Periodic Solutions for a Class of Impulsive Delay Differential Equations

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Abstract We study two coupled linear delay differential equations (DDEs) with additive impulses at regular time intervals. The equations are transformed to a DDE coupled to an ODE. Conditions are found for positive periodic solutions, and some examples are given for periodic solutions and for non-periodic solutions.

Keywords Delay differential equations · Impulses · Periodic solutions

1 Introduction

Periodic solutions to delay differential equations (DDE) have been studied by analogy to Floquet theory of ODE [1], by lower and upper solutions [2], by Lyapunov's second method and the contraction mapping principle [3], or by fixed point arguments [4–6]. In this work, we use the results of [4] to investigate the conditions for periodic solutions for the following linear DDE with impulses:

$$\frac{d}{dt}\mathbf{x}(t) + \mathbf{A}(t) \cdot \mathbf{x}(t) + \mathbf{B}(t) \cdot \mathbf{x}(t - r) = 0 \qquad (t \geq t_{in}) \tag{1}$$

$$\mathbf{x}\left({t_k}^+\right) - \mathbf{x}\left(t_k\right) = \mathbf{I}_{(k)} \qquad t_k = t_0 + k \cdot T \qquad (k \in N) \tag{2}$$

where the constant time delay satisfies: $r > 0$, and t_0 is related to the initial time value t_{in} by $t_0 - r \geq t_{in}$. The impulses are assumed to be additive, as in [6]. The arrays in (1) are defined as follows:

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$$\mathbf{x}(t) \quad = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{3}$$

$$\mathbf{A}(t) = \begin{pmatrix} a_1(t) & -a_1(t) \\ -a_2(t) & a_2(t) \end{pmatrix} \tag{4}$$

$$\mathbf{B}(\mathbf{t}) \ = b_1 \cdot h(t) \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{5}$$

Here, b_1 is a constant, and the time interval T is the common period of the functions h(t), $a_1(t)$, $a_2(t)$.

This is given together with the initial condition

$$\begin{aligned} x_1(t) &= \phi_1(t) \\ x_2(t) &= \phi_2(t) \end{aligned} \tag{6}$$

for $\quad t_0 - r < t < t_0 \quad$, with $x_1(t_0) = m_1 \qquad x_2(t_0) = m_2$.

2 Solutions for the DDE

In order to simplify the treatment of the coupled equations presented above, we define the transformation

$$\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{7}$$

Then Eq. (1) leads to

$$\frac{d}{\mathrm{d}t} y_1(t) + (a_1(t) - a_2(t)) \cdot y_2(t) + 2 \cdot b_1 \cdot h(t) \cdot y_1(t-r) = 0 \tag{8}$$

$$\frac{d}{\mathrm{d}t} y_2(t) + (a_1(t) + a_2(t)) \cdot y_2(t) = 0 \tag{9}$$

There are still two coupled functions, but now only one function, $y_1(t)$, satisfies a DDE, whereas $y_2(t)$ satisfies an ODE. Impulses can be considered for each of these functions. The initial conditions for the two functions are

$$\begin{aligned} y_1(t) &= \tfrac{1}{2} (\phi_1(t) + \phi_2(t)) \\ y_2(t) &= \tfrac{1}{2} (\phi_1(t) - \phi_2(t)) \end{aligned} \tag{10}$$

for $\quad t_0 - r < t < t_0 \quad$, with $y_1(t_0) = \frac{1}{2}(m_1 + m_2) \qquad y_2(t_0) = \frac{1}{2}(m_1 - m_2)$

Using the notation

$$a(t) = (a_1(t) + a_2(t))$$

the function $y_2(t)$ is calculated (for $t_0 < t$) as

$$y_2(t) = exp\left\{-\int_{t_0}^{t} a(s)\cdot \mathrm{d}s\right\}\cdot y_2(t_0) \tag{11}$$

Note that if the initial conditions include $m_1 = m_2$, then $y_2(t_0) = 0$ so the function remains zero for all times. We shall assume here that $m_1 \neq m_2$ so that $y_2(t)$ is not trivial.

