves well then so does $\mathcal{F}_{t,k}$ for every integer $\mathsf{k} \geqslant 0$.*

Proof. We leave it as an exercise to check that $\mathcal{F}_{t,0}$ behaves well.

HINT: $\mathcal{F}_{t,0} = \varnothing$.

We proceed by induction on k and assume that $\mathcal{F}_{t,k-1}$ behaves well.

Let $(Q, \leqslant)$ be a well quasi - order. Let (D_i, r_i) be a sequence of rooted digraphs in $\mathcal{F}_{t,k}$ and let $\phi_i : V(D_i) \to Q$.

Let S_i be a minimal set of cutvertices x of D_i that root some branch $(B, x) \in \mathcal{F}^b_t \cup \mathcal{F}_{t,k-1}$ and that covers all branches of D_i that are in $\mathcal{F}_{t,k-1}$ — that is — for every branch (B, x) of D_i that is in $\mathcal{F}_{t,k-1}$ there is a $(B', x') \in \mathcal{F}^b_t \cup \mathcal{F}_{t,k-1}$ with $B \subseteq B'$ and $x' \in S_i$.

By the induction assumption and by Higman's lemma the branches of D_i that are in $\mathcal{F}^b_t \cup \mathcal{F}_{t,k-1}$ are well - behaved.

Let D'_i be the digraph obtained from D_i by removing the internal vertices of branches at vertices $x \in S_i$ that are in $\mathcal{F}^b_t \cup \mathcal{F}_{t,k-1}$.

Label the vertices of D'_i with elements of a well quasi - order $(Q', \leqslant')$ as follows.

1. if $x \notin S_i$ then label x with $\phi_i(x)$
2. if $x \in S_i$ then label x with a pair $(\phi_i(x), \phi_i(B))$ where B is the union of branches at x that are in $\mathcal{F}^b_t \cup \mathcal{F}_{t,k-1}$.

Quasi - order pairs by the Cartesian product of the components.

If D'_i is biconnected then it is in $\mathcal{F}^b_t$ (since it is in $\mathcal{F}_{t,k}$). So — since $\mathcal{F}^b_t$ behaves well — if there are an infinite number of D'_i that are biconnected then we are done. — Henceforth — we assume that all elements of the sequence (D'_i) have cutvertices.

The following claim is easily checked. When x is a cutvertex of D_i then all threads that run between r_i and x are directed paths and they all run in the same direction.

HINT: Let x be a cutvertex such that some thread $r_i \rightsquigarrow x$ is not directed. By definition of S_i $(B, x) \notin \mathcal{F}_{t,k}$ so there is a $(k-1)$ - alternating path in B that ends in x. Then there is a k - alternating path that ends in r_i. This contradicts that $(D_i, r_i) \in \mathcal{F}_{t,k}$.

Every block of D'_i has at most two cutvertices and the block that contains r_i contains at most one cutvertex of D'_i. To see that use Exercise 4.20.

It follows that (D'_i, r_i) is an $\mathcal{F}^b_t$ - series parallel tree. We show that the set of $\mathcal{F}^b_t$ - series parallel trees behaves well. Let $\mathcal{F}'$ be the set of one way series parallel - triples (B, x, y) with $(B, x) \in \mathcal{F}^b_t$ and $y \in V(B) \setminus x$. These series parallel triples are in A_t. The set of all the truncations of the element of $\mathcal{F}'$ are in A_t (see Lemma 4.176). Thus $\mathcal{F}'$ and all truncations are well - similated. By Lemma 4.175 this proves the claim. □

Exercise 4.109

Let $\mathcal{F}$ be a collection of rooted digraphs and assume that $\mathcal{F}$ behaves well. For $s \in \mathbb{N}$ let $\mathcal{F}^s$ be the collection of rooted digraphs (D, r) for which there exists

$$X \subseteq V(D) \quad |X| \leqslant s \quad r \in X \quad \text{and} \quad (D - X, r') \in \mathcal{F} \quad \text{for some } r' \in D \setminus X. \tag{4.21}$$

Show that $\mathcal{F}^s$ behaves well.

Hint: Let $(Q, \leqslant)$ be a well quasi - order and let $((D_i, r_i))$ be a sequence in $\mathcal{F}^s$ and write

$$X_i = \{u_{i,1}, u_{i,2}, \cdots, u_{i,s}\} \quad \text{where } u_{i,1} = r_i.$$

For $x \in V(D_i - X_i)$ define

$$\phi'_i(x) = (\phi_i(x), a_1, b_1, \cdots, a_s, b_s),$$

where a_ℓ is the number of edges $u_{i,\ell} \to x$ and b_ℓ is the number of edges $x \to u_{i,\ell}$. Define a useful well quasi - order to label the vertices of $D_i - X$.

Recall: $\mathcal{F}^b_t$ is the collection of rooted digraphs (D, r) that have no t - alternating thread and of which the underlying graph is biconnected. For $t = 1$ these are obtained from a directed path or cycle by multiplication of edges.

Lemma 4.184. $\mathcal{F}^b_t$ *behaves well for all* $t \in \mathbb{N}$.

Proof. We prove this by induction on t. For $t = 1$ the claim is proved in Exercise 4.101 on Page 262.

Assume that $\mathcal{F}^b_{t-1}$ behaves well. By Exercise 4.183 and Lemma 4.108 $\mathcal{F}^*_{t-1}$ behaves well.

Let $(Q, \leqslant)$ be a well quasi - order; let $((D_i, r_i))$ be a sequence in $\mathcal{F}^b_t$ and let $\phi_i : V(D_i) \to Q$. We show that there exist $j < j'$ and a strong immersion $\eta : (D_j, r_j) \to (D_{j'}, r_{j'})$ which satisfies $\phi_j(x) \leqslant \phi_{j'}(\eta(x))$ for all $x \in V(D_j)$.

Assume that for infinitely many i $D_i = X_i \cup Y_i$ for one way series parallel triples (X_i, s_i, t_i) and (Y_i, t_i, s_i). Then we are done by Lemma 4.176. So — by the summary on Page 274 — we may assume that every D_i has a collection of separations S_i that satisfy

Remove any D_i that is a union of one way series parallel triples. We may assume that this removes only a finite number of elements from the sequence. So we are left with an infinite sequence; which we simply call (D_i).

- $(A, B) \in S_i$ is a series parallel separation of D_i
- if $(A_1, B_1) \in S_i$ and $(A_2, B_2) \in S_i$ then $A_1 \subseteq B_2$ and $A_2 \subseteq B_1$
- if (A, B) is a series parallel separation of D_i then there exists $(A', B') \in S_i$ with $A \subseteq A'$.

S_i is the set of maximal series parallel separations of D_i.

By Lemma 4.179 there exist $N \in \mathbb{N}$ and $Z_i \subseteq V(D_i)$ with $|Z_i| \leqslant N$ which hits every $(t-1)$ - alternating path P for which there is no series parallel separation (A, B) with $P \subseteq A$.

Let's get started.

Let $(A, B) \in S_i$. Replace A with a handle which is a directed P_5 that runs between the two terminals of A and that has a multiplicity on its edges. The multiplicity of the end - edges are the degrees in A of the two terminals. The multiplicity of the two middle edges is the number of edge - disjoint directed paths in A that run between the two terminals.

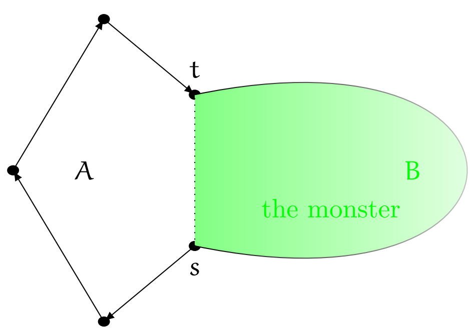


Figure 4.21: Replacement of A with a handle. Multiplicities are not shown.

Let D'_i be obtained from D_i by replacing each A_i of the separations $(A_i, B_i) \in S_i$ that satisfies $|V(A_i) \setminus V(B_i)| \geqslant 2$ with a handle.

WE NEED TO SUPPLY D'_i WITH A ROOT. If $r_i \in B_i$ then let $r'_i = r_i$ and if $r_i \in A_i \setminus B_i$ then let r'_i be an arbitrary vertex of D'_i.

Let Z'_i be the following set of vertices in D'_i.

1. the vertices of Z_i that are in D'_i
2. r'_i
3. the two terminals s and t for every one way series parallel triple (A, s, t) for which
   - $A \cup B = D_i$
   - $(A, B) \in S_i$
   - $V(A \cap B) = \{s, t\}$
   - $|V(A) \setminus V(B)| \geqslant 2$
   - Z_i has a vertex in $A \setminus B$.

... for which A has been replaced by a handle and for which $Z_i \cap (A \setminus B) \neq \varnothing$.

Then $Z'_i \subseteq V(D_i) \cap V(D'_i)$ and $|Z_i| \leqslant 2 \cdot |Z_i| + 1 \leqslant 2 \cdot N + 1$.

NOTICE THAT Z'_i hits every $(t-1)$ - alternating path in D'_i.

EXERCISE! **Hint:** Assume there is a $(t-1)$ - alternating path P' in D'_i that misses Z'_i. Replace parts in handles by threads in D_i and construct an $(t-1)$ - alternating path in D_i that misses Z_i.

Let $D^*_i = D'_i - Z'_i$ and let r^*_i be an arbitrary vertex of D^*_i. By the previous observation $(D^*_i, r^*_i) \in \mathcal{F}^*_{t-1}$.

Let $\mathcal{F}^*$ be the set of rooted digraphs (D, r) which satisfy

1. D has a set Z of vertices with $r \in Z$ and $|Z| \leqslant 2 \cdot N + 1$
2. $(D - Z, r') \in \mathcal{F}^*_{t-1}$ for some $r' \in V(D) \setminus Z$.

Then $\mathcal{F}^*$ behaves well and $(D'_i, r'_i) \in \mathcal{F}^*$.

Exercise 4.109 shows that $\mathcal{F}^*$ behaves well.

TO EXPLOIT THE FACT that $\mathcal{F}^*$ behaves well the vertices of (D'_i, r'_i) are now labeled with elements of a well quasi - order — say — $\phi'_i : V(D'_i) \to Q'$. First we supply Q' with an element that is incomparable to all others. This element of Q' is used to label r'_i. For vertices x of D'_i that are in D_i define $\phi'_i(x) = \phi_i(x)$.

It remains to label the vertices of handles that replace one way series parallel triples; say (A, s, t) in D_i. The midpoint of such a handle is labeled as $((A, s, t), \phi_i)$.

