Research on Solutions Stability for Dynamic Switched Time-Delayed Systems

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Abstract In this paper, we study the stability for switched systems using linear differential subsystems with time delays. We have used Lyapunov functions to study our results. The results are new in the literature.

Keywords Common Lyapunov function, Uniformly asymptotically stability · Delay systems · Switched systems

1 Introduction

We understand switched time-delayed systems as systems described by a set of differential equations with constant time delay that function on the finite time intervals, switching while maintaining continuity and again by differential equations with time delay [1–4].

Functioning of that system is described by the set of equations

$$\dot{x}(t) = f_k(x(t), x(t-\tau), t), k \in K, t_k < t < t_{k+1}, x(t_k + 0) = g_k(x(t_k - 0), t_k - 0).$$

Let us assume that the value of delay is lower than the functioning time of the subsystem of this kind, so the switched systems has solutions conformant to the uniqueness and continuously dependent on initial conditions.

Definition 1.1 Zero solution of the switched system is called stable by Lyapunov if for an arbitrary solution $x(t)$ for any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for an arbitrary solution $x(t)$, the equation $|x(t)| < \varepsilon, t > t_0$, is correct whenever $|x(t_0)| < \delta(\varepsilon)$.

Definition 1.2 Zero solution for switched system is called asymptotically stable if it is stable by Lyapunov and $\lim\limits_{t \to +\infty} |x(t)| = 0$.

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2 Stability of the Solutions for Switched Systems with Linear Differential Subsystems with Time Delay

Let us review the usage of the Lyapunov common function method during the investigation into the stability of the switched systems described by the differential subsystems preserving the continuity in the switching points.

Let $S(A, B) = \{S_i(A_i, B_i), i \in N\}$ be the set of the dynamical subsystems $S_i(A_i, B_i)$, which are systems of linear differential equations [5–7]

$$\dot{x}_i(t) = A_i x(t) + B_i x(t - \tau), i \in N_i,$$

functioning over the time intervals $t \in T_i$, $T_i : t_{i-1} \leq t < t_i$. At the moment $t = t_i$, the switching to $i + 1$ subsystem occurs

$$\dot{x}(t) = A_{i+1} x(t) + B_{i+1}(t - \tau), i \in N_1.$$

And the functioning of this subsystem while preserving the continuity condition occurs on the interval $t_i \leq t < t_{i+1}$. Further dynamic process occurs likewise.

It is said that the solution $x(t) \equiv 0$ of a differential switched system $S(A, B)$ is stable if for an arbitrarily set systems $S_i(A, B)$, $i = 0, 1, 2, \ldots$ and time intervals T_s : $t_s \leq t < t_{s+1}$, $s = 0, 1, 2, \ldots$ for an arbitrary $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that for any solution of the system $S(A, B)$ with initial conditions $|x(s)| < \delta(\varepsilon)$, $-\tau \leq s \leq 0$ with $t > 0$, the condition $|x(t)| < \varepsilon$ is met. Furthermore, if $\lim\limits_{t \to +\infty} |x(t)| = 0$, then zero solution will be asymptotically stable.

Let us define the conditions for zero solution of a system $S(A, B)$ be asymptotically stable. Investigation into the stability will be carried out by the method of the Lyapunov function and the Razumikhin condition. There is the following result.

Theorem 2.1 *For a zero solution of a differential switched system $S(A, B)$ to be uniformly asymptotically stable, it is enough for all its subsystems $S_i(A_i, B_i)$ that the common Lyapunov function should exist.*

Let us see the switched systems described only by subtractional equations. Let $S(C) = \{S_i(C), i \in N\}$ be a system consisting of a set of subtractional subsystems

$$S_i(C) : x(k + 1) = C_i x(k),$$

which function over the integer intervals $T_i = \left[k_{i-1} + 1, k_{i-1} + 2, \ldots, k_i\right]$. Subtractional switched system $S(C)$ is a dynamic system which is composed of a system of subtractional equations functioning over the intervals T_i, $i = 0, 1, 2, \ldots N_2$.

We can say that the solution $x(k) \equiv 0$ of a subtractional switched system $S(C)$ is stable on switchings, if for an arbitrary preset subsystems $S_i(C)$, time intervals T_i and arbitrary $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that for an arbitrary solution of a $S(C)$ system with starting conditions $|x(0)| < \delta(\varepsilon)$, while $k = 1, 2, 3, \ldots$, the condition

$|x(k)| < \varepsilon$ will be met. If, additionally, $\lim\limits_{k\to+\infty} |x(k)| = 0$, then zero solution will be asymptotically stable.

Theorem 2.2 *For a zero solution $x(k) \equiv 0$ of a switched system $S(C)$ to be uniformly asymptotically stable, it is enough that for all its subsystems $S_i(C)$, the common Lyapunov function should exist.*

Finally, let us consider the system $S(A, B, C) = \{S_i(A, B), S_j(C), i \in N_1, j \in N_2\}$, which consists of a set of subsystems $S_i(A, B)$, which are systems of linear differential equations with time delay [4],

$$\dot{x}(t) = A_j x(t) + B_i x(t-\tau), t \in N_j,$$

functioning over time intervals $T_i : t_i \leq t < t_{i+1}$, $i = 0, 1, 2, ...,$ and system of subtractional equations

$$x(t_j + 0) = B_j x(t_j - 0), j \in N_2.$$

At moments of time $t = t_j$, the switchings occur due to subtractional subsystems principles.