Proposition 1 *For the solution of Eq. (9), if*

$$\int_{t^+}^{t+T} a(s)\cdot \mathrm{d}s = 0$$

then the solution is periodic. Otherwise, if the solution is modified by adding for each t_k (k = 1, 2, ...) the impulse

$$I_{2(k)} = \{1 - exp\{-\int_{t_k^+}^{t_k+T} a(s)\cdot \mathrm{d}s\}\cdot y_2\left(t_k^+\right)\},$$

the resulting modified solution is periodic. If the function a(t) is continuous in the interval $[t_k, t_k + T]$, then the solution y(t) is bounded.

The periodicity is checked by the evolution of the solution between t_k and $t_{k+1} = t_k + T$:

$$y_2(t_k + T) = exp\{-\int_{t_k^+}^{t_k+T} a(s)\cdot \mathrm{d}s\}\cdot y_2\left(t_k^+\right) \tag{12}$$

In the trivial case where

$$\int_{t^+}^{t+T} a(s)\cdot \mathrm{d}s = 0,$$

the solution for $y_2(t)$ is already periodic, without any need for impulses. If

$$\int_{t^+}^{t+T} a(s)\cdot \mathrm{d}s > 0$$

and no impulses are applied, then the solution tends to zero for $t \to \infty$, so that the zero solution is stable, but there is no periodicity. If

$$\int_{t^+}^{t+T} a(s)\cdot \mathrm{d}s < 0$$

and no impulses are applied, the solution diverges as $t \to \infty$. In the last two cases, if the additive impulse

$$I_{2(k)} = \{1 - exp\{-\int_{t_k^+}^{t_k+T} a(s)\cdot \mathrm{d}s\}\cdot y_2\left(t_k^+\right)\} \tag{13}$$

is applied at the times $t_k = k \cdot T$, i.e.,

$$y_2\left(t_k{}^+\right) - y_2\left(t_k\right) = y_2\left(t_k\right) + I_{2(k)}, \tag{14}$$

then $y_2(t)$ is periodic. If the function $a(t)$ is continuous in each interval $[t_k, t_k + T]$, then $y_2(t)$ is bounded.

Equation (8) for $y_1(t)$ will be re-written as

$$\frac{d}{dt}y_1(t) + b(t) \cdot y_1(t-r) = -\left(a_1(t) - a_2(t)\right) \cdot y_2(t) \tag{15}$$

where $b(t) \equiv 2 \cdot b_1 \cdot h(t)$. In [4], Schauder's fixed point theorem is used in order to prove that if there exists a continuous function w(t) such that

$$\int_t^{t+T} b(s) \cdot w(s) \cdot \mathrm{d}s = 0 \qquad (for \quad t - r \geq t_0) \tag{16}$$

and also

$$\int_{t-r}^{t} b(s) \cdot w(s) \cdot \mathrm{d}s = ln\left(w(t)\right), \tag{17}$$

then there is a positive periodic solution to the homogeneous part of Eq. (15). A way to construct the solution is given in [4]. If this periodic solution is denoted by $y_0(t)$, then the solution to full Eq. (15) is

$$y_1(t) = y_0(t) + \int_{t_0}^{t} X(t,s) \cdot \{-\left(a_1(s) - a_2(s)\right)\} \cdot y_2(s) \cdot \mathrm{d}s \tag{18}$$

where $X(t, s)$ is the fundamental solution to Eq. (15) [7]. This solution evolves over one period of $y_0(t)$ as

$$\begin{aligned} &y_1(t+T) - y_0(t+T) = \\ &\qquad y_1(t) - y_0(t) + \int_{t^+}^{t+T} X(t,s) \cdot \{-\left(a_1(s) - a_2(s)\right)\} \cdot y_2(s) \cdot \mathrm{d}s \end{aligned} \tag{19}$$

Proposition 2 *For the equation as Eq.* (15) *above, if*

$$\int_{t^+}^{t+T} X(t,s) \cdot \{-\left(a_1(s) - a_2(s)\right)\} \cdot y_2(s) \cdot \mathrm{d}s = 0,$$

then the solution to the equation is positive and periodic. If the functions b(t) and a(t) are continuous in each interval $[t_k, t_k + T]$ *(with at most a finite number of finite discontinuities), then the solution is bounded.*

Note: If the integral in Eq. (19) is not zero, stability for Eq. (15) can hold if: (a) the integral tends to zero as $t \to \infty$ and (b) the equation for $y_0(t)$ is stable. The stability of $y_0(t)$ can be checked as in [8]. However, if the integral in Eq. (18) diverges for $t \to \infty$, then the equation for $y_1(t)$ is not stable, even if the equation for $y_0(t)$ is stable.