Fix a partition $\{S, T\}$ of $V(A)$ with $s \in S$ and $t \in T$ and such that the crossing edges form a minimum cut. Label the two neighbors of the midpoint in the handle with the two truncations of which the vertices are labeled by ϕ_i.

Lemma 4.176 shows that one way series parallel triples without k - alternating threads and their truncations are well - simulated.

$\mathcal{F}^*$ behaves well; (D'_i) is a sequence in $\mathcal{F}^*$ and $\phi'_i : V(D'_i) \to Q'$ for a well quasi - order $(Q', \leqslant')$. — Thus — there exist $j < j'$ and a strong immersion $\eta' : V(D'_j) \to V(D'_{j'})$ which respects $(Q', \leqslant')$. It follows easily that there is a strong immersion $\eta : (D_j, r_j) \to (D_{j'}, r_{j'})$ which respects $(Q, \leqslant)$.

This proves the lemma. □

Hooray! We're done.

Hip, hip!

Theorem 4.185. *Let $k \in \mathbb{N}$ and let (D_i) be a sequence of digraphs without k - alternating path. Let $(Q, \leqslant)$ be a well quasi - order and for $i \in \mathbb{N}$ let $\phi_i : V(D_i) \to Q$. Then there exist $j < j'$ and a strong immersion $\eta : D_j \to D_{j'}$ such that for all $x \in V(D_j)$*

$$\phi_j(x) \leqslant \phi_{j'}(\eta(x)).$$

Proof. By Lemma 4.184 $\mathcal{F}^b_t$ behaves well and so $\mathcal{F}^*_t$ behaves well (by Exercise 4.108 and Lemma 4.183).

This proves the theorem. □

4.15.8 Exercise

A permutation graph is an intersection graph of a set of straight line - segments with their endpoints on two parallel lines.

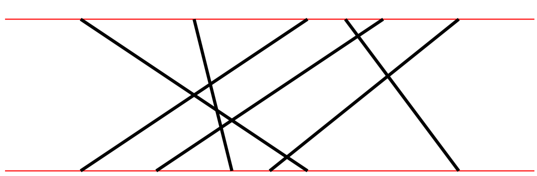


Figure 4.22: The figure shows a permutation diagram. Crossing line segments represent adjacent vertices in the permutation graph.

A graph G is a permutation graph if and only if G and $\bar{G}$ are comparability graphs — so — a permutation graph can be represented as a tournament with a 2-coloring of its edges such that every color is transitive.

Exercise 4.110

Let $(P_i)_{i \in \mathbb{N}}$ be a sequence of permutation graphs. Show that there exist $j < j'$ such that P_j immerses strongly in $P_{j'}$.

Show that the class of AT - free graphs is well quasi - ordered by strong immersions.

A graph is AT - free if it has no asteroidal triple — that is — if it has no three vertices of which every pair is connected by a path that avoids the closed neighborhood of the third.

4.16 Asteroidal sets

Definition 4.186. Let G be a graph. A set $A \subseteq V$ is an asteroidal set if for each vertex $a \in A$ the set $A \setminus \{a\}$ is contained in a component of $G - N[a]$.

Asteroidal sets with 3 vertices are called asteroidal triples. — For example — consider a claw and subdivide every edge one time. The set of leaves of this tree is an asteroidal triple. Another example is an independent set in C_6 (or the simplicials in a 3-sun). Gallai presents a list of the minimal graphs that have an asteroidal triple.

The concept was used by Lekkerkerker and Boland to characterize interval graph in the following manner.

> A graph is an interval graph if and only if it is chordal and has no asteroidal triples.

4.16.1 AT *- free graphs*

In this section we have a look at the structure of graphs that do not have an asteroidal triple. Clearly (by the characterization of Lekkerkerker and Boland) interval graphs are graphs without asteroidal triple. Another example of a class of graphs that are AT-free is the class of permutation graphs.

All complements of comparability graphs are AT-free. To see that, use the fact that cocomparability graphs are intersection graphs of continuous functions $f : [0, 1] \to \mathbb{R}$. When 3 functions pairwise don't intersect then one is between the other two — and so — its closed neighborhood hits every path that runs between the outer two.

4.16.2 Independent set in AT*-free graphs*

Computing ω is NP-complete on AT-free graphs. That is so because α is NP-complete on triangle-free graphs. In this section we show that there is a polynomial algorithm to compute the independence number α on AT-free graphs.

The algorithm computes (recursively) the following numbers.

Exercise 4.111

Let G be a graph. Then

$$\alpha(G) \quad = \quad 1 + \max_{x \in V} \sum_i \alpha(C_i)$$

where the C_i are the components of $G - N[x]$.

Let G be AT-free and let x and y be nonadjacent vertices in G. The <u>interval</u> $I(x, y)$ is the set of all vertices that are between x and y.

Exercise 4.112

Let G be AT-free. Let $x \in V$ and let C be a component of $G - N[x]$.

$$\alpha(C) \quad = \quad 1 + \max_{y \in C} \left\{ \alpha(I(x,y)) + \sum_i \alpha(C_i) \right\}$$

where the C_i are the components of $G - N[y]$ that are contained in C.

THE FINAL STEP IS TO DECOMPOSE THE INTERVALS.

Exercise 4.113

Let G be AT-free and let $I(x,y)$ be an interval in G. When $I = \varnothing$ then $\alpha(I) = 0$. Otherwise

$$\alpha(I) \quad = \quad 1 + \max_{s \in I} \alpha(\, I(x,s)\,) + \alpha(\, I(y,s)\,) + \sum_i \alpha(C_i)$$

where $C_1, \cdots$ are the components of $G - N[s]$ that are contained in I.

Exercise 4.114

Prove the following theorem.

Theorem 4.187. *There exists an* $O(n^4)$ *algorithm to compute the independence number in* AT*-free graphs.*

Exercise 4.115

Show that the computation of the clique number ω is NP - complete on AT - free graphs.

Hint: The independence number α is NP - complete for the class of triangle - free graphs.

4.16.3 Exercise

AT - FREE GRAPHS ARE χ - BOUNDED — that is — there exists a function $\mathsf{f} : \mathbb{N} \to \mathbb{N}$ which satisfies

$$\chi \quad \leqslant \quad \mathsf{f}(\,\omega\,) \qquad \text{for all } \mathsf{AT}\text{ - free graphs.}$$

That is so because AT - free graphs do not contain a subdivision of a claw as an induced subgraph. Kierstead and Penrice showed (in 1994) that the class of graphs without subdivision of a claw is χ - bounded.

Remark 4.188. The Gyárás - Sumner conjecture suggests that for every tree T the class of graphs that do no contain T as an induced subgraph is χ - bounded.

Remark 4.189. WE ARE NOT AWARE of any hereditary class of graphs which is χ - bounded but not polynomially so.

Exercise 4.116

Let $\mathsf{k} \in \mathbb{N}$. Show that there is a polynomial - time algorithm to check if $\chi \leqslant \mathsf{k}$ for AT - free graphs.

A CONFLICT - FREE COLORING of a graph G is a coloring of its vertices such that every closed neighborhood has a uniquely colored vertex. Let $\kappa(\mathsf{G})$ denote the mimimal number of colors needed in a conflict - free coloring of G.

Exercise 4.117

Show that tK_2 - free graphs satisfy $\kappa \leqslant 2 \cdot \mathsf{t} - 1$.

Exercise 4.118

Show that circle graphs satisfy

$$\kappa \quad \leqslant \quad 28 \cdot \omega.$$

Hint: First show that permutation graphs satisfy $\kappa \leqslant 4$. To see that make use of the fact that permutation graphs have a shortest path that is dominating. Davies and McCarty show that the vertex set of a circle graph can be partitioned into 7ω parts that induce permutation graphs.

Exercise 4.119

Prove or disprove: there exists $k \in \mathbb{N}$ such that every circle graph can be colored conflict free with k colors.

Remark 4.190. For any graph H the class of graphs that do not contain H as a vertex minor has bounded rankwidth if and only if H is a circle graph.

J. Geelen, O. Kwon, R. McCarty, and P. Wollan, The grid theorem for vertex - minors. Manuscript on arXiv: 1909.08113, 2020.

4.16.4 Bandwidth of AT*-free graphs*

Definition 4.191. A layout of a graph G is an ordering of its vertices $L : V \leftrightarrow [n]$. The width of L is 0 if $E = \varnothing$ and otherwise it is

$$\max \{ |L(x) - L(y)| \mid \{x, y\} \in E \}.$$

The bandwidth of G is the minimal width of a layout of G. [127]

[127] When a graph has small bandwidth then the rows and columns of its adjacency matrix can be permuted so that all 1s appear in a narrow band around the diagonal.

In this section we prove the following theorem.

Theorem 4.192. *There exists a linear - time algorithm to approximate the bandwidth of* AT*-free graphs with worst - case performance ratio 6.*

TO PROVE THE PERFORMANCE RATIO WE NEED A LOWER BOUND.

Lemma 4.193. *Let* G *be a graph. Let* $\{x, y\} \in E$. *Then*

$$\mathsf{bw}(G) \geqslant \frac{1}{3} \cdot (\,|N(x) \cup N(y)| - 1\,).$$

Proof. Let L be an optimal layout. Assume $L(x) < L(y)$. Consider the three sets S_1, S_2, and S_3:

$$\begin{aligned} S_1 &= \{z \in N \mid L(z) \leqslant L(x)\} \\ S_2 &= \{z \in N \mid L(x) \leqslant L(z) \leqslant L(y)\} \\ S_3 &= \{z \in N \mid L(z) \geqslant L(y)\} \end{aligned}$$

where $N = N(x) \cup N(y)$.

Then

$$\mathsf{bw}(G) \geqslant \max_i |S_i| - 1 \quad \text{and} \quad \sum_i |S_i| = |N| + 2.$$

There must exist i such that $|S_i| - 1 \geqslant \frac{1}{3}(|N| - 1)$.

This proves the lemma. □

Definition 4.194. A caterpillar is a tree with a dominating path.

The vertices of the caterpillar that are not in the dominating path are called the feet of the caterpillar. [128]

[128] These animals should not be confused with (but look similar to) another kind of animals called ‘centipedes’ — that is — ‘100-feet.’ (In Dutch they have a 1000 feet.)

Exercise 4.120

A tree is a caterpillar if and only if it does not contain the tree obtained from a claw by subdividing each edge one time.
Show that a tree is AT-free if and only if it is a caterpillar.

Lemma 4.195. *There exists an* $O(n)$ *algorithm to approximate the bandwidth of caterpillars within a factor* $3/2$.