The stability conditions for a zero solution of a combined system $S(A, B) = \{S_i(A), S_j(B), i \in N_1, j \in N_2\}$ have a more strict form.

Theorem 2.3 *For a zero solution of subtractional switched system $S(A, B, C)$ to be asymptotically stable it is enough that for its differential and subtractional subsystems, a common Lyapunov functions should exist $V_{dif}(x)$, $V_{ras}(x)$. It is also necessary to have monotonic decrease at the break points*

$$V_{dif}(x(t_k - 0)) > V_{ris}(x(t_k - 0)) > V_{ris}(x(t_k + 0)) > V_{dif}(x(t_k + 0)).$$

Notice 2.1 It is very difficult to verify the condition formulated in Theorem 2.3. Therefore, we should formulate another, more strict but easier to verify stability condition.

Theorem 2.4 *For a zero solution of a switched system $S(A, B, C)$ to be asymptotically stable, it is enough that for its differential and subtractional subsystems, a common Lyapunov function $V_{obsh}(x)$ should exist.*

Let us consider the constructive conditions of a time-delayed switched system stability. It is known [5] that for a time-delayed linear systems

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \tag{1}$$

there are the following stability conditions.

Theorem 2.5 *Let $A + B$ is an asymptoticalle stable matrix and there exists positively defined matrix H such that the following is true*

$$\lambda_{\min}(C) - 2\,|HB|\left(1 + \sqrt{\varphi(H)}\right) > 0, \varphi(H) = \lambda_{\max}(H)/\lambda_{\min}(H),$$

where $\lambda_{\max}(H)$, $\lambda_{\min}(H)$—extremal eigenvalues of a symmetrical positively defined matrix H, which is a solution for the Lyapunov matrix equation

$$(A + B)^T H + H(A + B) = -C, \tag{2}$$

for any positively defined matrix C. Then zero solution for time-delayed system (2) is asymptotically stable for any time delay $\tau > 0$.

Theorem 2.6 *Let $A + B$ be asymptotically stable matrix. Then for $\tau < \tau_0$,*

$$\tau_0 = \frac{\lambda_{\min}(C)}{2\,|HB|\,(|A| + |B|)\sqrt{1 + \varphi(H)}}$$

time-delayed system (2) will also be asymptotically stable.

On the grounds of the aforementioned auxiliary statements, we shall formulate the stability conditions for switched systems, whose differential part is described by the linear time-delayed systems in the form of (2).

Theorem 2.7 *Let there be symmetrical positively defined matrix H, for which matrices $C_i = -(A_i + B_i)^T H + H(A_i + B_i)$, $i \in N_i$, are also positively defined and the following equation is true*

$$\lambda_{\min}(C_i) - 2\,|HB_i|\left(1 + \sqrt{\varphi(H)}\right) > 0, i \in N_i.$$

Then the switched system $S(A, B)$ will be asymptotically stable with any time delay $\tau > 0$.

We shall get the stability conditions dependent on time delay.

Theorem 2.8 *Let there be symmetrical positively defined matrix H, for which matrices $C_i = -(A_i + B_i)^T H + H(A_i + B_i)$, $i \in N_i$, are also positively defined. Then the switched system $S(A, B)$ will be asymptotically stable with time delay $\tau < \tau_0$,*

$$\tau_0 = \min\left\{\frac{\lambda_{\min}(C_i)}{2\,|HB_i|\,(|A_i| + |B_i|)\sqrt{1 + \sqrt{\varphi(H)}}}\right\}, i \in N_i.$$

References

1. Azbelev N.V., Maksimov, Rakhmatullina L.F. Introduction to the theory of functional-differential equations. – M.: Nauka, 277 s
2. Samojlenko A.N., Perestryuk N.A. Differential equations with impulse action. – K., Vy'shhya shkola, 1987. – 286 s

3. Branicky M.S., Hebbar R. Fast marching for hybrid control. In: Proceedings of the 38th IEEE Conference on Decision and Control (1999)
4. Berezansky, L., Idels, L., Troib, L.: Global dynamics of the class on nonlinear nonautonomous systems with time-varying delays. Nonlinear Anal. **74**(18), 7499–7512 (2011)
5. Khusainov D.Ya., Shaty'rko A.V.: Lyapunov's method of functions in the study of stability of differential functional systems. - Kiev, Izd.-vo Kievskogo universiteta (1997). - 236 s
6. Hale, J.K., Verduyn Lunel Sjoerd M. Introduction to Functional Differential Equations. Springer, New-York-Berlin-Ytidelberg-London-Paris (1991)
7. Kolmanovski V., Myshkis A.: Applied Theory of Funktional Differential Equations. Kluver Academic Publishers, Dordrecht-Boston-London (1992)