The original Equation (1) is solved (for $t_k < t \leq t_k + T$) by

$$x_1(t) = exp\{-\int_{t_k}^{t} a(s) \cdot \mathrm{d}s\} \cdot y_2(t_k) + y_0(t) + \int_{t_k}^{t} X(t,s) \cdot \{-(a_1(s) - a_2(s))\} \cdot y_2(s) \cdot \mathrm{d}s \tag{20}$$

$$x_2(t) = -exp\{-\int_{t_k}^{t} a(s) \cdot \mathrm{d}s\} \cdot y_2(t_k) + y_0(t) + \int_{t_k}^{t} X(t,s) \cdot \{-(a_1(s) - a_2(s))\} \cdot y_2(s) \cdot \mathrm{d}s \tag{21}$$

where the properties of the individual terms (y_2 and y_1) determine the properties of the original variables $x_1(t), x_2(t)$.

3 Examples

3.1 Example 1

Consider a delay of $r = 6\pi$ and the following functions:

$h(t) = \cos(t)$

$a_1(t) = c_0 + c_1 \cdot \cos(t)$, $a_2(t) = c_2 \cdot \cos(t)$ where c_0, c_1, c_2 are constants. For initial conditions, let us choose $m_1 \neq m_2$, so that $y_2(t_0) \neq 0$ and take $t_0 = 0$. Then for $0 < t$,

$$y_2(t) = exp\{-c_0 \cdot (t - t_0) - (c_1 + c_2) \cdot (\sin(t) - \sin(t_0))\} \cdot y_2\left(0^+\right) \tag{22}$$

As for $y_1(t)$, the solution for the homogenous equation of Eq. (15) can be obtained by choosing $w(t) = 1$, and then Eqs. (16) and (17) become

$$\int_{t}^{t+2\cdot\pi} 2b_1 \cdot \cos(s) \cdot \mathrm{d}s = 0$$

$$\int_{t-6\cdot\pi}^{t} 2b_1 \cdot \cos(s) \cdot \mathrm{d}s = 0$$

The solution for the homogeneous equation of $y_1(t)$ is

$$y_0(t) = exp\{2 \cdot b_1 \cdot (\sin(t_0) - \sin(t))\}$$

so that the fundamental solution is

$$X(t,s) = exp\{2 \cdot b_1 \cdot (\sin(s) - \sin(t))\}$$

The integral in Eq. (18) is

$$\int_{0^+}^{2\pi} X(t,s) \cdot \{-c_0 - (c_1 - c_2) \cdot \cos(s)\} \cdot y_2(s) \cdot \mathrm{d}s$$

3.1.1 Case 1.A

If $c_1 = c_2$, the only contribution to this integral in the $a_1 - a_2$ term is from c_0. Substituting in Eq. (18), one gets

$$y_1(t) = y_0(t) + \int_{t_0}^{t} X(t,s) \cdot \{-c_0\} \cdot exp\{-c_0 \cdot (s - t_0) - 2 \cdot c_1 \cdot \sin(s)\} \cdot \mathrm{d}s \cdot y_2(0) \tag{23}$$

The integral term $J \equiv y_1(t) - y_0(t)$ is equal (for $t_0 = 0$) to

$$J = -c_0 \cdot exp\{-2 \cdot b_1 \cdot \sin(t)\} \cdot \int_{t_0}^{t} exp\{-c_0 \cdot s + \sin(s) \cdot (2b_1 - 2c_1)\} \cdot \mathrm{d}s \cdot y_2(t_0) \tag{24}$$