Proof. Let $[b_1 \cdots b_\ell]$ be the dominating path of a caterpillar T and let d_i be the number of feet attached to b_i. Define

$$L(b_i) \quad = \quad \left\lfloor \frac{d_i}{2} \right\rfloor + \sum_{j<i} d_j.$$

For feet z adjacent to b_i let

$$L(z) \quad \in \quad \left\{ L(b_i) - \left\lfloor \frac{d_i}{2} \right\rfloor, \quad \cdots \quad , L(b_i) + \left\lceil \frac{d_i}{2} \right\rceil \right\}.$$

The width of L is

$$\max_i \; L(b_{i+1}) - L(b_i) \quad = \quad \max_i \; \left\lceil \frac{d_i}{2} \right\rceil + \left\lfloor \frac{d_{i+1}}{2} \right\rfloor.$$

By Lemma 4.193

$$\mathsf{bw}(T) \quad \geqslant \quad \frac{1}{3} \cdot \max_i \; d_i + d_{i+1} + 1.$$

This implies that the width of L is at most $\frac{3}{2} \cdot \mathsf{bw}(T)$.

This proves the lemma. □

Exercise 4.121

Prove the following lemma.

Lemma 4.196. *Let G be a connected AT-free graph. There exists a spanning caterpillar T such that any adjacent pair in G is at distance at most 4 in T. This caterpillar can be found in linear time.* [129]

[129] The caterpillar T has $V(T) = V(G)$ (it spans V). The graph G is a (spanning) subgraph of T^4.

HINT: Use the fact that a connected AT-free graph has a dominating pair — that is — a pair of vertices such that any path that connects them is a dominating path.

WE ARE READY TO PROOF THEOREM 4.192.

Proof. Let G be AT-free. Let T be a spanning caterpillar such that adjacent vertices in G are at distance at most 4 in T. Let L be a layout of T of width at most $\frac{3}{2} \cdot \mathsf{bw}(T)$.

Use L as a layout for G. We have

$$\begin{aligned}\mathsf{width}(\mathsf{G},\mathsf{L}) &\leqslant \mathsf{width}(\mathsf{T}^4,\mathsf{L}) \\ &\leqslant 4\cdot\mathsf{width}(\mathsf{T},\mathsf{L}) \\ &\leqslant 4\cdot\frac{3}{2}\mathsf{bw}(\mathsf{T}) \\ &\leqslant 6\cdot\mathsf{bw}(\mathsf{G}).\end{aligned}$$

This proves the theorem. □

Remark 4.197. There exists an $O(m+n\log n)$ algorithm to compute the bandwidth on caterpillars. Alternatively there exists a $O(n^3)$ algorithm that approximates the bandwidth of AT-free graphs within a factor 2.

The bandwidth problem remains NP-complete on cobipartite graphs (which are AT-free). (For cobipartite graphs the bandwidth equals the treewidth of the graph.)

Another way to approximate the bandwidth of AT-free graphs is via the computation of a minimal triangulation. Let G be AT-free and let H be a minimal triangulation of G. Then the bandwidth of H is at most twice the bandwidth of G. To see that observe that AT-free graphs have no induced C_6. It follows that in any minimal separator S of G two nonadjacent vertices of S have a common neighbor in G. This shows that any two adjacent vertices of H that are not adjacent in G have a common neighbor in G. CONSEQUENTLY $\mathsf{bw}(H)\leqslant 2\cdot\mathsf{bw}(G)$. — Finally — every minimal triangulation of G is an interval graph and there is an $O(n^2)$ algorithm to compute the bandwidth of interval graphs. [130]

[130] D. Kleitman and R. Vohra, *Computing the bandwidth of interval graphs*. SIAM Journal on Discrete Mathematics **3** (1990), pp. 373–375.

Exercise 4.122

A graph is AT-free if and only if every minimal triangulation is an interval graph.

4.16.5 Dominating pairs

A connected graph with at least two vertices and without asteroidal triples has a dominating pair — that is — a pair of vertices s and t with the property that every $s \rightsquigarrow t$ - path in the graph is a dominating set.

4.16.6 Antimatroids

Let V be the set of vertices of a graph G. A betweenness relation in G is a collection of rooted sets $\mathcal{K} = \{ (K, r) \}$ where $K \subseteq V$ and $r \in K$. A betweenness relation $\mathcal{K}$ defines a convexity: a set $C \subseteq V$ is convex if

$$K \setminus r \subseteq C \quad \Rightarrow \quad r \in C$$

for every betweenness $(K, r) \in \mathcal{K}$.

Definition 4.198. Let V be a finite set and let $\mathcal{C}$ be a collection of subsets of V. The set system $(V, \mathcal{C})$ is a convex geometry if

1. $\varnothing \in \mathcal{C}$ and $V \in \mathcal{C}$
2. if $A \in \mathcal{C}$ and $B \in \mathcal{C}$ then $A \cap B \in \mathcal{C}$
3. if $A \in \mathcal{C}$ and $A \neq V$ then there exists $x \in V \setminus A$ such that $A \cup x \in \mathcal{C}$.

Chang et al. proved the following characterization of AT - free graphs.

Theorem 4.199. *There exists a betweenness relation such that the collection of convex sets in a graph is a convex geometry if and only if the graph has no asteroidal triple.*

The betweenness relation consists of rooted sets with three pairwise nonadjacent vertices for which there is a path from the root to each end that avoids the neighborhood of the other end.

When some vertex is between two others then it is one of the following.

1. the nose of a bull
2. a root of a 6 - chain
3. a midpoint of P_5
4. a pendant, adjacent to the midpoint of P_5.

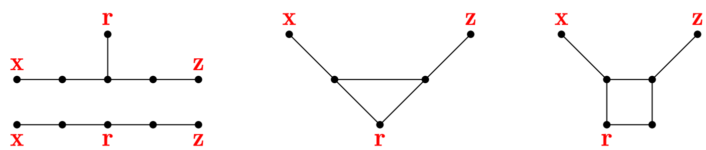


Figure 4.23: The figure shows P_5, P_5 with a pendant, the bull and the 6-chain. The 'root' r is the element of the betweenness that is between the two 'ends' x and z.

An AT - free order is a shelling sequence of the convex geometry; it repeatedly removes vertices from the graph that are not between two others.

Algorithm 9: Compute an AT - free order

$\alpha \leftarrow \varnothing$;

while $\alpha \neq V$ **do**

Choose $x \in V \setminus \alpha$ such that
there is no betweenness (K, x) $K \subseteq V \setminus \alpha$ with root x;

$\alpha \leftarrow \alpha x$

end while

Example 4.200. Consider a shelling of a poset which eliminates elements that have no descendants. The sequences are the words of an antimatroid. Poset antimatroids have a betweenness relation with only two elements — namely — the cover - relation of the poset.

If the poset is a rooted tree then the antimatroid is the collection of elimination orders which remove leaves until there are no more vertices. The betweenness relation is the parent relation of the tree.

Exercise 4.123

Let G be a chordal graph. The collection of simplicial elimination orders of G is an antimatroid. Please describe a concise betweenness relation that defines this antimatroid.

If a graph has no root of a P_5, bull or 6 - chain then it is AT - free and any order of the vertices is an AT - free order.

Lemma 4.201. *Let G be a graph and assume that G is prime with respect to modular decomposition. If G has no induced P_5, bull or 6 - chain then any independent set in G has at most two elements.*

Proof. Assume G is prime and has no induced P_5 or bull. Maffray shows that either G is the complement of a graph without triangles or G has no house (that is the complement of P_5) or C_5. [131]

[131] The refences are listed on Page 294.

Fouquet and Vanherpe show that if a graph is prime and has no C_5, P_5, house or bull then it is a chain graph or the complement of a chain graph. In our case the graph has no 6-chain. This leaves complements of chain graphs (which includes the 4-chain P_4). — In any case — the complements are graphs without triangle. □

4.16.7 Totally balanced matrices

Let H be a hypergraph. Its incidence matrix is the 0/1 - matrix of which the rows are indexed by the vertices of H and the columns are indexed by the hyperedges of H. An entry (x, e) of this matrix is 1 if the vertex x is in the edge e.

Definition 4.202. A hypergraph is totally balanced if the incidence matrix does not contain a submatrix of size at least 3 with no identical columns and with each row sum and column sum equal to 2.

Lemma 4.203. *Let* G *be connected and* AT *- free. Let* $\{s, t\}$ *be a dominating pair and let* P *be a shortest* $s \rightsquigarrow t$ *- path in* G*. Let* H *be the hypergraph with vertex set* $V(G)$ *and the following edges. For each* P_3 *in* P *the union of the closed neighborhoods is an edge of* H*. Then* H *is totally balanced.*

Proof. It is sufficient to to show that the hyperedges are a path decomposition of G — in other words — the graph becomes an interval graph if we make clique of all hyperedges.

Both endpoints of an edge are in the closed neighborhood of a P_3 in P — otherwise there is a cycle of length at least 6.

Clearly each vertex of G is in a consecutive set of hyperedges. □

Remark 4.204. Strongly chordal graphs are chordal graphs without a sun. They have a simple elimination order; that is a simplicial elimination order that avoids taking out the nose of a bull, or a midpoint of P_5, or a pendant to a midpoint of P_5.
A net is a graph that consists of a clique and an independent set both of size at least 3 and a perfect matching between them. A net is the smallest strongly chordal graph that has a nose of a bull between any two of its ends. (Three leaves in a net are an asteroidal triple.)

A vertex is simple if for any two vertices x and y in its closed neighborhood

$$N[x] \subseteq N[y] \quad \text{or} \quad N[y] \subseteq N[x].$$

Exercise 4.124

The simple elimination orders of an interval graph are the words of an antimatroid. Describe a betweenness relation that defines this antimatroid.

Exercise 4.125

A paired dominating set in a graph is a dominating set that has a perfect matching.

1. Show that every graph without isolated vertices has a paired dominating set.

2. Show that there is a greedy algorithm to compute a paired dominating set in AT - free graphs of smallest size.

HINT: Let P be a dominating shortest path. This defines a path - decomposition as in Lemma 4.203. Prove that there is a minimum paired dominating set with a perfect matching of which every edge hits P. (See Figures 4.24, 4.25 and 4.26.)

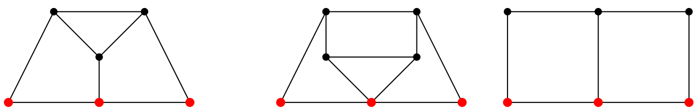


Figure 4.24: The figure shows a betweenness involving a P_3 in P. (The red vertices are vertices of the shortest path P.)

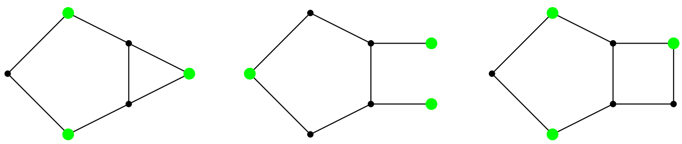


Figure 4.25: The absence of asteroidal triples limit the neighborhood of a C_5

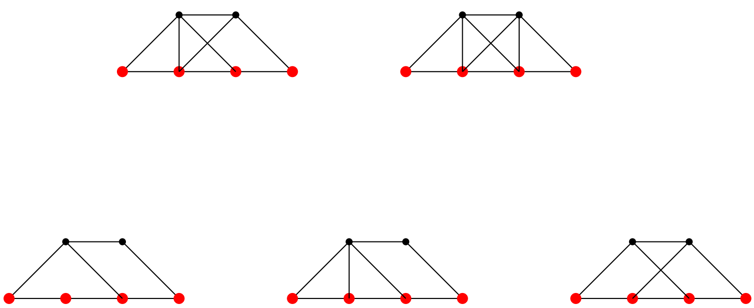


Figure 4.26: The figure shows the connections between two P_4s.

Further reading

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4.16.8 Triangle graphs

Circle graphs are the intersection graphs of chords of a circle.

Elmallah and Stewart introduced the class of k-polygon graphs. These graphs are the intersection graphs of chords in a k-sided polygon. Elmallah and Stewart show that k-polygon graphs can be recognized in polynomial time and that the domination problem can be solved in $O(n^{4k^2+3})$ time on k-polygon graphs.

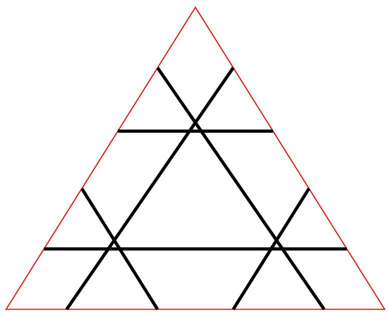

Figure 4.27: The figure shows some chords in a triangle. It is the model of a 3-sun.

Exercise 4.126

Define a betweenness which generates an antimatroid on triangle graphs. Design a greedy algorithm to compute γ on triangle graphs (γ is the domination number).

4.17 Sensitivity

In 2019 Hao Huang proved the sensitivity conjecture!

In this chapter we take a look at the proof.

A decision tree is an algorithm that evaluates a Boolean function $f : \{0,1\}^n \to \{0,1\}$ by a sequence of queries. A query reads one bit x_i of an input $x \in \{0,1\}^n$. The choice of a query depends on the outcome of previous queries.

Let T be a rooted binary, tree. The leaves are labeled as 0 or 1. Internal nodes (including the root) are labeled with variables x_i. Given input $x \in \{0,1\}^n$ the tree is evaluated as follows. If the root is a leaf then output its label 0 or 1. Otherwise query the value of the root variable x_i. If it is 0 then evaluate the left subtree — otherwise — evaluate the right subtree. The tree T 'computes' a Boolean function $f : \{0,1\}^n \to \{0,1\}$ if the algorithm described above gives output $f(x)$ for all $x \in \{0,1\}^n$. The depth of f is the smallest depth of a tree that evaluates f.

The depth of a decision tree is the largest number of queries made by the algorithm to evaluate $f(x)$ (over all $x \in \{0,1\}^n$). The depth $D(f)$ of a Boolean function f is the smallest depth over all decision trees that compute f.

Definition 4.205. Let $f : \{0,1\}^n \to \{0,1\}$ be a Boolean function. The sensitivity of f at input $x \in \{0,1\}^n$ is the number of i's for which a flip of the i^{th} element of x changes the value of $f(x)$. The sensitivity $s(f)$ of f is the largest sensitivity at input x over all $x \in \{0,1\}^n$.

Make sure you understand this definition properly: $\{0,1\}$ is an alphabet. Elements of $\{0,1\}^n$ are words of length n with letters in $\{0,1\}$. For $S \subseteq [n]$ let x^S be the word obtained from $x = x_1 \cdots x_n$ by flipping the value of x_i for $i \in S$. The sensitivity of f at x is the number of $i \in [n]$ for which $f(x) \neq f(x^{\{i\}})$.

In 2019 Hao Huang proved the following theorem.

Theorem 4.206 (The sensitivity theorem)**.** *Let* $f : \{0,1\}^n \to \{0,1\}$ *be a Boolean function. Then* $s(f) \leqslant D(f)$ *and there exists a constant* c *such that*

$$D(f) \quad = \quad O\,(\,s(f)^c\,)\,.$$

Exercise: Show that $D(f) \geqslant s(f)$.

4.17.1 What happened earlier ...

For $x \in \{0,1\}^n$ and $S \subseteq [n]$ let x^S be the word obtained from x by flipping all bits in S.

Definition 4.207. The block sensitivity of a Boolean function f at $x \in \{0,1\}^n$ is the maximal number of disjoint subsets $B \subseteq [n]$ for which $f(x^B) \neq f(x)$. The block sensitity of f is the largest block sensitivity of $f(x)$ over all $x \in \{0,1\}^n$.

Noam Nisan showed (in 1989) the following sandwich

$$s(f) \quad \leqslant \quad bs(f) \quad \leqslant \quad D(f) \quad = \quad O(bs(f)^4).$$

Exercise: Show that $bs(f) \geqslant s(f)$.

Hao Huang proves the following theorem. — Notice — that this proves the sensitivity theorem.

Exercise: Show that this theorem proves the sensitivity theorem.

Theorem 4.208. *For every Boolean function*

$$s(f) \leqslant bs(f) \leqslant s(f)^4.$$

We take a look at the proof — but first — let's do something else.

4.17.2 Cauchy's interlace lemma

Let A be a real symmetric $n \times n$ matrix. Then all eigenvalues are real numbers. A principle submatrix B is a submatrix of A on the same subset of rows and columns. Cauchy's interlace lemma says that the eigenvalues of A and B interlace which is defined as in the lemma.

The eigenvalues of A and B are like shoelaces. Shoelaces 'interlace' (that is; they 'twine') to tie up your shoe.
When $m = n - 1$ then

$$\lambda_1 \geqslant \mu_1 \geqslant \lambda_2 \geqslant \cdots \cdots \geqslant \mu_{n-1} \geqslant \lambda_n$$

Lemma 4.209. *Let A be a real symmetric $n \times n$ - matrix. Let B be a $m \times m$ principal submatrix of A. Let $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of A and let $\mu_1 \geqslant \cdots \geqslant \mu_m$ be the eigenvalues of B. Then for $i \in [m]$:*

$$\lambda_i \geqslant \mu_i \geqslant \lambda_{i+n-m}.$$

4.17.3 Hypercubes

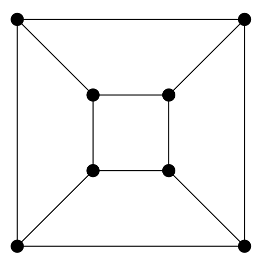

Figure 4.28: A hypercube

For a proof of the following lemma see eg the monograph by Brouwer and Haemers on spectra of graphs.

Lemma 4.210. *The spectrum of the hypercube Q_n consists of the numbers $n - 2i$ with multiplicity $\binom{n}{i}$ for $i = 0 \cdots n$.*

In his paper Huang Huo proves the following theorem.

Exercise: Prove Theorem 4.211 by using Lemma 4.210 and interlacing.

Hint: $\Delta(H) \geqslant \lambda_1(H)$.

Theorem 4.211 (The hypercube theorem)**.** *Let* H *be an induced subgraph of the hypercube* Q_n *with* $2^{n-1}+1$ *vertices. Then the largest degree in* H *satisfies*

$$\Delta(H) \quad \geqslant \quad \sqrt{n}$$

and — this inequality is tight when n *is a square.*

To prove Theorem 4.211 let's start with two exercises.

Exercise 4.127

Define a sequence (A_n) of $\{0,-1,+1\}$ - matrices as follows.

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A_n = \begin{pmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{pmatrix}.$$

Show that the eigenvalues of A_n are $\sqrt{n}$ and $-\sqrt{n}$ both of multiplicity 2^{n-1}.

Hint: A_n satisfies $A_n^2 = n \cdot I$.

If we flip all -1's in A_n to $+1$ we get the adjacency matrix of the hypercube Q_n.

Remark: An $n \times n$ conference matrix C has zeros on the diagonal and $+1$ or -1 everywhere else and satisfies $C^T C = (n-1)I$. Van Lint and Seidel show that a symmetric conference matrix can only exist if $n = 2 \mod 4$ and $n-1$ is a sum of two squares.

Exercise 4.128

Let G be a graph. Let A be a symmetric matrix whose rows and columns are indexed by $V(G)$ and which has entries in $\{0,-1,+1\}$ such that

$$A(x,y) \neq 0 \qquad \Rightarrow \qquad \{x,y\} \in E(G)$$

Then $\Delta(G) \geqslant \lambda_1(A)$. [132]

[132] $\lambda_1(A)$ is the largest eigenvalue of A and $\Delta(G)$ is the largest degree of a vertex in G.

The proof of the hypercube theorem

Proof. Let A_n be the $\{0,-1,+1\}$ - matrix as defined in Exercise 4.127. Change all the -1 - entries in A_n to $+1$. Then the matrix becomes the adjacency matrix of the hypercube Q_n. — So — we may assume that there is a 1-1 correspondence between rows and columns of A_n and Q_n.

Remark: A matrix H is Hadamard if all its entries are $+1$ or -1 and $HH^T = n \cdot I$. When H is hadamard then so is $\left(\begin{smallmatrix} H & H \\ H & -H \end{smallmatrix}\right)$.

Let H be an induced subgraph of Q_n with at least $2^{n-1}+1$ vertices. Let A_H be the principal submatrix of A_n whose rows and columns are indexed by the vertices of H. By Exercise 4.128

$$\Delta(H) \quad \geqslant \quad \lambda_1(A_H).$$

By Cauchy's interlace lemma:

$$\lambda_1(A_H) \quad \geqslant \quad \lambda_{2^{n-1}}(A_n) \quad = \quad \sqrt{n}. \tag{4.22}$$

This proves the theorem. □

It is easy to see that the inequality (4.22) is tight: Let H be the subgraph of Q_n induced by all vertices of even weight and one vertex of odd weight. Then H is a union of the star $K_{1,n}$ and isolated vertices. The largest eigenvalue is $\sqrt{n}$.

There are $\sum_i \binom{n}{2i} = 2^{n-1}$ vertices of even weight. They form an independent set in Q_n. Every vertex in Q_n of odd weight has a neighborhood of size n which is a set of vertices that all have even weight.