The result of the integral is a non-periodic function, so calculating the integral between the limits: t_k and $t_k + T$ will not give zero. In the special case $b_1 = c_1$, the integral term is much simpler, but still the result is not periodic.Thus, the function $y_1(t)$ is not periodic, unlike $y_0(t)$. Then the original variables $x_1(t)$ and $x_2(t)$ are a combination of a periodic part (that of $y_2(t)$ and $y_0(t)$) and a non-periodic part (J). If one adds an impulse to $y_1(t)$:

$$I_{1(k)} = \{y_1(t_k) - y_0(t_k)\} - \{y_1(t_k + T) - y_0(t_k + T)\},$$

this will correct the value of the function only for a single time point. Due to the dependence on the time delay, the behavior of the function for the next time interval will in general be different from that in the previous interval, so $y_1(t)$ will remain non-periodic. Therefore, in both cases, $b_1 \neq c_1$ and $b_1 = c_1$, the solution diverges for $t \to \infty$.

3.1.2 Case 1.B

Now assume $c_1 \neq c_2$ and $c_0 = 0$. Now the integral term is equal to

$$\begin{aligned} J &= \int_{t_0}^{t} X(t,s) \cdot \{-(c_1 - c_2) \cdot \cos(s)\} \cdot exp\{-(c_1 + c_2) \cdot \sin(s)\} \cdot \mathrm{d}s \cdot y_2(0) \\ &= -(c_1 - c_2) \cdot exp\{-2 \cdot b_1 \cdot \sin(t)\} \cdot \\ &\quad \int_{t_0}^{t} \cos(s) exp\{\sin(s) \cdot (2b_1 - (c_1 + c_2))\} \cdot \mathrm{d}s \cdot y_2(t_0) \end{aligned} \tag{25}$$

If $2b_1 \neq c_1 + c_2$, then the expression above is equal to

$$\begin{aligned} J = &-(c_1 - c_2) \cdot exp\{-2 \cdot b_1 \cdot \sin(t)\} \cdot \\ &\frac{1}{(2b_1 - (c_1 + c_2)))} \cdot \\ &\{exp\{(2 \cdot b_1 - (c_1 + c_2)) \cdot \sin(t)\} - exp\{(2 \cdot b_1 - (c_1 + c_2)) \cdot \sin(t_0)\}\} \cdot y_2(t_0) \end{aligned}$$

and this is a periodic function, so that also $y_1(t)$ is periodic. If $2b_1 = c_1 + c_2$, then the expression is

$$\begin{aligned} J &= -(c_1 - c_2) \cdot exp\{-2 \cdot b_1 \cdot \sin(t)\} \cdot \int_{t_0}^{t} \cos(s) \cdot \mathrm{d}s \cdot y_2(t_0) \\ &= -(c_1 - c_2) \cdot exp\{-2 \cdot b_1 \cdot \sin(t)\} \cdot (\sin(t) - \sin(t_0)) \end{aligned}$$

which is also periodic. Thus, regardless of the value of b_1, the solution is periodic, both for $x_1(t)$ and for $x_2(t)$.

3.2 Example 2

With the time delay: $r = \frac{\pi}{2}$, consider the following functions:

$$h(t) = -\sin(t) \cdot exp\{2b_1 \cdot (\sin(t) - \cos(t))\}$$

and $a_1(t)$, $a_2(t)$ as in the previous example. Then $y_2(t)$ is the same as above, and for $y_1(t)$, we define the function $w(t) = exp\{2b_1 \cdot (\cos(t) - \sin(t))\}$.

Then Eqs. (16) and (17) become

$$-\int_{t}^{t+2\cdot\pi} 2b_1 \cdot \sin(s) \cdot \mathrm{d}s = 0$$

$$-\int_{t-\frac{\pi}{2}}^{t} 2b_1 \cdot \sin(s) \cdot \mathrm{d}s = 2b_1 \cdot (\cos(t) - \sin(t))$$

Now the solution for the homogenous equation of Eq. (15) is

$$y_0(t) = exp\{2 \cdot b_1 \cdot (\cos(t_0) - \cos(t))\}$$

so that the fundamental solution is

$$X(t, s) = exp\{2 \cdot b_1 \cdot (\cos(s) - \cos(t))\}$$

3.2.1 Case 2.A

If $c_1 = c_2$, the only contribution to this integral is from c_0.