4.17.4 Möbius inversion

Let $f : \{0,1\}^n \to \mathbb{R}$ be a map. We show that f can be represented as a polynomial in n variables $x_1, \cdots, x_n$.

The elements of the domain $\{0,1\}^n$ are in 1-1 correspondence with subsets of $[n]$. Let $(P, \leqslant)$ be the poset with $P = 2^{[n]}$ and $\leqslant$ the subset - relation.

Exercise 4.129

Prove Lemma 4.212 below.

Hint: This lemma is 'the principle of inclusion - exclusion.'

Lemma 4.212 (Möbius inversion of the hypercube)**.** *Let* $f : P \to \mathbb{R}$ *be a map and let* $g : P \to \mathbb{R}$ *be defined as follows*

$$g(x) \quad = \quad \sum_{y \leqslant x} f(y).$$

Then

$$f(x) \quad = \quad \sum_{y \leqslant x} g(y) \cdot (-1)^{n(x)-n(y)}, \tag{4.23}$$

where $n(\cdot)$ *denotes the number of elements in the specified subset.*

The right hand - side of Equation 4.23 can be written as a multilinear polynomial in n variables $x_1, \cdots, x_n$: write $x \in P$ as $x = (x_1, \cdots, x_n)$ where each $x_i \in \{0, 1\}$. Then

$$(-1)^{n(x)} \quad = \quad (-1)^{\sum x_i} \quad = \quad \prod (-1)^{x_i} \quad = \quad \prod (1 - 2x_i).$$

In their paper Gotsman and Linial write $x \in P$ as

$$x = (x_1, \cdots, x_n) \quad \text{where} \quad x_i = \begin{cases} -1 & \text{if } i \in x \\ +1 & \text{if } i \notin x. \end{cases}$$

C. Gotsman and N. Linial, The equivalence of two problems on the cube, *Journal of Combinatorial Theory, Series A* **61** (1992), pp. 142–146.

and they rewrite (4.23) as

$$f(x) \quad = \quad \sum_{y \in P} \left(\alpha_y \cdot \prod_{i \in y} x_i \right) \quad = \quad \sum_{y \in P} \alpha_y \cdot (-1)^{n(x \cap y)}. \quad (4.24)$$

When $f : \{-1, +1\}^n \to \{-1, +1\}$ is a Boolean map then all of the 2^n coefficients satisfy $-1 \leqslant \alpha_x \leqslant +1$.

Gotsman and Linial call the coefficient α_x the Fourier transform of f at x.

Definition 4.213. The degree of a Boolean map $f : \{-1, +1\}^n \to \{-1, +1\}$ is

$$\delta(f) \quad = \quad \max\{\, |x| \mid \alpha_x \neq 0 \,\}.$$

4.17.5 The equivalence theorem

In their paper Gotsman and Linial prove the equivalence theorem.

Let G be an induced subgraph of Q_n. Define

$$\Gamma(G) \quad = \quad \max \quad \{\, \Delta(G),\, \Delta(Q_n - G) \,\}.$$

Theorem 4.214 (The equivalence theorem)**.** *Let* $h : \mathbb{N} \to \mathbb{R}$ *be a monotone map. The following two statements are both true or both false.*

1. *if* G *is an induced subgraph of* Q_n *and* $|V(G)| \neq 2^{n-1}$ *then* $\Gamma(G) \geqslant h(n)$

2. *any Boolean function* $f : 2^{[n]} \to \{-1, +1\}$ *satisfies* $s(f) > h(\delta(f))$.

Proof. Identify an induced subgraph G of Q_n with a Boolean function:

$$g(x) = \begin{cases} +1 & \text{if } x \in V(G) \\ -1 & \text{if } x \notin V(G). \end{cases}$$

Exercise 4.130

Show that

$$d_G(x) = n - s(g(x)).$$

Let $E(g) = \frac{1}{2^n} \cdot \sum_{x \in 2^{[n]}} g(x)$.

Exercise 4.131

Show that the statements in the lemma are equivalent with the following pair of statements.

1'. if g is a Boolean function and $E(g) \neq 0$ then there exists $x \in V(Q_n)$ with $s(g(x)) \leqslant n - h(n)$

2'. for any Boolean function f: if $s(f) < h(n)$ then $\delta(f) < n$.

The equivalence of $1.'$ and $2.'$ is shown as follows. Let

$$g(x) = f(x) \cdot \prod_1^n x_i.$$

f and g represent induced subgraphs with vertex sets that partition $V(Q_n)$.

Call the coefficients α_x ($x \in 2^{[n]}$) in (4.24), for the boolean functions f and g: $\hat{f}_x$ and $\hat{g}_x$.

Exercise 4.132

(a) Show that $s(g(x)) = n - s(f(x))$

(b) Show that the coefficients $\hat{f}_x$ and $\hat{g}_x$ of Equation (4.24) for the Boolean functions f and g satisfy

$$\hat{g}_x = \hat{f}_{\bar{x}} \qquad \text{where } \bar{x} = [n] - x$$

(c)

$$E(g) \quad = \quad \hat{g}_{\varnothing} \quad = \quad \hat{f}_{[n]}.$$

<u>$1.' \Rightarrow 2.'$:</u>
Assume $\delta(f) = n$. Then $\hat{f}_{[n]} \neq 0$ — so — $E(g) \neq 0$. By $1.'$ there exists $x \in 2^{[n]}$ such that $s(g(x)) \leqslant n - h(n)$. This implies that there exists $x \in 2^{[n]}$ such that $s(f(x)) \geqslant h(n)$.

<u>$2.' \Rightarrow 1.'$:</u>
Assume that for all $x \in 2^{[n]}$ $s(g(x)) > n - h(n)$. Then $s(f) < h(n)$. By $2.'$ $\delta(f) < n$ — and so —

$$\hat{f}_{[n]} \quad = \quad \hat{g}_{\varnothing} \quad = \quad E(g) \quad = \quad 0.$$

This proves the lemma. □

We omit the proof of the following lemma. It was proved by Tal :

A. Tal, Properties and applications of Boolean function composition *Electronic Colloquium on Computational Complexity*, Report No. 163, 2012.

Lemma 4.215 (Tal)**.** *The block sensitivity and degree of a Boolean functions satisfy*

$$bs(f) \quad \leqslant \quad \delta(f)^2.$$

We'll add a proof later on ... maybe ...

We now prove Theorem 4.208 (on Page 298).

Proof. In the equivalence theorem take $h(n) = \sqrt{n}$. Then the first item (1.) holds true since one of G or $Q_n - G$ has at least $2^{n-1} + 1$ vertices. We conclude

$$s(f) \quad \geqslant \quad \sqrt{\delta(f)}.$$

We obtain (by Tal's lemma) :

$$s(f)^4 \quad \geqslant \quad \delta(f)^2 \quad \geqslant \quad bs(f).$$

This proves Theorem 4.208. □

4.17.6 Further reading

The sensitivity conjecture stems from this paper. The paper shows (Lemma 7) that $\mathsf{bs}(f) \leqslant 2 \cdot \delta(f)^2$. (Tal's lemma removes the factor 2.)

N. Nisan and M. Szegedy, On the degree of Boolean functions as real polynomials, *Computational Complexity* **4** (1994), pp. 301–313.

The following paper makes a probabilistic approach.

J. Bourgain, J. Kahn, G. Kalai, Y. Katznelson and N. Linial, The influence of variables in product spaces, *Israel Journal of Mathematics* **77** (1992), pp. 55–64.

A lot about eigenvalues and about interlacing techniques can be found in the following publications.

Andries E. Brouwer and Willem H. Haemers, *Spectra of Graphs*, Universitext, Springer, 2011.

Willem H. Haemers, Interlacing eigenvalues and graphs, *Linear Algebra and its Applications* 227/228, (1995), pp. 593–616.

The paper below is a classic on Möbius functions.

Gian - Carlo Rota, On the foundations of combinatorial theory: **I** Theory of Möbius functions, *Z. Wahrscheinlichkeitstheorie* **2** (1964), pp. 340–368.

4.18 Homomorphisms

Definition 4.216. Let G and H be two graphs. A homomorphism

$$G \to H$$

is a map $h : V(G) \to V(H)$ with the property

$$\forall_{e \in E(G)} \quad h(e) \in E(H),$$

where — for a set $A \subseteq V(G)$ we write

$$h(A) = \{ h(a) \mid a \in A \}. \qquad (4.25)$$

Thus, a homomorphism is a map that sends edges to edges.[133]

[133] and dust to dust ...

Exercise 4.133

Define $\preceq_{\mathsf{hom}}$ as the quasi–order defined on graphs by

$$G \preceq_{\text{hom}} H \quad \text{if there exists a homomorphism } G \to H.$$

Show that $\preceq_{\mathsf{hom}}$ is not a well quasi–order.
Hint: Show that the sequence of odd cycles

$$C_3, C_5, C_7, \cdots$$

is an infinite <u>antichain</u> —that is— no two elements are comparable under $\preceq_{\mathsf{hom}}$. BTW, how about the even cycles?

Exercise 4.134

Show that for any graph G and $k \in \mathbb{N}$

$$G \to K_k \quad \Leftrightarrow \quad \chi(G) \leqslant k \tag{4.26}$$

$$K_k \to G \quad \Leftrightarrow \quad \omega(G) \geqslant k. \tag{4.27}$$

4.18.1 Retracts

Definition 4.217. Let G and H be graphs. The graph H is a <u>retract</u> of G if there exist homomorphisms [134]

$$\rho : G \to H \quad \text{and} \quad \gamma : H \to G \quad \text{such that} \quad \rho \circ \gamma = \mathrm{id}_H$$

where id_H is the identity map $V(H) \to V(H)$.

[134] The maps ρ and γ are called the retraction and co-retraction.

When H is a retract of G then H is isomorphic to an induced subgraph of G. [135] Since there are homomorphisms in two directions,

[135] However a copy of H in G is not necessarily a retract of G. (When H is a retract then a proper coloring of H extends to a proper coloring of G.)

G and H have the same clique number, chromatic number and odd girth. A graph G retracts to K_k if and only if $\chi(G) = \omega(G) = k$.

For any graph H to check if there is a homomorphism $G \to H$ is polynomial when H is bipartite and it is NP-complete otherwise. It follows that, for any graph H, checking if H is a retract of a graph G is NP-complete, unless H is bipartite. The question whether a graph G has a homomorphism to itself which is not the identity is NP-complete.

4.18.2 Retracts in threshold graphs

Theorem 4.218. *Let G and H be threshold graphs. There exists a linear-time algorithm to check if H is a retract of G.*

Proof. Assume that H is a retract of G and let ρ and γ be the retraction and co-retraction.

Assume that G has a universal vertex, say x_1. Then H must have a universal vertex as well, since a retract of a connected graph is connected. Let y_1 be a universal vertex of H. Let $y_i = \rho(x_1)$. Since ρ is a homomorphism it preserves edges, and since x_1 is universal in G, ρ maps no other vertex of G to y_i. Notice also that $\gamma(y_i) = x_1$ since $\rho \circ \gamma = id_H$ and ρ maps no other vertex to y_i.

Assume that $y_i \neq y_1$. Let $\gamma(y_1) = x_\ell$. Then $x_\ell \neq x_1$ since γ preserves edges and so

$$\{y_1, y_i\} \in E(H) \quad \Rightarrow \quad \{\gamma(y_1), \gamma(y_i)\} = \{x_\ell, x_1\} \in E(G) \quad \Rightarrow \quad x_\ell \neq x_1.$$

Furthermore, since y_1 is universal, γ maps no other vertex of H to x_ℓ. Of course, since $\rho \circ \gamma = id_H$, $\rho(x_\ell) = y_1$.

We claim that y_i is universal in H, and therefore exchangeable with y_1. Assume not and let $y_s \in V(H)$ be another vertex of H not adjacent to y_i. Let $\gamma(y_s) = x_p$. Then $x_p \neq x_1$ since $\rho \circ \gamma = id_H$ and $\rho(x_1) = y_i \neq y_s$. Now, since ρ is a homomorphism,

$$\{x_1, x_p\} \in E(G) \quad \Rightarrow \quad \{\rho(x_1), \rho(x_p)\} = \{y_i, y_s\} \in E(H),$$

which is a contradiction. Therefore, we may assume that $y_i = y_1$. — That is — from now on we assume that

$$\rho(x_1) = y_1 \quad \text{and} \quad \gamma(y_1) = x_1.$$

This proves that, when G is connected then H is a retract of G if and only if $H - y_1$ is a retract of $G - x_1$. By the way, notice that if $|V(H)| = 1$ then H can be a retract of G only if G is an independent set, so this case is easy to check.

Finally, assume that G is not connected. Since G has no induced $2K_2$, all components, except possibly one, have only one vertex. The number of components of H can be at most equal to the number of components of G, since ρ maps components in G to components of H, and $\rho \circ \gamma = id_H$, and so any two components of H are mapped by γ to different components of G.

First assume that H is also disconnected. Let $x_1, \ldots, x_a$ be the isolated vertices of G and let $y_1, \ldots, y_b$ be the isolated vertices of H. Let $\rho(x_i) = y_i$ and $\gamma(y_i) = x_i$ for $i \in \{1, \ldots, b\}$ and let $\rho(x_{b+1}) = \cdots = \rho(x_a) = y_b$. Now, H is a retract of G if and only if $H - \{x_1, \ldots, x_b\}$ is a retract of $G - \{x_1, \ldots, x_a\}$.

If H is connected, with at least two vertices, then let y_1 be a universal vertex and let $\rho(x_1) = \cdots = \rho(x_a) = y_1$. If H is a retract of G then G must have exactly one component with at least two vertices, since G is a threshold graph and ρ is a homomorphism. Let x_u be the universal vertex of that component and define $\rho(x_u) = y_1$ and $\gamma(y_1) = x_u$. In this case, H is a retract if and only if $H - y_1$ is a retract of $G - \{x_1, \ldots, x_a, x_u\}$.

An elimination ordering, which eliminates successive isolated and universal vertices in a threshold graph, can be obtained in linear time.

This proves the theorem. $\square$

4.18.3 Retracts in cographs

In this section we show that the retract - problem is NP-complete on cographs.

Recall that a graph G is perfect when $\omega(G') = \chi(G')$ for every induced subgraph G' of G. By the perfect graph theorem a graph is perfect if and only if it has no odd hole or odd antihole. This implies that cographs are perfect. Perfect graphs are recognizable in polynomial time. For a graph G, when $\omega(G) = \chi(G)$ one can compute this value in polynomial time via Lovász theta function.

Lemma 4.219. *Assume that* $\omega(H) = \chi(H)$*. There is a homomorphism* $G \to H$ *if and only if* $\chi(G) \leqslant \omega(H)$*.*

Proof. Write $\omega = \omega(H) = \chi(H)$. First assume that there is a homomorphism $\phi : G \to H$. There is a homomorphism $f : H \to K_\omega$ since H is ω-colorable. Then $f \circ \phi : G \to K_\omega$ is a homomorphism, and so G has an ω-coloring. This implies that $\chi(G) \leqslant \omega$.

Assume $\chi(G) \leqslant \omega$. There is a homomorphism $G \to K_k$, where $k = \chi(G)$. Since K_k is an induced subgraph of H, there is also a homomorphism $K_k \to H$. This implies that G is homomorphic to H — ie — $G \to H$.

This proves the lemma. □

Corollary 4.220. *When* G *and* H *are perfect one can check in polynomial time whether there is a homomorphism* $G \to H$*.*

Retracts — like general homomorphisms — constitute a transitive relation. We provide a short proof of this for completeness sake.

Lemma 4.221. *Let* A *be a retract of* G *and let* B *be a retract of* A*. Then* B *is a retract of* G*.*

Proof. Let ρ_1 and γ_1 be a retraction and co-retraction from G to A and let ρ_2 and γ_2 be a retraction and co-retraction from A to

B. Since all four maps ρ_1, ρ_2, γ_1 and γ_2 are homomorphisms, the following two maps are homomorphisms as well.

$$\rho_2 \circ \rho_1 : G \to B \quad \text{and} \quad \gamma_1 \circ \gamma_2 : B \to G. \tag{4.28}$$

Furthermore,

$$(\rho_2 \circ \rho_1) \circ (\gamma_1 \circ \gamma_2) = \rho_2 \circ \mathrm{id}_A \circ \gamma_2 = \rho_2 \circ \gamma_2 = \mathrm{id}_B. \tag{4.29}$$

This proves that B is a retract of G. □

Throughout the remainder of this section it is assumed that G and H are cographs. Note that, using the cotree, $\omega(G)$ and $\chi(G)$ can be computed in linear time when G is a cograph.

Lemma 4.222. *Assume* H *is disconnected; denote the components of* H *as*

$$H_1 \quad \cdots \quad H_t.$$

Assume that H *is a retract of a graph* G*. Then there is an ordering of the components of* G*, say* $G_1, \cdots, G_s$ *such that*

(a) $s \geqslant t$*, and*

(b) G_i *retracts to* H_i*, for every* $i \in \{1, \ldots, t\}$*, and*

(c) *for every* $j \in \{t+1, \ldots, s\}$*, there is a homomorphism* $G_j \to H$*.*

Proof. No connected graph has a disconnected retract since the homomorphic image of a connected graph is connected. To see that, notice that a homomorphism $\phi : G \to H$ is a vertex coloring of G, where the vertices of H represent colors. By that we mean that, for each $v \in V(H)$, the pre-image $\phi^{-1}(v)$ is an independent set in G or $\varnothing$. One obtains the image $\phi(G)$ by identifying vertices in G that receive the same color. When G is connected, this 'quotient graph' on the color classes is also connected, which is easy to prove by means of contradiction.

Assume that G retracts to H. Then we may assume that $H_1, \ldots, H_t$ are induced subgraphs of components $G_1, \ldots, G_t$ of G and that each G_i retracts to H_i. For the remaining components G_j, where $j > t$, there is then a homomorphisms $G_j \to H$.

Notice that, for $j > t$, we can check if there is a homomorphism $G_j \to H$ by checking if $G_j \oplus H_k$ retracts to H_k for some $1 \leqslant k \leqslant t$ — or, equivalently (since cographs are perfect) — if $\omega(G_j) \leqslant \omega(H_k)$ for some $1 \leqslant k \leqslant t$. □

Remark 4.223. Assume that we are given, for each pair G_i and H_j whether G_i retracts to H_j or not. Then, to check if G retracts to H, we may consider a bipartite graph B defined as follows. One color class of B has the components of G as vertices and the other color class has the components of H as vertices. There is an edge between G_i and H_j whenever G_i retracts to H_j. To check if G retracts to H, we can let an algorithm compute a maximum matching in B. There is a retraction only if the matching exhausts all components of H and if $\omega(G) = \omega(H)$.

A cocomponent of a graph G is a subset of vertices which induces a component of the complement $\bar{G}$.

Lemma 4.224. *Assume G is connected and assume that G retracts to H. Then H is also connected. Let $G_1, \cdots, G_t$ be the subgraphs of G induced by the cocomponents of G. There is a partition of the cocomponents of H such that the subgraphs of H induced by the parts of the partition can be ordered $H_1, \cdots, H_t$ such that G_i retracts to H_i for $i \in \{1, \cdots, t\}$.*

Proof. Every subgraph G_i of G, induced by a cocomponent, some subgraphs induced by cocomponents of H. Thus the parts of $V(H)$ that are the images of the subgraphs induced by cocomponents of G form a partition of the cocomponents of H. □

Theorem 4.225. *Let G and H be cographs. The problem to decide whether H is a retract of G is NP-complete.*

Proof. We reduce the 3-partition problem to the retract problem on cotrees. The 3-partition problem is the following. Let m and B be integers. Let S be a multiset of $3m$ positive integers, $a_1, \ldots, a_{3m}$. Determine if there is a partition of S into m subsets $S_1, \ldots, S_m$, such that the sum of the numbers in each subset is B. Without loss of generality we assume that each number is strictly between $B/4$ and $B/2$, which guarantees that in a solution each subset contains exactly three numbers that add up to B.

The 3-partition problem is strongly NP-complete, that is, the problem remains NP-complete when all the numbers in the input are represented in unary.

In our reduction, the cotree for the graph H has a root which is labeled as a join-node $\otimes$. The root has $3m$ children, one for each number a_i. For simplicity we refer to the children as a_i, $i \in \{1, \ldots, 3m\}$. Each child a_i has a union node $\oplus$ as the root. The root of each a_i-child has two children, one is a single leaf and the other is a join-node $\otimes$ with a_i leaves. This ends the description of H.

The cotree for the graph G has a join-node $\otimes$ as a root and this has m children. The idea is that each child corresponds with one set of a 3-partition of S. The subtrees for all the children are identical. It has a union-node $\oplus$ as the root. Consider all triples $\{i, j, k\}$ for which $a_i + a_j + a_k = B$. For each such triple create one child, which is the join of three cotrees, one for a_i, one for a_j and one for a_k in the triple. The subtree for a_i is a union of two subtrees. As in the cotree for the pattern H, one subtree is a single leaf, and the other subtree is the join of a_i leaves. The other two subtrees, for the numbers a_j and a_k in the triple are similar.