Substituting in Eq. (17), one gets

$$y_1(t) = y_0(t) + \int_{t_0}^{t} X(t, s) \cdot \{-c_0\} \cdot exp\{-c_0 \cdot (t - t_0) - 2 \cdot c_1 \cdot \sin(s)\} \cdot \mathrm{d}s \cdot y_2(0) \tag{26}$$

The integral term $J \equiv y_1(t) - y_0(t)$ is equal to

$$J = - c_0 \cdot exp\{-c_0 \cdot (t - t_0) - 2 \cdot b_1 \cdot \cos(t)\} \cdot \int_{t_0}^{t} exp\{(2b_1 \cdot \cos(s) - 2c_1 \cdot \sin(s))\} \cdot \mathrm{d}s \cdot y_2(t_0) \tag{27}$$

The result of the integral is a non-periodic function, so calculating the integral between the limits: t_k and $t_k + T$ will not give zero. In fact, for this case,

$$\int_{0}^{2\pi} exp\{(2b_1 \cdot \cos(s) - 2c_1 \cdot \sin(s))\} \cdot \mathrm{d}s = 2\pi I_0(\sqrt{4 \cdot (b_1)^2 + 4 \cdot (c_1)^2})$$

where $I_0(x)$ is the modified Bessel function of order zero.

Thus, the function $y_1(t)$ is not periodic, unlike $y_0(t)$. Then the original variables $x_1(t)$ and $x_2(t)$ are a combination of a periodic part (that of $y_2(t)$ and $y_0(t)$) and a non-periodic part (J).

3.2.2 Case 2.B

Now assume $c_1 \neq c_2$ and $c_0 = 0$. Now the integral term is equal to

$$\begin{aligned} J &= \int_{t_0}^{t} X(t, s) \cdot \{-(c_1 - c_2) \cdot \cos(s)\} \cdot exp\{-(c_1 + c_2) \cdot \sin(s)\} \cdot \mathrm{d}s \cdot y_2(0) \\ &= -(c_1 - c_2) \cdot exp\{-2 \cdot b_1 \cdot \cos(t)\} \cdot \\ &\quad \int_{t_0}^{t} \cos(s) exp\{(2b_1 \cdot \cos(s) - (c_1 + c_2) \cdot \sin(s))\} \cdot \mathrm{d}s \cdot y_2(t_0) \end{aligned} \tag{28}$$

The result of the integration is not a periodic function. Also, for the special case $(c_1 + c_2) = 0$, the result is not periodic, and in that case,

$$\int_0^{2\pi} \cos(s) \cdot exp\{2b_1 \cdot \cos(s)\} \cdot \mathrm{d}s = 2\pi I_1(2b_1)$$

where $I_1(x)$ is the modified Bessel function of order one. Thus, regardless of the value of b_1, the solution is not periodic, both for $x_1(t)$ and for $x_2(t)$. The solutions diverge for $t \to \infty$.

3.3 Example 3

With the time delay: $r = \frac{\pi}{2}$ consider the following functions:

$$h(t) = -\sin(t) \cdot exp\{2b_1 \cdot (\sin(t) - \cos(t))\}$$

and

$a_1(t) = c_0 + c_1 \cdot \sin(t)$, $a_2(t) = c_2 \cdot \sin(t)$ where c_0, c_1, c_2 are constants. Then for $0 < t$,

$$y_2(t) = exp\{-c_0 \cdot (t - t_0) + (c_1 + c_2) \cdot \cos(t)\} \cdot y_2\left(0^+\right) \tag{29}$$

and for $y_1(t)$, we define the function $w(t) = exp\{2b_1 \cdot (\cos(t) - \sin(t))\}$.

Then Eqs. (16) and (17) are the same as in Example 2 above,

and also the solution for the homogenous equation of Eq. (15) and consequently the fundamental solution are the same as in Example 2 above.

3.3.1 Case 3.A

If $c_1 = c_2$, the only contribution to this integral is from c_0.

Substituting in Eq. (17), one gets

$$y_1(t) = y_0(t) + \int_{t_0}^{t} X(t, s) \cdot \{-c_0\} \cdot exp\{-c_0 \cdot (s - t_0) + 2 \cdot c_1 \cdot \cos(s)\} \cdot \mathrm{d}s \cdot y_2(0) \tag{30}$$

The integral term $J \equiv y_1(t) - y_0(t)$ is equal to

$$J = -c_0 \cdot exp\{c_0 \cdot (t_0) - 2 \cdot b_1 \cdot \cos(t)\} \cdot \int_{t_0}^{t} exp\{c_0 \cdot s + (2b_1 \cdot \cos(s) + 2c_1 \cdot \cos(s))\} \cdot \mathrm{d}s \cdot y_2(t_0) \tag{31}$$

Thus, the function $y_1(t)$ is not periodic, unlike $y_0(t)$.

3.3.2 Case 3.B

Now assume $c_1 \neq c_2$ and $c_0 = 0$. Now the integral term is equal to

$$\begin{aligned} J &= \int_{t_0}^{t} X(t,s) \cdot \{-(c_1 - c_2) \cdot \sin(s)\} \cdot exp\{-(c_1 + c_2) \cdot \cos(s)\} \cdot \mathrm{d}s \cdot y_2(0) \\ &= -(c_1 - c_2) \cdot exp\{-2 \cdot b_1 \cdot \cos(t)\} \cdot \\ &\quad \int_{t_0}^{t} \sin(s) exp\{(2b_1 \cdot \cos(s) + (c_1 + c_2) \cdot \cos(s))\} \cdot \mathrm{d}s \cdot y_2(t_0) \end{aligned} \tag{32}$$

If $2b_1 + c_1 + c_2 \neq 0$, then the expression above is equal to

$$\begin{aligned} J &= +(c_1 - c_2) \cdot exp\{-2 \cdot b_1 \cdot \sin(t)\} \cdot \\ &\quad \frac{1}{(2b_1 + (c_1 + c_2)))} \cdot \\ &\quad \{exp\{(2 \cdot b_1 + (c_1 + c_2)) \cdot \cos(t)\} - exp\{(2 \cdot b_1 + (c_1 + c_2)) \cdot \cos(t_0)\}\} \cdot y_2(t_0) \end{aligned}$$

and this is a periodic function, so that also $y_1(t)$ is periodic. If $2b_1 = c_1 + c_2$, then the expression is

$$\begin{aligned} J &= -(c_1 - c_2) \cdot exp\{-2 \cdot b_1 \cdot \cos(t)\} \cdot \int_{t_0}^{t} \sin(s) \cdot \mathrm{d}s \cdot y_2(t_0) \\ &= +(c_1 - c_2) \cdot exp\{-2 \cdot b_1 \cdot \cos(t)\} \cdot (\cos(t) - \cos(t_0)) \end{aligned}$$

which is also periodic. Thus, regardless of the value of b_1, the solution is periodic, both for $x_1(t)$ and for $x_2(t)$.

References

1. Hale, J.K., Verdyun-Lunel, S.M.: Introduction to Functional Differential Equations. Springer, New York (1993)
2. Graef, J.R., Kong, L.: Periodic solutions of first order functional differential equations. Appl. Math. Lett. **24**, 1981–1985 (2011)
3. Li, X., Bohner, M., Wang, C.-K.: Impulsive differential equations: periodic solutions and applications. Automatica **52**, 173–178 (2015)
4. Olach, R.: Positive periodic solutions of delay differential equations. Appl. Math. Lett. **26**, 1141–1145 (2013)
5. Faria, T., Oliviera, J.J.: Existence of positive periodic solutions for scalar delay differential equations with and without impulses. J. Dyn. Diff. Equat. **31**, 1223–1245 (2019)

6. Federson, M., Györi, I., Mesquita, J.G., Táboas, P.: A delay differential equations with an impulsive self-support condition. J. Dyn. Diff. Equ. (2019). https://doi.org/10.1007/s10884-019-09750-5
7. Agarwal, R.P., Berezansky, L., Braverman, E., Domoshnitsky, A.: Nonoscillation Theory of Functional Differential Equations with Applications. Springer, New York (2012)
8. Berezansky, L., Braverman, E.: On exponential stability of a linear delay differential equation with an oscillating coefficient. Appl. Math. Lett. **22**, 1833–1837 (2009)