Let T_H and T_G be the cotrees for H and G as constructed above. Say T_H and T_G have roots r_H and r_G. When the graph H is a retract of G then the a_i-children of r_H are partitioned into triples, such that there is a bijection between these triples, say $\{a_i, a_j, a_k\}$ and a branch in the cotree of G. Each $\oplus$-node which is the root of a child of r_G must have exactly one $\{a_i, a_j, a_k\}$-child that corresponds with the triple. Notice that, by the construction, all subgraphs induced by remaining components of the $\oplus$-node have maximal cliques of size B. Therefore, all other children of the $\oplus$-node are homomorphic to the one child which corresponds to the triple $\{a_i, a_j, a_k\}$.

It now follows from the Lemma above that there is a 3-partition if and only if the graph H is a retract of G.

This completes the proof. $\square$

4.19 Products

Let G and H be graphs. The categorical product (or tensor product) is a graph denoted as $G \times H$. [136] The vertices of $G \times H$ are

$$V(G \times H) \quad = \quad \{\,(g, h) \mid g \in V(G) \quad \text{and} \quad h \in V(H)\,\}.$$

Two vertices — say (g_1, h_1) and (g_2, h_2) — are adjacent if

$$\{g_1, g_2\} \in E(G) \quad \text{and} \quad \{h_1, h_2\} \in E(H).$$

[136] The categorical product also goes by the name of 'tensor product' or 'direct product', 'Kronecker product' and more. This is just one way to define a graph product.

Hedetniemi made the following CONJECTURE (some 50 years ago). For any two graphs G and H

$$\chi(G \times H) \quad = \quad \min\{\,\chi(G),\, \chi(H)\,.\} \tag{4.30}$$

The right hand - side is an upperbound. To see that let f be a coloring of G. Let f' be defined as

$$\text{for} \quad g \in V(G) \quad \text{and} \quad h \in V(H): \quad f'(g, h) \quad = \quad f(g).$$

Then f' is a proper coloring of $G \times H$.

The 'fractional version' of Hedetniemi's conjecture is true: X. Zu, *The fractional version of Hedetniemi's conjecture is true*, European Journal of Combinatorics **32** (2011), pp. 1168–1175.

Yaroslav Shitov produced a COUNTEREXAMPLE to the conjecture in 2019.

Y. Shitov. Counterexamples to Hedetniemi's conjecture. Manuscript on ArXiv: 1905:02167, 2019.

Theorem 4.226. *When* G *and* H *are perfect then* (4.30) *holds true.*

Proof. CLEARLY [137]

$$\omega(G \times H) \quad \geqslant \quad \min\{\,\omega(G),\, \omega(H)\,\}.$$

[137] Exercise !

When we assume that G and H are perfect then

$$\chi(\mathsf{G}) = \omega(\mathsf{G}) \quad \text{and} \quad \chi(\mathsf{H}) = \omega(\mathsf{H}).$$

The claim easily follows from the following observation.

$$\chi(\mathsf{G}\times\mathsf{H}) \geqslant \omega(\mathsf{G}\times\mathsf{H}) \geqslant \min\{\omega(\mathsf{G}), \omega(\mathsf{H})\} = \min\{\chi(\mathsf{G}), \chi(\mathsf{H})\} \geqslant \chi(\mathsf{G}\times\mathsf{H}).$$

This proves the theorem. □

4.19.1 Categorical products of cographs

Much less is known about the independence number in the categorical product of graphs. Clearly we have that [138]

$$\alpha(\mathsf{G}\times\mathsf{H}) \geqslant \max\{\alpha(\mathsf{G})\cdot|\mathsf{V}(\mathsf{H}), \alpha(\mathsf{H})\cdot|\mathsf{V}(\mathsf{G})|,\} \qquad (4.31)$$

but this lower bound is not sharp not even for threshold graphs. [139]

[138] Exercise !

[139] P. Jha and S. Klavžar, *Independence in direct - product graphs*, Ars Combinatoria **50**, 1998.

Cographs are perfect. But the product of two cographs is not necessarily perfect. As an example let G be isomorphic to the paw (see Figure 2.8 on Page 80) and let H be isomorphic to K_3. Then $\mathsf{G}\times\mathsf{H}$ contains C_5 as an induced subgraph. [140]

[140] Check !

Ravindra and Parthasarathy showed that $\mathsf{G}\times\mathsf{H}$ is perfect if and only if one of the following holds. [141]

1. G or H is bipartite
2. G and H contain no odd holes and no paws.

[141] G. Ravindra and K. Parthasarathy, *Perfect product graphs*, Discrete Mathematics **20** (1977), pp. 177–186.

Exercise 4.135

(a) Let G and H be complete multipartite. Then $\mathsf{G}\times\mathsf{H}$ is perfect.

(b) If G and H are complete multipartite then

$$\alpha(G \times H) \quad = \quad \max\{\,\alpha(G) \cdot |V(H)|,\ \alpha(H) \cdot |V(G)|\,\}.$$

Exercise 4.136

Let G and H be cographs and assume that G is diconnected. Say $G = G_1 \oplus G_2$ (G is the union of G_1 and G_2). Show that

$$\alpha(G \times H) \quad = \quad \alpha(G_1 \times H) + \alpha(G_2 \times H).$$

Exercise 4.137

Let G and H be cographs and assume that both are connected. Say $G = G_1 \otimes G_2$ and $H = H_1 \otimes H_2$ (G and H are joins of the constituents). Show that

$$\alpha(G \times H) \quad = \quad \max\{\,\alpha(G_1 \times H),\ \alpha(G_2 \times H),\ \alpha(G \times H_1),\ \alpha(G \times H_2)\,\}.$$

WE CAN NOW LEAVE THE PROOF OF THE FOLLOWING THEOREM AS AN EXERCISE.

Theorem 4.227. *There exists an* $O(n^2)$ *algorithm to compute* $\alpha(G \times H)$ *when* G *and* H *are cographs.*

Exercise 4.138

Show that there is a polynomial - time algorithm to compute the independence number of $G \times H$ when G and H are splitgraphs.

4.19.2 Tensor capacity

Definition 4.228. The independence ratio of a graph G is defined as

$$r(G) \quad = \quad \frac{\alpha(G)}{|V(G)|}.$$

Exercise 4.139

Show that
$$r(G \times H) \quad \geqslant \quad \max\{r(G),\, r(H)\}.$$

HINT: Use (4.31).

WRITE G^k FOR $G \times \cdots \times G$ where G is $k-1$ times multiplied by itself. Notice that $r(G^k)$ is non-decreasing and it is at most 1. Therefore $\lim_{k\to\infty} r(G^k)$ exists. Call this limit the tensor capacity of the graph.

Definition 4.229. Let G be a graph. The tensor capacity of G is
$$\Theta(G) \quad = \quad \lim_{k\to\infty} r(G^k).$$

It can be shown that the computation of the tensor capacity is NP-complete.

Let G be a graph. Define
$$a(G) \quad = \quad \max \frac{|I|}{|I| + |N(I)|}$$
where I varies over the independent sets in G. Define
$$a^*(G) = \begin{cases} a(G) & \text{if } a(G) \leqslant 1/2 \\ 1 & \text{if } a(G) > 1/2. \end{cases}$$

Tóth proved the following theorem. [142]

[142] Á. Tóth, *Answer to a question of Alon and Lubetzky about the ultimate categorical independence ratio*. Manuscript on arXiv: 1112.6172, 2011.

Theorem 4.230.
$$\Theta(G) \quad = \quad a^*(G).$$
Equivalently for any graph
$$a^*(G^2) \quad = \quad a^*(G).$$

Let G and H be graphs. Tóth showed that
$$\Theta(G \oplus H) = \Theta(G \times H) = \max\{\Theta(G),\, \Theta(H)\}.$$

Theorem 4.231. *There exists a polynomial - time algorithm to compute the tensor capacity for cographs.*

Proof. By Tóth's result, it is sufficient to show that $a(G)$ can be computed.

Consider a cotree. For each node in the cotree the algorithm computes a table with numbers $\ell(k)$

$$\ell(k) \quad = \quad \min\{\, |N(I)| \mid I \text{ is an independent set and } |I| = k\,\}.$$

The value $a(G)$ is then obtained from the table at the root as

$$a(G) \quad = \quad \max_k \frac{k}{k + \ell(k)}.$$

Assume that G is disconnected — say $G = G_1 \oplus G_2$. Let ℓ_1 and ℓ_2 denote the tables for G_1 and G_2. Then

$$\ell(k) \quad = \quad \min\{\, \ell_1(k_1) + \ell_2(k_2) \mid k_1 + k_2 = k\,\}.$$

Let G be connected — say $G = G_1 \otimes G_2$. In that cae we have

$$\ell(k) \quad = \quad \min\{\, \ell_1(k) + |V(G_2)|, |V(G_1)| + \ell_2(k)\,\}.$$

This proves the theorem. □

Exercise 4.140

Show that there is an $O^*(3^{n/3})$ algorithm to compute the tensor capacity of a graph.

HINT: Use Moon and Moser.

Remark 4.232. It can be shown that the tensor capacity can be computed in time $O(3^{k+1} \cdot n^3)$ for graphs of treewidth at most k.

It is NP-complete to determine $\alpha(G \times K_4)$ when G is a planar graph of maximal degree 3.

4.19.3 Cartesian products

THE CARTESIAN PRODUCT $G \square H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and (g_1, h_1) adjacent to (g_2, h_2) if

$$\begin{aligned} g_1 &= g_2 \quad \text{and} \quad \{h_1, h_2\} \in E(H) \quad \text{or} \\ h_1 &= h_2 \quad \text{and} \quad \{g_1, g_2\} \in E(G). \end{aligned}$$

Independence domination

Definition 4.233. Let G be a graph. A set $B \subseteq V$ dominates a set $A \subseteq V$ when

$$A \subseteq \bigcup_{x \in B} N[x].$$

The minimal cardinality of a set B that dominates a set A is denoted as $\gamma(A)$.

Definition 4.234. The independence domination number of a graph G is

$$\gamma^i(G) = \max\{\gamma(A) \mid A \text{ an independent set}\}.$$

CLEARLY — for any graph $\gamma \geqslant \gamma^i$.

Here $\gamma = \gamma(G)$ is the domination number of G: it is the smallest cardinality of a dominating set D — a set D that satisfies $N[x] \cap D \neq \varnothing$ for every $x \in V$.

Vizing's CONJECTURE states

$$\gamma(G \square H) \geqslant \gamma(G) \cdot \gamma(H).$$

Aharoni and Szabó proved in 2009 that Vizing's conjecture holds true for chordal graphs. — Furthermore — they show that for all graphs G and H

$$\gamma(G \square H) \geqslant \gamma^i(G) \cdot \gamma(H) \quad \text{and} \quad \gamma^i(G \square H) \geqslant \gamma^i(G) \cdot \gamma^i(H).$$

A graph is chordal if it has no induced cycle of length more than 3. Computing γ for chordal graphs is NP-complete.

PROGRESS towards proving the conjecture was made by Suen and Tarr in 2012. They proved

$$\gamma(G \Box H) \geqslant \frac{1}{2} \cdot \gamma(G) \cdot \gamma(H) + \min\{\gamma(G), \gamma(H)\}.$$

4.19.4 Independence domination in cographs

WHEN G IS A COGRAPH it is either the join or the union of two cographs — say

$$G = G_1 \otimes G_2 \quad \text{or} \quad G = G_1 \oplus G_2.$$

Exercise 4.141

When G is a cograph with at least two vertices then

$$\gamma(G) = \begin{cases} \min\{\gamma(G_1), \gamma(G_2), 2\} & \text{if } G = G_1 \otimes G_2 \\ \gamma(G_1) + \gamma(G_2) & \text{if } G = G_1 \oplus G_2. \end{cases}$$

Exercise 4.142

When G is a cograph then $\gamma^i(G)$ is the number of components of G.

Exercise 4.143

Let $k \in \mathbb{N}$. Design a polynomial - time algorithm to compute γ^i for graphs of rankwidth $\leqslant k$.

Wing-Kai Hon, T. Kloks, H. Liu, S. Poon and Yue-Li Wang, *On independence domination*. Manuscript on arXiv: 1304.6450, 2013.

Exercise 4.144

Show that there is a polynomial - time algorithm to compute γ^i for permutation graphs.

4.19.5 $\theta_e(K_n \times K_n)$

EVERY GRAPH IS THE INTERSECTION GRAPH OF A COLLECTION OF SUBSETS OF A SET U. By that we mean that every vertex is represented by a subset of U and two vertices are adjacent precisely when the subsets have a nonempty intersection.

For example interval graphs are intersection graphs.

Definition 4.235. For a graph G let θ_e denote the minimal size of a set U such that G is the intersection graph of a collection of subsets of U.

Show that θ_e is finite for any graph. Hint: Number the edges of the graph $e_1, \cdots, e_m$. For each vertex x let $S_x = \{ j \mid x \in e_j \}$.

For the reason given below the parameter θ_e is called the edge clique cover - number of the graph.

Exercise 4.145

Show that θ_e is the minimal size of a set of cliques that has the property that every edge is contained in at least one of them.

The tensor product $K_n \times K_n$

The tensor product $K_n \times K_n$ has the following set of vertices

$$V = \{ (i,j) \mid i \in [n] \quad \text{and} \quad j \in [n] \}.$$

Two vertices (i,j) and (k,ℓ) are adjacent if and only if

$$i \neq k \quad \text{and} \quad j \neq \ell.$$

Exercise 4.146

Show that any graph satisfies

$$\theta_e \geqslant m/\tbinom{\omega}{2}.$$

Corollary 4.236.

$$\theta_e(K_n \times K_n) \quad \geqslant \quad n(n-1).$$

Exercise 4.147

Let p be prime and let $u \leqslant p$. Show that

$$\theta_e(K_p \times K_u) \quad = \quad p(p-1).$$

The following theorem characterizes those $n \in \mathbb{N}$ for which $K_n \times K_n$ has an edge clique - cover with $n(n-1)$ cliques.

Theorem 4.237. $\theta_e(K_n \times K_n) = n(n-1)$ *if and only if there exists a projective plane of order* n.

We omit the proof. The theorem is reminiscent of a result of De Bruin and Erdős concerning $\theta_e(K_n)$.

Wing-Kai Hon, Ton Kloks, Hsiang-Hsuan Liu and Yue-Li Wang, *Edge clique - covers of the tensor product*, Theoretical Computer Science **607** (2015), pp. 68–74.

Definition 4.238. Let $n \in \mathbb{N}$. A projective plane of order n is a set of n^2+n+1 points and n^2+n+1 lines such that [143]

P1. every line has $n+1$ points on it

P2. every point is on $n+1$ lines

P3. any two lines intersect in exactly one point

P4. any two points lie on exactly one line.

[143] Lines are sets of points. We say that a point lies on a line if the line contains it.

The Fano plane is a projective plane of order two. A projective plane of order 6 does not exist. The case $n = 10$ was ruled out by computer calculations. The existence of a projective plane of order 12 is open. There exists a projective plane of order n when n is a prime power.

Corollary 4.239. *For every n which is the power of a prime number*

$$\theta_e(K_n \times K_n) \quad = \quad n(n-1).$$

Exercise 4.148

For $n \geqslant 2$

$$\theta_e(K_n \times K_n) \quad \leqslant \quad (2n-1)\cdot(2n-2).$$

HINT: Let p be the smallest prime $\geqslant n$. Bertrand's postulate says that $p \leqslant 2n-1$.

Remark 4.240. When the Riemann hypothesis holds true then

$$\lim_{n\to\infty} \frac{\theta_e(K_n \times K_n)}{n(n-1)} \quad = \quad 1.$$

4.20 Outerplanar Graphs

An embedding of a graph G in the plane is a drawing of G such that no two edges intersect. A plane graph is already embedded.

The maximal regions — bounded by the edges of the graph — are called faces. The unbounded region is unique, and it is called the outerface .

Definition 4.241. A planar graph is outerplanar if it can be embedded in the plane so that all its vertices lie on the same face. Customarily, this face is called the exterior.

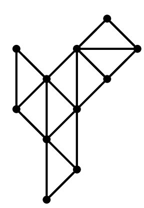

Figure 4.29: A MOP

Exercise 4.149

Show that the class of outerplanar graphs is closed under taking minors. The obstruction set is

$$\mathcal{O} = \{K_4, K_{2,3}\}.$$

Exercise 4.150

A recursive definition of a MOP is the following. [144]

(i) A graph consisting of a single edge is a MOP,

(ii) If G is a MOP, then a new MOP is constructed by adding a vertex and making it adjacent to the endpoints of an edge that is not a minimal separator of G.

A maximal outerplanar graph is an outerplanar graph with an inclusion – wise maximal set of edges. — Thus — adding an edge destroys the outerplanarity.

Lemma 4.242. *Any outerplanar graph has treewidth at most 2.*

Proof. Any outerplanar graph embeds in a MOP. A maximal outerplanar graph has treewidth at most two. □

4.20.1 k – Outerplanar Graphs

A parametrization of the class of all planar graphs is obtained by partitioning its vertices into outerplanar layers.

Definition 4.243. Let G be a plane graph. Its layers, say

$$L_1, L_2, \cdots$$

form a partition of $V(G)$ where L_i is the set of vertices in the outerface of

$$G - \bigcup_{j=1}^{i-1} L_j .$$

The graph G is k – outerplanar if it has a plane embedding with at most k nonempty layers. [145]

Bodlaender proved the following generalization of Lemma 4.242.

Lemma 4.244. *A k – outerplanar graph has treewidth* $3k-1$. [146]

[144] Show that every MOP is Hamiltonian.

[145] Computing the (smallest) outerplanarity of a graph is NP-complete.

[146] See Figure 4.29. Contract the outerface to a MOP, such that each component of the remaining layers is contained in a triangle. Continue this process for the remaining layers. This adds one triangle per layer and so, the clique number is bounded by 3k.

4.20.2 Courcelle's Theorem

Courcelle proved in 1990 the following theorem.

Theorem 4.245. *Any problem that can be formulated in* MS_2 *can be solved in linear time for graphs of bounded treewidth.*

Courcelle's theorem — as above — is based on Bodlaender's linear-time algorithm to recognize graphs of treewidth at most k. The class of graphs that have treewidth at most k is minor-closed and characterized by a finite obstruction set — say $\mathcal{T}_k$. This implies that bounded treewidth can be formulated in monadic second-order logic. By Theorem 4.245, $\mathsf{tw}(G) \leqslant k$ can be tested in linear time for all $k \in \mathbb{N}$. [147]

[147] Bodlaender's algorithm constructs an embedding; for that, the obstruction set is not needed.

4.20.3 Approximations for Planar Graphs

Baker showed that many optimization problems on planar graphs can be approximated — to an arbitrary degree of accuracy — by a linear-time algorithm.

This elegant method is best explained via an example.

4.20.4 Independent Set in Planar Graphs

To compute α for planar graphs is NP-complete. [148] Baker's method provides an efficient approximation scheme.

[148] On the other hand, to compute $\omega(G)$ is polynomial.

Theorem 4.246. *For every* $k \in \mathbb{N}$ *there exists a linear-time algorithm that approximates* $\alpha(G)$ *— for planar graphs — within a factor* $\frac{k}{k+1}$.

Proof. Let G be a plane graph. Number its layers consectively as

$$L_1, L_2, \cdots \quad \text{where } L_1 \text{ is the outerface.}$$

For $1 \leqslant i \leqslant k$ define the graph G_i as the graph induced by

$$\bigcup_{j \neq i \bmod k} L_j .$$

The k graphs G_i are k-outerplanar since the missing layers form separators. Thus — by Theorem 4.245 — the independence numbers $\alpha(G_i)$ can be computed in linear time.[149]

Let M be a maximum independent set in G. Then

$$\alpha(G) = \sum_i L_i \cap M \tag{4.32}$$

$$= \frac{1}{k-1} \cdot \sum_{i=1}^{k} \sum_{j \neq i \bmod k} L_j \cap M \tag{4.33}$$

$$\leqslant \frac{1}{k-1} \cdot \sum_{i=1}^{k} \alpha(G_i) \tag{4.34}$$

$$\leqslant \frac{1}{k-1} \cdot k \cdot \max\{\alpha(G_i) \mid 1 \leqslant i \leqslant k\} \tag{4.35}$$

$$\Rightarrow \max_i \alpha(G_i) \geqslant \frac{k-1}{k} \cdot \alpha(G). \tag{4.36}$$

This proves the theorem.[150] □

[149] The problem of the independence number can be formulated in monadic second-order logic.

[150] Dash it all! I should've started with k + 1! I *always* do that!

4.21 Graph isomorphism

COMING SOON!

We regret that we can not present the beautiful graph isomorphism test of Lásló Babai in this book.

Must - reads on graph isomorphism

Grohe, M. and D. Neuen, Recent advances on the graph isomorphism problem. Manuscript on ArXiv: 2011.01366, 2021.